Applying homological algebra to a problem on symbolic powers

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Background

Symbolic Power

The n-th **symbolic power** of a radical ideal I in a regular ring R is

$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(R/I)} (I^n R_P \cap R).$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.

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(3) If I is generated by a regular sequence, then $I^n = I^{(n)}$ for all n.

(4) In general, $I^n \neq I^{(n)}$.

Containment Problem (Schenzel)

When is $I^{(b)} \subseteq I^a$?

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Does the question make sense?

For every a there exists a b such that $I^{(b)} \subseteq I^a$ if and only if the I-adic and I-symbolic topologies are equivalent.

Theorem (Swanson, 2000)

Let I be a radical ideal in a noetherian local ring. If the I-adic and I-symbolic topologies are equivalent, there exists a constant k such that $I^{(kn)} \subseteq I^n$ for all n.

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a regular ring R and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

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Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

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$$\text{In fact, } P^{(3)} \subseteq P^2.$$

Question (Huneke, 2000)

Let P be a height 2 prime in a regular local ring. Is $P^{(3)} \subseteq P^2$?

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Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Theorem (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]} \subseteq I^q.$$

Notation: $I^{[q]} = (f^q \mid f \in I)$.

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Harbourne's Conjecture

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Dumnicki, Szemberg, Tutaj-Gasińska, 2015

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \not\subseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

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When does Harbourne's Conjecture hold?

- For general points in $\mathbb{P}^2$ (Harbourne–Huneke), $\mathbb{P}^3$ (Dumnicki).
- If R/I is an F-pure ring (G–Huneke).
  Eg, when I is a squarefree monomial ideal, or when R/I is a determinantal ring or a Veronese ring.

An homological question

Huneke's Question

If P is a prime of height 2 in a regular local ring, is $P^{(3)} \subseteq P^2$?

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If P is a prime of height 2 in a $k[\![x, y, z]\!]$, is $P^{(3)} \subseteq P^2$?

Theorem (–)

Let k be a field of characteristic not 3, let a, b and c be integers, and let P be the defining ideal of $k[\![t^a, t^b, t^c]\!]$. Then

$$P^{(3)} \subseteq P^2.$$

Monomial space curves

Let k be a field. The kernel of the map

$$k[\![x,y,z]\!] \longrightarrow k[\![t^a,t^b,t^c]\!] \subseteq k[\![t]\!]$$

is a prime ideal of height 2, generated by the maximal minors of

$$\begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}.$$

And the ideals are...

We want to study the height 2 ideals $I = I_2(M) \subseteq R = k[\![x, y, z]\!]$ generated by the 2×2 minors of

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

We write

$$I = \left(\underbrace{a_2 b_3 - a_3 b_2}_{f_1}, \underbrace{a_3 b_1 - a_1 b_3}_{f_2}, \underbrace{a_1 b_2 - a_2 b_1}_{f_3} \right)$$

Fermat configurations

The ideal

$$I = \big(x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3)\big) = I_2 \begin{pmatrix} x^2 & y^2 & z^2 \\ yz & xz & xy \end{pmatrix}$$

has $I^{(3)} \not\subseteq I^2$ in any characteristic except 2.

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Alexandra Seceleanu found conditions that imply $I^{(3)} \not\subseteq I^2$. We are going to follow her strategy to find conditions that imply $I^{(a)} \subseteq I^b$.

We know the symbolic powers of our ideals!

For all $n \geqslant 1$, $I^{(n)} = (I^n : \mathfrak{m}^\infty) = \bigcup_{k \geqslant 1} \left(I^n : \mathfrak{m}^k\right)$. So

$$\mathrm{H}^0_{\mathfrak{m}}(R/I^n) = I^{(n)}/I^n.$$

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An homological criterion

For $a \geqslant b$, consider $I^a \subseteq I^b$ and $R/I^a \twoheadrightarrow R/I^b$. TFAE:

- $I^{(a)} \subseteq I^b$.
- $\mathrm{H}^0_{\mathfrak{m}}(R/I^a) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(R/I^b)$ vanishes.
- $\mathrm{Ext}^3_R(R/I^b, R) \longrightarrow \mathrm{Ext}^3_R(R/I^a, R)$ vanishes.
- $\mathrm{Ext}^2_R(I^b, R) \longrightarrow \mathrm{Ext}^2_R(I^a, R)$ vanishes.

An homological criterion

$I^{(a)} \subseteq I^b$ if and only if $\operatorname{Ext}^2_R(I^b, R) \longrightarrow \operatorname{Ext}^2_R(I^a, R)$ vanishes.

One possible approach

$$\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & I^b \arrow[r] & 0 \\
0 \arrow[r] & G_2 \arrow[r] \arrow[u, dashed] & G_1 \arrow[r] \arrow[u, dashed] & G_0 \arrow[r] \arrow[u, dashed] & I^a \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Rees algebra

The Rees algebra of I is the graded algebra

$$\mathcal{R}(I) = \bigoplus I^n t^n \subseteq R[t].$$

There is a graded map

$$\begin{aligned} R[T_1, T_2, T_3] &\twoheadrightarrow \mathcal{R}(I) \\ T_i &\longmapsto f_i t \end{aligned}$$

but determining what the kernel of this map is can be a very difficult task. Thankfully, things are easy in our setting.

When

$$I = I_2 \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

we have

$$\mathcal{R}(I) \cong R[T_1, T_2, T_3]/(a_1 T_1 + a_2 T_2 + a_3 T_3, b_1 T_1 + b_2 T_2 + b_3 T_3).$$

This is a complete intersection, so the Koszul complex is a resolution of $\mathcal{R}(I)$ over $R[T_1, T_2, T_3]$. The strand in degree n gives a resolution of I^n.

$$0 \longrightarrow R^{\binom{n}{2}} \longrightarrow R^{\binom{n+1}{2}} \bigoplus R^{\binom{n+1}{2}} \longrightarrow R^{\binom{n+2}{2}} \longrightarrow I^n \longrightarrow 0$$

$$0 \longrightarrow R^{\binom{n}{2}} \longrightarrow R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} \longrightarrow R^{\binom{n+2}{2}} \longrightarrow I^n \longrightarrow 0$$

The Euler operator

The differential $D = f_1 \frac{\partial}{\partial T_1} + f_2 \frac{\partial}{\partial T_2} + f_3 \frac{\partial}{\partial T_3}$ on $R[T_1, T_2, T_3]$ induces the map $n\iota : I^n t^n \to I^{n-1} t^{n-1}$, where ι is the map induced by the inclusion $I^n \subseteq I^{n-1}$.

$$\begin{tikzcd}
0 \arrow[r] & R^{\binom{n-1}{2}} \arrow[r] & R^{\binom{n}{2}} \oplus R^{\binom{n}{2}} \arrow[r] & R^{\binom{n+1}{2}} \arrow[r] & I^{n-1} \arrow[r] & 0 \\
0 \arrow[r] & R^{\binom{n}{2}} \arrow[r] \arrow[u, "D_{n-2}"] & R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} \arrow[r] \arrow[u, "D_{n-1}"'] & R^{\binom{n+2}{2}} \arrow[r] \arrow[u, "D_n"'] & I^n \arrow[r] \arrow[u, "n\,\iota"'] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 & R^{\binom{b}{2}} \arrow[l] \arrow[d, "C"'] & R^{\binom{b+1}{2}} \oplus R^{\binom{b+1}{2}} \arrow[l] \arrow[d] & F_0 \arrow[l] \arrow[d] & I^b \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & R^{\binom{a}{2}} \arrow[l] & R^{\binom{a+1}{2}} \oplus R^{\binom{a+1}{2}} \arrow[l, "E"'] & R^{\binom{a+2}{2}} \arrow[l] & I^a \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$I^{(a)} \subseteq I^b$ if and only if all the columns of C are in the image of E.

We need to solve an explicit linear algebra question.

Theorem (Seceleanu)

The containment $I^{(3)} \subseteq I^2$ is equivalent to

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \operatorname{im} \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}.$$

$$I = I_2 \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Lemma (-)

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \operatorname{im} E \Leftrightarrow \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{im} E \Leftrightarrow \begin{pmatrix} 0 \\ f_2 \\ 0 \end{pmatrix} \in \operatorname{im} E \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ f_3 \end{pmatrix} \in \operatorname{im} E.$$

Theorem (G–Huneke–Mukundan)

Let k be a field of characteristic not 3, and $I \subseteq k[\![x, y, z]\!]$ be the height 2 ideal generated by the maximal minors of

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

If $I_1(M)$ is generated by 5 or less elements, then $I^{(3)} \subseteq I^2$.

Fermat configurations

For $R = k[x, y, z]$, and

$$\begin{aligned} I &= \big(x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3)\big) \\ &= I_2 \begin{pmatrix} x^2 & y^2 & z^2 \\ yz & xz & xy \end{pmatrix} \end{aligned}$$

Fun fact: if we switch the order of the entries, we get an ideal I with $I^{(3)} \subseteq I^2$.

Theorem (–)

Let k be a field of characteristic not 2, 3 or 5, let a, b and c be integers, and let P be the defining ideal of $k[\![t^a, t^b, t^c]\!]$. Then

$$P^{(3)} \subseteq P^2 \text{ and } P^{(5)} \subseteq P^3.$$

Theorem (–)

Let k be a field of characteristic not 2 nor 3, $a \leqslant b \leqslant c$ integers, $a = 3$ or 4, and let P be the defining ideal of $k[\![t^a, t^b, t^c]\!]$. Then

$$P^{(4)} \subseteq P^3.$$

As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The defining ideal P of $k[t^9, t^{11}, t^{14}]$ has $P^{(4)} \not\subseteq P^3$, but according to Macaulay2 computations,

$$P^{(2\times 4-2=6)} \subseteq P^{4},$$

so $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Obrigada!