Problems for the midterm

In the midterm, there will be some new problems you have never seen before, together with at least one problem from the problem sets, and at least one problem from the list below.

Problem 1 (January 2021)**.** Let R be a domain and M an R-module. We say that a subset S of M is **maximally linearly independent** if it is linearly independent and every subset T of M properly containing S is not linearly independent. Recall that we say a module M is **torsion** if for every $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$.

a) Let S be a linearly independent set of M and let N be the submodule generated by S. Show that S is maximally linearly independent if and only if M/N is torsion.

b) Suppose that for every module M, every maximally linearly independent set of M generates M. Show that R must be a field.

Problem 2 (May 2019)**.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z}(7, 11)$. Show that M is torsion-free.

Problem 3 (January 2016)**.** Let I be an ideal in a commutative ring R, and let $f\colon M \to N$ be an R-module homomorphism.

a) Let $p\colon M \to M/IM$ and $q\colon N \to N/IN$ be the canonical projection maps. Prove that there is a unique R-module homomorphism $\overline{f} : M/IM \to N/IN$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads, "p"'] & N \arrow[d, two heads, "q"] \\ M/IM \arrow[r, "\overline{f}"'] & N/IN. \end{tikzcd}$$

b) Recall that I^2 is the ideal generated by all elements of the form ab where $a, b \in I$. Show that if $I^2 = 0$ and $\overline{f}$ is surjective, then f is surjective.

Problem 4 (June 2008)**.** Let R be a commutative ring and M a nonzero R-module. An R-submodule N of M is called **maximal** if $N \neq M$ and there are no proper R-submodules of M properly containing N.

a) Show that finitely generated R-module M has a maximal submodule.

b) Show that if N is a maximal submodule of N, then $M/N \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R.