

Math 412

Final Exam review problems

True or false. Justify!

- 1) $D_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. **FALSE**
- 2) $D_3 \cong \mathcal{S}_3$. **TRUE**
- 3) $D_4 \cong \mathcal{S}_4$. **FALSE**
- 4) $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$. **FALSE**
- 5) $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$. **TRUE**
- 6) An element g of a group G can satisfy $g^{30} = e$ and have order 6. **TRUE**
- 7) An element g of a group G can satisfy $g^{30} = e$ and have order 7. **FALSE**
- 8) Every normal subgroup is the kernel of some group homomorphism. **TRUE**
- 9) If a group G contains an element of infinite order, then G is infinite. **TRUE**
- 10) If a group G contains a nontrivial element of finite order, then G is finite. **FALSE**
- 11) If every element in a group G has finite order, then G is finite. **FALSE**
- 12) There are no nontrivial simple abelian groups. **FALSE**
- 13) There exists a group G of order 8 acting on a set X such that some $x \in X$ has orbit of cardinality 5. **FALSE**
- 14) There exists a group G of order 12 acting on a set X such that some $x \in X$ has orbit of cardinality 6 and stabilizer of order 4. **FALSE**
- 15) When a group G acts on a set X , a point $x \in X$ is fixed if and only if $\text{Stab}(x) = G$. **TRUE**
- 16) For every $n \geq 5$, \mathcal{A}_n is simple. **TRUE**
- 17) The group of units in \mathbb{Z} is cyclic. **TRUE**
- 18) The group of units in \mathbb{Z}_{51} is cyclic. **TRUE**
- 19) The group of units in \mathbb{Z}_8 is cyclic. **FALSE**
- 20) The subgroup of $SL_2(\mathbb{R})$ generated by $\begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$ is isomorphic to \mathbb{Z}_n . **TRUE**
- 21) There always exists a group homomorphism between any two groups. **FALSE**
- 22) If H is an abelian subgroup of the (possibly nonabelian) group G , then H is normal. **FALSE**
- 23) If H is a subgroup of an abelian group G , then G/H is abelian. **TRUE**
- 24) If a group G of order 14 acts on a set with 14 elements, it's possible the total number of orbits is 3. **FALSE**
- 25) When the Klein-4 group acts on a set of 11 elements, there are at most 4 orbits. **FALSE**
- 26) The center of an abelian group G is the set of all elements of G . **FALSE**
- 27) The center of a group G is an abelian group. **TRUE**
- 28) The center of a group is always a normal subgroup. **TRUE**
- 29) If every proper subgroup of a group G is cyclic, then G is cyclic. **FALSE**
- 30) There exists a surjective group homomorphism $\mathbb{Z}_7 \rightarrow \mathbb{Z}_5$. **FALSE**
- 31) There exists an injective group homomorphism $\mathcal{S}_7 \rightarrow \mathcal{S}_8$. **TRUE**
- 32) There exists an injective group homomorphism $\mathbb{Z}_7 \rightarrow \mathbb{Z}_8$. **FALSE**
- 33) When a nontrivial group acts on itself by conjugation, there is always a fixed point. **TRUE**
- 34) When a nontrivial group acts on itself by left multiplication, there is always a fixed point. **FALSE**
- 35) If an action of the group G on the set X has at least one fixed point, then the action is faithful. **FALSE**
- 36) If an action of the group G on the set X has no fixed points, the action is faithful. **FALSE**
- 37) The quotient group G/K is a subset of G . **FALSE**
- 38) The elements gK and hK are equal in G/K if and only if $g = h$. **FALSE**

- 39) If G is a group of order n and $k|n$, there is an element of G of order k . **FALSE**
- 40) If G is a group of order n and $k|n$, there is a subgroup of G of order k . **FALSE**
- 41) Every element in S_{123} is a product of elements of order 2. **TRUE**
- 42) Every element in S_{123} is a product of elements of order 3. **FALSE**
- 43) There exists a field \mathbb{F} such that both $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \times)$ are cyclic. **TRUE**
- 44) Every quotient of a nonabelian group is nonabelian. **FALSE**
- 45) A subgroup of order 2 is always normal. **FALSE**
- 46) Every subgroup of index 2 is normal. **TRUE**
- 47) Every subgroup of index 3 is normal. **FALSE**
- 48) For a ring homomorphism $f: R \rightarrow S$, $f^\times: R^\times \rightarrow S^\times$ given by $f^\times(u) = f(u)$ is a group homomorphism. **TRUE**
- 49) The image of a group homomorphism $G \rightarrow H$ with G abelian is always an abelian subgroup of H . **TRUE**
- 50) If there exists a nontrivial group homomorphism $G \rightarrow H$ with G abelian, then H is abelian. **FALSE**
- 51) Suppose that G acts on itself by conjugation. Then not every point is a fixed point. **FALSE**
- 52) The quotient group \mathbb{Q}/\mathbb{Z} is a finite group. **FALSE**
- 53) Let \mathbb{F} be a finite field with a nonidentity element g satisfying $g^2 = 1$. Then $|\mathbb{F}|$ is odd. **TRUE**
- 54) There exists a surjective group homomorphism $S_5 \rightarrow \mathbb{Z}_3$. **FALSE**
- 55) If $g^{18} = e$, then the order of g is 18. **FALSE**
- 56) A group of order 400 can have an element of order 19. **FALSE**
- 57) The intersection of two normal subgroups is a normal subgroup. **TRUE**
- 58) If R is a finite ring, R^\times is a cyclic group. **FALSE**
- 59) Given any ring $(R, +, \times)$, $(R, +)$ is always a group. **TRUE**
- 60) Given any ring $(R, +, \times)$, (R, \times) is always a group. **FALSE**
- 61) The rings $\mathbb{Q}[x]/(x^2)$ and $\mathbb{Q}[x]/(x^2 + 1)$ are isomorphic. **FALSE**
- 62) S_3 is a cyclic group. **FALSE**
- 63) There are two nonisomorphic cyclic groups of order 20. **FALSE**
- 64) There exists a subgroup of S_5 that is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. **FALSE**
- 65) Every 4-cycle in S_{103} is odd. **TRUE**
- 66) S_{120} has no subgroup isomorphic to D_{60} . **FALSE**
- 67) Every group of order 12 contains an element of order 4. **FALSE**
- 68) Every group of order 120 contains an element of order 3. **TRUE**
- 69) Let R be the subgroup of all rotations in D_4 . Then $D_4/R \cong \mathbb{Z}_3^\times$. **TRUE**
- 70) Given a group G and $x \in G$, x defines a group homomorphism $G \rightarrow G$ by $g \mapsto gxg^{-1}$. **FALSE**
- 71) If F is a field and R is a nonzero ring, every ring homomorphism $F \rightarrow R$ is injective. **TRUE**
- 72) If G and H are two groups of the same order, then $G \cong H$. **FALSE**
- 73) Every group of order 29 is simple. **TRUE**
- 74) The image of a group homomorphism is always a normal subgroup. **FALSE**
- 75) The kernel of a group homomorphism is always a normal subgroup. **TRUE**
- 76) Every finite group G is isomorphic to a subgroup of S_n for some n . **TRUE**
- 77) Every quotient of a domain is a domain. **FALSE**
- 78) Every quotient of a field is a field. **TRUE**
- 79) In any group G , the product of elements of finite order always has finite order. **FALSE**
- 80) Every nontrivial group has at least two subgroups. **TRUE**
- 81) Every nontrivial group has at least two normal subgroups. **TRUE**
- 82) Every ring homomorphism $M_2(\mathbb{R}) \rightarrow R$ to a nontrivial ring R is injective. **TRUE**
- 83) There are exactly 2 ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2$ to \mathbb{Z}_4 . **FALSE**
- 84) Every subgroup of an abelian group is abelian. **TRUE**
- 85) There are no group homomorphisms $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4$. **FALSE**

- 86) There are no group homomorphisms $\mathbb{Z}_n \rightarrow \mathbb{Z}$. **FALSE**
- 87) In \mathbb{Z} , if $n = p_1 \cdots p_t = q_1 \cdots q_s$, for primes p_i, q_j , then $s = t$ and $p_1 = q_1, \dots, p_s = q_s$. **FALSE**
- 88) In general, the fastest way to find the gcd of two large integers is to factor them into primes. **FALSE**
- 89) The equation $[a]_n x = [b]_n$ has a solution in \mathbb{Z}_n if and only if $\gcd(a, n) = 1$. **FALSE**
- 90) The system of equations $7|(x+3)$ and $11|(x-1)$ has a solution modulo 77. **TRUE**
- 91) The system of equations $3|x$ and $6|(x-1)$ has a solution modulo 18. **FALSE**
- 92) If $n|a$ and $m|a$, then $nm|a$. **FALSE**
- 93) Given any ring R , there exists exactly one ring homomorphism $\mathbb{Z} \rightarrow R$. **TRUE**
- 94) Given any ring R , there exists exactly one ring homomorphism $R \rightarrow \mathbb{Z}$. **FALSE**
- 95) Given any ring R , there exists exactly one ring homomorphism $\mathbb{Z}_n \rightarrow R$. **FALSE**
- 96) Given any ring R , there exists exactly one ring homomorphism $R \rightarrow \mathbb{Z}_n$. **FALSE**
- 97) Every element in \mathbb{Z} is a unit. **FALSE**
- 98) The additive inverse of $[5]_{77}$ in \mathbb{Z}_{77} is $[149]_{77}$. **TRUE**
- 99) The multiplicative inverse of $[5]_{77}$ in \mathbb{Z}_{77} is $[108]_{77}$. **TRUE**
- 100) Every nonzero ring contains at least two ideals. **TRUE**
- 101) Every domain is a field. **FALSE**
- 102) Every field is a domain. **TRUE**
- 103) The zero ring is a domain. **FALSE**
- 104) There always exists a ring homomorphism between any two rings. **FALSE**
- 105) Any commutative ring that has only two ideals is a field. **TRUE**
- 106) The kernel of any ring homomorphism is an ideal. **TRUE**
- 107) The kernel of any ring homomorphism is a subring. **FALSE**
- 108) The image of any ring homomorphism is an ideal. **FALSE**
- 109) The image of any ring homomorphism is a subring. **TRUE**
- 110) If R is a commutative ring and $(g) = R$, then g is a unit. **TRUE**
- 111) If R is a domain, then $R[x]$ is a domain. **TRUE**
- 112) If F is a field, then $F[x]$ is a field. **FALSE**
- 113) Every reducible polynomial of degree 4 in $F[x]$ for a field F has a root in F . **FALSE**
- 114) Every reducible polynomial of degree 3 in $F[x]$ for a field F has a root in F . **FALSE**
- 115) If $p(x) \in \mathbb{Z}_2[x]$ has degree 3, then $\mathbb{Z}_2[x]/(p(x))$ has 4 elements. **FALSE**
- 116) If $p(x) \in F[x]$ for some field F is irreducible, then $\gcd(p(x), f(x))$ is 1 or p . **TRUE**
- 117) If F is a field, the remainder of dividing $f(x)$ by $x - a$ is $f(a)$. **TRUE**
- 118) Modern algebra is fun! **TRUE**
- 119) The ring $\mathbb{Z}_n[x]$ is a domain. **FALSE**
- 120) $\mathbb{Z}_{12} \times \mathbb{Z}_5 \cong \mathbb{Z}_{60}$ as rings. **TRUE**
- 121) $\mathbb{Z}_{10} \times \mathbb{Z}_6 \cong \mathbb{Z}_{60}$ as rings. **FALSE**
- 122) If f and g differ by a unit in $F[x]$, where F is a field, then $(f, g) = 1$. **TRUE**
- 123) If $uf + vg = 4$ in $\mathbb{Q}[x]$, then $f + (g)$ is a unit in $\mathbb{Q}[x]/(g)$. **TRUE**
- 124) In $R[x]$, the product of two monic polynomials can be zero. **FALSE**
- 125) If F is a field, the map $F[x] \rightarrow F$ sending each polynomial to its constant term is a ring homomorphism. **TRUE**
- 126) $x^3 + 2$ is a unit in $\mathbb{Z}_5[x]/(x^4 - x^2)$. **TRUE**
- 127) The quotient ring $\mathbb{R}[x]/(x^3 - x - 6)$ is a field.
- 128) Every ideal is the kernel of some ring homomorphism. **TRUE**
- 129) Any subring of a domain is a domain. **TRUE**
- 130) Any subring of a field is a field. **FALSE**
- 131) $2^3 \equiv 2^7 \pmod{5}$. **TRUE**
- 132) Every integer is congruent to the sum of its digits modulo 11. **FALSE**
- 133) An element of a commutative ring R cannot be both a unit and a zerodivisor. **TRUE**

- 134) A subset of a ring that is also a ring is a subring. **FALSE**
- 135) \mathbb{Z}_n is a domain if and only if it is a field. **TRUE**
- 136) If $ua + vb = n$ for some $a, b, u, v \in \mathbb{Z}$, then $(a, b) = n$. **FALSE**
- 137) If $ua + vb = 1$ for some $a, b, u, v \in \mathbb{Z}$, then $(a, b) = 1$. **TRUE**
- 138) Every element in \mathbb{Z}_{11} is invertible. **FALSE**
- 139) In \mathbb{Z}_{77} , $(a) = (b)$ if and only if $a = b$. **FALSE**
- 140) Every ideal in \mathbb{Z}_{123} is principal. **TRUE**
- 141) In $\mathbb{Z}[x]$, $(a, b) = (\gcd(a, b))$. **FALSE**
- 142) If R and S are domains, then $R \times S$ is a domain. **FALSE**
- 143) In any ring R , $ab = 0$ implies $a = 0$ or $b = 0$. **FALSE**
- 144) In any ring R , we can cancel addition. **TRUE**
- 145) In any ring R , we can cancel multiplication. **FALSE**
- 146) On the set of real numbers, $r \sim s$ if and only if $|r| = |s|$ defines an equivalence relation. **TRUE**
- 147) If a is even and b is odd, (a, b) is even. **FALSE**
- 148) If $a|b$ and $b|c$, then $a|c$. **TRUE**
- 149) If I and J are ideals in a ring R , $I \cup J$ is an ideal in R . **FALSE**
- 150) There is a bijection between ideals in R containing I and ideals in R/I . **TRUE**
- 151) Every prime ideal is maximal. **FALSE**
- 152) If I is a prime ideal, then R/I is a field. **FALSE**
- 153) If R is a field, then R has at most two ideals. **TRUE**
- 154) If p is a prime and a is any integer, $a^{p-1} \equiv 1 \pmod{p}$. **FALSE**
- 155) If p is a prime and a is any integer, $a^p \equiv a \pmod{p}$. **TRUE**