Problem Set 5

Problem 1. Let R be a domain and Q be its fraction field. Let $T(-)$ denote the torsion functor we introduced in Problem Set 3.

a) Show that $T(M) = \operatorname{Tor}_1^R(M, Q/R)$.[1]

b) Show that for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of R-modules gives rise to an exact sequence[2]

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow (Q/R) \otimes_R A \longrightarrow (Q/R) \otimes_R B \longrightarrow (Q/R) \otimes_R C \longrightarrow 0.$$

c) Show that the right derived functors of T are $R^1T = (Q/R) \otimes_R -$ and $R^iT = 0$ for all $i \leqslant 2$.

Problem 2. Let I be an ideal in R. Show that

$$\operatorname{Ext}_R^n(I, M) \cong \operatorname{Ext}_R^{n+1}(R/I, M)$$

for all $n \geqslant 1$ and all R-modules M.

Problem 3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $r \in R$ and M and N be finitely generated R-modules.

a) Show that the map $\operatorname{Ext}_R^i(M, N) \to \operatorname{Ext}_R^i(M, N)$ induced by $M \xrightarrow{r} M$ is the map given by multiplication by r.

b) Show that if r is regular on M and $\operatorname{Ext}_R^i(M/rM, N) = 0$ for $i \gg 0$, then $\operatorname{Ext}_R^i(M, N) = 0$ for $i \gg 0$.

Problem 4. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

a) Show that for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules,

$$\operatorname{depth}(A) \geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(C) + 1\}.$$

b) Given any finitely generated R-module M over a Cohen-Macaulay ring R, show that there exists $n \geqslant 1$ such that either $\operatorname{pdim}(M) < n$ or $\operatorname{depth}(\Omega_n M) = \dim(R)$.

Problem 5. Here are two fun but unrelated problems about regular rings.

a) Show that every principal ideal domain is a regular ring.

b) Solve the Localization Problem that baffled mathematicians for decades: if R is a regular local ring, then R_P is a regular local ring for every prime P.

[1] Hint: you want to look at some long exact sequence for Tor.
[2] Hint: apply the Snake Lemma to some nice diagram.

Problem 6. Let R be a ring, M an R-module.

a) Show that M is injective if and only if $\operatorname{Ext}^1_R(R/I, M) = 0$ for every ideal I.

b) Let E be any injective resolution of M, $C_0 := \operatorname{coker}(M \longrightarrow E^0)$, and $C_n := \operatorname{coker}(E^{n-1} \longrightarrow E^n)$. Show that for all $i \geqslant 2$ and every R-module N,

$$\operatorname{Ext}^i_R(N, M) \cong \operatorname{Ext}^1_R(N, C_{i-2}).$$

c) Show that $\operatorname{injdim}_R(M) \leqslant n$ if and only if $\operatorname{Ext}^{n+1}_R(R/I, M) = 0$ for every ideal I.

Problem 7. Consider the ring $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$ and the 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

Let P be the ideal in $S = \mathbb{Q}[x, y, z]$ defining the curve $\{(t^{13}, t^{42}, t^{73}) \mid t \in \mathbb{Q}\}$.

To solve this problem, you are not allowed to use any additional Macaulay2 packages besides the `Complexes` package and the ones that are automatically loaded with Macaulay2.

a) Find $\operatorname{pdim}_S(S/P)$ and $\operatorname{depth}(S/P)$.

b) Is P generated by a regular sequence?

c) Find $\operatorname{pdim}_R(M)$ and $\operatorname{depth}(M)$.

d) Is R a regular ring? Is it Cohen-Macaulay?

A complex C in $\operatorname{Ch}(R)$ is **split** if there are R-module homomorphisms $s_n\colon C_n \longrightarrow C_{n+1}$ such that the differential ∂ satisfies $\partial = \partial s \partial$. A complex is **split exact** if it is both exact and split.

Problem 8. Let R be a ring. Our goal is to find the projective objects in $\operatorname{Ch}(R)$.

a) Show that if C is a split complex, then the short exact sequence

$$0 \longrightarrow Z_n(C) \longrightarrow C_n \xrightarrow{d_n} B_{n-1}(C) \longrightarrow 0$$

must split.

b) If C is a split exact complex, show that C is the direct sum of complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow B_n(C) \xrightarrow{=} B_n(C) \longrightarrow 0 \longrightarrow \cdots$$

c) Show that every complex of the form

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{=} P \longrightarrow 0 \longrightarrow \cdots$$

with P projective is a projective object in $\operatorname{Ch}(R)$.

d) Show that a complex P is a projective object in $\operatorname{Ch}(R)$ if and only if P is a split exact complex of projectives.[3]

[3] When P is projective, consider the short exact sequence that comes with the cone of 1_P.