Measure-theoretic Probability

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1. Introduction

The goal of this set of notes is to understand the rudiments of measure theory and specifically its relation to probability and statistics. The layout of these notes follows the form of definition or theorem followed by lots of examples and proofs. I pulled and solved several problems from different reference texts below - some answers may not be complete or wrong - let me know about changes. To learn along with these notes, I suggest the following:

1. Grab two measure theory references - a problem book/site (hopefully with complete solutions) and a reference text. For problems, I found many analysis qualifier exams with solutions online and used several of the below mentioned texts. The complete reference should really be one of the following: Folland's "Real Analysis: Modern Techniques and their Applications", Bogachev's "Measure Theory Volume 1", Halmos' "Measure Theory", Rudin's "Real and Complex Analysis", Schilling's "Measures, Integrals and Martingales", or maybe (but seriously pushing it) Billingsley's "Probability and Measure". Personally, I used the first few chapters of Halmos for topics on algebras, σ-algebras, rings and measures. Then transitioned to Bogachev and Folland for talk of measures (outer, inner, and extensions) and related theorems, then started using Rudin for learning measurable mappings and integration. For martingales and convergence, Billingsley and Schilling's books were good enough for starting. All of these can be found online if you know the site libgen.

2. Grab a reference probability text with measure theory and lucid examples/problems: Billingsley's "Probability and Measure", Durrett's "Probability Theory and Examples", or Çinlar's "Probability and Stochastics". These can also easily be found online.

3. How to learn with this set of notes: generate your own list of examples after reading definitions and theorems from the reference text - look for edge cases and counterexamples to really understand the limits of the concept. If you get stuck creating examples, that is the purpose of this document - to provide you with ideas and get you back on track. By creating many examples, you'll begin to pick up on various tactics and build on those learned in real analysis. Measure theory is much more difficult and I recommend constantly relearning the basic concepts, doing problem sets, and posting your work to problems online for feedback or asking a professor you know.

2. Algebras, σ-Algebras, and Measures

This section will introduce the concept of algebras, σ-algebras, as well as constructions and extensions of measures. Reference/readings: Halmos Chapter 1 and 2, Bogachev Chapter 1, Folland Chapter 1 and 2, Billingsley Chapter 1 Sections 2, 2 and Chapter 2 Sections 10-12.

Definition In probability theory, Ω denotes the set of all the possible outcomes, ω, of an experiment.

Examples:

- In observing the number of heads in one toss of a coin, $\Omega = \{0, 1\}$
- In observing the number of cars passing by a stop sign, $\Omega = \mathbb{N}$
- In observing the position of an airplane, $\Omega \subset \mathbb{R}^3$

Definition Let Ω be an arbitrary nonempty space. A class $\mathcal{F}$ of subsets of Ω is called a field or algebra (doesn't matter) if it contains Ω itself and is closed under the formation of complements and **finite** unions.

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_1, \ldots, A_n \in \mathcal{F}$, then $\bigcup\limits_{i=1}^{n} A_i \in \mathcal{F}$

Examples:

- Let $\Omega = \mathbb{N}$. Then the power set, $\mathcal{P}(\mathbb{N})$, is an algebra. The power set includes all subsets by construction - so $\Omega \in \mathcal{P}(\mathbb{N})$ and any union (finite or infinite) of members of Ω is contained in the power set. Additionally, the power set is closed under complements (again by definition) as $A \in \mathcal{P}(\mathbb{N}) \Rightarrow A^c \in \mathcal{P}(\mathbb{N})$.
- Let $\Omega = [0, 1]$. Then the set, $S = \{[m, \frac{n}{n+1}] : n \in \mathbb{N}, m \leq \frac{n}{n+1}\}$, is not an algebra since while it is closed under finite intersections, it fails to satisfy the first two properties since $\Omega = [0, 1]$ is not contained in the set as 1 is not in S. There are many more gaps in this class of subsets, but to show it is not an algebra, it suffices to find just one. This example helps show that (1) isn't really necessary in the presence of (2) (i.e. (2) implies (1) but not the converse unless Ω is a singleton (set of one element)).
- Let $\Omega = \{0, 1\}$ and $S = \{\{0\}, \{1\}, \{0, 1\}\}$. Then $\Omega \in S$ and S is closed under unions, but the complement of $\{0, 1\}$ is the empty set which isn't contained in S and so S is not an algebra.
- A class $\mathcal{F}$ of subsets of Ω is called a semi-algebra (or semi-field) if
  1. $\emptyset \in \mathcal{F}$
  2. $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$
  3. For $A \in \mathcal{F}$, there exist finitely many (possibly empty) disjoint sets $E_1, \ldots, E_n \in \mathcal{F}$ such that $\Omega - A = \bigcup_{i=1}^{n} E_i$.

  Every algebra is a semi-algebra (but not the reverse direction - take $\Omega = \{0, 1, 2\}$ with $\mathcal{F} = \{\{0\}, \{1\}, \{2\}, \Omega, \emptyset\}$). To see that every algebra is a semi-algebra, clearly $\Omega^c = \emptyset \in \mathcal{F}$. If $A, B \in \mathcal{F}$ then A^c and B^c are in $\mathcal{F}$ so $A^c \cup B^c = (A \cap B)^c \in \mathcal{F}$ so $A \cap B \in \mathcal{F}$ by closure under complementation. For 3., $\Omega - A = A^c \cup \emptyset$ and so $\mathcal{F}$ is also a semi-algebra.

- Every algebra/field is a **ring**. A ring is a class of sets closed under finite unions and differences. Put simply, an algebra is a ring containing Ω.

- The smallest algebra containing a specified class of sets E is called the algebra generated by E and is defined as the intersection of all algebras containing E. Similarly, for rings, we let $\mathcal{R}(E)$ denote the ring generated by the class E. Later on, we will introduce the same concept for σ-algebras - denoted by $\sigma(E)$. As an exercise, try proving that if E is countable, then the algebra generated by E is also countable.

- If we fix $A \subset \Omega$, then define the class of sets $E = \{A\}$. The algebra generated by E is given by $\mathcal{F}(E) = \{\emptyset, \Omega, A, A^c\}$.

- Let $\Omega = \{0, 1\}$ and $S = \{\emptyset, \{0\}, \{1\}\}$. Then S is clearly closed under the formation of intersections and differences ($A - B = A \cap B^c$); however, $\{0\} \cup \{1\} = \Omega \notin S$ so it is not closed under finite unions nor possess the space itself.

- Let Ω be an uncountable set. Then the class of countable subsets of Ω does not form an algebra, but does form a ring. To see that it doesn't satisfy the former, note that the finite union or intersection of countable sets is still countable and so the class could never contain the entire space Ω which is uncountable. The properties of a ring are met using the fact that intersections of countable sets are countable and the intersection of a countable and uncountable set is countable.

- Halmos 5.A: If E is any class of sets, then every set in $\mathcal{R}(E)$ may be covered by a finite union of sets in E. Here $\mathcal{R}(E) = \cap_n R_n$ where $E \subset R_n$ for all n - this is referred to as the ring generated by the class E. The proof of this statement emphasizes a recurring theme - to show that all $A \in \mathcal{R}(E)$ have property P, we define X to be the class of sets in $\mathcal{R}(E)$ with property P. Now, if we can show that $E \subset X$ and that X is a ring, then since $\mathcal{R}(E)$ is the smallest ring containing E, we can conclude that $\mathcal{R}(E) \subset X$. In our case, $X = \{A : A = \bigcup_n B_n, B_n \in E\}$ and we just have to show this 1) contains E, 2) is closed under intersections, and 3) is closed under differences.

- Billingsley 2.3(a): Suppose that $\Omega \in \mathcal{F}$ and $A, B \in \mathcal{F}$ implies $A - B = A \cap B^c \in \mathcal{F}$. Then $\mathcal{F}$ is clearly nonempty as $\Omega \in \mathcal{F}$. $\mathcal{F}$ is closed under complementation as $\Omega, A \in \mathcal{F}$ implies that $\Omega \cap A^c = A^c \in \mathcal{F}$ for $A \subset \Omega$. Then since the class is closed under complementation and finite intersections, the class is also closed under finite unions. Thus $\mathcal{F}$ is an algebra.

- Billingsley 2.3(b): Suppose that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed under the formation of complements and finite disjoint unions. Then $\mathcal{F}$ is not necessarily closed under finite unions i.e. there can be $A, B \in \mathcal{F} : A \cap B \neq \emptyset$ but $A \cup B \notin \mathcal{F}$. For a concrete example, look at $\Omega = \{0, 1, 2, 3\}$ with $\mathcal{F} = \{\{0, 1\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{0, 1, 2, 3\}, \emptyset\}$.

- Billingsley 2.4(a): Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be classes of sets in a common space Ω. Suppose that $\mathcal{F}_n$ are algebras satisfying $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Then we will show that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a algebra. If $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, then since the classes are increasing, there is some index j such that $A \in \mathcal{F}_j$ and $A \in \mathcal{F}_k$ for $k > j$. Then $A^c \in \bigcup_{n=k}^{\infty} \mathcal{F}_n = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ so it is closed under the formation of complements. A similar argument argument can be applied to guarantee the closure under finite unions. This sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is known as a filtration in the study of stochastic processes/probability - will be prevalent later on in these notes.

- Billingsley 2.5(b): Define

$$X = \left\{ \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} A_{ij} | \left(\bigcap_{j=1}^{n} A_{pj} \cap \bigcap_{j=1}^{n} A_{qj} \right) = \emptyset \text{ for } p \neq q,\ A_{ij} \in \mathcal{A} \text{ or } A_{ij}^c \in \mathcal{A} \right\}$$

Then define $f(\mathcal{A})$ as the field generated by a class $\mathcal{A} \subset \Omega$. Then since $f(\mathcal{A})$ is a field containing $\mathcal{A}$, it is closed under complementation and finite unions and intersections of all elements in $\mathcal{A}$. The set X above is a finite union of finite intersections of elements (and complements of elements) in $\mathcal{A}$ and thus must be contained in $f(\mathcal{A})$ (i.e. $X \subset f(\mathcal{A})$). To show that $X = f(\mathcal{A})$, now it suffices to show now that X is a field containing $\mathcal{A}$ and by definition of the minimal field, $f(\mathcal{A})$, $f(\mathcal{A}) \subset X$. Let $M, N \in X$ with

$$M = \bigcup_{i=1}^{m_1} \bigcap_{j=1}^{n_1} A_{ij} \text{ and } N = \bigcup_{i=1}^{m_2} \bigcap_{j=1}^{n_2} A'_{ij}$$

Then

$$M \cap N = \left(\bigcup_{i=1}^{m_1} \bigcap_{j=1}^{n_1} A_{ij} \right) \cap \left(\bigcup_{i=1}^{m_2} \bigcap_{j=1}^{n_2} A'_{ij} \right)$$

$$= \left(\bigcup_{i=1}^{m_2} \left[\left(\bigcap_{j=1}^{n_1} A_{1j} \right) \cap \left(\bigcap_{j=1}^{n_2} A'_{ij} \right) \right] \right) \cup \cdots \cup \left(\bigcup_{i=1}^{m_2} \left[\left(\bigcap_{j=1}^{n_1} A_{m_1 j} \right) \cap \left(\bigcap_{j=1}^{n_2} A'_{ij} \right) \right] \right).$$

Now, note that since $\bigcap_{j=1}^{n_2} A'_{pj} \cap \bigcap_{j=1}^{n_2} A'_{qj} = \emptyset$, that for all $t \in \{1, \ldots, m_1\}$

$$\left(\bigcap_{j=1}^{n_1} A_{tj} \cap \bigcap_{j=1}^{n_2} A'_{pj} \right) \cap \left(\bigcap_{j=1}^{n_1} A_{tj} \cap \bigcap_{j=1}^{n_2} A'_{qj} \right) = \bigcap_{j=1}^{n_1} A_{tj} \cap \left(\bigcap_{j=1}^{n_2} A'_{pj} \cap \bigcap_{j=1}^{n_2} A'_{qj} \right) = \bigcap_{j=1}^{n_1} A_{tj} \cap (\emptyset) = \emptyset$$

so each of the m_2 sets of intersections are pairwise disjoint - so $M \cap N$ is a class of set of the form described in X. Thus $M \cap N \in X$ - X is closed under finite intersections. To show it's closed under complementation,

$$M^c = \left(\bigcup_{i=1}^{m_1} \bigcap_{j=1}^{n_1} A_{ij} \right)^c = \bigcap_{i=1}^{m_1} \bigcup_{j=1}^{n_1} A_{ij}^c = \bigcap_{i=1}^{m_1} A_{i1}^c \cup (A_{i2}^c \cap A_{i1}) \cup \cdots \cup (A_{in_1}^c \cap A_{in_1 - 1} \cap \cdots \cap A_{i1})$$

where the terms are all clearly still in $\mathcal{A}$ (or their complement is) and $A_{i1}^c \cap (A_{i2}^c \cap A_{i1}) = \emptyset$ and this is true for all these terms under the union (they're pairwise disjoint) - thus this term is contained in X. As X is closed under finite intersections, M^c is contained in X. Finally, we have shown that X is an algebra (X is nonempty so we only needed two conditions to check) and so $f(\mathcal{A}) = X$.

- If A is any class of sets in the space X, then every set in $\mathcal{F}(A)$, the algebra generated by the class A, may be covered by a finite union of sets in A. To prove this, we employ the good set principle: let

$$\mathcal{C} = \left\{ E \in X : E \subset \bigcup_n A_n,\ A_n \in A \right\}$$

and we show this is an algebra containing A which proves the statement.

Definition A class $\mathcal{F}$ of subsets of Ω is a σ-algebra if it is a algebra and is also closed under the formation of countable unions. We say that Ω is a measurable space if such a class exists.

Examples:

- Let $\mathcal{F}$ be a σ-algebra in a space Θ. If $\theta \in \Theta$ but not in $\mathcal{F}$, then we can construct $\sigma(\mathcal{F} \cup \theta)$, the σ-algebra generated by union of the class $\mathcal{F}$ and the set θ, by

$$\{(A \cap \theta) \cup (B \cap (\Theta \setminus \theta)) |\, A, B \in \mathcal{F}\}$$

To prove that this set, let's call it X, is indeed the σ-algebra generated by the union of $\mathcal{F}$ and θ, it suffices to show that $X \subset \sigma(\mathcal{F} \cup \theta)$ and $\sigma(\mathcal{F}, \theta) \subset X$. Since $\sigma(\mathcal{F}, \theta)$ is the minimal σ-algebra, we prove the latter inequality by showing that X is a σ-algebra. Let $A, B = \Omega$, then $(\Omega \cap \theta) \cup (\Omega \cap \theta^c) = \Omega$. It is also closed under countable unions using induction on the following tactic (as $\mathcal{F}$ is closed under countable intersections): let $I = (A \cap \theta) \cup (B \cap \theta^c)$ and $J = (C \cap \theta) \cup (D \cap \theta^c)$ for $A, B, C, D \in \mathcal{F}$. Then

$$I \cap J = ((A \cap \theta) \cup (B \cap \theta^c)) \cap ((C \cap \theta) \cup (D \cap \theta^c))$$

$$= ((A \cap \theta) \cap (C \cap \theta)) \cup ((A \cap \theta) \cap (D \cap \theta^c)) \cup ((B \cap \theta^c) \cap (C \cap \theta)) \cup ((B \cap \theta^c) \cap (D \cap \theta^c))$$

$$= ((A \cap C) \cap \theta) \cup ((B \cap D) \cap \theta^c)$$

which is in X since $(A \cap C) \in \mathcal{F}$ and $(B \cap D) \in \mathcal{F}$ as $\mathcal{F}$ is closed under countable union. As $\mathcal{F}$ is closed under complementation, we clearly have that $I^c = (A^c \cap \theta) \cup (B^c \cap \theta^c) \in X$. Thus X is an algebra and $\sigma(\mathcal{F} \cup \theta) \subset X$. The reverse direction is obvious as X consists of a union of intersections of elements in $\sigma(\mathcal{F} \cup \theta)$, so it has to be contained in the algebra, thus $X = \sigma(\mathcal{F} \cup \theta)$.

- A space X is called measurable if there exists a σ-algebra, $\mathcal{F}$, in X. It's more formal to call a measurable space by the tuple $(X, \mathcal{F})$ since the σ-algebras are usually not unique.

- Halmos 5.D (variant - I consider σ-algebras, not rings here): Let E be any class of sets and $A \subset \Omega$. Then we will prove that

$$\sigma(E) \cap A = \sigma(E \cap A).$$

Clearly $\sigma(E \cap A) \subset \sigma(E) \cap A$ since $E \cap A \in \sigma(E) \cap A$ and $\sigma(E \cap A)$ is the minimal σ-algebra containing $E \cap A$. To prove the reverse direction, define the following class of sets

$$\mathcal{L} = \{B \in \Omega : B \cap A \in \sigma(E \cap A)\}.$$

We will prove this is a σ-algebra containing $\sigma(E)$. For any $A \in \mathcal{L}$,

$$(E \cap A) \cup (A - E) = A \in \sigma(E \cap A)$$

and since $A = \Omega \cap A$, we have that $\Omega \in \mathcal{L}$. If $B \in \mathcal{L}$, then

$$B^c \cap A = ((B \cap A) \cup (A - B))^c \cap (B \cap A) \cup (A - B) \in \sigma(E \cap A)$$

so $\mathcal{L}$ is closed under complementation. Finally, if $B_1, \cdots \in \mathcal{L}$, then

$$\bigcup_n B_n \cap A = A \cap \bigcup_n B_n \in \sigma(E \cap A)$$

so $\bigcup_n B_n \in \mathcal{L}$ which implies $\mathcal{L}$ is a σ-algebra. Since for all $B \in E$, we have $B \cap A \in E \cap A \subset \sigma(E \cap A)$, we know $E \subset \mathcal{L}$ and thus $\sigma(E) \subset \mathcal{L}$. Note that $\mathcal{L} \cap A = \sigma(E \cap A)$, so

$$\sigma(E) \cap A \subset \mathcal{L} \cap A = \sigma(E \cap A)$$

which proves the reverse direction we needed (that $\sigma(E) \cap A \subset \sigma(E \cap A)$ so $\sigma(E) \cap A = \sigma(E \cap A)$).

- Let $\Omega = [0, 1)$ and S be the set of all finite, disjoint unions of subintervals of Ω of the form $\bigcup_{i=1}^{n} [a_i, b_i)$ and the empty set. Then for $A = [a_1, b_1) \cup \cdots \cup [a_n, b_n) \in S$, $A^c = [0, a_1) \cup \cdots \cup [b_n, 1)$ which lies in S as S includes **all** finite, disjoint unions of subintervals. Similarly, let $B = [a'_1, b'_1) \cup \cdots \cup [a'_m, b'_m) \in S$, then

$$A \cup B = (A^c \cap B^c)^c = \big(([0, a_1) \cup \cdots \cup [b_n, 1)) \cap \big([0, a'_1) \cup \cdots \cup [b'_m, 1)\big)\big)^c$$

$$= ([0, a_1) \cap B^c) \cup \cdots \cup ([b_n, 1) \cap B^c)^c$$

where each intersection is clearly either empty or still a half-open interval and these must also be disjoint since otherwise, there would exist a $[b_i, a_i) \cap [b_j, a_j) \neq \emptyset$ so A^c wouldn't be disjoint but it's closed under complementation - a contradiction. Finally, the complement of the union of these is in S since we already showed it's closed under complementation. Hence S is an algebra. However, it's not a σ-algebra! Define $a_n = \frac{1}{n}$ and $b_n = 1$, then

$$\lim_{n \to \infty} \bigcup_{i=1}^{n} [\frac{1}{i}, 1) = (0, 1) \notin S.$$

- Let $\mathcal{A}$ be a σ-algebra on X and $f : X \to Y$. The class of sets $\mathcal{F} = \big\{A \mid f^{-1}(A) \in \mathcal{A},\ A \subset Y\big\}$ is a σ-algebra on Y (sometimes called the pull back σ-algebra). Why is it a σ-algebra? Well, if f is bijective (both Rudin and Bogachev just define these as surjective maps - I guess it's implicit), then $f^{-1}(Y) = X$, so $Y \in \mathcal{F}$. If $f^{-1}(A_1) \in \mathcal{A}, f^{-1}(A_2) \in \mathcal{A}, \ldots,$ clearly

$$f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$$

which is contained in $\mathcal{A}$ as it's closed under countable intersections. For complementation, we have

$$f^{-1}(A^c) = f^{-1}(Y \cap A^c) = f^{-1}(Y) \cap f^{-1}(A^c) = X \cap f^{-1}(A)^c$$

which is in $\mathcal{A}$ as it's a σ-algebra. Thus $\mathcal{F}$ is a σ-algebra on Y.

- Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. A mapping $f : X \to Y$ is measurable if $f^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{G}$. The mapping, f, is Borel measurable if f is continuous (Y is a topological space) and $f^{-1}(A) \in \mathcal{F}$ for every open set $A \in \mathcal{G}$. Rudin makes the great point in "Real and Complex Analysis" by noting the very interesting analogue between measurable functions in measurable spaces to continuous functions in metric spaces. That is, a function is continuous in a metric or topological space iff every open set gets mapped to an open set while a measurable function is one where every measurable set gets mapped to another measurable set.

- Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, $h : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$. We will show that $h = g \circ f$ is also measurable. To do so, we must show that $(g \circ f)^{-1}(A) = f^{-1} \circ g^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}$. This is trivial if we remember the invariance of domain theorem, called Theorem 4.8 in Rudin's "Principles of Mathematical Analysis". That is, since g is continuous, we know that $g^{-1}(A)$ is open for every open set $A \in \mathbb{R}$. Hence

$$h^{-1}(A) = f^{-1} \circ g^{-1}(A) = f^{-1}(B)$$

  for some open subset $B \in \mathbb{R}$. Now since f is measurable and $B \in \mathcal{B}$, we know $f^{-1}(B) \in \mathcal{B}$ and so the composition of a Borel measurable function with a continuous function is also Borel measurable.

- As mentioned above, the power set of a space is clearly a σ-algebra for that space - it is in fact the largest σ-field. On a similar note, the smallest σ-algebra of a space is the space itself and the empty set (rather useless).

- Let $\mathcal{F}$ be a σ-algebra. If $A, B \in \mathcal{F}$, then $A^c, B^c \in \mathcal{F}$ so $A - B = A \cup B^c \in \mathcal{F}$. It then also follows that the symmetric difference of any two elements of $\mathcal{F}$ is also contained in the algebra: $A \Delta B = (A - B) \cup (B - A) \in \mathcal{F}$.

- Let $\Omega = (0, 1]$ and $\mathcal{A}$ denote the class of subintervals of $(0, 1]$. We will define $\sigma(\mathcal{A})$ to denote the class of intersections of $\sigma-$algebras containing $\mathcal{A}$. Let $\mathcal{F} = \bigcap(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ be these σ-algebras. Then clearly if $\mathcal{A} \in \mathcal{F}$, then $\mathcal{A} \in \mathcal{F}_1, \ldots, \mathcal{A} \in \mathcal{F}_n$ implies that $\mathcal{A}^c \in \mathcal{F}_1, \ldots, \mathcal{F}_n$ implies that $\mathcal{A}^c \in \bigcap(\mathcal{F}_1, \ldots, \mathcal{F}_n)$. Similarly, it's obvious that it's also closed under countable unions and nonempty by construction so $\sigma(\mathcal{A})$ is a σ-algebra. It's also easy to see that this is the smallest σ-algebra containing $\mathcal{A}$.

- Let X be a topological space. Consider the σ-algebra, $\mathcal{B}$, generated by the class of open sets in X. We say the members of this σ-algebra are the Borel sets of X. As the σ-algebra is closed under complementation, we know closed sets are Borel sets and similarly the countable union of closed sets and the countable intersection of open sets. For continuous mappings $f : X \to Y$, we know that $f^{-1}(V)$ is open for every open set V in Y and thus $f^{-1}(V) \in \mathcal{B}$ - in this case, we call f a Borel function/Borel measurable.

- Let $\mathcal{F}$ be a σ-field over Ω. We will show the cardinality of $\mathcal{F}$ is restricted to be finite or that of the continuum. Assume the contrary, that the cardinality of $\mathcal{F}$ is countably infinite. Now let us construct a sequence of sets and σ-fields. For any $A \in \mathcal{F}$, where $A \notin \{\emptyset, \Omega\}$, define
$$A \cap \mathcal{F} = \{A \cap X | X \in \mathcal{F}\}.$$
This is a σ-field - let $X = \emptyset$, then $A \cap \mathcal{F} = \emptyset$. It is closed under complementation as
$$A \cap (A \cap X)^c = (A \cap A^c) \cup (A \cap X^c) = A \cap X^c$$
It is also closed under countable unions as
$$\bigcup_{i=1}^{\infty} A \cap X_i = A \cap \bigcup_{i=1}^{\infty} X_i$$
where $\bigcup\limits_{i=1}^{\infty} X_i \in \mathcal{F}$ as $\mathcal{F}$ is a σ-field. Similarly, $U^c \cap \mathcal{F}$ is a σ-field. Since $\mathcal{F} = (U \cap \mathcal{F}) \cup (U^c \cap \mathcal{F})$, either $U \cap \mathcal{F}$ or $U^c \cap \mathcal{F}$ has infinite cardinality. Denote this σ-field by $\mathcal{F}_U$ and now we repeat this - pick $U_1 \in \mathcal{F}_U$ s.t. $U_1 \notin \{\emptyset, U\}, \ldots$. Continuing this process gives us a decreasing sequence of subsets of Ω: $\Omega \supset U \supset U_1 \supset \cdots$ and a sequence of countably infinite σ-fields: $\mathcal{F} \supset \mathcal{F}_U \supset \mathcal{F}_{U_1} \supset \cdots$. Let $B_i = U_i \setminus U_{i-1}$, so then B_i is a sequence of disjoint subsets of Ω. Then define $f : \mathcal{P}(\mathbb{N}) \to \mathcal{F}$ by $f(X) = \bigcup\limits_{i \in X} B_i$ which is a bijection as the B_i are disjoint and nonempty which finishes the contradiction so $\mathcal{F}$ must have cardinality of the continuum.

- Suppose that $\Omega_0 \subset \Omega$. We will show that if $\mathcal{F}$ is a σ-algebra in Ω, then $\mathcal{F}_0 = [A \cap \Omega_0 : A \in \mathcal{F}]$ is a σ-algebra in Ω_0. Clearly $\Omega_0 \in \mathcal{F}_0$ if we let $A = \Omega$. If $A \cap \Omega_0 \in \mathcal{F}_0$, then $(A \cap \Omega_0)^c = \Omega_0 \cap (A^c \cup \Omega^c) = \Omega_0 \cap A^c \in \mathcal{F}_0$ as $A^c \in \mathcal{F}$. To show it's closed under countable unions, note that if $(A_1 \cap \Omega_0), (A_2 \cap \Omega_0), \cdots \in \mathcal{F}_0$, then
$$\bigcup_{n=1}^{\infty} (A_n \cap \Omega_0) = \Omega_0 \cap \bigcup_{n=1}^{\infty} A_n$$
and so $\mathcal{F}_0$ is indeed a σ-algebra on Ω_0. This shows us that if we have a subset of a space that has a σ-algebra, then the elements of that smaller σ-algebra consist of the larger σ-algebra intersected with that particular subset.

- Halmos 6.1c: Let E be the class of all sets which contain exactly two points. If Ω is finite, then $\sigma(E)$ is clearly the class of sets which are finite and cofinite. The only exception to this is when the cardinality of Ω is 2, in which case the generated σ-algebra is the trivial algebra consisting of Ω and the empty set. If Ω is infinite, we will show that $\sigma(E)$ is the class of all countable and co-countable subsets of Ω. Let $\mathcal{F}$ be the class of countable and co-countable subsets of Ω. We will show it is a σ-algebra first. Clearly Ω is co-countable as the empty set is countable. $\mathcal{F}$ is closed under complementation by construction as we include both countable and co-countable subsets in the class. Finaly, arbitrary unions of countable sets remains countable and so we see that $\mathcal{F}$ is indeed a σ-algebra. It remains to check that $\mathcal{F}$ contains E. This is trivially satisfied as all pairs of points are countable and contained in $\mathcal{F}$ - hence $\sigma(E) \subset \mathcal{F}$. Conversely, for any $A \in \mathcal{F}$ if A is countable then if $|A| = 1$, A is the intersection of two pairs of points in $\sigma(E)$ and if $|A| > 1$, A is the union of a countable collection of pairs of points. If A is co-countable, then we can just take the complement of a countable collection of pairs of points which is contained in $\sigma(E)$. Thus $\sigma(E) = \mathcal{F}$. A similar argument can be extended for the class of all n-tuples of points in Ω.

- If $\mathcal{F}$ is a non empty class of sets in X containing X and closed under the formation of differences and countable intersections, then $\mathcal{F}$ is a σ-algebra. Note that since $\Omega \in \mathcal{F}$, we have that $\Omega - A = A^c \in \mathcal{F}$ so $\mathcal{F}$ is closed under complementation. Similarly, for $A_1^c, A_2^c, \cdots \in \mathcal{F}$ we have that $\bigcap_n A_n^c = \bigcup_n A_n$ so it is closed under countable unions and consequently a σ-algebra.

- Let $\Omega = \mathbb{R}^3$. Let a subset E of Ω be called a cylinder if whenever $(x, y, z) \in E$, then $(x, y, t) \in E$ for every $t \in \mathbb{R}$. Let $\mathcal{A}$ denote the class of all cylinders. *Topic will be skipped: do problems 6.5d from Halmos and 2.16 from Billingsley*

Definition The Borel σ-algebra of $\mathbb{R}^n$ is the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ generated by all open sets. For any subset $A \subset \mathbb{R}^n$, let $\mathcal{B}(A) = \{E \cap B : B \in \mathcal{B}(\mathbb{R}^n)\}$ denote the σ-algebra generated by the intersection of E with open sets in $\mathbb{R}^n$.

Examples:

- Billingsley 2.10(a): A σ-algebra is countably generated, or separable, if it is generated by some countable class of sets. The σ-algebra $\mathcal{B}(\mathbb{R}^n)$ of Borel sets is countably generated.

- Bartle 2.B: The Borel algebra is also generated by the class of all half-open intervals $(a, b]$ in $\mathbb{R}$. To show this, let $\mathcal{B}_c(\mathbb{R})$ denote the Borel sets generated by clopen/half-open intervals and note that any open interval can be written as a countable union of clopen sets :
$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$$
and so $(a, b) \in \mathcal{B}_c$. Thus each element of $\mathcal{B}(\mathbb{R})$ can be written as a countable union, intersection, and/or complement of open sets so $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}_c(\mathbb{R})$. Similarly, we have that
$$(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$$
so then $(a, b] \in \mathcal{B}(\mathbb{R})$ and so all clopen intervals can be written as countable intersections, unions, and/or complements of open intervals so $\mathcal{B}_c(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$.

- From this previous exercise, we can generalize (with very similar arguments) an equivalent formulation of the Borel σ-algebra in $\mathbb{R}$ by any of the following classes of sets
  - Open Intervals
  - Closed Intervals
  - Clopen Intervals (either $(a, b]$ or $[a, b)$)
  - Intervals of the form $(-\infty, a)$ or (a, ∞) where a is chosen from a dense subset of $\mathbb{R}$
  - Intervals of the form $(-\infty, a]$ or $(a, \infty]$ where again a is chosen from a dense subset of $\mathbb{R}$
  - Intervals with only irrational or only rational endpoints
- We can also extend the above class of Borel sets to $\mathbb{R}^k$ by considering the bounded rectangles (using clopen, open, etc. types of intervals)

$$[x = (x_1, \dots, x_k) : a_i < x_i < b_i, i = 1, \dots, k]$$

Definition A class $\mathcal{F}$ of subsets of a nonempty space Ω is a π-system if it is closed under finite intersections:

$$A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$$

Examples:

- Let $\Omega = [0, 1]$. Then $\mathcal{F} = \{[a, b] : a, b \in \mathbb{Q} \cap [0, 1]\}$ forms a π-system.
- Let $\Omega = (-1, 1)$. Then $\mathcal{M} = \{(-a, a) : |a| < 1\}$ forms a π-system but not it is not closed under countable intersections with $a_n = \frac{1}{n}$ so then $(-a_n, a_n) \to \{0\}$ which is clopen.
- Filters, rings, algebras, and σ-algebras are trivially π-systems
- Let $\Omega = \{1, 2, 3\}$. $\mathcal{W} = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \emptyset\}$ is closed under finite unions but not intersections and so it does not define a π-system.
- Let $\Omega = \mathbb{R}$. Then $\mathcal{D} = \{(a, \infty) : a \geq 0\}$ is a π-system

Definition A class $\mathcal{G}$ of subsets of a nonempty space Ω is a λ-system (or Dynkin's system) if it is closed under complementation and countable disjoint unions:

1. $\Omega \in \mathcal{G}$
2. $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$
3. $A_1, A_2, \dots, \in \mathcal{G}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ imply $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$

Examples:

- Let $(\Omega, \mathcal{H})$ be a measurable space. Let μ and γ be two probability measures on $\mathcal{H}$. $\mathcal{Y} = \{A \in \mathcal{Y} : \mu(A) = \gamma(A)\}$ is a lambda system since clearly 1. $\mu(\Omega) = \gamma(\Omega) = 1$, so $\Omega \in \mathcal{Y}$, 2. $A \in \mathcal{Y} \Rightarrow \mu(A^c) = \mu(\Omega) - \mu(A) = 1 - \mu(A) = 1 - \gamma(A) = \gamma(A^c)$, and 3. If $A_1, A_2, \cdots \in \mathcal{Y}$ are pairwise disjoint, then $\mu(A_i) = \gamma(A_i)$ for all i and

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \gamma(A_i) = \gamma(\bigcup_{i=1}^{\infty} A_i)$$

  so $\mathcal{Y}$ is indeed a λ-system. Notice that we could have used any other measures that assign the same measure to the space ($\mu(\Omega) = \gamma(\Omega) = a$ for $a < \infty$).

- More examples will come here when I start talking about probability applications.

Dynkin's π-λ Theorem: If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system, then $\mathcal{P} \subset \mathcal{L}$ implies that

$$\sigma(\mathcal{P}) \subset \mathcal{L}.$$

Proof: I will follow the style of Billingsley with an ordered list, but less cryptic.

1. We will first prove the following lemma: if $\mathcal{F}$ is a π-system and a λ-system, then $\mathcal{F}$ is a σ-algebra. The first two properties are immediately deduced and the third can be found by using a recurring argument we have seen: for $A_1, A_2, \cdots \in \mathcal{F}$, let $B_1 = A_1$ and $B_n = A_n \setminus \{A_1 \cup \cdots \cup A_{n-1}\}$. Then for all n, we have $\bigcup_n A_n = \bigcup_n B_n \in \mathcal{F}$ and so $\mathcal{F}$ is closed under countable unions and a σ-algebra as desired.

2. Now, define $\mathcal{L}^*$ to be the λ-system generated by the intersection of all λ-systems containing the class $\mathcal{P}$. To prove the π-λ theorem, it now suffices to show that $\mathcal{L}^*$ is a π-system as the lemma above shows that if $\mathcal{L}$ is a π and λ system, then $\sigma(\mathcal{P}) \subset \mathcal{L}^* \subset \mathcal{L}$.

3. To prove $\mathcal{L}^*$ is a π-system, first let $A \in \Omega$ and define

$$\mathcal{L}_A = \{B \subset \Omega : A \cap B \in \mathcal{L}^*\}.$$

   If $A \in \mathcal{L}^*$, then $\mathcal{L}_A$ is a λ-system. Why? First, let $B = \Omega$, so since $\Omega \cap A = A \in \mathcal{L}^*$, $\Omega \in \mathcal{L}_A$. Second, for $B \in \mathcal{L}_A$, notice that

$$A \cap B^c = (A^c \cup B)^c = (\Omega \cap (A^c \cup B))^c = (A^c \cup (A \cap B))^c$$

   which is clearly in $\mathcal{L}^*$ since the two sets are disjoint so the complement of this disjoint union is in $\mathcal{L}^*$, so then $B^c \in \mathcal{L}_A$ and so it's closed under complementation. Finally, if $B_1, \cdots \in \mathcal{L}_A$ are pairwise disjoint, then $(A \cap B_i) \cap (A \cap B_j) = \emptyset$ so as $\mathcal{L}^*$ is a λ-system, we have that $\bigcup_n A \cap B_n = A \cap \bigcup_n B_n \in \mathcal{L}^*$ so then $\bigcup_n B_n \in \mathcal{L}_A$. Hence we have shown that $\mathcal{L}_A$ is a λ-system provided $A \in \mathcal{L}^*$. Now we defer to two claims that help us deduce that $\mathcal{L}^*$ is a π-system.

   (a) Claim 1: if $A \in \mathcal{P}$, then $\mathcal{L}^* \subset \mathcal{L}_A$. To prove this: if $A \in \mathcal{P} \subset \mathcal{L}^*$, then for all $B \in \mathcal{P}$ we have that $A \cap B \in \mathcal{P} \subset \mathcal{L}^*$ as $\mathcal{P}$ is a π-system and so $B \in \mathcal{L}_A$. This implies that all of $\mathcal{P}$ is contained in $\mathcal{L}_A$ and as $\mathcal{L}^*$ is the minimal λ-system containing $\mathcal{P}$, we also have that $\mathcal{L}^* \subset \mathcal{L}_A$ as desired.

(b) Claim 2: if $B \in \mathcal{L}^*$, then $\mathcal{L}^* \subset \mathcal{L}_B$. To prove this: for $B \in \mathcal{L}^*$, we know from the previous claim that for all $A \in \mathcal{P}$, $\mathcal{L}^* \subset \mathcal{L}_A$ and so $B \in \mathcal{L}^* \subset \mathcal{L})_A$. This implies that $B \cap A \in \mathcal{L}^*$ which also shows that $A \in \mathcal{L}_B$. As this is true for all $A \in \mathcal{P}$, we have that $\mathcal{P} \subset \mathcal{L}_B$ and from our previous claim we know that since $B \in \mathcal{L}^*$, $\mathcal{L}_B$ is a λ-system and a superset of $\mathcal{L}^*$ ($\mathcal{L}^* \subset \mathcal{L}_B$). Similarly, for $A \in \mathcal{L}^*$, we have that $\mathcal{L}^* \subset \mathcal{L}_A$ and since $B \in \mathcal{L}^*$, $B \in \mathcal{L}_A$ so $A \cap B \in \mathcal{L}^*$. Thus $\mathcal{L}^*$ is closed under finite intersections as desired and a π-system.

4. From the first lemma, these last two claims combined with the first lemma show us that $\mathcal{L}^*$ is a π and λ system and thus a σ-algebra containing $\mathcal{P}$. By definition of $\sigma(\mathcal{P})$ being the minimal σ-algebra containing $\mathcal{P}$, we conclude that

$$\sigma(\mathcal{P}) \subset \mathcal{L}^* \subset \mathcal{L}$$

Examples:

- Corollary: If P_1 and P_2 are probability measures on $\sigma(\mathcal{P})$, where $\mathcal{P}$ is a π-system and $P_1(x) = P_2(x)$ for all $x \in \mathcal{P}$, then $P_1(x) = P_2(x)$ for all $x \in \sigma(\mathcal{P})$. Proof: As we generated a class of example λ-systems above, we know that $\mathcal{L} = \{x \in \sigma(\mathcal{P}) : P_1(x) = P_2(x)\}$ is a λ-system. Thus, since $\mathcal{P} \subset \sigma(\mathcal{P})$ and $\mathcal{P} \subset \mathcal{L}$, we employ the result of the π-λ theorem which tells us that $\sigma(\mathcal{P}) \subset \mathcal{L}$ and since the probability measures agree on $\mathcal{L}$, they agree on the subset $\sigma(\mathcal{P})$ as desired.

- Billingsley 3.13: We will show that it is impossible to drop the assumption that $\mathcal{P}$ is a π-system as in the above corollary.

- Halmos's montone class theorem is equivalent to the π-λ theorem. It states: if $\mathcal{F}_0$ is a field and $\mathcal{M}$ is a monotone class, then $\mathcal{F}_0 \subset \mathcal{M}$ implies that $\sigma(\mathcal{F}_0) \subset \mathcal{M}$ where a monotone class is a class of subsets closed under the formation of countable monotone unions and intersections. In my opinion, it's a waste of time to go over both extensively, so I won't include a proof of this claim nor much discussion of the theorem.

Definition A set function μ on a field $\mathcal{F}$ in Ω is a measure if it satisfies these conditions (countable additivity and non-negativity):

1. $\mu(A) \in [0, \infty]$ for $A \in \mathcal{F}$

2. $\mu(0) = 0$

3. If $A_1, A_2, \dots$ is a disjoint sequence of $\mathcal{F}$-sets and if $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Examples:

- A set A in a field $\mathcal{F}$ is said to have finite measure if $\mu(A) < \infty$

- A set A in a field $\mathcal{F}$ is said to have σ-finite measure if there exist a sequence $\{A_n\}$ of sets in $\mathcal{F}$ such that $A \subset \bigcup_n A_n$ and $\mu(A_n) < \infty$ for all n.

- If the measure of every set in $\mathcal{F}$ is finite (or σ-finite), we say that μ is totally finite (or totally σ-finite) on $\mathcal{F}$. We will reserve the terms finite and σ-finite for measures defined on rings.

- If $A \in \mathcal{F}$, $E \subset A$, and $\mu(E) = 0$ imply that $B \in \mathcal{F}$ then we say that μ is complete. The completion of a measure is denoted by a bar over the measure ($\overline{\mu}$).

- Schilling 4.19: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{H} \subset \mathcal{F}$ be a sub-σ-algebra (fancy term for subclass that is also a σ-algebra). Denote by $\nu \equiv \mu|_{\mathcal{H}}$ the restriction of μ to $\mathcal{H}$. Clearly as $\mathcal{H}$ is a subset of $\mathcal{F}$, we have 1. $\nu(A) \in [0, \infty]$ provided μ is a measure, 2. $\Omega \in \mathcal{H} \Rightarrow \Omega^c \in \mathcal{H}$ so $\mu(\emptyset) = \nu(\emptyset) = 0$, and 3. if $A_1, A_2, \cdots \in \mathcal{H}$ are pairwise disjoint, then $\nu\left(\bigcup_n A_n\right) = \sum_n \nu(A_n)$ which shows that the restriction of μ is indeed a measure still. Provided μ is a probability measure on $\mathcal{F}$ ($\mu(\Omega) = 1$), we have $\nu(\Omega) = 1$ and so the restriction is still a probability measure. Finally, if μ is σ-finite, we do not have that ν is necessarily σ-finite. For an example, consider the typical Lebesgue measure endowed on the Borel σ-algebra generated by open intervals in $\mathbb{R}$. Then take $\mathcal{H}$ to be the sub-σ-algebra, $\{\emptyset, (-\infty, 0), [0, \infty), \mathbb{R}\}$, which is not σ-finite.

- Halmos 7.1/Billingsley 10.1: If μ is non-negative and countably additive set function defined on $\mathcal{F}$ and if $\mu(E) < \infty$ for at least one E in $\mathcal{F}$, then $\mu(\emptyset) = 0$. Proof: since $E \cap \emptyset = \emptyset$, then $\mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset) \iff \mu(E) = \mu(E) + \mu(\emptyset) \iff \mu(\emptyset) = 0$ since $\mu(E) < \infty$.

- Halmos 7.2: If E is a non-empty class of sets and μ is a measure on $\mathcal{F}(E)$, the algebra generated by the class E, such that $\mu(A) < \infty$ for all $A \in E$, then μ is finite on $\mathcal{F}(E)$. Proof: for all $A \in \mathcal{F}(E)$, we have since $\mathcal{F}(E)$ is an algebra, that every set can be covered by a finite union of sets $E_1, \ldots, E_n \in E$ such that $A \subset \bigcup_{i=1}^{n} E_i$. Thus using basic properties of measures, we arrive at

$$\mu(A) \leq \mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu(E_i) < \infty$$

  which proves μ is finite on the algebra. This holds for rings, but does not hold for all σ-algebras as some sets may not be covered by finite unions which wouldn't ensure $\sum \mu(E_i)$ is finite.

- Halmos 7.4: Suppose that μ is a measure on a σ-algebra $\mathcal{F}$ and that E is a set in $\mathcal{F}$ of σ-finite measure. If D is a disjoint class of sets in S, then $\mu(E \cap D) \neq 0$ for at most countably many sets $A \in D$. This proof follows nearly the same structure of the following result in real analysis: if $\{x_\alpha\}_{\alpha \in \Gamma}$ is a collection of real numbers $x_i > 0$ such that $\sum_{\alpha \in \Gamma} x_i < \infty$, then $x_i = 0$ for all but countably many $\alpha \in \Gamma$. The proof relies on constructing a set $A_n = \{\alpha \in \Gamma : x_i > \frac{1}{n}\}$ which contains a finite number of points since

$$\infty > S = \sum_{\alpha \in \Gamma} x_i \geq \sum_{\alpha \in A_n} \frac{1}{n} = \frac{|A_n|}{n} \iff |A_n| = Sn$$

and since $\{\alpha \in \Gamma : x_i > 0\} = \bigcup_{n \in \mathbb{N}} A_n$ we have our result as the countable union of finite sets is countable. The only difference is that we let $A_n = \{A \in D : \mu(A \cap E) > \frac{1}{n}\}$ and later use the property of countable additivity.

- Let $\mathcal{F} = \mathcal{P}(\Omega)$ be the σ-algebra of an arbitrary Ω. Now, the counting measure is defined the number of points in A for $A \in \mathcal{F}$ so that $\mu(A) = \infty$ if A has an infinite number of points. Provided that $\Omega = \bigcup_n A_n$ for some countable or finite sequence of $\mathcal{F}$-sets satisfying $\mu(A_n) < \infty$, then μ is a σ-finite measure. If we let $\mathcal{F} = \mathcal{P}(\mathbb{N})$ with $A_n = \{1, \ldots, n\}$, then $A_n \in \mathcal{F}$ for all n, $A_n \uparrow \mathbb{N}$, and $\mu(A_n) = n < \infty$ so here μ is σ-finite over the power set of naturals. The same cannot be said if the domain is $\mathbb{R}$ as any countable collection of sets spanning $\mathbb{R}$ must contain an uncountable number of points and thus have infinite measure.

- Rather trivially, if we consider the same example with $\mathcal{F} = \mathcal{P}(\mathbb{N})$ on $\Omega = \mathbb{N}$, but this time define A_n as the number of points in A_n plus some constant $c > 0$, then $\mu(A_n) \in [0, \infty]$ but $\mu(0) = 1 \neq 0$ and it's not even finitely additive so μ is not a measure.

- If $A \subset B$, then $\mu(B) = \mu(A) + \mu(B - A) \geq \mu(A)$

- For a measure μ on the σ-algebra $\mathcal{F}$, we have that $\mu(A_n) \to \mu(A)$ for $A_n \uparrow A$ with $A_n, A \in \mathcal{F}$. Why? Well let $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ so then $A_n = \bigcup_{i=1}^{k} B_i$ and

  $$A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} B_j = \bigcup_{i=1}^{\infty} B_i.$$

  From this, we have

  $$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \mu(A_1) + \sum_{i=2}^{\infty} \mu(B_i) = \lim_{n \to \infty} \mu(A_n)$$

  since the infinite sum telescopes.

- Let us talk about the Lebesgue measure on intervals. Consider the Borel σ-algebra, $\mathcal{B}(\mathbb{R})$, with $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}^+$ and $\mu([a, b)) = b - a$. Why is this a measure? First off, $\mu(\emptyset) = 0$ and $\mu(\mathbb{R}) = \infty$. To prove countable additivity we'll utilize two lemmas.

  1. Lemma: for a finite, disjoint set of intervals $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$ where $A_i = [a_i, b_i)$ and $\bigcup_n A_n \subset A = [a, b)$, we have that

     $$\sum_n \mu(A_n) \leq \mu(A).$$

     Proof: $\sum_n \mu(A_n) = \sum_n b_n - a_n \leq \max_n(b_n) - \min_n(a_n) \leq b - a$.

2. Lemma: for a countable set of intervals $A_1, \cdots \in \mathcal{B}(\mathbb{R})$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$ where $A_i = [a_i, b_i)$ and $A = [a, b)$, then

$$\mu(A) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Proof: Let $B = [a, b+\epsilon]$ where $\epsilon > 0$, then since B is closed and bounded, we know it is compact by the Heine Borel theorem. This implies there exists a finite subcover of open sets $U_1, \ldots, U_n$ of the form $U_i = (a_i, b_i + \frac{\epsilon}{2^i})$. This shows that $[a, b+\epsilon) \subset [a, b+\epsilon] \subset \bigcup_{i=1}^{n} U_i$. As this holds for arbitrary ϵ, we have $[a, b) \subset \bigcup_{i=1}^{n} U_i$ which proves the lemma.

For $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$ with $A = \bigcup_n A_n$ and the A_i are disjoint, we have combining lemma 1 and 2 that

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

As this holds for all n, we can deduce that μ is indeed countably additive which also proves that the Lebesgue set function defined on the Borel σ-algebra is actually a measure. This argument can be easily extended from $\mathbb{R}$ to $\mathbb{R}^n$ as well.

- The Lebesgue measure is the unique measure defined on $\sigma(\mathcal{B}(\mathbb{R}))$ such that for all $a, b \in \mathbb{R}$ with $a \leq b$, we have $\mu([a, b)) = b - a$. To prove this claim, we will utilize the π-λ theorem. Note that the class of intervals of the form $[a, b)$ is clearly a π-system as it's closed under intersections. Let us define a new measure on the class of semi-open intervals: $\lambda([a, b)) = a - b$ for $a < b$. Now, we know there's a corollary we proved as a direct result of the π-λ theorem which tells us that if two probability measures agree on a class, they agree on the σ-algebra generated by the class. Hence we want to tweak our measures above by considering for $\in \mathbb{N}$, the functions $\lambda_n([a, b)) = \lambda([n, n+1] \cap [a, b))$ and $\mu_n([a, b)) = \mu([n, n+1] \cap [a, b))$. These define probability measures for fixed n as $\mu, \lambda \leq (n + 1 - n) = 1$ for all sets. As the two clearly agree on the π-system, we know they agree on the σ-algebra generated by the class of semi-open intervals which proves the uniqueness of the Lebesgue measure on $\sigma(\mathcal{B}(\mathbb{R}))$.

- Schilling 5.12(i): Let $\mathcal{G}$ be a algebra in X. Let $\mathcal{A} = \sigma(\mathcal{G})$ and μ be a finite measure on $(X, \mathcal{A}, \mu)$. We will show that for every $\epsilon > 0$ and $A \in \mathcal{A}$, there is some $G \in \mathcal{G}$ such that $\mu(A \Delta G) < \epsilon$. To do so, let us show the following is a λ-system:

$$\mathcal{L} = \{A \in \mathcal{A} : \forall\, \epsilon > 0 \;\exists\, G \in \mathcal{G} \;: \mu(A \Delta G) < \epsilon\}.$$

To do so, note that $X \in \mathcal{L}$ if we let $A = G = X$, then $\mu(X \Delta X) = 0 < \epsilon$. Now if $A \in \mathcal{L}$, then $\mu(A \Delta G) < \epsilon$ for some $G \in \mathcal{G}$. To show A^c is well approximated and contained in $\mathcal{L}$, note that if we take G^c, we get

$$\mu(A^c \Delta G^c) = \mu\left(A^c \setminus G^c \cup G^c \setminus A^c\right) = \mu(G \setminus A \cup A \setminus G) < \epsilon$$

hence $\mathcal{L}$ is closed under complementation. Finally, let $A_1, A_2, \dots \in \mathcal{L}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$. As μ is finite, for $\epsilon > 0$, there is some N :

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) < \infty \iff \sum_{i>N} \mu(A_i) < \epsilon.$$

As $\mathcal{G}$ is finitely additive, for $A_1, \dots, A_N$, there is a corresponding sequence $G_1, \dots, G_N \in \mathcal{G}$ such that $\mu(A_i \Delta G_i) < \epsilon$ for $i \in \{1, \dots, n\}$. Thus as measures are countably subadditive, we have

$$\mu\left(\bigcup_{i=1}^{N} A_i \Delta G_i\right) \leq \sum_{i=1}^{N} \mu(A_i \Delta G_i) < N\epsilon.$$

Combining these two inequalities gives us that for pairwise disjoint $A_1, A_2, \dots \in \mathcal{A}$, there is some N such that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i \Delta G_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i \Delta G_i) = \sum_{i=1}^{N} \mu(A_i \Delta G_i) + \sum_{i=N+1}^{\infty} \mu(A_i \Delta G_i) < (N+1)\epsilon$$

which proves that $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{L}$ and thus that $\mathcal{L}$ is a λ-system. Clearly $\mathcal{G} \subset \mathcal{L}$ since the symmetric difference of an element with itself is always zero which is less than arbitrary ϵ. As $\mathcal{G}$ is an algebra, it is implicitly a π-system and so we employ the π-λ theorem which tells us that $\sigma(\mathcal{G}) \subset \mathcal{L}$ and so we have proven the statement.

- Halmos 9.5/Billingsley 10.5: If μ is a measure on a ring R and if $\{E_n\}$ is a sequence of sets in R for which $\bigcap_{n=N}^{\infty} E_i \in R$ for all n and $\liminf_n E_n \in R$, then

$$\mu(\liminf_n E_n) \leq \liminf_n \mu(E_n)$$

Proof: $\mu\left(\liminf\limits_n E_n\right) = \mu\left(\bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} E_n\right)$. Then since $\bigcap\limits_{n=N}^{\infty} E_n$ is an increasing sequence, we have by continuity of measure that

$$\mu\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} E_n\right).$$

As $\bigcap\limits_{n=N}^{\infty} E_n \subset E_n$ for $n \geq N$, we have by monotonocity of measure that $\mu\left(\bigcap\limits_{n=N}^{\infty} E_n\right) \leq \mu(E_n)$ and so

$$\lim_{N\to\infty} \mu\left(\bigcap_{n=N}^{\infty} E_n\right) \leq \lim_{N\to\infty} \inf_{n\geq N} \mu(E_n) = \liminf_n \mu(E_n).$$

- Billingsley 10.3: On the σ-field of all subsets of $\Omega = \mathbb{N}$, put $\mu(A) = \sum_{k\in A} 2^{-k}$ if A is finite and $\mu(A) = \infty$ otherwise. μ is finitely additive trivially - take $A_1, \dots, A_n \in \mathcal{P}(\mathbb{N})$ to be pairwise disjoint. Then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k\in\cup_n A_n} 2^{-k} = \sum_{k\in A_1} 2^{-k} + \dots + \sum_{k\in A_n} 2^{-k} = \sum_{i=1}^{n} \mu(A_i).$$

The set function is not countably additive however. Take $A_1 = \{1\}, A_2 = \{2\}, \dots,$ then the sets are disjoint and their union is $\mathbb{N}$, so $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$ however,

$$\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} 2^{-i} = 1 \neq \infty.$$

- Halmos 9.6: If μ is a measure on a ring R, $\bigcup_{n=N}^{\infty} E_n \in R$ for $N \in \mathbb{N}$, $\limsup_n E_n \in R$, and $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then

  $$\mu\left(\limsup_n E_n\right) = 0.$$

  Proof:

  $$\mu\left(\limsup_n E_n\right) = \mu\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} E_n\right) = \mu\left(\lim_{N\to\infty}\bigcup_{n=N}^{\infty} E_n\right) = \lim_{N\to\infty} \mu(\bigcup_{n=N}^{\infty} E_n).$$

  Then by subadditivity of measure and since $\sum_n \mu(E_n) < \infty$,

  $$\lim_{N\to\infty} \mu(\bigcup_{n=N}^{\infty} E_n) \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) = 0.$$

- Halmos 9.10: Prove there exists a countable union of compact sets E in a complete, separable metric space X where $\mu(X) = \mu(E) = 1$.

- A measure space $(X, \mathcal{F}, \mu)$ is complete if $A \subset B$, $B \in \mathcal{F}$, and $\mu(B) = 0$ together imply that $A \in \mathcal{F}$ (and thus $\mu(A) = 0$). This will be heavily used in the study of stochastic processes and integration and I intend on appending more material on this when I reach the topic.

Definition A set function is a real-valued function defined on some class of subsets of Ω. A set function P on an algebra $\mathcal{F}$ is a probability measure if it satisfies these conditions:

1. $P(A) \in [0, 1]$ for $A \in \mathcal{F}$
2. $P(0) = 0$ and $P(\Omega) = 1$
3. If $A_1, A_2, \dots$ is a disjoint sequence of $\mathcal{F}$-sets and if $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$, then

   $$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Examples:

- Let $\Omega = \{1, 2, \dots, n\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$. Then $P : \Omega \to \mathbb{R}$ defined by $P(A) = \frac{|A|}{n}$ for $A \in \mathcal{F}$ is a probability measure on $\mathcal{F}$. Why? Well $P(\emptyset) = 0$, $P(\Omega) = \frac{n}{n} = 1$, and for a disjoint sequence of sets in $\mathcal{F}$ (for $k \leq n$), $A_1, \dots, A_k$, we have that

$$\sum_{i=1}^{k} P(A_i) = \frac{|A_1|}{n} + \cdots + \frac{|A_k|}{n} = \frac{|A_1 + \cdots + A_k|}{n} = P\left(\bigcup_{i=1}^{k} A_i\right)$$

since $|A_1| + |A_2| = |A_1 + A_2|$ if the two sets are disjoint. This is an example of a discrete probability measure - a measure over a finite/countably infinite probability space.

- Billingsley 2.12(a): Let $\mathcal{F}$ be the field consisting of the finite and the cofinite sets in an infinite Ω and define P on $\mathcal{F}$ by taking $P(A)$ to be 0 or 1 as A is finite or cofinite (*a set A is cofinite if A^c is finite*). We will show that P is finitely additive . P is not well-defined if Ω is finite as that would imply that for $A \in \mathcal{F}$, A is both finite and cofinite so we assume Ω is infinite. If $A_1, \dots, A_n$ are disjoint, finite sets in Ω, then $P(A_1 \cup \cdots \cup A_n) = 0 = P(A_1) + \cdots P(A_n)$. If $A_1, \dots, A_n$ are disjoint, but not all finite, then we'll prove first that there is a unique A_k that is cofinite. Assume the contrary, that there is a A_j that is also cofinite with $A_j \cap A_k = \emptyset$. Then by De Morgan's, $(A_j \cap A_k)^c = A_j^c \cup A_k^c = \Omega$ - a contradiction as Ω is infinite, but A_j^c and A_k^c were assumed to be finite. Thus we have that for the disjoint set $A_1, \dots, A_n$ with one element cofinite:

$$P(A_1 \cup \cdots \cup A_n) = 1 = P(A_j) + \sum_{i \neq j} P(A_i)).$$

- Billingsley 2.12(b,d) With the same P as before, we will show that if Ω is countably infinite, then P cannot be countably additive. If Ω is countably infinite, then there is a bijective mapping $f : \mathbb{N} \to \Omega$ such that for each $\omega \in \Omega$, there's a corresponding unique natural number k so that $\omega = f(k)$. By the Axiom of Union,

$$\mathbb{N} = \bigcup_{k=1}^{\infty} \{k\}$$

so then

$$P(\Omega) = P(f(\mathbb{N})) = P\left(\bigcup_{k=1}^{\infty} f(k)\right) = 1,$$

since $P(\Omega) = 1$ by definition, but

$$\sum_{k=1}^{\infty} P(f(k)) = 0$$

as each $f(k)$ is finite so P is not countably additive over a countable sample space. Part (d) of the problem follows this reasoning - the union of countable sets is still countable and any countable set union an uncountable set is uncountable.

- Inclusion-Exclusion Principle (you can find your favorite proof via Google):

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

- Let P be a probability measure on a field $\mathcal{F}$, A_n be a sequence of increasing subsets in $\mathcal{F}$, and $A = \bigcup\limits_{n=1}^{\infty} A_n$ (i.e. $A_n \uparrow A$). Define $B_{n+1} = A_{n+1} \setminus A_n$ so then $B_i \cap B_j = \emptyset$ for $i \neq j$ and clearly $\bigcup\limits_{i=1}^{\infty} A_i = \bigcup\limits_{i=1}^{\infty} B_i$, so then by countable additivity of P, we have

$$\sum_{i=1}^{\infty} P(B_) = P(\bigcup_{i=1}^{\infty} B_i) = P(\bigcup_{i=1}^{\infty} A_i) = P(A)$$

- Billingsley 2.15: Let $\Omega = \mathbb{N}$ and $\mathcal{F}$ denote the class of all subsets of $\mathbb{N}$. Define

$$P_n(A) = \frac{1}{n} \#\{m : 1 \leq m \leq n, m \in A\}$$

where n denotes the first n integers.Then the density of A is given by

$$D(A) = \lim_{n \to \infty} P_n(A)$$

provided the limit exists. Let $\mathcal{D}$ denote the class of sets having density.

(a) We will show that D is finitely, but not countably additive over $\mathcal{D}$. The latter is clear if we let $A_n = n$, so that $D\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = 1$ while $\sum\limits_{n=1}^{\infty} D(A_n) = 0$ as each A_n has cardinality one. To show that it's finitely additive, suppose we have a collection $A_1, \ldots, A_n$ of disjoint subsets of $\mathcal{F}$. Then

$$D\left(\bigcup_{i=1}^{k} A_i\right) = \lim_{n \to \infty} \frac{\#\left[m : 1 \leq m \leq n,\ m \in \cup_{i=1}^{k} A_i\right]}{n} = \lim_{n \to \infty} \sum_{i=1}^{k} \frac{\#\left[m : 1 \leq m \leq n,\ m \in A_i\right]}{n}$$

$$= \sum_{i=1}^{k} D(A_i)$$

since we can separate the cardinality of disjoint sets and obviously exchange the limit and sum here because the sum is finite.

(b) Let $A = \emptyset$, then $D(\emptyset) = \lim\limits_{n \to \infty} \frac{\#[m : m \leq n,\ m \in \emptyset]}{n} = 0$ and so $\emptyset \in \mathcal{D}$. In a similar trivial fashion, $\Omega \in \mathcal{D}$. It is closed under complementation as

$$D(A^c) = \lim_{n \to \infty} P_n(A^c) = \lim_{n \to \infty} \frac{\#\left[n - m : 1 \leq m \leq n,\ m \in A\right]}{n}$$

which is clearly well defined. Again, similar arguments show it is closed under proper differences and finite disjoint unions. To show it is not closed under the formation of countable disjoint unions, note that if we let A consist of the odd positive integers and B of the even positive integers. Then $D(A \cup B) = 1 \neq D(A) + D(B) = 2$

- If $\mathcal{F}$ is a σ-algebra in Ω and P is a probability measure on $\mathcal{F}$, the triple $(\Omega, \mathcal{F}, P)$ is called a probability measure space, or simply a probability space. A support of P is any $\mathcal{F}$-set A for which $P(A) = 1$

- Let $(X, \mathcal{F}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{F}$ for all $A \in \mathcal{F}$ such that $\mu(A) < \infty$. . Let $\tilde{\mathcal{F}}$ denote the class of all locally measurable sets in a measure space $(X, \mathcal{F}, \mu)$. If $\mathcal{F} = \tilde{\mathcal{F}}$, then μ is called saturation.

- Billingsley 3.2(a): Let P be a probability measure on a field $\mathcal{F}_0$ and for every subset A of Ω define $P^*(A)$ as the induced outer measure of P and define also by P the extension of P to $\sigma(\mathcal{F}_0)$. We will show that

$$P^*(A) = \inf\left\{P(B) : A \subset B,\ B \in \sigma(\mathcal{F}_0)\right\}$$

and the infimum is always achieved. It's obvious that we can always cover A since $\Omega \in \mathcal{F}_0$ and so the infimum can always be achieved (as it's taken over the sequences of sets that cover A). To show this is equivalent to our earlier notion of the outer measure induced by a probability measure (the infimum of sums of $P(A_n)$), notice that for a sequence $\{A_n\}_{n\in\mathbb{N}}$ covering A, we can construct a disjoint set $B_n = A_n \setminus \bigcup\limits_{i=1}^{n-1} A_i$ which satisfies the conditions for countable additivity (and so $P(A) = P(\bigcup A_n) = \sum P(A_n)$) and so we see they are equivalent. This argument is very similar if we desire to show

$$P_*(A) = \sup\{P(C) : C \subset A,\ C \in \mathcal{F}_0\}$$

is equivalent to

$$P_*(A) = 1 - P^*(A).$$

Definition An outer measure is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ with these three properties:

1. $\mu^*(\emptyset) = 0$
2. μ^* is monotone: $A \subset B$ implies $\mu^*(A) \le \mu^*(B)$
3. μ^* is countably subadditive: $\mu^*(\bigcup_n A_n) \le \sum_n \mu^*(A_n)$

Examples:

- Occasionally (in Bogachev's book at least), the outer measure is alternatively defined as follows: for any nonnegative set function μ that is defined on a certain class $\mathcal{A}$ of subsets in a space X that contains X itself, the formula

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A},\ E \subset \bigcup_{n=1}^{\infty} A_n\right\}$$

defines a new set function defined for all $E \subset X$ and we call μ^* the outer measure induced by the measure μ.

- We will show that $\mu^* \leq \mu$. In a space X, define $\mu : \mathcal{F} \to [0, \infty]$ and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ where $\mathcal{F}$ is a sub σ-algebra of the power set. Let $A \in \mathcal{F}$, then take $B_1 = A$ and $B_k = \emptyset$ for $k > 1$ so $A \subset \bigcup_n B_n$ and thus

$$\mu^*(A) = \inf_{\{A_n\} \subset \mathcal{F},\, A \subset \bigcup_n A_n} \sum_n \mu(A_n) \leq \sum_n \mu(B_n) = \mu(A).$$

For an example where $\mu^*(A) < \mu(A)$, let $\Omega = \mathbb{N}$ and

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is cofinite} \end{cases}.$$

Let $A = \Omega$ and $A_1 = \{1\}, A_2 = \{2\}, \ldots$ so that $A = \bigcup_n A_n$. Then $\mu^*(A) = \sum_n \mu(A_n) = 0$, but $\mu(A) = \infty$.

- We will prove that the outer measure is countably additive i.e. that

$$\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$$

Proof: If $\mu^*(A_n) = \infty$ for some n, then the inequality is trivially satisfied. Otherwise, let $\epsilon > 0$ and suppose $\mu^*(A_n)$ is finite for all n. Then for each n, there is a collection of sets $\{B_{n,k}\}_{k \in \mathbb{N}}$ such that $A_n \subset \bigcup_k B_{n,k}$ and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_k \mu(B_{n,k}).$$

If no such covering that satisfies this inequality exists, then

$$\sum_k \mu(B_{n,k}) > \mu^*(A_n) + \frac{\epsilon}{2^n}$$

for all $\{B_{n,k}\}_{k \in \mathbb{N}}$ covering A_n which implies that $\mu^*(A_n)$ is not in fact the greatest lower bound. Then since $\mu^* < \mu$ (see above) and by countable subadditivity and monotonicity ($A_n \subset \cup_k B_{n,k}$) of the measure μ, we have

$$\mu^*\left(\bigcup_n A_n\right) \leq \mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n) \leq \sum_n \mu\left(\bigcup_k B_{n,k}\right) \leq \sum_n \sum_k \mu(B_{n,k})$$

$$\leq \sum_n \mu^*(A_n) + \frac{\epsilon}{2^n} = \sum_n \mu^*(A_n) + \epsilon.$$

- Let $\Omega = \{0, 1\}$ with $\mathcal{F} = \{\emptyset, \Omega\}$, the trivial σ-algebra, then set $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$ so μ is a measure on $\mathcal{F}$. μ^* is not additive however on the power set of Ω - $\mu^*(\{0\}) = \mu^*(\{1\}) = 1$ so $\mu^*(\{0\}) + \mu^*(\{1\}) = 2 \neq \mu^*(\{0\} \cup \{1\}) = 1$ since Ω is the only subset of $\mathcal{F}$ that contain $\{0\}$ and $\{1\}$.

- Differences between measures and outer measures - 1. the domain of measures is a σ-algebra while the domain of outer measures is a power set (Halmos generalizes the latter set functions to hereditary σ-rings - see appendix), 2. measures are countably additive while outer measures only possess subadditivity (outer measures need not even be finitely additive).

- An outer measure μ^* on $\mathcal{P}(X)$ is a metric outer measure if

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

whenever $\rho(E, F) > 0$ and ρ is some metric on X.

- Halmos 12.8: If X is a metric space, p is a positive real, and $E \subset X$, then the p-dimensional Hausdorff outer measure of E is defined to be the number

$$\mu_p^*(E) = \sup_{\epsilon > 0} \inf \left\{ \sum_{i=1}^{\infty} diam(E_i)^p : E = \bigcup_{i=1}^{\infty} E_i,\ diam(E_i) < \epsilon \right\}.$$

We will show that this set function is a metric outer measure. To do so, let $E, F \in X$ such that $\rho(E, F) > 0$ and $\mathcal{A}$ denote the cover of $E \cup F$ such that

$$\mathcal{A} = E \cup F = \bigcup_{i=1}^{\infty} E_i$$

where $diam(E_i) < \epsilon$ for all $i \in \mathbb{N}$. As $\rho(E, F) > 0$, there is no element of $\mathcal{A}$ contained in the intersection of E and F since otherwise, the distance between the sets would be zero. Thus, we can partition $\mathcal{A}$ into two sub-families $\mathcal{G}$ and $\mathcal{H}$ such that $\mathcal{G} = \{A \in \mathcal{A} : E \cap A \neq\}$ and $\mathcal{H} = \{A \in \mathcal{A} : F \cap A \neq\}$. Then $\mathcal{G}$ covers E and $\mathcal{H}$ covers F which sets inside both families still arbitrarily small in diameter. As they are both disjoint, we get

$$\sum_{G \in \mathcal{G}} diam(G)^p + \sum_{H \in \mathcal{H}} diam(H)^p = \sum_{A \in \mathcal{A}} diam(A)^p$$

from which it trivially follows that $\mu_p^*(E) + \mu_p^*(F) = \mu_p^*(E \cup F)$.

- A set $E \subset X$ is μ^*-measurable if for all $A \subset X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Definition Let $A \subset \mathcal{P}(X)$ be an algebra. The function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a premeasure if

1. $\mu_0(\emptyset) = 0$

2. If $A_1, \dots \in \mathcal{A}$ such that $\bigcup_n A_n \in \mathcal{A}$, then

$$\mu_0\left(\bigcup_n A_n\right) = \sum_n A_n$$

Examples:

- Premeasures are finitely additive just like measures - set $A_j = \emptyset$ for $j > n$. Trival examples of premeasures are measures.

- Let's call this the premeasure theorem: If μ_0 is a premeasure on a semiring $\mathcal{A}$ and μ^* is the outer measure induced by by the premeasure on $\mathcal{A}$, then $\mu^*|\mathcal{A} = \mu_0$ and every set in $\mathcal{A}$ is μ^*-measurable. To prove the first claim, let $A \in \mathcal{A}$. If A is covered by $\bigcup\limits_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{A}$, then we can form a disjoint sequence of sets containing A by letting
$$B_n = A \cap \left(A_n \setminus \bigcup_{i=1}^{n-1} A_i \right)$$
so then
$$A = \bigcup_{i=1}^{\infty} B_i$$
and $B_i \cap B_j = \emptyset$. This implies that
$$\mu^*(A) = \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0 \left(\bigcup_{i=1}^{\infty} B_i \right) = \mu_0 \left(\bigcup_{i=1}^{\infty} A_i \right) = \mu_0(A)$$
which proves the first claim. To show $\mathcal{A} \subset \mathcal{M}(\mu^*)$, let $A \in \mathcal{A}$ and $E \subset X$. If $\mu^*(E) = \infty$, then it trivially holds that $A \in \mathcal{M}(\mu^*)$. Otherwise if $\mu^*(E) < \infty$ and $\epsilon > 0$, then there exists a sequence of sets $\{A_n\}_{n \in \mathbb{N}}$ covering E with $A_i \in \mathcal{A}$ for all $i \in \mathbb{N}$ and satisfying
$$\mu^*(E) + \epsilon > \sum_{i=1}^{\infty} \mu_0(A_n).$$
Now, note that
$$\bigcup_{i=1}^{\infty} A_i \cap A = E \cap A,$$
and
$$\bigcup_{i=1}^{\infty} A_i \cap A^c = \bigcup_{i=1}^{\infty} A_i \setminus A = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} C_{ij}$$
where the C_{ij} are pairwise disjoint for $j \in \{1, \dots, n_i\}$. Finally, combining these together and noting that $\mu^* \le \mu_0$ (outer measure was induced by the premeasure), we obtain
$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \le \mu_0(A \cap E) + \mu_0(A^c \cap E) \le \mu_0 \left(A \cap \bigcup_{i=1}^{\infty} A_i \right) + \mu_0 \left(A^c \cap \bigcup_{i=1}^{\infty} A_i \right)$$
$$\le \mu_0 \left(A \cap \bigcup_{i=1}^{\infty} A_i \right) + \mu_0 \left(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} C_{ij} \right) \le \sum_{i=1}^{\infty} \mu_0(A \cap A_i) + \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \mu_0(C_{ij}).$$

From here note that

$$A_i = (A \cap A_i) \cup (A^c \cap A_i) = (A \cap A_i) \cup (\bigcup_{j=1}^{n_i} C_{ij})$$

where $(A \cap A_i) \cap (A^c \cap A_i) = \emptyset$, so by additivity of μ_0, we get that

$$\sum_{i=1}^{\infty} \mu_0(A \cap A_i) + \sum_{i=1}^{\infty}\sum_{j=1}^{n_i} \mu_0(C_{ij}) = \sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \epsilon$$

and since ϵ was arbitrary, we have proven that $A \in \mathcal{M}(\mu^*)$ and so $\mathcal{A} \subset \mathcal{M}(\mu^*)$.

Carathéodory's Extension Theorem: If μ^* is an outer measure on X, the collection $\mathcal{M}(\mu^*)$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}(\mu^*)$ is a complete measure.

Proof: $X \in \mathcal{M}(\mu^*)$ as $\mu^*(A \cap X) + \mu^*(A \cap \emptyset) = \mu^*(A)$. By symmetry, we see that if $E \in \mathcal{M}(\mu^*)$, then $E^c \in \mathcal{M}(\mu^*)$ as well. To show it is closed under finite unions, let $B, C \in \mathcal{M}(\mu^*)$ and $A \subset X$, so

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

Since the domain of μ^* is the powerset, we know both $A \cap B$ and $A \cap B^c$ are subsets of X and so we can apply the definition of μ^*-measurability with C to get

$$\mu^*(A) = \mu^*(A \cap B \cap C) + \mu^*(A \cap B \cap C^c) + \mu^*(A \cap B^c \cap C) + \mu^*(A \cap B^c \cap C^c).$$

As

$$B \cup C = (B \cap C) \cup (B \cap C^c) \cup (B^c \cap C),$$

and so by subadditivity of μ^*, we get

$$\mu^*(A \cap (B \cup C)) \leq \mu^*(A \cap B \cap C) + \mu^*(A \cap B \cap C^c) + \mu^*(A \cap B^c \cap C).$$

Subbing this latter inequality into the first and noticing that $B^c \cap C^c = (B \cup C)^c$ gives us that

$$\mu^*(A) \geq \mu^*(A \cap (B \cup C)) + \mu^*(A \cap (B \cup C)^c).$$

As the reverse inequality is covered by the subadditivity of μ^*, the two are equivalent and so $B \cup C \in \mathcal{M}(\mu^*)$. To show it is closed under countable unions, let $\{E_n\}_{n \in \mathbb{N}} \in \mathcal{M}(\mu^*)$. Define $B_j = E_j \setminus \bigcup_{i=1}^{j-1} E_i$ so that $E_i \cap E_j = \emptyset$ for $i \neq j$ and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i.$$

Then for all $A \subset X$,

$$\mu^*\left(A \cap \bigcup_{i=1}^{n} B_i\right) = \mu^*\left(A \cap \bigcup_{i=1}^{n} B_i \cap B_n\right) + \mu^*\left(A \cap \bigcup_{i=1}^{n} B_i \cap B_n^c\right) = \mu^*(A \cap B_n) + \mu^*\left(A \cap \bigcup_{i=1}^{n-1} B_i\right)$$

which iteratively gives us that

$$\mu^*\left(A \cap \bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu^*(A \cap B_i).$$

Since we showed that $\mathcal{M}(\mu^*)$ is closed under finite unions and by the monotonicity of μ^* ($\bigcup_1^n B_i^c \supset \bigcup_1^\infty B_i^c$), we have

$$\begin{aligned}
\mu^*(A) &= \mu^*\left(A \cap \bigcup_{i=1}^{n} B_i\right) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right) \\
&= \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right) \\
&\geq \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right).
\end{aligned}$$

As $n \to \infty$,

$$\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right) \\
&\geq \mu^*\left(A \cap \bigcup_{i=1}^{\infty} B_i\right) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right).
\end{aligned}$$

The reverse inequality is again given by subadditivity of μ^* and so we get that

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i \in \mathcal{M}(\mu^*)$$

and so $\mathcal{M}(\mu^*)$ is a σ-algebra. To prove the second half of the theorem, that μ^* is a complete measure with respect to $\mathcal{M}(\mu^*)$, note that if $\mu^*(B) = 0$, we have for all $A \subset X$, that

$$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A \cap B^c) \leq \mu^*(A)$$

which implies $B \in \mathcal{M}(\mu^*)$ and so μ^* restricted to $\mathcal{M}(\mu^*)$ is complete. Additionally, we have by definition of an outer measure, that $\mu^*(\emptyset) = 0$ and μ^* is monotonic and so to prove μ^* is a measure, it suffices to show that it is countably additive. Note that we have already iteratively established that $\mu^*(A \cap \bigcup\limits_{i=1}^{\infty} B_n) = \sum\limits_{i=1}^{\infty} \mu^*(A \cap B_i)$ and so if we take $A = \bigcup\limits_{i=1}^{\infty} B_i$ we get that

$$\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu^*\left(A \cap \bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu^*(A \cap B_i) = \sum_{i=1}^{\infty} \mu^*(B_i)$$

which implies that μ^* is countably additive and indeed a measure restricted to $\mathcal{M}(\mu^*)$.

Examples:

- Halmos 11.6: (Inclusion-Exclusion for outer measure) If μ^* is an outer measure on X and if E and F are two sets in $\mathcal{P}(X)$ of which at least one is μ^*-measurable, then we'll prove
$$\mu^*(E) + \mu^*(F) = \mu^*(E \cup F) + \mu^*(E \cap F).$$
Proof: Without loss of generality, assume F is μ^*-measurable (if they're both μ^*-measurable, the same steps apply). By definition, we know
$$\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c)$$
so then plugging this in to the LHS gives
$$\mu^*(E)+\mu^*(F) = \mu^*(E\cap F)+\mu^*(E\cap F^c)+\mu^*(F) \geq \mu^*(E\cap F)+\mu^*((E\cap F^c)\cup F) = \mu^*(E\cap F)+\mu^*(E\cup F).$$
To prove the reverse inequality, note that
$$\mu^*(E \cap F) = \mu^*(E) - \mu^*(E \cap F^c)$$
and by monotonicity
$$\mu^*(E \cup F) \geq \mu^*(F).$$
Combining these two and working with the RHS, we get
$$\mu^*(E\cup F)+\mu^*(E\cap F) = \mu^*(E\cup F)+\mu^*(E)-\mu^*(E\cap F^c) \geq \mu^*(E\cup F)+\mu^*(E) \geq \mu^*(F)+\mu^*(E).$$

- Hahn-Kolmogorov Theorem: A premeasure μ_0 on a semiring $\mathcal{A}$ extends to a measure μ on $\sigma(\mathcal{A})$. If η is another measure on $\sigma(\mathcal{A})$ that extends μ_0, then $\eta(A) \leq \mu(A)$ for all $A \in \sigma(\mathcal{A})$ with equality when A has finite measure. If μ_0 is σ-finite, it follows by the π-λ theorem that μ is the unique extension of μ_0 to a measure on $\sigma(\mathcal{A})$. To prove this theorem, it is quite simple since we have done the heavy lifting proving the premeasure theorem and Caratheodory's Extension theorem. Notice that from the premeasure theorem, $\mathcal{A} \subset \mathcal{M}(\mu^*)$ and from Caratheodory's Extension theorem, $\mathcal{M}(\mu^*)$ is a σ-algebra, so by definition of the minimal σ-algebra, we have that $\sigma(\mathcal{A}) \subset \mathcal{M}(\mu^*)$ and μ^* is a complete measure on $\sigma(\mathcal{A})$.

- Tao 6.15: Let $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ be a σ-finite premeasure, let $\mu : \sigma(\mathcal{B}_0) \to [0, \infty]$ be the Hahn-Kolmogorov extension of μ_0 and let $\mu' : \mathcal{C} \to [0, \infty]$ be another countably additive extension of μ_0. We will show that μ and μ' agree on $\sigma(\mathcal{B}_0) \cap \mathcal{C}$. First, notice that for $E \in \mathcal{C}$, there exists a sequence of sets, $\{A_n\}_{n\in\mathbb{N}}$, covering E where the $A_n \in \mathcal{B}_0$. By countable subadditivity, we know that
$$\mu'(E) \leq \sum_n \mu'(A_n) = \sum_n \mu_0(A_n)$$
which implies that $\mu'(E)$ is a lower bound for the set
$$\mathcal{A} = \left\{ \sum_n \mu_0(A_n) : E \subset \bigcup_n A_n \right\}.$$

By construction, we know that μ is defined as the outer measure μ^* which is equivalent to the infimum of the set $\mathcal{A}$ and so $\mu(E)$ is additionally a lower bound albiet the greatest lower bound, thus $\mu(E) \geq \mu'(E)$. To show uniqueness, i.e. that $\mu(E) = \mu'(E)$ for $E \in \sigma(\mathcal{B}_0)$, notice that the two measures agree on $\mathcal{B}_0$ which is a semiring and thus closed under intersections and thus a π-system. By the π-λ theorem, we can employ the λ-system

$$\mathcal{L} = \left\{A \in \sigma(\mathcal{B}_0) : \mu(A) = \mu'(A)\right\}$$

which contains $\mathcal{B}_0$ and so $\sigma(\mathcal{B}_0) \subset \mathcal{L}$ which implies that the two measures agree on the generated σ-algebra.

- Folland 1.4.18: Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ denote the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ denote the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be the premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

  (a) We will prove that for all $E \subset X$, $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$. To do so, note that there always exists a cover for any subset of X in $\mathcal{A}_\sigma$ since $X \in \mathcal{A}_\sigma$ contains X itself. Thus for $\epsilon > 0$ and $E \subset X$, let $\{A_n\}_{n\in\mathbb{N}}$ denote a countable cover for E where $A_n \in \mathcal{A}_\sigma$ for all n and $\sum_n \mu_0(A_n) \leq \mu^*(E) + \epsilon$ (this inequality is guaranteed by definition of the infimum). Thus since the induced outer measure agrees with the premeasure on $\mathcal{A}$ which is a superset of $\mathcal{A}_\sigma$ and letting $A = \bigcup_n A_n \in \mathcal{A}_\sigma$, we have

  $$\mu^*(A) = \mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n) = \sum_n \mu_0(A_n) \leq \mu^*(E) + \epsilon.$$

  (b) We will prove that E is μ^*-measurable iff there exists a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$. For ($\Rightarrow$), we know E is μ^*-measurable iff for all $A \subset X$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Thus our claim reduces to showing there exists a $B \subset X$ such that $\mu^*(B \cap E^c) = 0$. By part (a), we can define a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$ where $A_n \in \mathcal{A}_\sigma$, $E \subset A_n$, and $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$. Defining $B_n = \bigcap_n A_n$ and taking the limit as $n \to \infty$ we see by monotonicity of measure that

  $$\mu^*(B) = \mu^*\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \lim_{n\to\infty} \mu^*(A_n) \leq \mu^*(E)$$

  and also that $E \subset B$ since $E \subset B_n$ for all n. Note that this limiting intersection is contained in $\mathcal{A}_{\sigma\delta}$ and by hypothesis, we have that

  $$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) \iff \mu^*(E) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c).$$

  Here we see that $B \cap E = E$ since $E \subset B$, so

  $$\mu^*(E) \geq \mu^*(E) + \mu^*(B \cap E^c) \iff \mu^*(B \cap E^c) = 0$$

  since $\mu^* \geq 0$. For ($\Leftarrow$), we know using the hypotheses that

  $$\mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(B \cap E) = \mu^*(E) \leq \mu^*(B)$$

  since $E \subset B$ and μ^* is monotonic. The reverse direction follows by subadditivity of outer measure, and so we get equality which implies that E is μ^*-measurable.

- Caratheodory's Extension theorem also shows us that given any measure space $(\Omega, \mathcal{F}, \mu)$, we can increase the σ-algebra to make the space complete (all subsets of a set of measure zero are also measure zero). The proof of this is a direct result of the theorem, we can show that the induced outer measure of the measure restricted to $\mathcal{M}(\mu^*)$, $\mu^*|\mathcal{M}(\mu^*)$ formed in the proof of the theorem is complete. If $A \subset B$ and $B \in \mathcal{M}(\mu^*)$, then by μ^*-measurability, we know

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E) + \mu^*(B) = \mu^*(E)$$

where the first inequality followed since $E \cap A \subset A \subset B$ and $E \cap A^c \subset E$ (monotonicity of outer measure). To prove equality, the reverse inequality follows by subadditivity, and so $A \in \mathcal{M}(\mu^*)$ which proves the result.

- Billingsley 3.5(a): As mentioned above, a probability space $(\Omega, \mathcal{F}, P)$ has a complete extension - that is, a complete probability space $(\Omega, \mathcal{F}_1, P_1)$ such that $\mathcal{F} \subset \mathcal{F}_1$ and P_1 agrees with P on $\mathcal{F}$. Let us suppose that $(\Omega, \mathcal{F}_2, P_2)$ is a second complete extension. We will show by example that in a space $\Omega = \{a, b\}$ that P_1 and P_2 need not agree on $\mathcal{F}_1 \cap \mathcal{F}_2$. First let $\mathcal{F}$ be the trivial σ-algebra - $\mathcal{F} = \{\Omega, \emptyset\}$. Then define

$$\mathcal{F}_1 = \mathcal{F}_2 = \{\{a\}, \{b\}, \Omega, \emptyset\}$$

where $P_1(\Omega) = P_2(\Omega) = 1$ and $P_1(\emptyset) = P_2(\emptyset) = 0$ so they agree with P on $\mathcal{F}$. However these two clearly don't have to agree on $\mathcal{F}_1 \cap \mathcal{F}_2$ since we can let $P_1(\{a\}) = 1$ and $P_1(\{b\}) = 0$ while $P_2(\{a\}) = 0$ and $P_2(\{b\}) = 0$. In fact we could even introduce a third measure that doesn't agree with either - as we extend this into bigger and less trivial spaces, the number of possibilities of non-unique complete extensions rises as well.

- Billingsley 3.5(b): Again let $(\Omega, \mathcal{F}, P)$ be a probability space. We will show that there is a **unique** minimal complete extension: Let $\mathcal{F}^+$ consist of the sets A for which there exist $\mathcal{F}$-sets B and C such that $A\Delta B \subset C$ and $P(C) = 0$. We will first show this is a σ-algebra. If $A \in \mathcal{F}^+$, then $A\Delta B \subset C$ and $P(C) = 0$ for some $B, C \in \mathcal{F}$. Since $\mathcal{F}$ is a σ-algebra, it also possesses B^c and so since $A^c\Delta B^c = A\Delta B \subset C$, $A^c \in \mathcal{F}^+$ which implies $\mathcal{F}^+$ is closed under complementation. To show it is closed under countable unions. let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}^+$-sets. As they are $\mathcal{F}^+$-sets, there is a corresponding sequence of $\mathcal{F}$-sets $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ such that $A_i\Delta B_i \subset C_i$ and $P(C_i) = 0$ for all i. Notice that this implies that

$$\bigcup_{i=1}^{\infty} A_i\Delta B_i \subset \bigcup_{i=1}^{\infty} C_i.$$

For all $n \in \mathbb{N}$, we have that

$$\bigcup_{i=1}^{n} A_i \Delta \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \backslash \bigcup_{i=1}^{n} B_i \cup \bigcup_{i=1}^{n} B_i \backslash \bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} A_i \backslash B_i \cup \bigcup_{i=1}^{n} B_i \backslash A_i = \bigcup_{i=1}^{n} A_i\Delta B_i \subset \bigcup_{i=1}^{n} C_i$$

and since this holds for all n, the limiting union $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}^+$ which implies closure under countable unions and thus that $\mathcal{F}^+$ is a σ-algebra. Now, for all $A \in \mathcal{F}^+$, define

$P^+(A) = P(B)$. We will show that P^+ is indeed a probability measure on $\mathcal{F}^+$ and that $(\Omega, \mathcal{F}^+, P^+)$ is complete. First, if we let $A = B = \emptyset$, then $A\Delta B \subset C$ for any C of measure 0 and so $P^+(\emptyset) = P(\emptyset) = 0$. Similarly, if $A = \Omega$, then we can let $B = \Omega$ so $A\Delta B = \emptyset$ while $P^+(A) = P(B) = 1$. To show countable additivity, let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint sequence of $\mathcal{F}^+$-sets with corresponding sequences $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$-sets, then it suffices to construct a sequence of disjoint $\mathcal{F}$-sets using the two above from which we use that P is a probability measure to conclude additivity. If $A_i \subset A_j$ and $A_j \in \mathcal{F}^+$, then there exists a $B_j \in \mathcal{F}$ such that $A_j \Delta B_j \subset C_j$. Let $B_i = B_j$ and $C_i = C_j \cup B_j$, then

$$A_i \Delta B_j = A_i \setminus B_j \;\cup\; B_j \setminus A_i \subset A_j \setminus B_j \;\cup\; B_j \setminus A_i \subset A_j \setminus B_j \;\cup\; B_j \setminus A_j \;\cup\; B_j$$

$$= A_j \Delta B_j \;\cup\; B_j \subset C_j \;\cup\; B_j.$$

As P is a probability measure, it follows that

$$P(C_j \cup B_j) \leq P(C_j) + P(B_j) = P(C_j) + P^+(A) = 0$$

and so we have as desired that $A_i \in \mathcal{F}$ which implies that $(\Omega, \mathcal{F}^+, P^+)$ is complete. Finally, I will show that if $(\Omega, \mathcal{F}_1, P_1)$ is any other complete extension of $(\Omega, \mathcal{F}, P)$, then $\mathcal{F}^+ \subset \mathcal{F}_1$ and $P_1(A) = P^+(A)$ for all $A \in \mathcal{F}^+$. Suppose $(\Omega, \mathcal{F}_1, P_1)$ is any other complete extension. For any $A \in \mathcal{F}^+$, we know there exist $B, C \in \mathcal{F}$ such that $A\Delta B \subset C$ where $P(C) = 0$. By definition, $(\Omega, \mathcal{F}_1, P_1)$ is complete and $B, C \in \mathcal{F} \subset \mathcal{F}_1$, so $A\Delta B \subset C$ must be contained in $\mathcal{F}_1$. As σ-algebras are closed under symmetric differences, this implies that $(A\Delta B)\Delta B = A$ is in $\mathcal{F}_1$ which proves that $\mathcal{F}^+$ is the minimal σ-algebra for the complete space.

3. Appendix

Useful Set Theoretic Operations: Let $(A_n)_{n\in\mathbb{N}}$ be a family of subsets of a set X and let $f : X \to E$ be an arbitrary mapping. Then

1. $X \setminus \bigcup_{n\in\mathbb{N}} A_n = X \cap \left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = X \cap \left(\bigcap_{n\in\mathbb{N}} A_n^c\right) = \bigcap_{n\in\mathbb{N}} X \cap A_n^c = \bigcap_{n\in\mathbb{N}} X \setminus A_n$

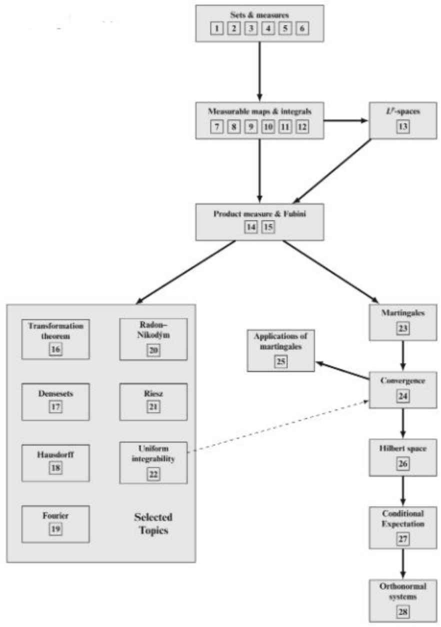
2. $X \setminus \bigcap_{n\in\mathbb{N}} A_n = X \cap \left(\bigcap_{n\in\mathbb{N}} A_n\right)^c = X \cap \left(\bigcup_{n\in\mathbb{N}} A_n^c\right) = \bigcup_{n\in\mathbb{N}} X \setminus A_n$

3. $f\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} f(A_n)$

4. $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n)$

5. Symmetric Difference (when discussing outer measures, this pops up quite a bit):

$$A + B = A\Delta B = (A - B) \cup (B - A)$$



(i) This is how one should progress through topics in measure theory and its relation to probability/statistics

Axiom of Choice: Informally, the axiom states that if you have a collection of sets, then there is a way to select one element from each set. Formally, there's a great article if you have time by a professor at UToronto: http://www.math.toronto.edu/ivan/mat327/docs/notes/11-choice.pdf

Hereditary Set: A nonempty class of sets $\mathcal{H}$ is called hereditary if, whenever $E \in \mathcal{H}$ and $F \subset E$, then $F \in \mathcal{H}$.

Semiring: A class $\mathcal{A}$ of subsets of Ω is a semiring provided

1. $\in \mathcal{A}$
2. $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$
3. If $A, B \in \mathcal{A}$ where $A \subset B$, then there exist $C_1, \ldots, C_n \in \mathcal{A}$ such that $B \cap A^c = \bigcup_{k=1}^{n} C_k$ and $C_i \cap C_j =$ for $i \neq j$.

Diameter (From Baby Rudin) Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $\rho(p, q)$, with $p, q \in E$. The supremum of S is called the diameter of E and is denoted by $diam(E)$.