Linear isogeometric Timoshenko beam formulation

Dr.-Ing. Andreas Apostolatos

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1 Timoshenko Beam Theory

A relaxation to the Bernoulli hypotheses, is applied by the Timoshenko beam theory. In this setting, plane cross sections remain plane throughout the deformation, but the cross sections do not remain necessarily orthogonal to the mid-line. Consequently, a linear distribution of the stresses along the beam height is expected, whereas the development of shear forces is counterbalancing the shear deformation that takes place. For an illustration of the concept see Fig. (1).

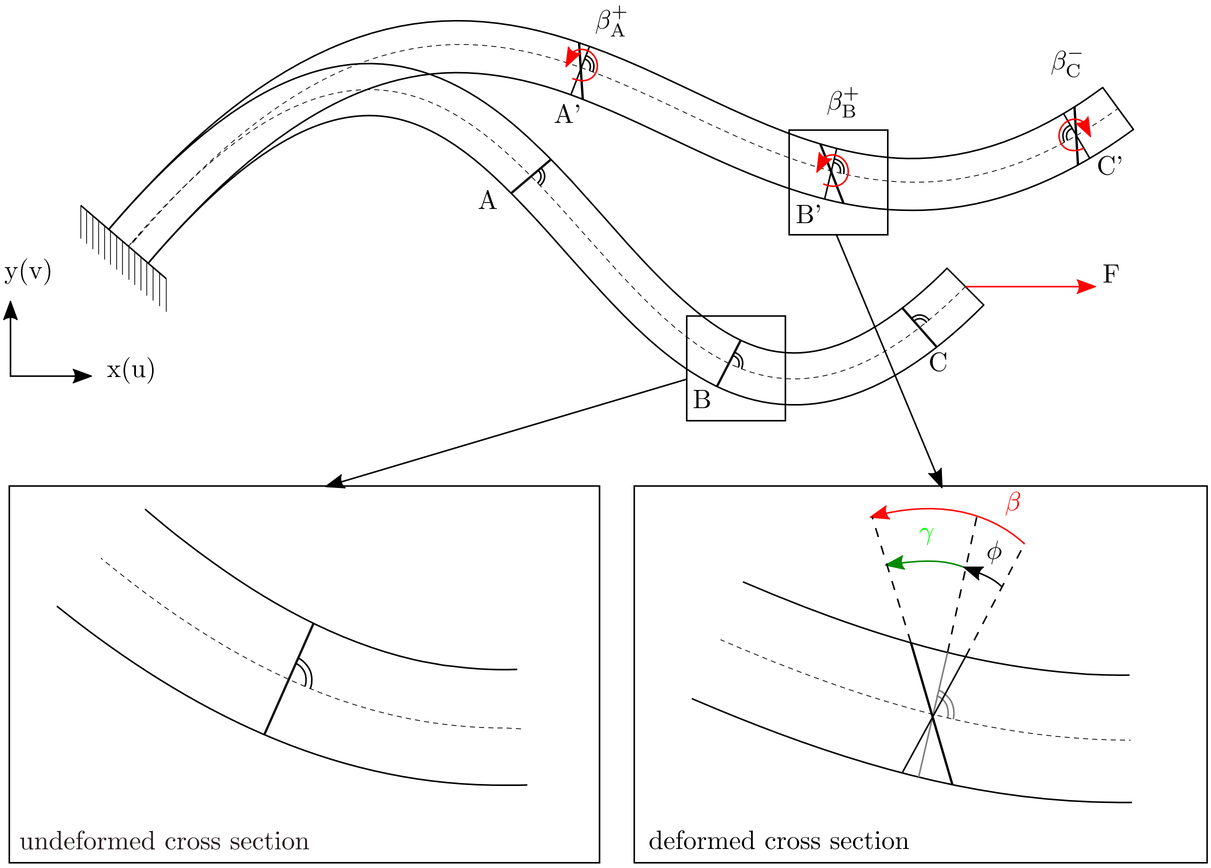

Figure 1: Cross sectional deformation in the Timoshenko setting

The definitions given in the isogeometric Euler-Bernoulli beam formulation, hold here as well, with the only difference that in the deformed configuration, $\mathbf{a}_2$ is not any more orthogonal to $\mathbf{a}_1$. Let the total cross sectional rotation to be denoted by β then:

$$\beta = \gamma + \phi \,, \tag{1}$$

where ϕ is the rigid body rotation of the cross section given by (see also Fig. (2)):

$$\phi = \frac{1}{|\mathbf{a}_{1,1}|}\mathbf{a}_{1,1} \cdot \mathbf{n} \,, \tag{2}$$

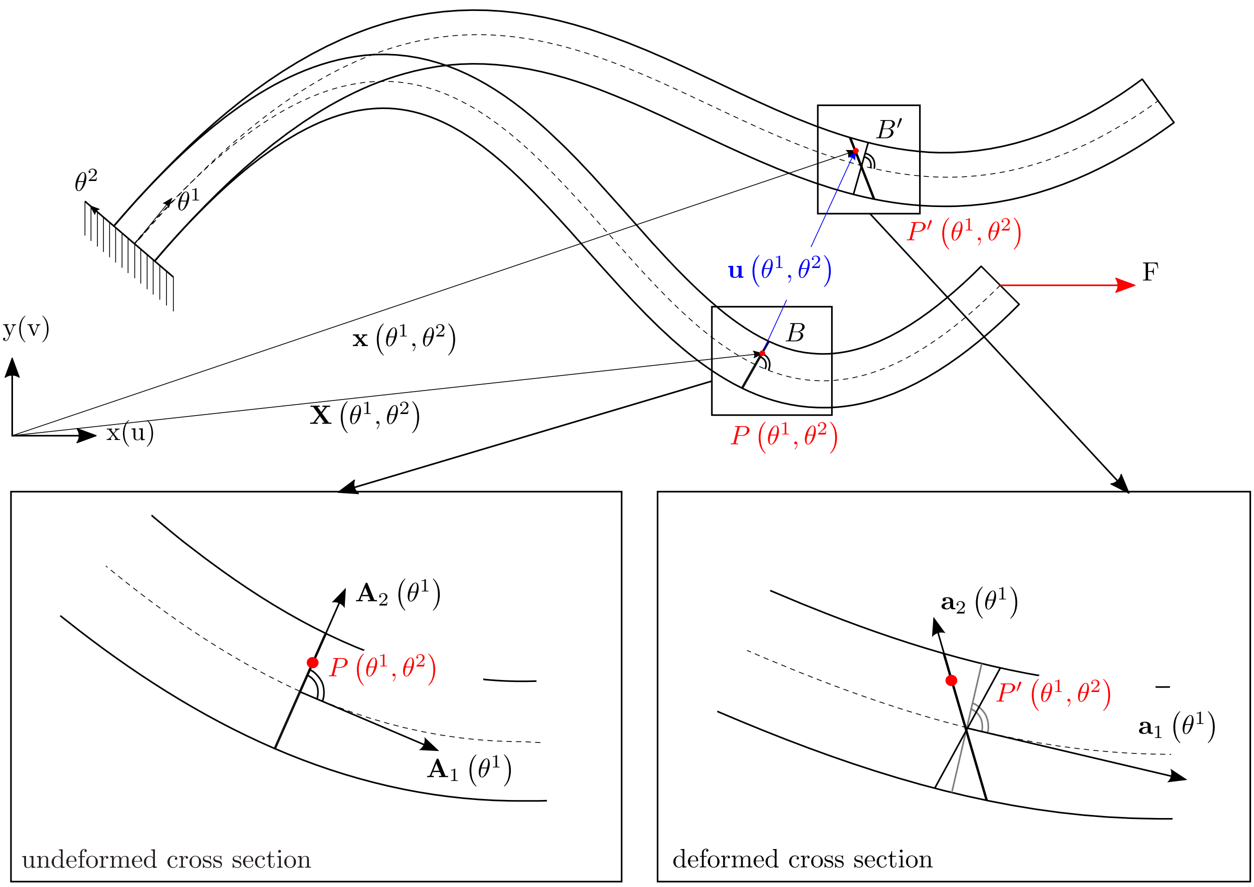

Figure 2: The continuum's position vectors for the Timoshenko placement of the problem

and γ the shear deformation of the cross section, i.e. the actual strain. Moreover $\mathbf{n}$ is the unit normal vector of the deformed curve and we write;

$$\gamma = \widehat{(\mathbf{a}_2, \mathbf{n})} \tag{3}$$

for the angle between the deformed cross section of the beam and the unit normal of the deformed mid-line of the beam. All the angles, are considered positive counter-clockwise. It is also important to identify how $\mathbf{A}_2\left(\theta^1\right)$ is chosen for the undeformed configuration. In this setting it is assumed:

$$\mathbf{A}_2\left(\theta^1\right) := \frac{1}{\|\mathbf{A}_1\|_2}\left[\begin{array}{c} -A_{1(y)} \\ A_{1(x)} \end{array}\right] \quad \forall\theta^1 \in [0, L] \ . \tag{4}$$

The latter has the following implication:

$$\mathbf{A}_{2,1}\cdot\mathbf{A}_1 = \frac{1}{\|\mathbf{A}_1\|_2}\left[\begin{array}{c} -A_{1(y),1} \\ A_{1(x),1} \end{array}\right]\cdot\left[\begin{array}{c} A_{1(x)} \\ A_{1(y)} \end{array}\right] =$$
$$= -\frac{1}{\|\mathbf{A}_1\|_2}\left[\begin{array}{c} A_{1(x),1} \\ A_{1(y),1} \end{array}\right]\cdot\left[\begin{array}{c} -A_{1(y)} \\ A_{1(x)} \end{array}\right] = -\mathbf{A}_{1,1}\cdot\mathbf{A}_2 \ .$$

Using the above identity, the metric coefficient G_{11} for the undeformed configuration can be written as:

$$G_{11}\left(\theta^1,\theta^2\right) = A_{11} - 2\theta^2\mathbf{A}_{1,1}\cdot\mathbf{A}_2 \ , \tag{5}$$

and the respective metric coefficient for the deformed case:

$$g_{11}\left(\theta^1,\theta^2\right) = a_{11} + 2\theta^2 \mathbf{a}_1 \cdot \mathbf{a}_{2,1} \ . \tag{6}$$

The metric coefficients G_{12} and g_{12} are thus given by:

$$G_{12}\left(\theta^1,\theta^2\right) = \theta^2 \mathbf{A}_{2,1} \cdot \mathbf{A}_2 \ , \tag{7a}$$
$$g_{12}\left(\theta^1,\theta^2\right) = \mathbf{a}_1 \cdot \mathbf{a}_2 + \theta^2 \mathbf{a}_{2,1} \cdot \mathbf{a}_2 \ , \tag{7b}$$

and the difference in the formulas is due to the orthogonality of the mid-line base vectors at the undeformed configuration. The normal strain ϵ_{11} is defined as for the Euler-Bernoulli beam formulation, whereas the curvature change is given by:

$$\kappa\left(\theta^1\right) = \mathbf{A}_{1,1} \cdot \mathbf{A}_2 + \mathbf{a}_1 \cdot \mathbf{a}_{2,1} \tag{8}$$

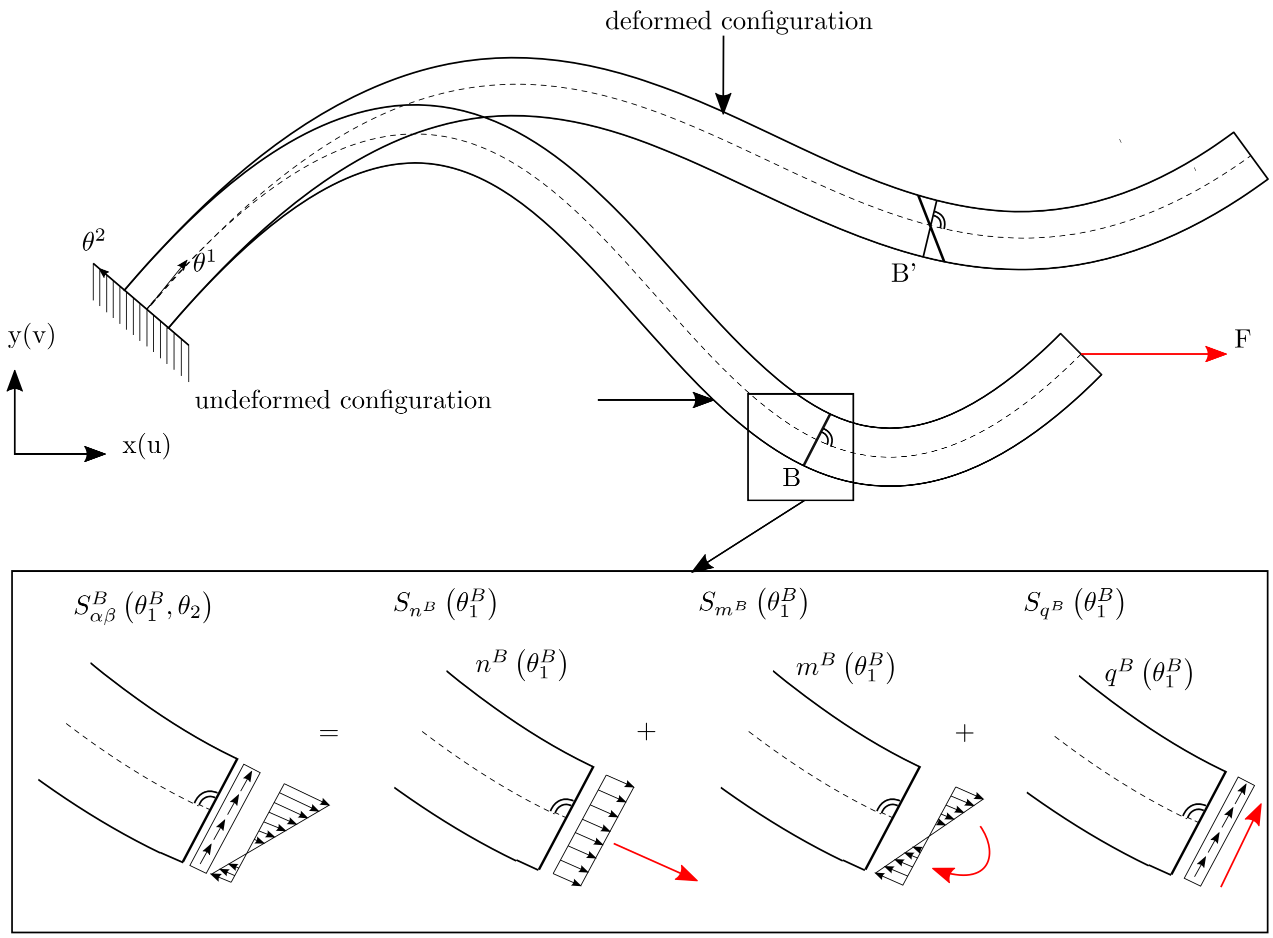

Figure 3: The stress resultants in the Timoshenko setting

In the case of the Timoshenko beam theory the strain tensor becomes 2-dimensional. Thus, for the shear part of the strain tensor one gets:

$$\epsilon_{12}\left(\theta^1,\theta^2\right) = \frac{1}{2}\left(a_{12} + \theta^2\left(\mathbf{a}_{2,1} \cdot \mathbf{a}_2 - \mathbf{A}_{2,1} \cdot \mathbf{A}_2\right)\right) \ . \tag{9}$$

According to (1), one can write:

$$\mathbf{a}_2 = \overline{\mathbf{R}}\,\mathbf{A}_2 \;, \tag{10}$$

where $\overline{\mathbf{R}}$ is the so called rotation matrix;

$$\overline{\mathbf{R}} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \;. \tag{11}$$

A close look into (9) reveals that the shear strain consists of one constant and one linear part. The constant part is attributed to the so-called transverse shear deformation γ, and the linear part to the warping[1]:

$$\gamma\left(\theta^1\right) = 2\epsilon_{12} = a_{12}\left(\theta^1\right) \;. \tag{12}$$

Having defined each strain component, we can determine the stress resultants. Proceeding in a similar way as in Bernoulli beam theory, the normal force and the bending moment are given by the corresponding equations (only the definition of κ changes in the latter case), whereas given that $C^{1211} = \frac{E}{1+\nu}$ the shear force can be computed as follows (see also Fig. (3)):

$$\begin{aligned} q\left(\theta^1\right) &= \alpha \int_{-\frac{h}{2}}^{+\frac{h}{2}} S^{12}\, b_{\text{cs}}\, d\theta^2 = \alpha \int_{-\frac{h}{2}}^{+\frac{h}{2}} C^{1211} \epsilon_{12}\, b_{\text{cs}}\, d\theta^2 = \\ &= \alpha \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{E}{2(1+\nu)} \gamma\, b_{\text{cs}}\, d\theta^2 = \frac{1}{\|\mathbf{A}_1\|_2} \alpha G\, b\, h\, \gamma\left(\theta^1\right) \quad [\text{KN}] \;. \end{aligned} \tag{13}$$

where α is the shear correction factor[1], G is the shear modulus given by $G = \frac{E}{2(1+\nu)}$, b_{cs} and h are the width and the height of the beam, respectively.

Finally one can write an explicit formula concerning the principle of virtual work for the Timoshenko beam case:

$$\delta W_{\text{int}} = \int_{\Omega \subset \mathbb{R}^3} \mathbf{S} : \delta \mathbf{E}\, d\Omega = \int_L \begin{bmatrix} n \\ m \\ q \end{bmatrix} \cdot \delta \begin{bmatrix} \epsilon \\ \kappa \\ \gamma \end{bmatrix} dL = \int_L n\delta\epsilon + m\delta\kappa + q\delta\gamma\; dL \;. \tag{14}$$

2 NURBS-based Timoshenko Beam Element

Based on the equations derived in section 1, will be formulated the stiffness matrix for a linear NURBS-based isogeometric Timoshenko beam element. We consider once more, that the curve is affected by $n \in \mathbb{N}$ control points as in the previous section, namely $\left\{\hat{\mathbf{X}}_i\right\}_{i=1}^{n}$. Recall the rotation matrix given in (11), its linearized form is:

$$\overline{\mathbf{R}}\left(\theta^1\right) \overset{\text{lin}}{=} \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \;, \tag{15}$$

since we assume small deformations i.e. $\sin\beta \approx \beta$ and $\cos\beta \approx 1$. Degrees of freedom, that appear in a power of 2 or higher will be neglected in a linear setting and thus we have:

$$\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}} = \begin{bmatrix} 1+\beta^2 & 0 \\ 0 & 1+\beta^2 \end{bmatrix} \overset{\text{lin}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2 \;.$$

Recall, that the mid-line base vector $\mathbf{A}_2$ is normalized to unit length which has the following implication:

$$\mathbf{A}_2 \cdot \mathbf{A}_{2,1} = \frac{1}{2}\left(\mathbf{A}_2 \cdot \mathbf{A}_2\right)_{,1} = \frac{1}{2}\left(\|\mathbf{A}_2\|_2^2\right)_{,1} = \|\mathbf{A}_2\|_2 \|\mathbf{A}_2\|_{2,1} = 0 \;.$$

[1] It will be shown in section (2), equation (16), that under linear assumptions the warping of the cross section vanishes in the discrete element formulation

[1] The shear correction factor balances the shear deformation under energetic considerations, since in the Timoshenko setting, quadratic distribution of the shear stresses along the beam's height cannot be recovered

Using the two last relations we get:

$$\mathbf{A}_2^{\mathrm{T}}\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}}\mathbf{A}_{2,1} = \mathbf{A}_2 \cdot \mathbf{A}_{2,1} = 0 \,.$$

Moreover, linearisation of $\overline{\mathbf{R}}_{,1}$ yields:

$$\overline{\mathbf{R}}_{,1}\left(\theta^1\right) = \begin{bmatrix} -\beta_{,1}\sin\beta & -\beta_{,1}\cos\beta \\ \beta_{,1}\cos\beta & -\beta_{,1}\sin\beta \end{bmatrix} \overset{\text{lin}}{=} \begin{bmatrix} -\beta_{,1}\beta & -\beta_{,1} \\ \beta_{,1} & -\beta_{,1}\beta \end{bmatrix} \overset{\text{lin}}{=} \begin{bmatrix} 0 & -\beta_{,1} \\ \beta_{,1} & 0 \end{bmatrix} \,.$$

Then it also holds:

$$\mathbf{A}_2^{\mathrm{T}}\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}}_{,1}\mathbf{A}_2 = \mathbf{A}_2^{\mathrm{T}} \begin{bmatrix} \beta\beta_{,1} & -\beta_{,1} \\ \beta_{,1} & \beta\beta_{,1} \end{bmatrix} \mathbf{A}_2 \overset{\text{lin}}{=} \mathbf{A}_2^{\mathrm{T}} \begin{bmatrix} 0 & -\beta_{,1} \\ \beta_{,1} & 0 \end{bmatrix} \mathbf{A}_2 = 0 \,.$$

Using the above identities (in the linear setting), equation (9) becomes:

$$\gamma\left(\theta^1\right) = 2\epsilon_{12}\left(\theta^1\right) = a_{12} + \theta^2\left(\mathbf{A}_2^{\mathrm{T}}\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}}_{,1}\mathbf{A}_2 + \mathbf{A}_2^{\mathrm{T}}\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}}\mathbf{A}_{2,1}\right) = a_{12}\left(\theta^1\right) \tag{16}$$

What is needed, is the variation of all the strains with respect to the vector of the degrees of freedom:

$$\hat{\mathbf{u}}^{\mathrm{T}} = \begin{bmatrix} \hat{u}_1 & \hat{v}_1 & \hat{\beta}_1 & \cdots & \hat{u}_n & \hat{v}_n & \hat{\beta}_n \end{bmatrix} \,. \tag{17}$$

Variation of the base vector $\mathbf{a}_1$ with respect to $\hat{\mathbf{u}}$ yields:

$$\mathbf{a}_{1,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} \frac{\partial R^{1,p}}{\partial\theta^1} & 0 & 0 & \cdots & \frac{\partial R^{n,p}}{\partial\theta^1} & 0 & 0 \\ 0 & \frac{\partial R^{1,p}}{\partial\theta^1} & 0 & \cdots & 0 & \frac{\partial R^{n,p}}{\partial\theta^1} & 0 \end{bmatrix} \,. \tag{18}$$

Using (18), the variation of the normal strain ϵ with respect to $\hat{\mathbf{u}}$ is:

$$\begin{gathered} \epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) \overset{\text{lin}}{=} \mathbf{A}_1 \cdot \mathbf{a}_{1,\hat{\mathbf{u}}} \Rightarrow \\ \epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} A_{1(x)}\frac{\partial R^{1,p}}{\partial\theta^1} & A_{1(y)}\frac{\partial R^{1,p}}{\partial\theta^1} & 0 & \cdots & A_{1(x)}\frac{\partial R^{n,p}}{\partial\theta^1} & A_{1(y)}\frac{\partial R^{1,p}}{\partial\theta^1} & 0 \end{bmatrix} \,. \end{gathered} \tag{19}$$

Once more, a vector operator for the normal strain variation is introduced:

$$\mathcal{L}_\epsilon : C_0^1\left[0,1\right] \rightarrow \left(C\left[0,1\right]\right)^3 \quad \mathcal{L}_\epsilon f := \begin{bmatrix} A_{1(x)}\frac{\partial f}{\partial\theta^1} & A_{1(y)}\frac{\partial f}{\partial\theta^1} & 0 \end{bmatrix} \quad \forall f \in C_0^1\left[0,1\right] \,. \tag{20}$$

Then, equation (19) reads as:

$$\epsilon_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \begin{bmatrix} \left(\mathcal{L}_\epsilon R^{1,p}\right)\left(\theta^1\right) & \cdots & \left(\mathcal{L}_\epsilon R^{n,p}\right)\left(\theta^1\right) \end{bmatrix} \,. \tag{21}$$

Next, with the variation of the change in curvature κ with respect to $\hat{\mathbf{u}}$ is derived (for the definition of κ refer to (8)):

$$\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \left(\mathbf{a}_1 \cdot \mathbf{a}_{2,1}\right)_{,\hat{\mathbf{u}}} = \mathbf{a}_{2,1}^{\mathrm{T}}\mathbf{a}_{1,\hat{\mathbf{u}}} + \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_{2,1,\hat{\mathbf{u}}} \,. \tag{22}$$

To find an expression for $\kappa_{,\hat{\mathbf{u}}}$, we need to treat each term of (22) separately. That is:

$$\mathbf{a}_{2,1}^{\mathrm{T}}\left(\theta^1\right) = \mathbf{A}_{2,1}^{\mathrm{T}}\overline{\mathbf{R}}^{\mathrm{T}} + \mathbf{A}_2^{\mathrm{T}}\overline{\mathbf{R}}_{,1}^{\mathrm{T}} \,.$$

Linearization of the latter equation with respect to the degrees of freedom yields:

$$\mathbf{a}_{2,1}^{\mathbf{T}}\left(\theta^1\right) \overset{\text{lin}}{=} \mathbf{A}_{2,1}^{\mathrm{T}}\mathbf{I}_2 + \mathbf{A}_2^{\mathrm{T}}\mathbf{0}_2 = \mathbf{A}_{\mathbf{2},\mathbf{1}}\left(\theta^1\right) \,. \tag{23}$$

where we have used the following linearization rules (see also remark 1):

$$
\overline{\mathbf{R}} = \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \overset{\text{lin}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2 \ ,
$$

$$
\overline{\mathbf{R}}_{,1} = \begin{bmatrix} 0 & -\beta_{,1} \\ \beta_{,1} & 0 \end{bmatrix} \overset{\text{lin}}{=} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}_2 \ .
$$

The derivative $\mathbf{A}_{2,1}$ is given by:

$$
\mathbf{A}_{2,1}\left(\theta^1\right) = \left(\frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} -A_{1(y)} \\ A_{1(x)} \end{bmatrix} \right)_{,1} = \frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} -A_{1(y),1} + \frac{\|\mathbf{A}_1\|_{2,1}}{\|\mathbf{A}_1\|_2} A_{1(y)} \\ A_{1(x),1} - \frac{\|\mathbf{A}_1\|_{2,1}}{\|\mathbf{A}_1\|_2} A_{1(x)} \end{bmatrix} \ . \tag{25}
$$

Evidently, the derivative $\|\mathbf{A}_1\|_{2,1}$ is also needed, therefore:

$$
\begin{gathered}
\left\|\mathbf{A}_1\left(\theta^1\right)\right\|_{2,1} = \left(\sqrt{\mathbf{A}_1 \cdot \mathbf{A}_1}\right)_{,1} = \\
= \tfrac{1}{\|\mathbf{A}_1\|_2} \mathbf{A}_1 \cdot \mathbf{A}_{1,1} = \tfrac{1}{\|\mathbf{A}_1\|_2} \left(A_{1(x),1} A_{1(x)} + A_{1(y),1} A_{1(y)} \right) \ .
\end{gathered} \tag{26}
$$

Hence we have the expression:

$$
\mathbf{A}_{2,1}\left(\theta^1\right) = \begin{bmatrix} A_{2(x),1}\left(\theta^1\right) \\ A_{2(y),1}\left(\theta^1\right) \end{bmatrix} \ , \tag{27}
$$

where:

$$
A_{2(x),1}\left(\theta^1\right) = -\frac{1}{\|\mathbf{A}_1\|_2} A_{1(y),1} + \alpha A_{1(y)} \ , \tag{28a}
$$

$$
A_{2(y),1}\left(\theta^1\right) = \frac{1}{\|\mathbf{A}_1\|_2} A_{1(x),1} - \alpha A_{1(x)} \ , \tag{28b}
$$

$$
\alpha\left(\theta^1\right) := \frac{1}{\|\mathbf{A}_1\|_2^3} \left(A_{1(x),1} A_{1(x)} + A_{1(y),1} A_{1(y)} \right) \ . \tag{28c}
$$

We conclude the derivations for the first term of equation (22), with:

$$
\mathbf{a}_{2,1}^{\mathrm{T}} \mathbf{a}_{1,\hat{\mathbf{u}}} \overset{\text{lin}}{=} \begin{bmatrix} A_{2(x),1} \frac{\partial R^{1,p}}{\partial \theta^1} & A_{2(y),1} \frac{\partial R^{1,p}}{\partial \theta^1} & 0 & \cdots & A_{2(x),1} \frac{\partial R^{n,p}}{\partial \theta^1} & A_{2(y),1} \frac{\partial R^{n,p}}{\partial \theta^1} & 0 \end{bmatrix} \ . \tag{29}
$$

Next, the second term of (22) is derived. As a first step we have:

$$
\mathbf{a}_{2,1,\hat{\mathbf{u}}}\left(\theta^1\right) = \left(\overline{\mathbf{R}} \mathbf{A}_2\right)_{,1,\hat{\mathbf{u}}} = \left(\overline{\mathbf{R}}_{,1} \mathbf{A}_2\right)_{,\hat{\mathbf{u}}} + \left(\overline{\mathbf{R}} \mathbf{A}_{2,1}\right)_{,\hat{\mathbf{u}}} \ . \tag{30}
$$

The first term of equation (30) writes:

$$
\begin{gathered}
\left(\overline{\mathbf{R}}_{,1} \mathbf{A}_2\right)_{,\hat{\mathbf{u}}} = \left(\frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} 0 & -\beta_{,1} \\ \beta_{,1} & 0 \end{bmatrix} \begin{bmatrix} -A_{1(y)} \\ A_{1(x)} \end{bmatrix} \right)_{,\hat{\mathbf{u}}} = \frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} -\beta_{,1} A_{1(x)} \\ -\beta_{,1} A_{1(y)} \end{bmatrix}_{,\hat{\mathbf{u}}} \Rightarrow \\
\left(\overline{\mathbf{R}}_{,1} \mathbf{A}_2\right)_{,\hat{\mathbf{u}}} = -\frac{1}{\|\mathbf{A}_1\|_2} \begin{bmatrix} 0 & 0 & A_{1(x)} \frac{\partial R^{1,p}}{\partial \theta^1} & \cdots & 0 & 0 & A_{1(x)} \frac{\partial R^{n,p}}{\partial \theta^1}\left(\theta^1\right) \\ 0 & 0 & A_{1(y)} \frac{\partial R^{1,p}}{\partial \theta^1}\left(\theta^1\right) & \cdots & 0 & 0 & A_{1(y)} \frac{\partial R^{n,p}}{\partial \theta^1}\left(\theta^1\right) \end{bmatrix} \ .
\end{gathered} \tag{31a}
$$

The second term of equation (30) writes:

$$
\begin{gathered}
\left(\overline{\mathbf{R}} \mathbf{A}_{2,1}\right)_{,\hat{\mathbf{u}}} = \left(\begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} A_{2(x),1} \\ A_{2(y),1} \end{bmatrix} \right)_{,\hat{\mathbf{u}}} = \begin{bmatrix} A_{2(x),1} - \beta A_{2(y),1} \\ \beta A_{2(x),1} + A_{2(y),1} \end{bmatrix}_{,\hat{\mathbf{u}}} \Rightarrow \\
\left(\overline{\mathbf{R}} \mathbf{A}_{2,1}\right)_{,\hat{\mathbf{u}}} = \begin{bmatrix} 0 & 0 & -A_{2(y),1} R^{1,p}\left(\theta^1\right) & \cdots & 0 & 0 & -A_{2(y),1} R^{n,p}\left(\theta^1\right) \\ 0 & 0 & A_{2(x),1} R^{1,p}\left(\theta^1\right) & \cdots & 0 & 0 & A_{2(x),1} R^{n,p}\left(\theta^1\right) \end{bmatrix}
\end{gathered} \tag{32a}
$$

Substituting (31a) and (32a) into (30) yields:

$$\mathbf{a}_{2,1,\hat{\mathbf{u}}}\left(\theta^1\right) = \left[\begin{array}{ccc} 0 & 0 & -\frac{1}{\|\mathbf{A}_1\|_2} A_{1(x)} \frac{\partial R^{1,p}}{\partial \theta^1} - A_{2(y),1} R^{1,p}\left(\theta^1\right) \quad \cdots \\ 0 & 0 & -\frac{1}{\|\mathbf{A}_1\|_2} A_{1(y)} \frac{\partial R^{1,p}}{\partial \theta^1} + A_{2(x),1} R^{1,p}\left(\theta^1\right) \quad \cdots \end{array}\right.$$
$$\left.\begin{array}{cccc} \cdots & 0 & 0 & -\frac{1}{\|\mathbf{A}_1\|_2} A_{1(x)} \frac{\partial R^{n,p}}{\partial \theta^1} - A_{2(y),1} R^{n,p}\left(\theta^1\right) \\ \cdots & 0 & 0 & -\frac{1}{\|\mathbf{A}_1\|_2} A_{1(y)} \frac{\partial R^{n,p}}{\partial \theta^1} + A_{2(x),1} R^{n,p}\left(\theta^1\right) \end{array}\right] . \tag{33}$$

Then, the second term of equation (22) becomes:

$$\mathbf{a}_1^{\mathrm{T}} \mathbf{a}_{2,1,\hat{\mathbf{u}}} \overset{\text{lin}}{=} \mathbf{A}_1^{\mathrm{T}} \mathbf{a}_{2,1,\hat{\mathbf{u}}} \Rightarrow$$

$$\mathbf{a}_1^{\mathrm{T}} \mathbf{a}_{2,1,\hat{\mathbf{u}}} \overset{\text{lin}}{=} \left[\begin{array}{c} 0 \quad 0 \quad -\|\mathbf{A}_1\|_2 \frac{\partial R^{1,p}}{\partial \theta^1} + b R^{1,p}\left(\theta^1\right) \quad \cdots \\ 0 \quad 0 \quad -\|\mathbf{A}_1\|_2 \frac{\partial R^{n,p}}{\partial \theta^1} + b R^{n,p}\left(\theta^1\right) \end{array}\right] . \tag{34}$$

In the latter equation, the following abbreviation is used:

$$b\left(\theta^1\right) := A_{2(x),1} A_{1(y)} - A_{1(x)} A_{2(y),1} . \tag{35}$$

Substituting equations (29) and (34) into (22) we get the matrix expression:

$$\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \left[\begin{array}{c} A_{2(x),1} \frac{\partial R^{1,p}}{\partial \theta^1} \quad A_{2(y),1} \frac{\partial R^{1,p}}{\partial \theta^1} \quad -\|\mathbf{A}_1\|_2 \frac{\partial R^{1,p}}{\partial \theta^1} + b R^{1,p}\left(\theta^1\right) \quad \cdots \\ \cdots \quad A_{2(x),1} \frac{\partial R^{n,p}}{\partial \theta^1} \quad A_{2(y),1} \frac{\partial R^{n,p}}{\partial \theta^1} \quad -\|\mathbf{A}_1\|_2 \frac{\partial R^{n,p}}{\partial \theta^1} + b R^{n,p}\left(\theta^1\right) \end{array}\right] . \tag{36}$$

Finally, the vector operator $\mathcal{L}_\kappa$ is defined as follows:

$$\mathcal{L}_\kappa : \quad C_0^1[0,1] \to \left(C[0,1]\right)^3 : \ \mathcal{L}_\kappa\, f := \begin{bmatrix} A_{2(x),1} \frac{\partial f}{\partial \theta^1} \\ A_{2(y),1} \frac{\partial f}{\partial \theta^1} \\ -\|\mathbf{A}_1\|_2 \frac{\partial f}{\partial \theta^1} + b\, f \end{bmatrix}^{\mathrm{T}} \quad \forall f \in C_0^1[0,1] , \tag{37}$$

to get a short, closed form expression for the variation of κ with respect to $\hat{\mathbf{u}}$:

$$\kappa_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \left[\ \left(\mathcal{L}_\kappa R^{1,p}\right)\left(\theta^1\right) \cdots \left(\mathcal{L}_\kappa R^{n,p}\right)\left(\theta^1\right)\ \right] , \tag{38}$$

As a last step, the variation of γ (see also equation (3)) with respect to $\hat{\mathbf{u}}$ is determined. Substituting (16) into (3) and differentiating with respect to $\hat{\mathbf{u}}$ we get:

$$\gamma_{,\hat{\mathbf{u}}}\left(\theta^1\right) = \mathbf{A}_2^{\mathrm{T}} \overline{\mathbf{R}}^{\mathrm{T}} \mathbf{a}_{1,\hat{\mathbf{u}}} + \mathbf{a}_1^{\mathrm{T}} \left(\overline{\mathbf{R}} \mathbf{A}_2\right)_{,\hat{\mathbf{u}}} , \tag{39}$$

where the first term in equation (39) writes:

$$\begin{gathered} \mathbf{A}_2^{\mathrm{T}} \overline{\mathbf{R}}^{\mathrm{T}} \mathbf{a}_{1,\hat{\mathbf{u}}} \overset{\text{lin}}{=} \mathbf{A}_2^{\mathrm{T}} \mathbf{I}_2 \mathbf{a}_{1,\hat{\mathbf{u}}} = \\ = \frac{1}{\|\mathbf{A}_1\|_2} \left[\begin{array}{cccccc} -A_{1(y)} \frac{\partial R^{1,p}}{\partial \theta^1} & A_{1(x)} \frac{\partial R^{1,p}}{\partial \theta^1} & 0 & \cdots & -A_{1(y)} \frac{\partial R^{n,p}}{\partial \theta^1} & A_{1(x)} \frac{\partial R^{n,p}}{\partial \theta^1} \quad 0 \end{array}\right] . \end{gathered} \tag{40}$$

Then, the second term of equation obtains the vector form:

$$\begin{gathered} \mathbf{a}_1^{\mathrm{T}} \left(\overline{\mathbf{R}} \mathbf{A}_2\right)_{,\hat{\mathbf{u}}} \overset{\text{lin}}{=} \mathbf{A}_1^{\mathrm{T}} \left(\overline{\mathbf{R}} \mathbf{A}_2\right)_{,\hat{\mathbf{u}}} = \\ = \|\mathbf{A}_1\|_2 \left[\ 0 \quad 0 \quad -R^{1,p}\left(\theta^1\right) \quad \cdots \quad 0 \quad 0 \quad -R^{n,p}\left(\theta^1\right)\ \right] . \end{gathered} \tag{41}$$

Substituting the identities (40) and (41) into equation (39) we finally get:

$$\gamma_{,\hat{\mathbf{u}}}\left(\theta^1\right) \overset{\text{lin}}{=} \left[\ \left(\mathcal{L}_\gamma R^{1,p}\right)\left(\theta^1\right) \quad \cdots \quad \left(\mathcal{L}_\gamma R^{n,p}\right)\left(\theta^1\right)\ \right] . \tag{42}$$

Here the vector operator $\mathcal{L}_\gamma$ is defined as:

$$\mathcal{L}_\gamma : C_0^1\left[0,1\right] \rightarrow \left(C\left[0,1\right]\right)^3 : \quad \mathcal{L}_\gamma\, f := \begin{bmatrix} -\frac{1}{\|\mathbf{A}_1\|_2} A_{1(y)} \frac{\partial f}{\partial \theta^1} \\ \frac{1}{\|\mathbf{A}_1\|_2} A_{1(x)} \frac{\partial f}{\partial \theta^1} \\ -\|\mathbf{A}_1\|_2\, f \end{bmatrix}^{\mathrm{T}} \quad \forall f \in C_0^1\left[0,1\right] \ . \tag{43}$$

Summarizing the internal virtual work (14) obtains the matrix-vector form:

$$\delta W_{\text{int}} = \left(\delta \hat{\mathbf{u}}\right)^{\mathrm{T}} \int_L C^{\text{axial}} \epsilon_{,\hat{\mathbf{u}}} \otimes \epsilon_{,\hat{\mathbf{u}}} + C^{\text{bending}} \kappa_{,\hat{\mathbf{u}}} \otimes \kappa_{,\hat{\mathbf{u}}} + C^{\text{shear}} \gamma_{,\hat{\mathbf{u}}} \otimes \gamma_{,\hat{\mathbf{u}}}\; dL\; \hat{\mathbf{u}} \tag{44}$$

Where the individual stiffness matrices are:

$$\mathbf{K}^{\text{axial}} = \int_L C^{\text{axial}} \epsilon_{,\hat{\mathbf{u}}} \otimes \epsilon_{,\hat{\mathbf{u}}}\; dL \ , \quad C^{\text{axial}} = E\, h\, b \ , \tag{45a}$$

$$\mathbf{K}^{\text{bending}} = \int_L C^{\text{bending}} \kappa_{,\hat{\mathbf{u}}} \otimes \kappa_{,\hat{\mathbf{u}}}\; dL \ , \quad C^{\text{bending}} = \frac{E\, h^3\, b}{12} \ , \tag{45b}$$

$$\mathbf{K}^{\text{shear}} = \int_L C^{\text{shear}} \gamma_{,\hat{\mathbf{u}}} \otimes \gamma_{,\hat{\mathbf{u}}}\; dL \ , \quad C^{\text{shear}} = \alpha\, G\, b\, h \ , \tag{45c}$$

$$\mathbf{K} = \mathbf{K}^{\text{axial}} + \mathbf{K}^{\text{bending}} + \mathbf{K}^{\text{shear}} \ . \tag{45d}$$

Remark 1. It is important to identify the variational index of the Bernoulli and Timoshenko problem. For this reason, let us introduce a vector containing all the shape functions:

$$\mathbf{N}\left(\theta^1\right) = \begin{bmatrix} R^{1,p}\left(\theta^1\right) & \cdots & R^{n,p}\left(\theta^1\right) \end{bmatrix}^{\mathrm{T}} \ . \tag{46}$$

Then, we can rewrite the corresponding equation from the Bernoulli beam theory herein as follows:

$$\mathbf{A}_1^{\mathrm{T}}\left(\theta^1\right) = \begin{bmatrix} A_{1(x)} & A_{1(y)} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_x \cdot \mathbf{N} & \mathcal{L}_y \cdot \mathbf{N} \end{bmatrix} \ , \tag{47}$$

where $\mathcal{L}_x,\ \mathcal{L}_y$ are first order vector operators defined as:

$$\mathcal{L}_x := \begin{bmatrix} \hat{X}^1 \frac{\partial}{\partial \theta^1} & \cdots & \hat{X}^n \frac{\partial}{\partial \theta^1} \end{bmatrix}^{\mathrm{T}} \ , \tag{48a}$$

$$\mathcal{L}_y := \begin{bmatrix} \hat{Y}^1 \frac{\partial}{\partial \theta^1} & \cdots & \hat{Y}^n \frac{\partial}{\partial \theta^1} \end{bmatrix}^{\mathrm{T}} \ . \tag{48b}$$

Additionally, let us introduce the second order differential operators $\mathcal{L}_{xx},\ \mathcal{L}_{yy}$:

$$\mathcal{L}_{xx} := \begin{bmatrix} \hat{X}^1 \frac{\partial^2}{\partial(\theta^1)^2} & \cdots & \hat{X}^n \frac{\partial^2}{\partial(\theta^1)^2} \end{bmatrix}^{\mathrm{T}} \ , \tag{49a}$$

$$\mathcal{L}_{yy} := \begin{bmatrix} \hat{Y}^1 \frac{\partial^2}{\partial(\theta^1)^2} & \cdots & \hat{Y}^n \frac{\partial^2}{\partial(\theta^1)^2} \end{bmatrix}^{\mathrm{T}} \ . \tag{49b}$$

Evidently the norm $\|\mathbf{A}_1\|_2$ can be obtained as usual:

$$\|\mathbf{A}_1\|_2\left(\theta^1\right) = \left(\left(\mathcal{L}_x \cdot \mathbf{N}\right)^2 + \left(\mathcal{L}_y \cdot \mathbf{N}\right)^2\right)^{\frac{1}{2}} \ . \tag{50}$$

It can be seen, that the formulation of the Bernoulli beam element involves the operators $\mathcal{L}_\epsilon$, $\mathcal{L}_\kappa$ defined in the corresponding equations. Note, that since $\mathcal{L}_\kappa$ is a second order operator and all others are first order ones, the Bernoulli problem has a variational index equal to 2, i.e. shape functions have to lie in the C_0^2-space.

The formulation of the Timoshenko problem involves directly the operators $\mathcal{L}_\epsilon$, $\mathcal{L}_\kappa$, $\mathcal{L}_\gamma$, $\mathcal{L}_x$, $\mathcal{L}_y$, defined in equations (20), (37), (43), (48a), (48b), respectively. At first glance, it seems that C_0^1-basis functions are sufficient, since all operators $\mathcal{L}_\epsilon$, $\mathcal{L}_\kappa$, $\mathcal{L}_\gamma$ are first order ones.

However, the involvement of the terms $A_{2(x),1}$ and $A_{2(y),1}$ in the element formulation (see equation (37)) enables additional features in the problem, which we analyse subsequently: The equations (28a)-(28b) can be rewritten in the following form:

$$A_{2(x),1} = -\frac{\mathcal{L}_{yy} \cdot \mathbf{N}}{\sqrt{\left(\mathcal{L}_x \cdot \mathbf{N}\right)^2 + \left(\mathcal{L}_y \cdot \mathbf{N}\right)^2}} + \frac{\left(\left(\mathcal{L}_{xx} \cdot \mathbf{N}\right)\left(\mathcal{L}_x \cdot \mathbf{N}\right) + \left(\mathcal{L}_{yy} \cdot \mathbf{N}\right)\left(\mathcal{L}_y \cdot \mathbf{N}\right)\right)\mathcal{L}_y \cdot \mathbf{N}}{\left(\left(\mathcal{L}_x \cdot \mathbf{N}\right)^2 + \left(\mathcal{L}_y \cdot \mathbf{N}\right)^2\right)^{\frac{3}{2}}} ,$$

$$A_{2(y),1} = \frac{\mathcal{L}_{yy} \cdot \mathbf{N}}{\sqrt{\left(\mathcal{L}_x \cdot \mathbf{N}\right)^2 + \left(\mathcal{L}_x \cdot \mathbf{N}\right)^2}} - \frac{\left(\left(\mathcal{L}_{xx} \cdot \mathbf{N}\right)\left(\mathcal{L}_x \cdot \mathbf{N}\right) + \left(\mathcal{L}_{yy} \cdot \mathbf{N}\right)\left(\mathcal{L}_y \cdot \mathbf{N}\right)\right)\mathcal{L}_x \cdot \mathbf{N}}{\left(\left(\mathcal{L}_x \cdot \mathbf{N}\right)^2 + \left(\mathcal{L}_y \cdot \mathbf{N}\right)^2\right)^{\frac{3}{2}}} ,$$

by also using the norm $\|\mathbf{A}_1\|_2$ defined in an operator form in equation (50). Evidently, the second order operators $\mathcal{L}_{xx}$, $\mathcal{L}_{yy}$ defined in (49a) and (49b) are also involved in the element formulation. If the beam is initially straight then the terms $A_{2(x),1}$, $A_{2(y),1}$ vanish (see equations (28a)-(28c)) and the problem preserves variational index equal to 1.

Nevertheless, if the parametric geometry of the beam attains some curvature, then it is clear that the basis functions are restricted into the C_0^2-space, i.e. the variational index of the problem is 2 as in the Bernoulli case, and the use of low order basis functions becomes not feasible.

Note that for moderately curved beams, the terms $A_{2(x),1}$, $A_{2(y),1}$ and b (see equations (28a), (28b) and (35)) can be neglected, and in this case the variational index of the problem is yet 1. For highly curved beams, this assumption is incorrect and may lead to wrong results.