# Linear isogeometric Euler-Bernoulli beam formulation

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## 1 Bernoulli beam theory

One of the simplest models to describe the behaviour of a bending beam is the Bernoulli beam theory. Accordingly, it is assumed that plane cross sections remain plane throughout the deformation and additionally cross sections initially orthogonal to the mid-line, remain orthogonal to the mid-line after the deformation as well (see Fig. (1)).

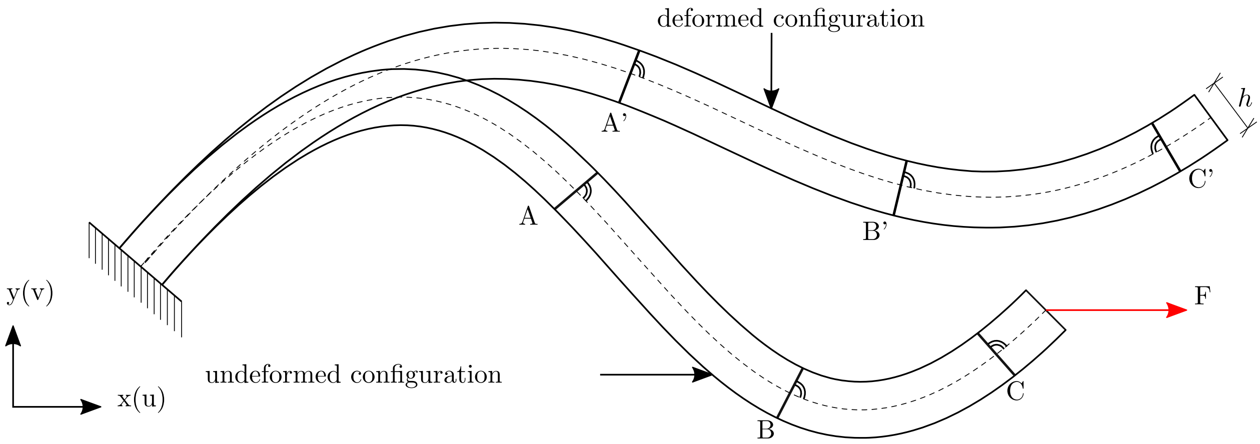


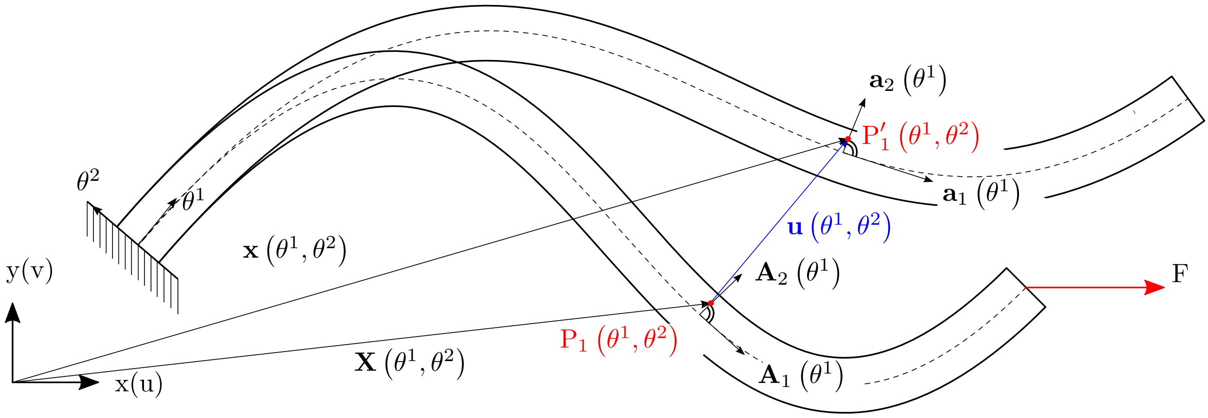

Figure 1: Normal to the midline cross sections remain normal throughout deformation

Assuming that plane cross sections remain plane, we neglect the warping of the cross section, which in turn leads to a linear distribution of the stresses over the beam height. Additionally, by assuming that initially orthogonal to the mid-line cross sections remain orthogonal, we neglect the shear deformation, which in turn increases the variational index of the problem to 2. As a consequence, the least requirement of the basis functions is to be $C^2$-continuous inside the elements and $C^1$-continuous across the element borders.

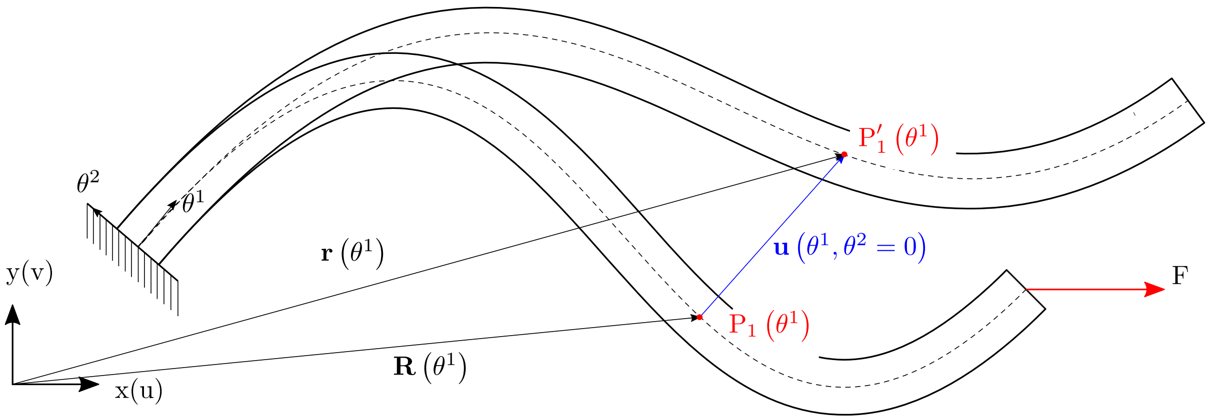
The beam under study can resist axial forces through normal reaction forces, and lateral forces through bending moments. In this setting, two degrees of freedom are to be considered; Horizontal and vertical displacements $u$ and $v$, respectively.

In what follows, we will employ the *Total Lagrangian* (TL) formulation of the problem, i.e. stress and strain tensors will be regarded with respect to the reference configuration. The 2-D continuum will be described by the position vector $\mathbf{X}$ (resp. $\mathbf{x}$ for the deformed configuration) whereas the mid-line position vector will be denoted as $\mathbf{R}$ (resp. $\mathbf{r}$). The convective contravariant coordinates $\left\{\theta^i\right\}_{i=1,2}$ are considered to be orthogonal to each other one in the direction of the mid-line, and the other in the direction of the height of the beam at each point (see Fig. (2)).

Evidently, the following relations hold:




(a) The continuum's position vectors




(b) The mid-line reduction position vectors

Figure 2: Covariant description of the Bernoulli problem

$$\mathbf{R}\left(\theta^1\right)=\mathbf{X}\left(\theta^1,\theta^2=0\right) \tag{1a}$$
$$\mathbf{r}\left(\theta^1\right)=\mathbf{x}\left(\theta^1,\theta^2=0\right) \tag{1b}$$

The position vectors are defined as:

$$\mathbf{X}\left(\theta^1,\theta^2\right)=\mathbf{R}\left(\theta^1\right)+\theta^2\mathbf{A}_2\,,\quad \theta^1\in[0,1]\,,\ \theta^2\in\left[-\frac{1}{2}h,\frac{1}{2}h\right]\ , \tag{2a}$$
$$\mathbf{x}\left(\theta^1,\theta^2\right)=\mathbf{r}\left(\theta^1\right)+\theta^2\mathbf{a}_2\,,\quad \theta^1\in[0,1]\,,\ \theta^2\in\left[-\frac{1}{2}h,\frac{1}{2}h\right]\ , \tag{2b}$$

where the $\{\mathbf{A}_\alpha\}_{\alpha=1,2}$ and $\{\mathbf{a}_\alpha\}_{\alpha=1,2}$ are the middle line base vectors. Additionally the base vectors $\mathbf{A}_2$ and $\mathbf{a}_2$ are of unit length and orthogonal to the base vectors $\mathbf{A}_1$ and $\mathbf{a}_1$, respectively. The base vectors of the continuum are defined as follows:

$$\mathbf{G}_\alpha=\mathbf{X}_{,\alpha}\ , \tag{3a}$$
$$\mathbf{g}_\alpha=\mathbf{x}_{,\alpha}\ . \tag{3b}$$

Next the *Green-Lagrange* strain tensor is defined as:

$$E_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right) \ . \tag{4}$$

Note that in our case, $E_{12} = E_{21} = E_{22} = 0$, i.e. the strain tensor reduces to a scalar quantity. Thus, one needs to compute the metric coefficients $g_{11}$ and $G_{11}$:

$$\begin{aligned} G_{11}\left(\theta^1,\theta^2\right) = \mathbf{G}_1 \cdot \mathbf{G}_1 = \mathbf{X}_{,1} \cdot \mathbf{X}_{,1} = \left(\mathbf{A}_1 + \theta^2 \mathbf{A}_{2,1}\right) \cdot \left(\mathbf{A}_1 + \theta^2 \mathbf{A}_{2,1}\right) = \\ = A_{11} + 2\theta^2 \mathbf{A}_{2,1} \cdot \mathbf{A}_1 + \left(\theta^2\right)^2 \mathbf{A}_{2,1} \cdot \mathbf{A}_{2,1} \ , \end{aligned} \tag{5a}$$

$$\begin{aligned} g_{11}\left(\theta^1,\theta^2\right) = \mathbf{g}_1 \cdot \mathbf{g}_1 = \mathbf{x}_{,1} \cdot \mathbf{x}_{,1} = \left(\mathbf{a}_1 + \theta^2 \mathbf{a}_{2,1}\right) \cdot \left(\mathbf{a}_1 + \theta^2 \mathbf{a}_{2,1}\right) = \\ = a_{11} + 2\theta^2 \mathbf{a}_{2,1} \cdot \mathbf{a}_1 + \left(\theta^2\right)^2 \mathbf{a}_{2,1} \cdot \mathbf{a}_{2,1} \ . \end{aligned} \tag{5b}$$

Under the assumption of a slender beam (i.e. $h << L$), and considering that $\left|\theta^2\right| \leq \frac{h}{2}$, one can safely neglect the term $\left(\theta^2\right)^2 \mathbf{A}_{2,1} \cdot \mathbf{A}_{2,1}$ in (5a) $\big($resp. for equation (5b)$\big)$, which results in:

$$G_{11}\left(\theta^1,\theta^2\right) = A_{11} + 2\theta^2 \mathbf{A}_{2,1} \cdot \mathbf{A}_1 \ , \tag{6a}$$

$$g_{11}\left(\theta^1,\theta^2\right) = a_{11} + 2\theta^2 \mathbf{a}_{2,1} \cdot \mathbf{a}_1 \ . \tag{6b}$$

The curvature of $\mathbf{R}\left(\theta^1\right)$ $\big($resp. for $\mathbf{r}\left(\theta^1\right)\big)$ is given by:

$$B\left(\theta^1\right) = -\mathbf{A}_{2,1} \cdot \mathbf{A}_1 \ , \tag{7a}$$

$$b\left(\theta^1\right) = -\mathbf{a}_{2,1} \cdot \mathbf{a}_1 \ . \tag{7b}$$

Using the definitions (7a)-(7b) equation (6b) can be rewritten as:

$$G_{11}\left(\theta^1,\theta^2\right) = A_{11} - 2\theta^2 B \ , \tag{8a}$$

$$g_{11}\left(\theta^1,\theta^2\right) = a_{11} - 2\theta^2 b \ . \tag{8b}$$

The strain tensor summarizes into:

$$E_{11}\left(\theta^1,\theta^2\right) = \overbrace{\frac{1}{2}\left(a_{11} - A_{11}\right)}^{\epsilon} + \theta^2 \overbrace{\left(B - b\right)}^{\kappa} \ . \tag{9}$$

In the equation (9), $\epsilon$ denotes the axial strain, whereas $\kappa$ denotes the change in the curvature.
A close look at the equation (9) reveals that the linearized strain consists of two parts: one constant which is attributed to the axial stiffness and one linear which is attributed to the bending stiffness. For an illustration of the idea see figure (3).

As a next step, a pre-integration over the beam's height is performed, in order to find expressions for the stress resultants. That is, for the normal force:

$$n\left(\theta^1\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} S^{11}\, b_{\mathrm{cs}}\, d\theta^2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} C^{1111} E_{11}\, b_{\mathrm{cs}}\, d\theta^2 = E\, h\, b_{\mathrm{cs}} \epsilon = \frac{1}{\|\mathbf{A}_1\|_2^2} E\, h\, b_{\mathrm{cs}} \epsilon \ \ [\mathrm{kN}] \ , \tag{10}$$

and the bending moment:

$$m\left(\theta^1\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} S^{11} \theta^2\, b_{\mathrm{cs}}\, d\theta^2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} C^{1111} E_{11} \theta^2\, b_{\mathrm{cs}}\, d\theta^2 = \frac{1}{\|\mathbf{A}_1\|_2^2} E\, I_z\, \kappa \ \left[\mathrm{kN\, m^2}\right] \ , \tag{11}$$

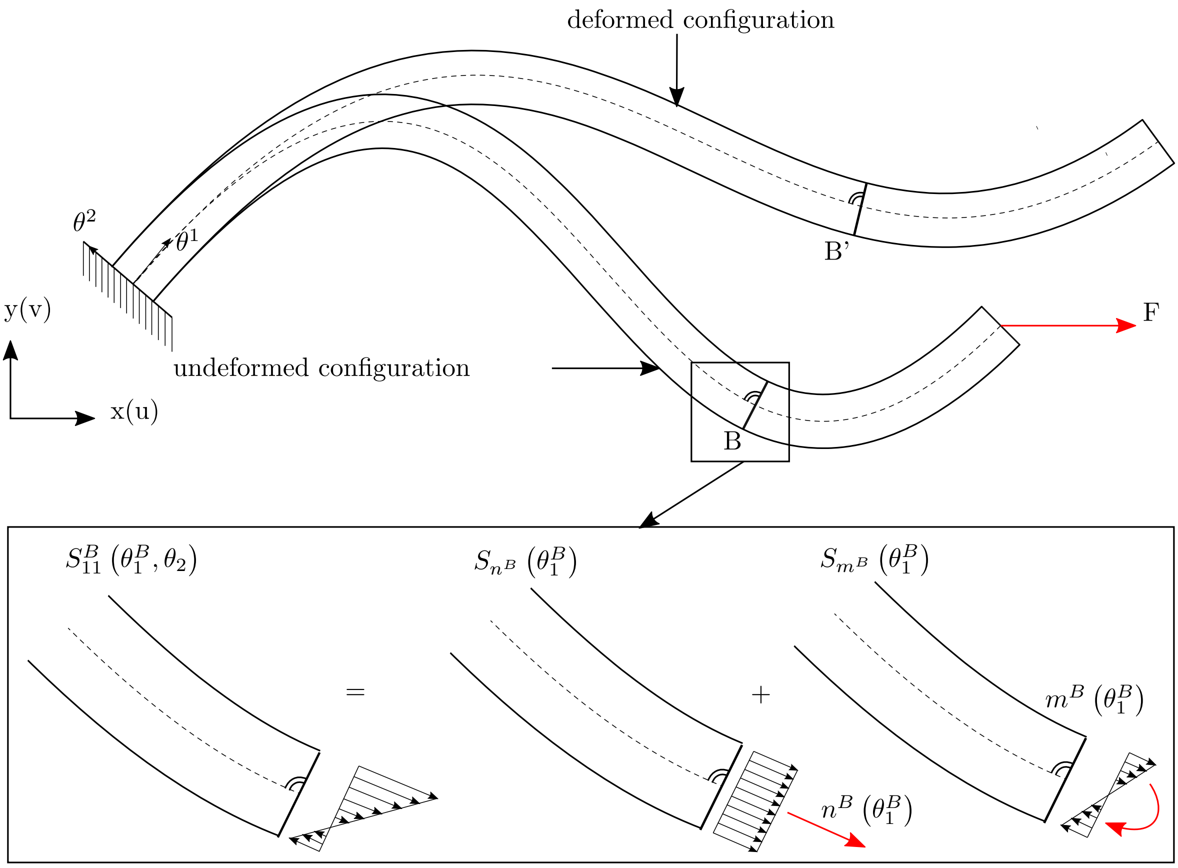



Figure 3: Normal to the mid-line cross sections remain normal throughout deformation

where $I_z$ is the moment of inertia which is for a rectangular cross section equal to $\frac{h^3\, b_{\mathrm{cs}}}{12}$ and $E$ is the Young's modulus. Note that in equations (10)-(11) transformation to the physical space has been achieved by dividing with the length of the middle line base vector $\mathbf{A}_1$ Then, the internal virtual work can be written as:

$$\delta W_{int} = \int_{\Omega\subset\mathbb{R}^3} \mathbf{S} : \delta\mathbf{E}\, d\Omega = \int_L \left[\begin{array}{c} n \\ m \end{array}\right] \cdot \delta \left[\begin{array}{c} \epsilon \\ \kappa \end{array}\right] dL = \int_L \quad n\,\delta\epsilon + m\,\delta\kappa\, dL\ , \tag{12}$$

were $b_{\mathrm{cs}}$ is the width of the beam. The transformation of the integral contained in equation (12) from volume to line integral is performed via the differential transformation $d\Omega = dA_{\mathrm{cs}}\, dL$, which in turn for a rectangular cross section simplifies to $d\Omega = b_{\mathrm{cs}}\, d\theta^2\, dL$.

# 2 NURBS-based Bernoulli Beam Element

In this section, an expression for the stiffness matrix of the NURBS-based isogeometric Bernoulli beam element will be derived. For this purpose consider that the element under study is affected by a set of control points $\left\{\hat{\mathbf{X}}_i\right\}_{i=1}^n$, $n \in \mathbb{N}$. The given curvilinear geometry is expressed in terms of NURBS basis functions as:

$$\mathbf{R}\left(\theta^1\right) = \left[ \begin{array}{c} \sum_{i=1}^{n} R^{i,p}\left(\theta^1\right)\hat{X}^i \\ \sum_{i=1}^{n} R^{i,p}\left(\theta^1\right)\hat{Y}^i \end{array} \right] \Rightarrow$$

$$\mathbf{R}\left(\theta^1\right) = \left[ \begin{array}{ccccc} R^{1,p}\left(\theta^1\right) & 0 & \cdots & R^{n,p}\left(\theta^1\right) & 0 \\ 0 & R^{1,p}\left(\theta^1\right) & \cdots & 0 & R^{n,p}\left(\theta^1\right) \end{array} \right] \left[ \begin{array}{c} \hat{X}^1 \\ \hat{Y}^1 \\ \vdots \\ \hat{X}^n \\ \hat{Y}^n \end{array} \right]. \tag{13}$$

Moreover the vector of the control variables $\hat{\mathbf{X}}$ is defined as:

$$\hat{\mathbf{X}} := \left[ \begin{array}{ccccc} \hat{X}^1 & \hat{Y}^1 & \cdots & \hat{X}^n & \hat{Y}^n \end{array} \right]^{\mathrm{T}}. \tag{14}$$

In *isogeometric* analysis setting, we make use of the isoparametric element concept, i.e. the NURBS ansatz of the displacement field is as follows:

$$\mathbf{u}\left(\theta^1, \theta^2 = 0\right) = \mathbf{u}\left(\theta^1\right) = \left[ \begin{array}{c} \sum_{i=1}^{n} R^{i,p}\hat{u}_i \\ \sum_{i=1}^{n} R^{i,p}\hat{v}_i \end{array} \right] \Rightarrow$$

$$\mathbf{u}\left(\theta^1\right) = \left[ \begin{array}{ccccc} R^{1,p}\left(\theta^1\right) & 0 & \cdots & R^{n,p}\left(\theta^1\right) & 0 \\ 0 & R^{1,p}\left(\theta^1\right) & \cdots & 0 & R^{n,p}\left(\theta^1\right) \end{array} \right] \left[ \begin{array}{c} \hat{u}_1 \\ \hat{v}_1 \\ \vdots \\ \hat{u}_n \\ \hat{v}_n \end{array} \right]. \tag{15}$$

Therefore, using (13) and (15), one obtains:

$$\mathbf{r}\left(\theta^1\right) = \mathbf{R}\left(\theta^1\right) + \mathbf{u}\left(\theta^1\right) = \left[ \begin{array}{c} \sum_{i=1}^{n} R^{i,p}\left(\theta^1\right)\left(\hat{X}^i + \hat{u}_i\right) \\ \sum_{i=1}^{n} R^{i,p}\left(\theta^1\right)\left(\hat{Y}^i + \hat{v}_i\right) \end{array} \right] \Rightarrow$$

$$\mathbf{r}\left(\theta^1\right) = \left[ \begin{array}{ccccc} R^{1,p}\left(\theta^1\right) & 0 & \cdots & R^{n,p}\left(\theta^1\right) & 0 \\ 0 & R^{1,p}\left(\theta^1\right) & \cdots & 0 & R^{n,p}\left(\theta^1\right) \end{array} \right] \left[ \begin{array}{c} \hat{X}^1 + \hat{u}_1 \\ \hat{Y}^1 + \hat{v}_1 \\ \vdots \\ \hat{X}^n + \hat{u}_n \\ \hat{Y}^n + \hat{v}_n \end{array} \right]. \tag{16}$$

The base vectors $\mathbf{A}_2$ and $\mathbf{a}_2$ can be determined by the following relations:

$$\mathbf{A}_2\left(\theta^1\right) = \frac{1}{\|\mathbf{A}_1\|_2} \left[ \begin{array}{c} -A_{1(y)} \\ A_{1(x)} \end{array} \right], \tag{17a}$$

$$\mathbf{a}_2\left(\theta^1\right) = \frac{1}{\|\mathbf{a}_1\|_2} \left[ \begin{array}{c} -a_{1(y)} \\ a_{1(x)} \end{array} \right], \tag{17b}$$

given that:

$$\mathbf{A}_1\left(\theta^1\right) = \left[ \begin{array}{c} A_{1(x)} \\ A_{1(y)} \end{array} \right] = \left[ \begin{array}{c} \sum_{i=1}^{n} \frac{\partial R^{i,p}}{\partial \theta^1}\hat{X}^i \\ \sum_{i=1}^{n} \frac{\partial R^{i,p}}{\partial \theta^1}\hat{Y}^i \end{array} \right], \tag{18a}$$

$$\mathbf{a}_1\left(\theta^1\right) = \left[ \begin{array}{c} a_{1(x)} \\ a_{1(y)} \end{array} \right] = \left[ \begin{array}{c} \sum_{i=1}^{n} \frac{\partial R^{i,p}}{\partial \theta^1}\left(\hat{X}^i + \hat{u}^i\right) \\ \sum_{i=1}^{n} \frac{\partial R^{i,p}}{\partial \theta^1}\left(\hat{Y}^i + \hat{v}^i\right) \end{array} \right]. \tag{18b}$$

Now the variation of the axial strain $\epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right)$ and the change in curvature $\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right)$ with respect to $\hat{\mathbf{u}}$ should be obtained. This step is equivalent to obtaining the B-operator matrix in the classical FEA. However in the frame of IGA where we use NURBS as an approximating space, this is done in a much more elegant way. At first, the variation $\epsilon_{,\hat{\mathbf{u}}}$ is derived:

$$\epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \frac{1}{2} a_{11,\hat{\mathbf{u}}} = \mathbf{a}_1 \cdot \mathbf{a}_{1,\hat{\mathbf{u}}} \ . \tag{19}$$

**Remark 2.1.** In the remainder of the study, linearization of the tensorial quantities will be employed whenever small deformations are to be assumed. Products between tensors often result into products between the unknown degrees of freedom, which in the case of Isogeometric Analysis are the control variables.

These products can be physically understood as the influence of the geometry change on the deformation, and hence non-linear expressions with respect to the degrees of freedom are to be obtained.

In the linear setting, i.e. within the small deformations assumption, we neglect the dependency of the geometry on the degrees of freedom. Therefore if $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{d})$ is a tensor of arbitrary dimension and $\mathbf{d}$ the underlying primary field, then linearization of $\boldsymbol{\sigma}$ is to be understood as follows:

$$\boldsymbol{\sigma}^{\text{lin}}(\mathbf{d}) = \boldsymbol{\sigma}(\mathbf{d} = \mathbf{0}) \ . \tag{20}$$

Moving towards a linear formulation, i.e. following remark 2.1, $\mathbf{a}_1(\hat{\mathbf{u}}) \overset{\text{lin}}{=} \mathbf{a}_1(\hat{\mathbf{u}} = \mathbf{0}) = \mathbf{A}_1$, equation (19) writes:

$$\epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) \overset{\text{lin}}{=} \mathbf{A}_1 \cdot \mathbf{a}_{1,\hat{\mathbf{u}}} \ . \tag{21}$$

The mid-axis basis vector $\mathbf{a}_1$ is given by:

$$\mathbf{a}_1\left(\theta^1\right) = \mathbf{r}_{,1}\left(\theta^1\right) = \begin{bmatrix} \frac{\partial R^{1,p}}{\partial \theta^1} & 0 & \cdots & \frac{\partial R^{n,p}}{\partial \theta^1} & 0 \\ 0 & \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & 0 & \frac{\partial R^{n,p}}{\partial \theta^1} \end{bmatrix} \left(\hat{\mathbf{X}} + \hat{\mathbf{u}}\right) \ , \tag{22}$$

i.e. as a vector depending on the location and movement of the control points. Therefore variation of $\mathbf{a}_1$ with respect to $\hat{\mathbf{u}}$ yields:

$$\mathbf{a}_{1,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} \frac{\partial R^{1,p}}{\partial \theta^1} & 0 & \cdots & \frac{\partial R^{n,p}}{\partial \theta^1} & 0 \\ 0 & \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & 0 & \frac{\partial R^{n,p}}{\partial \theta^1} \end{bmatrix} \ . \tag{23}$$

Substituting equation (23) into equation (21) delivers:

$$\epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} A_{1(x)} \frac{\partial R^{1,p}}{\partial \theta^1} & A_{1(y)} \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & A_{1(x)} \frac{\partial R^{n,p}}{\partial \theta^1} & A_{1(y)} \frac{\partial R^{n,p}}{\partial \theta^1} \end{bmatrix} \ . \tag{24}$$

**Definition 2.1.** Let $D$ to be a bounded domain. A function $\varphi : D \to \mathbb{R}$ is said to have a *compact support* in $D$ if there exists a compact subset $K$ of $D$ such that $\varphi(\mathbf{x}) = 0 \ \forall \mathbf{x} \in D \setminus K$.

The set of all functions $\varphi \in C^m(D)\,,\ m \in \mathbb{N} \cup \{\infty\}$ with compact support in $D$ will be denoted as $C_0^m(D)$.

Note that in equation (24), the values of $A_{1(x)}$ and $A_{1(y)}$ can be fully determined by the relation (18a). For clarity, we introduce the vector operator:

$$\mathcal{L}_\epsilon : C_0^1[0,1] \to \left(C[0,1]\right)^2 \quad \mathcal{L}_\epsilon f := \begin{bmatrix} A_{1(x)} \frac{\partial f}{\partial \theta^1} & A_{1(y)} \frac{\partial f}{\partial \theta^1} \end{bmatrix} \quad \forall f \in C_0^1[0,1] \ . \tag{25}$$

Then relation (24) can be rewritten as:

$$\epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} \left(\mathcal{L}_\epsilon R^{1,p}\right)\left(\theta^1\right) & \cdots & \left(\mathcal{L}_\epsilon R^{n,p}\right)\left(\theta^1\right) \end{bmatrix} \ . \tag{26}$$

We proceed to determine the variation of the change in the curvature with respect to the degrees of freedom $\kappa_{,\hat{\mathbf{u}}}$ (for the definition of $\kappa$ see equations (9), (7a)-(7b)):

$$\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right) = -\left(\mathbf{a}_{1,1} \cdot \mathbf{a}_2\right)_{,\hat{\mathbf{u}}} = -\left(\mathbf{a}_2\, \mathbf{a}_{1,1,\hat{\mathbf{u}}} + \mathbf{a}_{1,1}\, \mathbf{a}_{2,\hat{\mathbf{u}}}\right) \ . \tag{27}$$

Once more, we need to employ the following linearizations (see remark 2.1):

$$\mathbf{a}_2\left(\theta^1\right) \overset{\text{lin}}{=} \mathbf{A}_2\left(\theta^1\right) \;, \tag{28a}$$
$$\mathbf{a}_{1,1}\left(\theta^1\right) \overset{\text{lin}}{=} \mathbf{A}_{1,1}\left(\theta^1\right) \;. \tag{28b}$$

In this way, the linear form of (27) writes:

$$\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right) \overset{\text{lin}}{=} -\left(\mathbf{A}_2\,\mathbf{a}_{1,1,\hat{\mathbf{u}}} + \mathbf{A}_{1,1}\,\mathbf{a}_{2,\hat{\mathbf{u}}}\right) \;. \tag{29}$$

We define the not unit normal vector of the deformed configuration as:

$$\widetilde{\mathbf{a}_2}\left(\theta^1\right) := \|\mathbf{a}_1\|_2\,\mathbf{a}_2 \;, \tag{30}$$

and in this way, the equation (17b) can be rewritten as:

$$\mathbf{a}_2\left(\theta^1\right) = \frac{1}{\|\mathbf{a}_1\|_2}\widetilde{\mathbf{a}_2} \;. \tag{31}$$

Variation of $\mathbf{a}_2$ (see eq. (31)) with respect to the degrees of freedom results in:

$$\mathbf{a}_{2,\hat{\mathbf{u}}}\left(\theta^1\right) = \frac{1}{\|\mathbf{a}_1\|_2^2}\left(\|\mathbf{a}_1\|_2\widetilde{\mathbf{a}_2}_{,\hat{\mathbf{u}}} - \widetilde{\mathbf{a}_2}\|\mathbf{a}_1\|_{2,\hat{\mathbf{u}}}\right) \overset{\text{lin}}{=} \frac{1}{\|\mathbf{A}_1\|_2}\left(\widetilde{\mathbf{a}_2}_{,\hat{\mathbf{u}}} - \mathbf{A}_2\|\mathbf{a}_1\|_{2,\hat{\mathbf{u}}}\right) \;. \tag{32}$$

The not unit normal vector is then given by:

$$\widetilde{\mathbf{a}_2}\left(\theta^1\right) = \begin{bmatrix} -\sum_{i=1}^{n}\frac{\partial R^{i,p}}{\partial\theta^1}\left(\hat{Y}^i + \hat{v}_i\right) \\ \sum_{i=1}^{n}\frac{\partial R^{i,p}}{\partial\theta^1}\left(\hat{X}^i + \hat{u}_i\right) \end{bmatrix} \;, \tag{33}$$

and its variation with respect to the degrees of freedom:

$$\widetilde{\mathbf{a}_2}_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} 0 & -\frac{\partial R^{1,p}}{\partial\theta^1} & \cdots & 0 & -\frac{\partial R^{n,p}}{\partial\theta^1} \\ \frac{\partial R^{1,p}}{\partial\theta^1} & 0 & \cdots & \frac{\partial R^{n,p}}{\partial\theta^1} & 0 \end{bmatrix} \;. \tag{34}$$

Variation of the base vector length $\|\mathbf{a}_1\|_2$ with respect to $\hat{\mathbf{u}}$, provides us with:

$$\|\mathbf{a}_1\|_{2,\hat{\mathbf{u}}}\left(\theta^1\right) = \left(\sqrt{\mathbf{a}_1\cdot\mathbf{a}_1}\right)_{,\hat{\mathbf{u}}} = \frac{1}{\|\mathbf{a}_1\|_2}\mathbf{a}_1^{\mathrm{T}}\mathbf{a}_{1,\hat{\mathbf{u}}} \overset{\text{lin}}{=} \frac{1}{\|\mathbf{A}_1\|_2}\mathbf{A}_1^{\mathrm{T}}\mathbf{a}_{1,\hat{\mathbf{u}}} \;. \tag{35}$$

Equation (35) in a matrix form, using also (23), writes:

$$\|\mathbf{a}_1\|_{2,\hat{\mathbf{u}}}\left(\theta^1\right) = \frac{1}{\|\mathbf{A}_1\|_2}\begin{bmatrix} A_{1(x)}\frac{\partial R^{1,p}}{\partial\theta^1} & A_{1(y)}\frac{\partial R^{1,p}}{\partial\theta^1} & \cdots & A_{1(x)}\frac{\partial R^{n,p}}{\partial\theta^1} & A_{1(y)}\frac{\partial R^{n,p}}{\partial\theta^1} \end{bmatrix} \;. \tag{36}$$

The matrix expression of the second term in equation (32) writes:

$$\mathbf{A}_2\,\|\mathbf{a}_1\|_{2,\hat{\mathbf{u}}} = \tfrac{1}{\|\mathbf{A}_1\|_2^2}\begin{bmatrix} -A_{1(x)}A_{1(y)}\frac{\partial R^{1,p}}{\partial\theta^1} & -A_{1(y)}^2\frac{\partial R^{1,p}}{\partial\theta^1} & \cdots & -A_{1(x)}A_{1(y)}\frac{\partial R^{n,p}}{\partial\theta^1} & -A_{1(y)}^2\frac{\partial R^{n,p}}{\partial\theta^1} \\ A_{1(x)}^2\frac{\partial R^{1,p}}{\partial\theta^1} & A_{1(x)}A_{1(y)}\frac{\partial R^{1,p}}{\partial\theta^1} & \cdots & A_{1(x)}^2\frac{\partial R^{n,p}}{\partial\theta^1} & A_{1(x)}A_{1(y)}\frac{\partial R^{n,p}}{\partial\theta^1} \end{bmatrix} \;. \tag{37}$$

To simplify expression (32), we set:

$$d_1\left(\theta^1\right) := \frac{1}{\|\mathbf{A}_1\|_2^2} A_{1(x)} A_{1(y)} \ , \tag{38a}$$

$$d_2\left(\theta^1\right) := \frac{1}{\|\mathbf{A}_1\|_2^2} A_{1(y)}^2 - 1 \ , \tag{38b}$$

$$d_3\left(\theta^1\right) := 1 - \frac{1}{\|\mathbf{A}_1\|_2^2} A_{1(x)}^2 \ . \tag{38c}$$

In this way, and by carrying out the operations needed in equation (32) we obtain:

$$\mathbf{a}_{2,\hat{\mathbf{u}}}\left(\theta^1\right) = \frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} d_1 \frac{\partial R^{1,p}}{\partial \theta^1} & d_2 \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & d_1 \frac{\partial R^{n,p}}{\partial \theta^1} & d_2 \frac{\partial R^{n,p}}{\partial \theta^1} \\ d_3 \frac{\partial R^{1,p}}{\partial \theta^1} & -d_1 \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & d_3 \frac{\partial R^{n,p}}{\partial \theta^1} & -d_1 \frac{\partial R^{n,p}}{\partial \theta^1} \end{bmatrix} . \tag{39}$$

We will compute each term of (29) separately and we start with the second term:

$$\mathbf{A}_{1,1}\, \mathbf{a}_{2,\hat{\mathbf{u}}} = \frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} \alpha_1 \frac{\partial R^{1,p}}{\partial \theta^1} & \alpha_2 \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & \alpha_1 \frac{\partial R^{n,p}}{\partial \theta^1} & \alpha_2 \frac{\partial R^{n,p}}{\partial \theta^1} \end{bmatrix} , \tag{40}$$

were in the latter equation we have set the scalar functions $\{\alpha_i\}_{i=1,2}$ as:

$$\alpha_1\left(\theta^1\right) := \frac{1}{\|\mathbf{A}_1\|_2} \left( \frac{\partial A_{1(x)}}{\partial \theta^1} d_1 + \frac{\partial A_{1(y)}}{\partial \theta^1} d_3 \right) , \tag{41a}$$

$$\alpha_2\left(\theta^1\right) := \frac{1}{\|\mathbf{A}_1\|_2} \left( \frac{\partial A_{1(x)}}{\partial \theta^1} d_2 - \frac{\partial A_{1(y)}}{\partial \theta^1} d_1 \right) . \tag{41b}$$

Then it is clear that:

$$\mathbf{a}_{1,1,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} \frac{\partial^2 R^{1,p}}{\partial (\theta^1)^2} & 0 & \cdots & \frac{\partial^2 R^{n,p}}{\partial (\theta^1)^2} & 0 \\ 0 & \frac{\partial^2 R^{1,p}}{\partial (\theta^1)^2} & \cdots & 0 & \frac{\partial^2 R^{n,p}}{\partial (\theta^1)^2} \end{bmatrix} . \tag{42}$$

Note that at this point (42), Bernoulli's second hypothesis makes impossible the use of low order (i.e. $C^1$-continuous) basis functions. The first term of equation (29) in a matrix form writes:

$$\begin{aligned} \mathbf{A}_2 \cdot \mathbf{a}_{1,1,\hat{\mathbf{u}}} = \tfrac{1}{\|\mathbf{A}_1\|_2} \Big[ & -A_{1(y)} \frac{\partial^2 R^{1,p}}{\partial (\theta^1)^2} \quad A_{1(x)} \frac{\partial^2 R^{1,p}}{\partial (\theta^1)^2} \quad \cdots \\ & -A_{1(y)} \frac{\partial^2 R^{n,p}}{\partial (\theta^1)^2} \quad A_{1(x)} \frac{\partial^2 R^{n,p}}{\partial (\theta^1)^2} \Big] . \end{aligned} \tag{43}$$

By defining the vector operator:

$$\mathcal{L}_\kappa : C_0^2\,[0,1] \to \left(C\,[0,1]\right)^2 \quad \mathcal{L}_\kappa f := \begin{bmatrix} -\alpha_1 \frac{\partial f}{\partial \theta^1} + \frac{A_{1(y)}}{\|\mathbf{A}_1\|_2} \frac{\partial^2 f}{\partial (\theta^1)^2} \\ -\alpha_2 \frac{\partial f}{\partial \theta^1} - \frac{A_{1(x)}}{\|\mathbf{A}_1\|_2} \frac{\partial^2 f}{\partial (\theta^1)^2} \end{bmatrix}^{\mathrm{T}} \quad \forall f \in C_0^2\,[0,1] \ , \tag{44}$$

(see also definition 2.1 for the notation $C_0^2\,[0,1]$) one obtains a closed form linear expression for $\kappa_{,\hat{\mathbf{u}}}$, namely:

$$\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} \left(\mathcal{L}_\kappa R^{1,p}\right)\left(\theta^1\right) & \cdots & \left(\mathcal{L}_\kappa R^{n,p}\right)\left(\theta^1\right) \end{bmatrix} . \tag{45}$$

Now an expression for the stiffness matrix of the Bernoulli beam element using the internal virtual work can be derived. Recall equation (12), and consider its linear dependency on the local control variables vector:

$$\begin{aligned} \delta W_{\text{int}} &= \int_L \left(\epsilon_{,\hat{\mathbf{u}}} \delta \hat{\mathbf{u}}\right)^{\mathrm{T}} C^{\text{axial}}\, \epsilon_{,\hat{\mathbf{u}}} \hat{\mathbf{u}} + \left(\kappa_{,\hat{\mathbf{u}}} \delta \hat{\mathbf{u}}\right)^{\mathrm{T}} C^{\text{bending}}\, \kappa_{,\hat{\mathbf{u}}} \hat{\mathbf{u}} \quad dL = \\ &= \left(\delta \hat{\mathbf{u}}\right)^{\mathrm{T}} \int_L C^{\text{axial}}\, \epsilon_{,\hat{\mathbf{u}}} \otimes \epsilon_{,\hat{\mathbf{u}}} + C^{\text{bending}}\, \kappa_{,\hat{\mathbf{u}}} \otimes \kappa_{,\hat{\mathbf{u}}} \quad dL\; \hat{\mathbf{u}} \ . \end{aligned} \tag{46}$$

In equation (46) the symbol $\otimes$ stands for the dyadic product between two vectors in $\mathbb{R}^n$, and the material constants are defined with respect to equations (10) and (11), i.e.:

$$C^{\text{axial}} = E\,h\,b\;, \tag{47a}$$

$$C^{\text{bending}} = \frac{E\,h^3\,b}{12}\;. \tag{47b}$$

Consequently, the stiffness matrix can be divided into two parts; one attributed to axial elongation and one attributed to bending:

$$\mathbf{K}^e = \mathbf{K}^{\text{axial}} + \mathbf{K}^{\text{bending}}\;, \tag{48}$$

were each stiffness matrix is defined as:

$$\mathbf{K}^{\text{axial}} = \int_L \quad C^{\text{axial}}\,\epsilon_{,\hat{\mathbf{u}}} \otimes \epsilon_{,\hat{\mathbf{u}}} \quad dL\;, \tag{49a}$$

$$\mathbf{K}^{\text{bending}} = \int_L \quad C^{\text{bending}}\,\kappa_{,\hat{\mathbf{u}}} \otimes \kappa_{,\hat{\mathbf{u}}} \quad dL\;. \tag{49b}$$

Note that the operator $\mathcal{L}_\epsilon$ defined in equation (25) is a first order operators, whereas operator $\mathcal{L}_\kappa$ defined in (44) is a second order one, raising the variational index of the problem to 2.