Symbolic powers of ideals defining F-pure rings

joint work with Craig Huneke

Michigan Commutative Algebra Seminar

January 12, 2016

<u>Setup</u> R RLR, containing a field

$I \subseteq R$ radical ideal

$h =$ max height of an associated prime of I

<u>Symbolic Powers of I</u>:

P prime $\rightsquigarrow P^{(n)} = P^n R_P \cap R = \{ f \in R : fs \in P^n, \text{ for some } s \notin P \}$

$$I^{(n)} = \bigcap_{P \in \operatorname{Ass}(R/I)} \left(I^n R_P \cap R \right)$$

<u>Note</u>: $I^n \subseteq I^{(n)}$. In general, $I^{(n)} \not\subseteq I^n$.

<u>Example</u> (Macaulay) $P = (\underbrace{xz - y^2}_{f}, \underbrace{x^3 - yz}_{g}, \underbrace{x^2y - z^2}_{h}) \subseteq k[x,y,z]$

Can show that $fh - g^2 = xq \Rightarrow q \in P^{(2)}$, but $q \notin P^2$ for degree reasons

P is a prime, ht $P = 2$ $\rightsquigarrow P = \operatorname{Ker}(R \to k[t], x \mapsto t^3, y \mapsto t^4, z \mapsto t^5)$

Example not in polynomial ring: $p = (x,y)$ in $R = \dfrac{k[x,y,z]}{(x^n - yz)}$: $y \notin p^2$, $y \in p^{(n)}$

<u>Check</u>: $yz = x^n \in p^n$, $z \notin p \Rightarrow y \in p^{(n)}$

<u>Question</u> When is $I^{(m)} \subseteq I^n$?

<u>Theorem</u> (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002)

$$I^{(hn)} \subseteq I^n.$$

<u>Comments</u>: ELS first proved the theorem in char 0

HH used char $p>0$ techniques (tight closure), reduction to char p.

this does not completely answer the question.

<u>Example</u> (*) $h=2 \rightsquigarrow P^{(4)} \subseteq P^2$ by the theorem

But actually $P^{(3)} \subseteq P^2$

<u>Conjecture</u> (Harbourne, ≤ 2008) $I^{(hn-h+1)} \subseteq I^n$

<u>Note</u>: When $h=2$, the Conjecture says $I^{(2n-1)} \subseteq I^n$

<u>Note</u> there are examples of star configurations of points with $I^{(hn-h)} \not\subseteq I^n$.

<u>Facts</u>: the conjecture holds for:

- I in char 2 (Huneke)
- Monomial ideals in any characteristic
- General points in $\mathbb{P}^2$ (Harbourne) and $\mathbb{P}^3$ (Dumnicki)

- there exist ideals of points in $k[x,y,z]$ with $h=2$, $I^{(3)} \not\subseteq I^2$.
(first example found by Dumnicki, Szemberg, Tutaj-Gasińska)

However, there are good reasons to believe the conjecture holds for nice classes of ideals in char $p>0$:

<u>theorem</u> (HH, 2007) R/I F-pure, $h=2 \Rightarrow I^{(3)} \subseteq I^2$

<u>theorem</u> (Takagi–Yoshida, 2007, Hochster–Huneke, 2007)

$$R/I \text{ F-pure} \Rightarrow I^{(hn-1)} \subseteq I^n$$

In particular, Harbourne's Conjecture holds when $h=2$.

So what is an F-pure ring?

R of char $p>0$, reduced (no nilpotents), F-finite

R is <u>F-pure</u> if $M \otimes R \xrightarrow{1 \otimes F} M \otimes R$ is injective $\forall$ R-mod M

<u>Fedder's</u> <u>Criterion</u> (83) (R,m) RLR, char $p>0$, $I \subseteq R$ ideal.

$$R/I \text{ F-pure} \Leftrightarrow I^{[q]} : I \not\subseteq m^{[q]} \quad \forall q=p^e \gg 0$$

theorem (–, Huneke) R/I F-pure $\Rightarrow \forall n \geq 1$ $I^{(hn-h+1)} \subseteq I^n$.

<u>Idea of proof</u>: show that $(I^{[q]} : I) \subseteq (I^n : I^{(hn-h+1)})^{[q]}$, $q \gg 0$.

this uses several technical lemmas by Hochster–Huneke.

<u>Remark</u>: Stanley–Reisner rings (over a field of char $p>0$) are F-pure, so we recover the result for (squarefree) monomial ideals.

<u>Remark</u> Is this containment

But the proof is.

Can we specialize R/I more and get tighter containments

R of char $p>0$, reduced (no nilpotents), F-finite

R is <u>F-pure</u> if $M \otimes R \xrightarrow{1 \otimes F} M \otimes R$ is injective $\forall$ R-mod M

R is strongly F-regular if for c not in a minimal prime of R

$\exists\, \phi : R^{1/q} \longrightarrow R$ $\phi(c^{1/q}) = 1$ for all/some/large enough $q = p^e$.

Glassbrenner's Criterion (96)

(R,m) RLR, char $p>0$, I radical ideal.

R/I strongly F-regular $\Rightarrow c(I^{[q]}:I) \not\subseteq m^{[q]}$ $\forall q \gg 0$, $\forall c \notin$ min primes I

theorem (–, Huneke) R/I strongly F-regular $\Rightarrow I^{(hn-n+1)} \subseteq I^{n+1}$

or $I^{((h-1)n+1)} \subseteq I^{n+1}$ for all n.

Corollary P prime of height 2, R/P strongly F-regular

then $I^{(n)} = I^n$ $\forall n \geq 1$

Example (Singh) $I = I_2\left(\begin{bmatrix} a^2 & b & d \\ c & a^2 & b^m - d \end{bmatrix}\right) \subseteq k[a,b,c,d] = R$

R/I strongly F-regular $\Rightarrow$ $I^{(n)} = I^n$ $\forall n \geq 2$.

Proof of theorem (sketch)

Claim $(I^{d} : I^{(d)})(I^{[q]} : I) \subseteq (I\, I^{(d-h+1)} : I^{(d)})^{[q]}$ $\forall q$.
$\forall d \geq h-1$.

If the claim holds, take $t \in (I^{d} : I^{(d)})$ not in any minimal prime of I. then $t(I^{[q]} : I) \subseteq (I\, I^{(d-h+1)} : I^{(d)})^{[q]}$

Use the criterion!

then use induction.