Maths and Voting: Gerrymandering and Apportionment

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Abstract

In this paper, we analyse the apportionment and districting steps of electing representatives to a legislative body with a particular focus on the United States of America. We first explore the process where representatives, and therefore seats, are apportioned amongst the states. We look at past and present Apportionment methods, before examining how Pairwise Bias can be used to evaluate these methods, leading up to our proof of the Balinksi-Young Impossibility theorem. We then move on to the second section of our paper where we look at attempts to quantify gerrymandering and its effect on the districting process. Here we analyse compactness and symmetry measures together with the efficiency gap before using them to assess the redistricting cycle in Indiana after the 2010 census.

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Chapter 1

Introduction

1.1 The United States Congress

The legislative branch of government in the United States of America consists of two bodies, the House of Representatives and the Senate [1]. These bodies are charged with drawing up legislation on behalf of the United States of America and combine to form the United States Congress. The Senate consists of 100 senators, two for each state with term lengths of six years [2]. These term lengths are staggered so that approximately one-third of the senates seats are reelected every two years with the senators elected by popular vote over each state as a whole.

The other branch of Congress, the House of Representatives, consists of 435 representatives [3] elected to two-year terms, every even year. These seats are not distributed uniformly as in the Senate, rather, they are apportioned in relation to state population. More populous states receive a larger number of representatives than less populous states. The most populous state, California, with a current population of approximately 40 million [4], receives 53 seats. However, the least populated, Wyoming [4] receives only one seat.

Once the apportionment has been carried out, states must then elect officials to represent the views of their populations. The vast majority of states split themselves into districts corresponding to the number of representatives that they are apportioned. In each district, a popular vote takes place to determine the representative of that district. There are a few states such as Alaska, Delaware, Montana, North Dakota, South Dakota, Vermont and Wyoming which use one at-large [5] district as they are only entitled to one representative.

1.2 Organisation of the Paper

In this paper, we analyse and evaluate the process of apportionment and districting with a primary focus on the congressional elections for the United States House of Representatives.

In our second chapter, we start with an Introduction to Apportionment where we look at the different methods that have been suggested and implemented in the United States. We describe these methods and examine some of their inconsistencies.

Firstly, the basic rules of apportionment are addressed and the first definition of the Quota Rule is also given. This is a key rule since if this is broken, then a method cannot be deemed to be completely fair all of the time.

In the second section of this chapter, we address the Alabama, Population and New States paradoxes and how they highlight weaknesses in different apportionment methods.

Next, we discuss how a method of apportionment can be deemed to be biased or exhibit favouritism towards certain states. In doing so, we can gain insight into whether or not a method is fair enough to be used in congress.

The fourth section of our Apportionment chapter discusses the Balinski-Young Impossibility theorem. This is a major theorem in the topic of apportionment as it proves that no method can be completely fair.

We then apply the techniques learned to more recent census data by analysing the different methods' apportionments. The favouritism of these methods towards larger and smaller states is also investigated as this is a key measure in determining fairness.

We then move on to the third chapter and second theme of our paper: Gerrymandering. The first section of this chapter is our Introduction to Gerrymandering where we explain how a districting plan may be gerrymandered through the terminology of Wasted Votes and the process of Packing & Cracking.

The following three sections are all methods of measuring how gerrymandered a mapping may be, these are Compactness, Symmetry and the Efficiency Gap.

Compactness derives from the intuitive notion that a compactly shaped district is preferred to a sprawling one and we use this concept to create measures that rank districting plans in order of their compactness.

Symmetry refers to the principle that there should be parity between parties such that a party winning a certain proportion of the votes necessarily leads to them winning a certain number of seats irrespective of which party it is. We then survey a series of measures that are constructed so as to rank districting plans according to how closely they abide by this principle.

We then discuss the efficiency gap which tries to quantify the number of wasted votes as this may be indicative of an attempt to try and pack and crack the districts.

Finally, we use measures from each of these three sections on the two different districting plans used for congressional elections in Indiana before and after the 2010 census and the redistricting cycle that accompanies it.

Chapter 2

Apportionment

2.1 Introduction to Apportionment

Apportionment, in a political context, is the act of distributing the seats of a legislative body to those entitled to representation. This chapter will mainly focus on the U.S. congressional apportionment, in which seats in the United States House of Representatives are divided among the 50 states. The idea is that each state gets a number of seats roughly proportional to its proportion of the population of the country. There have been many different methods of apportionment over the years but there has yet to be a method devised with universal approval.

We are interested in two of the main types of methods, natural quota methods and divisor methods. For these apportionment methods, we are interested in investigating the relationship between states' populations and their number of representatives.

If we are apportioning h seats between S states with s_i and p_i representing state i and its population respectively, we calculate the natural quota q_i of state i using the formula:

$$q_i = \frac{p_i h}{\sum_j p_j}. \tag{2.1}$$

Rounding the natural quota down to the nearest integer value, $\lfloor q_i \rfloor$, gives us the lower quota and rounding it up to the nearest integer value, $\lceil q_i \rceil$, gives us the upper quota which we will be denoting $\underline{q_i}$ and $\overline{q_i}$ respectively. We shall also denote the apportionment of state i by a_i.

There are rules that need to be followed when we consider natural quotas, for example, the Quota Rule [6] which we will look at in greater detail later. Divisor methods, on the other hand, use different denominators or modified divisors when calculating the quota.

When investigating these apportionment methods, we come across certain inconsistencies or paradoxes. We will explore three main examples of these, namely the Alabama, Population and New-States paradoxes, and identify when these occur.

Finally, we will be delving into the proof of the renowned Balinski Young Impossibility Theorem that is vital when considering apportionment methods. The theorem states that there is no apportionment method that simultaneously satisfies the quota rule and does not fall prey to the apportionment paradoxes. Ultimately, this theorem shows that it is impossible for an apportionment method to be completely fair.

2.1.1 Apportionment Rules

There are many laws that methods of apportionment must follow to be deemed fair enough. Article 1, section 2, clause 3 of the United States Constitution states that "each State shall have at Least one Representative" [7]. This is equivalent to saying $a_i \geq 1 \,\forall\, i \in S$, so that all states, including those with proportionally small population sizes, are still awarded representation.

Next, we consider the Quota Rule which states that "the number of seats that should be allocated should be between the upper and lower quotas" [6]. That is, for a given state i,

$$\underline{q_i} \leq a_i \leq \overline{q_i}. \tag{2.2}$$

We know that $\underline{q_i}$, $\overline{q_i} \in \mathbb{N}$ are consecutive, positive integers and since $a_i \in \mathbb{N}$ as well, we can see that in order for an apportionment method to satisfy the quota rule, $a_i = \underline{q_i}$ or $a_i = \overline{q_i}$.

2.1.2 Past Methods

Throughout history, many different methods have been suggested and implemented in the United States, however, every method displays inconsistencies rendering them potentially unfair.

Within this section, we will consider examples of apportionment methods some of which were deemed fair enough to be implemented in legislation and others that were not. The two main types of apportionment are the divisor and natural quota methods.

Natural Quota Methods

A natural quota method is an apportionment method in which the house size is decided before the apportionment takes place. Then, the natural quota calculated by (2.1) would be manipulated in some way such that a_i is an integer value. The way in which this natural quota is manipulated is what leads to the different methods seen below.

Hamilton

Firstly, we will consider the method of Alexander Hamilton which was also known as the largest remainder method. This method was invented by Hamilton in 1792 and "was first adopted to apportion the U.S. House of Representatives every ten years" [8].

The method of Hamilton is quite simple and, as with all natural-quota methods, the house size is to be fixed before the apportionment occurs. First, we need to calculate the natural quota, using (2.1), for each state and rank each state in order from smallest to largest remainder. Next, round every state's quota down to the nearest integer value (lower quota) and then assign the remaining seats yet to be apportioned to the "states having the largest fractions or remainders" [9, pg. 17]. A worked example of this method is shown in Table 2.1. We see that upon assigning each state their lower quota of seats, there are 2 remaining to be assigned. Hence we award them to states E and C.

State	Population	q_i	Remainder rank	$\lfloor q_i \rfloor$	a_i
A	152	1.013	5	1	1
B	347	2.313	4	2	2
C	54	0.360	2	0	1
D	502	3.347	3	3	3
E	445	2.967	1	2	3
Total	1500	10	-	8	10

Table 2.1: Example of Hamilton's method for a house size $h = 10$.

Lowndes

Unlike most other methods explored in this paper, "Lowndes's method has never been used to apportion Congress" [10]. This method works similar to the previously discussed method of Alexander Hamilton. However, instead of ranking the remainders of each states' natural quotas, we divide each remainder by the lower quota and then rank those proportions. We then follow Hamilton's method again to assign the remaining seats to the highest-ranked states first. An example of this method is shown in Table 2.2.

State	Population	q_i	Remainder r_i	$\lfloor q_i \rfloor$	$\frac{r_i}{\lfloor q_i \rfloor}$	$\frac{r_i}{\lfloor q_i \rfloor}$ rank	a_i
A	152	1.013	0.013	1	0.013	5	1
B	347	2.313	0.313	2	0.157	3	2
C	54	0.360	0.360	0	-	1	1
D	502	3.347	0.347	3	0.116	4	3
E	445	2.967	0.967	2	0.484	2	3
Total	1500	10	-	8	-	-	10

Table 2.2: Example of Lowndes' method for a house size $h = 10$.

In this example, state C has a lower quota of zero, hence we cannot calculate a value for $\frac{r_C}{\lfloor q_C \rfloor}$. However, we assign this state rank 1 since, according to Article 1, section 2, clause 3 of the United States Constitution [7], every state requires representation. Therefore, we assign one of the extra seats to this state.

A smaller state has a lower natural quota in comparison to a larger state, hence when we calculate $\frac{r_i}{\lfloor q_i \rfloor}$, smaller states will tend to have a higher rank than the larger states. Therefore, we can theorize that Lowndes' method favours smaller states.

Divisor Methods

A state's quota is calculated by dividing its population by a given divisor. The natural quota is calculated using the standard divisor which is given by $\frac{\sum_j p_j}{h}$. Opposed to natural quota methods, divisor methods use divisors other than the standard divisor. We call these modified divisors and they are used to calculate the modified quotas in the same way the standard divisor is used to find the natural quotas. Different divisor methods use different modified divisors and deal with the modified quotas differently.

Jefferson

Jefferson's method was used in Congress from 1791 until 1842 [11] and the first step is to find the standard divisor. The natural quotas are then calculated and the lower quotas are summed. If the sum of the lower natural quotas are equal to the house size we are done and each state is apportioned its lower natural quota. If the sum is not equal to the house size, we must reduce our divisor until it is. An example is shown in Table 2.3

State	Pop.	q_i	$\underline{q}_i$	q_i^*	$\underline{q}_i^*$	a_i
A	152	1.013	1	1.250	1	1
B	347	2.313	2	2.892	2	2
C	104	0.693	0	0.867	0	0
D	527	3.513	3	4.392	4	4
E	370	2.467	2	3.083	3	3
Total	1500	10	8	-	10	10

Table 2.3: Jefferson using modified divisor 120.

The standard divisor is given by: $\frac{\sum_j p_j}{h} = \frac{1500}{10} = 150$ which is used to calculate our lower natural quotas. The lower quotas only summed to 8 so we had to reduce our divisor until the sum equalled our house size of 10. We see the modified divisor, 120, works so is used to calculate our modified quotas, q_i^*. The sum of our lower modified quotas is now equal to 10 so each state is apportioned their modified lower quotas.
Note that State C was apportioned no seats. As we read earlier this violates the Constitution of the United States so would be considered an injustice. So to be used in the US, Jefferson's method would have to be adjusted so each state receives at least one representative.

D'Hondt

The D'Hondt method is an iterative method in which, at each iteration, the state with the highest quota is given a seat. The quota is calculated, at each iteration, by,

$$q_i = \frac{p_i}{s_i + 1}, \tag{2.3}$$

where q_i is the quota of state i, p_i is the population of state i and s_i is the number of seats already won by state i. So if a state has the highest quota and wins a seat, in the next iteration its value of s will increase by 1. Below we have an example where our final apportionment is given in Table 2.4 and our quotas and seat numbers at each iteration are shown in Table 2.5

State	Pop.	a_i
A	152	1
B	347	2
C	104	0
D	527	4
E	370	3

Table 2.4: D'Hondt method's apportionment.

i	1	2	3	4	5	6	7	8	9	10
q_A	152	152	152	152	152	152	152	152	76	76
s_A	0	0	0	0	0	0	0	1	1	1
q_B	347	347	347	173.5	173.5	173.5	173.5	115.7	115.7	115.7
s_B	0	0	1	1	1	1	2	2	2	2
q_C	104	104	104	104	104	104	104	104	104	104
s_C	0	0	0	0	0	0	0	0	0	0
q_D	527	263.5	263.5	263.5	175.7	175.7	131.75	131.75	131.75	105.4
s_D	1	1	1	2	2	3	3	3	4	4
q_E	370	370	185	185	185	123.3	123.3	123.3	123.3	123.3
s_E	0	1	1	1	2	2	2	2	2	3

Table 2.5: Iterations of D'Hondt Method

Again, State C is apportioned no seats, so if applied to the US, D'Hondt's method would need to be adjusted such that each state receives at least one representative.

Webster

Webster's method begins as in Jefferson's method, by finding the standard divisor and natural quotas. Then, each state's natural quota is rounded to the nearest integer and summed. If the sum of the rounded natural quotas is equal to the house size we are done and each state is apportioned its rounded natural quota. If the sum is not equal to the house size we must use a modified divisor to calculate the modified quotas. Unlike Jefferson's method, the first step towards finding an appropriately modified divisor is not necessarily to reduce the divisor. If the sum is less than the house size we reduce the divisor and if it is greater than the house size, we increase it.

State	Pop.	q_i	$\lfloor q_i \rceil$	a_i
A	152	1.013	1	1
B	347	2.313	2	2
C	104	0.693	1	1
D	527	3.513	4	4
E	370	2.467	2	2
Total	1500	10	10	10

Table 2.6: Webster's Method.

In Table 2.6, we see an example of Webster's method. Here, our rounded natural quotas sum to the house size of 10 so we do not need to adjust our divisor, therefore, we apportion each state its rounded natural quota.

Huntington-Hill

The Huntington-Hill method, also known as the method of equal proportions, is the method currently used in the US House of Representatives and has been since 1941 [4]. It is an iterative method that uses the following formula to calculate state quotas at each iteration,

$$q_i = \frac{p_i}{\sqrt{s_i(s_i + 1)}}, \tag{2.4}$$

where once again q_i is the quota of state i, p_i is the population of state i and s_i is the number of seats already won by state i. Below, we have an example where the final apportionment is shown in Table 2.7 and the quotas and seat numbers at each iteration are shown in Table 2.8.

State	Pop.	a_i
A	152	1
B	347	2
C	104	0
D	527	4
E	370	3

Table 2.7: Huntington-Hill Apportionment.

Again State C is apportioned no seats which would not be politically acceptable in the US and therefore this method would need to be adjusted so that each state is apportioned at least one seat.

i	1	2	3	4	5	6	7	8	9	10
q_A s_A	152 0	152 0	152 0	152 0	152 0	152 0	152 0	152 0	152 1	107.5 1
q_B s_B	347 0	347 0	347 0	347 1	245.4 2	141.7 2	141.7 2	141.7 2	141.7 2	141.7 2
q_C s_C	104 0	104 0	104 0	104 0	104 0	104 0	104 0	104 0	104 0	104 0
q_D s_D	527 1	372.6 2	215.1 2	215.1 2	215.1 2	215.1 2	215.1 3	152.1 4	117.8 4	117.8 4
q_E s_E	370 0	370 0	370 1	216.6 1	216.6 1	216.6 2	151.1 2	151.1 2	151.1 2	151.1 3

Table 2.8: Iterations of Huntington-Hill Method

2.2 Paradoxes

There are a number of apportionment paradoxes that we will consider in the following section, namely the Alabama Paradox, the Population Paradox and the New States Paradox. Examples of these phenomena have been seen in the past in the United States of America.

In 1850, a new method of apportionment was enacted and was known as the "Vinton's method" [8]. However, this method is more commonly known as the Largest Remainder Method and is the same method proposed by Alexander Hamilton (See Section 2.1.2) that "was vetoed by President Washington" [12, pg. 378] in 1792. Although we will be mainly focusing on this method, all Natural Quota methods fall prey to all of the apportionment paradoxes in this section.

2.2.1 Alabama Paradox

The concept of the Alabama Paradox is quite simple: firstly, we assume that all states have a fixed population (i.e. p_i cannot change $\forall i \in S$). Then, we consider the Alabama Paradox to occur when the house size, h, increases and the apportionment of any state i, a_i, decreases. Clearly, in order for the method in question to be proportional, this cannot take place. We notice that this phenomenon arises when we consider Hamilton's (alias Vinton's or Largest Remainder) method when analysing the data from the 1870 census.

Definition 2.2.1 (Alabama Paradox)**.** Consider fixed state populations p_i for all states, then the Alabama Paradox occurs when an increase in the house size results in an electoral district or political party losing a seat. [8].

Once the census of 1870 had been conducted, the Census Bureau wished to investigate the effect of different house sizes on the apportionment produced by Hamilton's method. They did so "for all House sizes between 241 and 300" [9, pp. 38]. Upon this analysis, it was noticed that if the house size was 270, then the state of Rhode Island would be apportioned two seats. However, if the house size increased to 280, Rhode Island is now only apportioned one seat. It is clear to see from Table 2.9 that even though the natural quota for Rhode Island increased, under the Largest Remainder method the apportionment decreased.

State	Population	$q, h = 270$	a	$q, h = 280$	a
Rhode Island	217353	1.539	2	1.597	1

Table 2.9: Table to show the Alabama Paradox on the state of Rhode Island based on the 1870 US census.

However, this was not an isolated incident. In 1880, the chief clerk of the census office, W. Seaton, noticed that if the house size was increased from 299 to 300, then Alabama would unjustly lose out on their seats.

As we can see in Table 2.10, when the house size is $h = 299$, Alabama has its natural quota rounded up whereas both Illinois and Texas have theirs rounded down. However, when we increase the house size to $h = 300$, then each natural quota is increasing by 0.33%. Hence it is clear to see that for the smaller states, their natural quotas increase by a smaller amount than the larger states and this is shown by the increase column in Table 2.10.

State	Population	q_i	a_i	q_i	a_i	Increase
Alabama	$1,262,505$	7.646	8	7.671	7	0.025
Texas	$1,591,749$	9.640	9	9.672	10	0.032
Illinois	$3,077,871$	18.640	18	18.702	19	0.062
Total	$49,371,340$		299		300	

Table 2.10: Table to show the effects of the Alabama Paradox on 1880 US Census data [4].

Now, upon the increase of the house size, the natural quotas for Texas and Illinois are both greater than for Alabama so the new seat goes to Illinois and Alabama loses one of their seats to Texas. Despite all of the quotas increasing in the same proportion, Hamilton's method has once again displayed the Alabama Paradox.

There is a third time in which the Alabama paradox was apparent. In the analysis of the 1990 census data, the state of Maine's "delegation kept hopping up and down between 3 and 4 seats" [9, pg. 40] as the house size was increased gradually between 350 and 400.

In light of these examples, the United States Congress voted, in 1901, to reject Hamilton's method (known as Vinton's method at the time), over three decades after the emergence of the Alabama Paradox.

2.2.2 Population Paradox

The Alabama Paradox is not the only apportionment phenomenon to grace US history. Another paradox appeared in the decennial reapportionment of 1900-1901 and was named the Population Paradox. In the 1900 census data, "Virginia's population was growing 60% faster than Maine's" [13], yet during the reapportionment process Maine gained one extra seat and Virginia lost a seat. We can now begin to formulate a definition for this Population paradox.

Definition 2.2.2 (Population Paradox)**.** Consider a given house size h and a set of states S, then the Population Paradox occurs when a state with a lower population growth rate gains a seat from a state with a higher population growth rate. [8]

In a more general sense, if p_i increases and $p_j,\ i \neq j$ remains the same, then the population paradox occurs if state i loses a seat to any state j. We can see an example in Table 2.11.

State	Population A	$q_i^{(A)}$	$a_i^{(A)}$	Population B	$q_i^{(B)}$	$a_i^{(B)}$
A	150	8.333	8	150	8.251	8
B	**78**	4.333	**4**	**78**	4.290	**5**
C	173	9.611	10	181	9.956	10
D	204	11.333	11	204	11.221	11
E	**295**	16.389	**17**	**296**	16.282	**16**
Total	900		50	909		50

Table 2.11: Table to show the Population Paradox on 5 arbitrary states.

Here, the populations for states C and E increased from set A to B, whereas the others remained constant. However, state E loses a seat to state B, even though p_B did not change and p_E increased by 1. The difference between $q_B^{(A)}$ and $q_B^{(B)}$ is 0.043, whereas the difference between $q_E^{(A)}$ and $q_E^{(B)}$ is much greater at 0.107. The total population has increased, so when calculating the natural quotas using (2.1), the denominator has increased. Therefore, the larger states' natural quotas will decrease by a larger amount than a smaller states' natural quota.

2.2.3 New States Paradox

Finally, we will consider the New states Paradox. This paradox occurs upon the introduction of new states, hence increasing the total population in (2.1). We will consider 1907, the year when President Roosevelt "admitted Oklahoma as the forty-sixth state" [14]. In this era, Webster's method of apportionment was being used by the US government. However, if they had used Hamilton's method on this data, they would have observed a strange phenomenon which is now known as the new states paradox.

Definition 2.2.3 (New States Paradox)**.** When a new state is introduced, the New States Paradox occurs if a state previously in the union loses representation despite experiencing no population variation. [8]

In 1907 Oklahoma added $1,000,000$ people to the total US population. After this addition, their natural quota was 5.175 and they were allocated an apportionment of $a_{OK} = 5$. The house size was subsequently increased from 386 to 391 to accommodate for this.

State	Population (before OK)	q_i	a_i	Population (after OK)	q_i	a_i
Oklahoma	-	-	-	$1,000,000$	5.175	5
New York	$7,264,183$	37.606	38	$7,264,183$	37.589	37
Maine	$694,466$	3.595	3	$694,466$	3.594	4
Total	$74,562,608$		386	$75,562,608$		391

Table 2.12: Table to show the New States Paradox on data [15] from the 1907 reapportionment using the Largest Remainder Method.

In Table 2.12, we see that the natural quota for both the states of New York and Maine decreases. However, since New York is a larger state than Maine (i.e. $p_{NY} > p_{ME}$), q_{NY} actually decreases by a larger amount than q_{ME} and hence, Maine overtakes New York in terms of priority when apportioning the seats. This is an example of the New States Paradox.

2.3 Fairness

In the early 20th century, the method of apportionment was a big point of debate for both members of Congress and the academic community. Apportionment was now being studied as a mathematical problem with the main focus being on how states with large and small populations are treated differently. At the forefront of this topic, Walter Francis Willcox and Edward Vermilye Huntington argued bitterly over which method of apportionment was the fairest, both claiming the other's method exhibited bias. Willcox argued in favour of Webster's method and Huntington was in favour of Hill's method. Huntington made slight adjustments to Hill's method to create the Huntington-Hill method which has been used in the United States congressional apportionment since 1941. [9] The decision to choose the Huntington-Hill method is not fully understood since "the Constitution does not specify what fairness criteria should be used in comparing two different proposed ways to solve an apportionment problem." [16] In this chapter, we will give definitions of bias and find out which methods exhibit it and which do not. We will only consider divisor methods, as we will later see divisor methods are the only methods that do not fall prey to any of the paradoxes.

2.3.1 Methods of Apportionment

For any method of apportionment, we must have, for any s-vector, $\underline{p}$, and integer $h \geqslant 0$, an apportionment of h among s. A single-valued function will not work for this. The reason a single-valued function is not sufficient can be seen in the following scenario. Suppose two identically populated states must share an odd number of seats between them, call this $2k+1$. This leaves us with two options for the apportionment: we can give one of the states k seats and the other $k+1$ or vice-versa. Clearly both options are as fair as the other so a fair method must include both options as solutions.

As we have just seen a method of apportionment must be a multiple-valued function, M, that consists of a set of apportionments of h among s for any s-vector $\underline{p} \geqslant 0$ and integer $h \geqslant 0$. A particular M-solution is a single-valued function, $f(\underline{p}, h) = \underline{a} \in M(\underline{p}, h)$ that breaks every draw, like the one we just saw, in a random fashion [17].

Proportionality only concerns the size of populations. Proportionality means that whenever a problem can be perfectly solved with integers, it is. We say that a method of apportionment, M, is proportional if it satisfies the following two properties, [9]

- M is weakly proportional.
- As the house size increases solutions should tend towards the ideal of proportionality.

The first property means that if whenever an apportionment, $\underline{a}$ is proportional to population, $\underline{p}$, then $\underline{a}$ is the only M-apportionment for $\underline{p} > \underline{0}$ when $h = \sum a_i$.

The second property means that if $\underline{a'}$ is an M-apportionment for population, $\underline{p} > \underline{0}$ and if $\underline{a}$ is integer-valued and proportional to $\underline{a'}$ with $\sum a_i < \sum a'_i$ then $\underline{a}$ should be the unique apportionment in $M(\underline{p}, \sum a_i)$. So for example if two states split 100 seats between them using method M, each state getting 50 seats each then the only way they could split 50 seats between them using the same method would be for both states to get 25 seats.

Since we want our method of apportionment to be proportional, there are some properties it should exhibit. One of these is homogeneity, a method is homogeneous if all the M-apportionments for $\underline{p} \geqslant 0$ and $h \geqslant 0$ are the same as all the M-apportionments for $\lambda\underline{p}$ and h for any positive, rational λ.

A method must also be symmetric, meaning it only takes population into account. This means if we take a permutation of the populations, $\underline{p} = (p_1, p_2, ..., p_s)$, and apply our method of apportionment to it, we would just get the corresponding permutation of our apportionment, $\underline{a} = (a_1, a_2, ..., a_s)$.

2.3.2 Pairwise Bias

When talking about bias in a method, we will be referring to whether or not it favours smaller states or larger states. First, we must ask what is bias? It is, in essence, a systematic tendency to favour some states over others. This means it is not enough to look at one apportionment and conclude that the method is biased, we have to see a trend. Since in practice no two states can be expected to have the same ratio of representation to population.

We say that a method, M', favours small states relative to another method, M, if for any M-apportionment, $\underline{a}$, and any M'-apportionment, $\underline{a'}$, for populations $\underline{p}$ and house size h, $p_i < p_j$ implies either $a'_i \geqslant a_i$ or $a'_j \leqslant a_j$. [9]

There is also an absolute meaning of favouritism. For any apportionment method that apportions a_1 and a_2 to states with populations p_1 and p_2 respectively with

$p_1 > p_2 > 0$ we say that the method favours the larger state if $\frac{a_1}{p_1} > \frac{a_2}{p_2}$ and favours the smaller state if $\frac{a_1}{p_1} < \frac{a_2}{p_2}$ [9].

In any implementation of a method of apportionment some states must be favoured and others disadvantaged. To show a method is biased we would have to show that it favours small or large states over many implementations. We can test many different pairs of states to see if a method is biased. There are different ways in which we can do this.

A simple method is to consider a pair of populations $p_1 > p_2 > 0$. Two states with these populations could divide any number of seats, h, between them, and since we are only considering divisor methods, the way in which the two states would share the seats is independent of any other states that may be a part of the same apportionment. Due to the fact both p_1 and p_2 are integers, there must exist a perfect house size h^* where both states quotas are integer values.

We can then, for any pair of populations and any method, M, count the number of apportionments favouring the smaller states, $S(p_1, p_2)$, and the number of apportionments favouring the larger states, $L(p_1, p_2)$ over all the apportionments (a_1, a_2) such that $a_1 + a_2 \leqslant h^*$. If $L(p_1, p_2) = S(p_1, p_2)$ for every pair of populations, we can say that the method, M, is pairwise unbiased on populations [9].

A benefit of this method of measuring bias is that no assumption has to be made on the distribution of the populations, however, this method is not very realistic. As an empirical test it has its limits as the number of seats shared between two states will typically be much smaller than the number of seats, h^*.

A more realistic way to check if a method is pairwise bias is to consider a pair of integer apportionments, $a_1 > a_2 > 0$. We then ask if the populations, (p_1, p_2), have the M-apportionment (a_1, a_2), how likely is it the small state is favoured?

To answer this we can take a probabilistic model where $(p_1, p_2) = \underline{p} > \underline{0}$ are uniformly distributed in the positive quadrant. Given an integer apportionment, $\underline{a} = (a_1, a_2) > \underline{0}$ and a method M, the set of $\underline{p}'s$ which lead to $\underline{a}$ being an M-apportionment is unbounded. To define a bounded subset of the sample space we use the fact that M is a divisor method.

Now, we choose any $x > 0$ to represent a hypothetical district size. We define $R_x(\underline{a})$ to be the set of all populations, $\underline{p}$, that yield the M-apportionment, $\underline{a}$ using the divisor x:

$$R_x(\underline{a}) = \{\underline{p} > \underline{0} : d(a_i) \geqslant \frac{p_i}{x} \geqslant d(a_i - 1)\}$$

where $d(a)$ is the dividing point in the interval of quotients $[a, a + 1]$ for each non-negative integer a. We use the convention $d(-1) = 0$.

A divisor method, M, is pairwise unbiased on apportionments if for every $a_1 > a_2 > 0$ and every $x > 0$, the probability that state 1 is favoured is the same as the probability of state 2 being favoured. [17]

Theorem 2.3.1. Webster's is the unique proportional divisor method which is pairwise unbiased on apportionments. [17]

Proof. Suppose we have a divisor method, M, and an apportionment $a_1 > a_2 > 0$. The ray defined by $\frac{a_1}{p_1} = \frac{a_2}{p_2}$ splits $R_x(\underline{a})$ into two sets: populations that favour state 1 and populations that favour state 2.
These two sets have equal measure - simply put, the ray bisects the rectangle if and only if it passes through its centre, $\underline{c} = (c_1, c_2)$.
We have that,

$$c_1 = c(d(a_1 - 1) + d(a_1))/2 \text{ and } c_2 = x(d(a_2 - 1) + d(a_2))/2.$$

This implies that M is pairwise unbiased if and only if the following condition is satisfied,

$$\frac{d(a_1 - 1) + d(a_1)}{d(a_2 - 1) + d(a_2)} = \frac{a_1}{a_2}, \quad \forall a_1 > a_2 > 0. \tag{2.5}$$

In particular, Webster's method is pairwise unbiased on apportionments. Conversely, suppose that M satisfies 2.5, we set $a_1 = a \geqslant 2$ and $a_2 = 1$. This gives us,

$$(d(a - 1) + d(a))/a = d(0) + d(1),$$

and since $a \leqslant d(a) \leqslant a + 1 \forall a$, taking the limit as a goes to infinity implies $d(0) + d(1) = 2$.
It follows that $d(2a) = 2a + d(0)$and $d(2a + 1) = 2a + 2 - d(0)$.
Now, if we use our assumption of proportionality we can show the following:

$$\frac{d(b)}{d(b-1)} \geqslant \frac{d(b+1)}{d(b)}, \forall b > 0. \tag{2.6}$$

If we substitute $b = 2a + 1$ into 2.5 we get the following,

$$\frac{2a + 2 - d(0)}{2a + d(0)} \geqslant \frac{2a + 2 + d(0)}{2a + 2 - d(0)}, \text{or } 4a + 4 \geqslant (8a + 6)d(0).$$

We once again take the limit of a as it goes to infinity and see that $d(0) \leqslant 1/2$.
If instead we substitute $b = 2a$ into 2.6 we find,

$$(8a + 2)d(0) \geqslant 4a \implies d(0) \geqslant 1/2.$$

$\therefore d(0) = 1/2$ and we see that M is Webster's method. □

2.3.3 Analysing bias of more than two states

In this section, we will see how different sized states are treated by an apportionment method. Again we have an apportionment, $\underline{a} > \underline{0}$, for population $\underline{p}$. We will group all the states into two disjoint sets, Large and Small, such that $a_i > a_j$ for all states $i \in$ Large and $j \in$ Small.

We say the apportionment, $\underline{a}$, favours the smaller states if,

$$\frac{\Sigma_{Small} a_i}{\Sigma_{Small} p_i} > \frac{\Sigma_{Large} a_j}{\Sigma_{Large} p_j}, \tag{2.7}$$

and favours the larger states if,

$$\frac{\Sigma_{Large} a_j}{\Sigma_{Large} p_j} > \frac{\Sigma_{Small} a_i}{\Sigma_{Small} p_i}. \tag{2.8}$$

We say that a divisor method is unbiased if for any apportionment, $\underline{a}$, and divisor $x > 0$ and for any disjoint sets of larger and smaller states the probability given $\underline{p} \in R_x(\underline{a})$ that $(\underline{p}; \underline{a})$ favours small states is the same as the probability that it favours larger states.

Theorem 2.3.2. Webster's method is the only unbiased proportional divisor method. [17]

Proof. Suppose we have a method, M, which is a proportional divisor method. Given an apportionment $\underline{a} > \underline{0}$ and divisor $x > 0$, let L and S be any disjoint sets of larger and smaller states respectively. The hyperplane $\frac{\sum_L a_i}{\sum_L p_i} = \frac{\sum_S a_i}{\sum_S p_i}$ splits $R_x(\underline{a})$ into the populations that favour L and the populations that favour S.
These two sets have equal measure - simply put the hyperplane bisects the rectangular solid $R_x(\underline{a})$ iff it goes through its centre. The coordinates of the centre of $R_x(\underline{a})$ are given by: $c_i = [d(a_i - 1) + d(a_i)]/2$.
This implies that M is unbiased iff $\forall a > 0$ we have the following:

$$\frac{\sum_L a_i}{\sum_L d(a_i - 1) + d(a_i)} = \frac{\sum_S a_i}{\sum_S d(a_i - 1) + d(a_i)}. \tag{2.9}$$

Conversely, if we suppose that M satisfies $\forall \underline{a} > \underline{0}$, then in particular it satisfies it whenever $\underline{a} = (a+1, a), a > 0$. From here we can conclude in the same way as in the proof of 2.3.1 that the method M must be Webster's method. $\square$

2.4 The Balinski-Young Impossibility Theorem

In 1983, two Mathematicians Michel Balinski and Peyton Young developed a theorem noting that there does not exist an apportionment method satisfying the Quota Rule that is free from the apportionment paradoxes [9]. More specifically, there is no method of apportionment that satisfies the following three properties:

- Satisfies the Alabama Paradox,
- Satisfies the Population Paradox,
- Avoids violation of the aforementioned Quota Rule.

In order to prove the Balinski-Young Impossibility theorem, we need to first define the key principles embedded within the theorem, namely house and population monotonicity, and determine how they relate to the Paradoxes.

2.4.1 Monotonicity

In order for an apportionment method to be as fair as possible, it should take into account the 3 main parameters which would directly influence the apportionment result if they were to change. These are:

- Population of each state, p_i,
- House size, h,
- Number of States, S.

Of these 3 parameters, both house size and the number of states very rarely change whereas the populations are constantly wavering. In 1929, the United States Congress introduced a cap on the size of the house of 435 [18]. Therefore, we will define an apportionment method $\mathbf{M}(\mathbf{p})$ as a function with fixed house size and number of states, but a parameter $\mathbf{p} \in \mathbb{R}^S$. Here, the S-vector $\mathbf{p}$ represents the populations of each state.

Population Monotonicity

We begin to outline our first mathematical approach to the principle of population monotonicity in Definition 2.4.1 below.

Definition 2.4.1. (Population Monotonicity) An apportionment method, $\mathbf{M}$ is population monotone when, for states $i, j \in S$, p_i increases and all $p_j (j \neq i)$ remain the same, and i's apportionment does not decrease. [9, pp. 106].

However, a more general approach to this principle would be "as the conditions of a problem change ... apportionment should respond accordingly" [9, pp. 117]. In this instance, the conditions are house size, the population of the states in question and the number of States, all of which are factors when considering the fairness of the methods with respect to the outlined paradoxes.

We can define the idea of population monotonicity more precisely if we compare the two population vectors $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^s$. If we look at states i and j which have populations $p_i, p_j \in \mathbf{p}$ and $p'_i, p'_j \in \mathbf{p}'$, we want to explore how the seats are apportioned as state i grows larger with respect to state j. Due to population monotonicity, we can infer [9, pp. 108],

$$\frac{p'_i}{p'_j} \geq \frac{p_i}{p_j} \Longleftrightarrow \begin{cases} a'_i \geq a_i \text{ or } a'_j \leq a_j \\ \frac{p'_i}{p'_j} = \frac{p_i}{p_j} \end{cases} . \tag{2.10}$$

Clearly, in the second case, the populations change directly proportionally to each other, so their apportionments do not change. However, the first case says that as the population of state i increases with respect to the population of state j, its apportionment cannot decrease.

Theorem 2.4.2. If an apportionment method is a divisor method, it must satisfy population monotonicity.

Proof. Suppose we have two sets of populations $\mathbf{p} = (p_1, \ldots, p_n),\ \mathbf{p}' = (p'_1, \ldots, p'_n)$ with respective apportionments $\mathbf{a} = (a_1, \ldots, a_n),\ \mathbf{a}' = (a'_1, \ldots, a'_n)$. For a divisor method, $\mathbf{M}$, if $a'_i < a_i$ and $a'_j > a_j$, then state i's modified divisor, $\frac{p_i}{d}$ must have decreased and state j's modified divisor, $\frac{p_j}{d}$ increased as we go from populations $\mathbf{p}$ to $\mathbf{p}'$.

Therefore, we have $\frac{p'_i}{d'} < \frac{p_i}{d}$ and $\frac{p'_j}{d'} > \frac{p_j}{d}$. Upon rearranging these formulae, we can obtain the equations

$$p'_i < \frac{d'}{d} p_i \quad \text{and} \quad p'_j > \frac{d'}{d} p_j. \tag{2.11}$$

We need to investigate the size of the fraction $\frac{d'}{d}$, in order to do this, we need to split it into two cases: $\frac{d'}{d} \geq 1$, $\frac{d'}{d} < 1$.

Let us consider the first case where $\frac{d'}{d} \geq 1$. This would imply that we have $p'_j > p_j$. On the other hand, if we consider the case where $\frac{d'}{d} < 1$, then this implies that $p'_i < p_i$. Therefore, we have that either $p'_i < p_i$ or $p'_j > p_j$. Since $a'_i < a_i$ and $a'_j > a_j$ implies that either $p'_i < p_i$ or $p'_j > p_j$, the method is population monotone by (2.10). □

However, we can analyse further and examine another approach to monotonicity. The guiding principle behind monotonicity is that a state with a larger population should have greater representation than a state with a smaller population and hence we need to introduce the concept of rank.

Definition 2.4.3. (Order-Preserving Criterion) An apportionment method is said to satisfy the order-preserving criterion if "whenever $a_i > a_j$, it follows that $p_i > p_j$" [19].

We can investigate the divisor method's abilities to rank the population of states in Proposition 2.4.4.

Proposition 2.4.4. A population monotone apportionment method must follow the order-preserving criterion.

Proof. Let us consider a set of populations $\mathbf{p} \in \mathbb{R}^s$ where s is the number of states and consider two states $i, j \in S$. These states have populations p_i, p_j respectively and let us suppose that $p_j > p_i$.

Now if we were to construct a second set of populations, $\mathbf{p}' \in \mathbb{R}^s$, by swapping the populations of states i and j such that $p'_i = p_j,\ p'_j = p_i,\ p'_k = p_k\ \ \forall\, k \neq i, j$, we can see that state i has increased in population and state j has decreased. Since we have a population monotone method, we must have that either $a_i < a'_i$ or $a_j > a'_j$. We can see that in swapping the populations of states i and j, we have also swapped their rank within the apportionment method. Therefore, it follows the order-preserving criterion. □

House Monotonicity

If, for a state $i \in S$, its population p_i remains the same yet the total house size, h, increases, then it would be unfair for this to result in state i losing a seat during the apportionment process. Therefore, we need to define another key principle of monotonicity: House monotonicity.

Definition 2.4.5. A method of apportionment is house monotone if an increase in the house size does not result in the number of seats apportioned to any state decreasing. [8]

In other words, if h increases and p_i remains the same $\forall\, i \in S$, then a_i cannot decrease.

Theorem 2.4.6. If an apportionment method is population monotone, that implies that it must also be house monotone.

Proof. In order for a method to be population monotone, the natural quotas must be proportional to each other with respect to population, i.e. for 2 states A and B with two sets of populations: $\mathbf{p} = (p_A, p_B)$ and $\mathbf{p}' = (p'_A, p'_B)$, if $\frac{p'_A}{p'_B} \geq \frac{p_A}{p_B}$ then $a'_i \geq a_i$ or $a'_j \leq a_j$ by (2.10). Therefore, as the value of h increases, if the populations remain fixed these inequalities remain and so the method is house monotone. □

Upon combining Theorem 2.4.6 and Proposition 2.4.4, we can clearly see that all population monotone methods are also house monotone. Additionally, we can see that these methods must also satisfy the order-preserving criterion.

2.4.2 Relating Monotonicity to the Paradoxes

Now that we have defined the fundamental principles of monotonicity, we can begin to relate these to the previously discussed apportionment paradoxes. Firstly, we will consider the relation of monotonicity to the Alabama Paradox from Definition 2.2.1.

Theorem 2.4.7. If a method does not fall prey to the Alabama paradox, then it must be house monotone.

Proof. By the definition of house monotonicity, as the house size increases then the apportionment of each state cannot decrease. However, the Alabama paradox occurs when the house size increases yet a state loses out on a seat. Therefore, from the definition of house monotonicity, any method for which the Alabama paradox doesn't occur must be house monotone. □

Now that we have explored the link between house monotonicity and the Alabama paradox, we can look at the next two paradoxes: the Population paradox from Definition 2.2.2; and the New States paradox from Definition 2.2.3.

Theorem 2.4.8. If a method does not fall prey to the Population and New states paradoxes, then it must be population monotone.

Proof. In order for a method to be population monotone, as one states' population, p_i, increases and all p_j remain the same for $i \neq j$, a_i cannot decrease.

If a method does not evade the population paradox, then as a states' population increases, it loses out on a seat to a state whose population has not increased as much. Therefore, in order to avoid this, we need a population monotone method.

Similarly, in order for a method to show the New States paradox, as a new state needs representation in congress, then other states lose out on seats, even when the house size is increased to account for the new state. It is clear to see that if we have a population monotone method, then this cannot occur as they are fixed in population so their apportionment cannot decrease.

Therefore, population monotone methods are the only methods that do not fall prey to the aforementioned paradoxes. □

Upon combining Theorems 2.4.8 & 2.4.7, we can deduce the following:

Theorem 2.4.9. In order for a method to avoid all three paradoxes, it must be population monotone.

2.4.3 Proof of the Balinski-Young Impossibility Theorem

Now that we have discussed house and population monotonicity and their relation to the paradoxes, we can rewrite the previously outlined Balinski-Young Impossibility Theorem. Previously, we stated this as: there does not exist an apportionment method satisfying the Quota Rule that is free from the apportionment paradoxes [9]. However, Theorem 2.4.9 allows us to rewrite the Theorem as below.

Theorem 2.4.10. It is impossible for an apportionment method to simultaneously satisfy the quota rule and be population monotone.

Proof. We want to prove this theorem by contradiction. Thus, let us consider an apportionment method **M** which is population monotone and satisfies the quota rule for a total house size to be apportioned h.

For any $\epsilon \in \mathbb{R}$ sufficiently small, we have two different sets of populations:

1. $\mathbf{p}_\epsilon = (5+\epsilon, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}-\epsilon, b_5, \ldots, b_s)$
2. $\mathbf{p'}_\epsilon = (4-\epsilon, 2-\frac{\epsilon}{2}, \frac{1}{2}+\frac{\epsilon}{2}, \frac{1}{2}+\epsilon, b_5, \ldots, b_s)$

where $b_5, \ldots, b_s$ are positive integers with sum $h-7$. Clearly $\sum_i p_i = \sum_i p'_i = h$, so we can see that $p_i = q_i$ and $p'_i = q'_i$.

If we consider the first set of populations, $\mathbf{p}_\epsilon$, then by the quota rule assumption we determine that $a_1 = 5$ and $a_i = b_i \ \forall i \geq 5$. Therefore, $a_2 + a_3 + a_4 = h - 5 - (h-7) = 2$ so one state must get 0 seats. Due to the method $\mathbf{M}$ being population monotone, we can apply the approach from Proposition 2.4.4 to see that states 2 and 3 have larger populations than state 4 and hence have higher order. Therefore, the state that receives 0 seats is state 4.

Now if we consider the other set of populations, $\mathbf{p}'_\epsilon$, we have that $a'_1 = 4, a'_2 = 2$ and $a'_i = b_i \ \forall i \geq 5$. This means that $a'_3 + a'_4 = h - 4 - 2 - (h-7) = 1$, thus either state 3 or 4 will have an apportionment of 0. Using the same approach as earlier, we can apply the approach from Proposition 2.4.4 and infer that it is state 3 that does not receive a seat.

Therefore, $a'_1 < a_1$ and $a'_4 > a_4$. Since our method $\mathbf{M}$ is population monotone, we can employ (2.10) to see that

$$\frac{p'_1}{p'_4} < \frac{p_1}{p_4} \iff \frac{4-\epsilon}{\frac{1}{2}+\epsilon} < \frac{5+\epsilon}{\frac{2}{3}-\epsilon} \implies \epsilon > \frac{1}{61}.$$

This is false for sufficiently small ϵ. That is a contradiction, therefore the result follows. $\square$

2.5 2000 & 2010: Apportionment

Now that we have delved into the mathematics of apportionment methods and their fairness, we will apply these to two real examples. After obtaining the populations from the 2000 and 2010 U.S. census data sets [4], we apply four of the apportionment methods we have evaluated thus far. These methods are:

- Hamilton's (aka, the largest remainder method),
- Lowndes',
- Jefferson's,
- Webster's.

We chose these methods since we wish to see two Natural Quota and two divisor methods and also, all Jefferson's, Hamilton's and Webster's have, at one point, been used by the U.S. congress.

2.5.1 2000

We apply the four aforementioned methods of apportionment to the 2000 census data to obtain Figure 2.1. Rather than focusing on the whole of the United States, we will turn our attention to some key states in particular.

In Figure 2.1, the four red columns represent the four apportionments by each of the methods. In order to calculate each one, we need to calculate: Natural Quota, q_i; Jefferson's modified quota, $q*_{i,J}$; and Webster's modified quota, $q*_{i,W}$.

In order to calculate the modified quotas, we must use the modified divisors $613,500$ and $643,500$ for Jefferson's and Webster's method respectively.

State	Populations	Quota	L Quota	Remainder	R Rank	a (Hamilton)	Remainder/ L Quota	Remainder/ L Quota Rank	a (Lowndes)		Quota*	L Quota*	a (Jefferson)	Quota**	Rounded Quota**	a (Webster)
Alabama	4,447,092	6.8880	6	0.8880	9	7	0.1480	17	7		7.2487	7	7	6.9108	7	7
Alaska	626,932	0.9710	0	0.9710	3	1		1	1		1.0219	1	1	0.9743	1	1
Arizona	5,130,632	7.9467	7	0.9467	5	8	0.1352	18	8		8.3629	8	8	7.9730	8	8
Arkansas	2,673,400	4.1408	4	0.1408	45	4	0.0352	37	4		4.3576	4	4	4.1545	4	4
California	33,871,648	52.4629	52	0.4629	26	53	0.0089	47	52		55.2105	55	55	52.6366	53	53
Colorado	4,301,261	6.6621	6	0.6621	19	7	0.1104	20	7		7.0110	7	7	6.6842	7	7
Connecticut	3,405,565	5.2748	5	0.2748	37	5	0.0550	30	5		5.5510	5	5	5.2923	5	5
Delaware	783,596	1.2137	1	0.2137	41	1	0.2137	13	2		1.2773	1	1	1.2177	1	1
Florida	15,982,349	24.7546	24	0.7546	16	25	0.0314	40	24		26.0511	26	26	24.8366	25	25
Georgia	8,186,446	12.6798	12	0.6798	17	13	0.0566	29	12		13.3438	13	13	12.7217	13	13
Hawaii	1,211,537	1.8765	1	0.8765	10	2	0.8765	7	2		1.9748	1	1	1.8827	2	2
Idaho	1,293,941	2.0042	2	0.0042	50	2	0.0021	49	2		2.1091	2	2	2.0108	2	2
Illinois	12,419,231	19.2358	19	0.2358	39	19	0.0124	45	19		20.2432	20	20	19.2995	19	19
Indiana	6,080,485	9.4179	9	0.4179	28	9	0.0464	31	9		9.9111	9	9	9.4491	9	9
Iowa	2,926,324	4.5325	4	0.5325	24	5	0.1331	19	5		4.7699	4	4	4.5475	5	5
Kansas	2,688,418	4.1640	4	0.1640	44	4	0.0410	33	4		4.3821	4	4	4.1778	4	4
Kentucky	4,041,733	6.2601	6	0.2601	38	6	0.0434	32	6		6.5880	6	6	6.2809	6	6
Louisiana	4,468,976	6.9219	6	0.9219	7	7	0.1536	15	7		7.2844	7	7	6.9448	7	7
Maine	1,274,923	1.9747	1	0.9747	2	2	0.9747	5	2		2.0781	2	2	1.9812	2	2
Maryland	5,296,324	8.2033	8	0.2033	42	8	0.0254	42	8		8.6330	8	8	8.2305	8	8
Massachussetts	6,349,097	9.8339	9	0.8339	11	10	0.0927	24	10		10.3490	10	10	9.8665	10	10
Michigan	9,938,444	15.3934	15	0.3934	31	15	0.0262	41	15		16.1996	16	16	15.4444	15	15
Minnesota	4,919,479	7.6196	7	0.6196	22	8	0.0885	25	8		8.0187	8	8	7.6449	8	8
Mississippi	2,844,646	4.4060	4	0.4060	29	4	0.1015	21	5		4.6367	4	4	4.4206	4	4
Missouri	5,595,210	8.6663	8	0.6663	18	9	0.0833	26	9		9.1201	9	9	8.6950	9	9
Montana	902,195	1.3974	1	0.3974	30	1	0.3974	11	2		1.4706	1	1	1.4020	1	1
Nebraska	1,711,263	2.6505	2	0.6505	20	3	0.3253	12	3		2.7893	2	2	2.6593	3	3
Nevada	1,998,257	3.0950	3	0.0950	47	3	0.0317	39	3		3.2571	3	3	3.1053	3	3
New Hampshire	1,235,786	1.9141	1	0.9141	8	2	0.9141	6	2		2.0143	2	2	1.9204	2	2
New Jersey	8,414,297	13.0327	13	0.0327	48	13	0.0025	48	13		13.7152	13	13	13.0758	13	13
New Mexico	1,819,046	2.8175	2	0.8175	12	3	0.4087	9	3		2.9650	2	2	2.8268	3	3
New York	18,976,457	29.3921	29	0.3921	32	29	0.0135	44	29		30.9315	30	30	29.4894	29	29
North Carolina	8,049,310	12.4674	12	0.4674	25	13	0.0389	34	12		13.1203	13	13	12.5086	13	13
North Dakota	642,196	0.9947	0	0.9947	1	1		1	1		1.0468	1	1	0.9980	1	1
Ohio	11,353,140	17.5846	17	0.5846	23	18	0.0344	38	17		18.5055	18	18	17.6428	18	18
Oklahoma	3,450,647	5.3446	5	0.3446	33	5	0.0689	27	5		5.6245	5	5	5.3623	5	5
Oregon	3,421,397	5.2993	5	0.2993	35	5	0.0599	28	5		5.5768	5	5	5.3169	5	5
Pennsylvania	12,281,054	19.0218	19	0.0218	49	19	0.0011	50	19		20.0180	20	20	19.0848	19	19
Rhode Island	1,048,319	1.6237	1	0.6237	21	2	0.6237	8	2		1.7088	1	1	1.6291	2	2
South Carolina	4,012,012	6.2141	6	0.2141	40	6	0.0357	36	6		6.5395	6	6	6.2347	6	6
South Dakota	754,844	1.1692	1	0.1692	43	1	0.1692	14	2		1.2304	1	1	1.1730	1	1
Tennessee	5,689,272	8.8120	8	0.8120	13	9	0.1015	22	9		9.2735	9	9	8.8411	9	9
Texas	20,851,820	32.2968	32	0.2968	36	32	0.0093	46	32		33.9883	33	33	32.4038	32	32
Utah	2,233,169	3.4589	3	0.4589	27	3	0.1530	16	4		3.6400	3	3	3.4703	3	3
Virginia	7,078,513	10.9637	10	0.9637	4	11	0.0964	23	11		11.5379	11	11	11.0000	11	11
Vermont	608,808	0.9430	0	0.9430	6	1		1	1		0.9924	0	0	0.9461	1	1
Washington	5,894,121	9.1292	9	0.1292	46	9	0.0144	43	9		9.6074	9	9	9.1595	9	9
West Virginia	1,808,308	2.8008	2	0.8008	14	3	0.4004	10	3		2.9475	2	2	2.8101	3	3
Wisconsin	5,363,673	8.3076	8	0.3076	34	8	0.0385	35	8		8.7427	8	8	8.3352	8	8
Wyoming	493,782	0.7648	0	0.7648	15	1		1	1		0.8049	0	0	0.7673	1	1
Total	280,849,375	435	409			435			435		457.7822	435	435		435	435

Figure 2.1: Apportionment by four different methods for 2000 census data [4].

The first two states that we are going to be investigating are California and Wyoming. These are the largest and smallest states by population, and hence are good indicators of whether a method could potentially be biased towards larger or smaller states.

For our largest state, California, we see that Jefferson's method awards 55 seats whereas the other methods assign a much lower number. In addition to this, we can see that Jefferson's method is the only one to assign no seats to Wyoming which is in violation of the U.S. Constitution [7]. Therefore, we have an indication that Jefferson's method is favouring the larger states in this example.

Using the techniques discussed previously, we can determine whether this hypothesis is true by considering two disjoint sets of large and small states. This gives us:

$L =$ {California, Texas, New York, Florida, Illinois, Pennsylvania, Ohio, Michigan, New Jersey, Georgia, North Carolina, Virginia, Massachusetts, Indiana, Washington, Tennessee, Missouri, Wisconsin, Maryland, Arizona, Minnesota, Louisiana, Alabama, Colorado, Kentucky},

$S =$ {South Carolina, Oklahoma, Oregon, Connecticut, Iowa, Mississippi, Kansas, Arkansas, Utah, Nevada, New Mexico, West Virginia, Nebraska, Idaho, Maine, New Hampshire, Hawaii, Rhode Island, Montana, Delaware, South Dakota, North Dakota, Alaska, Vermont, Wyoming}.

Using (2.8), we obtain the inequality,

$$\frac{373}{234980064} > \frac{62}{45869311} \iff 1.587 \times 10^{-6} > 1.352 \times 10^{-6}$$

Therefore, we can confirm that for this example, Jefferson's method favours larger states. This has many implications with the most noticeable being that Jefferson's method violates the U.S. constitution on multiple occasions. Wyoming has already been mentioned, however, it is also prevalent in Vermont.

We can also investigate whether Lowndes' method is "more favourable to smaller states" [10] as it is notoriously known for. Using the same disjoint sets L & S as before and (2.7), the inequality,

$$\frac{360}{234980064} < \frac{75}{45869311} \iff 1.532 \times 10^{-6} < 1.635 \times 10^{-6}$$

clearly holds. Therefore, our hypothesis of Lowndes' method favouring the smaller states in this example has been confirmed.

2.5.2 2010

We will now also apply the same four methods of apportionment to the U.S. census data from 2010. In doing so, we generate Figure 2.2.

Alabama	4,779,736	6.7435	6	0.7435	14	7	0.1239	16	7		7.1339	7	7	6.7798	7	7
Alaska	710,231	1.0020	1	0.0020	50	1	0.0020	49	1		1.0600	1	1	1.0074	1	1
Arizona	6,392,017	9.0182	9	0.0182	49	9	0.0020	50	9		9.5403	9	9	9.0667	9	9
Arkansas	2,915,918	4.1139	4	0.1139	43	4	0.0285	33	4		4.3521	4	4	4.1361	4	4
California	37,253,956	52.5599	52	0.5599	17	53	0.0108	44	52		55.6029	55	55	52.8425	53	53
Colorado	5,029,196	7.0955	7	0.0955	45	7	0.0136	41	7		7.5063	7	7	7.1336	7	7
Connecticut	3,574,097	5.0425	5	0.0425	46	5	0.0085	45	5		5.3345	5	5	5.0696	5	5
Delaware	897,934	1.2669	1	0.2669	35	1	0.2669	14	2		1.3402	1	1	1.2737	1	1
Florida	18,801,310	26.5259	26	0.5259	18	27	0.0202	37	26		27.9912	27	27	26.4982	26	26
Georgia	9,867,653	13.9218	13	0.9218	4	14	0.0709	23	14		14.7278	14	14	13.9967	14	14
Hawaii	1,360,301	1.9192	1	0.9192	6	2	0.9192	4	2		2.0303	2	2	1.9295	2	2
Idaho	1,567,582	2.2116	2	0.2116	37	2	0.1058	18	3		2.3397	2	2	2.2235	2	2
Illinois	12,830,632	18.1022	18	0.1022	44	18	0.0057	47	18		19.1502	19	19	18.1995	18	18
Indiana	6,483,803	9.1477	9	0.1477	40	9	0.0164	40	9		9.6773	9	9	9.1969	9	9
Iowa	3,046,355	4.2980	4	0.2980	31	4	0.0745	21	5		4.5468	4	4	4.3211	4	4
Kansas	2,853,118	4.0253	4	0.0253	47	4	0.0063	46	4		4.2584	4	4	4.0470	4	4
Kentucky	4,339,367	6.1222	6	0.1222	42	6	0.0204	36	6		6.4767	6	6	6.1551	6	6
Louisiana	4,533,372	6.3959	6	0.3959	28	6	0.0660	25	6		6.7662	6	6	6.4303	6	6
Maine	1,328,361	1.8741	1	0.8741	10	2	0.8741	5	2		1.9826	1	1	1.8842	2	2
Maryland	5,773,552	8.1456	8	0.1456	41	8	0.0182	38	8		8.6172	8	8	8.1894	8	8
Massachussetts	6,547,629	9.2378	9	0.2378	36	9	0.0264	34	9		9.7726	9	9	9.2874	9	9
Michigan	9,883,640	13.9444	13	0.9444	3	14	0.0726	22	14		14.7517	14	14	14.0193	14	14
Minnesota	5,303,925	7.4831	7	0.4831	22	8	0.0690	24	7		7.9163	7	7	7.5233	8	8
Mississippi	2,967,297	4.1864	4	0.1864	38	4	0.0466	30	4		4.4288	4	4	4.2089	4	4
Missouri	5,988,927	8.4495	8	0.4495	25	8	0.0562	27	8		9.0012	9	9	8.5102	9	9
Montana	989,415	1.3959	1	0.3959	29	1	0.3959	9	2		1.4767	1	1	1.4034	1	1
Nebraska	1,826,341	2.5767	2	0.5767	16	3	0.2884	12	3		2.7259	2	2	2.5906	3	3
Nevada	2,700,551	3.8101	3	0.8101	12	4	0.2700	13	4		4.0307	4	4	3.8306	4	4
New Hampshire	1,316,470	1.8573	1	0.8573	11	2	0.8573	6	2		1.9649	1	1	1.8673	2	2
New Jersey	8,791,894	12.4041	12	0.4041	27	12	0.0337	32	12		13.1222	13	13	12.4708	12	12
New Mexico	2,059,179	2.9052	2	0.9052	7	3	0.4526	8	3		3.0734	3	3	2.9208	3	3
New York	19,378,102	27.3397	27	0.3397	30	27	0.0126	43	27		28.9225	28	28	27.4867	27	27
North Carolina	9,535,483	13.4532	13	0.4532	24	13	0.0349	31	13		14.2321	14	14	13.5255	14	14
North Dakota	672,591	0.9489	0	0.9489	2	1			1		1.0039	1	1	0.9540	1	1
Ohio	11,536,501	16.2763	16	0.2763	34	16	0.0173	39	16		17.2187	17	17	16.3638	16	16
Oklahoma	3,751,351	5.2926	5	0.2926	32	5	0.0585	26	5		5.5990	5	5	5.3211	5	5
Oregon	3,831,074	5.4051	5	0.4051	26	5	0.0810	20	6		5.7180	5	5	5.4341	5	5
Pennsylvania	12,702,379	17.9212	17	0.9212	5	18	0.0542	28	17		18.9588	18	18	18.0176	18	18
Rhode Island	1,052,567	1.4850	1	0.4850	21	2	0.4850	7	2		1.5710	1	1	1.4930	1	1
South Carolina	4,625,364	6.5257	6	0.5257	19	7	0.0876	19	7		6.9035	6	6	6.5608	7	7
South Dakota	814,180	1.1487	1	0.1487	39	1	0.1487	15	2		1.2152	1	1	1.1549	1	1
Tennessee	6,346,105	8.9534	8	0.9534	1	9	0.1192	17	9		9.4718	9	9	9.0016	9	9
Texas	25,145,561	35.4767	35	0.4767	23	36	0.0136	42	35		37.5307	37	37	35.6675	36	36
Utah	2,763,885	3.8994	3	0.8994	8	4	0.2998	11	4		4.1252	4	4	3.9204	4	4
Virginia	8,001,024	11.2883	11	0.2883	33	11	0.0262	35	11		11.9418	11	11	11.3490	11	11
Vermont	625,741	0.8828	0	0.8828	9	1			1		0.9339	0	0	0.8876	1	1
Washington	6,724,540	9.4873	9	0.4873	20	10	0.0541	29	9		10.0366	10	10	9.5384	10	10
West Virginia	1,852,994	2.6143	2	0.6143	15	3	0.3072	10	3		2.7657	2	2	2.6284	3	3
Wisconsin	5,686,986	8.0235	8	0.0235	48	8	0.0029	48	8		8.4880	8	8	8.0666	8	8
Wyoming	563,626	0.7952	0	0.7952	13	1			1		0.8412	0	0	0.7995	1	1
Total	308,323,813	435	412			435			435			435	435		435	435

Figure 2.2: Apportionment by four different methods for 2010 census data [4].

Similar to Figure 2.1, the four red columns represent the four different apportionments and we need to once again calculate the same three values: Natural Quota; Jefferson's modified quota, $q*_{i,J}$; and Webster's modified quota, $q*_{i,W}$. In order to calculate these modified quotas, we use the modified divisors $670,000$ and $705,000$ respectively.

Once again, investigating the apportionment of the largest and smallest states is a good indication of the potential bias of methods for certain data sets. In fact, we see that both California and Wyoming have the same apportionment in 2000 and 2010. Therefore, we wish to test the same assumptions of favouritism as before. In order to do this, we split the states into two disjoint sets of smaller and larger states. These sets are:

$L =$ {California, Texas, New York, Florida, Illinois, Pennsylvania, Ohio, Michigan, New Jersey, Georgia, North Carolina, Virginia, Massachusetts, Indiana, Washington, Tennessee, Missouri, Wisconsin, Maryland, Arizona, Minnesota, Louisiana, Alabama, Colorado, South Carolina},

$S =$ {Kentucky, Oklahoma, Oregon, Connecticut, Iowa, Mississippi, Kansas, Arkansas, Utah, Nevada, New Mexico, West Virginia, Nebraska, Idaho, Maine, New Hampshire, Hawaii, Rhode Island, Montana, Delaware, South Dakota, North Dakota, Alaska, Vermont, Wyoming}.

By using (2.8) & (2.7) again, we can determine whether or not Jefferson's method is favourable toward larger states. Using (2.8), the inequality

$$\frac{371}{257943287} > \frac{64}{50380526} \Longleftrightarrow 1.438 \times 10^{-6} > 1.270 \times 10^{-6}$$

which clearly holds. Therefore, our hypothesis of Jefferson's method favouring larger states in this data set is confirmed. However, if we consider the difference between the two values in each of our inequalities, (i.e. let $D = |\frac{\sum_L a_i}{\sum_L p_i} - \frac{\sum_S a_i}{\sum_S p_i}|$) then we can gain a measure into how much favouritism a method exhibits. For our 2000 census data, we have

$$D_{2000} = 1.587 \times 10^{-6} - 1.352 \times 10^{-6} = 2.357 \times 10^{-7},$$

and for our 2010 census data, we have

$$D_{2010} = 1.438 \times 10^{-6} - 1.270 \times 10^{-6} = 1.680 \times 10^{-7}.$$

As we can see, the difference is greater for our data from the 2000 census data. Therefore, we can see that Jefferson's method exhibits more favouritism towards large states in 2000 than 2010.

We also wish to investigate Lowndes' method for favouritism towards smaller states. Using (2.7), we see that the inequality

$$\frac{358}{257943287} > \frac{77}{50380526} \Longleftrightarrow 1.387 \times 10^{-6} < 1.528 \times 10^{-6}$$

holds and our theory is confirmed. Therefore, Lowndes' method exhibits favouritism towards smaller states in this example. If we were once again to consider the difference between these values, D, then for the 2000 census data,

$$D_{2000} = 1.635 \times 10^{-6} - 1.532 \times 10^{-6} = 1.030 \times 10^{-7},$$

and for our 2010 data,

$$D_{2010} = 1.528 \times 10^{-6} - 1.387 \times 10^{-6} = 1.405 \times 10^{-7}.$$

Since $D_{2010} > D_{2000}$, Lowndes' method favours the smaller states more in our 2010 example.

2.5.3 Progression from 2000 to 2010

When we are considering the data from two consecutive decennial censuses, it is interesting to consider the progression from one to another. In order to highlight some anomalies in our data, we will focus on the states Indiana, Nebraska and Missouri. The reason that these states are chosen is that if we ranked each state by population growth, these states are all of adjacent rank to one another.

State	Population	$a_{i,H}$	$a_{i,L}$	$a_{i,J}$	$a_{i,W}$	Population increase (%)
Indiana	$(6080485, 6483803)$	$(9, 9)$	$(9, 9)$	$(9, 9)$	$(9, 9)$	6.6
Nebraska	$(1711263, 1826341)$	$(3, 3)$	$(3, 3)$	$(2, 2)$	$(3, 3)$	6.7
Missouri	$(5595210, 5988927)$	$(9, 8)$	$(9, 8)$	$(9, 9)$	$(9, 9)$	7.0

Table 2.13: Apportionment in (2000, 2010) for the four different methods on three key states from Figures 2.1 & 2.2.

In Table 2.13, we can actually see an example of the population paradox which we previously discussed, see Definition 2.2.2. For both Lowndes' and Hamilton's method, although Missouri has a higher population growth rate than both Indiana and Nebraska, Missouri loses seats from 2000 to 2010 whereas the other two states do not. This is clearly an example of an injustice in the form of the population paradox. We can also apply Theorems 2.4.2 and 2.4.9 to see why both of the divisor methods (Jefferson's and Webster's) do not fall prey to this paradox.

In conclusion, we have been able to show that Jefferson's method exhibits favouritism toward larger states and Lowndes' method is favourable to smaller states in our chosen census years. In addition, Jefferson's method exhibits more favouritism for the 2000 data set whereas Lowndes' method exhibits more favouritism for the 2010 data. We have also been able to identify another example of the population paradox in a much more current situation than we have been able to see before.

Chapter 3

Gerrymandering

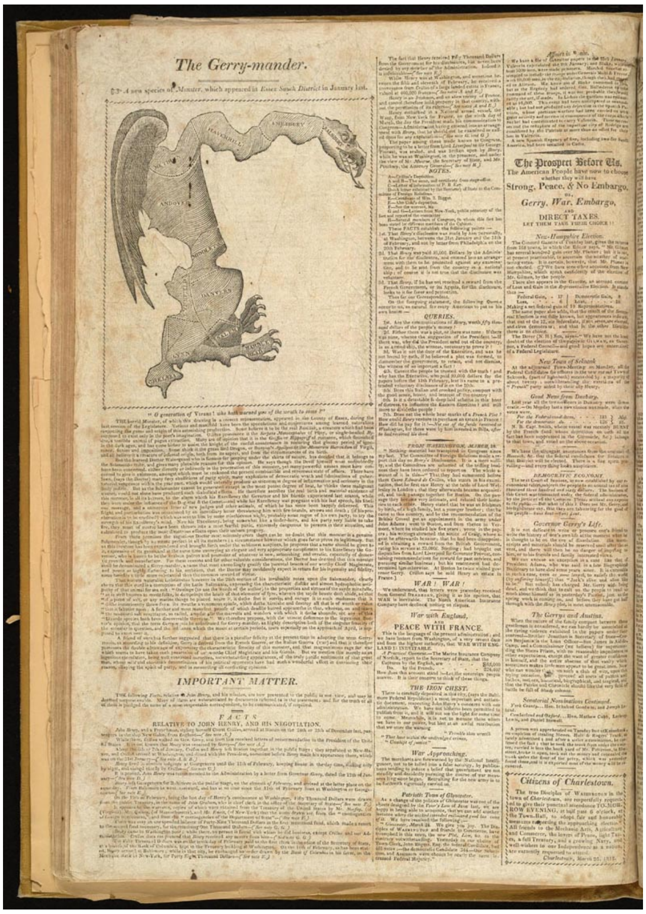
After the apportionment of the 435 United States congressional seats has taken place, a number of representatives have been assigned to each state, the states themselves must then decide how to split these representatives to best represent the views of their people at the federal level. States in the USA use single-member districts that appoint one Member of Congress to represent their district and state [20]. Most states have their congressional districts drawn by the party in power and passed through the state legislature, therefore, giving the ruling party the ability to shape the districts in their state.

There are states that task redistricting commissions with drawing up their congressional districts [21]. Such as Arizona, California, Colorado, Hawaii, Idaho, Michigan, New Jersey, and Washington. These commissions, with the exception of New Jersey, are nonpartisan and gerrymandering in these states should be limited.

It is important to note that described above is the process for congressional districting plans. Each state repeats the above process to elect members of its own state congress that represent the views of the people in the state's version of the United States Congress. These are known as state districting plans and are also passed through the state legislature unless the state uses a redistricting commission.

3.1 Introduction to Gerrymandering

Gerrymandering takes its name from the Essex district in the Boston area in 1812 passed into law by the governor, signatory of the declaration of independence and future vice president of the United States of America, Elbridge Gerry [22]. Under pressure from members of his Republican party as the opposition Federalist party were gaining support with their challenges of the Republican foreign policy, Governor Gerry reluctantly signed the districting plan into law. A local newspaper *the Boston Gazette* ran an article likening the district's unusual shape to a salamander, birthing the word Gerrymander (see Figure 3.1).


The Gerry-mander.

IMPORTANT MATTER.

The Prospect Before Us.

Citizens of Charlestown.

Figure 3.1: Boston Gazzete 26th March 1812 page 2 [22]

Gerrymandering is the act of shaping district boundaries to provide an advantage to one party. Parties utilise packing and cracking to this end by looking to optimise wasted votes across the state. In this section, We will go on to discuss all of this terminology.

3.1.1 Wasted Votes

For simplicity, we consider a two-party election system. We have the intuitive understanding that a vote is wasted when it does not help to elect the chosen candidate [23]. More precisely the wasted votes for any given election are:

i The votes cast for the losing party.

ii The votes cast for the winning party that are above the required majority threshold.

The following notation has been derived with slight modification from Duchin [24]. Say that we have parties A and B in a statewide election over n districts where the set of districts is denoted $\mathcal{D} = \{d_1, d_2, ..., d_n\}$. Denote by v_i^j the total number of votes for party j in district d_i then $V_i := v_i^A + v_i^B$ is the total number of votes in district d_i and let $V^j := \sum_{i=1}^{n} v_i^j$ be the total statewide votes for party j, then $V = V^A + V^B$ is the total number of votes statewide.

We can then provide formal definitions for the concepts introduced by McGhee and Stephanopolous in [25] as lost and surplus votes.

Definition 3.1.1 (Surplus Votes)**.** Consider a party j in district d_i. We then define,

$$S_i^j := \begin{cases} 0 & v_i^j \leq \frac{V_i}{2} \\ v_i^j - \frac{V_i}{2} & v_i^j > \frac{V_i}{2} \end{cases}$$

to be the surplus votes for party j in district d_i. It follows that $S^j = \sum_{i=1}^n S_i^j$ is the total surplus votes for party j.

Definition 3.1.2 (Losing Votes)**.** Consider a party j in district d_i. We then define,

$$L_i^j := \begin{cases} v_i^j & v_i^j \leq \frac{V_i}{2} \\ 0 & v_i^j > \frac{V_i}{2} \end{cases}$$

as the losing votes for a party j in district d_i and we define $L^j := \sum_{i=1}^n L_i^j$ to be the total losing votes for party j.

We then use these definitions to define Wasted Votes.

Definition 3.1.3 (Wasted Votes)**.** The wasted votes for party j in district d_i is defined to be $W_i^j := S_i^j + L_i^J$ and then the total wasted votes for party j across the state is,

$$W^j := \sum_{i=1}^n W_i^j = \sum_{i=1}^n (S_i^j + L_i^j).$$

We make the remark that in a two-party election, with parties A and B, over a state with districts $\mathcal{D} = \{d_1, d_2, ..., d_n\}$ and total number of votes V. The total wasted votes is

$$W := W^A + W^B = V/2.$$

The proof of this remark is not technical and follows easily from the definitions however we include it for illustrative purposes.

Proof. Without loss of generality, we can assume that party A wins the election in district d_i so that $v_i^A > \frac{v_i}{2}$. Then, the total number of wasted votes over district d_i is given by

$$W_i = W_i^A + W_i^B = S_i^A + L_i^A + S_i^A + S_i^B.$$

When we appeal to the definitions above, it is clear that $S_i^B = 0$ and $L_i^A = 0$ since party A wins the election. We also have that $S_i^A = v_i^A - \frac{V_i}{2}$, $L_i^B = v_i^B$. Therefore,

$$W_i = v_i^A - \frac{V_i}{2} + v_i^B.$$

Noting that $V_i = v_i^A + v_i^B$, the above simplifies to,

$$W_i = \frac{V_i}{2}.$$

Finally, we sum over all districts to attain the desired result,

$$W = \sum_{i=1}^{n} W_i = \frac{1}{2}\sum_{i=1}^{n} V_i = \frac{V}{2}.$$

$\square$

This disenfranchisement is an often cited fault in the first-past-the-post election system. Since half of the votes in any election do not go to help elect the chosen candidate, they are wasted.

We now consider a simple two-party election to provide an explicit example of the definitions given in this section.

Example 3.1
Consider the following two-party election results in a state with 5 districts and votes distributed as follows,

	District 1	District 2	District 3	District 4	District 5
Party A	20	40	53	14	4
Party B	12	62	23	24	10

We calculate, $V = 262, S^A = 19$, $S^B = 19$ and $L^A = 58, L^B = 35$. So that the total votes wasted are $W = S^A + L^A + S^B + L^B = 131$. We note that $W = V/2$ as expected.

3.1.2 Packing and Cracking

We are now in a position to define packing and cracking [26]. We will also look at some examples of districts that display these characteristics.

Packing is the practice of grouping together large numbers of opposition voters into districts so as to maximise the number of surplus wasted votes for that party in that district. Packing in this manner hampers the party's ability to win other districts in the state. As a packer, we seek to maximise S_i^j for the opposition party in a given district.

Cracking is the practice of breaking up groups of opposition voters and spreading them among a number of districts so as to reduce their influence. As a cracker, we seek to maximise L_i^j for the opposition party in a given district.

In Figure 3.2, we have a simple model of a state with 25 voters that require to be split into 5 equally populated districts. In this election there are 15 blue voters and 10 red voters, suggesting a convincing victory for the blue party. However, the red party can maximise wasted surplus votes for its opposition by packing voters

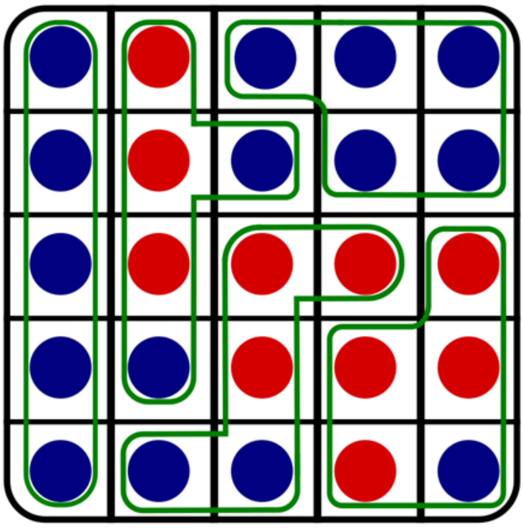


Figure 3.2: Gerrymandering Example

into two districts (on the left and the top right). Narrowly winning the remaining districts, the red party can claim an election victory with a seat majority.

3.1.3 The Supreme Court & Gerrymandering

It was not until 1983 in the supreme court case *Karcher v Daggett* [27] from New Jersey that the courts first considered a case of Gerrymandering and ultimately struck down the districting plan for violating the constitution. This proved to be the first time that the courts would signal stepping in on a gerrymandering issue. It was not long before the supreme court had another case *Davis v Bandemer* [28] in 1986, this time from Indiana. In this case, the justices argued that partisan gerrymandering was justiciable but could not agree on a specific standard while Justice Powell noted that he believed a 'totality-of-the-circumstances' test could be applied that would consider a number of measures.

For the next twenty years, fifty cases were brought before the supreme court challenging congressional and state house districting plans, none of which saw a plan struck down on partisan gerrymandering grounds [23].

However, in 2006, the case *LULAC vs Perry* [29] was brought before the supreme court and the partisan bias measure that we will meet in section 3.3.5 featured heavily in the arguments and was one that the Justices made a point of mentioning. The Justices split along partisan lines with Justice Anthony Kennedy as the swing voter between the liberal and conservative sides of the court. Ultimately, he joined the liberal wing to help the court rule that District 23 had to be redrawn as it discriminated against Latino voters.

Interesting to law professors McGhee and Stephanopoulos were Justice Kennedy's remarks about the partisan bias measure and the alterations required to formulate a corresponding measure that could be used in the courts. He disliked partisan bias due to its reliance on hypothetical election scenarios, "We are wary of adopting a constitutional standard that invalidates a map based on unfair results that would occur in a hypothetical state of affairs." [30]. He also required that the measure gives a reliable standard from which comparisons can be drawn, "A successful claim attempting to identify unconstitutional acts of partisan gerrymandering must do what appellants' sole-motivation theory explicitly disavows: show a burden, as measured by a reliable standard." [31].

It was these remarks by Justice Kennedy that prompted McGhee and Stephanopoulos to create the efficiency gap that we will meet in section 3.4. The efficiency gap has not been used in a gerrymandering case in the supreme court since its publication in 2014, however, its creators believe it provides the best opportunity to see a positive result against partisan gerrymandering. We will come to meet the measures mentioned above and more in the remainder of this chapter.

3.2 Compactness

Compactness centres around the intuitive idea that irregularly shaped districts are an indicator that gerrymandering has taken place. Using compactness has historically been the first place to start for students of Gerrymandering [32]. In fact, as we discussed earlier, the origins of the word Gerrymander stem from the infamous salamander shaped district which became emblematic of the process due to its lack of compactness. The idea that the shape of a district can help identify gerrymandering and the degree to which it occurs has also been adopted by the legal system. For example in the 1993 *Shaw vs. Reno* supreme court case one of the justices Sandra Day O'Connor declared in respect to districting plans that 'appearance does matter' and measures of compactness have been used in multiple gerrymandering cases since then [33].

A compactness measure works by attributing to any given district a value based on how compact the district is. These scores enable you to compare the overall compactness of different districting plans to one another. Theoretically, you could then take the compactness measure for all possible plans and see where a proposed plan fits in this spectrum to make a judgement about whether it has been gerrymandered.

Niemi classifies compactness measures into three distinct classes: perimeter, dispersion and population [34].

3.2.1 Perimeter Measures

The Perimeter Only Measure is the most basic perimeter measure where the compactness of a districting plan is simply equal to the sum of the perimeters of its districts. This is distinctive for the measures we will look because it only assigns a value to the districting plan as a whole as opposed to independently assigning compactness scores for each district and then taking the average of these [35]. This measure is highly simple and interpretable however it leaves open the possibility of gerrymandering in urban areas because lengthened borders there could be easily offset by slight changes to rural borders.

Perimeter-Area Measures derive from the notion that a circle is the most compact shape. One example of such a measure is called the Polsby-Popper method:

$$\text{Per} = \frac{A_D^P}{A_C^P} = \frac{4\pi \cdot A_D^P}{P^2}$$

where A_D^P is the area of a district with perimeter length P and A_C^P is the area of a circle with perimeter length P. This measure is bounded by 0, where the district would be a straight line, and 1, where it would be a circle [36]. However clearly no districting plan will ever maximise compactness under this criterion as it is impossible to create a plan with adjoining districts such that all the districts are circles [34].

However, compactness rankings using this measure can lead to undesirable outcomes. Simply looking at the 5^{th} Congressional district of Oklahoma and the 21^{st} Congressional district of Florida, in Figure 3.3, it appears as though the latter has been drawn in a far more natural way. However, using the Polsby-Popper compactness measure we get [37]:

$$\text{Per}_{OK5} = \frac{4\pi \cdot 5430.68}{505.68^2} = 0.26688, \qquad \text{Per}_{FL26} = \frac{4\pi \cdot 12680.75}{886.13^2} = 0.20293$$

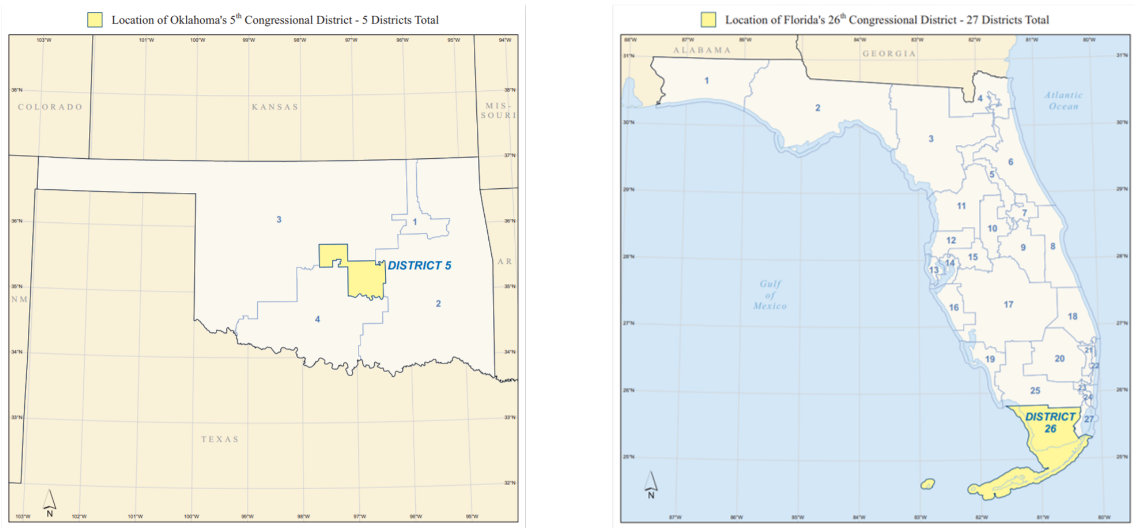


Figure 3.3: Oklahoma's 5th district (left) and Florida's 26th district (right) [38][39]

Therefore this compactness measure assigns a higher value to the Oklahoma district despite our intuition suggesting that the Florida district has been drawn in a more natural way. The compactness value is artificially low due to the effect of the varying coastline adding to the perimeter value as well as the shape of the land, where the district line would have to be drawn, not being a naturally compact shape. The fact that the Oklahoma district is made out of mostly straight lines also helps reduce its perimeter giving it a higher compactness score.

3.2.2 Dispersion Measures

Width-Length Measures are one simple type of dispersion measure e.g

$$\text{Dis}_1 = \frac{W}{L}$$

Where W and L are the respective width and length of the district's circumscribing rectangle with a minimised perimeter. These measures, though simple, are highly dependent on extreme points [34].

Moment-of-inertia Measures assess the average squared distance of all points in a district from the centre [40]. This means they are less susceptible to misrepresenting a small number of extreme points as they are also affected by the frequency of such points.

Then there are also Dispersion Measures which compare district area with the area of a compact figure, one typical example is:

$$\text{Dis}_2 = \frac{A_D}{A_{CC}}$$

Where A_D is the area of the district and A_{CC} is the area of the minimum circumscribing circle [41]. This particular example again uses the idea that a circle is the most compact shape however another convex shape could be used so that the idealized districts under this criterion fit together.

Take the Lake of the Woods County on the northern border of Minnesota seen in Figure 3.4. The county comprises of an area of land below the 49^{th} parallel with an area of approximately 3053.1km^2 and the Angle Township area consisting of land and water covering an area of 1543.27km^2 [42].

The Angle Township area makes up only 2 percent of the population for the county and its borders exactly overlay the national borders of the USA [42]. Therefore gerrymandering in this area could not have occurred and if we suppose it had its effect would be negligible due to the small population. Using the CensusViewer tool which draws data from the 2010 census we can calculate the radius of the minimum circumscribing circle of the county with (62.593km) and without the Angle Township area (41.185km) [43]. Therefore we can calculate the Dis_2 measure for these two areas:

Compactness for whole county $= \frac{4596.37}{\pi \cdot 62.593^2} \approx 0.37$

Compactness for county without Angle Township area $= \frac{3053.1}{\pi \cdot 41.185^2} \approx 0.57$

As this measure produces values between 0 and 1 this is quite a marked change considering that this redistricting affects such a small number of people. We would hope an ideal measure would be able to take into account that adding an area of largely unpopulated land to a district would only lead to small changes to the value given. Dispersion measures alone are therefore not sufficient in determining the extent of gerrymandering and measures taking into account the population within districts should be considered.

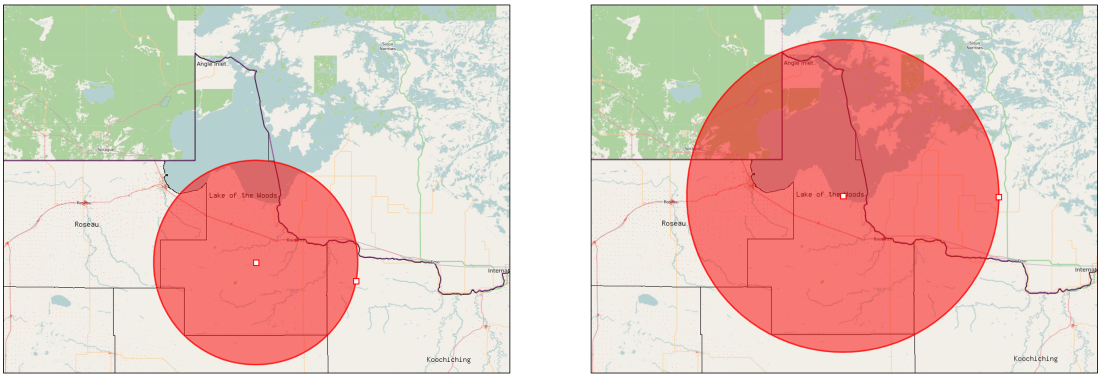


Figure 3.4: Lake of the Woods County without Angle Township area (left) and with (right) with their corresponding circumscribing circles

3.2.3 Population Measures

One category of Population Measures compare the population of a district to the population within a figure circumscribing that district e.g.

$$\text{Pop} = \frac{P_D}{P_{CC}}$$

Where P_D is the population inside the district and P_{CC} is the population inside the smallest circle circumscribing the district [34].

This measure is quite intuitive and by looking just at the population it better accounts for instances where a district is irregularly shaped but has not been gerrymandered, such as when there is a natural boundary like a river. Again we have that there are similar arguments for and against using a circle as the convex shape to circumscribe the district. One issue with this measure is that adding unpopulated areas to the mapping can dramatically change the value assigned to the district despite the compactness of the population remaining unchanged.

This can be seen through the example of Stark County in Illinois:

From Figure 3.5 you can see that the county can be split into a main body, with the bulk of the area, and an adjoining, smaller rectangle in the north containing the Elmira and Osceola Townships [44]. The smaller and larger segments together do seem to be intuitively slightly less compact than if the larger segment were on its own however we would not expect the difference to be large. As well as this by using the census data from 2010 we can see that the population for this smaller section makes up only 20% of the total population of the county. Given these two factors, we would hope that a population compactness measure would give similar values to the entire county and the bottom segment on its own. However, when we calculate the Pop measure for the county with and without the northern segment we get [45]:

Compactness for whole county $= \frac{5994}{24080} \approx 0.249$

Compactness for county without northern region $= \frac{5994-1333}{8106} \approx 0.575$

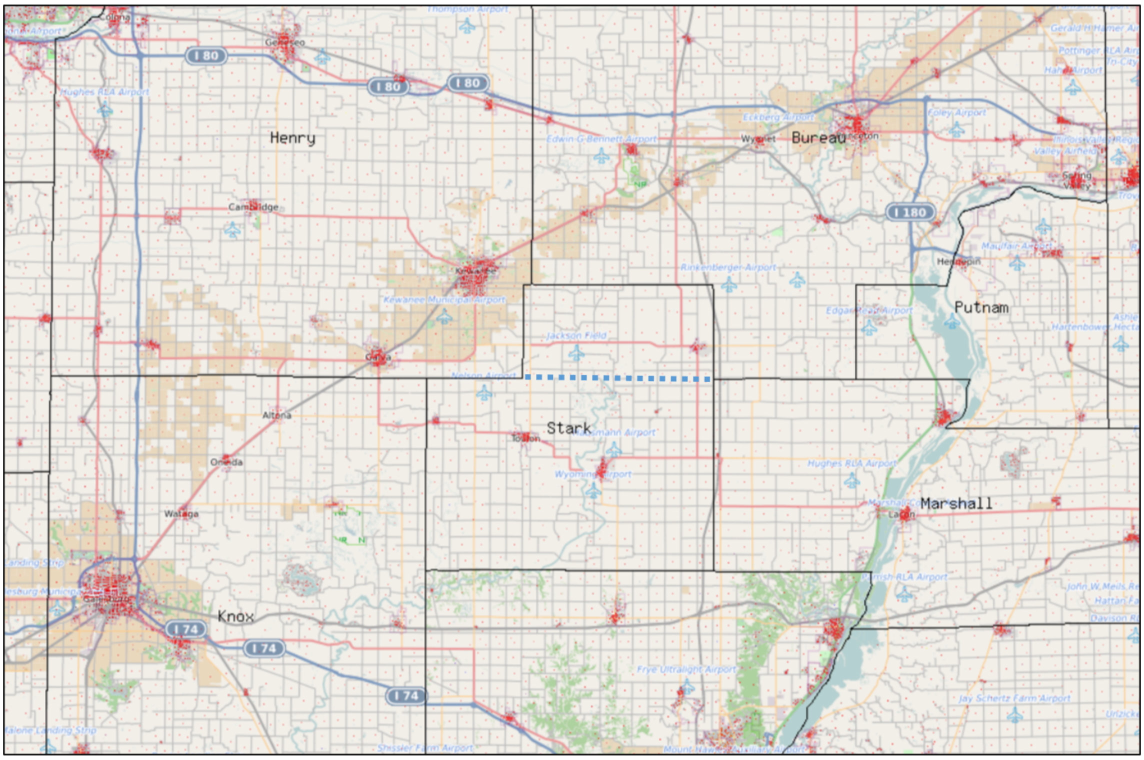


Figure 3.5: Stark County with added dotted line

The reasons for this fairly large difference in compactness values can be seen by the figure below.

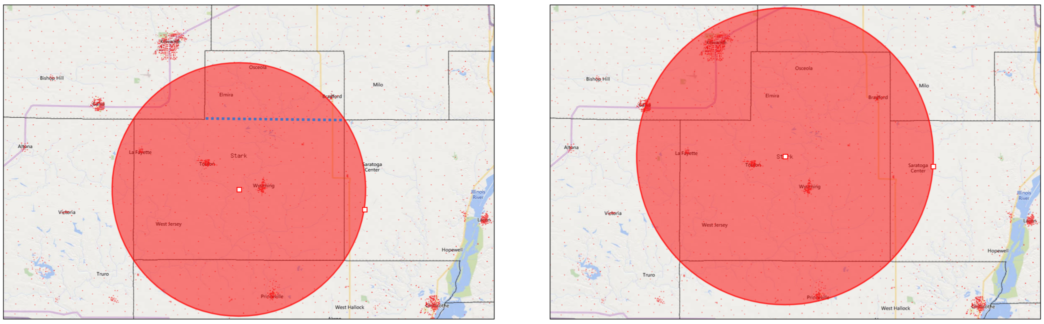


Figure 3.6: Stark County with containing Elmira and Osceola Townships (right) and without (left)

The enlargement of the mapping from the left to right pictures causes the circumscribing circle to enlarge so that it contains quite a dense population centre at the city Kewanee. This massively affects the value given by this population measure even though the original mapping remains unchanged and that the area of land added to the mapping contains only a small population. Ideally, our population-based compactness measure would be sensitive to things like this.

Moment-of-inertia Population Measures address this issue by utilising the dispersion measure described earlier by weighting each point in the district by the population in the surrounding area [34].

Population measures are more arduous to calculate, especially moment-of-inertia measures, and they tend not to fit so well with our natural instincts for what constitutes a compact district.

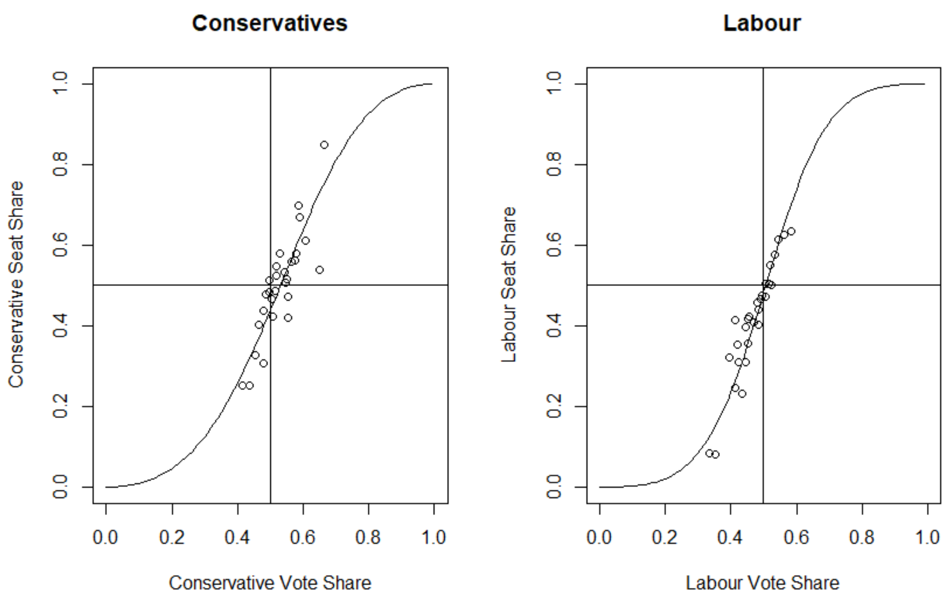
3.3 Symmetry

An appealing idea in a so-called fair election is that equivalent outcomes should be seen under opposite results. That is, if an equivalent vote share had been obtained by the opposite party, we would see the same number of seats distributed to that party.

This is the idea underpinning the analysis by way of *Partisan Symmetry*, that if party A wins x% of the available seats with y% of the votes across districts, then party B should also win x% of the seats with y% of the votes across districts if this election outcome was observed. [30]

3.3.1 Cube Law

In the early 1900s, officials began attempting to understand seat share results which initially appeared to be wholly unrelated to the vote share that parties received. A first attempt to explain this relationship between vote share and the resulting seat share in parliament resulted in the formula of the cube law [46]. The law was first observed in 1910 by James Parker Smith and later popularised in the 1950s.



Party	$\hat{\beta}_0$	$\hat{\beta}_1$	r^2	p
Conservatives	-0.25 (0.07)	2.00 (0.25)	0.72	44%
Labour	-0.07 (0.06)	2.74 (0.23)	0.85	48%

Figure 3.7: UK Election Results 1918-2019: Logit Model

If we have a first-past-the-post election system with two parties A and B, we let Y_A be the percentage of seats that party A holds and X_A be the percentage of votes commanded by party A then the cube law states,

$$\frac{Y_A}{1-Y_A} = \left(\frac{X_A}{1-X_A}\right)^3.$$

The proposed "law" was suggested as it fitted previous election results. However, as the law possessed no theoretical backing it was rejected by Tufte [47], who instead proposed the following model,

$$\ln\left(\frac{Y_A}{1-Y_A}\right) = \beta_0 + \beta_1 \ln\left(\frac{X_A}{1-X_A}\right)$$

that is equivalent to the cube law when $\beta_0 = 0$ and $\beta_1 = 3$. We calculated the parameter estimates in this model for the period 1918-2019. In Figure 3.7 we see that the cube law fits the Labour party's election results more closely, however, still leaves much to be desired.

3.3.2 Linear Relationship

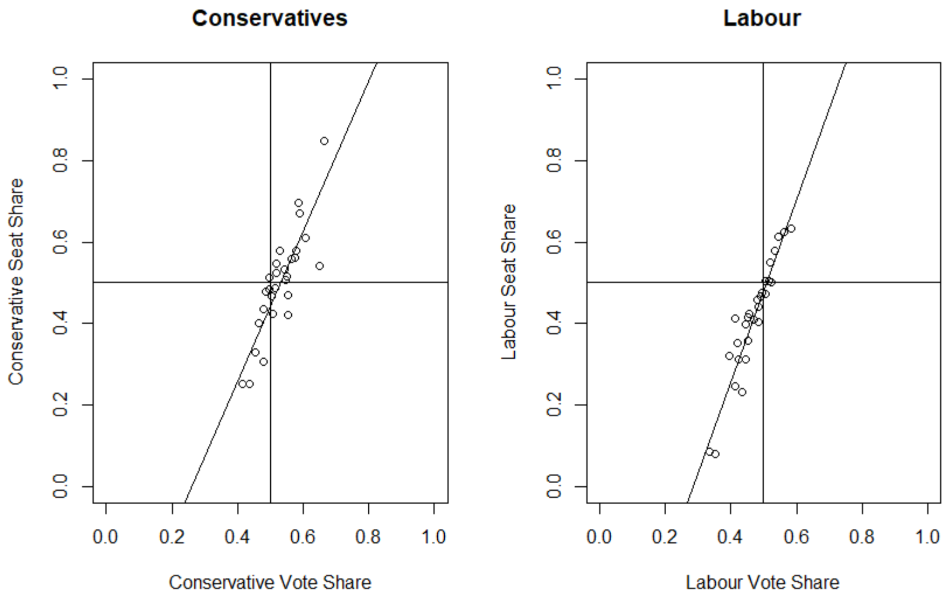
We then investigate the relationship between vote and seat percentages by plotting the vote share of parties against the resulting fraction of seats in the legislature. If symmetry indeed holds, when plotting the linear model

$$Y = \beta_0 + \beta_1 X$$

we expect to see that $\beta_0 = 0$ and $\beta_1 = 1$ (Where Y is the seat share and X the vote share). We also expect that our fitted model would pass through the point $(0.5, 0.5)$.

The investigation by way of a linear model was proposed in [47] by Tufte. He investigated the period 1945-1970 and found $\hat{\beta}_1 = 2.83$ for the Labour party. In 3.8, we extend this work to the period 1918-2019 [48] where we find $\hat{\beta}_1 = 2.24$ for the Labour party. Therefore, we see historical results have deviated largely from the symmetry standard in the United Kingdom, indicating that a different relationship exists.

In 3.8, we let p denote the seat percentage predicted by our model when the vote share equals 50%. The linear model is limited in that the points $(0, 0)$ and $(1, 1)$ do not lie on the model. In practice, we are not worried about this as elections generally do not see such parliamentary majorities formed.



Party	$\hat{\beta}_0$	$\hat{\beta}_1$	r^2	p
Conservatives	-0.48 (0.12)	1.84 (0.23)	0.72	44%
Labour	-0.64 (0.07)	2.24 (0.15)	0.89	48%

Figure 3.8: UK Election Results 1918-2019: Linear Model

3.3.3 Partisan Symmetry

We now turn our attention to investigating the relationship between votes and seats in single election results. We will use *Uniform Partisan Swing* [49], that is starting with an actual election outcome - represented by the blue dot - and adjusting the vote count district by district until each district flips its allegiance to the opposite party. The full code to plot these figures can be found in Appendix A.

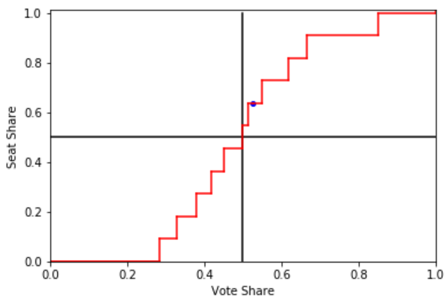
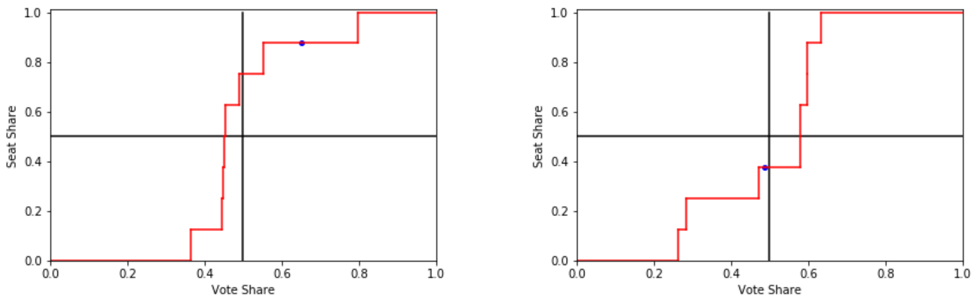


Figure 3.9: Virginia 2020 Seats-Votes plot

In Figure 3.9, we show the seats-vote plot for the 2020 United States House of Representatives election in Virginia, derived from election results in [50], from the democratic party's perspective. Where the realised election result was a 52.4% vote share for the democratic party, resulting in the winning of seven of the available eleven seats or a 63.6% seat share.

We create this plot by adjusting the vote share in all districts so as to move the number of votes for a particular party over the 50% mark and hence deliver their seat to the other party. We repeat this adjustment process until both parties have achieved all of the votes statewide. We will discuss the mathematical details in the next section. The plot represents how the seats in the state are distributed given specific vote shares achieved by either party.



(a) Maryland (b) Wisconsin

Figure 3.10: Seats-vote plots for 2020 House of Representatives elections

We can see that, at first viewing, the seats-votes plot in Figure 3.9 appears to be roughly symmetric about the centre point (0.5,0.5). We will see that Virginia's seats-vote plot gives a reasonable representation of the symmetry standard that we desire.

In Figure 3.10 we see two examples of states which do not have such symmetric seats-vote curves. These curves, generated from the 2020 House of Representatives elections in Maryland [51] and Wisconsin [52] show signs that the states' current districting plans may be gerrymandered. In Wisconsin, the democratic party requires almost 60% of the vote to achieve a seat majority while in Maryland, the democratic party need only 45% to achieve a majority.

3.3.4 Constructing the Seats-Votes Curve

Now we will derive the seats-votes curve. From chapter 2, we have that v_i^{A} is the total number of votes for party A in district d_i and $V_i = v_i^{\mathrm{A}} + v_i^{\mathrm{B}}$ is the total number of votes in district d_i, then we denote by $v_i = \frac{v_i^{\mathrm{A}}}{V_i}$ the vote share for party A in district d_i.

Note, this implies that $v_i^{\mathrm{A}} = V_i v_i$ so the vote share for party A over the whole state is,

$$\mathcal{V}_0 = \frac{\sum_{i=0}^{n} V_i v_i}{\sum_{i=0}^{n} V_i}$$

Using the same construction as DeFord et al in their paper on partisan symmetry, we denote the result of the election by a vote share vector of length n, where n is the number of districts and hence seats, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$[53]. The elements of $\mathbf{v}$ are redefined so that they are in ascending order i.e. $0 \leq v_1 \leq \cdots \leq v_n \leq 1$.

We also can denote the seat outcome of the election over the whole state by $s_0 = \#\{i|v_i > \frac{1}{2}\}$. This being the number of districts in which party A holds a majority of votes.

Now, we construct the seats-vote plot using linear uniform partisan swing[53]. The technique involves varying the vote share in each district by some amount t so that we have a modified vote share vector $\mathbf{v}_t = (v_1 + t, v_2 + t, \ldots, v_n + t)$ represents the modified vote share vector. We want to pick all values of t so that our modified vote share vector varies from the election outcome where party A receives every vote to the election outcome where party A receives no votes. Our vote share vector is in ascending order, therefore we have $t \in [-v_n, 1 - v_1]$. Finally, We want our modified vote shares to lie between 0 and 1, so we will denote the modified vote shares by,

$$v_i(t) := \begin{cases} 1 & v_i + t > 1 \\ 0 & v_i + t < 0 \\ v_i + t & \text{otherwise} \end{cases}.$$

Then the modified vote share vector can be written $\mathbf{v}_t = (v_1(t), v_2(t), \ldots, v_n(t))$

Note that to achieve a vote share of $v_i(t) = v_i + t$ in district d_i, the number of votes for party A becomes $(v_i + t)V_i = v_i(t)V_i$. Then under this value of t, uniform partisan swing gives a new vote share over the state for party A,

$$\mathcal{V}_t = \frac{\sum_{i=0}^{n} v_i(t)V_i}{\sum_{i=0}^{n} V_i}.$$

Then new seat share over the state for party A is,

$$s_t = \frac{\#\{i|v_i(t) > \frac{1}{2}\}.}{n}$$

The black dot in the seats-votes curves represents the point $(\mathcal{V}_0, s_0)$, which is the realised election outcome. The seats-vote curve is the points $(\mathcal{V}_t, s_t)$ for all values of t which we call γ.

We wish to define the flip values for a vote share vector $\mathbf{v}$ as the values t_j for $j \in \{1, 2, \ldots, n\}$ such that one of the districts in the vote share vector has a vote share equal to $\frac{1}{2}$.

The values $t_1, \ldots, t_n$ are given as the vote shares required to reach a 50% vote share in each of the districts,

$$t_1 := \frac{1}{2} - v_n, \quad t_2 := \frac{1}{2} - v_{n-1}, \quad \ldots \quad t_n := \frac{1}{2} - v_1. \tag{3.1}$$

Note that we have written these values of t such that they are in ascending order.

Then we can write the flip points as $\mathcal{V}_{t_j}$ for all values of j. Therefore, the flip point vector is given by $\mathbf{v}_{\text{flip}} = (\mathcal{V}_{t_1}, \ldots, \mathcal{V}_{t_n})$.

It is important to stress at this point that the seats-vote curve represents hypothetical election outcomes derived from our realised election outcome $(\mathcal{V}_0, s_0)$.

3.3.5 Symmetry Measurement Techniques

Now, we move on to methods that measure the extent of deviation from our symmetry standard. Our first measure is the area between the seats-vote curve and its reflection about the point $\times$, denoted PG.

Definition 3.3.1. Given a seats-votes curve γ and a vote share vector $\mathbf{v}$, the Partisan Gini score is the area between γ and its reflection about the point $\times$.

$$\mathrm{PG}(\mathbf{v}) = \int_0^1 |\gamma(x) - \gamma(1-x) + 1| dx$$

In Figure 3.11, we have the seats-votes curves for both the Republican and Democratic party. The shaded area between the curves is the area measured by the Partisan Gini score and we desire $PG = 0$ in our symmetric case. Clearly, Wisconsin's plan falls far short of the symmetry standard. See Appendix B for a selection of seats-votes curves for different states.

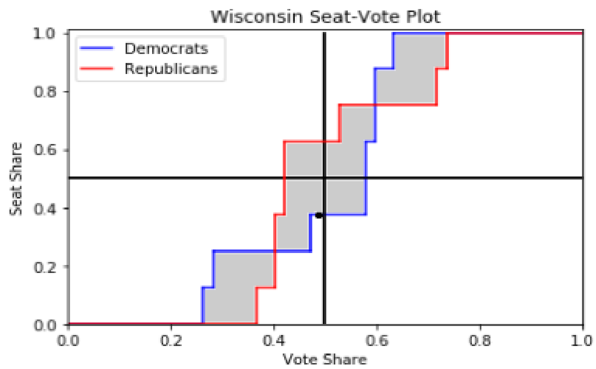


Figure 3.11: Seats-vote plots for Wisconsin 2020 U.S. House of Representatives elections

The second measure, $\mathrm{PB}(\mathbf{v})$, is the vertical distance from the curve γ to the centre point $\times$, representing the seat bias for party A. The seat share above or below the symmetry standard that party A would have achieved had they obtained half of the votes under linear uniform partisan swing.

Definition 3.3.2. Given a seats-vote curve γ, the Partisan Bias (PB) score is $\gamma(\frac{1}{2}) - \frac{1}{2}$

The Partisan Bias score featured in the case *LULAC vs Perry* and nearly won the backing of the Justices as a sole measure to identify partisan gerrymandering. It then became the basis for the Efficiency Gap that we will meet in the next section.

For the next measure, we will first need to define the median element of the vote share vector,

$$\mathbf{v}_{\text{med}} := \begin{cases} v_{(n+1)/2} & \text{n is odd} \\ \frac{1}{2}(v_{n/2} + v_{(n+1)/2}) & \text{n is even} \end{cases}$$

Definition 3.3.3. We can then define the mean-median score of a vote share vector $\mathbf{v} = (v_1, \ldots, v_n)$ as

$$\text{MM}(\mathbf{v}) = \mathbf{v}_{\text{med}} - \bar{\mathbf{v}}$$

This is another measure that indicates and provides a numerical value of the degree to which there is an asymmetry between vote share and seat share. It is optimised at zero when the mean and median scores of the vote share vector are equal. Positive values of MM($\mathbf{v}$) indicate that the districting plan benefits party A while negative values indicate that it disadvantages the party. For example if 'the median is 53 and the mean is 55...the bias runs two points against Party A' [54].

DeFord and his colleagues remark that the mean-median measure is equal to the horizontal displacement from a point on the seats-vote curve γ, where the seat share is equal to $\frac{1}{2}$, to the centre point $\times$ [53, p. 4]. However, we can show that this is not generally true:

Take the case where there is an odd number of districts, n, then it is not possible for party A to have half the number of seats.

Therefore the seats-votes curve crosses the axis where seat-share $= \frac{1}{2}$ at a single point when the number of seats won by Party A increases from $\frac{n-1}{2}$ to $\frac{n+1}{2}$. This occurs at the flip point $\mathcal{V}_{t_{\frac{n+1}{2}}}$ which corresponds to the flip value:

$$\text{T} = t_{\frac{n+1}{2}} = \frac{1}{2} - v_{\frac{n+1}{2}} = \frac{1}{2} - \mathbf{v}_{\text{med}}$$

Hence we have that the horizontal displacement from the centre to the only point where γ crosses the horizontal axis is given by $\frac{1}{2} - \mathcal{V}_{\text{T}}$. Substituting in our value of T we get:

$$\begin{aligned} \frac{1}{2} - \mathcal{V}_{\text{T}} &= \frac{1}{2} - \frac{\sum_{i=0}^{n} v_i(\text{T})V_i}{\sum_{i=0}^{n} V_i} \\ &= \frac{1}{2} - \frac{\sum_{i=0}^{n}(v_i + \frac{1}{2} - \mathbf{v}_{\text{med}})V_i}{\sum_{i=0}^{n} V_i} \\ &= \frac{1}{2} - \frac{\sum_{i=0}^{n} v_i V_i}{\sum_{i=0}^{n} V_i} - \frac{(\frac{1}{2} - \mathbf{v}_{\text{med}})\sum_{i=0}^{n} V_i}{\sum_{i=0}^{n} V_i} \\ &= \frac{1}{2} - \mathcal{V}_0 - \frac{1}{2} + \mathbf{v}_{\text{med}} \\ &= \mathbf{v}_{\text{med}} - \mathcal{V}_0 \end{aligned}$$

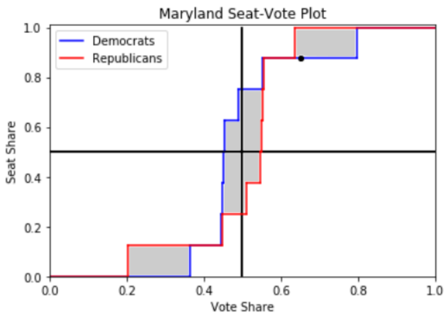
Therefore we have that for every plan with an odd number of districts this horizontal distance on a seats-votes plot is only equal to MM($\mathbf{v}$) when $\mathcal{V}_0 = \bar{\mathbf{v}}$ which is not generally true.

We can remark that under the simplifying assumption that each district has the same number of voters ($V_1 = \ldots = V_k = V$) then we have $\mathcal{V}_0 = \frac{\sum v_i V_i}{\sum V_i} = \frac{\sum v_i}{k} = \bar{\mathbf{v}}$ in which case the horizontal distance between the centre point on the graph and γ is equal to MM($\mathbf{v}$).

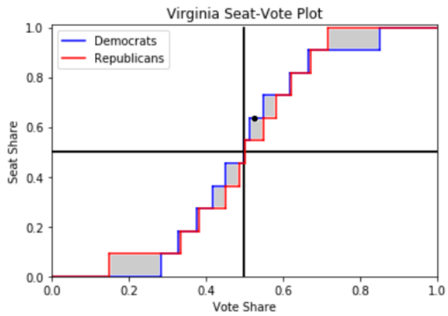
This is likely the reason why this imprecision is repeated by DeFord et al throughout their paper.

We can then use these three measures to evaluate and compare the overall symmetry of different districting plans and how close they come to the idealized scores of:

$$\text{PG} = \text{PB} = \text{MM} = 0$$



(a) Maryland



(b) Virginia

	PG	MM	PB
Virginia	0.0383	0.0180	0.0455
Maryland	0.0801	0.0326	0.25
Wisconsin	0.1123	-0.0856	-0.125

Figure 3.12: Seats-vote plots for 2020 U.S. House of Representatives elections

From Figure 3.12 we see the seats-votes plots for the same states as in section 3.3.3, this time for both parties, with the shaded region indicating the area between the curves. The figure includes the corresponding symmetry scores for each of the plans. We can see that Virginia, whose plot looks the most symmetric, scores the lowest in each of the measures, as we would expect. Therefore, we can conclude that Virginia has the most symmetric plan of the three states in this election.

It is interesting to note that the positive scores for Virginia and Maryland indicate plans favouring the Democratic party while the negative scores for Wisconsin indicate a plan favouring the Republican Party. These scores also agree with our earlier intuition in section 3.3.3 when considering only where the curves cross the line corresponding to a seat share of 50%.

3.4 Efficiency Gap

The Efficiency Gap, formulated by Mcghee and Stephanopoulos [23], set out to improve upon the partisan bias measure from the last section by meeting the demands of Justice A. Kennedy. The law professors looked to provide a simple metric that relies on the realised election outcome and gives a clear threshold to identify gerrymandering.

3.4.1 Definition

The Efficiency Gap is a calculation of the proportion of total votes that are wasted in favour of one party and is equal to zero in an ideal districting plan.

Definition 3.4.1. If the wasted votes for party A and B over the state are W^A and W^B respectively and V is the total statewide votes, then the efficiency gap is,

$$EG := \frac{W^A - W^B}{V}.$$

Here, $EG \in [-0.5, 0.5]$ by construction. If $EG > 0$ the districting plan favours party B while if $EG < 0$ then the districting plan favours party A. The authors calculated efficiency gaps for all congressional and state house districting plans from 1972 to 2012 and inferred an acceptable threshold of below two seats for congressional plans and 8% for state house plans. Scores above this threshold indicate a plan scoring in the lowest 14% of plans, 1.5 standard deviations from the mean [23]. In this paper, we deal solely with congressional plans and so we will use the two-seat threshold.

We convert the efficiency gap into seats by multiplying the magnitude of the gap by the number of districts in that state, therefore $EG_s := n|EG|$.

Example 2.1 Here we return to example 3.1 and calculate the efficiency gap for this five district state example. We have,

$$EG = \frac{W^A - W^B}{V} = \frac{(19 + 58) - (19 + 35)}{262} = 0.088.$$

Then we have $EG_s = 0.44$, which is well within the congressional two-seat threshold, however, would be considered evidence of gerrymandering in a state house election.

3.4.2 Problems

We now investigate an issue with the efficiency gap as in [24].In addition to the notation of section 3.1.1, we define I_i^j as the indicator function which returns 1 if party j wins district d_i, so that $I^j = \sum_{i=1}^{n} I_i^j$ is the number of seats won by party j over the state. We can now define the statewide vote lean τ and seat lean σ favouring either of the parties,

$$\tau := \frac{V^A - V^B}{V}, \qquad \sigma := \frac{I^A - I^B}{I^A + I^B}.$$

These leans are simply the difference between the seat and vote share for parties A and B. In a fair districting plan, we would expect that τ and σ are approximately equal. Now, if we make the simplifying assumption that the voter turnout is equal across districts, that is, $V_i = \frac{V}{I^A + I^B}$. Then,

$$W^j = \sum_{i=1}^{n} W_i^j = \sum_{i=1}^{n} (v_i^j - I_i^j \frac{V_i}{2}) = V^j - I^j \frac{V}{2(I^A + I^B)}.$$

Where the second equality is the definition of wasted votes as the sum of surplus and losing votes. Now we can rewrite the EG formula,

$$EG = \frac{W^A - W^B}{V} = \frac{(V^A - I^A \frac{V}{2(I^A+I^B)}) - (V^B - I^B \frac{V}{2(I^A+I^B)})}{V}$$

$$EG = \frac{V^A - V^B}{V} - \frac{1}{2}\frac{I^A - I^B}{I^A + I^B} = \tau - \frac{1}{2}\sigma.$$

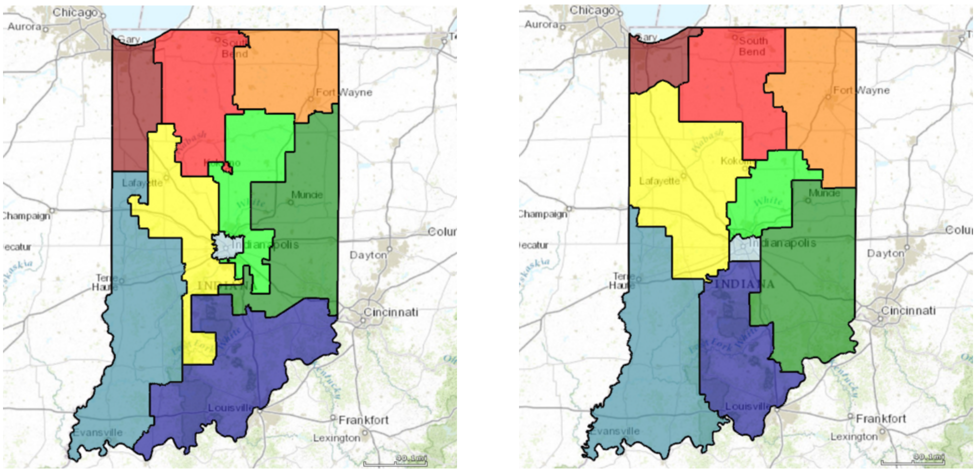
The result is striking, that the efficiency gap is simply the statewide vote lean favouring A minus half the seat lean favouring A. Unrelated to how the voters are distributed and districts are drawn. We can take this analysis further by considering our supposedly desirable outcome of $EG = 0$ which implies $\sigma = 2\tau$, any election satisfying this will be considered fair under the efficiency gap measure.

If we consider the election outcome where party A wins 60% of the seats with a 60% vote share, we see that $EG = \tau - \frac{1}{2}\sigma = 0.2 - 0.1 = 0.1 > 0.08$ that would identify the state house plan as gerrymandered. This is not desirable since we would like to see approximate proportionality from the vote share to seat share in an election.

We can also consider an election where each district has vote shares of 75% to party A and 25% to party B, then party A wins every seat. Intuitively, we would consider this to be a highly unfair plan, however, our plan will pass with an Efficiency Gap of zero, since $\tau = \frac{1}{2}$ and $\sigma = 1$.

3.5 Indiana: Gerrymandering

In this section, we will continue the case study from section 2.5. Now we will be investigating the state from a gerrymandering perspective by considering two districting plans using the measures we have discussed in this chapter. We will compare the districting plan for Indiana used in congressional elections from 2002 to 2010 with the plan used in the elections from 2012 to 2020. We will then conclude whether we consider the updated plan to be an improvement from a gerrymandering perspective.



(a) 2002-2011 (b) 2012-2021

Figure 3.13: Indiana's Congressional Districts [55]

We selected Indiana as a small state that is computationally easy to work with due to the smaller number of districts. And also because its districting plan changes significantly over the cycle of redistricting as can be seen in Figure 3.13. We note that the new districting plan was used for the first time in the 2012 election.

Compactness

We'll first look at the two districting plans in terms of their compactness.

	Pop		Dis_2	
	2001 - 2010	2011 - 2020	2001 - 2010	2011 - 2020
district 1	0.344662	0.628213	0.351853	0.469408
district 2	0.522983	0.685587	0.458956	0.576991
district 3	0.716879	0.577768	0.642836	0.447509
district 4	0.232482	0.290487	0.20565	0.469649
district 5	0.308829	0.441962	0.296934	0.440877
district 6	0.154983	0.152302	0.31322	0.48686
district 7	0.67205	0.738245	0.536325	0.615288
district 8	0.189864	0.324943	0.244684	0.322089
district 9	0.235731	0.356414	0.329727	0.421931
Average	0.375385	0.466213	0.375576	0.472289

Table 3.1: Indiana Compactness Scores

As you can see from Figure 3.13 the state borders in the south of Indiana between Illinois and Kentucky are extremely jagged while the state borders between the northern districts of Indiana and the other states are largely straight lines. This massively restricts the usefulness of perimeter measures in Indiana as the southern districts are unfairly penalised.

We have therefore calculated the previously defined population and dispersion measures Pop and Dis_2 respectively for each of the districts in the two districting plans.

$$\mathrm{Dis}_2 = \frac{A_D}{A_{CC}}, \qquad \mathrm{Pop} = \frac{P_D}{P_{CC}}$$

As you can see from Table 3.1 there is broad agreement between the two measures that the new districting plan is more compact than the previous one. In fact, district 3 is the only district that appears to be less compact in the latest plan.

The only point of disagreement between the two measures is in district 5 where there is a marginal decrease in the value assigned by the population measure. This strong overall agreement between the two measures provides good evidence that the districting plan has become more compact across this cycle of redistricting.

Efficiency Gap

We can see the efficiency Gap calculations for each of the elections in Table 3.2 over both of the districting plans. We note that the mean efficiency gap is much greater in the second districting plan at 1.33 seats rather than the first at 0.53 seats. This average efficiency gap is below the national average of 1.58 seats for districting plans over the period 2012-2020 [23].

Year	EG	EG_s	Year	EG	EG_s
2002	0.056	0.50	2012	0.201	1.81
2004	0.130	1.17	2014	0.087	0.78
2006	-0.060	0.54	2016	0.159	1.43
2008	0.021	0.19	2018	0.184	1.66
2010	0.028	0.25	2020	0.110	0.99

Table 3.2: Indiana Efficiency Gap Scores

We also note that the efficiency gap is positive in all but one election over the 18 years. This indicates that the republican party wasted less votes than the democratic party in each of these elections. However, the efficiency gap never exceeds the standard of 2 seats set by McGhee and Stephanopolous.

Symmetry

We will use the seats-vote curves derived in section 3.3.4 to investigate all of the elections in the period 2002-2020. We can then look for patterns in the graphs and their symmetry measures to identify potential gerrymandering. In figures 3.14 and 3.15 we can see that the seats-votes plots for the first districting plan appear more symmetrical, which is reflected in their lower PG scores from Table 3.3a to Table 3.3b.

The MM measure in the first plan flips from positive to negative over the course of the elections and has a small positive mean value, indicating a slight advantage

	PG	MM	PB
Indiana 2002	0.028	0.036	0.056
Indiana 2004	0.060	-0.022	-0.056
Indiana 2006	0.024	0.025	0.056
Indiana 2008	0.036	0.065	0.056
Indiana 2010	0.027	-0.026	-0.056
Mean	0.035	0.016	0.011

(a) Indiana 1st Districting Plan

	PG	MM	PB
Indiana 2012	0.068	-0.014	-0.167
Indiana 2014	0.097	-0.045	-0.167
Indiana 2016	0.134	-0.084	-0.167
Indiana 2018	0.090	-0.015	-0.167
Indiana 2020	0.076	-0.053	-0.167
Mean	0.093	-0.0422	-0.167

(b) Indiana 2nd Districting Plan

Table 3.3: Indiana Symmetry Measures

to the democratic party. Whereas the second districting plan has a negative MM score for each of the elections and a mean MM score much larger in magnitude than that of the first plan, indicating a plan that advantages the republican party. The negative MM in each of the election years can be seen clearly in 3.15 since the democratic curve in each of the seats-votes curves passes to the right of the centre point. Indicating that the democratic party required a greater than 50% vote share to secure a majority of seats.

Finally, the Partisan Bias score indicates that with a 50% vote share, the democratic party secure much fewer seats under the second districting plan than the first. This indicates that the districting plan moved from favouring the republican party to the democratic party.

Conclusion

In conclusion, the increasing mean efficiency gap identifies the second plan as being further from the symmetry standard, however not sufficiently for us to say that that the districting plan is gerrymandered. The symmetry scores point to the districting plan becoming less symmetric over the cycle of redistricting, however, they give no threshold above which we can denote that a plan is gerrymandered. While the compactness measures state that the plan becomes more compact over the cycle of redistricting.

Therefore although two out of our three measures indicate that the plan worsens, neither of them provides enough evidence to conclude that gerrymandering has occurred over this cycle of redistricting.

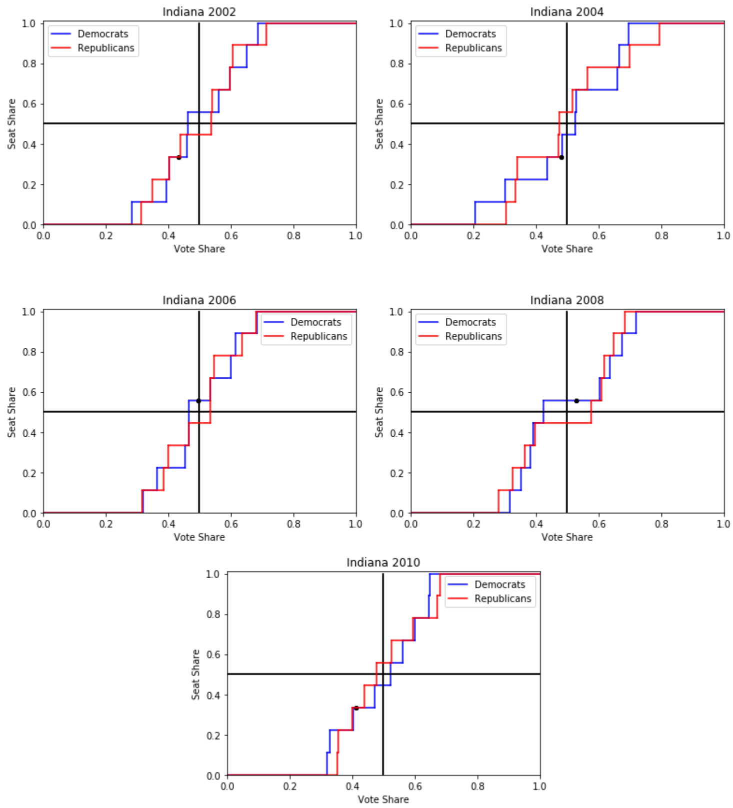


Figure 3.14: Seats-Votes Curves for 2003-2013 Districting Plan

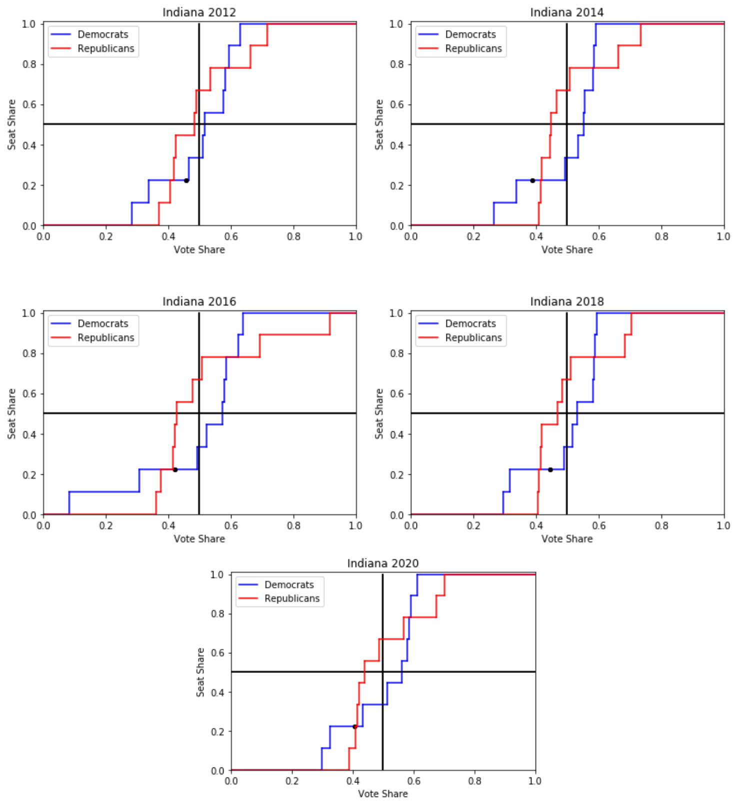


Figure 3.15: Seats-Votes Curves for 2013-2020 Districting Plan

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Appendix A

Seats-Votes Curves Code

What follows is the Python code that generates the seats vote plots throughout chapter 3.

The code includes functions to generate the seats-votes plot, calculate the symmetry measures and efficiency gap for a state with a_votes for the democratic party and b_votes for the republican party.

For the code in an easily readable format please see [56].

```
import matplotlib.pyplot as plt
import numpy as np

def flipPoints(a_votes, b_votes):
    """
    Returns the array of points where the step function of the Seats-Vote plot steps
    a_votes: an array of votes for party A indexed by district
    b_votes: an array of votes for party B indexed by district
    """

    # Create copies of both arrays to adjust and an array to store the flip points
    a_votes_adjusted = np.copy(a_votes)
    b_votes_adjusted = np.copy(b_votes)
    flip_points = []

    # Calculate V_0
    a_vote_share_adjusted = (np.sum(a_votes_adjusted)) / (np.sum(a_votes_adjusted) + np.sum(b_votes_adjusted))

    # Loop through the states adding votes to party a until they have all votes
    while a_vote_share_adjusted != 1:
        for i in range(num_districts):
            if b_votes_adjusted[i] != 0:
                a_votes_adjusted[i] += 1
                b_votes_adjusted[i] -= 1

            if a_votes_adjusted[i] == b_votes_adjusted[i] or a_votes_adjusted[i] == b_votes_adjusted[i] + 1:
                flip_points.append(a_vote_share_adjusted)

        a_vote_share_adjusted = (np.sum(a_votes_adjusted)) / (np.sum(a_votes_adjusted) + np.sum(b_votes_adjusted))

    # Create copies of both arrays to adjust
    a_votes_adjusted = np.copy(a_votes)
    b_votes_adjusted = np.copy(b_votes)

    # Calculate V_0
    a_vote_share_adjusted = (np.sum(a_votes_adjusted)) / (np.sum(a_votes_adjusted) + np.sum(b_votes_adjusted))

    # Loop through the states removing votes from party a until they have no votes
    while a_vote_share_adjusted != 0:
        for i in range(num_districts):
            if a_votes_adjusted[i] != 0:
                a_votes_adjusted[i] -= 1
                b_votes_adjusted[i] += 1

            if a_votes_adjusted[i] == b_votes_adjusted[i] or a_votes_adjusted[i] == b_votes_adjusted[i] + 1:
                flip_points.append(a_vote_share_adjusted)

        a_vote_share_adjusted = (np.sum(a_votes_adjusted)) / (np.sum(a_votes_adjusted) + np.sum(b_votes_adjusted))
    # Return the flip points
    return np.sort(flip_points)

def SeatsVotePlot(a_votes, b_votes, plot_reverse = True):
```

```
    """
    Constructs a Vote-Seat Plot for the given election results
    a_votes: an array of votes for party A indexed by district
    b_votes: an array of votes for party B indexed by district
    """
    # Create arrays to help with plotting
    x = np.linspace(0, 1, 1000)
    const = np.empty(1000)
    const.fill(0.5)

    # Plot the basic empty plot
    plt.xlim(left = 0, right = 1)
    plt.ylim(bottom = 0, top =1.01)
    plt.xlabel("Vote Share")
    plt.ylabel("Seat Share")
    plt.axis

    # Plot the realised election result
    plt.plot(a_vote_share, a_seat_share, 'ko', markersize = 4)

    # Plot the lines seatshare=1/2, voteshare=1/2
    plt.plot(x,const, 'black')
    plt.plot(const,x, 'black')

    # Use the function flipPoints to find the flip points
    flips = flipPoints(a_votes, b_votes)
    flips.sort()

    # Access the global variables defined in main
    global color1
    global Label1
    global once

    # For each of the flip points plot the seats-vote curve
    for i in range(len(flips)-1):
        constant = np.empty(1000)
        constant.fill((i+1)/num_districts)
        plt.plot(np.linspace(flips[i], flips[i+1],1000),constant,color1)

        vconstant = np.empty(1000)
        vconstant.fill(flips[i])
        plt.plot(vconstant, np.linspace(i/num_districts,(i+1)/num_districts,1000),color1)

    # Plot the last verticle line that does not get plotted by the previous for loop
    ones = np.empty(1000)
    zeros = np.empty(1000)
    last_verticle = np.empty(1000)
    ones.fill(1)
    zeros.fill(0)
    last_verticle.fill(flips[-1])
    plt.plot(last_verticle,np.linspace((num_districts-1)/num_districts,1,1000) ,color1)
    plt.plot(np.linspace(0,flips[0],1000),zeros, color1)
    plt.plot(np.linspace(flips[-1],1,1000),ones, color1, label=Label1)

    # If plot_reverse is true, run the function again to plot the republican
    # seats-vote curve
    if plot_reverse:
        if once == 0:
             once += 1
             color1 = 'r'
             Label1 = 'Republicans'
             SeatsVotePlot(b_votes,a_votes)

    # Add a title and legend
    plt.title(Title)
    plt.legend()
    return

def SymmetryMeasures(a_votes, b_votes):
    # Initialize the measures as zero
    PB=0
    PG=0
    MM=0

    # Find the flip points for both parties
    dem_flips = (flipPoints(a_votes, b_votes))
    con_flips = (flipPoints(b_votes, a_votes))

    # Construct the vote vector
    vote_vector = a_votes / (a_votes + b_votes)
    vote_vector = np.sort(vote_vector)
    n = len(vote_vector)

    # Find the mean and median votes
    MM = str(round(np.median(vote_vector)-np.mean(vote_vector),10))

    # Calculate PB - a warning is printed if the flips is close to 0.5
    for i in range(1,n):
        if dem_flips[i-1] < 0.5 and dem_flips[i] > 0.5:
            gamma_half = i / n
            if dem_flips[i] - 0.5 < 0.01:
                print('Check that PB makes sense here, there is a flip within 1% of 50%')
    PB = gamma_half - 0.5

    # Calculate PG
    for i in range(len(dem_flips)):
```

```
        PG += abs(dem_flips[i]-con_flips[i]) * (1/num_districts)

    # Return the measures
    return PG, MM, PB

def EfficiencyGap(a_votes, b_votes):
    # Initialize the variables as zero
    EG = 0
    wasted_a = 0
    wasted_b = 0

    # for each district, calculate the wasted votes for each party
    for i in range(len(a_votes)):
        if a_votes[i] > b_votes[i]: # if party a wins district i
            wasted_b += b_votes[i]
            wasted_a += a_votes[i] - ((a_votes[i]+b_votes[i]) / 2)
        else:                       # if party b wins district i
            wasted_a += a_votes[i]
            wasted_b += b_votes[i] - ((a_votes[i]+b_votes[i]) / 2)

    # Calculate the Efficiency Gap
    EG = (wasted_a - wasted_b) / (sum(a_votes) + sum(b_votes))

    return EG


if __name__ == '__main__':
    ##########################################################################
    # 2020 House of Representatives ##########################################
    ##########################################################################

    # Body of Project Text

    #Virginia
    #Title='Virginia Seat-Vote Plot'
    #a_votes = np.array([186923,185733,233326,241142,190315,134729,230893,301454,0,268734,280725])
    #b_votes = np.array([260614,165031,107299,149625,210988,246606,222623,95365,271851,206253,111380,])

    #Virginia Even Test
    #Title='Virginia Seat-Vote Plot Even Test'
    #a_votes = np.array([186922,185733,233325,241141,190314,134728,230893,301453,0,268733,280724])
    #b_votes = np.array([260614,165031,107299,149625,210988,246606,222623,95365,271850,206253,111380,])

    #Wisconsin
    #Title='Wisconsin Seat-Vote Plot'
    #a_votes = np.array([163170,318523,199870,232668,175902,164239,162761,149558])
    #b_votes = np.array([238271,138306,189524,70769,265434,238874,252048,268173])

    #Maryland
    #Title='Maryland Seat-Vote Plot'
    #a_votes = np.array([143877,224836,260358,282119,274210,215540,237084,274716])
    #b_votes = np.array([250901,106335,112117,71671,123525,143599,92825,127157])

    ##########################################################################
    # Indiana - Case Study
    ##########################################################################

    # Districting Plan A

    # Indiana 2002
    #Title = 'Indiana 2002'
    #a_votes = np.array([90443,86253,50509,41314,45283,63871,77478,88763,96654])
    #b_votes = np.array([41909,95081,92566,112760,129442,118436,64379,98952,87169])

    # Indiana 2004
    #Title = 'Indiana 2004'
    #a_votes = np.array([178406,115513,76232,77574,228718,85123,121303,121522,140772])
    #b_votes = np.array([82858,140496,171389,190445,82637,182529,97491,145576,142197])

    # Indiana 2006
    #Title = 'Indiana 2006'
    #a_votes = np.array([104195,103561,80357,66986,64362,76812,74750,131019,110454])
    #b_votes = np.array([40146,88300,95421,111057,133118,115266,64304,83704,100469])

    # Indiana 2008
    #Title = 'Indiana 2008'
    #a_votes = np.array([199954,187416,112309,129038,123357,94265,172650,188693,181281])
    #b_votes = np.array([76647,84455,155693,192526,234705,180608,92645,102769,120529])

    # Indiana 2010
    #Title = 'Indiana 2010'
    #a_votes = np.array([99387,91341,61267,53167,60024,56647,86011,76265,95353])
    #b_votes = np.array([65558,88803,116140,138732,146899,126027,55213,117259,118040])

    # Districting Plan B

    # Indiana 2012
    #Title = 'Indiana 2012'
    #a_votes = np.array([187743,130113,92363,93015,125347,96678,162122,122325,132848])
    #b_votes = np.array([91291,134033,187872,168688,194570,162613,95828,151533,165332])

    # Indiana 2014
    #Title = 'Indiana 2014'
    #a_votes = np.array([86579,55590,39771,47056,49756,45509,61443,61384,55016])
    #b_votes = np.array([51000,85583,97892,94998,105277,102187,46887,103344,101594])
```

```
# Indiana 2016
#Title = 'Indiana 2016'
#a_votes = np.array([207515,102401,66023,91256,123849,79135,158739,93356,130627])
#b_votes = np.array([0,164355,201396,193412,221957,204920,94456,187702,174791])

# Indiana 2018
#Title = 'Indiana 2018'
#a_votes = np.array([159611,103363,86610,87824,137142,79430,141139,86895,118090])
#b_votes = np.array([85594,125499,158927,156539,180035,154260,76457,157396,153271])

# Indiana 2020
Title = 'Indiana 2020'
a_votes = np.array([185180,114967,104762,112984,191226,91103,176422,95691,122566])
b_votes = np.array([132247,183601,220989,225531,208212,225318,106146,214643,222057])

# Initialize some basic variables
once = 0
color1 = 'b'
Label1 = 'Democrats'
num_districts = len(a_votes)
a_votes_sum = np.sum(a_votes)
b_votes_sum = np.sum(b_votes)
total_votes = a_votes_sum + b_votes_sum
seats_won_a = np.zeros(num_districts)

# Calculate the number of seats won by party a
for i in range(num_districts):
    if a_votes[i] > b_votes[i]:
        seats_won_a[i] = 1

# Calculate the vote share vectors
a_vote_share = a_votes_sum / total_votes
a_seat_share = np.sum(seats_won_a) / num_districts

# Print the election year and state
print(Title)

# Print the efficiency Gap for party a
print(EfficiencyGap(a_votes, b_votes))

# Print the flip points for party a
print((flipPoints(a_votes, b_votes)))

# Print the symmetry measures for party a
print(SymmetryMeasures(a_votes, b_votes))

# Plot the seats-vote curve
SeatsVotePlot(a_votes, b_votes, plot_reverse = False)
```

Appendix B

Seats-Votes Curve: Examples

Figure B.1: Seats-Vote Curves for 2020 US House of Representatives Elections