

# Introduction to Modern Algebra II

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# Contents

<b>1</b>	<b>Modules</b>	<b>3</b>
1.1	Basic assumptions . . . . .	3
1.2	Modules: definition and examples . . . . .	5
1.3	Submodules and restriction of scalars . . . . .	8
1.4	Module homomorphisms and isomorphisms . . . . .	10
1.5	Module generators, bases and free modules . . . . .	17
<b>2</b>	<b>Vector spaces and linear transformations</b>	<b>22</b>
2.1	Classification of vector spaces and dimension . . . . .	22
2.2	Linear transformations and homomorphisms between free modules . . . . .	28
2.3	Change of basis . . . . .	31
2.4	A warning on the differences between vector spaces and general free modules	33
<b>3</b>	<b>Finitely generated modules over PIDs</b>	<b>34</b>
3.1	Every module is a quotient of a free module . . . . .	34
3.2	Presentations for finitely generated modules over noetherian rings . . . . .	36
3.3	Classification of finitely generated modules over PIDs . . . . .	42
<b>4</b>	<b>Canonical forms for endomorphisms</b>	<b>50</b>
4.1	Rational canonical form . . . . .	50
4.2	The Cayley-Hamilton Theorem . . . . .	53
4.3	Jordan canonical form . . . . .	57
<b>5</b>	<b>Field Extensions</b>	<b>61</b>
5.1	Definition and first properties . . . . .	62
5.2	Algebraic and transcendental extensions . . . . .	67
5.3	Algebraically closed fields and algebraic closure . . . . .	71
5.4	Splitting fields . . . . .	75
5.5	Separability . . . . .	79
<b>6</b>	<b>Galois theory</b>	<b>83</b>
6.1	Group actions on field extensions . . . . .	84
6.2	Automorphism groups of finite field extensions . . . . .	88
6.3	The Fundamental Theorem of Galois Theory . . . . .	90
6.4	Solvable polynomials and solvable groups . . . . .	97

6.5	The primitive element theorem . . . . .	104
6.6	The proof of Artin's Theorem . . . . .	106
<b>Index</b>		<b>109</b>

# Chapter 1

## Modules

Modules are a generalization of the concept of a vector space to any ring of scalars. But while vector spaces make for a great first example of modules, many of the basic facts we are used to from linear algebra are often a little more subtle over a general ring. These differences are features, not bugs. We will introduce modules, study some general linear algebra, and discuss the differences that make the general theory of modules richer and even more fun.

### 1.1 Basic assumptions

In this class, all rings have a multiplicative identity, written as 1 or  $1_R$  if we want to emphasize that we are referring to the ring  $R$ . This is what some authors call *unital rings*; since for us all rings are unital, we will omit the adjective. Moreover, we will think of 1 as part of the structure of the ring, and thus require it be preserved by all natural constructions. As such, a subring  $S$  of  $R$  must share the same multiplicative identity with  $R$ , meaning  $1_R = 1_S$ . Moreover, any ring homomorphism must preserve the multiplicative identity. To clear any possible confusion, we include below the relevant definitions.

**Definition 1.1.** A **ring** is a set  $R$  equipped with two binary operations,  $+$  and  $\cdot$ , satisfying:

- (1)  $(R, +)$  is an abelian group with identity element denoted 0 or  $0_R$ .
- (2) The operation  $\cdot$  is associative, so that  $(R, \cdot)$  is a semigroup.
- (3) For all  $a, b, c \in R$ , we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

- (4) there is a multiplicative identity, written as 1 or  $1_R$ , such that  $1 \cdot a = a = a \cdot 1$  for all  $a \in R$ .

To simplify notation, we will often drop the  $\cdot$  when writing the multiplication of two elements, so that  $ab$  will mean  $a \cdot b$ .

**Definition 1.2.** A ring  $R$  is a **commutative ring** if for all  $a, b \in R$  we have  $a \cdot b = b \cdot a$ .

**Definition 1.3.** A ring  $R$  is a **division ring** if  $1 \neq 0$  and  $R \setminus \{0\}$  is a group under  $\cdot$ , so every nonzero  $r \in R$  has a multiplicative inverse. A **field** is a commutative division ring.

**Definition 1.4.** A commutative ring  $R$  is a **domain**, sometimes called an **integral domain** if it has no zerodivisors:  $ab = 0 \Rightarrow a = 0$  or  $b = 0$ .

For some familiar examples,  $M_n(R)$  (the set of  $n \times n$  matrices) is a ring with the usual addition and multiplication of matrices,  $\mathbb{Z}$  and  $\mathbb{Z}/n$  are commutative rings,  $\mathbb{C}$  and  $\mathbb{Q}$  are fields, and the real Hamiltonian quaternion ring  $\mathbb{H}$  is a division ring.

**Definition 1.5.** A **ring homomorphism** is a function  $f: R \rightarrow S$  satisfying the following:

- $f(a + b) = f(a) + f(b)$  for all  $a, b \in R$ .
- $f(ab) = f(a)f(b)$  for all  $a, b \in R$ .
- $f(1_R) = 1_S$ .

Under this definition, the map  $f: \mathbb{R} \rightarrow M_2(\mathbb{R})$  sending  $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$  preserves addition and multiplication but not the multiplicative identities, and thus it is not a ring homomorphism.

**Exercise 1.** For any ring  $R$ , there exists a unique homomorphism  $\mathbb{Z} \rightarrow R$ .

**Definition 1.6.** A subset  $S$  of a ring  $R$  is a **subring** of  $R$  if it is a ring under the same addition and multiplication operations and  $1_R = 1_S$ .

So under this definition,  $2\mathbb{Z}$ , the set of even integers, is not a subring of  $\mathbb{Z}$ ; in fact, it is not even a ring, since it does not have a multiplicative identity!

**Definition 1.7.** Let  $R$  be a ring. A subset  $I$  of  $R$  is an **ideal** if:

- $I$  is nonempty.
- $(I, +)$  is a subgroup of  $(R, +)$ .
- For every  $a \in I$  and every  $r \in R$ , we have  $ra \in I$  and  $ar \in I$ .

The final property is often called **absorption**. A **left ideal** satisfies only absorption on the left, meaning that we require only that  $ra \in I$  for all  $r \in R$  and  $a \in I$ . Similarly, a **right ideal** satisfies only absorption on the right, meaning that  $ar \in I$  for all  $r \in R$  and  $a \in I$ .

When  $R$  is a commutative ring, the left ideals, right ideals, and ideals over  $R$  are all the same. However, if  $R$  is not commutative, then these can be very different classes.

One key distinction between unital rings and nonunital rings is that if one requires every ring to have a 1, as we do, then the ideals and subrings of a ring  $R$  are very different creatures. In fact, the *only* subring of  $R$  that is also an ideal is  $R$  itself. The change lies in what constitutes a subring; notice that nothing has changed in the definition of ideal.

**Remark 1.8.** Every ring  $R$  has two **trivial ideals**:  $R$  itself and the zero ideal  $(0) = \{0\}$ .

A **nontrivial ideal**  $I$  of  $R$  is an ideal that  $I \neq R$  and  $I \neq (0)$ . An ideal  $I$  of  $R$  is a **proper ideal** if  $I \neq R$ .

## 1.2 Modules: definition and examples

**Definition 1.9.** Let  $R$  be a ring with  $1 \neq 0$ . A **left  $R$ -module** is an abelian group  $(M, +)$  together with an action  $R \times M \rightarrow M$  of  $R$  on  $M$ , written as  $(r, m) \mapsto rm$ , such that for all  $r, s \in R$  and  $m, n \in M$  we have the following:

- $(r + s)m = rm + sm$ ,
- $(rs)m = r(sm)$ ,
- $r(m + n) = rm + rn$ , and
- $1m = m$ .

A **right  $R$ -module** is an abelian group  $(M, +)$  together with an action of  $R$  on  $M$ , written as  $M \times R \rightarrow M$ ,  $(m, r) \mapsto mr$ , such that for all  $r, s \in R$  and  $m, n \in M$  we have

- $m(r + s) = mr + ms$ ,
- $m(rs) = (mr)s$ ,
- $(m + n)r = mr + nr$ , and
- $m1 = m$ .

By default, we will be studying left  $R$ -modules. To make the writing less heavy, we will sometimes say  **$R$ -module** rather than left  $R$ -module whenever there is no ambiguity.

**Remark 1.10.** If  $R$  is a commutative ring, then any left  $R$ -module  $M$  may be regarded as a right  $R$ -module by setting  $mr := rm$ . Likewise, any right  $R$ -module may be regarded as a left  $R$ -module. Thus for commutative rings, we just refer to modules, and not left or right modules.

**Lemma 1.11** (Arithmetic in modules). *Let  $R$  be a ring with  $1_R \neq 0_R$  and  $M$  be an  $R$ -module. Then  $0_R m = 0_M$  and  $(-1_R)m = -m$  for all  $m \in M$ .*

*Proof.* Let  $m \in M$ . Then

$$0_R m = (0_R + 0_R)m = 0_R m + 0_R m.$$

Since  $M$  is an abelian group, the element  $0_R m$  has an additive inverse,  $-0_R m$ , so adding it on both sides we see that

$$0_M = 0_R m.$$

Moreover,

$$m + (-1_R)m = 1_R m + (-1_R)m = (1_R - 1_R)m = 0_R m = 0_M,$$

so  $(-1_R)m = -m$ . □

Typically, one first encounters modules in an undergraduate linear algebra course: the vector spaces from linear algebra are modules over fields. Later we will see that vector spaces are much simpler modules than modules over other rings. So while one might take linear algebra and vector spaces as an inspiration for what to expect from a module, be warned that this perspective can often be deceiving.

**Definition 1.12.** Let  $F$  be a field. A **vector space** over  $F$  is an  $F$ -module.

We will see more about vector spaces soon. Note that many of the concepts we will introduce have special names in the case of vector spaces. Here are some other important examples:

**Lemma 1.13.** Let  $M$  be a set with a binary operation  $+$ . Then

- (1)  $M$  is an abelian group if and only if  $M$  is a  $\mathbb{Z}$ -module.
- (2)  $M$  is an abelian group such that  $nm := \underbrace{m + \cdots + m}_{n \text{ times}} = 0_M$  for all  $m \in M$  if and only if  $M$  has a  $\mathbb{Z}/n$ -module structure.

*Proof.* First, we show 1). If  $M$  is a  $\mathbb{Z}$ -module, then  $(M, +)$  is an abelian group by definition of module. Conversely, if  $(M, +)$  is an abelian group then there is a unique  $\mathbb{Z}$ -module structure on  $M$  given by the formulas below. The uniqueness of the  $\mathbb{Z}$  action follows from the identities below in which the right hand side is determined only by the abelian group structure of  $M$ . The various identities follow from the axioms of a module:

$$\begin{cases} i \cdot m = (\underbrace{1 + \cdots + 1}_i) \cdot m = \underbrace{1 \cdot m + \cdots + 1 \cdot m}_i = \underbrace{m + \cdots + m}_i & \text{if } i > 0 \\ 0 \cdot m = 0_M \\ i \cdot m = -(-i) \cdot m = -(\underbrace{m + \cdots + m}_{-i}) & \text{if } i < 0. \end{cases}$$

We leave it as an exercise to check that this  $\mathbb{Z}$ -action really satisfies the module axioms.

Now we show 2). If  $M$  is a  $\mathbb{Z}/n$  module, then  $(M, +)$  is an abelian group by definition, and  $nm = \underbrace{m + \cdots + m}_n = \underbrace{[1]_n \cdot m + \cdots + [1]_n \cdot m}_n = [0]_n m = 0_M$ .

Conversely, there is a unique  $\mathbb{Z}/n$ -module structure on  $M$  given by the formulas below, which are analogous to the ones above:

$$\begin{cases} [i]_n \cdot m = (\underbrace{[1]_n + \cdots + [1]_n}_i) \cdot m = \underbrace{[1]_n \cdot m + \cdots + [1]_n \cdot m}_i = \underbrace{m + \cdots + m}_i & \text{if } i > 0 \\ 0 \cdot m = 0_M \\ [i]_n \cdot m = -(-[i]_n) \cdot m = -(\underbrace{m + \cdots + m}_{-i}) & \text{if } i < 0. \end{cases}$$

These formulas are well-defined, meaning they are independent of the choice of representative for  $[i]_n$ , because of the assumption that  $nm = 0_M$ . Again checking that this  $\mathbb{Z}/n$ -action really satisfies the module axioms is left as an exercise.  $\square$

The proposition above says in particular that any group of the form

$$G = \mathbb{Z}^\ell \times \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_m$$

is a  $\mathbb{Z}$ -module, and if  $\ell = 0, m \geq 1$  and  $d_i \mid n$  for  $1 \leq i \leq m$  then  $G$  is also a  $\mathbb{Z}/n$ -module. In particular, the Klein group is a  $\mathbb{Z}/2$ -module.

In contrast to vector spaces, for  $M$  a module over a ring  $R$ , it can happen that  $rm = 0$  for some  $r \in R$  and  $m \in M$  such that  $r \neq 0_R$  and  $m \neq 0_M$ . For example, in the Klein group  $K_4$  viewed as a  $\mathbb{Z}$ -module we have  $2m = 0$  for all  $m \in K_4$ .

**Example 1.14.** (1) The trivial  $R$ -module is  $0 = \{0\}$  with  $r0 = 0$  for any  $r \in R$ .

- (2) If  $R$  is any ring, then  $R$  is a left and right  $R$ -module via the action of  $R$  on itself given by its internal multiplication.
- (3) If  $I$  is a left (respectively, right) ideal of a ring  $R$  then  $I$  is a left (respectively, right)  $R$ -module with respect to the action of  $R$  on  $I$  by internal multiplication.
- (4) If  $R$  is a subring of a ring  $S$ , then  $S$  is an  $R$ -module with respect to the action of  $R$  on  $S$  by internal multiplication in  $S$ .
- (5) If  $R$  is a commutative ring with  $1 \neq 0$ , then  $R[x_1, \dots, x_n]$  is an  $R$ -module for any  $n \geq 1$ . This is a special case of (4).
- (6) If  $R$  is a commutative ring and  $G$  is a group, then  $R[G]$  is an  $R$ -module. This is a special case of (4).
- (7) If  $R$  is a commutative ring, let  $M_n(R)$  denote set of  $n \times n$  matrices with entries in  $R$ . Then  $M_n(R)$  is an  $R$ -module for  $n \geq 1$ , with the  $R$ -action given by multiplying all the entries of the given matrix by the given element of  $R$ .
- (8) The **free module** over  $R$  of rank  $n$  is the set

$$R^n = \left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mid r_i \in R, 1 \leq i \leq n \right\}$$

with componentwise addition and multiplication by elements of  $R$ , as follows:

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r'_1 \\ \vdots \\ r'_n \end{bmatrix} = \begin{bmatrix} r_1 + r'_1 \\ \vdots \\ r_n + r'_n \end{bmatrix} \quad \text{and} \quad r \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} rr_1 \\ \vdots \\ rr_n \end{bmatrix}.$$

We will often write the elements of  $R^n$  as  $n$ -tuples  $(r_1, \dots, r_n)$  instead. Notice that  $R$  is the free  $R$ -module of rank 1.

- (9) More generally, given a collection of  $R$ -modules  $\{A_i\}$ , the abelian group

$$\bigoplus_i A_i = \{(a_i)_i \mid a_i \in A_i, a_i = 0 \text{ for all } i \text{ but finitely many}\}$$

is an  $R$ -module with the  $R$ -action  $r(a_i) := (ra_i)$ .



### 1.3 Submodules and restriction of scalars

**Definition 1.15.** Let  $R$  be a ring and let  $M$  be a left  $R$ -module. An  $R$ -**submodule** of  $M$  is a subgroup  $N$  of  $M$  satisfying  $rn \in N$  for all  $r \in R$  and  $n \in N$ .

The submodules of an  $R$ -module  $M$  are precisely the subsets of  $M$  which are modules in their own right, via the same  $R$ -action as we are considering for  $M$ .

**Exercise 2.** Show that if  $N$  is a submodule of  $M$ , then  $N$  is an  $R$ -module via the restriction of the action of  $R$  on  $M$  to the subset  $N$ .

**Example 1.16.** Every  $R$ -module  $M$  has two **trivial submodules**:  $M$  itself and the **zero module**  $0 = \{0_M\}$ . A submodule  $N$  of  $M$  is **nontrivial** if  $N \neq M$  and  $N \neq 0$ .

**Lemma 1.17** (One-step test for submodules). *Let  $R$  be a ring with  $1 \neq 0$  and let  $M$  be a left  $R$ -module. A nonempty subset  $N$  of  $M$  is an  $R$ -submodule of  $M$  if and only if  $rn + n' \in N$  for all  $r \in R$  and  $n, n' \in N$ .*

*Proof.* The One-step Test for subgroups says that if for all  $n, n' \in N$  we have  $n' - n \in N$ , then  $N$  is a subgroup of  $M$ . By Lemma 1.11, by taking  $r = -1$  we get  $rn + n' = n' - n$ , and by assumption this is an element of  $N$ . Therefore,  $N$  is a subgroup of  $M$ . As a consequence,  $0_M \in N$ . By taking  $n' = 0_M$ , we see that for all  $n \in N$  and all  $r \in R$  we have  $rn = rn + n' \in N$ , and thus we can now conclude that  $N$  is a submodule of  $M$ .  $\square$

**Example 1.18.** Let  $R$  be a ring and let  $M$  be a subset of  $R$ . Then  $M$  is a left (respectively, right)  $R$ -submodule of  $R$  if and only if  $M$  is a left (respectively, right) ideal of  $R$ .

**Exercise 3.** Let  $R$  be a ring and let  $A$  and  $B$  be submodules of an  $R$ -module  $M$ . Then the **sum** of  $A$  and  $B$ ,

$$A + B := \{a + b \mid a \in A, b \in B\},$$

and  $A \cap B$  are both  $R$ -submodules of  $M$ .

**Exercise 4.** Let  $R$  be a commutative ring with  $1 \neq 0$ , let  $I$  be an ideal of  $R$  and let  $M$  be an  $R$ -module. Show that

$$IM := \left\{ \sum_{k=1}^n j_k m_k \mid n \geq 0, j_k \in I, m_k \in M \text{ for } 1 \leq k \leq n \right\}$$

is a submodule of  $M$ .

**Example 1.19.** When  $R$  is a field, the submodules of a vector space  $V$  are precisely the subspaces of  $V$ . When  $R = \mathbb{Z}$ , then the class of  $R$ -modules is simply the class of all abelian groups, by Lemma 1.13. The submodules of a  $\mathbb{Z}$ -module  $M$  coincide with the subgroups of the abelian group  $M$ .

**Definition 1.20.** Let  $R$  be a ring with  $1 \neq 0$  and let  $M$  be an  $R$ -module. Given elements  $m_1, \dots, m_n \in M$ , the **submodule generated by**  $m_1, \dots, m_n$  is the subset of  $M$  given by

$$Rm_1 + \dots + Rm_n := \{r_1 m_1 + \dots + r_n m_n \mid r_1, \dots, r_n \in R\}.$$

**Exercise 5.** Let  $R$  be a ring with  $1 \neq 0$  and  $M$  be an  $R$ -module. Given  $m_1, \dots, m_n \in M$ , the submodule generated by  $m_1, \dots, m_n$  is indeed a submodule of  $M$ . Moreover, this is the smallest submodule of  $M$  that contains  $m_1, \dots, m_n$ , meaning that every submodule of  $M$  containing  $m_1, \dots, m_n$  must also contain  $Rm_1 + \dots + Rm_n$ .

**Definition 1.21.** Let  $R$  be a ring with  $1 \neq 0$ . An  $R$ -module  $M$  is **cyclic** if there exists an element  $m \in M$  such that

$$M = Rm := \{rm \mid r \in R\}.$$

Given an  $R$ -module  $M$ , the ring  $R$  is often referred to as the **ring of scalars**, by analogy to the vector space case. Given an action of a ring of scalars on a module, we can sometimes produce an action of a different ring of scalars on the same set, producing a new module structure.

**Lemma 1.22** (Restriction of scalars). *Let  $\phi: R \rightarrow S$  be a ring homomorphism. Any left  $S$ -module  $M$  may be regarded via **restriction of scalars** as a left  $R$ -module with  $R$ -action defined by  $rm := \phi(r)m$  for any  $m \in M$ . In particular, if  $R$  is a subring of a ring  $S$ , then any left  $S$ -module  $M$  may be regarded via restriction of scalars as a left  $R$ -module with  $R$ -action defined by the action of the elements of  $R$  viewed as elements of  $S$ .*

*Proof.* Let  $r, s \in R$  and  $m, n \in M$ . One checks that the axioms in the definition of a module hold for the given action using properties of ring homomorphisms. For example:

$$(r + s)m = \phi(r + s)m = (\phi(r) + \phi(s))m = \phi(r)m + \phi(s)m = rm + sm.$$

The remaining properties are left as an exercise. □

Note that the second module structure on  $M$  obtained via restriction of scalars is induced by the original module structure, so the two are related. In general, one can give different module structures on the same abelian group over different, possibly unrelated, rings.

**Example 1.23.** If  $I$  is an ideal of a ring  $R$ , applying restriction of scalars along the quotient homomorphism  $q: R \rightarrow R/I$  tells us that any left  $R/I$ -module is also a left  $R$ -module. In particular, applying this to the  $R/I$ -module  $R/I$  makes  $R/I$  a left and right  $R$ -module by restriction of scalars along the quotient homomorphism. Thus the  $R$ -action on  $R/I$  is given by

$$r \cdot (a + I) := ra + I.$$

**Example 1.24.** Given any ring  $R$  there exists a unique ring homomorphism  $\mathbb{Z} \rightarrow R$ , by Exercise 1. Thus any  $R$ -module can be given the structure of a  $\mathbb{Z}$ -module by restriction of scalars along this unique map. Note also that a module over any ring is in particular an abelian group, so we can always regard any  $R$ -module as a  $\mathbb{Z}$ -module by forgetting the  $R$ -action and focusing only on the abelian group structure. These two constructions – the restriction of scalars to  $\mathbb{Z}$  and the *forgetful functor*<sup>1</sup> – actually coincide.

The next example explains why restriction of scalars is called a *restriction*.

**Example 1.25.** Let  $R$  be a subring of  $S$ , and let  $i: R \rightarrow S$  be the inclusion map, which must by definition be a ring homomorphism. Applying restriction of scalars to an  $S$ -module  $M$  via  $i$  is the same as simply *restricting* our scalars to the elements of  $R$ .

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<sup>1</sup>This is a concrete abstract nonsense construction that we will discuss in Homological Algebra next Fall.

## 1.4 Module homomorphisms and isomorphisms

**Definition 1.26.** Given  $R$ -modules  $M$  and  $N$ , an  $R$ -**module homomorphism** from  $M$  to  $N$  is a function  $f: M \rightarrow N$  such that for all  $r \in R$  and  $m, n \in M$  we have

- $f(m + n) = f(m) + f(n)$
- $f(rm) = rf(m)$ .

**Remark 1.27.** The condition  $f(m + n) = f(m) + f(n)$  says that  $f$  is a homomorphism of abelian groups, and the condition  $f(rm) = rf(m)$  says that  $f$  is  $R$ -linear, meaning that it preserves the  $R$ -action. Since  $f$  is a homomorphism of abelian groups, it follows that  $f(0) = 0$  must hold.

**Definition 1.28.** Let  $M$  and  $N$  be vector spaces over a field  $F$ . A **linear transformation** from  $M$  to  $N$  is an  $F$ -module homomorphism  $M \rightarrow N$ .

**Example 1.29.** Let  $R$  be a commutative ring and  $M$  be an  $R$ -module. For each  $r \in R$ , the multiplication map  $\mu_r: M \rightarrow M$  given by  $\mu_r(m) = rm$  is a homomorphism of  $R$ -modules: indeed, by the definition of  $R$ -module we have

$$\mu_r(m + n) = r(m + n) = rm + rn = \mu_r(m) + \mu_r(n),$$

and

$$\mu_r(sm) = r(sm) = (rs)m = (sr)m = s(rm) = s\mu_r(m).$$

**Definition 1.30.** An  $R$ -module homomorphism  $h: M \rightarrow N$  is an  $R$ -**module isomorphism** if there is an  $R$ -module homomorphism  $g: N \rightarrow M$  such that  $h \circ g = \text{id}_N$  and  $g \circ h = \text{id}_M$ . We say  $M$  and  $N$  are **isomorphic**, denoted  $M \cong N$ , if there exists an isomorphism  $M \rightarrow N$ .

To check that an  $R$ -module homomorphism  $f: M \rightarrow N$  is an isomorphism, it is sufficient to check that it is bijective.

**Exercise 6.** Let  $f: M \rightarrow N$  be a homomorphism of  $R$ -modules. Show that if  $f$  is bijective, then its set-theoretic inverse  $f^{-1}: N \rightarrow M$  is an  $R$ -module homomorphism. Therefore, every bijective homomorphism of  $R$ -modules is an isomorphism.

One should think of a module isomorphism as a relabelling of the names of the elements of the module. If two modules are isomorphic, that means that they are *essentially the same*, up to renaming the elements.

**Definition 1.31.** Let  $f: M \rightarrow N$  be a homomorphism of  $R$ -modules. The **kernel** of  $f$  is

$$\ker(f) := \{m \in M \mid f(m) = 0\}.$$

The **image** of  $f$ , denoted  $\text{im}(f)$  or  $f(M)$ , is

$$\text{im}(f) := \{f(m) \mid m \in M\}.$$

**Exercise 7.** Let  $R$  be a ring with  $1 \neq 0$ , let  $M$  be an  $R$ -module, and let  $N$  be an  $R$ -submodule of  $M$ . Then the inclusion map  $i: N \rightarrow M$  is an  $R$ -module homomorphism.

**Exercise 8.** If  $f: M \rightarrow N$  is an  $R$ -module homomorphism, then  $\ker(f)$  is an  $R$ -submodule of  $M$  and  $\text{im}(f)$  is an  $R$ -submodule of  $N$ .

**Definition 1.32.** Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -modules. Then  $\text{Hom}_R(M, N)$  denotes the set of all  $R$ -module homomorphisms from  $M$  to  $N$ , and  $\text{End}_R(M)$  denotes the set  $\text{Hom}_R(M, M)$ . We call  $\text{End}(M)$  the **endomorphism ring** of  $M$ , and elements of  $\text{End}(M)$  are called **endomorphisms** of  $M$ .

The endomorphism ring of an  $R$ -module  $M$  is called that because it *is* a ring, with multiplication given by composition of endomorphisms, 0 given by the zero map (the constant equal to 0), and 1 given by the identity map. However, two homomorphisms from  $M$  to  $N$  are not composable unless  $M = N$ , so  $\text{Hom}_R(M, N)$  is not a ring.

When  $R$  is commutative,  $\text{Hom}_R(M, N)$  is, however, an  $R$ -module; let us describe its  $R$ -module structure. Given  $f, g \in \text{Hom}_R(M, N)$ ,  $f + g$  is the map defined by

$$(f + g)(m) := f(m) + g(m),$$

and given  $r \in R$  and  $f \in \text{Hom}_R(M, N)$ ,  $r \cdot f$  is the  $R$ -module homomorphism defined by

$$(r \cdot f)(m) := r \cdot f(m) = f(rm).$$

The zero element of  $\text{Hom}_R(M, N)$  is the **zero map**, the constant equal to  $0_N$ .

**Lemma 1.33.** *Let  $M$  and  $N$  be  $R$ -modules over a commutative ring  $R$ . Then the addition and multiplication by scalars defined above make  $\text{Hom}_R(M, N)$  an  $R$ -module.*

*Proof.* There are many things to check, including:

- The addition and the  $R$ -action are both well-defined: given  $f, g \in \text{Hom}_R(M, N)$  and  $r \in R$ , we always have  $f + g, rf \in \text{Hom}_R(M, N)$ .
- The axioms of an  $R$ -module are satisfied for  $\text{Hom}_R(M, N)$ .

We leave the details as exercises. □

We will see later that for an  $n$ -dimensional vector space  $V$  over a field  $F$ , there is an isomorphism of vector spaces  $\text{End}_F(V) \cong M_n(F)$ . This says that every linear transformation  $T: V \rightarrow V$  corresponds to some  $n \times n$  matrix. However, the story for general  $R$ -modules is a lot more complicated.

**Lemma 1.34.** *For any commutative ring  $R$  with  $1 \neq 0$  and any  $R$ -module  $M$  there is an isomorphism of  $R$ -modules  $\text{Hom}_R(R, M) \cong M$ .*

Before we write a formal proof, it helps to think about *why* this theorem is true. What does it mean to give an  $R$ -module homomorphism  $f: R \rightarrow M$ ? More precisely, what information do we need to determine such an  $f$ ? Do we need to be given the values of  $f(r)$  for every  $r \in R$ ? Since  $f$  is a homomorphism of  $R$ -modules, for any  $r \in R$  we have

$$f(r) = f(r \cdot 1) = rf(1),$$

so the value of  $f(1)$  *completely determines* which  $R$ -module homomorphism we are talking about. On the other hand, we can choose *any*  $m \in M$  to be the image of 1, since thanks to the axioms for modules, the function

$$f(r) := rm$$

is a well-defined  $R$ -module homomorphism for any  $m \in M$ . In summary, to give an  $R$ -module homomorphism  $R \rightarrow M$  is the same as choosing an element  $m \in M$ , and  $\text{Hom}_R(R, M) \cong M$ .

*Proof.* Let  $f: M \rightarrow \text{Hom}_R(R, M)$  be given for each  $m \in M$  by  $f(m) = \phi_m$  where  $\phi_m$  is the map defined by  $\phi_m(r) = rm$  for all  $r \in R$ . Now we have many things to check:

- $f$  is well-defined, meaning that for any  $m \in M$ , its image  $f(m) = \phi_m$  is an element of  $\text{Hom}_R(R, M)$ , since

$$\phi_m(r_1 + r_2) = (r_1 + r_2)m = r_1m + r_2m = \phi_m(r_1) + \phi_m(r_2)$$

$$\phi_m(r_1r_2) = (r_1r_2)m = r_1(r_2m) = r_1\phi_m(r_2)$$

for all  $r_1, r_2 \in R$ .

- $f$  is an  $R$ -module homomorphism, since

$$\phi_{m_1+m_2}(r) = r(m_1 + m_2) = rm_1 + rm_2 = \phi_{m_1}(r) + \phi_{m_2}(r)$$

$$\phi_{r'm}(r) = r(r'm) = (rr')m = r'(rm) = r'\phi_m(r)$$

- $f$  is injective, since  $\phi_m = \phi_{m'}$  implies in particular that  $\phi_m(1_R) = \phi_{m'}(1_R)$ , which by definition of  $\phi_-$  means that  $m = m'$ .
- $f$  is surjective, since for  $\psi \in \text{Hom}_R(R, M)$  we have  $\psi(r) = \psi(r1_R) = r\psi(1_R)$  for all  $r \in R$ , so  $\psi = \phi_{\psi(1_R)}$ .

This shows that  $f$  is an  $R$ -module isomorphism. □

**Definition 1.35.** Let  $R$  be a commutative ring with  $1_R \neq 0_R$ . An  **$R$ -algebra** is a ring  $A$  with  $1_A \neq 0_A$  together with a ring homomorphism  $f: R \rightarrow A$  such that  $f(R)$  is contained in the center of  $A$ .

Given an  $R$ -algebra  $A$ , the  $R$ -algebra structure on  $A$  induces a natural  $R$ -module structure: given elements  $r \in R$  and  $a \in A$ , the  $R$ -action is defined by

$$r \cdot a := f(r)a,$$

where the product on the right is the multiplication in  $A$ . Similarly, we get a natural right  $R$ -module structure on  $A$ , and since by definition  $f(R)$  is contained in the center of  $A$ , we obtain what is called a *balanced bimodule* structure on  $A$ . We will discuss these further in Homological Algebra next Fall.

**Example 1.36.** Let  $R$  be a commutative ring with  $1_R \neq 0_R$ . The ring  $R[x_1, \dots, x_n]$  together with the inclusion map  $R \hookrightarrow R[x_1, \dots, x_n]$  is an  $R$ -algebra. More generally, any quotient of  $R[x_1, \dots, x_n]$  is an  $R$ -algebra.

The ring of matrices  $M_n(R)$  with the homomorphism  $r \mapsto rI_n$  is also an  $R$ -algebra, as is the group ring  $R[G]$  for any group  $G$  with the inclusion of  $R$  into  $R[G]$  given by  $r \mapsto re_G$ .

**Lemma 1.37.** *Let  $R$  be a commutative ring with  $1 \neq 0$  and let  $M$  be an  $R$ -module. Then  $\text{End}_R(M)$  is an  $R$ -algebra, with addition and  $R$ -action defined as above, and multiplication defined by composition  $(fg)(m) = f(g(m))$  for all  $f, g \in \text{End}_R(M)$  and all  $m \in M$ .*

*Proof.* There are many things to check here, including that:

- The axioms of a (unital) ring are satisfied for  $\text{End}_R(M)$ .
- There is a ring homomorphism  $f: R \rightarrow \text{End}_R(M)$  such that  $f(1_R) = 1_{\text{End}_R(M)} = \text{id}_M$  and  $f(R) \subseteq Z(\text{End}_R(M))$ .

We will just check the last item and leave the others as exercises. Define  $f: R \rightarrow \text{End}_R(M)$  by  $f(r) = r \text{id}_M$ . Notice that this is the map  $\mu_r$  from Example 1.29. Then

$$f(r + s) = (r + s) \text{id}_M = r \text{id}_M + s \text{id}_M = f(r) + f(s)$$

and

$$f(rs) = (rs) \text{id}_M = (r \text{id}_M) \circ (s \text{id}_M) = f(r)f(s)$$

show that  $f$  is a ring homomorphism. Moreover,  $\text{id}_M \in Z(\text{End}_R(M))$ , and one can check easily that  $\mu_r \in \text{End}_R(M)$ : given any other  $g \in \text{End}_R(M)$ , and any  $m \in M$ , since  $g$  is  $R$ -linear we have

$$(g \circ \mu_r)(m) = g(\mu_r(m)) = g(rm) = rg(m) = (\mu_r \circ g)(m).$$

This shows that  $f(R) \subseteq \text{End}_R(M)$ . □

**Remark 1.38.** Let  $R$  be a commutative ring with  $1 \neq 0$  and let  $M$  be an  $R$ -module. Then  $M$  is also an  $\text{End}_R(M)$ -module with the action  $\phi m = \phi(m)$  for any  $\phi \in \text{End}_R(M)$ ,  $m \in M$ .

**Definition 1.39.** Let  $R$  be a ring, let  $M$  be an  $R$ -module, and let  $N$  be a submodule of  $M$ . The quotient module  $M/N$  is the quotient group  $M/N$  with  $R$  action defined by

$$r(m + N) := rm + N$$

for all  $r \in R$  and  $m + N \in M/N$ .

**Lemma 1.40.** *Let  $R$  be a ring, let  $M$  be an  $R$ -module, and let  $N$  be a submodule of  $M$ . The quotient module  $M/N$  is an  $R$ -module, and the quotient map  $q: M \rightarrow M/N$  is an  $R$ -module homomorphism with kernel  $\ker(q) = N$ .*

*Proof.* Among the many things to check here, we will only check the well-definedness of the  $R$ -action on  $M$ , and leave the others as exercises. To check well-definedness, consider  $m + N = m' + N$ . Then  $m - m' \in N$ , so  $r(m - m') \in N$  by the definition of submodule. This gives that  $rm - rm' \in N$ , hence  $rm + N = rm' + N$ . □

**Definition 1.41.** Given an  $R$ -module  $M$  and a submodule  $N$  of  $M$ , the map  $q: M \rightarrow M/N$  is the **canonical quotient map**, or simply the canonical map from  $M$  to  $N$ .

**Example 1.42.** If  $R$  is a field, quotient modules are the same thing as quotient vector spaces. When  $R = \mathbb{Z}$ , recall that  $\mathbb{Z}$ -modules are the same as abelian groups, by Lemma 1.13. Quotients of  $\mathbb{Z}$ -modules coincide with quotients of abelian groups.

**Theorem 1.43.** Let  $N$  be a submodule of  $M$ , let  $T$  be an  $R$ -module, and let  $f: M \rightarrow T$  be an  $R$ -module homomorphism. If  $N \subseteq \ker f$ , then the function

$$\begin{aligned} M/N &\xrightarrow{\bar{f}} T \\ m + N &\mapsto f(m) \end{aligned}$$

is a well-defined  $R$ -module homomorphism. In fact,  $\bar{f}: M/N \rightarrow T$  is the unique  $R$ -module homomorphism such that  $\bar{f} \circ q = f$ , where  $q: M \rightarrow M/N$  denotes the canonical map.

We can represent this in a more visual way by saying that  $\bar{f}$  is the unique  $R$ -module homomorphism that makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & T \\ \downarrow q & \searrow \exists! \bar{f} & \\ M/N & & \end{array}$$

commute.

*Proof.* By 817, we already know that  $\bar{f}$  is a well-defined homomorphism of groups under  $+$  and that it is the unique one such that  $\bar{f} \circ q = f$ . It remains only to show  $\bar{f}$  is an  $R$ -linear map:

$$\bar{f}(r(m + N)) = \bar{f}(rm + N) = f(rm) = rf(m) = r\bar{f}(m + N).$$

where the third equation uses that  $f$  preserves scaling.  $\square$

**Theorem 1.44** (First Isomorphism Theorem). Let  $N$  be an  $R$ -module and let  $h: M \rightarrow N$  be an  $R$ -module homomorphism. Then  $\ker(h)$  is a submodule of  $M$  and there is an  $R$ -module isomorphism  $M/\ker(h) \cong \text{im}(h)$ .

*Proof.* If we forget the multiplication by scalars in  $R$ , by the First Isomorphism Theorem for Groups, we know that there is an isomorphism of abelian groups under  $+$ , given by

$$\begin{aligned} \bar{h}: M/\ker(h) &\xrightarrow{\cong} \text{im}(h) \\ m + \ker(h) &\mapsto h(m). \end{aligned}$$

It remains only to show this map preserves multiplication by scalars. And indeed:

$$\begin{aligned} \bar{h}(r(m + \ker(h))) &= \bar{h}(rm + \ker(h)) && \text{by definition of the } R\text{-action on } M/\ker(h) \\ &= h(rm) && \text{by definition of } \bar{h} \\ &= rh(m) && \text{since } h \text{ is an } R\text{-module homomorphism} \\ &= r\bar{h}(m + \ker(h)) && \text{by definition of } h. \end{aligned}$$

**Theorem 1.45** (Second Isomorphism Theorem). *Let  $A$  and  $B$  be submodules of  $M$ , and let  $A + B = \{a + b \mid a \in A, b \in B\}$ . Then  $A + B$  is a submodule of  $M$ ,  $A \cap B$  is a submodule of  $A$ , and there is an  $R$ -module isomorphism  $(A + B)/B \cong A/(A \cap B)$ .*

*Proof.* By Exercise 3,  $A + B$  and  $A \cap B$  are submodules of  $M$ . By the Second Isomorphism Theorem for Groups, there is an isomorphism of abelian groups

$$\begin{aligned} h: A/(A \cap B) &\cong \rightarrow (A + B)/B \\ a + (A \cap B) &\mapsto a + B \end{aligned}$$

It remains only to show  $h$  preserves multiplication by scalars:

$$h(r(a + (A \cap B))) = h(ra + A \cap B) = ra + B = r(a + B) = rh(a + (A \cap B)). \quad \square$$

**Theorem 1.46** (Third Isomorphism Theorem). *Let  $A$  and  $B$  be submodules of  $M$  with  $A \subseteq B$ . Then there is an  $R$ -module isomorphism  $(M/A)/(B/A) \cong M/B$ .*

*Proof.* From 817, we know that  $B/A$  is a subgroup of  $M/A$  under  $+$ . Given  $r \in R$  and  $b + A \in B/A$  we have  $r(b + A) = rb + A$  which belongs to  $B/A$  since  $rb \in B$ . This proves  $B/A$  is a submodule of  $M/A$ . By the Third Isomorphism Theorem for Groups, there is an isomorphism of abelian groups

$$\begin{aligned} (M/A)/(B/A) &\cong \rightarrow M/B \\ (m + A) + B/A &\mapsto m + B \end{aligned}$$

and it remains only to show this map is  $R$ -linear:

$$\begin{aligned} h(r((m + A) + B/A)) &= h(r(m + A) + B/A) = h((rm + A) + B/A) \\ &= rm + B = r(m + B) \\ &= rh((m + A) + B/A). \end{aligned} \quad \square$$

**Theorem 1.47** (Lattice Isomorphism Theorem). *Let  $R$  be a ring, let  $N$  be a  $R$ -submodule of an  $R$ -module  $M$ , and let  $q: M \rightarrow M/N$  be the quotient map. Then the function*

$$\begin{aligned} \{R\text{-submodules of } M \text{ containing } N\} &\xrightarrow{\Psi} \{R\text{-submodules of } M/N\} \\ K &\mapsto K/N \end{aligned}$$

*is a bijection, with inverse defined by*

$$\Psi^{-1}(T) := q^{-1}(T) = \{a \in M \mid a + N \in T\}$$

*for each  $R$ -submodule  $T$  of  $M/N$ . Moreover,  $\Psi$  and  $\Psi^{-1}$  preserve sums and intersections of submodules.*



*Proof.* From 817, we know there is a bijection between the set of subgroups of  $M$  and that contain  $N$  and subgroups of the quotient group  $M/N$ , given by the same map  $\Psi$ . We just need to prove that these maps send submodules to submodules. If  $K$  is a submodule of  $M$  containing  $N$ , then by the [Third Isomorphism Theorem](#) we know that  $K/N$  is a submodule of  $M/N$ . If  $T$  is a submodule of  $M/N$ , then  $\pi^{-1}(T)$  is an abelian group, by 817. For  $r \in R$  and  $m \in \pi^{-1}(T)$ , we have  $\pi(m) \in T$ , and hence  $\pi(rm) = r\pi(m) \in T$  too, since  $T$  is a submodule. This proves  $\pi^{-1}(T)$  is a submodule.  $\square$

We come to a very important class of examples which will help us study linear transformations using module theory.

**Lemma 1.48** ( $F[x]$ -modules). *Let  $F$  be a field. There is a bijection*

$$\{V \text{ an } F[x]\text{-module}\} \longleftrightarrow \{V \text{ an } F\text{-vector space and } T \in \text{End}_F(V)\}.$$

*Proof.* If  $V$  is an  $F[x]$  module then  $V$  is an  $F$ -vector space by restriction of scalars along the inclusion  $F \hookrightarrow F[x]$ . Let  $T : V \rightarrow V$  be defined by  $T(v) = xv$ . To show that  $T \in \text{End}_F(V)$ , note that for any  $c \in F$  and  $v, v_1, v_2 \in V$  the axioms of the  $F[x]$ -module give us

$$T(v_1 + v_2) = x(v_1 + v_2) = xv_1 + xv_2 = T(v_1) + T(v_2) \text{ and } T(cv) = x(cv) = c(xv).$$

Conversely, let  $V$  be an  $F$ -vector space and  $T \in \text{End}_F(V)$ . We claim that the action of  $F[x]$  on  $V$  given by

$$f(x)v = (f(T))(v)$$

satisfies the axioms for a module (exercise!). Alternatively, we can explain this module structure in a more conceptual way, as follows. Consider the evaluation homomorphism  $\varphi : F[x] \rightarrow \text{End}_F(V)$ ,  $\varphi(f(x)) = f(T)$ . Since  $V$  is an  $\text{End}_F(V)$ -module by Remark 1.38, then  $V$  is also an  $F[x]$ -module by restriction of scalars along  $\phi$ ; the  $F[x]$  action is the one we described above:

$$f(x)v = \varphi(f)(v) = (f(T))(v)$$

Finally, one can check that the two constructions above are inverse to each other.  $\square$

**Notation 1.49.** We shall denote the  $F[x]$ -module structure on an  $F$ -vector space  $V$  induced by  $T \in \text{End}_F(V)$  by  $V_T$ .

**Example 1.50.** The proposition above says that if we fix an  $F$ -vector space  $V$  then any linear transformation  $T$  gives a different  $F[x]$  module structure on  $V$ . For example,

- for  $T = 0$  the  $F[x]$  module  $V_0$  carries an action given by scaling by the constant coefficient of  $f$ , that is if  $f(x) = a^n x^n + \cdots + a_0$  then

$$f(x)v = (f(0))v = a_0 v \text{ for all } f \in F[x].$$

- for  $T$  the “shift operator” that takes  $T(e_i) = e_{i-1}$ , where  $e_i$  is the  $i$ -th standard basis

vector, the  $F[x]$  module  $V_T$  has the action  $x^m$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_{n-m} \\ v_{n-m+1} \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_{m+1} \\ \vdots \\ v_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

## 1.5 Module generators, bases and free modules

**Definition 1.51.** Let  $M$  be an  $R$ -module. A **linear combination** of finitely many elements  $a_1, \dots, a_n$  of  $M$  is an element of  $M$  of the form  $r_1 a_1 + \dots + r_n a_n$  for some  $r_1, \dots, r_n \in R$ .

**Definition 1.52.** Let  $R$  be a ring with  $1 \neq 0$  and let  $M$  be an  $R$ -module. For a subset  $A$  of  $M$ , the submodule of  $M$  **generated by**  $A$  is

$$RA := \{r_1 a_1 + \dots + r_n a_n \mid n \geq 0, r_i \in R, a_i \in A\}.$$

We say  $M$  is **generated by**  $A$  if  $M = RA$ . If  $M$  is an  $F$ -vector space, we say that  $M$  is **spanned** by a set  $A$  instead of generated by  $A$ .

A module  $M$  is **finitely generated** if there is a finite subset  $A$  of  $M$  that generates  $M$ . If  $A = \{a\}$  has a single element, the module  $RA = Ra$  is called *cyclic*.

**Exercise 9.** Let  $M$  be an  $R$ -module and let  $A \subseteq M$ . Then  $RA$  is the smallest submodule of  $M$  containing  $A$ , that is

$$RA = \bigcap_{A \subseteq N, N \text{ submodule of } M} N.$$

**Exercise 10.** Being finitely generated and being cyclic are  $R$ -module isomorphism invariants.

**Example 1.53.** Let  $R$  be a ring with  $1 \neq 0$ .

- (1)  $R = R1$  is cyclic.
- (2)  $R \oplus R$  is generated by  $\{(1, 0), (0, 1)\}$ .
- (3)  $R[x]$  is generated as an  $R$ -module by the set  $\{1, x, x^2, \dots, x^n, \dots\}$  of monic monomials in the variable  $x$ .
- (4) Let  $M = \mathbb{Z}[x, y]$ .  $M$  is generated by
  - $\{1, x, y\}$  as a ring,
  - $\{1, y, y^2, \dots, y^n, \dots\}$  as an  $\mathbb{Z}[x]$ -module, and
  - $\{x^i y^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$  as a group ( $\mathbb{Z}$ -module).

**Lemma 1.54.** Let  $R$  be a ring with  $1 \neq 0$ , let  $M$  be an  $R$ -module, and let  $N$  be an  $R$ -submodule of  $M$ .

- (1) If  $M$  is finitely generated as an  $R$ -module, then so is  $M/N$ .
- (2) If  $N$  and  $M/N$  are finitely generated as  $R$ -modules, then so is  $M$ .

*Proof.* The proof of (2) will be a problem set question. To show (1), note that if  $M = RA$  then  $M/N = R\bar{A}$ , where  $\bar{A} = \{a + N \mid a \in A\}$ .  $\square$

**Definition 1.55.** Let  $M$  be an  $R$ -module and let  $A$  be a subset of  $M$ . The set  $A$  is **linearly independent** if whenever  $r_1, \dots, r_n \in R$  and  $a_1, \dots, a_n$  are distinct elements of  $A$  satisfying  $r_1 a_1 + \dots + r_n a_n = 0$ , then  $r_1 = \dots = r_n = 0$ . Otherwise  $A$  is **linearly dependent**.

**Definition 1.56.** A subset  $A$  of an  $R$ -module  $M$  is a **basis** of  $M$  if  $A$  is linearly independent and generates  $M$ . An  $R$ -module  $M$  is a **free  $R$ -module** if  $M$  has a basis.

We will later see that over a field, every module is free. However, when  $R$  is not a field, there are  $R$ -modules that are not free; in fact, *most* modules are not free.

**Example 1.57.** Here are some examples of free modules:

- (1) If we think of  $R$  as a module over itself, it is free with basis  $\{1\}$ .
- (2) The module  $R \oplus R$  is free with basis  $\{(1, 0), (0, 1)\}$ .
- (3) The  $R$ -module  $R[x]$  is free, and  $\{1, x, x^2, \dots, x^n, \dots\}$  is a basis.
- (4) Let  $M = \mathbb{Z}[x, y]$ . Then  $\{1, y, y^2, \dots, y^n, \dots\}$  is a basis for the  $\mathbb{Z}[x]$ -module  $M$ , and  $\{x^i y^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$  is a basis for the  $\mathbb{Z}$ -module  $M$ .

**Example 1.58.**  $\mathbb{Z}/2$  is not a free  $\mathbb{Z}$ -module. Indeed suppose that  $A$  is a basis for  $\mathbb{Z}/2$  and  $a \in A$ . Then  $2a = 0$  so  $A$  cannot be linearly independent, a contradiction.

**Lemma 1.59.** *If  $A$  is a basis of  $M$  then every nonzero element  $0 \neq m \in M$  can be written uniquely as  $m = r_1 a_1 + \dots + r_n a_n$  with  $a_i$  distinct elements of  $A$  and  $r_i \neq 0$ .*

*Proof.* Suppose that if  $m \neq 0$  and  $A_1, A_2$  are finite subsets of  $A$  such that

$$m = \sum_{a \in A_1} r_a a = \sum_{b \in A_2} s_b b$$

for some  $r_a, s_b \in R$ . Then

$$\sum_{a \in A_1 \cap A_2} (r_a - s_a) a + \sum_{a \in A_1 \setminus A_2} r_a a - \sum_{a \in A_2 \setminus A_1} s_a a = 0.$$

Since  $A$  is a linearly independent set, we conclude that  $r_a = s_a$  for  $a \in A_1 \cap A_2$ ,  $r_a = 0_R$  for  $a \in A_1 \setminus A_2$ , and  $s_a = 0_R$  for  $a \in A_2 \setminus A_1$ . Set

$$B := \{a \in A_1 \cap A_2 \mid r_a \neq 0_R\}.$$

Then

$$m = \sum_{a \in B} r_a a$$

is the unique way of writing  $m$  as a linear combination of elements of  $A$  with nonzero coefficients.  $\square$

**Theorem 1.60.** *Let  $R$  be a ring,  $M$  be a free  $R$ -module with basis  $B$ ,  $N$  be any  $R$ -module, and let  $j : B \rightarrow N$  be any function. Then there is a unique  $R$ -module homomorphism  $h : M \rightarrow N$  such that  $h(b) = j(b)$  for all  $b \in B$ .*

*Proof.* We have two things to prove: existence and uniqueness.

*Existence:* By Lemma 1.59, any  $0 \neq m \in M$  can be written uniquely as

$$m = r_1 b_1 + \cdots + r_n b_n$$

with  $b_i \in B$  distinct and  $0 \neq r_i \in R$ . Define  $h: M \rightarrow N$  by

$$\begin{cases} h(r_1 b_1 + \cdots + r_n b_n) = r_1 j(b_1) + \cdots + r_n j(b_n) & \text{if } r_1 b_1 + \cdots + r_n b_n \neq 0 \\ h(0_M) = 0_N \end{cases}$$

One can check that this satisfies the conditions to be an  $R$ -module homomorphism (exercise!).

*Uniqueness:* Let  $h: M \rightarrow N$  be an  $R$ -module homomorphism such that  $h(b_i) = j(b_i)$ . Then in particular  $h: (M, +) \rightarrow (N, +)$  is a group homomorphism and therefore  $h(0_M) = 0_N$  by properties of group homomorphisms. Furthermore, if  $m = r_1 b_1 + \cdots + r_n b_n$  then

$$h(m) = h(r_1 b_1 + \cdots + r_n b_n) = r_1 h(b_1) + \cdots + r_n h(b_n) = r_1 j(b_1) + \cdots + r_n j(b_n)$$

by the definition of homomorphism, and because  $h(b_i) = j(b_i)$ .  $\square$

**Corollary 1.61.** *If  $A$  and  $B$  are sets of the same cardinality, and fix a bijection  $j: A \rightarrow B$ . If  $M$  and  $N$  are free  $R$ -modules with bases  $A$  and  $B$  respectively, then there is an isomorphism of  $R$ -modules  $M \cong N$ .*

*Proof.* Let  $g: M \rightarrow N$  and  $h: N \rightarrow M$  be the module homomorphisms induced by the bijection  $j: A \rightarrow B$  and its inverse  $j^{-1}: B \rightarrow A$ , which exist by Theorem 1.60. We will show that  $h$  and  $g$  are inverse homomorphisms. First, note that  $g \circ h: N \rightarrow N$  is an  $R$ -module homomorphism and  $(g \circ h)(b) = g(j^{-1}(b)) = j(j^{-1}(b)) = b$  for every  $b \in B$ . Since the identity map  $\text{id}_N$  is an  $R$ -module homomorphism and  $\text{id}_N(b) = b$  for every  $b \in B$ , by the uniqueness in Theorem 1.60 we have  $g \circ h = \text{id}_N$ . Similarly, one shows that  $h \circ g = \text{id}_M$ .  $\square$

The corollary gives that, up to isomorphism, there is only one free module with basis  $A$ , provided such a module exists. But does a free module generated by a given set  $A$  exist? It turns out it does.

**Definition 1.62.** Let  $R$  be a ring and let  $A$  be a set. The free  $R$ -module generated by  $A$ , denoted  $F_R(A)$  is the set of formal sums

$$\begin{aligned} F_R(A) &= \{r_1 a_1 + \cdots + r_n a_n \mid n \geq 0, r_i \in R, a_i \in A\} \\ &= \left\{ \sum_{a \in A} r_a a \mid r_a \in R, r_a = 0 \text{ for all but finitely many } a \right\}, \end{aligned}$$

with addition defined by

$$\left( \sum_{a \in A} r_a a \right) + \left( \sum_{a \in A} s_a a \right) = \sum_{a \in A} (r_a + s_a) a$$

and  $R$ -action defined by

$$r \left( \sum_{a \in A} r_a a \right) = \sum_{a \in A} (r r_a) a.$$

**Exercise 11.** This construction  $F_R(A)$  results in an  $R$ -module, which is free with basis  $A$ , and  $F_R(A) \cong \bigoplus_{a \in A} R$ .

**Theorem 1.63** (Uniqueness of rank over commutative rings). *Let  $R$  be a commutative ring with  $1 \neq 0$  and let  $M$  be a free  $R$ -module. If  $A$  and  $B$  are both bases for  $M$ , then  $A$  and  $B$  have the same cardinality, meaning that there exists a bijection  $A \rightarrow B$ .*

*Proof.* You will show this in the next problem set (at least in the case where  $M$  has a finite basis).  $\square$

**Definition 1.64.** Let  $R$  be a commutative ring with  $1 \neq 0$  and let  $M$  be a free  $R$ -module. The **rank** of  $M$  is the cardinality of any basis of  $M$ .

**Example 1.65.** Let  $R$  be a commutative ring with  $1 \neq 0$ . The rank of  $R^n$  is  $n$ . Note that by Corollary 1.61, any free  $R$ -module of rank  $n$  must be isomorphic to  $R^n$ .

Earlier, we described the  $R$ -module structure on the direct sum of  $R$ -modules; this is how we construct  $R^n$ , by taking the direct sum of  $n$  copies of the  $R$ -module  $R$ . This construction can also be described as the direct product of  $n$  copies of  $R$ . However, the direct sum and direct product are two different constructions.

**Definition 1.66.** Let  $R$  be a ring. Let  $\{M_a\}_{a \in J}$  be a collection of  $R$ -modules. The **direct product** of the  $R$ -modules  $M_a$  is the Cartesian product

$$\prod_{a \in J} M_a := \{(m_a)_{a \in J} \mid m_a \in M_a\}$$

with addition defined by

$$(m_a)_{a \in J} + (n_a)_{a \in J} := (m_a + n_a)_{a \in J}$$

and  $R$ -action defined by

$$r(m_a)_{a \in J} = (rm_a)_{a \in J}.$$

The **direct sum** of the  $R$ -modules  $M_a$  is the  $R$ -submodule  $\bigoplus_{a \in J} M_a$  of the direct product  $\prod_{a \in J} M_a$  given by

$$\bigoplus_{a \in J} M_a = \{(m_a)_{a \in J} \mid m_a = 0 \text{ for all but finitely many } a\}.$$

**Exercise 12.** The direct sum and the direct product of an arbitrary family of  $R$ -modules are  $R$ -modules.

**Example 1.67.** Suppose that  $|A| = n < \infty$ . Let  $M_1, \dots, M_n$  be  $R$ -modules. The direct product module  $M_1 \times \dots \times M_n$  is the abelian group  $M_1 \times \dots \times M_n$  with ring action given by  $r(m_1, \dots, m_n) = (rm_1, \dots, rm_n)$  for all  $r \in R$  and  $m_i \in M_i$ . Comparing the definitions we see that

$$M_1 \times \dots \times M_n = M_1 \oplus \dots \oplus M_n.$$

If  $M_i = R$  for  $1 \leq i \leq n$ , then we denote  $R^n = \underbrace{R \times \dots \times R}_n = \underbrace{R \oplus \dots \oplus R}_n$ .

It is useful to talk about maps from the factors/summands to the direct product/ direct sum and conversely.

**Definition 1.68.** For  $i \in J$  the *inclusion of the  $i$ -th factor* into a direct product or direct sum is the map

$$\iota_i: M_i \rightarrow \prod_{a \in J} M_a \text{ or } \iota_i: M_i \rightarrow \bigoplus_{a \in J} M_a, \iota_i(m) = (m_a)_{a \in J}, \text{ where } m_a = \begin{cases} m & \text{if } a = i \\ 0 & \text{if } a \neq i \end{cases}.$$

For  $i \in J$  the  $i$ -th *projection map* from a direct product or a direct sum module is

$$\pi_i: \prod_{a \in J} M_a \rightarrow M_i \text{ or } \pi_i: \bigoplus_{a \in J} M_a \rightarrow M_i, \pi_i((m_a)_{a \in J}) = m_i.$$

**Lemma 1.69.** *Projections from direct products or sums of  $R$ -module, inclusions into direct products or sums of  $R$ -modules, and products of  $R$ -module homomorphisms are  $R$ -module homomorphisms. Furthermore, inclusions are injective, projections are surjective, and*

$$\pi_i \circ \iota_i = \text{id}_{M_i}.$$

*Also,  $\iota_i(M_i)$  is an  $R$ -submodule of the direct product/sum which is isomorphic to  $M_i$ .*

Note, however, that  $\iota_i \circ \pi_i \neq \text{id}$ .

# Chapter 2

## Vector spaces and linear transformations

### 2.1 Classification of vector spaces and dimension

Recall that for a subset  $A$  of an  $F$ -vector space  $V$ , the **span** of  $A$ , denoted  $\text{span}(A)$ , is the subspace generated by  $A$ :

$$\text{span}(A) := \left\{ \sum_{i=1}^n c_i a_i \mid n \geq 0, c_i \in F, a_i \in A \right\}.$$

**Lemma 2.1.** *Suppose  $I$  is a linearly independent subset of an  $F$ -vector space  $V$  and  $v \in V \setminus \text{span}(I)$ , then  $I \cup \{v\}$  is also linearly independent.*

*Proof.* Let  $w_1, \dots, w_n$  be any list of distinct elements of  $I \cup \{v\}$  and suppose that  $\sum_i c_i w_i = 0$  for some  $c_i \in F$ . If none of the  $w_i$ 's is equal to  $v$ , then  $c_i = 0$  for all  $i$ , since  $I$  is linearly independent. Without loss of generality, say  $w_1 = v$ . If  $c_1 = 0$  then  $c_i = 0$  for all  $i$  by the same reasoning as in the previous case. If  $c_1 \neq 0$ , then

$$v = \sum_{i \geq 2} \frac{c_i}{c_1} w_i \in \text{span}(I),$$

contrary to assumption. This proves that  $I \cup \{v\}$  is a linearly independent set.  $\square$

To prove that every vector space has a basis, we will need to use Zorn's Lemma. Before we recall what Zorn's Lemma says, let's recall some notation:

**Definition 2.2.** A **poset** is a set  $S$  with an order relation  $\leq$  such that for all elements  $x, y, z \in S$  we have

- $x \leq x$ ,
- if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ , and
- if  $x \leq y$  and  $y \leq x$  then  $x = y$ .

A **totally ordered** set is a poset  $(T, \leq)$  such that for all  $x, y \in T$  either  $x \leq y$  or  $y \leq x$ .

**Example 2.3.** Given a set  $X$ , the collection  $\mathcal{P}(X)$  of all subsets of  $X$  forms a poset with  $\leq$  defined to be set containment  $\subseteq$ . Unless  $X$  is empty or a singleton, the poset  $\mathcal{P}(X)$  is not totally ordered.

**Definition 2.4.** Let  $(\mathcal{A}, \leq)$  be a **poset**, meaning that  $\mathcal{A}$  is a set with a partial order  $\leq$ . A subset  $\mathcal{B}$  of  $\mathcal{A}$  is **totally ordered** if for all  $b, b' \in \mathcal{B}$  either  $b \leq b'$  or  $b' \leq b$ ; a totally ordered subset of  $\mathcal{A}$  is sometimes called a **chain**. We say a subset  $\mathcal{B}$  of  $\mathcal{A}$  has an **upper bound** in  $\mathcal{A}$  if there exists an element  $u_B \in \mathcal{A}$  such that  $b \leq u_B$  for all  $b \in \mathcal{B}$ . We say  $\mathcal{A}$  has a **maximal element** if there exists  $m \in \mathcal{A}$  such that whenever  $x \in \mathcal{A}$  and  $m \leq x$  then  $m = x$ .

**Axiom 2.5** (Zorn's Lemma). If  $\mathcal{A}$  is a nonempty poset such that every totally ordered subset  $\mathcal{B} \subseteq \mathcal{A}$  has an upper bound in  $\mathcal{A}$ , then there is a maximal element  $m \in \mathcal{A}$ .

Some mathematicians refuse to accept Zorn's Lemma into their axiom system. We will at least pretend to be mathematicians who do. Fun fact: Theorem 2.6 is actually equivalent to the Axiom of Choice, meaning that if one replaces the Axiom of Choice in the ZFC axioms for set theory by Theorem 2.6, that does not change set theory – and one would then be able to deduce the Axiom of Choice.

If we accept Zorn's Lemma, we can now show that every vector space has a basis.

**Theorem 2.6** (Every vector space has a basis). *Let  $V$  be an  $F$ -vector space and assume  $I \subseteq S \subseteq V$  are subsets such that  $I$  is linearly independent and  $S$  spans  $V$ . Then there is a subset  $B$  with  $I \subseteq B \subseteq S$  such that  $B$  is a basis.*

Before we prove this theorem, note that a corollary of Theorem 2.6 is that every vector space has a basis; in particular, this says that every module over a field is free!

**Corollary 2.7.** *Every vector space  $V$  has a basis. Moreover, every linearly independent subset of  $V$  is contained in some basis, and every set of vectors that spans  $V$  contains some basis.*

*Proof.* For this first part, apply the theorem with  $I = \emptyset$  and  $S = V$ . For the second and third, use  $I$  arbitrary and  $S = V$  and  $I = \emptyset$  and  $S$  arbitrary, respectively.  $\square$

**Example 2.8.**  $\mathbb{R}$  has a basis as a  $\mathbb{Q}$ -vector space; just don't ask me what it looks like.

We will not prove Theorem 2.6. But before we give a formal proof, let's first give a heuristic proof. To so that, start with  $I$ . If  $\text{span}(I) = V$ , then  $B = I$  does the job. If not, then since  $\text{span}(S) = V$ , there must be a  $v \in S \setminus \text{span}(I)$ . Let  $I' := I \cup \{v\}$ . Then  $I' \subseteq S$  and, by Lemma 2.1,  $I'$  is linearly independent. If  $\text{span}(I') = V$ , we have found our  $B$ , and if not we construct  $I''$  from  $I'$  just as we constructed  $I'$  from  $I$ . At this point we would like to say that this process cannot go on for ever, and this is more-or-less true. But at least in an infinite dimensional setting, we need to use Zorn's Lemma to complete the proof rigorously.

*Proof of Theorem 2.6.* Let  $\mathcal{P}$  denote the collection of all subsets  $X$  of  $V$  such that  $I \subseteq X \subseteq S$  and  $X$  is linearly independent. We make  $\mathcal{P}$  into a poset by the order relation given by set containment  $\subseteq$ . We note that  $\mathcal{P}$  is not empty since, for example  $I \in \mathcal{P}$ .



Let  $\mathcal{T}$  be any nonempty chain in  $\mathcal{P}$ . Let  $Z = \bigcup_{Y \in \mathcal{T}} Y$ . We claim  $Z \in \mathcal{P}$ . Given  $z_1, \dots, z_m \in Z$ , for each  $i$  we have  $z_i \in Y_i$  for some  $Y_i \in \mathcal{T}$ . Since  $\mathcal{T}$  is totally ordered, one of  $Y_1, \dots, Y_m$  contains all the others and hence contains all the  $z_i$ 's. Since  $Y_i$  is linearly independent, this shows  $z_1, \dots, z_m$  are linearly independent. Thus  $Z$  is linearly independent. Since  $\mathcal{T}$  is non-empty,  $Z \supseteq I$  and hence  $Z \in \mathcal{P}$ . It is an upper bound for  $\mathcal{T}$  by construction.

By Zorn's Lemma,  $\mathcal{P}$  has a maximal element  $B$ , which we claim is a basis for  $V$ . Note that  $B$  is linearly independent and  $I \subseteq B \subseteq S$  by construction. We need to show that it spans  $V$ . Suppose not. Since  $S$  spans  $V$ , if  $S \subseteq \text{span}(B)$ , then  $\text{span}(B)$  would have to be all of  $V$ . So, there is at least one  $v \in S$  such that  $v \notin \text{span}(B)$ , and set  $X := B \cup \{v\}$ . Clearly,  $I \subset X \subseteq S$  and, by Lemma 2.1,  $X$  is linearly independent. This shows that  $X$  is an element of  $\mathcal{P}$  that is strictly bigger than  $B$ , contrary to the maximality of  $B$ .  $\square$

**Corollary 2.9.** *Let  $F$  be a field and  $W$  be a subspace of the  $F$ -vector space  $V$ . Then every basis of  $W$  extends to a basis of  $V$ , that is, if  $B$  is a basis of  $W$  then there exists a basis  $\tilde{B}$  of  $V$  such that  $B$  is a subset of  $\tilde{B}$ .*

*Proof.* Apply Corollary 2.7 with  $B = I$  and  $S = V$ . Since  $B$  is a basis of  $W$ ,  $B$  is linearly independent, and  $B$  remains linearly independent when regarded as a subset of  $V$ .  $\square$

**Remark 2.10.** It is *not* true that, with the notation of the previous Corollary, if  $\tilde{B}$  is a basis of  $V$  then there exists a basis  $B$  of  $W$  such that  $B$  is a subset of  $\tilde{B}$ . For instance, take  $F = \mathbb{R}$ ,  $V = \mathbb{R}^2$ ,  $\tilde{B} = \{(1, 0), (0, 1)\}$  and  $W$  the subspace spanned by  $(1, 1)$ .

**Definition 2.11.** A vector space is **finite dimensional** if there is spanned by a finite subset.

Thanks to Theorem 2.6, this is equivalent to the property that it has a finite basis. In the language of modules, a finite dimensional vector space is just a finitely generated  $F$ -module.

The following is an essential property of vector spaces that eventually will allow us to compare bases in terms of size.

**Lemma 2.12** (Exchange Property). *Let  $B$  be a basis for the vector space  $V$  and consider any finite set of linearly independent vectors  $C = \{c_1, \dots, c_m\}$  in  $V$ . Then there are distinct vectors  $b_1, \dots, b_m$  in  $B$  such that  $(B \setminus \{b_1, \dots, b_m\}) \cup C$  is also a basis of  $V$ .*

*Proof.* Using induction on  $k$ , we will show that for each  $k$  with  $0 \leq k \leq m$  there are distinct vectors  $b_1, \dots, b_k$  in  $B$  such that  $(B \setminus \{b_1, \dots, b_k\}) \cup \{c_1, \dots, c_k\}$  is also a basis of  $V$ . In the base case,  $k = 0$ , there is nothing to show. The terminal case,  $k = m$ , gives us the desired statement.

For the inductive step, assume  $B' = (B \setminus \{b_1, \dots, b_k\}) \cup \{c_1, \dots, c_k\}$  is also a basis of  $V$ . Since  $c_{k+1} \in V$ , we can write

$$c_{k+1} = \sum_{i=1}^n \lambda_i b_i + \sum_{i=1}^k \mu_i c_i$$

for some scalars  $\lambda_i, \mu_i \in F$  and some elements  $b_i \in B \setminus \{b_1, \dots, b_k\}$ . Note that since  $C$  is linearly independent, at least one of the scalars  $\lambda_i$  is nonzero. Let  $i_0$  be such that  $\lambda_{i_0} \neq 0$ , and notice that solving for  $b_{i_0}$  from the displayed equation gives that  $b_{i_0} \in \text{span}(B'')$  where  $B'' = (B' \setminus \{b_{i_0}\}) \cup \{c_{k+1}\}$ . Now we can “replace”  $b_{i_0}$  by  $c_k$ , since the previous statement implies  $\text{span}(B'') = \text{span}(B') = V$  and moreover  $B''$  is linearly independent since otherwise  $B'$  would be linearly dependent.  $\square$

Next, we will show that all bases of the same vector space have the same cardinality. We will only prove this under the assumption that  $V$  is finite dimensional, though it is true even if  $V$  has infinite dimension.

**Theorem 2.13** (Dimension Theorem). *Any two bases of the same vector space have the same cardinality.*

*Proof of the finite dimensional case.* Suppose  $V$  is finite dimensional. Then it has a finite basis  $B$ . Let  $B'$  be any other basis, and note that we cannot yet assume  $B'$  is necessarily finite. Let  $\{c_1, \dots, c_m\}$  be any  $m$ -element subset of  $B'$  for any  $m$ . An immediate consequence of Lemma 2.12 is that  $m \leq |B|$ , since otherwise we could not find  $m$  distinct elements of  $B$  to replace the  $c_i$ 's by. Since every finite subset of  $B'$  has cardinality no larger than  $|B|$ , this proves that  $B'$  is finite and  $|B'| \leq |B|$ . By symmetry, we obtain  $|B| \leq |B'|$  too, hence equality follows.  $\square$

**Definition 2.14.** The **dimension** of a vector space  $V$ , denoted  $\dim_F(V)$  or  $\dim(V)$ , is the cardinality of any of its bases.

**Example 2.15.**  $\dim_F(F^n) = |\{e_1, e_2, \dots, e_n\}| = n$ .

**Theorem 2.16** (Classification of finitely generated vector spaces). *Let  $F$  be a field.*

- (1) *Every finitely generated vector space over  $F$  is isomorphic to  $F^n$  for  $n = \dim_F(V)$ .*
- (2) *For any  $m, n \in \mathbb{Z}_{\geq 0}$ ,  $F^m \cong F^n$  if and only if  $m = n$ .*

*Proof.* To show (1), let  $V$  be a finite dimensional  $F$ -vector space. Then  $F$  has a finite spanning set  $S$  and by Theorem 2.6 there is a basis  $B \subseteq S$  for  $V$ . Notice that  $B$  is necessarily finite and  $V = FB$ . Set  $|B| = n$  and  $B = \{b_1, \dots, b_n\}$ . By Theorem 1.60, there is a linear transformation  $f: F^n \rightarrow V$  such that  $f(e_i) = b_i$  as well as a linear transformation  $g: V \rightarrow F^n$  such that  $g(b_i) = e_i$ . Then both  $f \circ g: V \rightarrow V$  and  $g \circ f: F^n \rightarrow F^n$  are linear transformation which agree with the identity map on a basis. Hence by the uniqueness part of Theorem 1.60 we have  $f \circ g = \text{id}_V$  and  $g \circ f = \text{id}_{F^n}$ . Therefore, these maps are the desired isomorphisms.

To show (2), let  $\varphi: F^m \cong F^n$  be a vector space isomorphism and let  $B$  be a basis of  $F^m$ . We claim that  $\varphi(B)$  is a basis for  $F^n$ . Indeed, if

$$\sum_{i=1}^m c_i \varphi(b_i) = 0 \quad \text{then} \quad \varphi\left(\sum_{i=1}^m c_i b_i\right) = 0, \quad \text{so} \quad \sum_{i=1}^m c_i b_i = 0$$

since  $\varphi$  is injective. But  $B$  is linearly independent, so we must have  $c_i = 0$  for all  $1 \leq i \leq m$ . If  $v \in F^n$ , then since  $B$  spans  $F^m$  we have

$$\varphi^{-1}(v) = \sum_{i=1}^m c_i b_i$$

for some  $c_i$ . Thus

$$v = \sum_{i=1}^m c_i \varphi(b_i),$$

which shows  $\varphi(B)$  spans  $F^n$ . By the [Dimension Theorem](#), we have

$$\dim_F(F^n) = n = |\varphi(B)| = |B| = m. \quad \square$$

**Remark 2.17.**

- (1) The same proof as in part (1) of Theorem 2.16 above shows that every finitely generated free  $R$ -module is isomorphic to  $R^n$  for some  $n \geq 0$ .
- (2) Part (2) of the [Classification Theorem](#) can be extended to modules over commutative rings as stated in Theorem 1.63; this is a problem in Problem Set 3.
- (3) The [Classification Theorem](#) yields that dimension is an isomorphism invariant.

**Corollary 2.18.** *Two finite dimensional vector spaces  $V$  and  $V'$  over the same field  $F$  are isomorphic if and only if  $\dim_F(V) = \dim_F(V')$ .*

*Proof.* By Theorem 2.16,  $V$  and  $V'$  are both of the form  $V \cong F^m$  and  $V' \cong F^n$ , while  $F^m \cong F^n$  if and only if  $m = n$ .  $\square$

A word on infinite-dimensional vector spaces.

**Example 2.19.** Consider the vector space  $F[x]$ . This cannot be a finite dimensional vector space. For instance, if  $\{f_1, \dots, f_n\}$  were a basis, then setting

$$M = \max_{1 \leq j \leq n} \{\deg(f_j)\}$$

we see that the element  $x^{M+1}$  is not in the span of  $\{f_1, \dots, f_n\}$ . We can find a basis for this space though. Consider the collection  $B = \{1, x, x^2, \dots\}$ . This set is linearly independent and spans  $F[x]$ , thus it forms a basis for  $F[x]$ . This basis is *countable*, so  $\dim_F(F[x]) = \aleph_0 = |\mathbb{N}|$ .

**Example 2.20.** Consider the real vector space

$$V := \mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$$

This space can be identified with sequences  $\{a_n\}$  of real numbers. One might be interested in a basis for this vector space. At first glance, the most obvious choice for a basis would be  $E = \{e_1, e_2, \dots\}$ . It turns out that  $E$  is the basis for the direct sum  $\bigoplus_{i \in \mathbb{N}} \mathbb{R}$ . However, it is immediate that this set does not span  $V$ , as  $v = (1, 1, \dots)$  can not be represented as a finite linear combination of these elements. Since  $v$  is not in  $\text{span}(E)$ , then by Lemma 2.1 we know that  $E \cup \{v\}$  is a linearly independent set. However, this new set  $E \cup \{v\}$  does not span  $V$  either, as  $(1, 2, 3, 4, \dots)$  is not in the span of  $E \cup \{v\}$ . We know that  $V$  has a basis, but it can be shown that no countable collection of vectors forms a basis for this space, and in fact  $\dim_{\mathbb{R}}(\mathbb{R}^{\mathbb{N}}) = |\mathbb{R}|$ .

We now deduce some formulas that relate the dimensions of various vector spaces.

**Theorem 2.21.** *Let  $W$  be a subspace of a vector space  $V$ . Then*

$$\dim(V) = \dim(W) + \dim(V/W).$$

Here the dimension of a vector space is understood to be either a nonnegative integer or  $\infty$ , and the arithmetic of the formula is understood to follow the rules  $n + \infty = \infty = \infty + \infty$  for any  $n \in \mathbb{Z}_{\geq 0}$ . We leave the proof for Problem Set 4.

**Example 2.22.** Consider the vector space  $V = \mathbb{R}^2$  and its subspace  $W = \text{span}\{e_1\}$ . Then the quotient vector space  $V/W$  is, by definition,

$$V/W = \{(x, y) + W \mid (x, y) \in \mathbb{R}^2\}.$$

Looking at each coset we see that

$$(x, y) + W = (x, y) + \text{span}\{e_1\} = \{(x, y) + (a, 0) \mid a \in \mathbb{R}\} = \{(t, y) \mid t \in \mathbb{R}\},$$

so  $(x, y) + W$  is geometrically a line parallel to the  $x$ -axis and having the  $y$ -intercept  $y$ . It is intuitively natural to identify such a line with its intercept, which gives a map

$$V/W \rightarrow \text{span}\{e_2\} \quad (x, y) + W \mapsto (0, y).$$

It turns out that this map is a vector space isomorphism, hence

$$\dim(V/W) = \dim(\text{span}\{e_2\}) = 1$$

and we can check that

$$\dim(W) + \dim(V/W) = 1 + 1 = 2 = \dim(V).$$

If  $V$  and  $W$  are both infinite dimensional vector spaces, it can happen that  $V/W$  is finite dimensional but also that it is infinite dimensional.

**Example 2.23.** Let  $V = F[x]$ , which we saw in Example 2.19 is an infinite dimensional vector space over  $F$ . Fix a polynomial  $f$  with  $\deg(f) = d$ , and note that the ideal  $(f)$  of  $F[x]$  generated by  $f$  is also an  $F$ -vector subspace of  $F[x]$  via restriction of scalars. We will show later that  $\dim(F[x]/(f)) = d$ . In contrast, the subspace  $E$  of all even degree polynomials in  $F[x]$  together with the zero polynomial, then  $\dim(F[x]/E) = \infty$ .

**Definition 2.24.** Let  $T: V \rightarrow W$  be a linear transformation. The **nullspace** of  $T$  is  $\ker(T)$ . The **rank** of  $T$  is  $\dim(\text{im}(T))$ .

**Corollary 2.25** (Rank-Nullity Theorem). *Let  $f: V \rightarrow W$  be a linear transformation. Then*

$$\dim(\ker(f)) + \dim(\text{im}(f)) = \dim(V).$$

*Proof.* By the [First Isomorphism Theorem for modules](#) we have  $V/\ker(f) \cong \text{im}(f)$ , thus

$$\dim(V/\ker(f)) = \dim(\text{im}(f)).$$

By Theorem 2.21, we have

$$\dim(V) = \dim(\ker(f)) + \dim(V/\ker(f)).$$

Thus

$$\dim(V) = \dim(\ker(f)) + \dim(V/\ker(f)) = \dim(\ker(f)) + \dim(\text{im}(f)).$$

□

## 2.2 Linear transformations and homomorphisms between free modules

**Exercise 13.** If  $W$  is a free  $R$ -module with basis  $C = \{c_1, \dots, c_m\}$  and  $w \in W$ , then  $w$  can be written *uniquely* as  $w = \sum_{j=1}^m a_j c_j$  with  $a_1, \dots, a_m \in R$ .

**Definition 2.26** (The matrix of a homomorphism between free modules). Let  $R$  be a commutative ring with  $1 \neq 0$ . Let  $V$  be a finitely generated free  $R$ -module of rank  $n$ , and let  $W$  be a finitely generated free  $R$ -module of rank  $m$ . Let  $B = \{b_1, \dots, b_n\}$  and  $C = \{c_1, \dots, c_m\}$  be *ordered* bases of  $V, W$ . Given an  $R$ -module homomorphism  $f : V \rightarrow W$ , we define elements  $a_{ij} \in R$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$  by the formulas

$$f(b_j) = \sum_{i=1}^m a_{ij} c_i. \quad (2.2.1)$$

The matrix

$$[f]_B^C = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

is said to **represent** the homomorphism  $f$  with respect to the bases  $B$  and  $C$ .

**Remark 2.27.** By Exercise 13, the coefficients  $a_{j,i}$  in equation 2.2.1 are uniquely determined by the  $f(b_i)$  and the elements of  $C$ . The coefficients  $a_{j,i}$  corresponding to  $f(b_i)$  form the  $i$ th column of  $[f]_B^C$ . Note that  $[f]_B^C$  is an  $m \times n$  matrix with entries in  $R$ .

**Definition 2.28.** Let  $V$  and  $W$  be finite  $F$ -vector spaces of dimension  $n$  and  $m$  with ordered bases  $B$  and  $C$  respectively and let  $f : V \rightarrow W$  be a linear transformation. The matrix  $[f]_B^C$  is called the **matrix of the linear transformation**  $f$  with respect to the bases  $B$  and  $C$ .

**Example 2.29.** If  $\text{id}_V : V \rightarrow V$  is the identity automorphism of an  $n$ -dimensional free  $R$ -module  $V$ , then for any basis  $B$  of  $V$  we have  $\text{id}_V(b_i) = b_i$  for all  $i$  and hence

$$[\text{id}_V]_B^B = I_n.$$

**Example 2.30.** Let  $P_3$  denote the the  $F$ -vector space of polynomials of degree at most 3 (including the zero polynomial) and consider the linear transformation  $d : P_3 \rightarrow P_3$  given by taking the derivative  $d(f) = f'$ . Let  $B = \{1, x, x^2, x^3\}$ . Then

$$[d]_B^B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Example 2.31.** Let  $F$  be a field and consider a linear transformation  $f: V \rightarrow W$ , where  $V = F^n$  and  $W = F^m$ . Consider also the standard ordered bases  $B$  and  $C$ , i.e.  $b_i = e_i \in V$  and  $c_i = e_i \in W$ . Then for any

$$v = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix} = \sum_i l_i b_i$$

in  $V$  we have

$$f\left(\sum_i l_i b_i\right) = \sum_i l_i f(b_i).$$

Each  $f(b_i)$  can be written uniquely as a linear combination of the  $c_j$ 's as in (2.2.1):

$$f(b_i) = \sum_j a_{j,i} c_j.$$

Then we get

$$f(v) = \sum_i l_i \left( \sum_j a_{j,i} c_j \right) = \sum_j \left( \sum_i a_{j,i} l_i \right) c_j.$$

In other words, we have

$$f(v) = \begin{bmatrix} \sum_i a_{1,i} l_i \\ \vdots \\ \sum_i a_{m,i} l_i \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix} = [f]_B^C \cdot v.$$

Then for any

$$v = \sum_i l_i b_i$$

in  $V$  we have

$$f\left(\sum_i l_i b_i\right) = \sum_i l_i f(b_i).$$

Each  $f(b_i)$  is uniquely expressible as a linear combination of the  $c_j$ 's, say

$$f(b_i) = \sum_j a_{j,i} c_j.$$

Then we get

$$f(v) = \sum_i l_i \left( \sum_j a_{j,i} c_j \right) = \sum_j \left( \sum_i a_{j,i} l_i \right) c_j.$$

In other words, we have

$$f(v) = [f]_B^C \cdot v$$

where

$$[f]_B^C = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

and  $[f]_B^C \cdot v$  denote the usual rule for matrix multiplication.

This says that any linear transformation  $f : F^n \rightarrow F^m$  is given by multiplication by a matrix, since we noticed above that  $f(v) = [f]_B^C \cdot v$ . The same type of statement holds for free modules over commutative rings, and we will show it below in Theorem 2.32.

**Theorem 2.32.** *Let  $R$  be a commutative ring with  $1 \neq 0$ . Let  $V$  and  $W$  be finitely generated free  $R$ -modules of ranks  $n$  and  $m$  respectively. Fixing ordered bases  $B$  for  $V$  and  $C$  for  $W$  gives an isomorphism of  $R$ -modules*

$$\text{Hom}_R(V, W) \cong M_{m,n}(R) \quad f \mapsto [f]_B^C.$$

If  $V = W$ , so that in particular  $m = n$ , and  $B = C$ , then the above map is an  $R$ -algebra isomorphism  $\text{End}_R(V) \cong M_n(R)$ .

*Proof.* Let  $\varphi : \text{Hom}_R(V, W) \rightarrow M_{m,n}(R)$  be defined by  $\varphi(f) = [f]_B^C$ . We need to check that  $\varphi$  is a homomorphism of  $R$ -modules, which translates into  $[f + g]_B^C = [f]_B^C + [g]_B^C$  and  $[\lambda f]_B^C = \lambda [f]_B^C$  for any  $f, g \in \text{Hom}_R(V, W)$  and  $\lambda \in R$ . Let  $A = [f]_B^C$  and  $A' = [g]_B^C$ . Then

$$(f + g)(b_i) = f(b_i) + g(b_i) = \sum_j a_{j,i} c_j + \sum_j a'_{j,i} c_j = \sum_j (a_{j,i} + a'_{j,i}) c_j$$

gives  $[f + g]_B^C = A + A'$  and

$$(\lambda f)(b_i) = \lambda \left( \sum_j a_{j,i} c_j \right) = \sum_j (\lambda a_{j,i}) c_j$$

gives  $[\lambda f]_B^C = \lambda A$ . We leave the proof that for  $f, g \in \text{End}_R(V)$  we have  $[f \circ g]_B^B = [f]_B^B [g]_B^B$  as an exercise.

Finally, the argument described in Example 2.31 also works for any ring  $R$ , and it can be adapted for any two chosen basis  $B$  and  $C$ , showing that  $\varphi$  is a bijection.  $\square$

**Corollary 2.33.** *For any field  $F$  and finite  $F$ -vector spaces  $V$  and  $W$  of dimension  $n$  and  $m$  respectively,  $\dim(\text{Hom}_F(V, W)) = mn$ .*

*Proof.* The isomorphism  $\text{Hom}_F(V, W) \cong M_{m,n}(F)$  gives

$$\dim(\text{Hom}_F(V, W)) = \dim(M_{m,n}(F)) = mn.$$

$\square$

## 2.3 Change of basis

**Definition 2.34.** Let  $V$  be a finitely generated free module over a commutative ring  $R$ , and let  $B$  and  $C$  be bases of  $V$ . Let  $\text{id}_V$  be the identity map on  $V$ . Then  $[\text{id}_V]_B^C$  is a matrix called the **change of basis matrix** from  $B$  to  $C$ .

In Theorem 2.39 we will show that  $[\text{id}_V]_B^C$  is invertible with inverse  $([\text{id}_V]_B^C)^{-1} = [\text{id}_V]_C^B$ .

**Example 2.35.** Consider the subspace  $V = P_2$  of  $F[x]$  of all polynomials of degree up to 2, and the bases  $B = \{1, x, x^2\}$  and  $C = \{1, x - 2, (x - 2)^2\}$  of  $V$ . We calculate the change of basis matrix. We have

$$\begin{aligned}\text{id}_V(1) &= 1, \\ \text{id}_V(x) &= 2 \cdot 1 + 1 \cdot (x - 2), \\ \text{id}_V(x^2) &= 4 \cdot 1 + 4 \cdot (x - 2) + 1 \cdot (x - 2)^2.\end{aligned}$$

Thus, the change of basis matrix is given by  $[\text{id}_V]_B^C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ .

**Lemma 2.36.** If  $V, W, U$  are finitely generated free  $R$ -modules spaces with ordered bases  $B$ ,  $C$ , and  $D$ , and if  $f: V \rightarrow W$  and  $g: W \rightarrow U$  are  $R$ -module homomorphisms, then

$$[g \circ f]_D^B = [g]_D^C \cdot [f]_C^B.$$

*Proof.* Given  $v \in V$ , we have

$$(f \circ g)(v) = f(g(v)) = f([g]_B^C v) = [f]_C^D ([g]_B^C v) = ([f]_C^D [g]_B^C) v,$$

$$\text{so } [f \circ g]_B^B = [f]_B^B [g]_B^B. \quad \square$$

**Definition 2.37.** Let  $V$  be a finitely generated free module over a commutative ring  $R$ . Two  $R$ -module homomorphisms  $f, g: V \rightarrow V$  are **similar** if there is a bijective linear transformation  $h: V \rightarrow V$  such that  $g = h \circ f \circ h^{-1}$ . Two  $n \times n$  matrices  $A$  and  $B$  with entries in  $R$  are **similar** if there is an invertible  $n \times n$  matrix  $P$  such that  $B = PAP^{-1}$ .

**Remark 2.38.** For elements  $A, B \in \text{GL}_n(R)$ , the notions of similar and conjugate are the same.

**Theorem 2.39.** Let  $V, W$  be finitely generated free modules over a commutative ring  $R$ , let  $B$  and  $B'$  be bases of  $V$ , let  $C$  and  $C'$  be bases of  $W$ , and let  $f: V \rightarrow W$  be a homomorphism. Then

$$[f]_{B'}^{C'} = [\text{id}_W]_C^{C'} [f]_B^C [\text{id}_V]_{B'}^B, \quad (2.3.1)$$

In particular, if  $g: V \rightarrow V$  is an  $R$ -module homomorphism, then  $[g]_B^B$  and  $[g]_{B'}^{B'}$  are similar.

*Proof.* Since  $f = \text{id}_W \circ f \circ \text{id}_V$ , by Lemma 2.36 we have

$$[f]_{B'}^{C'} = [\text{id}_W]_C^{C'} [f]_B^C [\text{id}_V]_{B'}^B.$$



Setting  $V = W$ ,  $B = C$ ,  $B' = C'$ , and  $f = \text{id}_V$  in (2.3.1) we have  $[\text{id}_V]_{B'}^{B'} = [\text{id}_V]_{B'}^{B'} [\text{id}_V]_B^B [\text{id}_V]_B^{B'}$ . Notice that  $[\text{id}_V]_B^B = [\text{id}_V]_{B'}^{B'} = I$  is the identity matrix, so the previous formula says that

$$I = [\text{id}_V]_{B'}^{B'} I [\text{id}_V]_B^B.$$

Setting  $P = [\text{id}_V]_{B'}^{B'}$ , we notice that the previous identity gives  $P^{-1} = [\text{id}_V]_B^B$ .

Now set  $V = W$ ,  $B = C$ ,  $B' = C'$  and  $f = g$  in (2.3.1) to obtain

$$[g]_{B'}^{B'} = [\text{id}_V]_{B'}^{B'} [g]_B^B [\text{id}_V]_B^{B'} = P [g]_B^B P^{-1}.$$

□

We now come to certain special changes of basis and their matrices:

**Definition 2.40.** Let  $R$  be a commutative ring with  $1 \neq 0$ , let  $M$  be a free  $R$ -module of finite rank  $n$ , and let  $B = \{b_1, \dots, b_n\}$  be an ordered basis for  $M$ . An **elementary basis change operation** on the basis  $B$  is one of the following three types of operations:

1. Replacing  $b_i$  by  $b_i + rb_j$  for some  $i \neq j$  and some  $r \in R$ ,
2. Replacing  $b_i$  by  $ub_i$  for some  $i$  and some unit  $u$  of  $R$ ,
3. Swapping the indices of  $b_i$  and  $b_j$  for some  $i \neq j$ .

**Definition 2.41.** Let  $R$  be a commutative ring with  $1 \neq 0$ . An **elementary row operation** on a matrix  $A \in M_{m,n}(R)$  is one of the following three types of operations:

1. Adding an element of  $R$  times a row of  $A$  to a different row of  $A$ .
2. Multiplying a row of  $A$  by a unit of  $R$ .
3. Interchanging two rows of  $A$ .

**Definition 2.42.** Let  $R$  be a commutative ring with  $1 \neq 0$ . An **elementary matrix** over  $R$  is an  $n \times n$  matrix obtained from  $I_n$  by applying a single elementary row operation:

1. For  $r \in R$  and  $1 \leq i, j \leq n$  with  $i \neq j$ , let  $E_{i,j}(r)$  be the matrix with 1s on the diagonal,  $r$  in the  $(i, j)$  position, and 0 everywhere else.
2. For  $u \in R^\times$  and  $1 \leq i \leq n$  let  $E_i(u)$  denote the matrix with  $(i, i)$  entry  $u$ ,  $(j, j)$  entry 1 for all  $j \neq i$ , and 0 everywhere else.
3. For  $1 \leq i, j \leq n$  with  $i \neq j$ , let  $E_{(i,j)}$  denote the matrix with 1 in the  $(i, j)$  and  $(j, i)$  positions and in the  $(l, l)$  positions for all  $l \notin \{i, j\}$ , and 0 in all other entries.

**Remark 2.43.** Let  $E$  be an  $n \times n$  elementary matrix.

- $E$  is the change of basis matrix  $[\text{id}_V]_{B'}^B$ , where  $B$  is any basis of  $V$  and  $B'$  is the basis obtained from  $B$  by the corresponding elementary basis change operation.
- If  $A \in M_{n,q}(R)$ , then the product matrix  $EA$  is the result of performing the corresponding elementary row operation on  $A$ .
- If  $B \in M_{m,n}(R)$ , then the product matrix  $BE$  is the result of performing the corresponding elementary column operation on  $B$ .

## 2.4 A warning on the differences between vector spaces and general free modules

Many of the nice theorems we showed about vector spaces, basis, and dimension do not extend well to general free modules over a commutative ring. Most importantly, Theorem 2.6, which says that for a vector space  $V$  every linearly independent set can be extended to a basis and every set that spans  $V$  contains a basis, does not hold in general, even in simple cases.

**Example 2.44.** Let  $R = \mathbb{Z}$  and consider the free  $R$ -module  $R$ . The set  $\{2\}$  is linearly independent but it is not a basis for  $R$ ; given any other element  $n \in R$ ,  $\{2, n\}$  is necessarily linearly dependent, since  $n \cdot 2 - 2 \cdot n = 0$ . Thus we cannot extend  $\{2\}$  to a basis of this free module.

Conversely the set  $\{2, 3\}$  spans the free module  $R = \mathbb{Z}$ , but it is not linearly independent, and the subsets  $\{2\}$  and  $\{3\}$  do not generate the entire free module  $R$ .

The failure of Theorem 2.6 leads to the failure of other properties we might expect. For example, one can show that Theorem 2.6 implies that if  $W$  is a subspace of  $V$ , then  $\dim(W) \leq \dim(V)$ . But when  $R$  is a general commutative ring, submodules of free modules do not have to be free, so we can't even talk about dimension; and if we count the number of generators needed, even cyclic modules might have submodules that are not cyclic.

**Example 2.45.** Let  $R$  be a ring and  $I$  be an ideal that is not principal. For example, we can take  $R = k[x, y]$  with  $k$  a field and  $I = (x, y)$ . Then the  $R$ -module  $R$  is cyclic, while the submodule  $I$  needs at least 2 generators.

Moreover, submodules of free modules are not necessarily free:

**Example 2.46.** Let  $k$  be a field and  $R = k[x]/(x^2)$ . The submodule  $I = (x)$  of the free module  $R$  is not free: it is cyclic, generated by  $x$ , but  $\text{ann}(I) = (x)$  is nontrivial, and thus  $I$  is not free.

# Chapter 3

## Finitely generated modules over PIDs

We have seen that every module over a field is free. In contrast, whenever  $R$  is a commutative ring that is not a field, we can always construct modules that are not free. We will see that, however, every module is still a quotient of a free module. Describing that quotient explicitly is to give a presentation for the module, similarly to how we gave presentations for groups. We will study the particular case of finitely generated modules over PIDs in more detail.

### 3.1 Every module is a quotient of a free module

**Lemma 3.1.** *Given any ring  $R$  with  $1 \neq 0$ , any direct sum of copies of  $R$  is always a free  $R$ -module.*

*Proof.* Suppose that  $F = \bigoplus_{i \in \Lambda} R$  is a direct sum of copies of  $R$  indexed by some set  $\Lambda$ . The tuples

$$e_i = (a_j)_{j \in \Lambda} \quad \text{with } a_j = 0 \text{ for all } j \neq i \text{ and } a_i = 1$$

generate  $F$ , since we can write any element as

$$(c_i)_{i \in \Lambda} = \sum_{i \in \Lambda} c_i e_i.$$

Notice that by definition  $c_i \neq 0$  for only finitely many  $i$ , so the sum on the right has finitely many nonzero terms. Moreover, the  $e_i$  are linearly independent, and thus they form a basis for  $F$ .  $\square$

We will show in the next chapter that every when  $R$  is a field, every  $R$ -module is free. In contrast, we will also see that if  $R$  is a commutative ring that is not a field, there always exists an  $R$ -module that is not free – in fact, given a ring  $R$  that is not a field, one can give a very concrete recipe for building nonfree modules.

However, even though not all modules are free, what is true is that every  $R$ -module can be written as a quotient of a free module, as follows. Given a module  $M$ , first take a generating set for  $M$ , say  $\Gamma = \{m_i\}_{i \in \Lambda}$ . Notice that a generating set always exists: for example, we can take  $\Gamma = M$ , though of course that is a bit of an overkill, since it's quite likely that some elements can be obtained from linear combinations of others.

Next, we construct a free module on the set  $\Lambda$ ; more precisely, we take a free module on as many generators as generators for  $M$  that we picked. Now the map

$$\begin{aligned} \bigoplus_{i \in \Lambda} R &\xrightarrow{\pi} M \\ (r_i) &\mapsto \sum_{i \in \Lambda} r_i m_i. \end{aligned}$$

Notice this map actually makes sense: the tuples  $(r_i)$  have only finitely many nonzero entries, and thus  $\sum_{i \in \Lambda} r_i m_i$  is a (finite) linear combination of our chosen generators. Moreover, since we chose the  $m_i$  to be generators for  $M$ , this map  $\pi$  is surjective. It is also easy to check that it is an  $R$ -module homomorphism: in fact, this is the  $R$ -module homomorphism we would get from Theorem 1.60 by setting  $e_i \mapsto m_i$ .

By the [First Isomorphism Theorem](#),

$$M \cong \bigoplus_{i \in \Lambda} R / \ker \pi.$$

This shows the following:

**Theorem 3.2.** *Every  $R$ -module is a quotient of a free  $R$ -module.*

Notice that the map  $\pi$  we constructed above depends on a choice of generating set for  $M$ . Given the map  $\pi$  corresponding to the set of generators  $\Gamma = \{m_i\}$ , each element in  $\ker(\pi)$  is a **relation** among the generators for  $M$ : the tuple  $(r_i)$  is a relation for the generators  $\{m_i\}$  if

$$\sum_{i \in \Lambda} r_i m_i = 0.$$

A nonzero relation among the  $m_i$  tells us that the set  $\{m_i\}$  is linearly dependent. Thus we see that

$$\pi \text{ is injective} \iff \{m_i\} \text{ is linearly independent} \iff \{m_i\} \text{ is a basis for } M.$$

In particular, the existence of such a map  $\pi$  that is injective is equivalent to  $M$  being free. Since  $\pi$  is always surjective (as long as  $\{m_i\}$  forms a generating set for  $M$ , we can now rephrase this as

$$\pi \text{ is an isomorphism} \iff \{m_i\} \text{ is a basis for } M.$$

The module  $M$  is free if and only if we can find a basis for  $M$ , thus  $M$  is free if and only if  $M$  is isomorphic to a direct sum of copies of  $R$ . Since we have already shown that a direct sum of copies of  $R$  is free, we conclude the following:

**Theorem 3.3.** *An  $R$ -module is free if and only if it is isomorphic to a direct sum of copies of  $R$ .*

## 3.2 Presentations for finitely generated modules over noetherian rings

Writing a given  $R$ -module  $M$  as a quotient of a free module is giving a **presentation** for  $M$ . In 817, we studied presentations for groups; these consisted of a set of generators and a set (normal subgroup) of relations among these generators. Presentations are important for modules as well. In this case, the relations are encoded by a matrix, or equivalently by a homomorphism between a pair of free modules. We study below how the change of basis techniques can be applied to unravel the structure of a module starting with its presentation.

**Definition 3.4.** Let  $R$  be a commutative ring with  $1 \neq 0$ , let  $A \in M_{m,n}(R)$ , and let  $t_A : R^n \rightarrow R^m$  be the  $R$ -module homomorphism represented by  $A$  with respect to the standard bases. Notice that this homomorphism is given by the rule  $t_A(v) = Av$ . The  **$R$ -module presented by  $A$**  is the  $R$ -module  $R^m / \text{im}(t_A)$ .

The  $R$ -module  $M$  presented by  $A \in M_{m,n}(R)$  has  $m$  generators and  $n$  relations. Each row of  $A$  corresponds to a generator for  $M$ , while each column encodes a relation among those generators. More precisely, the relations among the  $m$  generators are themselves *generated* by the  $n$  generators of  $\text{im}(t_A)$ , which are the images of the standard basis of  $R^n$  by  $t_A$ .

**Example 3.5.** The  $\mathbb{Z}$ -module  $M = \mathbb{Z}/6$  is presented by

$$\mathbb{Z} \xrightarrow{6} \mathbb{Z},$$

since  $M \cong \mathbb{Z} / \text{im}(t_6) = \mathbb{Z} / (6)$ . Notice here we abused notation and wrote 6 instead of the  $1 \times 1$  matrix  $[6]$ .

**Example 3.6.** Let  $R = k[x, y]$ , where  $k$  is a field, and  $I = (x, y)$ . The  $R$ -module  $M = R/I$  has 1 generator,  $m = 1 + I$ , so we can write a presentation for  $M$  of the form  $F \xrightarrow{p} R$  for some free module  $F$  and some  $R$ -module homomorphism  $p$ . To find such an  $F$ , we need to ask about the relations among the generators of  $M$ . For any  $a \in I$ , we have the relation  $am = 0$ , so  $I$  is the **module of relations** for this presentation of  $M$ .

How many generators does the module of relations have? In this case, we need 2: the relations  $xm = 0$  and  $ym = 0$  generate *all* the relations, since for any  $a \in I$ , we can write  $a = rx + sy$  for some  $x, y \in R$ , and thus  $am = 0$  can be rewritten as  $r(xm) + s(ym) = 0$ , which is a linear combination of the two relations  $xm = 0$  and  $ym = 0$ . Finally, we have the following presentation for  $M$ :

$$R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R.$$

Indeed, the image of  $\begin{bmatrix} x & y \end{bmatrix}$  is  $(x, y)$ , and  $M \cong R/(x, y)$ .

Conversely, we might be given a matrix and ask about what module it represents; one thing to keep in mind is that some presentations might be inefficient, either by having more generators or more relations than necessary. We want to answer to key questions: given a presentation for a module, how to find a more efficient presentation; and how to decide if two different presentations actually give us isomorphic modules. Keeping these goals in mind, let's try a more elaborate example.

**Example 3.7.** Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 9 & 5 \\ 1 & -2 & 7 \\ 0 & 1 & 2 \end{bmatrix}.$$

What  $\mathbb{Z}$ -module  $M$  is presented by  $A$ ? Formally,  $M$  is the quotient module  $M = \mathbb{Z}^4 / \text{im}(t_A)$ , where  $t_A: \mathbb{Z}^3 \rightarrow \mathbb{Z}^4$  is defined by  $t_A(v) = Av$ . Since  $\mathbb{Z}^4$  is generated by its standard basis elements  $\{e_1, e_2, e_3, e_4\}$ , we deduce as in Lemma 1.54 that  $M = \mathbb{Z}^4 / \text{im}(t_A)$  is generated by the cosets of the  $e_i$ . To keep the notation short, we set  $m_i = e_i + \text{im}(t_A)$ .

Let  $N = \text{im}(t_A)$  and note that  $N$  is the submodule of  $\mathbb{Z}^4$  generated by the columns of  $A$ :

$$N = R \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\} = R\{2e_1 + 3e_2 + e_3, e_1 + 9e_2 - 2e_3 + e_4, 5e_2 + 7e_3 + 2e_4\}.$$

Since  $N$  maps to 0 under the quotient map  $q: \mathbb{Z}^4 \rightarrow M = \mathbb{Z}^4/N$ , the relations of  $M$  can be written as

$$\begin{cases} 2m_1 + 3m_2 + m_3 & = 0 \\ m_1 + 9m_2 - 2m_3 + m_4 & = 0 \\ 5m_2 + 7m_3 + 2m_4 & = 0. \end{cases}$$

We can now see that this is a rather inefficient presentation, since we can clearly use the first equation to solve for  $m_3 = -2m_1 - 3m_2$ . This implies that  $M$  can be generated using only  $m_1, m_2$  and  $m_4$ , that is

$$M = R\{m_1, m_2, m_3, m_4\} = R\{m_1, m_2, m_4\}.$$

This eliminates the first equation and the latter two become

$$\begin{cases} 5m_1 + 15m_2 + m_4 & = 0 \\ -14m_1 - 16m_2 + 2m_4 & = 0 \end{cases}$$

Now we can also eliminate  $m_4$ , i.e leaving just two generators  $m_1, m_2$  that satisfy

$$-24m_1 - 46m_2 = 0.$$

Another way to do this is to look at the matrix  $A$  and use elementary row operations to “make zeros” on the 1st and 2nd columns, as follows:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 9 & 5 \\ 1 & -2 & 7 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 5 & -14 \\ 0 & 15 & -16 \\ 1 & -2 & 7 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -24 \\ 0 & 0 & -46 \\ 1 & 0 & 13 \\ 0 & 1 & 0 \end{bmatrix}$$

Eliminating the generators  $m_3$  and  $m_4$  amounts to dropping the first two columns (which are the 3rd and 4th standard basis vectors) as well as the last two rows. As we will prove

soon, this shows that the  $\mathbb{Z}$ -module presented by  $A$  is isomorphic to the  $\mathbb{Z}$ -module presented by

$$B = \begin{bmatrix} -24 \\ -46 \end{bmatrix}.$$

We can go further. Set  $m'_1 := m_1 + 2m_2$ . Then  $m'_1$  and  $m_2$  also form a generating set of  $M$ . The relation on  $m_1, m_2$  translates to

$$-24m'_1 + 2m_2 = 0$$

given by the matrix

$$C = E_{1,2}(-2)B = \begin{bmatrix} -24 \\ 2 \end{bmatrix}.$$

Note that we have done a row operation (subtract twice row 1 from row 2) to get from  $B$  to  $C$ . Continuing in this fashion by subtracting 12 row 2 from row 1 we also form

$$D = E_{1,2}(12)C = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

The last matrix  $D$  presents the module  $M' = \mathbb{Z}^2 / \text{im}(t_D)$  with generators  $a, b$ , where

$$a = e_1 + \text{im}(t_D), \quad b = e_2 + \text{im}(t_D)$$

and relation  $2a = 0$ . This module  $M'$  is isomorphic to our original module  $M$ . As we will see, this proves  $M \cong \mathbb{Z} \oplus \mathbb{Z}/2$ . An explicit isomorphism between  $M'$  and  $\mathbb{Z} \oplus \mathbb{Z}/2$  is given by sending  $\mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2$  by the unique  $\mathbb{Z}$ -module homomorphism defined by

$$e_1 \mapsto (1, 0) \text{ and } e_2 \mapsto (0, [1]_2).$$

Now notice that the kernel of this homomorphism is the submodule  $(2e_2)\mathbb{Z} = \text{im}(t_D)$ . Then the first isomorphism theorem gives  $M' = \mathbb{Z}^2 / \text{im}(t_D) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ .

**Lemma 3.8.** *Let  $R$  be a commutative ring with  $1 \neq 0$ ,  $A \in M_{m,n}(R)$  and  $B \in M_{m',n'}(R)$  for some  $m, n, m', n' \geq 1$ . Then  $A$  and  $B$  present isomorphic  $R$ -modules if  $B$  can be obtained from  $A$  by any finite sequence of operations of the following form:*

- (1) *an elementary row operation,*
- (2) *an elementary column operation,*
- (3) *deletion of the  $j$ th column and  $i$ th row of  $A$  if  $Ae_j = e_i$ , that is, if the  $j$ th column of  $A$  is the vector  $e_i$ ,*
- (4) *the reverse of 3: insertion of a row and column satisfying  $Ae_j = e_i$ ,*
- (5) *deletion of a column of all 0's,*
- (6) *the reverse of 5: insertion of a column of all 0s.*

*Proof.* It is sufficient to show that each individual operation gives an isomorphism, as the composition of isomorphisms is an isomorphism.

For operations (1) and (2), consider matrices  $A$  and  $A'$  where  $A'$  is obtained from  $A$  by the given elementary row/column operation, and set  $M = R^m / \text{im}(t_A)$  and  $M' = R^{m'} / \text{im}(t_{A'})$ . We need to prove that there is an isomorphism  $M \cong M'$ .

In case (1), where we have an elementary row operation, let  $E$  be the corresponding elementary matrix. Since  $A' = EA$ , the isomorphism  $E : R^n \rightarrow R^n$  maps  $\text{im}(A)$  bijectively onto  $\text{im}(A')$ . Thus  $Q$  induces an isomorphism

$$M = R^m / \text{im}(t_A) \xrightarrow{\cong} R^m / \text{im}(t_{A'}) = M'.$$

In case (2), where we have an elementary column operation, let  $E$  be the corresponding elementary matrix. Since  $A' = AE$  and since  $E$  is an isomorphism, we have

$$\text{im}(t_{A'}) = \text{im}(t_{AE}) = \text{im}(t_A \circ t_E) = \text{im}(t_A)$$

and so  $m = m'$  and  $M = R^m / \text{im}(t_A) = R^{m'} / \text{im}(t_{A'}) = M'$ . In fact, note that for this one we get equality, not merely an isomorphism.

For case (3), we have  $m' = m - 1$  and  $n' = n - 1$ . Since  $R^m$  is free, by the [UMP for free modules](#) there is a unique  $R$ -module homomorphism  $p : R^m \rightarrow R^{m-1}$  sending

$$\begin{aligned} e_1 &\mapsto e'_1, \dots, e_{i-1} \mapsto e'_{i-1} \\ e_i &\mapsto 0 \\ e_{i+1} &\mapsto e'_i, \dots, e_m \mapsto e'_{m-1} \end{aligned}$$

Similarly, there is a unique  $R$ -module homomorphism  $q : R^n \rightarrow R^{n-1}$  sending

$$\begin{aligned} e_1 &\mapsto e'_1, \dots, e_{j-1} \mapsto e'_{j-1}, \\ e_j &\mapsto 0, \\ e_{j+1} &\mapsto e'_j, \dots, e_n \mapsto e'_{n-1}. \end{aligned}$$

Here the elements  $e_i$  are part of a standard basis for  $R^n$  or for  $R^m$ , while the elements  $e'_i$  are part of a standard basis for  $R^{n-1}$  or for  $R^{m-1}$ . Then the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{A} & R^m \\ q \downarrow \cong & & \downarrow p \\ R^{n-1} & \xrightarrow{A'} & R^{m-1} \end{array}$$

commutes by the definition of  $A'$ . In particular,  $p(\text{im}(t_A)) \subseteq \text{im}(t_{A'})$  and so  $p$  induces an  $R$ -module homomorphism

$$\bar{p} : M \rightarrow M',$$

and we claim  $\bar{p}$  is bijective.

Since  $p$  is onto, so is  $\bar{p}$ . Suppose  $m \in \ker(\bar{p})$ . Then  $m = v + \text{im}(t_A)$  for some  $v \in R^m$  and  $p(v) \in \text{im}(t_{A'})$ . Say  $p(v) = A'w$ . Since  $q$  is onto,  $w = q(u)$  for some  $u$ . Then

$$p(v - Au) = p(v) - pA(u) = p(v) - A'q(u) = p(v) - A'w = p(v) - p(v) = 0,$$



and thus  $v - Au \in \ker(p)$ . Now, the kernel of  $p$  is clearly  $Re_i$ , so that  $v - Au = re_i$  for some  $r$ . Finally, since  $Ae_j = e_i$ , we have  $A(re_j) = re_i = v - Au$  and hence  $v = A(u + re_j)$ , which proves  $v = t_A(u + re_j) \in \text{im}(t_A)$  and hence that  $m = 0$ .

For (5), it is clear that the columns of  $A'$  generate the same submodule of  $R^m$  as do the columns of  $A$ , and thus  $M = M'$ .

Finally, for operations (4) and (6), since the isomorphism relation is reflexive, the statements of parts (3) and (5) show that parts (4) and (6) are true as well.  $\square$

Which modules have presentations? The discussion in Section 3.1 shows that the answer is every module. But if we want to make the presentation be finite (that is, so that the matrix describing the module has finitely many rows and columns) then we need to restrict ourselves to finitely generated modules. This in general does not suffice to guarantee that there will only be finitely many generators for the submodule of relations.

It might seem like no submodule of a finitely generated module could ever fail to itself be finitely generated, but indeed this happens!

**Example 3.9.** Let  $k$  be a field and  $R = k[x_1, x_2, \dots]$  be a polynomial ring in infinitely many variables. When we think of  $R$  as a module over itself, it is finitely generated – by the element 1. However, there are submodules of  $R$  that are not finitely generated: for example, the ideal  $(x_1, x_2, \dots)$  generated by all the variables.

**Definition 3.10** (Noetherian ring). A ring  $R$  is **noetherian** if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

eventually stabilizes: there is some  $N$  for which  $I_n = I_{n+1}$  for all  $n \geq N$ .

The following characterization of noetherian rings is the key to guaranteeing that submodules of finitely generated modules are also finitely generated.

**Theorem 3.11.** *A commutative ring  $R$  is noetherian if and only if every ideal of  $R$  is finitely generated.*

Many rings are noetherian.

**Example 3.12.**

- a) Every field  $k$  is noetherian, since  $(0)$  and  $k$  are the only ideals.
- b) If  $R$  is a principal ideal domain (PID), then by definition every ideal is generated by a single element, and hence  $R$  is noetherian.
- c) If  $R$  is noetherian, then you will show in Problem Set 5 that every quotient of  $R$  is also noetherian.

For more examples, the following famous theorem is useful.

**Theorem 3.13** (Hilbert Basis Theorem). *If  $R$  is a noetherian ring, then so is  $R[x_1, \dots, x_n]$  for all integers  $n \geq 1$ .*

In the interest of time, and since we really won't need it in this class, I will not give a proof of the Hilbert Basis Theorem. Combining the facts above together gives the following very nice fact:

**Corollary 3.14.** *Let  $k$  be a field and let  $I$  be an ideal in  $S = k[x_1, \dots, x_n]$  for some  $n \geq 1$ . Then the ring  $S/I$  is noetherian.*

This includes a large collection of the rings that are of most interest in the fields of commutative algebra and algebraic geometry. In contrast, above we saw an example of a ring that is not noetherian:

**Example 3.15.** Let  $k$  be any field and  $R = k[x_1, x_2, \dots]$  be the polynomial ring in arbitrarily many variables. Then  $R$  is not noetherian: the ideal  $(x_1, x_2, \dots)$  generated by all the variables is not finitely generated. Alternatively, we can see that the ascending chain of ideals

$$(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \dots$$

does not stop.

**Theorem 3.16.** *Let  $R$  be a commutative ring. If  $R$  is a noetherian ring, then every submodule of a finitely generated module is also finitely generated.*

*Proof.* We will first prove that for each  $n \geq 1$ , every submodule of  $R^n$  is finitely generated. The base case  $n = 1$  holds by Theorem 3.11, since a submodule of  $R^1$  is the same thing as an ideal of  $R$ . Assume  $n > 1$  and that every submodule of  $R^{n-1}$  is finitely generated. Let  $M$  be any submodule of  $R^n$ . Define

$$\pi: R^n \rightarrow R^1$$

to be the projection onto the last component of  $R^n$ . The kernel of  $\pi$  may be identified with  $R^{n-1}$ , and so  $N := \ker(\pi) \cap M$  is a submodule of  $R^{n-1}$ . By assumption,  $N$  is finitely generated. The image  $\pi(M)$  is a submodule of  $R^1$ , that is, an ideal of  $R$ , and so it too is finitely generated by Theorem 3.11. Furthermore, by the [First Isomorphism Theorem](#)  $M/\ker(\pi) \cong \pi(M)$ . By Lemma 1.54, we deduce that  $M$  is a finitely generated module.

Now let  $T$  be any finitely generated  $R$ -module and  $N \subseteq T$  any submodule. Since  $T$  is finitely generated, there exists a surjective  $R$ -module homomorphism  $q: R^n \rightarrow T$  for some  $n$ . Then  $q^{-1}(N)$  is a submodule of  $R^n$  and hence it is finitely generated by the case we already proved, say by element  $v_1, \dots, v_m \in q^{-1}(N)$ . We claim that  $q(v_1), \dots, q(v_m)$  generate  $N$ . Given any  $a \in N$ , since  $q$  is surjective we can find some  $b \in q^{-1}(N)$  such that  $q(b) = a$ . Since  $v_1, \dots, v_m$  generate  $q^{-1}(N)$ , we can find  $c_1, \dots, c_m \in R$  such that

$$b = c_1 v_1 + \dots + c_m v_m \implies c_1 q(v_1) + \dots + c_m q(v_m) = q(c_1 v_1 + \dots + c_m v_m) = q(b) = a. \quad \square$$

In fact, the converse of Theorem 3.16 is also true. More precisely, a commutative ring  $R$  is noetherian if and only if every submodule of a finitely generated module is also finitely generated.

**Remark 3.17.** Let  $R$  be a commutative ring. Note that  $R$  is a module over itself and a submodule of  $R$  is exactly the same thing as an ideal. This module  $R$  is always finitely generated as an  $R$ -module: 1 generates  $R$ , for example. If  $R$  is not noetherian, then by Theorem 3.11  $R$  has an ideal  $I$  that is not finitely generated. Then  $I$  is a submodule of a finitely generated module that fails to be finitely generated.

**Theorem 3.18.** *Any finitely generated module  $M$  over a noetherian ring  $R$  has a finite presentation given by an  $m \times n$  matrix  $A$ , that is, there is an isomorphism*

$$M \cong R^m / \text{im}(t_A),$$

where  $t_A: R^n \rightarrow R^m$  is the map on free modules  $t_A(v) = Av$  induced by  $A$ .

*Proof.* Let  $M$  be a finitely generated module over a noetherian ring. We start by following the general argument we described in Section 3.1: we choose a finite generating set  $y_1, \dots, y_m$  of  $M$  and obtain an  $R$ -module map  $\pi: R^m \rightarrow M$  that sends  $e_i$  to  $y_i$ , by using the [UMP for free modules](#). Since every element in  $M$  is given as a linear combination of the  $y_i$ , the map  $\pi$  is surjective. Notice, however, that this representation as a linear combination of the  $y_i$  is not necessarily unique, so  $\pi$  might have a nontrivial kernel.

Since  $R^m$  is finitely generated and  $R$  is noetherian, by Theorem 3.16 the submodule  $\ker(\pi)$  is also finitely generated, say by  $z_1, \dots, z_n$ . This too leads to a surjective  $R$ -module map  $g: R^n \rightarrow \ker(\pi)$  that sends  $e_i \mapsto z_i$ . The composition of  $g: R^n \rightarrow \ker(\pi)$  followed by the inclusion of  $\iota: \ker(\pi) \hookrightarrow R^m$  is an  $R$ -module homomorphism  $t = \iota \circ g: R^n \rightarrow R^m$  and hence by Theorem 2.32 we know  $t$  is given by a  $m \times n$  matrix  $A = [t]_B^C$  with respect to the standard bases of  $R^m$  and  $R^n$  respectively, meaning  $t = t_A$ .

It remains to show that  $M \cong R^m / \text{im}(t_A)$ . First note that since  $t_A = \iota \circ g$  and  $g$  is surjective we have

$$\text{im}(t_A) = \text{im}(\iota \circ g) = \iota(\text{im}(g)) = \iota(\ker(\pi)) = \ker(\pi).$$

By the [First Isomorphism Theorem](#) we now have

$$M = \text{im}(\pi) \cong R^m / \ker(\pi) = R^m / \text{im}(t_A). \quad \square$$

### 3.3 Classification of finitely generated modules over PIDs

Since any PID is a noetherian ring, any finitely generated module  $M$  over a PID has a finite presentation matrix  $A$ . We will discuss a canonical form for such a matrix  $A$  and the consequences it has on determining the isomorphism type of  $M$ .

**Theorem 3.19** (Smith Normal Form (SNF)). *Let  $R$  be a PID and let  $A \in M_{m,n}(R)$ . Then there exist invertible matrices  $P$  and  $Q$  such that  $M = PAQ = [a_{ij}]$  satisfies the following: all nondiagonal entries of  $M$  are 0, meaning  $a_{ij} = 0$  if  $i \neq j$ , and the diagonal entries of  $M$  satisfy*

$$a_{11} \mid a_{22} \mid a_{33} \mid \dots$$

Moreover, the number  $\ell$  of nonzero entries of  $M$  is uniquely determined by  $A$ , and the nonzero diagonal entries  $a_{11}, \dots, a_{\ell\ell}$  are unique up to multiplication by units.

**Remark 3.20.** Elementary row and column operations correspond to multiplication by elementary matrices, which are invertible, and that the composition of invertible matrices is invertible. So whenever we apply elementary row and column operations, we can translate it into multiplication by an invertible matrix on the left or the right, respectively.

To transform a matrix  $A$  into its Smith Normal Form, we will use a sequence of steps that all correspond to multiplication by invertible matrices. Many of those steps will actually be elementary row and column operations, which correspond to multiplication by an elementary matrix. Elementary matrices are invertible, and a product of invertible matrices is invertible, and so any finite sequence of elementary row and column operations can be described by multiplication by an invertible matrix. However, in general not every invertible matrix can be obtained as a product of elementary matrices. In fact, there are examples of PIDs  $R$  and matrices  $A$  for which the Smith Normal Form cannot be obtained by simply taking a sequence of elementary row and column operations. However, it is not easy to give such an example, in part because when our PID  $R$  is nice enough, the Smith Normal Form can in fact be obtained by simply taking a sequence of elementary row and column operations. This is the case for Euclidean domains: over such rings, the Euclidean Algorithm for finding the gcd of two elements works, and it's the key step we will need to find a Smith Normal Form. When  $R$  is a general PID, however, we need to work a little harder.

Before we prove Theorem 3.19, let's see how to classify modules over PIDs using the Smith Normal Form for their presentation matrix. First, we need a lemma on how to interpret the module presented by a matrix in Smith Normal Form; we leave the proof as an exercise.

**Lemma 3.21.** *Let  $R$  be a commutative ring with  $1 \neq 0$ , let  $m \geq n$ , let  $A = [a_{ij}] \in M_{m,n}(R)$  be a matrix such that all nondiagonal entries of  $A$  are 0, and let  $M$  be the  $R$ -module presented by  $A$ . Then  $M \cong R^{m-n} \oplus R/(a_{11}) \oplus \cdots \oplus R/(a_{nn})$ .*

**Theorem 3.22** (Classification of finitely generated modules over a PID using invariant factors). *Let  $R$  be a PID and let  $M$  be a finitely generated module. Then there exist  $r \geq 0$ ,  $k \geq 0$ , and nonzero nonunit elements  $d_1, \dots, d_k$  of  $R$  satisfying  $d_1 \mid d_2 \mid \cdots \mid d_k$  such that*

$$M \cong R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_k).$$

*Moreover  $r$  and  $k$  are uniquely determined by  $M$ , and the  $d_i$  are unique up to associates.*

*Proof.* By Theorem 3.18,  $M$  has a presentation matrix  $A$ . By Theorem 3.19,  $A$  can be put into Smith Normal Form  $B$ , where the diagonal entries of  $B$  are  $b_1, \dots, b_\ell$  and satisfy  $b_1 \mid b_2 \mid \cdots \mid b_\ell$ . Moreover,  $k$  is unique and the  $d_i$  are uniquely determined up to associates (ie, up to multiplication by units) by  $A$ , hence by  $B$ . By Theorem 3.18,  $M$  is isomorphic to the module presented by  $B$ . By Lemma 3.21, this is isomorphic to

$$M \cong R^r \oplus R/(b_1) \oplus \cdots \oplus R/(b_\ell).$$

Finally, some of these  $b_i$  might be units; let  $d_1 \mid \cdots \mid d_k$  be the nonunits among the  $b_i$ , and note that if  $u$  is a unit, then  $R/(u) \cong (0)$ . We conclude that

$$M \cong R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_k),$$

as desired. □

**Definition 3.23.** Let  $R$  be a PID, let  $r \geq 0, k \geq 0$ , and let  $d_1, \dots, d_k$  be nonzero nonunit elements of  $R$  satisfying  $d_1 \mid d_2 \mid \dots \mid d_k$ . Let  $M$  be any  $R$ -module such that

$$M \cong R^r \oplus R/(d_1) \oplus \dots \oplus R/(d_k).$$

We say  $M$  has **free rank**  $r$  and **invariant factors**  $d_1, \dots, d_k$ .

Notice that the invariant factors of  $M$  are only defined up to multiplication by units.

**Remark 3.24.** The classification theorem can be interpreted as saying that  $M$  decomposes into a free submodule  $R^r$  and a torsion submodule  $\text{Tor}(M) = R/(d_1) \oplus \dots \oplus R/(d_k)$ .

**Corollary 3.25** (Classification of finitely generated abelian groups). *Let  $G$  be a finitely generated abelian group. Then*

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \dots \oplus \mathbb{Z}/n_k$$

for some  $r \geq 0, k \geq 0$ , and  $n_i \geq 2$  for all  $i$ , satisfying  $n_{i+1} \mid n_i$  for all  $i$ . Moreover, the integers  $r, k$ , and  $n_1, \dots, n_k$  are uniquely determined by  $G$ .

**Example 3.26.** Consider the  $\mathbb{Z}$ -module  $M$  presented by the matrix

$$A = \begin{bmatrix} 1 & 6 & 5 & 2 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & 3 & 0 \end{bmatrix}.$$

We can obtain the Smith Normal Form as follows:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 6 & 5 & 2 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & 3 & 0 \end{bmatrix} \xrightarrow[R3 \rightarrow R3 - 3R1]{R2 \rightarrow R2 - 2R1} \begin{bmatrix} 1 & 6 & 5 & 2 \\ 0 & -11 & -11 & -4 \\ 0 & -18 & -12 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -11 & -11 & -4 \\ 0 & -18 & -12 & -6 \end{bmatrix} \\ &\xrightarrow{C2 \leftrightarrow C4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -11 & -11 \\ 0 & -6 & -12 & -18 \end{bmatrix} \xrightarrow[C4 \rightarrow C4 + 3C1]{C3 \rightarrow C3 + 2C2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -3 & 1 \\ 0 & -6 & 0 & 0 \end{bmatrix} \xrightarrow{C2 \leftrightarrow C4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & -6 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \xrightarrow{C3 \leftrightarrow C4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \xrightarrow{C3 \rightarrow -C3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}. \end{aligned}$$

Thus the Smith normal form of  $A$  is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix},$$

with invariant factor  $d_1 = 6$ . Notice that the two ones are not invariant factors: we only care about nonunits. Therefore we have

$$M \cong \mathbb{Z}/(1) \oplus \mathbb{Z}/(1) \oplus \mathbb{Z}/(6) \cong \mathbb{Z}/(6).$$

Here is a spinoff of the [classification theorem](#).

**Theorem 3.27** (Classification of finitely generated modules over a PID using elementary divisors). *Let  $R$  be a PID and let  $M$  be a finitely generated module. Then there exist  $r \geq 0$ ,  $s \geq 0$ , prime elements  $p_1, \dots, p_s$  of  $R$  (not necessarily distinct), and  $e_1, \dots, e_s \geq 1$  such that*

$$M \cong R^r \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_s^{e_s}).$$

*Moreover,  $r$  and  $s$  are uniquely determined by  $M$ , and the list  $p_1^{e_1}, \dots, p_s^{e_s}$  is unique up to associates and reordering.*

*Proof.* First, write  $M$  in invariant factor form  $M \cong R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_k)$ . Then write each invariant factor as a product of prime powers

$$d_i := \prod_{j=n_i}^{n_{i+1}} p_j^{e_j},$$

and recall that by the CRT we have

$$R/(d_i) \cong R/(p_{n_i}^{e_{n_i}}) \oplus \cdots \oplus R/(p_{n_{i+1}}^{e_{n_{i+1}}}).$$

Substituting into the invariant factor form gives the desired result. Uniqueness follows from the uniqueness of the invariant factor form and of the prime factorizations of each  $d_i$ .  $\square$

**Definition 3.28.** Let  $R$  be a PID, let  $r \geq 0$ ,  $s \geq 0$ ,  $p_1, \dots, p_s$  be prime elements of  $R$ , and let  $e_1, \dots, e_s \geq 1$ . Let  $M$  be the  $R$ -module  $M \cong R^r \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_s^{e_s})$ . The elements  $p_1^{e_1}, \dots, p_s^{e_s}$  of  $R$  are the **elementary divisors** of  $M$ .

Careful that a particular prime might appear repeatedly in the elementary divisors of a particular module.

**Example 3.29.** When  $R = \mathbb{Z}$  and  $M = \mathbb{Z}/(6)$ , we can write  $M \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$ , so the elementary divisors are 2 and 3.

**Corollary 3.30.** *Let  $G$  be a finitely generated abelian group. Then there exist  $r, s \geq 0$ , prime integers  $p_1, \dots, p_s$ , and positive integers  $e_i \geq 1$  such that*

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_s^{e_s}.$$

*Moreover,  $r$ ,  $p_i$ , and  $e_i$  are all uniquely determined by  $G$ .*

We have yet to show Theorem 3.19: every matrix over a PID has a Smith Normal Form. We will need a few auxiliary lemmas.

**Definition 3.31.** Let  $R$  be a PID and let  $a_1, \dots, a_n \in R$ . The **greatest common divisor** or **gcd** of  $a_1, \dots, a_n$ , denoted  $\gcd(a_1, \dots, a_n)$ , is a generator for the principal ideal  $(a_1, \dots, a_n)$ . Given a matrix  $A \in M_{m,n}(R)$ ,  $\gcd(A)$  is the gcd of the entries of  $A$ . We adopt the convention that  $\gcd(0, 0) = 0$  and thus if  $A$  is the matrix of all zeroes, then  $\gcd(A) = 0$ .

Notice that the greatest common divisor is only defined up to multiplication by a unit.

**Lemma 3.32.** *Let  $R$  be a PID. Let  $A \in M_{m,n}(R)$  be any matrix and let  $P \in M_m(R)$  and  $Q \in M_n(R)$  be invertible matrices. Then  $\gcd(A) = \gcd(PA) = \gcd(QA)$ . In particular, if  $B \in M_{m,n}(R)$  and  $B$  is obtained from  $A$  by a finite sequence of elementary row and column operations, then  $\gcd(A) = \gcd(B)$ .*

*Proof.* First, suppose that  $n = 1$ , meaning that  $A$  is a column, say

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \text{ and let } PA = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

We need to show that  $(a_1, \dots, a_m) = (b_1, \dots, b_m)$ . On the one hand, each  $b_i$  is a linear combination of the  $a_j$ , so  $(b_1, \dots, b_m) \subseteq (a_1, \dots, a_m)$ . On the other hand,

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = P^{-1}(PA) = P^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

so each  $a_j$  is a combination of the  $b_i$ , and  $a_j \in (b_1, \dots, b_m)$ . We conclude that we have an equality of ideals  $(a_1, \dots, a_m) = (b_1, \dots, b_m)$ , and thus multiplying a column vector by an invertible matrix does not change the greatest common divisor of the entries.

Now given  $A \in M_{m,n}(R)$ , if we denote the  $i$ th column of  $A$  by  $A_i$ , we have

$$PA = [PA_1 \quad \dots \quad PA_m].$$

Since the  $\gcd$  of each column remains the same, the  $\gcd$  of all the entries does not change.

To show that  $\gcd(AQ) = \gcd(A)$ , note that transposing a matrix does not change its  $\gcd$  nor the fact that it is invertible, so we can apply what we have already shown:

$$\gcd(AQ) = \gcd((AQ)^T) = \gcd(Q^T A^T) = \gcd(A^T) = \gcd(A).$$

Finally, applying elementary row and column operations corresponds to multiplying by an elementary matrix on the left or right, and elementary matrices are invertible.  $\square$

**Lemma 3.33.** *Let  $R$  be a PID and  $x, y \in R$ . There exists an invertible  $2 \times 2$  matrix  $P \in M_2(R)$  such that*

$$P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \gcd(x, y) \\ 0 \end{bmatrix}.$$

*Proof.* By definition of greatest common divisor,  $(x, y) = (\gcd(x, y))$ , so there exist  $a, b \in R$  such that  $ax + by = \gcd(x, y)$ . Write  $g := \gcd(x, y)$  and  $h = \gcd(a, b)$ . Then  $ax + by$  is a multiple of  $gh$ , but since  $ax + by = g$  and  $R$  is a domain, we conclude that  $h$  must be a unit, and  $(a, b) = (h) = (1)$ . In particular, we can find  $c, d \in R$  such that  $ad - bc = 1$ . Finally,  $bx + cy \in (x, y) = (g)$ , so

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ eg \end{bmatrix}.$$

Now we can apply the row operation that adds  $-e$  times the first row to the second row: by setting

$$P := \begin{bmatrix} 1 & 0 \\ -e & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c - ea & b - de \end{bmatrix}.$$

we get

$$P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}.$$

Finally, one can easily check that

$$P^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix}. \quad \square$$

By transposing the matrices in Lemma 3.33, we can show that there exists an invertible  $2 \times 2$  matrix  $Q$  such that

$$\begin{bmatrix} x & y \end{bmatrix} Q = \begin{bmatrix} \gcd(x, y) & 0 \end{bmatrix}.$$

**Exercise 14.** Show that for any  $i \geq 1$  and any commutative ring  $R$ , the ideal generated by the  $i \times i$  minors of a matrix with entries in  $R$  is unchanged by row and column operations.

We are now finally ready to show that every matrix over a PID can be put into Smith Normal Form.

**Theorem 3.19.** (Smith Normal Form) Let  $R$  be a PID and let  $A \in M_{m,n}(R)$ . There exist invertible matrices  $P$  and  $Q$  such that  $M = PAQ = [a_{ij}]$  satisfies the following: all nondiagonal entries of  $M$  are 0, meaning  $a_{ij} = 0$  if  $i \neq j$ , and the diagonal entries of  $M$  satisfy

$$a_{11} \mid a_{22} \mid a_{33} \mid \cdots.$$

Moreover, the number  $\ell$  of nonzero entries of  $M$  is uniquely determined by  $A$ , and the nonzero diagonal entries  $a_{11}, \dots, a_{\ell\ell}$  are unique up to multiplication by units.

*Proof.* Before we begin, note that we will apply a sequence of steps that correspond to multiplication by an invertible matrix on the left or right, and by Lemma 3.32, none of these steps will change the gcd.

To prove existence of such a matrix  $M$ , we claim we can multiply  $A$  on the right and the left by invertible matrices to transform it into a matrix of the form

$$\begin{bmatrix} g & 0 \\ 0 & B \end{bmatrix}$$

for some  $(n-1) \times (m-1)$  matrix  $B$ , where  $g = \gcd(A)$ . If our claim holds, then we are done: notice that  $g$  divides every entry of  $B$ , since  $\gcd(A) = \gcd(g, B)$  by Lemma 3.32, and so by applying this fact to  $B$ , and then over and over again, we arrive at a matrix of the desired form  $M$ . Notice moreover that if  $C$  is an invertible matrix, then so is

$$\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix}.$$



To construct a matrix in the form above, let  $a$  be the upper left  $(1, 1)$  entry of  $A$ . First, we are going to show that we can turn  $A$  into a matrix of the form

$$\begin{bmatrix} * & 0 \\ 0 & B \end{bmatrix}$$

with the same gcd as  $A$ . If  $a$  happens to divide all the entries on the first row and column, then we can simply apply elementary row and column operations to get to the desired form. Suppose there exists  $b$  on the first column such that  $a \nmid b$ . Then we may apply an elementary row operation switching rows so that  $b = a_{2,1}$  is on the first column, second row. By Lemma 3.33, we can now find an invertible matrix  $C \in M_2(R)$  such that

$$C \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \gcd(a, b) \\ 0 \end{bmatrix}.$$

Consider the  $m \times m$  matrix

$$P := \begin{bmatrix} C & 0 \\ 0 & I_{m-2} \end{bmatrix}.$$

Note that  $PA$  has  $(PA)_{1,1} = \gcd(a, b)$  and  $(PA)_{2,1} = 0$ , so this step replaces  $a$  by  $\gcd(a, b)$  and  $b$  by 0. By Lemma 3.32,  $\gcd(PA) = \gcd(A)$ . We can keep repeating this until the top left corner entry divides every entry on the first column, and this process must stop after at most  $m - 1$  steps, since there are only  $m$  elements on the first column.

Similarly, if there exists  $b$  on the first row that  $a$  does not divide, we can repeat this process by instead multiplying  $A$  on the left by an invertible matrix, until we zero out all the remaining entries on the first row and column. Finally, we arrive at a matrix of the form

$$\begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

If  $a = \gcd(A)$ , we are done. If not, then we can find some entry  $b = a_{i,j}$  such that  $a \nmid b$ . We can then add the  $j$ th column to the first column, which puts  $b$  into the first column without affecting  $a$ , since the remainder of the top row is zero. But this brings us back to the previous situation, and we have already shown that we can replace the top left corner by  $\gcd(a, b)$ .

At each step, we replace  $a$  by some  $c$  with is both a divisor of  $a$  and a multiple of  $\gcd(A)$ . Our ring  $R$  is a UFD, so there are finitely many factors of  $a/\gcd(A)$ , and this process must stop. This shows that we can eventually replace  $A$  by

$$\begin{bmatrix} \gcd(A) & 0 \\ 0 & B \end{bmatrix}.$$

Now it remains to show the uniqueness portion of the theorem. For any  $i$  and any matrix  $B$ , let  $\gcd_i(B)$  denote the gcd of all the  $i \times i$  minors of  $B$ . By Exercise 14,  $\gcd_i$  is unchanged by row and column operations, so  $\gcd_i(A) = \gcd_i(M)$ .

For a matrix of the form  $M$ , the only minors that are nonzero are those where the choices of columns and rows are the same, and hence the only nonzero  $i \times i$  minors of  $M$  are  $g_{s_1} \cdots g_{s_i}$  for some  $s_1 < \cdots < s_i$ . Since  $g_{s_1} \cdots g_{s_i}$  divide each other, it follows that

$$\gcd_i(A) = \gcd_i(M) = g_1 \cdots g_i.$$

In particular, the largest value of  $i$  such that some  $i \times i$  minor of  $A$  is nonzero is  $\ell$ . Also, we have

$$g_i = \frac{\gcd_i(A)}{\gcd_{i-1}(A)}.$$

This proves uniqueness, for it shows that  $\ell, g_1, \dots, g_\ell$  are all defined from  $A$  directly, without any choices.  $\square$

**Example 3.34.** Consider the PID  $R = k[x]$ , where  $k$  is any field, and the matrix

$$A = \begin{bmatrix} x-1 & 0 \\ 1 & x-2 \end{bmatrix}.$$

The first row has already been zeroed out, but unfortunately  $x-1$  does not divide 1. In this case, though, we can see that  $\gcd(A) = 1$ , so we can switch the first and second rows to get

$$\begin{bmatrix} 1 & x-2 \\ x-1 & 0 \end{bmatrix}.$$

Now we zero out the rest of the first row and first column using row and column operations:

$$\begin{bmatrix} 1 & x-2 \\ x-1 & 0 \end{bmatrix} \xrightarrow{R2 \rightarrow R2 - (x-1)R1} \begin{bmatrix} 1 & x-2 \\ 0 & -(x-1)(x-2) \end{bmatrix} \xrightarrow{C2 \rightarrow C2 - (x-2)C1} \begin{bmatrix} 1 & 0 \\ 0 & -(x-1)(x-2) \end{bmatrix}.$$

This is a Smith Normal Form. If we prefer to not have that negative sign, we can multiply the second row by  $-1$ , to obtain

$$\begin{bmatrix} 1 & x-2 \\ x-1 & 0 \end{bmatrix} \xrightarrow{R2 \rightarrow R2 - (x-1)R1} \begin{bmatrix} 1 & x-2 \\ 0 & -(x-1)(x-2) \end{bmatrix} \xrightarrow{C2 \rightarrow C2 - (x-2)C1} \begin{bmatrix} 1 & 0 \\ 0 & (x-1)(x-2) \end{bmatrix}.$$

There is only one invariant factor, which is  $(x-1)(x-2)$ . The  $k[x]$ -module  $M$  presented by  $A$  is

$$M \cong k[x]/((x-1)(x-2)).$$

If we prefer to write this in terms of elementary divisors, our module has two:  $x-1$  and  $x-2$ , and it is isomorphic to

$$M \cong k[x]/(x-1) \oplus k[x]/(x-2).$$

# Chapter 4

## Canonical forms for endomorphisms

### 4.1 Rational canonical form

Recall that given an  $F$ -vector space  $V$  with  $\dim_F(V) = n$  and an ordered basis  $B$  for  $V$  we have proven in Proposition 2.32 that  $\text{End}_F(V) \cong M_n(F)$  via the maps  $t \mapsto [t]_B^B$  and  $A \mapsto t_A$ . Recall also from Lemma 1.48 that to give a finitely generated module over  $F[x]$  is the same data as a finite dimensional vector space  $V$  and a linear transformation  $V \rightarrow V$ :

**Definition 4.1.** Let  $F$  be a field, let  $V$  be a finite dimensional vector space over  $F$ , and let  $t : V \rightarrow V$  be a linear transformation. The  $F[x]$ -module  $V_t$  is defined to be the vector space  $V$  with the unique  $F[x]$ -action satisfying  $xv = t(v)$  for all  $v \in V$ . That is,

$$(r_n x^n + \cdots + r_0)v = r^n t^n(v) + \cdots + r_0 v \quad \text{for all } r_n x^n + \cdots + r_0 \in F[x].$$

**Theorem 4.2.** Let  $F$  be a field, let  $V$  be an  $F$ -vector space of dimension  $n$ , let  $t : V \rightarrow V$  be a linear transformation, let  $B$  be an ordered basis for  $V$ , and let  $A = [t]_B^B$ . Then the matrix  $xI_n - A \in M_n(F[x])$  presents the  $F[x]$ -module  $V_t$ .

*Proof.* Let  $B = \{b_1, \dots, b_n\}$  be any basis for  $V$ , and note that  $B$  is a generating set for  $V_t$  as a module over  $F[x]$ . As we described in Section 3.1,  $V_t$  can then be written as a quotient of  $F[x]^n$ . More precisely, let  $e_1, \dots, e_n$  denote the standard  $F[x]$ -basis for the free  $F[x]$ -module  $F[x]^n$ , and let  $\pi : F[x]^n \rightarrow V_t$  be the surjective  $F[x]$ -module homomorphism sending  $e_i$  to  $b_i$ . That is,

$$\pi((g_1(x), \dots, g_n(x))) = \pi\left(\sum_{i=1}^n g_i(x)e_i\right) = \sum_{i=1}^n g_i(x)b_i = \sum_{i=1}^n g_i(t)b_i.$$

By the First Isomorphism Theorem, we have  $V_t \cong F[x]^n / \ker(\pi)$ . On the other hand, the matrix  $xI_n - A$  determines a map

$$t_{xI_n - A} : F[x]^n \rightarrow F[x]^n,$$

and to show that  $V_t \cong F[x]^n / \text{im}(t_{xI_n - A})$  it suffices to show that  $\text{im}(t_{xI_n - A}) = \ker(\pi)$ . Now

$$(\pi \circ t_{xI_n - A})(e_i) = \pi((xI_n - A)e_i) = (xI_n - A)\pi(e_i) = (xI_n - A)b_i = xb_i - Ab_i = t(b_i) - t(b_i) = 0.$$

This proves  $\text{im}(xI_n - a) \subseteq \ker(\pi)$ . It follows by Theorem 1.43 that there is a surjection of  $F[x]$ -modules

$$W := F[x]^n / \text{im}(xI_n - A) \twoheadrightarrow V_t.$$

We may also regard this as a surjection of  $F$ -vector spaces. Since  $\dim_F(V_t) = n$  and the map above is surjective, we have  $\dim_F(W) \geq n$ , which follows from the Rank Nulity Theorem. To establish that the map above is an isomorphism, it suffices to show that  $\dim_F(W) \leq n$ .

Denote by  $c_i = e_i + \text{im}(xI_n - A)$  the image of the standard basis of  $F[x]^n$  in  $W$ . The  $i$ th column of  $xI_n - A$  gives the relation  $xc_i = v_i$  in  $W$ , where  $v_i$  is the  $i$ -th column of  $A$ . It follows that  $p(x)c_i = p(A)c_i$  in  $W$  for any polynomial  $p(x)$ . Thus a typical element of  $W$ , given by  $\sum_i g_i(x)c_i$ , is equal to  $g_1(A)c_1 + \cdots + g_n(A)c_n$ . Such an expression belongs to the  $F$ -span of  $c_1, \dots, c_n$  in  $W$ ; that is,  $c_1, \dots, c_n$  span  $W$  as an  $F$ -vector space. Therefore, we have the desired inequality  $\dim_F(W) \leq n$ , which completes our proof.  $\square$

**Corollary 4.3.** *Suppose  $F$  is a field,  $V$  is an  $F$ -vector space, and  $t: V \rightarrow V$  is a linear transformation. There exist unique monic polynomials  $g_1 | \cdots | g_k \in F[x]$  of positive degree and an  $F[x]$ -module isomorphism*

$$V_t \cong F[x]/(g_1) \oplus \cdots \oplus F[x]/(g_k).$$

The polynomials  $g_1, \dots, g_k$  are both the invariant factors of the  $F[x]$ -module  $V_t$  and the entries on the diagonal of the Smith normal form of  $xI_n - [t]_B^B$  for any basis  $B$  of  $V$ .

*Proof.* Theorem 4.2 says that  $xI_n - [t]_B^B$  presents the  $F[x]$ -module  $V_t$ , and the remainder of the statement is an immediate application of the Classification of finitely generated modules over PIDs to this special case once we show that there is no free summand. Note that  $F[x]$  is an infinite dimensional vector space over  $F$ , while  $V_t$  is a finite dimensional vector space. If  $V_t$  had a free summand, then it would contain an infinite linearly independent set over  $F$ , and thus it could not be finite-dimensional.  $\square$

**Definition 4.4.** The polynomials  $g_1, \dots, g_k$  in Corollary 4.3 are called the **invariant factors** of the linear transformation  $t$ .

**Example 4.5.** Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{Q}).$$

Then

$$xI_2 - A = \begin{bmatrix} x-1 & -1 \\ 0 & x-1 \end{bmatrix}.$$

We could compute the invariant factors of  $t: \mathbb{Q}^2 \rightarrow \mathbb{Q}^2$  by appealing to the Smith Normal Form of  $xI_2 - A$ , but let us try another way. Let

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

be the Smith Normal Form of  $xI_2 - A$ . Recall from the proof of Theorem 3.19 that  $d_1$  is the gcd of the entries of  $xI_2 - A$  and  $d_1 d_2 = \det(xI_2 - A)$ . Thus  $d_2 = \det(xI_2 - A) = (x-1)^2$  and  $d_1 = 1$ . Therefore the only invariant factor of  $t_A$  is  $(x-1)^2$ .

You will show the following lemma in Problem Set 6:

**Lemma 4.6.** *For a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$  with  $n \geq 1$ , the classes of  $1, x, \dots, x^{n-1}$  form a basis for  $F[x]/(f(x))$  regarded as an  $F$ -vector space. Relative to this basis, the  $F$ -linear operator  $l_x : F[x]/(f(x)) \rightarrow F[x]/(f(x))$  defined by  $l_x(v) = xv$  is given by the following matrix:*

$$C(f) := \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ & & & -a_1 \\ & I_{n-1} & & \vdots \\ & & & -a_{n-1} \end{bmatrix}.$$

**Definition 4.7.** In the setup of Lemma 4.6, the matrix  $C(f)$  is called the **companion matrix** of the monic polynomial  $f$ .

**Definition 4.8.** Given square matrices  $A_1, \dots, A_m$  with entries in a ring  $R$ , not necessarily of the same size, we define  $A_1 \oplus \cdots \oplus A_m$  to be the block diagonal matrix

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_m \end{bmatrix}.$$

**Remark 4.9.** If  $f : V_1 \rightarrow W_1$  and  $g : V_2 \rightarrow W_2$  are linear transformations, then the map  $f \oplus g : V_1 \oplus V_2 \rightarrow W_1 \oplus W_2$  given by  $(f \oplus g)(a, c) = (f(a), g(c))$  is a linear transformation. If  $B_i$  is a basis for  $V_i$  and  $C_i$  is a basis for  $W_i$ , and  $\iota_i : A_i \hookrightarrow A_1 \oplus A_2$  are the natural inclusions, then  $\mathcal{B} = \iota_1(B_1) \cup \iota_2(B_2)$  is a basis for  $V_1 \oplus V_2$ ,  $\mathcal{C} = \iota_1(C_1) \cup \iota_2(C_2)$  is a basis for  $W_1 \oplus W_2$ , and

$$[f \oplus g]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} [f]_{B_1}^{C_1} & 0 \\ 0 & [g]_{B_2}^{C_2} \end{bmatrix}.$$

**Theorem 4.10** (Rational Canonical Form). *Let  $F$  be a field,  $V$  a finite dimensional  $F$ -vector space, and  $t : V \rightarrow V$  an  $F$ -linear transformation. There is a basis  $B$  of  $V$  such that*

$$[t]_B^B = C(g_1) \oplus \cdots \oplus C(g_k) = \begin{bmatrix} C(g_1) & 0 & 0 & \cdots & 0 \\ 0 & C(g_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & C(g_k) \end{bmatrix}$$

where  $g_1, \dots, g_k$  are the invariant factors of  $t$ , meaning they are monic polynomials of positive degree such that  $g_1 \mid g_2 \mid \cdots \mid g_k$ . Moreover, the polynomials  $g_1, \dots, g_k$  are unique.

*Proof.* By Corollary 4.3,  $V_t \cong \bigoplus_{i=1}^k F[x]/(g_i(x))$  for some unique  $g_i$  as in the statement. Set  $V_i = F[x]/(g_i(x))$  and note that  $V_t = V_1 \oplus \cdots \oplus V_k$ . The map  $l_x : V_t \rightarrow V_t$  given by multiplication by  $x$  preserves each summand in this decomposition:  $l_x(V_i) \subseteq V_i$ . Thus if we choose a basis  $B_i$  of each summand  $V_i$  and set  $B = \bigcup_{i=1}^k \iota_i(B_i)$ , by Remark 4.9,  $B$  is a basis of  $V_t$  and  $[t]_B^B = [t]_{V_1}^{B_1} \oplus \cdots \oplus [t]_{V_k}^{B_k}$ . The result now follows from Lemma 4.6.  $\square$

**Definition 4.11.** In the setup of Theorem 4.10, the matrix  $C(g_1) \oplus \cdots \oplus C(g_k)$  is called the **rational canonical form** (RCF) of the linear transformation  $t$ . The rational canonical form of a matrix  $A \in M_n(F)$  is defined to be the rational canonical form of the endomorphism  $t_A$  represented by  $A$  with respect to the standard basis of  $F^n$ .

**Example 4.12.** Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{Q})$  as in Example 4.5. Because the only invariant factor of  $xI_2 - A$  is  $(x - 1)^2$ , the Rational Canonical Form of  $t_A$  is

$$RCF(A) = C((x - 1)^2) = C(x^2 - 2x + 1) = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

We will later show that two matrices have the same Rational Canonical Form if and only if they are similar.

## 4.2 The Cayley-Hamilton Theorem

**Definition 4.13.** Let  $F$  be a field and let  $A \in M_n(F)$ . The **characteristic polynomial** of  $A$  is the polynomial  $c_A = \det(xI_n - A)$ .

**Definition 4.14.** Let  $V$  be an  $F$ -vector space of dimension  $n$ , and let  $t : V \rightarrow V$  be a linear transformation. The **characteristic polynomial** of  $t$ , denoted  $c_t$ , is the characteristic polynomial  $c_A$  for a matrix  $A = [t]_B^B$  with respect to some ordered basis  $B$  of  $V$ .

Characteristic polynomials are well-defined.

**Remark 4.15.** We need to check that the characteristic polynomial of a linear transformation is invariant under base changes. More precisely, we need to check that if we choose two different basis  $B$  and  $B'$  for  $V$ , then the matrices  $A = [t]_B^B$  and  $C = [t]_{B'}^{B'}$  have the same characteristic polynomial. First, recall that  $A$  and  $C$  are similar matrices, by Theorem 2.39, so  $C = PAP^{-1}$  for some invertible matrix  $P$ . Moreover, diagonal matrices are in the center of  $M_n(R)$ , meaning they commute with other matrices, and thus we have the following:

$$\begin{aligned} \det(xI_n - C) &= \det(xI_n - PAP^{-1}) \\ &= \det(P(xI_n - A)P^{-1}) \\ &= \det(P) \det(xI_n - A) \det(P^{-1}) \\ &= \det(xI_n - A). \end{aligned}$$

We conclude that  $A$  and  $B$  have the same characteristic polynomial.

**Remark 4.16.** For any matrices  $A$  and  $B$ ,  $c_{A \oplus B} = c_A c_B$ .

**Definition 4.17.** Let  $F$  be a field and let  $A \in M_n(F)$ . The **minimal polynomial** of  $A$ , denoted  $m_A$ , is the unique monic polynomial that generates the principal ideal

$$\{f(x) \in F[x] \mid f(A) = 0\}.$$

**Definition 4.18.** Let  $V$  be an  $F$ -vector space of dimension  $n$ , and let  $t : V \rightarrow V$  be a linear transformation. The **minimal polynomial** of  $t$ , denoted  $m_t$ , is the unique monic polynomial generating the ideal  $\text{ann}_{F[x]}(V_t)$  in the PID  $F[x]$ .

**Lemma 4.19.** Let  $F$  be a field. Let  $V$  be an  $F$ -vector space of dimension  $n$  with basis  $B$  and let  $t : V \rightarrow V$  be a linear transformation. The minimal polynomial  $m_A$  of  $A = [t]_B^B$  satisfies  $m_A = m_t$ .

*Proof.* Since  $m_A$  and  $m_t$  are both monic, it's sufficient to show  $\text{ann}_{F[x]}(V_t) = (m_A)$ . Indeed,

$$\begin{aligned}
 f \in \text{ann}_{F[x]}(V_t) &\iff f(x)v = 0 \text{ for all } v \in V_t \\
 &\iff f(A)v = 0 \text{ for all } v \in V_t \\
 &\iff \ker(f(A)) = V_t \\
 &\iff \text{rank}(f(A)) = 0 && \text{by the Rank-Nullity Theorem} \\
 &\iff f(A) = 0 \\
 &\iff f \in (m_A) && \text{by definition of } m_A. \quad \square
 \end{aligned}$$

**Remark 4.20.** If  $m(x)$  is the minimal polynomial of an endomorphism  $t$  and  $f(x)$  is another polynomial such that  $f(x)$  annihilates  $V_t$ , then  $f(x) \in \text{ann}(V_t) = (m(x))$ , and thus  $m(x) \mid f(x)$ .

Similarly, suppose that  $m(x)$  is the minimal polynomial of a matrix  $A$  and  $f(x)$  is another polynomial such that  $f(A) = 0$ . By Lemma 4.19, we know that  $m(x)$  is also the minimal polynomial of the linear transformation  $t : v \mapsto Av$ , and that  $f(x)$  also annihilates  $V_t$ . Thus we can also conclude that  $m(x) \mid f(x)$ .

**Lemma 4.21.** Let  $F$  be a field, let  $V$  be a finite dimensional  $F$ -vector space, and  $t : V \rightarrow V$  be a linear transformation with invariant factors  $g_1 \mid \cdots \mid g_k$ . Then  $c_t = g_1 \cdots g_k$  and  $m_t = g_k$ .

*Proof.* The product of the elements on the diagonal of the Smith Normal Form of  $xI_n - A$  is the determinant of  $xI_n - A$ . Thus the product of the invariant factors  $g_1 \cdots g_k$  of  $V_t$  is the characteristic polynomial  $c_t$  of  $t$ . Notice here that we chose our invariant factors  $g_1, \dots, g_k$  to be monic, so that  $g_1 \cdots g_k$  is monic, and thus actually equal to  $c_t$  (not just up to multiplication by a unit).

By Problem Set 5,  $\text{ann}_{F[x]}(V_t) = (g_k)$ , and since  $g_k$  is monic we deduce that  $m_t = g_k$ .  $\square$

We can now prove the famous Cayley-Hamilton theorem.

**Theorem 4.22** (Cayley-Hamilton). Let  $F$  be a field, and let  $V$  be a finite dimensional  $F$ -vector space. If  $t : V \rightarrow V$  is a linear transformation, then  $m_t \mid c_t$ , and hence  $c_t(t) = 0$ . Similarly, for any matrix  $A \in M_n(F)$  over a field  $F$  we have  $m_A \mid c_A$  and  $c_A(A) = 0$ .

*Proof.* Let  $A = [t]_B^B$  for some basis  $B$  of  $V$ . Note that the statements about  $A$  and  $t$  are equivalent, since by definition  $c_A = c_t$ , while  $m_A = m_t$  we have  $f(A) = 0$  if and only if  $f(t) = 0$ . So write  $m = m_A = m_t$  and  $c = c_A = c_t$ .

By Lemma 4.21,  $m = g_k$  and  $c = g_1 \cdots g_k$ , so  $m \mid c$ . By definition, we  $m(A) = 0$ . Since  $m \mid c$ , we conclude that  $c(A) = 0$ .  $\square$

**Remark 4.23.** As a corollary of the [Cayley-Hamilton Theorem](#), we obtain that the minimal polynomial of  $t: V \rightarrow V$  has degree at most  $n = \dim(V)$ , since  $m_t$  divides  $c_t$ , which is a polynomial of degree  $n$ .

**Lemma 4.24.** *Let  $F$  be a field and let  $V$  be a finite dimensional  $F$ -vector space. If  $t: V \rightarrow V$  is a linear transformation, then  $c_t \mid m_t^k$ .*

*Proof.* Since  $g_i \mid g_k$  for  $1 \leq i \leq k$ , we have  $c_t = g_1 \cdots g_k \mid g_k^k = m_t^k$ .  $\square$

It follows that  $c_t$  and  $m_t$  have the same roots, not counting multiplicities.

**Definition 4.25.** Let  $V$  be  $t: V \rightarrow V$  be a linear transformation over a field  $F$ . A nonzero element  $v \in V$  satisfying  $t(v) = \lambda v$  for some  $\lambda \in F$  is an **eigenvector** of  $t$  with **eigenvalue**  $\lambda$ . Similarly, given a matrix  $A \in M_n(F)$ , a nonzero  $v \in F^n$  satisfying  $Av = \lambda v$  for some  $\lambda \in F$  is an **eigenvector** of  $A$  with **eigenvalue**  $\lambda$ .

**Theorem 4.26.** *Let  $f \in F$ . The following are equivalent:*

- (1)  $\lambda$  is an eigenvalue of  $t$ .
- (2)  $\lambda$  is a root of  $c_t$ .
- (3)  $\lambda$  is a root of  $m_t$ .

*Proof.* By the [Cayley-Hamilton Theorem](#),  $m_t \mid c_t$ , and thus (3)  $\Rightarrow$  (2). On the other hand, by Lemma 4.24 we know that  $c_t \mid m_t^k$ , so if  $c_t(\lambda) = 0$  then  $m_t(\lambda)^k = 0$ , and since we are over a field, we conclude that  $m_t(\lambda) = 0$ . This shows (2)  $\Rightarrow$  (3).

Finally, to show that (1)  $\Leftrightarrow$  (2), notice that the scalar  $\lambda \in F$  is an eigenvalue of  $A$  if and only if there is a nonzero solution  $v$  to  $(\lambda I_n - A)v = 0$ . This happens if and only if  $\lambda I_n - A$  has a nontrivial kernel, or equivalently if  $\lambda I - A$  is not invertible. Thus  $\lambda \in F$  is an eigenvalue of  $A$  if and only if it is a root of its characteristic polynomial  $c_A(x) = \det(xI_n - A)$ , meaning  $c_A(\lambda) = 0$ .  $\square$

**Theorem 4.27.** *Let  $F$  be a field and let  $A, A' \in M_n(F)$ . The following are equivalent:*

- (1)  $A$  and  $A'$  are similar matrices.
- (2)  $A$  and  $A'$  have the same Rational Canonical Form.
- (3)  $A$  and  $A'$  have the same invariant factors.

*Proof.* To show (1)  $\Rightarrow$  (2), suppose  $A$  is similar to  $A'$ . Then there exists an invertible matrix  $P$  such that  $A' = PAP^{-1}$ , and thus

$$xI_n - A' = xI_n PAP^{-1} = P(xI_n - A)P^{-1}.$$

Thus the matrices  $xI_n - A$  and  $xI_n - A'$  are also similar. Moreover, by definition we see that similar matrices have the same Smith normal form, and thus  $A$  and  $A'$  have the Rational Canonical Form. The invariant factors can be read off of the Rational Canonical Form, and thus (2)  $\Rightarrow$  (3).

Finally, to show (3)  $\Rightarrow$  (1) notice that if  $A$  and  $A'$  have the same invariant factors then there is an isomorphism of  $F[x]$ -modules  $F_{t_A}^n \cong F_{t_{A'}}^n$ , which implies by a homework problem in Problem Set 6 that  $A$  and  $A'$  must be similar.  $\square$



**Example 4.28.** Let us find the minimal and characteristic polynomials of  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given as rotation by 60 degrees counter-clockwise. We could write this down as matrix and compute its characteristic polynomial, but a simpler way is to notice that  $T^3 = -I_2$ , and so  $T$  satisfies the polynomial  $x^3 + 1 = (x + 1)(x^2 - x + 1)$ . Its minimal polynomial must therefore divide  $x^3 + 1$ . Since  $x^3 + 1 = (x + 1)(x^2 - x + 1)$  and  $x^2 - x + 1$  is irreducible in  $\mathbb{R}[x]$ , we conclude that the minimal polynomial of  $T$ , which we know has degree at most 2, must be either  $x + 1$  or  $x^2 - x + 1$ . If  $m_T = x + 1$ , then  $T$  would be  $-I_2$ , which is clearly incorrect. So the minimal polynomial of  $T$  must be  $x^2 - x + 1$ . By [Cayley-Hamilton](#), this polynomial must divide the characteristic polynomial, and since the latter also has degree two, we conclude that

$$c_T(x) = x^2 - x + 1.$$

Since this is irreducible, in this example we have no choice for how to form the invariant factors: there must just be one of them,  $c_T(x)$  itself. So

$$C(x^2 - x + 1) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

is the rational canonical form of  $T$ .

**Example 4.29.** Let's find the minimal polynomial of

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By the [Cayley-Hamilton Theorem](#),  $m_A(x) \mid c_A(x)$ . The polynomial  $c_A(x)$  is easy to compute since this matrix is upper-triangular:

$$c_A(x) = \det(xI_4 - A) = (x - 1)^4.$$

So  $m_A(x) = (x - 1)^j$  for some  $j \leq 4$ . By brute-force, we verify that  $(A - I_4)^3 \neq 0$  and thus it must be the case that  $m_A(x) = c_A(x) = (x - 1)^4$ .

**Example 4.30.** Let's find the minimal polynomial of

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As in the previous example,  $c_A(x) = (x - 1)^4$  and so by the [Cayley-Hamilton Theorem](#)  $m_A(x) = (x - 1)^j$  for some  $j \leq 4$ . This time we notice that  $(A - I_4)^2 = 0$  and so, since  $(A - I_4) \neq 0$ , we have  $m_A(x) = c_A(x) = (x - 1)^2$ .

### 4.3 Jordan canonical form

We now turn to the Jordan canonical form. To motivate it, let us do an example.

**Example 4.31.** Let us consider

$$A = \begin{bmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 0 & 6 \end{bmatrix} = C((x-2)^3) \in M_3(\mathbb{Q}).$$

This means we can interpret this matrix as arising from the linear transformation  $l_x$  on

$$V = \mathbb{Q}[x]/(x-2)^3$$

given by multiplication by  $x$ . Recall that the basis that gives the matrix  $A$  is

$$B = \{\bar{1}, \bar{x}, \bar{x^2}\}$$

But notice that

$$B' = \{\overline{(x-2)^2}, \overline{x-2}, \bar{1}\}$$

is also a basis of  $V$ , and indeed seems like a more pleasing one. Let us calculate what the operator  $T$  does to this alternative basis. We could work this out by brute force, but a cleaner way is to first compute what the operator  $T' = T - 2\text{id}_V$  does. It is clear that  $T'$  is multiplication by  $x - 2$ , and hence  $T'$  sends each basis element to the previous one, except for the first which is sent to 0. That is the matrix of  $T'$  is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence the matrix for  $T$  is  $T' + 2I_3$ :

$$J_3(2) := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

This is a *Jordan Block*.

**Definition 4.32.** Let  $F$  be a field, let  $n > 0$ , and let  $r \in F$ . The **Jordan block**  $J_n(r)$  is the  $n \times n$  matrix over  $F$  with entries satisfying the following:

$$a_{ij} = \begin{cases} r & \text{if } i = j \\ 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus a Jordan block looks like

$$\begin{bmatrix} r & 1 & & & \\ & r & \ddots & & \\ & & \ddots & \ddots & \\ & & & r & 1 \\ & & & & r \end{bmatrix}.$$

**Theorem 4.33** (Jordan Canonical Form Theorem). *Let  $F$  be a field, let  $V$  be a finite dimensional vector space, and let  $t : V \rightarrow V$  be a linear transformation satisfying the property that the characteristic polynomial  $c_t$  of  $t$  factors completely into linear factors over  $F$ . Then there is an ordered basis  $B$  for  $V$  such that*

$$[t]_B^B = J_{e_1}(r_1) \oplus \cdots \oplus J_{e_s}(r_s) = \begin{bmatrix} J_{e_1}(r_1) & 0 & 0 & \cdots & 0 \\ 0 & J_{e_2}(r_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & J_{e_s}(r_s) \end{bmatrix}$$

such that each  $r_i \in F$  is a root of the characteristic polynomial  $c_t$  and each  $e_i \geq 1$ . Moreover, the polynomials  $(x - r_1)^{e_1}, \dots, (x - r_s)^{e_s}$  are the elementary divisors of the  $F[x]$ -module  $V_t$ , and this expression for  $[t]_B^B$  is unique up to ordering of the Jordan blocks.

*Proof.* The key point is the following: the assumption that  $c$  completely factors into linear terms guarantees that the elementary divisors of  $c$  are of the form  $(x - r)^e$ . The proof then follows along the lines of Example 4.31. First write  $V_t$  in terms of the elementary divisors, as follows

$$V_t \cong F[x]/((x - r_1)^{e_1}) \oplus \cdots \oplus F[x]/((x - r_s)^{e_s}).$$

Then pick bases  $B'_i = \{\overline{(x - r_i)^{e_i-1}}, \dots, \overline{x - r_i}, \overline{1}\}$  for each of the summands and set

$$B := \bigcup_{i=1}^s B'_i.$$

All that remains to show is that the matrix representing multiplication by  $x$  on each summand is  $J_{e_i}(r_i)$ . More precisely, we want to compute the matrix representing the  $F$ -linear transformation  $T : F[x]/((x - r)^e) \xrightarrow{x} F[x]/((x - r)^e)$  in the basis  $B = \{\overline{(x - r)^{e-1}}, \dots, \overline{x - r}, \overline{1}\}$ . Let  $T' := T - r \cdot \text{id}$ , and note that

$$T'(\overline{(x - r)^{e-1}}) = 0$$

and

$$T'(\overline{(x - r)^i}) = \overline{(x - r)^{i+1}} \text{ for all } i < e - 1.$$

Thus the first column of  $[T']_B^B$  is zero, and each of the remaining ordered basis vectors is taken to the previous basis vector, so that

$$[T']_B^B = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

Since  $T = T' + r \text{id}$ , we conclude that  $[T]_B^B$  is indeed given by the Jordan block  $J_e(r)$ , as desired.  $\square$

**Definition 4.34.** Let  $F$  be a field,  $V$  be a finite dimensional vector space, and let  $t : V \rightarrow V$  be a linear transformation satisfying the property that the characteristic polynomial  $c_t$  of  $t$  factors completely into linear factors and has elementary divisors  $(x - r_1)^{e_1}, \dots, (x - r_s)^{e_s}$ . The matrix  $J_{e_1}(r_1) \oplus \dots \oplus J_{e_s}(r_s)$  is a **Jordan canonical form** (JCF) of  $t$ .

A **Jordan canonical form** for  $A \in M_n(F)$  is a Jordan canonical form for the linear transformation  $t_A : F^n \rightarrow F^n$  such that  $A = [t]_E^E$  in the standard basis  $E$  of  $F^n$ .

The same matrix may fail to have a JCF when interpreted as a matrix with entries in a smaller field while it has a JCF when interpreted as a matrix with entries in a larger field.

**Example 4.35.** We revisit the example of the rotation by  $60^\circ$  but extend scalars to  $\mathbb{C}$ . That is, start with a matrix  $A$  with  $c_A(x) = x^2 - x + 1 = (x - w)(x - \bar{w})$  where  $w = \frac{1+\sqrt{3}i}{2}$ . Since the minimal polynomial of  $A$  is  $m_A = x^2 - x + 1$ , we deduced in Example 4.28 that the only invariant factor of  $A$  is  $x^2 - x + 1$ , and hence the RCF of  $A$  is  $C(x^2 - x + 1)$ . On the other hand, over  $\mathbb{C}$  the polynomial  $m_A$  factors, say as  $x^2 - x + 1 = (x - w)(x - \bar{w})$ , and thus by the CRT

$$\mathbb{C}[x]/(x^2 - x + 1) \cong \mathbb{C}[x]/(x - w) \oplus \mathbb{C}[x]/(x - \bar{w}).$$

Therefore,

$$A \sim C(x - w) \oplus C(x - \bar{w}) = \begin{bmatrix} w & 0 \\ 0 & \bar{w} \end{bmatrix}.$$

The latter matrix is the JCF of  $A$ , and in this case the JCF is a diagonal matrix. Notice that if we consider  $A \in M_2(\mathbb{R})$  then the characteristic polynomial fails to factor into linear factors. Hence  $A \in M_2(\mathbb{R})$  does not have a JCF.

**Definition 4.36.** Let  $F$  be a field, let  $V$  be a finite dimensional vector space, and let  $t : V \rightarrow V$  be a linear transformation. Then  $t$  is **diagonalizable** if there is a basis  $B$  for  $V$  such that the matrix  $[t]_B^B$  is a diagonal matrix. Let  $A \in M_n(F)$ . Then  $A$  is **diagonalizable** if  $A$  is similar to a diagonal matrix.

**Theorem 4.37.** Let  $F$  be a field, let  $V$  be a finite dimensional vector space, and consider a linear transformation  $t : V \rightarrow V$ . The following are equivalent:

- (1)  $t$  is diagonalizable.
- (2)  $t$  has a Jordan canonical form  $A$  and  $A$  is a diagonal matrix.
- (3)  $t$  has a Jordan canonical form and all elementary divisors are of the form  $x - r$  with  $r \in F$ .
- (4) Each invariant factor of  $t$  is a product of linear polynomials with no repeated linear factors.
- (5) The minimal polynomial of  $t$  is a product of linear polynomials with no repeated linear factors.

*Proof.* Note that a diagonal matrix is an example of a matrix in JCF. By the uniqueness of the JCF, (1) holds if and only if (2) holds. Moreover, the equivalence of (2) and (3) follows by definition. A matrix has a JCF if and only if its invariant factors factor completely. In this case, the elementary divisors are constructed by decomposing each invariant factor into powers of distinct linear polynomials. This gives that (3) holds if and only if (4) holds. Finally, since the minimal polynomial is one of the invariant factors and every other invariant factor divides it, we get the equivalence between (4) and (5).  $\square$

# Chapter 5

## Field Extensions

One motivation for studying field extensions is that we want to build fields in which certain polynomials have roots. Here is a classical example going back to Gauss: while over  $\mathbb{R}$  the polynomial  $f = x^2 + 1 \in \mathbb{R}[x]$  has no roots, if we want a field in which  $f$  does have a root we need to consider  $\mathbb{C} = \mathbb{R}(i) = \{a + bi \mid a, b \in \mathbb{R}\}$ .

Here's another example that has already come up in this class: the polynomial  $g = x^2 - x + 1 \in \mathbb{Q}[x]$ . We know that this has a root  $\omega = \frac{1+\sqrt{3}i}{2} \in \mathbb{C}$ . But if we look for the smallest field containing  $\mathbb{Q}$  in which  $x^2 - x + 1$  has a root we obtain the field  $\mathbb{Q}(\omega) = \{a + b\omega \mid a, b \in \mathbb{Q}\}$ .

So here's our goal: starting from a smaller field  $F$  and an irreducible polynomial  $f \in F[x]$ , we want to build a larger field  $L$ . One way to do this is to take a root  $a$  of  $f$  and adjoin it to  $F$  obtaining the field  $L = F(a)$ , which is the collection of all expressions that one can build using addition, subtraction, multiplication and division starting from the elements of  $F \cup \{a\}$ . Another way to build a larger field  $L$  from a smaller field  $F$  and an irreducible polynomial  $f \in F[x]$  is to let  $L = F[x]/(f(x))$ . We will show below that these two ways of creating larger fields are one and the same.

Throughout, we will need some results about irreducible polynomials from 817.

**Theorem 5.1.** *Let  $F$  be a field and  $f \in F[x]$ . An element  $\alpha \in F$  is a root of  $f$  if and only if  $f = (x - \alpha)g$  for some  $g \in F[x]$ .*

**Theorem 5.2** (Eisenstein's Criterion). *Suppose  $R$  is a domain and let  $n \geq 1$ , and consider the monic polynomial*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x].$$

*If there exists a prime ideal  $P$  of  $R$  such that  $a_0, \dots, a_{n-1} \in P$  and  $a_0 \notin P^2$ , then  $f$  is irreducible in  $R[x]$ .*

**Theorem 5.3** (Gauss' Lemma). *Let  $R$  be a UFD with field of fractions  $F$ . Regard  $R$  as a subring of  $F$  and  $R[x]$  as a subring of  $F[x]$  via the induced map  $R[x] \hookrightarrow F[x]$ . If  $f(x) \in R[x]$  is irreducible in  $R[x]$ , then  $f(x)$  remains irreducible as an element of  $F[x]$ .*

**Theorem 5.4.** *Let  $R$  be a UFD with field of fractions  $F$ . Regard  $R$  as a subring of  $F$  and  $R[x]$  as a subring of  $F[x]$  via the induced map  $R[x] \hookrightarrow F[x]$ . If  $f(x) \in R[x]$  is irreducible in  $F[x]$  and the gcd of the coefficients of  $f(x)$  is a unit, then  $f(x)$  remains irreducible as an element of  $R[x]$ .*

## 5.1 Definition and first properties

**Definition 5.5.** A **field extension** is an inclusion of one field  $F$  into a larger field  $L$ , making  $F$  into a subfield of  $L$ . We sometimes write  $F \subseteq L$  and sometimes  $L/F$  to signify that  $L$  is a field extension of  $F$ .

So a field extension is just another name for a subfield, but the emphasis is different. We think of  $F$  as coming first and  $L$  later.

**Remark 5.6.** If  $F \subseteq L$  is a field extension, then  $L$  is in particular an  $F$ -vector space. This is a special case of the more general fact that if  $\phi: R \rightarrow S$  is a ring homomorphism, then  $S$  is a left  $R$ -module via  $r \cdot s := \phi(r)s$  by restriction of scalars.

**Definition 5.7.** The **degree** of a field extension  $L/F$  is

$$[L : F] := \dim_F(L).$$

A field extension is **finite** if its degree is finite.

Here are some examples.

**Example 5.8.** Since  $\mathbb{C} = \mathbb{R}(i) = \{a + bi \mid a, b \in \mathbb{R}\}$ , we have  $[\mathbb{C} : \mathbb{R}] = 2$ .

**Example 5.9.** We have  $[\mathbb{R} : \mathbb{Q}] = \infty$ . In fact, to be more precise we should say that  $[\mathbb{R} : \mathbb{Q}]$  is the cardinality of  $\mathbb{R}$ , but in general we lump all infinite field extensions together when talking about degree, and just write  $[L : F] = \infty$ .

Now we show that for any field  $F$  and any nonconstant polynomial  $f$  with coefficients in  $F$ , there exists a field extension of  $F$  in which the polynomial  $f$  has at least one root.

**Theorem 5.10.** *Let  $F$  be a field,  $p \in F[x]$  with  $\deg(p) \geq 1$ , and  $L = F[x]/(p)$ . If  $p$  is irreducible, then*

(1)  $L/F$  is a field extension via the map

$$\begin{aligned} F &\hookrightarrow L \\ f &\mapsto f + (p). \end{aligned}$$

(2) The degree of the extension is  $[L : F] = \deg(p)$ .

(3) The element  $\bar{x} := x + (p) \in L$  is a root of  $p$  in  $L$ .

*Proof.* First, note that  $(p)$  is a nonzero principal ideal in  $F[x]$ . Recall that over a PID, ideals generated by an irreducible element are maximal. Since  $p$  is irreducible, we conclude that  $(p)$  is maximal, and thus  $F[x]/(p)$  is a field.

We regard  $L$  as a field extension of  $F$  via the canonical map  $F \rightarrow L$  sending  $f \in F$  to the coset of the constant polynomial  $f$ . This map is not technically an inclusion map, but since it is an injective map we can pretend that it is an inclusion by identifying  $F$  with its image under this map. Note that injectivity of this map follows from the fact that  $(p)$  is a

proper ideal of  $F[x]$ , and thus every nonzero constant  $a \in F$  is taken to a nonzero element in  $L = F[x]/(p)$ .

You showed in Problem Set 6 that if  $\deg(p) = n$ , then the classes of  $1, x, \dots, x^{n-1}$  modulo  $(p)$  form basis for  $L$  regarded as an  $F$ -vector space. Therefore,  $[L : F] = \deg(p)$ . Moreover, we can extend the inclusion  $F \subseteq L$  to an inclusion  $F[x] \subseteq L[x]$ , and thus we can regard  $p$  as belonging to  $L[x]$ . Setting  $\bar{x} = x + (p) \in L$ , the element  $\bar{x}$  is a root of  $p(x) \in L[x]$  since

$$p(\bar{x}) = p(x) + (p(x)) = 0_L. \quad \square$$

**Example 5.11.** The polynomial  $f(x) = x^2 + 1$  is irreducible over  $\mathbb{R}$ . Theorem 5.10 says that  $f$  has a root in the extension  $\mathbb{R}[x]/(x^2 + 1)$ , and indeed,  $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$ , where  $f$  factors completely into linear factors:  $f(x) = (x - i)(x + i)$ . In fact,  $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{R}[i]$ .

Now that we know that there exists a field extension of  $F$  in which  $p(x)$  has a root, we may wonder about the *smallest* such extension.

**Definition 5.12.** Let  $F \subseteq L$  be a field extension and  $\alpha \in L$ . We write  $F(\alpha)$  for the smallest subfield of  $L$  that contains all of  $F$  and  $\alpha$ .

In contrast with the previous definition, we will also consider the smallest *ring* containing  $F$  and  $\alpha$ .

**Remark 5.13.** Since the intersection of any two subfields of  $L$  is again a subfield,  $F(\alpha)$  exists and is given by

$$F(\alpha) = \bigcap_{\substack{E \text{ field} \\ F \cup \{\alpha\} \subseteq E \subseteq L}} E.$$

**Definition 5.14.** Let  $F \subseteq L$  be a field extension and  $\alpha \in L$ . We write

$$F[\alpha] := \{f(\alpha) \mid f \in F[x]\}.$$

**Remark 5.15.** Note that any subring of  $L$  containing  $F$  and  $\alpha$  must contain all products of  $\alpha$  and elements of  $F$ , and all linear combinations of such things. Thus  $F[\alpha]$  is the smallest subring of  $L$  containing  $F$  and  $\alpha$ . Note that our notation does not include  $L$ , since in fact  $F[\alpha]$  does not actually depend on the choice of  $L$  as long as  $L \ni \alpha$ .

Here is another way to describe this field  $F(\alpha)$ . We leave the proof for Problem Set 7.

**Lemma 5.16.** If  $F \subseteq L$  is a field extension and  $\alpha \in L$ , the field  $F(\alpha)$  is the fraction field of  $F[\alpha] = \{f(\alpha) \mid f \in F[x]\}$ : more precisely,

$$F(\alpha) = \left\{ \frac{g(\alpha)}{f(\alpha)} \mid g(x), f(x) \in F[x], f(\alpha) \neq 0 \right\}.$$

Soon we will give an even better description for  $F(\alpha)$  in the case where  $\alpha$  is the root of a polynomial  $p \in F[x]$ .

**Definition 5.17.** A field extension  $L/F$  is called **simple** if  $L = F(\alpha)$  for some element  $\alpha$  of  $L$ . We call such an  $\alpha$  a **primitive element** for the extension.



If  $L/F$  is a simple field extension, note that there might be many different elements  $\alpha \in L$  such that  $L = F(\alpha)$ . Thus primitive elements are not necessarily unique.

**Example 5.18.** The extension  $\mathbb{R} \subseteq \mathbb{C}$  is simple, and  $i$  is a primitive element:  $\mathbb{C} = \mathbb{R}(i)$ . For another choice of primitive element, take  $-i$ .

We can generalize this to adjoining a subset instead of a single element.

**Definition 5.19.** If  $F \subseteq L$  is a field extension and  $A$  is any subset of  $L$ , the **subfield generated by  $A$  over  $F$** , denoted  $F(A)$ , is the smallest subfield of  $L$  that contains all of  $F$ . If  $A = \{a_1, \dots, a_n\}$  is a finite set, we write  $F(a_1, \dots, a_n)$  for  $F(A)$ .

**Remark 5.20.** Again, since the intersection of any two subfields of  $L$  is again a subfield,  $F(A)$  exists and is given by

$$F(A) = \bigcap_{E \supseteq F, A} E.$$

**Example 5.21.** Regard  $\mathbb{Q}$  as a subfield of  $\mathbb{C}$  and let  $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Setting  $E = \mathbb{Q}(\sqrt{2})$ , we can also think of  $F$  as  $F = E(\sqrt{3})$ . We will see shortly that  $E = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ . In other words,  $E$  is the set of  $\mathbb{Q}$ -linear combinations of 1 and  $\sqrt{2}$ , so  $[E : \mathbb{Q}] = 2$ .

Since  $\sqrt{3}^2 \in \mathbb{Q} \subseteq E$ , every element in  $F$  can be rewritten as an  $E$ -linear combination of 1 and  $\sqrt{3}$ :

$$F = \{\alpha + \beta\sqrt{3} \mid \alpha, \beta \in E\} = \{(a + b\sqrt{2}) + (c + d\sqrt{2})\sqrt{3} \mid a, b, c, d \in \mathbb{Q}\}.$$

and On the other hand,  $E \neq F$ , so we conclude that  $[F : E] = 2$ .

We claim that  $F$  is in fact a simple extension of  $\mathbb{Q}$ ; more precisely, that  $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = F$ . Set  $\beta := \sqrt{2} + \sqrt{3}$ . Note that  $\beta^2 = 5 + 2\sqrt{6}$  and

$$\beta^3 = 5\sqrt{2} + 5\sqrt{3} + 4\sqrt{3} + 6\sqrt{2} = 11\sqrt{2} + 9\sqrt{3}.$$

So  $\frac{1}{2}(\beta^3 - 9\beta) = \sqrt{2}$ , and hence  $\sqrt{2} \in \mathbb{Q}(\beta)$ . Likewise,  $\sqrt{3} = -\frac{1}{2}(\beta^3 - 11\beta) \in \mathbb{Q}(\beta)$ . So  $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . This shows that  $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$  is simple and  $\sqrt{2} + \sqrt{3}$  is a primitive element of this field extension.

This example is an illustration of the Primitive Element Theorem, which we might or might not have time to prove this semester: every finite extension of  $\mathbb{Q}$  is simple.

Next we will show that if  $\alpha$  is a root of a given polynomial  $p(x) \in F[x]$ , then  $F(\alpha)$  is determined by  $p(x)$  up to isomorphism.

**Theorem 5.22.** Let  $L/F$  be a field extension and let  $p(x) \in F[x]$  be an irreducible polynomial. If  $p$  has a root  $\alpha \in L$ , then there is an isomorphism  $\phi$  with  $\phi|_F = \text{id}_F$  and

$$\begin{aligned} \frac{F[x]}{(p(x))} &\quad \simeq F(\alpha) \\ x + (p(x)) &\quad \mapsto \alpha \\ f(x) + (p(x)) &\quad \mapsto f(\alpha). \end{aligned}$$

*Proof.* Let  $\tilde{\phi} : F[x] \rightarrow F(\alpha)$  be the evaluation homomorphism that sends  $x \mapsto \alpha$ ; more precisely,  $\tilde{\phi}(f(x)) := f(\alpha)$ , and the restriction of this map to  $F$  is the identity on  $F$ . Since  $p(\alpha) = 0$ , we have  $(p(x)) \subseteq \ker(\tilde{\phi})$ , and since  $(p(x))$  is a maximal ideal and  $\ker(\tilde{\phi}) \neq F[x]$ , we conclude that  $(p(x)) = \ker(\tilde{\phi})$ .

Now by Theorem 1.43 we get an injective ring homomorphism

$$\phi : \frac{F[x]}{(p(x))} \rightarrow F(\alpha)$$

such that  $\phi(f(x) + (p(x))) = \tilde{\phi}(f(x)) = f(\alpha)$ .

It remains to be shown that  $\phi$  is surjective. We will actually show more, namely that  $\text{im}(\phi) = F[\alpha] = F(\alpha)$ . Note first that by the definition of  $\phi$  above, the image of  $\tilde{\phi}$  on  $F[x]$  is  $F[\alpha]$ . However, since  $\phi$  is injective the image of  $\tilde{\phi}$  is a field contained in  $F(\alpha)$ , and since the smallest field containing  $F[\alpha]$  is  $F(\alpha)$ , we must in fact have  $\text{im}(\tilde{\phi}) = F(\alpha)$ .  $\square$

Let's formalize the extra information we have obtained in the course of proving the theorem. First we used the following useful fact:

**Remark 5.23.** If  $\phi : F \rightarrow L$  is an injective ring homomorphism and  $F$  and  $L$  are fields then the image of  $\phi$  is a subfield of  $L$ .

**Corollary 5.24.** Let  $L/F$  be a field extension and let  $p(x) \in F[x]$  be irreducible having a root  $\alpha \in L$ . Then  $F[\alpha] = F(\alpha)$ .

**Corollary 5.25** (Uniqueness of  $F(\alpha)$ ). Let  $p(x) \in F[x]$  be irreducible and let  $\alpha$  and  $\beta$  be two roots of  $p(x)$  in some extensions  $L$  and  $K$  of  $F$ . Then  $F(\alpha) \cong F(\beta)$ , so that the two roots are algebraically indistinguishable.

**Example 5.26.** Taking  $p(x) = x^2 + 1 \in \mathbb{R}[x]$  with roots  $\alpha = i$  and  $\beta = -i$  in  $\mathbb{C}$ , we actually obtain equal fields  $\mathbb{R}(i) = \mathbb{C} = \mathbb{R}(-i)$ . But Corollary 5.25 gives that there is an interesting isomorphism  $\phi : \mathbb{C} \xrightarrow{\cong} \mathbb{C}$  that sends  $i$  to  $-i$ . In general, we have  $\phi(a + bi) = a - bi$  for  $a, b \in \mathbb{R}$ .

**Example 5.27.** Another example illustrating Corollary 5.25 is that  $\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(-\sqrt{2})$  are isomorphic fields. In fact, they are equal:  $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(-\sqrt{2})$ . But again Corollary 5.25 gives that there is an interesting isomorphism  $\phi : \mathbb{Q}(\sqrt{2}) \xrightarrow{\cong} \mathbb{Q}(-\sqrt{2}) = \mathbb{Q}(\sqrt{2})$  that sends  $\sqrt{2}$  to  $-\sqrt{2}$ . In general, we have  $\phi(a + b\sqrt{2}) = a - b\sqrt{2}$  for  $a, b \in \mathbb{Q}$ .

The two examples above preview the central idea of Galois theory.

**Example 5.28.** In Example 5.21, we showed that  $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ . We want to find a polynomial  $p \in \mathbb{Q}[x]$  such that  $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \cong \mathbb{Q}[x]/(p(x))$ . Set  $\beta = \sqrt{2} + \sqrt{3}$ .

Note that we have  $\beta^2 = 5 + 2\sqrt{6}$  and  $\beta^4 = 49 + 20\sqrt{6}$  and hence  $\beta^4 - 10\beta^2 + 1 = 0$ . So  $\beta$  is a root of  $x^4 - 10x^2 + 1$ . It can be shown that this polynomial is irreducible. How? First, Gauss' Lemma says that it is sufficient to show that it is irreducible in  $\mathbb{Z}[x]$ .

Suppose that  $f$  does factor. Then that factorization will be preserved when we go modulo  $p$  for any prime  $p$ . We will use this to show that  $f$  has no linear factors. When we go modulo

3, we claim that  $f$  has no roots: indeed, Fermat's Little Theorem says that  $a^3 = a$  for all  $a \in \mathbb{Z}/(3)$ , so our polynomial becomes

$$f(x) = x^4 - x^2 + 1 = x^2 - x^2 + 1 = 1.$$

Since there are no roots modulo 3, we conclude that  $f$  has no linear factors over  $\mathbb{Z}$  either. Thus if  $f$  factors over  $\mathbb{Z}$ , it must factor as a product of degree 2 polynomials, which we can assume to be minimal. Suppose

$$f(x) = (x^2 + ax + b)(x^2 + cx + d).$$

These coefficients must satisfy the following system of equations:

$$\begin{cases} a + c &= 0 \\ b + d + ac &= -10 \\ ad + bc &= 0 \\ bd &= 1. \end{cases}$$

The first equation tells us that  $a = -c$ , so  $0 = ad + bc = a(d - b)$ , and since  $\mathbb{Z}$  is a domain, we conclude that  $d = b$ . Moreover,  $b^2 = 1$ , so  $b \in \{-1, 1\}$ . Finally, we have

$$b + d + ac = -10 \implies a^2 = 10 \pm 2.$$

But neither 8 nor 12 are squares in  $\mathbb{Z}$ , so this is impossible.

The previous example partially illustrates a nice trick: to show that a polynomial over  $\mathbb{Q}$  is irreducible, we need only to show it is irreducible over  $\mathbb{Z}$ , and to do that, it is sufficient to show that the polynomial is irreducible modulo a prime. In what follows, we will be very interested in irreducible polynomials, and we might want to use this type of tricks. Before we move on, let's see another example of this technique.

**Example 5.29.** Consider the polynomial  $f(x) = x^4 - 10x^2 - 19 \in \mathbb{Q}[x]$ . We claim it is irreducible, and thanks to Gauss' Lemma it is sufficient to show that  $f$  is irreducible over  $\mathbb{Z}$ . If  $f$  is reducible over  $\mathbb{Z}$ , it must also be reducible over  $\mathbb{Z}/(p)$  for all primes, since going modulo  $p$  will preserve the fact that  $f$  factors. Modulo 3, our polynomial becomes

$$f(x) = x^4 + 2x^2 + 2.$$

Repeating the trick from Example 5.28, since  $x^4$  and  $x^2$  take the same values over  $\mathbb{Z}/(3)$ , we see that for any  $a \in \mathbb{Z}/(3)$  we have

$$f(a) = a^4 + 2a^2 + 2 = 3a^2 + 2 = 2 \neq 0.$$

Thus  $f$  has no roots modulo 3, and thus it has no linear factors. Thus if it is reducible, it must be a product of two degree 2 factors, say

$$f(x) = (x^2 + ax + b)(x^2 + cx + d).$$

These coefficients must satisfy the following system of equations:

$$\begin{cases} a + c &= 0 \\ b + d + ac &= 2 \\ ad + bc &= 0 \\ bd &= 2. \end{cases}$$

Since  $a = -c$ , we get  $a(d - b) = ad + bc = 0 \implies b = d$ . Thus the last equation tells us that  $b^2 = 2$ , but all squares modulo 3 are 0 or 1, so this is impossible.

## 5.2 Algebraic and transcendental extensions

**Definition 5.30.** For a field extension  $F \subseteq L$  and  $\alpha \in L$ , we say  $\alpha$  is **algebraic** over  $F$  if  $f(\alpha) = 0$  for some nonconstant polynomial  $f(x)$ . Otherwise,  $\alpha$  is **transcendental** over  $F$ .

**Example 5.31.** The element  $i \in \mathbb{C}$  is algebraic over  $\mathbb{R}$ , since  $i^2 + 1 = 0$ . In fact, every element of  $\mathbb{C}$  is algebraic over  $\mathbb{R}$ , and we will soon see why. In contrast, the numbers  $\pi$  and  $e$  of  $\mathbb{R}$  are transcendental over  $\mathbb{Q}$ , though these are both deep facts.

**Theorem 5.32.** Suppose  $L/F$  is a field extension and  $\alpha \in L$ .

- (1) The set  $I := \{f(x) \in F[x] \mid f(\alpha) = 0\}$  is an ideal of  $F[x]$ .
- (2)  $I = 0$  if and only if  $\alpha$  is transcendental over  $F$ . Equivalently,  $I \neq 0$  if and only if  $\alpha$  is algebraic over  $F$ .
- (3) If  $\alpha$  is algebraic over  $F$ , meaning  $I \neq 0$ , then the unique monic generator  $m_{\alpha,F}(x)$  of the ideal  $I$  is irreducible.
- (4) If  $\alpha$  is algebraic over  $F$ , then there is an isomorphism of fields

$$F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$$

sending  $F$  identically to  $F$  and sending  $x$  to  $\alpha$ .

- (5) The element  $\alpha$  is algebraic over  $F$  if and only if  $[F(\alpha) : F] < \infty$ . In this case,

$$[F(\alpha) : F] = \deg(m_{\alpha,F}(x)).$$

- (6) The element  $\alpha$  is transcendental over  $F$  if and only if  $[F(\alpha) : F] = \infty$ . In this case, there is an isomorphism of fields between  $F(\alpha)$  and the field of fractions of  $F[x]$ :

$$F(\alpha) \cong F(x) := \left\{ \frac{g(x)}{f(x)} \mid g \neq 0 \right\}$$

sending  $F$  identically to  $F$  and sending  $x$  to  $\alpha$ .

*Proof.* The set  $I$  is the kernel of the evaluation homomorphism that maps  $x \mapsto \alpha$ . This map is a ring homomorphism, and thus  $I$  must be an ideal of  $F[x]$ . The content of (2) follows by definition of algebraic and transcendental elements.

To show (3), assume  $I \neq 0$  and let  $p$  be its unique monic generator. Suppose  $p = fg$ . Since  $p(\alpha) = 0$  in  $F$  and  $F$  is a field (and thus a domain), either  $f(\alpha) = 0$  or  $g(\alpha) = 0$ . Therefore, either  $f(x) \in I$  or  $g(x) \in I$ . This proves  $(p)$  is a prime ideal and hence  $p$  is a prime element. Since  $F[x]$  is a PID, it follows that  $p$  is irreducible.

The statement of (4) is already Theorem 5.22.

Let's show (5). If  $\alpha$  is algebraic over  $F$ , then (4) shows that

$$[F(\alpha) : F] = \deg(m_{\alpha,F}(x)) < \infty.$$

For the converse, if  $[F(\alpha) : F] < \infty$ , then the infinite list  $1, \alpha, \alpha^2, \dots$  of elements of  $F(\alpha)$  must be  $F$ -linearly dependent. Thus  $a_0 + a_1\alpha + \dots + a_n\alpha^n = 0$  for some  $n$  and some  $a_0, \dots, a_n \in F$  not all zero. This shows  $\alpha$  is the root of a nonzero polynomial.

To show (6), the map  $\phi$  defined as in (4) is injective. Since the target is a field  $L$  and  $F[x]$  is an integral domain, by the UMP of the fraction field  $\phi$  can be extended to the field of fractions of  $F[x]$ , so there is a homomorphism of fields  $\tilde{\phi} : F(x) \rightarrow L$ . The image of this field map is

$$\left\{ \frac{g(\alpha)}{f(\alpha)} \mid g, f \in \mathbb{F}[x], f(x) \neq 0 \right\},$$

which is precisely  $F(\alpha)$  by Lemma 5.16. The map is injective since it is a field homomorphism that is not identically zero.  $\square$

**Definition 5.33.** Let  $F \subseteq L$  be a field extension and  $\alpha \in L$ , and consider the ideal

$$I = \{f(x) \in F[x] \mid f(\alpha) = 0\}$$

from the previous theorem. The unique monic generator  $m_{\alpha, F}(x)$  for  $I$  is called the **minimal polynomial** of  $\alpha$  over  $F$ .

**Remark 5.34.** Note that the minimal polynomial of  $\alpha$  over  $F$ , if it exists, must divide every polynomial in  $F[x]$  that has  $\alpha$  as a root. Also, it can be characterized as the monic polynomial in  $F[x]$  of least degree having  $\alpha$  as a root.

**Example 5.35.** Note that the minimal polynomial of  $i$  over  $\mathbb{R}$  is  $m_{i, \mathbb{R}}(x) = x^2 + 1$ .

**Theorem 5.36** (The Degree Formula). *Suppose  $F \subseteq L \subseteq K$  are field extensions. Then*

$$[K : F] = [K : L][L : F].$$

*In particular, the composition of two finite extensions of fields is again a finite extension.*

*Proof.* Let  $A \subseteq K$  be a basis for  $K$  as an  $L$ -vector space and let  $B \subseteq L$  be a basis for  $L$  as an  $F$ -vector space. Consider the subset of  $K$  given by

$$AB := \{ab \mid a \in A, b \in B\}.$$

First, we claim that  $AB$  is a basis of  $K$  as an  $F$ -vector space. For  $a \in K$ , we have  $a = \sum_i l_i a_i$  for some  $a_1, \dots, a_m \in A$  and  $l_1, \dots, l_m \in L$ . For each  $i$ ,  $l_i$  is an  $F$ -linear combination of a finite set of elements of  $B$ . Combining these gives that  $a$  is in the  $F$ -span of  $AB$ . To prove linear independence, it suffices to prove that if  $a_1, \dots, a_m$  and  $b_1, \dots, b_n$  be distinct elements of  $A$  and  $B$  respectively, then the set  $\{a_i b_j\}$  is linearly independent. Suppose  $\sum_{i,j} f_{i,j} a_i b_j = 0$  for some  $f_{i,j} \in F$ . Since the  $b_j$  are  $L$ -linearly independent and

$$\sum_{i,j} f_{i,j} a_i b_j = \sum_j \left( \sum_i f_{i,j} a_i \right) b_j$$

and  $f_{i,j} a_i \in L$ , we get that, for each  $j$ ,  $\sum_i f_{i,j} a_i = 0$ . Using now that the  $a_i$  are  $F$ -linearly independent, we have that for all  $j$  and all  $i$ ,  $f_{i,j} = 0$ . This proves that the set

$\{a_i b_j \mid i = 1, \dots, m, j = 1, \dots, n\}$  is linearly independent over  $F$ , and hence  $AB$  is linearly independent over  $F$ .

In particular, this shows that the elements of the form  $ab$  with  $a \in A$  and  $b \in B$  are all distinct, so  $|AB| = |A| \cdot |B|$ . Since  $AB$  is a basis for  $L$  over  $K$ , we conclude that

$$[K : F] = [K : L][L : F]. \quad \square$$

**Example 5.37.** In Example 5.21 we showed that  $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\beta)$  with  $\beta = \sqrt{2} + \sqrt{3}$ . We claim that  $m_{\beta, \mathbb{Q}}(x) = x^4 - x^2 + 1$ . By the Degree Formula, we have

$$[\mathbb{Q}(\beta) : \mathbb{Q}] = [\mathbb{Q}(\beta) : E][E : \mathbb{Q}] = 2 \cdot 2 = 4.$$

Thus  $m_{\beta, \mathbb{Q}}(x)$  has degree 4. We already know that  $\beta$  is a root of  $x^4 - 10x^2 + 1$ , hence this polynomial is divisible by the minimal polynomial of  $\beta$ . Since they are both monic and have degree 4, it must be that  $m_{\beta, \mathbb{Q}}(x) = x^4 - 10x^2 + 1$ . Arguing this way, there is no need to check this polynomial is irreducible; it must be by Theorem 5.32 (3).

**Definition 5.38.** A field extension  $F \subseteq L$  is **algebraic** if every element  $a \in L$  is algebraic over  $F$ .

**Definition 5.39.** We say an extension of fields  $F \subseteq L$  is **finite** if it has finite dimension.

Note: this is not a statement about the number of elements in the fields  $F$  and  $L$ .

**Example 5.40.** The extension  $\mathbb{R} \subseteq \mathbb{C}$  is finite, since  $[\mathbb{C} : \mathbb{R}] = 2$ .

**Lemma 5.41.** *Every finite extension of fields is algebraic.*

*First proof.* Let  $K \subseteq L$  be a finite field extension, and let  $a \in L$ . Since the extension is finite, any infinite set of elements in  $L$  must be linearly dependent over  $F$ . In particular, the set

$$\{a^n \mid n \geq 0\}$$

is linearly dependent. Does there exist  $n$  such that

$$\{1, a, a^2, \dots, a^n\}$$

is linearly dependent. Writing an equation of linear dependence, say

$$b_n a^n + \dots + b_1 a + b_0 = 0$$

for some  $b_i \in F$ , we might as well assume that  $b_n \neq 0$  (otherwise, replace  $n$  by the largest value of  $i$  such that  $b_i \neq 0$ ), and thus after multiplying by  $b_n^{-1}$  we conclude that we can write  $a^n$  in terms of the lower powers of  $a$ . In particular,  $a$  is algebraic over  $F$ .  $\square$

*Proof using the Degree formula.* Let  $K \subseteq L$  be a finite field extension, and let  $a \in L$ . By the Degree Formula, we have

$$[L : K] = [L : K(a)][K(a) : K],$$

and thus  $K \subseteq K(a)$  must be finite. By Theorem 5.32 (5),  $a$  must be algebraic over  $K$ .  $\square$

The converse is false, as shown by the following example:

**Example 5.42.** Let  $\overline{\mathbb{Q}}$  denote the set of complex numbers that are algebraic over  $\mathbb{Q}$ , which is by definition an algebraic extension of  $\mathbb{Q}$ . However, we claim that  $\overline{\mathbb{Q}}$  is not finite over  $\mathbb{Q}$ .

First, let  $p$  any prime integer,  $n > 0$  be any integer, and consider the polynomial  $x^n - p$  over  $\mathbb{Q}[x]$ . By applying [Eisenstein's Criterion](#) with the prime ideal  $(p)$ , we conclude that  $x^n - p$  is irreducible over  $\mathbb{Z}$ . By [Gauss' Lemma](#),  $x^n - p$  is also irreducible over  $\mathbb{Q}$ .

Now  $\overline{\mathbb{Q}}$  contains the subextension  $\mathbb{Q}(a)$ , where  $a$  is a root of  $x^n - p$ . Since  $x^n - p$  is irreducible over  $\mathbb{Q}$ , it is the minimal polynomial of  $a$  over  $\mathbb{Q}$ , and thus by Theorem 5.32 (5) we conclude that the degree of this extension is  $[\mathbb{Q}(a) : \mathbb{Q}] = n$ . Thus  $\mathbb{Q}$  contains subextensions of  $\mathbb{Q}$  of arbitrarily large degree. By the [Degree Formula](#) applied to  $\mathbb{Q} \subseteq \mathbb{Q}(a) \subseteq \overline{\mathbb{Q}}$ , if  $\mathbb{Q}$  had finite degree over  $\mathbb{Q}$  then that degree would be divisible by  $n$  for all  $n$ . We conclude that  $[\mathbb{Q} : \mathbb{Q}] = \infty$ .

**Theorem 5.43.** *Given field extensions  $F \subseteq L \subseteq E$ ,  $L/F$  and  $E/L$  are both algebraic if and only if  $E/F$  is algebraic.*

*Proof.* ( $\Leftarrow$ ) Suppose  $F \subseteq E$  is algebraic. Every element in  $L$  is in  $E$  as well, and thus it is algebraic over  $F$ ; thus  $F \subseteq L$  is algebraic. Moreover, any element  $\alpha \in E$  is algebraic over  $F$  by assumption, so it satisfies a polynomial with coefficients in  $F$ . But any polynomial with coefficients in  $F$  is also a polynomial with coefficients in  $L$ , and thus  $\alpha$  is algebraic over  $L$ .

( $\Rightarrow$ ) Fix  $\alpha \in E$ . We need to prove  $\alpha$  is a root of some monic polynomial with coefficients in  $F$ . This is surprisingly hard to prove directly, and in fact the proof we will give is rather nonconstructive.

Since  $\alpha$  is algebraic over  $L$ , it is a root of some polynomial  $a_n x^n + \cdots + a_1 x + a_0 \in L[x]$ . Note that this polynomial belongs to  $F(a_0, \dots, a_n)[x]$  too, and so  $\alpha$  is algebraic over  $F(a_0, \dots, a_n)$ .

Consider the chain of field extensions

$$F \subseteq F(a_0) \subseteq F(a_0, a_1) \subseteq \cdots \subseteq F(a_0, a_1, \dots, a_n) \subseteq F(a_0, \dots, a_n, \alpha).$$

Each  $a_i \in L$  is algebraic over  $F$  for all  $i$ , and  $\alpha$  is algebraic over  $F(a_0, a_1, \dots, a_n)$ , so each step in our tower of extensions consists of adding an algebraic element to the previous field. By Theorem 5.32, each step in this chain has finite dimension. By the [Degree Formula](#),

$$[F(a_0, \dots, a_n, \alpha) : F] = [F(a_0, \dots, a_n, \alpha) : F(a_0, \dots, a_n)] \cdots [F(a_0) : F]$$

is finite. Moreover, if we reorder the tower above to start from  $F \subseteq F(\alpha)$ , by the [Degree Formula](#) we have

$$[F(\alpha) : F][F(a_0, \dots, a_n, \alpha) : F(\alpha)] = [F(a_0, \dots, a_n, \alpha) : F] < \infty.$$

Therefore,  $[F(\alpha) : F]$  is finite. By Theorem 5.32 (5) again,  $\alpha$  is algebraic over  $F$ .  $\square$

In the proof of Theorem 5.43, we also showed the following corollary of the [Degree Formula](#):

**Corollary 5.44.** *If  $\alpha_1, \dots, \alpha_n$  are algebraic over  $F$ , then  $F \subseteq F(\alpha_1, \dots, \alpha_n)$  is a finite algebraic extension.*

### 5.3 Algebraically closed fields and algebraic closure

**Definition 5.45.** For any field extension  $F \subseteq L$ , we define the **algebraic closure** of  $F$  in  $L$  to be the set

$$\overline{F}_L = \{\alpha \in L \mid \alpha \text{ is algebraic over } F\}.$$

**Lemma 5.46.** For any field extension  $F \subseteq L$ , the set  $\overline{F}_L$  is a subfield of  $L$  that contains  $F$ . Moreover, it is the largest subfield of  $L$  that is algebraic over  $F$ .

*Proof.* First, note that every element  $a \in F$  satisfies the monic polynomial  $x - a$ , and thus  $F \subseteq \overline{F}_L$ , which is in particular nonempty. The claims that  $F \subseteq \overline{F}_L$  and that  $\overline{F}_L$  is the largest subfield of  $L$  that is algebraic over  $F$  follow from the definition of  $\overline{F}_L$ .

It remains to show that  $\overline{F}_L$  is a field: we need to show that  $\overline{F}_L$  is closed under addition, multiplication, and taking additive and multiplicative inverses. Let  $\alpha, \beta \in \overline{F}_L$ . Since  $\alpha$  and  $\beta$  are algebraic over  $F$  and consequently  $\beta$  is algebraic over  $F(\alpha)$ , we have that  $[F(\alpha) : F] < \infty$  and  $[F(\alpha, \beta) : F(\alpha)] < \infty$ . Thus by the [Degree Formula](#) the extension  $F(\alpha, \beta)/F$  is finite, and hence algebraic by [Lemma 5.41](#). It follows that every element of  $F(\alpha, \beta)$  is algebraic over  $F$ . In particular  $\alpha \pm \beta$ ,  $\alpha\beta$ , and  $\alpha^{-1}$  (if  $\alpha \neq 0$ ) are elements of  $F(\alpha, \beta) \subseteq \overline{F}_L$ .  $\square$

The notion of algebraic closure is closely related (pun intended) to being algebraically closed.

**Definition 5.47.** A field  $L$  is **algebraically closed** if every polynomial  $f(x) \in L[x]$  that is not a constant has a root in  $L$ .

This is equivalent to the condition that every nonconstant polynomial splits completely into linear factors.

**Example 5.48.** The Fundamental Theorem of Algebra says that any polynomial in  $\mathbb{C}[x]$  completely factors as a product of linear terms, thus  $\mathbb{C}$  is an algebraically closed field.

**Lemma 5.49.** If  $F \subseteq L$  is a field extension with  $L$  algebraically closed, then  $\overline{F}_L$  is also algebraically closed.

*Proof.* Let  $f \in \overline{F}_L[x]$  be a nonconstant polynomial. Since  $\overline{F}_L \subseteq L$ ,  $f \in L[x]$ , and thus  $f$  has a root in  $L$ , say  $\alpha \in L$ . Since  $\alpha$  satisfies a polynomial in  $\overline{F}_L[x]$ , it must then be algebraic over  $\overline{F}_L$ . Thus  $\overline{F}_L \subseteq \overline{F}_L(\alpha)$  is an algebraic extension, and  $F \subseteq \overline{F}_L \subseteq \overline{F}_L(\alpha)$  is a composition of two algebraic extensions. By [Theorem 5.43](#),  $F \subseteq \overline{F}_L(\alpha)$  is algebraic. By definition, this says that  $\alpha$  is algebraic over  $F$ , and thus  $\alpha \in \overline{F}_L$ . Therefore,  $f$  has a root over  $\overline{F}_L$ , and  $\overline{F}_L$  is algebraically closed.  $\square$

**Remark 5.50.** In contrast, if  $L/F$  is a field extension with  $L$  not algebraically closed, then  $\overline{F}_L$  need not be algebraically closed. For example, think of the extremal case when  $F = L$ , where we must have  $\overline{F}_L = F$ , which is not algebraically closed by assumption.

**Example 5.51.** [Lemma 5.49](#) shows that the field  $\overline{\mathbb{Q}}$  defined in [Example 5.42](#) is algebraically closed.

**Definition 5.52.** Given a field  $F$ , a field  $L$  is called an **algebraic closure** of  $F$  if  $F \subseteq L$  is an algebraic field extension and  $L$  is algebraically closed.



**Remark 5.53.** Let  $L$  be an algebraic closure of  $F$ . Since  $L$  is algebraically closed by definition, by Lemma 5.49 we conclude that  $\overline{F}_L$  is algebraically closed. On the other hand, since  $F \subseteq L$  is algebraic by definition, we conclude that  $\overline{F}_L = L$ . This explains why we say  $L$  is an *algebraic closure* of  $F$ .

**Example 5.54.**

- 1) Since  $[\mathbb{C} : \mathbb{R}] = 2$ , the extension  $\mathbb{R} \subseteq \mathbb{C}$  is finite, and thus by Lemma 5.41 the extension  $\mathbb{R} \subseteq \mathbb{C}$  must also be algebraic. Moreover,  $\mathbb{C}$  is algebraically closed by the Fundamental Theorem of Algebra. Thus  $\mathbb{C}$  is an algebraic closure of  $\mathbb{R}$ .
- 2) By Lemma 5.49, an algebraic closure inside an algebraically closed field is algebraically closed. Thus  $\overline{\mathbb{Q}}_{\mathbb{C}} = \{z \in \mathbb{C} \mid z \text{ is algebraic over } \mathbb{Q}\}$  is an algebraic closure of  $\mathbb{Q}$ .

Next we will show that every field has a unique algebraic closure, so we can talk about *the* algebraic closure of a field. To do that, we first need a lemma.

**Lemma 5.55.** *If  $L/F$  is an algebraic field extension and every nonconstant polynomial  $f(x) \in F[x]$  splits completely into linear factors in  $L[x]$ , then  $L$  is algebraically closed and hence is an algebraic closure of  $F$ .*

*Proof.* Suppose  $g(x) \in L[x]$  is not constant. We need to prove  $g$  has a root in  $L$ . We may form a (possibly trivial) algebraic extension  $L \subseteq E$  such that  $g(x)$  has a root  $\alpha$  in  $E$ . Note that  $E/F$  is algebraic and hence  $\alpha$  is algebraic over  $F$ . So  $\alpha$  is a root of some  $f(x) \in F[x]$ . But then  $f(x) = \prod_i (x - \beta_i) \in L[x]$  and it follows that  $\alpha$  must be one of the  $\beta_i$ , and hence belongs to  $L$ .  $\square$

We are going to use one more technical result, which will also be helpful to us later.

**Theorem 5.56.** *Let  $F$  be a field,  $f$  be an irreducible nonconstant polynomial, and consider a field isomorphism  $\theta: F \rightarrow F'$ . Consider the isomorphism  $\tilde{\theta}: F[x] \rightarrow F'[x]$  induced by  $\theta$ , and let  $f' = \tilde{\theta}(f) \in F'[x]$  be the polynomial corresponding to  $f$ . Let  $\alpha$  be any root of  $f$  in some field extension  $L$  of  $F$ , and  $\alpha'$  be any root of  $f'$  in some field extension  $L'$  of  $F'$ . Then there exists a field isomorphism*

$$\hat{\theta}: F(\alpha) \rightarrow F'(\alpha')$$

*that extends the map  $\theta$  and sends  $\alpha$  to  $\alpha'$ .*

*Proof.* The key point is that

$$F[x]/(f) \cong F(\alpha)$$

via a map that is the identity on  $F$  and sends  $x$  to  $\alpha$ , as we saw in Corollary 5.25. Thus we have

$$F(\alpha) \cong F[x]/(f) \cong F'[x]/(f') \cong F'(\alpha')$$

with the middle isomorphism induced by  $\theta$ . Tracking through these maps shows that it extends  $\theta$  and sends  $\alpha$  to  $\alpha'$ :

$$\alpha \mapsto x + (f) \mapsto x + (f') \mapsto \alpha'.$$

$\square$

We are now ready to show that every field has an algebraic closure, and that algebraic closures are unique up to isomorphism.

**Theorem 5.57** (Existence and uniqueness of algebraic closures). *For any field  $F$ , there exists an algebraic closure of  $F$ . If  $L$  and  $L'$  are two algebraic closures of the same field  $F$ , then there exists a field isomorphism  $\phi : L \xrightarrow{\cong} L'$  such that  $\phi|_F = \text{id}_F$ .*

*Proof of existence of algebraic closures.* First, we reduce the proof of existence to the following:

Claim: There is an algebraic field extension  $F \subseteq L$  such that every nonconstant polynomial in  $F[x]$  has at least one root in  $L$ .

Let us assume the claim holds. By using this fact repeatedly, we may form a tower of field extensions

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

such that, for all  $i$ , the extension  $F_i \subseteq F_{i+1}$  is algebraic and every nonconstant polynomial in  $F_i[x]$  has at least one root in  $F_{i+1}$ . At each step, we apply the claim to  $F_i$  to construct  $F_{i+1}$ .

Let  $E := \cup_i F_i$ . One can show  $E$  is a field and  $F \subseteq E$  is algebraic (exercise). Given  $f \in F[x]$ , by assumption  $f$  has a root  $\alpha$  in  $F_1$ , and hence  $f$  factors as  $f(x) = (x - \alpha)g(x)$  for  $g(x) \in F_1[x]$ . But then  $g$  has a root in  $F_2$  and hence factors in  $F_2[x]$ . Repeating this we see  $f$  splits completely into linear factors in  $F_n[x]$ , where  $n = \deg(f)$ , and thus  $f$  splits completely into linear factors in  $E[x]$ . By Lemma 5.55,  $E$  is an algebraic closure of  $F$ .

Proof of Claim: Let  $S$  be the collection of all nonconstant polynomials with coefficients in  $F$ , and for each  $f \in S$ , pick an indeterminate  $y_f$ . Now we form the rather large polynomial ring  $R = F[y_f \mid f \in S]$ . Let  $I$  be the ideal generated by  $f(y_f)$ . We claim that  $I$  is a proper ideal. If not, then  $1 \in I$ , so we would have an equation of the form

$$1 = g_1 f_1(y_{f_1}) + \cdots + g_m f_m(y_{f_m})$$

in  $R$ . There exists finite extension  $E$  of  $F$  in which each  $f_i$  has a root  $\alpha_i$ : by Theorem 5.10,  $f_i$  has a root  $\alpha_i$  in some extension of  $F$ , and  $F(\alpha_1, \dots, \alpha_n)$  is a finite extension of  $F$  by Corollary 5.44. Evaluating the above equation by setting  $y_{f_i} = \alpha_i$ , we get  $1 = 0$ , which is impossible. This shows that  $I$  must be a proper ideal.

Since  $I$  is a proper ideal, it is contained in some maximal ideal  $\mathfrak{m}$ . The quotient ring  $K := R/\mathfrak{m}$  is a field, and the composition  $F \hookrightarrow R \twoheadrightarrow K$  is a ring map  $F \rightarrow K$  between two fields, and thus must be injective. By a slight abuse of notation, we will think of this map as an actual inclusion. For any  $f \in S$ , in  $K$  we have  $f(y_f) = 0$ , so the image  $\overline{y_f} \in K$  of  $y_f \in R$  is a root of  $f$ . We have constructed a field extension  $F \subseteq K$  such that every  $f \in S$  has a least one root in  $K$ .

We are not quite done since it is not clear that  $K$  is algebraic over  $F$ . For each  $f \in S$ , pick a root  $\beta_f \in K$  of  $f$ . Let  $L = F(\beta_f \mid f \in S) \subseteq K$ . Then  $L$  is algebraic over  $F$  and every member of  $S$  has at least one root in  $L$ .  $\square$

*Proof of uniqueness of algebraic closures.* Suppose  $L$  and  $L'$  are two algebraic closures of  $F$ . Let  $S$  be the set of pairs  $(E, i)$  where  $E$  is a subfield of  $L$  that contains  $F$  and  $i : E \hookrightarrow L'$  is a ring homomorphism with  $i|_F = \text{id}_F$ . Make  $S$  into a poset by declaring that  $(E, i) \leq (E', i')$  whenever  $E \subseteq E'$  and  $i'|_E = i$ .

One can show (exercise!) that  $S$  satisfies the hypotheses of Zorn's Lemma, and hence it has a maximal element  $(E, i)$ . We claim  $E$  must equal  $L$ . If not, we can find  $\alpha \in L \setminus E$ . Let  $p(x) = m_{\alpha, E}$  and set  $E' := i(E)$ . So  $i$  maps  $E$  isomorphically onto  $E'$ . Let  $p'(x)$  be the polynomial in  $E'[x]$  corresponding to  $p(x)$  via  $i$ , and pick any root  $\alpha'$  of  $p'(x)$  in  $L'$ . By Theorem 5.56, there is an isomorphism  $E(\alpha) \rightarrow E'(\alpha')$  extending the isomorphism  $i$ . Since  $E(\alpha) \subseteq L$ , and by assumption  $\alpha \notin E$ , this contradicts the maximality of  $(E, i)$ .

Thus we have a field extension  $F \subseteq i(L) \subseteq L'$  with  $i(L) \cong L$  via an isomorphism that fixes  $F$ . It follows that  $i(L)$  is also an algebraic closure of  $F$ . Since  $L'/F$  is algebraic, we must have  $i(L) = L'$ . Thus  $i$  is surjective, and thus an isomorphism.  $\square$

We will then be able to talk about not just *an* algebraic closure of  $F$  but *the* algebraic closure of  $F$ , so we can simplify our notation a bit.

**Definition 5.58.** Given a field  $F$ , we will write  $\overline{F}$  for its algebraic closure inside an algebraically closed field extension of  $F$ .

By Theorem 5.57,  $\overline{F}$  is defined only up to isomorphism.

**Example 5.59.** The field  $\mathbb{C}$  is the algebraic closure of  $\mathbb{R}$ , so we write  $\overline{\mathbb{R}} = \mathbb{C}$ .

**Example 5.60.** In Example 5.42, we defined  $\overline{\mathbb{Q}}$  as the set of complex numbers that are algebraic over  $\mathbb{Q}$ . In our notation from this chapter, this is what we denote by  $\overline{\mathbb{Q}}_{\mathbb{C}}$ , the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . Since  $\mathbb{C}$  is an algebraically closed field, this is *the* algebraic closure of  $\mathbb{Q}$ , which explains our notation  $\overline{\mathbb{Q}}$  from Example 5.42. This field  $\overline{\mathbb{Q}}$  is sometimes called the **field of algebraic numbers**.

**Remark 5.61.** Earlier we used the notation  $\overline{F}_L$  to denote the algebraic closure of  $F$  in  $L$ . If  $L$  is an algebraically closed field, then by Lemma 5.49 we know that  $\overline{F}_L$  is also algebraically closed, and since it is by definition algebraic over  $F$ , then  $\overline{F}_L$  is *an* algebraic closure of  $F$ , so in fact this is *the* algebraic closure of  $F$  (defined only up to isomorphism).

In contrast, if  $L$  is not an algebraically closed field, we saw in Remark 5.50 that  $\overline{F}_L$  is not necessarily algebraically closed. In particular,  $\overline{F}_L$  is not necessarily an algebraic closure of  $F$ , and thus  $\overline{F}_L$  and  $\overline{F}$  might denote completely different things.

From now on, many constructions will happen inside an algebraic closure of a given field, and we will use the notation  $\overline{F}$ .

**Remark 5.62.** Every algebraically closed field is infinite. Indeed, if  $F$  is a finite field, say  $F = \{a_1, \dots, a_n\}$ , then the polynomial

$$(x - a_1) \cdots (x - a_n) + 1 \in F[x]$$

has no roots in  $F$ . In particular, the algebraic closure of any finite field is infinite.

## 5.4 Splitting fields

**Definition 5.63.** Let  $F$  be a field and let  $f \in F[x]$  be a nonconstant polynomial. A **splitting field** of  $f$  over  $F$  is a field extension  $F \subseteq L$  such that  $f$  splits completely into linear factors in  $L[x]$ , and  $f$  does not split completely into linear factors over any proper subfield of  $L$  that contains  $F$ .

A splitting field of  $f$  is given by adjoining all the roots of  $f$ .

**Lemma 5.64.** If  $F \subseteq E$  is a field extension such that  $f \in F[x]$  factors in  $E[x]$  as

$$f = c \prod_{i=1}^n (x - \alpha_i)$$

for some  $c, \alpha_1, \dots, \alpha_n \in E$ , then  $F(\alpha_1, \dots, \alpha_n)$  is a splitting field for  $f$  over  $F$ .

*Proof.* Note that  $c$  is just the coefficient of  $f$  in degree  $n$ , and thus  $c \in F$ . Thus  $f(x)$  also factors as

$$f(x) = c \prod_{i=1}^n (x - \alpha_i)$$

in  $F(\alpha_1, \dots, \alpha_n)[x]$ . Hence, given some splitting field  $L$  of  $f$  over  $F$ , by the minimality condition in the definition, we must have  $L \subseteq F(\alpha_1, \dots, \alpha_n)$ . However, the splitting field  $L$  must contain all roots of  $f$  in order for  $f$  to split completely in  $L[x]$ , so we also have  $F(\alpha_1, \dots, \alpha_n) \subseteq L$ .  $\square$

**Remark 5.65.** Note that there may be repetitions in the list  $\alpha_1, \dots, \alpha_n$ , but that does not affect the validity of anything here.

**Theorem 5.66** (Existence of splitting fields). *Let  $F$  be a field and  $f \in F[x]$  a nonconstant polynomial. There exists a splitting field  $L$  for  $f$  over  $F$ .*

*Proof.* Let  $\overline{F}$  be an algebraic closure of  $F$ , which exists by Theorem 5.57. Let  $\alpha_1, \dots, \alpha_m$  be the roots of  $f$  in  $\overline{F}$ . By construction,  $F(\alpha_1, \dots, \alpha_m)$  is a splitting field of  $f$ .  $\square$

**Example 5.67.**

- As a silly example, if  $f$  already splits into linear factors over  $F[x]$ , then  $F$  itself is the splitting field of  $f$  over  $F$ .
- The splitting field of  $f = x^2 + 1$  over  $\mathbb{R}$  is  $\mathbb{C}$ : the roots of  $f$  are  $i$  and  $-i$ , and  $\mathbb{R}(i, -i) = \mathbb{C}$ .
- Let  $q$  be any irreducible quadratic polynomial in  $\mathbb{R}[x]$ . You will show in problem set 10 that the splitting field of  $q$  is  $\mathbb{C}$ .

**Remark 5.68.** In general, to form a field extension given by adjoining all the roots of two polynomials  $g_1$  and  $g_2$  amounts to forming a splitting field of their product  $g_1 g_2$ . This naturally generalizes to any number of polynomials  $g_1, \dots, g_n$ : to adjoin all the roots of  $g_1, \dots, g_n$  is the same as forming the splitting field of  $g_1 \cdots g_n$ .

It seems intuitive that by adjoining all the roots of  $f \in F[x]$  to  $F$ , we will get a *unique* field (up to isomorphism). That is, it seems intuitive that splitting fields are unique up to isomorphism. This is indeed true, but the proof is a bit technical. We will actually show something a bit stronger.

**Theorem 5.69.** *Let  $F$  be a field,  $f$  be a nonconstant polynomial, and a field isomorphism  $\theta: F \rightarrow F'$ . Consider the isomorphism  $\tilde{\theta}: F[x] \rightarrow F'[x]$  induced by  $\theta$ , and let  $f' = \tilde{\theta}(f) \in F'[x]$  be the polynomial corresponding to  $f$ . Suppose  $L$  is a splitting field of  $f$  over  $F$  and  $L'$  is a splitting field of  $f'$  over  $F'$ . Then there is a field isomorphism  $\hat{\theta}: L \rightarrow L'$  extending  $\theta$ .*

*Proof.* We proceed by induction on the degree  $n$  of  $f$ . If  $f$  is linear, then so is  $f'$ , and in this case  $L = F$  and  $L' = F'$ . We have shown this already in Corollary 5.25.

Let  $p$  be any irreducible factor of  $f$ , and let  $\alpha \in L$  be any one of the roots of  $p$ . Let  $p' = \tilde{\theta}(p)$  be the irreducible polynomial in  $F'[x]$  that corresponds to  $p$ , and let  $\alpha'$  be any one of the roots of  $p'$ . By Theorem 5.56, there is an isomorphism

$$\phi: F(\alpha) \rightarrow F'(\alpha')$$

extending  $\theta$  and sending  $\alpha$  to  $\alpha'$ .

In  $F(\alpha)$ ,  $f$  factors as  $f = (x - \alpha)g$ , and in  $F'(\alpha')$ . Moreover, since  $\phi$  extends  $\theta$  and  $\phi(\alpha) = \alpha'$ , it follows that  $\phi(x - \alpha) = x - \alpha'$ . Therefore, the corresponding polynomial  $f' = \phi(f)$  factors as  $f' = (x - \alpha')g'$ , and we have

$$(x - \alpha')\phi(g) = \phi(x - \alpha)\phi(g) = \phi((x - \alpha)g) = \phi(f) = f' = (x - \alpha')g'.$$

Since  $F'$  is a domain, we conclude that  $\phi(g) = g'$ .

Since  $L$  is a splitting field of  $f$  over  $F$ ,  $f$  factors completely over  $L$  and thus so must  $g$ . Moreover, any other field  $E \supseteq F(\alpha)$  containing all the roots of  $g$  would also contain  $\alpha$ , and thus all the roots of  $f$ , and thus  $E = L$ . Thus  $L$  is a splitting field for  $g$  over  $F(\alpha)$ , and  $L'$  is a splitting field of  $g'$  over  $F'(\alpha')$ : Since  $\deg(g) < \deg(f) = n$ , it follows by induction that there is a field isomorphism  $\hat{\theta}: L \rightarrow L'$  that extends  $\phi$  and hence extends  $\theta$ .  $\square$

**Corollary 5.70** (Uniqueness of the splitting field of  $f(x)$  over the base field  $F$ ). *Any two splitting fields  $L$  and  $L'$  of  $f(x) \in F[x]$  over  $F$  are isomorphic via an isomorphism  $\phi: L \rightarrow L'$  that fixes  $F$ , i.e.  $\phi|_F = \text{id}_F$ .*

*Proof.* Apply part (2) of Theorem 5.69 to  $\theta = \text{id}_F$ .  $\square$

We will now be referring to *the* splitting field of  $F$ , rather than *a* splitting field, thanks to the uniqueness result above.

**Corollary 5.71.** *If  $L$  is the splitting field over  $F$  of an irreducible polynomial  $f(x) \in \mathbb{F}[x]$  and if  $\alpha, \beta \in L$  are any two roots of  $f$ , then there is a field automorphism  $s: L \rightarrow L$  such that  $s|_F = \text{id}_F$  and  $s(\alpha) = \beta$ .*

*Proof.* We basically already proved this, but since it is of large importance, let's do so again:

Since  $\alpha, \beta$  are roots of the same irreducible polynomial, by Corollary 5.25 there is an isomorphism  $\tau: F(\alpha) \rightarrow F(\beta)$  such that  $\tau|_F = \text{id}_F$  and  $\tau(\alpha) = \beta$ . We have two field maps, the inclusion  $F(\alpha) \hookrightarrow L$  and the composition of  $F(\alpha) \xrightarrow{\tau} F(\beta) \hookrightarrow L$ , and realize  $L$  as the splitting field of  $f$  over  $F(\alpha)$  in two different ways. Since splitting fields are unique, by Corollary 5.70, an isomorphism such as  $s$  exists.  $\square$

**Example 5.72.** Let  $L$  be the splitting field of  $x^3 - 2$  over  $\mathbb{Q}$ , so  $L = \mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3}\sqrt[3]{2}, e^{4\pi i/3}\sqrt[3]{2})$ . Corollary 5.71 says that there is a field automorphism  $s$  of  $L$  such that

$$s(e^{2\pi i/3}\sqrt[3]{2}) = e^{4\pi i/3}\sqrt[3]{2}.$$

In fact, complex conjugation gives such an isomorphism.

Corollary 5.71 also says that there is a field automorphism  $\tau$  of  $L$  such that

$$\tau(\sqrt[3]{2}) = e^{2\pi i/3}\sqrt[3]{2},$$

but it is not as clear what map this  $\tau$  is.

**Example 5.73.** The splitting field of  $f(x) = x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$  is

$$\mathbb{Q}(\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3}).$$

Note that we have shown the last equality in Example 5.21. This is an example where the splitting field of  $f \in F[x]$  is not the algebraic closure of  $F$ : we showed in Example 5.37 that  $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$ , while in Example 5.42 we showed that  $[\overline{\mathbb{Q}} : \mathbb{Q}] = \infty$ . Thus  $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subsetneq \overline{\mathbb{Q}}$ .

**Lemma 5.74.** *For every field  $F$  and every nonconstant polynomial  $f \in F[x]$  of degree  $n \geq 1$ , every splitting field  $L$  for  $f$  over  $F$  satisfies  $[L : F] \leq n!$ .*

*Proof.* By Corollary 5.70, splitting fields are unique up to isomorphism, so we just need to show that there exists a splitting field  $L$  for  $f$  over  $F$  with  $[L : F] \leq n!$ .

Intuitively, we just need to adjoin all the roots of  $f$ , which is possible since we already know we can adjoin a root of any polynomial. More formally, we start by showing that there is a field extension  $E/F$  such that  $f$  splits completely in  $E[x]$ , but without the minimality condition. Proceed by induction on the degree  $n$  of  $f$ . In the base case,  $n = 1$ , so  $f$  is linear and so  $E = F$  works.

Assume  $f$  has degree  $n > 1$ . We proved in Theorem 5.10 that there exists a field extension  $K$  of  $F$  such that  $f$  has a root  $\alpha$ . In  $K[x]$  we have  $f = (x - \alpha)g$  with  $\deg(g) = \deg(f) - 1 = n - 1$ . By induction, there is a field extension  $E$  of  $K$  with  $[E : K] \leq (n - 1)!$  in which  $g$  splits completely. Then  $f$  also splits completely in  $E$  and by the Degree Formula

$$[E : F] = [E : K][K : F] \leq (n - 1)!n = n!.$$

Finally, let

$$f(x) = \prod_i (x - \alpha_i)$$

be the factorization of  $f$  in  $E[x]$ , and set  $L = F(\alpha_1, \dots, \alpha_n)$ . By Lemma 5.64,  $L$  is a splitting field of  $f$  over  $F$ . By the Degree Formula,

$$[E : F(\alpha_1, \dots, \alpha_n)][F(\alpha_1, \dots, \alpha_n) : F] = [E : F] \leq n! \implies [F(\alpha_1, \dots, \alpha_n) : F] \leq n!. \quad \square$$

The degree of the splitting field of  $f$  can be  $n!$ , but it can also be much smaller.

**Example 5.75.** Let us find the splitting field  $L$  of  $x^3 - 2$  over  $\mathbb{Q}$ , and the degree of this field. Its roots in  $\mathbb{C}$  are  $\sqrt[3]{2}$ ,  $\zeta_3 \sqrt[3]{2}$ , and  $\zeta_3^2 \sqrt[3]{2}$ , where  $\zeta_3 = e^{\frac{2\pi i}{3}}$ . So

$$L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3 \sqrt[3]{2}, \zeta_3^2 \sqrt[3]{2}).$$

It is useful to simplify this a bit, by noting that

$$\zeta_3 = \frac{\zeta_3^2 \sqrt[3]{2}}{\zeta_3 \sqrt[3]{2}} \in L$$

and thus

$$L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3).$$

We know from Lemma 5.74 above that  $[L : \mathbb{Q}] \leq 3! = 6$ . We claim it is exactly 6. First, we have

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq L.$$

Moreover,  $x^3 - 2$  is irreducible over  $\mathbb{Q}$ , and  $\sqrt[3]{2}$  satisfies this polynomial, so it must be the minimal polynomial of  $\sqrt[3]{2}$  over  $\mathbb{Q}$ . Thus  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ . Note that  $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$  but  $\zeta_3$  is not real, so  $\mathbb{Q}(\sqrt[3]{2}) \subseteq L$  has degree at least two. The Degree Formula shows that

$$[L : \mathbb{Q}] = [L : \mathbb{Q}(\sqrt[3]{2})][\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] \geq 3 \cdot 2 = 6.$$

By Lemma 5.74,  $[L : \mathbb{Q}] \leq 6$ , so we conclude that  $[L : \mathbb{Q}] = 6$ .

**Example 5.76.** Let  $f(x) = x^n - 1 \in \mathbb{Q}[x]$ . Then  $f$  splits completely in  $\mathbb{C}[x]$ , and its  $n$  many roots are the  $n$ th roots of 1. One of these is  $\zeta_n := e^{2\pi i/n}$ . Notice that every other  $n$ th root of 1 is a power of this one. Thus  $\mathbb{Q}(\zeta_n)$  is the splitting field of  $x^n - 1$  over  $\mathbb{Q}$ . This is a somewhat special example: upon adjoining one of the roots of  $f$  we got all the others for free. This happens in other examples too, but it is certainly *not* a general principle.

In particular, we see that the degree of  $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_n)$  is at most  $n$ , far less than the bound of  $n!$  given by Lemma 5.74. In fact, it is at most  $n-1$ , since  $f$  factors as  $(x-1)(x^{n-1} + \cdots + x + 1)$ , and hence the minimum polynomial of  $\zeta_n$  is a divisor of  $x^{n-1} + \cdots + x + 1$ .

When  $n = p$  is prime, then one can show that  $x^{p-1} + \cdots + x + 1$  is irreducible, and hence it must equal the minimum polynomial of  $\zeta_p$ . So, in this case, the degree of  $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_p)$  is exactly  $p-1$ . However, the degree of  $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_n)$  can be smaller than  $n-1$  in general; for example, when  $n = 4$ ,  $\zeta_4 = i$  and  $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ . Note that  $x^3 + x^2 + x + 1$  factors as

$$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$$

and  $m_{i,\mathbb{Q}}(x) = x^2 + 1$ .

**Example 5.77.** In Problem Set 9, you showed that the splitting field  $F$  of  $f = x^4 + 5 \in \mathbb{Q}[x]$  is a degree 8 extension of  $\mathbb{Q}$ , and again  $8 < 4! = 24$ . This is also an example where adding one root does not give us the entire splitting field: since  $f$  is irreducible over  $\mathbb{Q}$ , which one can show via Eisenstein's Criterion, then for any root  $\alpha$  of  $f$  we must have  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ , but since  $[F : \mathbb{Q}] = 8$ , then  $\mathbb{Q}(\alpha)$  must not contain at least one of the other roots of  $f$ .

Here is another interesting feature of this example: let

$$\alpha_1 = e^{\pi i/4} \sqrt[4]{5}, \quad \alpha_2 = e^{3\pi i/4} \sqrt[4]{5}, \quad \alpha_3 = e^{5\pi i/4} \sqrt[4]{5}, \quad \alpha_4 = e^{7\pi i/4} \sqrt[4]{5},$$

and note that these are the four roots of  $f$ . You showed in Problem Set 9 that  $\mathbb{Q}(\alpha_1 + \alpha_4) \subseteq \mathbb{R}$ , but since  $\alpha_1, \dots, \alpha_4 \notin \mathbb{R}$ , this says that none of the roots (including  $\alpha_1$  and  $\alpha_2$  is in  $\mathbb{Q}(\alpha_1 + \alpha_4)$ ). This is in stark contrast with the example  $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$  from before.



## 5.5 Separability

**Definition 5.78.** Let  $R$  be a commutative ring. The **characteristic**  $\text{char}(R)$  of  $R$  is defined to be the smallest positive integer  $n$  such that

$$n \cdot 1_R = \underbrace{1_R + \dots + 1_R}_n = 0_R$$

if such an integer exists, and 0 otherwise.

**Example 5.79.** Here are some familiar examples:  $\text{char}(\mathbb{Z}) = 0$  and  $\text{char}(\mathbb{Z}/n) = n$ .

**Definition 5.80.** Given a field  $F$ , its **prime field** is the subfield of  $F$  generated by  $1_F$ . More precisely, the field

$$\text{Frac}(\{k1_F \mid k \in \mathbb{Z}\}).$$

You proved the following lemma in Problem Set 8:

**Lemma 5.81.** *Let  $F$  be a field.*

- a) *The prime field of  $F$  is isomorphic to exactly one of the fields  $\mathbb{Q}$  or  $\mathbb{Z}/p$ .*
- b) *The characteristic  $\text{char}(F)$  is either 0 or a prime number  $p$ .*

In prime characteristic  $p$ , the most important tool we have at our disposal is the Frobenius endomorphism  $x \mapsto x^p$ . This is a simple but very powerful tool. The fact that the  $p$ th power map is a ring homomorphism is known as the **Freshman's Dream**.

**Lemma 5.82** (Freshman's Dream). *If  $R$  is a commutative ring of prime characteristic  $p$ , then the function*

$$F: R \rightarrow R \\ c \mapsto F(c) = c^p$$

*is a ring homomorphism.*

*Proof.* Since

$$(a + b)^p = \sum_{k=0}^p \binom{p}{k} a^k b^{p-k}$$

and the binomial coefficients  $\binom{p}{k}$  are divisible by  $p$  for any  $1 \leq k \leq p-1$ , it follows that

$$(a + b)^p = a^p + b^p.$$

Because we also have  $(ab)^p = a^p b^p$  by commutativity of  $R$ , and  $F(1) = 1^p = 1$ , the function  $F$  is a ring homomorphism, as desired.  $\square$

**Remark 5.83.** Let  $R$  be a commutative ring of prime characteristic  $p$ . Since  $\text{End}(R)$  is closed under composition, the  $e$ th iterate of the Frobenius endomorphism  $F^e$ , given by

$$F^e = \underbrace{F \circ \dots \circ F}_{e \text{ times}}: R \rightarrow R \\ x \mapsto F^e(x) = x^{p^e}$$

is also a ring homomorphism.



We are now ready to talk about separability.

**Definition 5.84.** Let  $F$  be a field,  $f \in F[x]$  be a monic polynomial, and  $\alpha$  be a root of  $f$  in some field extension  $L$  of  $F$ . The **multiplicity** of  $\alpha$  in  $f$  is the number of times  $x - \alpha$  appears in the factorization

$$f = \prod_i (x - \beta_i)$$

of  $f$  in some (equivalently, any) splitting field of  $f$ .

**Definition 5.85.** A polynomial  $f \in F[x]$  is **separable** if the multiplicity of every root of  $f$  in  $\overline{F}$  is 1.

**Example 5.86.** The polynomial  $x^3 - 1$  is separable in  $\mathbb{R}[x]$  because it has 3 distinct roots in  $\mathbb{C}$ , namely 1,  $\zeta_3$ , and  $\zeta_3^2$ , but not in  $\mathbb{Z}/3[x]$ , since  $x^3 - [1]_3 = (x - [1]_3)^3$ .

**Definition 5.87.** For any field  $F$  and  $f = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$ , define its **derivative** to be

$$f' = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1.$$

**Remark 5.88.** The derivative is an  $F$ -linear map  $F[x] \rightarrow F[x]$ : indeed, for any  $f, g \in F[x]$ , we have

$$(f + g)' = f' + g' \quad \text{and} \quad (af)' = af'$$

for all  $a \in F$ .

**Remark 5.89.** If  $F$  is a field of characteristic 0, then every nonconstant polynomial  $f \in F[x]$  has  $f' \neq 0$ ; in fact,  $\deg(f') = \deg(f) - 1$ . In contrast, in prime characteristic  $p$  the condition  $f' = 0$  does not imply  $f$  is constant. For example,  $(x^p)' = p x^{p-1} = 0$ .

**Lemma 5.90** (Criteria for separability). *Let  $F$  be a field and  $f \in F[x]$ .*

- a) *Given a root  $\alpha$  of  $f$  in some field extension  $L$  of  $F$ , the multiplicity of  $\alpha$  in  $f$  is at least 2 if and only if  $f'(\alpha) = 0$ .*
- b) *A polynomial  $f$  is separable if and only if  $\gcd(f, f') = 1$  in  $F[x]$ .*
- c) *If  $f$  is irreducible in  $F[x]$ , then  $f$  is separable if and only if  $f' \neq 0$ .*

*Proof.* Let  $L$  be the splitting field of  $f$ .

- a) If  $f = (x - \alpha)^2 g(x)$  in  $L[x]$ , then  $f'(x) = 2(x - \alpha)g(x) + (x - \alpha)^2 g'(x)$ , so  $f'(\alpha) = 0$ .  
Conversely, if  $f = (x - \alpha)h(x)$  and  $h(\alpha) \neq 0$ , then  $f'(x) = h(x) + (x - \alpha)h'(x)$  does not have  $\alpha$  as a root.
- b) By 1),  $f$  is separable if and only if  $f$  has no common roots with  $f'$ . By a problem in Problem Set 9, we have  $\gcd(f, f') = 1$  if and only if  $f$  and  $f'$  have no common roots in  $\overline{F}$ .
- c) Since the degree of  $f'$  is strictly less than the degree of  $f$  and  $f$  is irreducible, we have that  $\gcd(f, f') \neq 1$  if and only if  $f' = 0$ .  $\square$

**Definition 5.91.** An algebraic field extension  $L/F$  is **separable** if for every  $\alpha \in L$  the minimal polynomial  $m_{\alpha,F}$  of  $\alpha$  over  $F$  is separable. An extension that is not separable is sometimes called **inseparable**.

**Exercise 15.** Let  $\alpha_1, \dots, \alpha_n$  be algebraic elements over  $F$ , and let  $L := F(\alpha_1, \dots, \alpha_n)$ . Show that the extension  $F \subseteq L$  is separable if and only if  $m_{\alpha_i,F}$  is separable for every  $i$ .

**Definition 5.92.** A field  $F$  is **perfect** if every algebraic extension of  $F$  is separable.

**Remark 5.93.** Every irreducible polynomial in  $F[x]$  is separable if and only if every algebraic field extension  $L/F$  is separable.

**Corollary 5.94** (Every field of characteristic zero is perfect). *Let  $F$  be a field of characteristic zero. Every irreducible polynomial in  $F[x]$  is separable and every algebraic field extension  $L/F$  is separable.*

*Proof.* For every  $\alpha \in L$ , its minimal polynomial  $m_{\alpha,F}$  is nonconstant. Since  $\text{char}(F) = 0$ ,  $m'_{\alpha,F} \neq 0$ . Since  $m_{\alpha,F}$  is irreducible in  $F[x]$ , Lemma 5.90 implies  $m_{\alpha,F}(x)$  is separable.  $\square$

**Example 5.95.** The characteristic zero extension  $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$  is algebraic, and thus separable by Corollary 5.94.

**Lemma 5.96.** *Let  $F$  be a field with prime characteristic  $\text{char}(F) = p$ .*

- a) *If  $b$  is an element of  $F$  that is not a  $p$ th power of an element of  $F$  and  $L$  is an algebraic extension of  $F$  that contains a root of  $x^p - b$ , then  $F \subseteq L$  is not separable.*
- b) *If every element of  $F$  is the  $p$ th power of another element of  $F$ , then every algebraic extension  $F \subseteq L$  is separable.*

*Proof.*

- a) In general, for such  $F$  and  $b$ , let  $\alpha$  be a root of  $x^p - b$  in some field extension of  $F$  and let  $L := F(\alpha)$ . We claim that  $F \subseteq L$  is not separable; specifically, we claim  $m := m_{\alpha,F}$  is not separable. Since  $\alpha$  is a root of  $x^p - b$ , we have  $m \mid x^p - b$ . In  $L[x]$ , by the [Freshman's Dream](#) we have

$$(x - \alpha)^p = x^p - \alpha^p = x^p - b.$$

It follows that  $m$  must divide  $(x - \alpha)^p$  in  $L[x]$  and hence  $m$  must have the form  $(x - \alpha)^i$  for some  $1 \leq i \leq p$ . But  $i \neq 1$  since  $\alpha \notin F$ . Thus  $\alpha$  is a multiple root of  $m$  and  $m$  is irreducible in  $F[x]$ .

- b) Given an irreducible polynomial  $q \in F[x]$ , if  $q' = 0$ , then we must have that  $q$  is a sum of terms of the form  $bx^{mp}$ , for some  $m \geq 0$  and  $b \in F$ . By assumption, for each such term, we have  $b = c^p$  for some  $c \in F$ , and thus each term of  $q$  has the form  $(cx^m)^p$ . By the [Freshman's Dream](#),  $q = g^p$  for some polynomial  $g \in F[x]$ . But this is impossible since  $q$  is irreducible. We conclude that  $q' \neq 0$ , which by Lemma 5.90 implies that  $q$  is separable. This shows that every irreducible polynomial over  $F$  is separable, and thus every algebraic extension over  $F$  is separable.  $\square$

**Remark 5.97.** Let  $F$  be a field of prime characteristic  $p$ . The condition that every element in  $F$  is a  $p$ th power of another element in  $F$  is equivalent to saying that the Frobenius map is surjective. We can write this more succinctly as  $F^p = F$ . Lemma 5.96 says that a field of prime characteristic  $p$  is perfect if and only if  $F^p = F$ .

**Example 5.98.** Let  $p$  be a prime and  $t$  be a variable, let  $F = \mathbb{Z}/p(t)$ , and consider the field  $L = F[z]/(z^p - t)$ . The element  $t \in F$  is not a  $p$ th power of any element in  $F$ , and  $F \subseteq L$  is an extension containing a root of the polynomial  $x^p - t$ . By Lemma 5.96,  $F \subseteq L$  is an inseparable extension. Moreover, the field  $F$  is not perfect.

**Theorem 5.99** (Finite fields are perfect). *Every algebraic field extension of a finite field is separable.*

*Proof.* Problem Set 10. □

We have shown that fields of characteristic 0 and fields  $K$  of characteristic  $p$  such that  $K = K^p$  are separable.

# Chapter 6

## Galois theory

An approximate definition of Galois Theory is the study of the symmetries enjoyed by the roots of a polynomial. As a simple example, the polynomial  $x^2 + 1 \in \mathbb{R}[x]$  has two roots, and there are essentially indistinguishable from an algebraic point of view — which root is  $\sqrt{-1}$  and which is the negative of it? It makes no difference, really.

For another example, consider  $p(x) = x^3 - 2 \in \mathbb{Q}[x]$ , which has three roots. As we will soon learn, these roots of  $x^3 - 2$  are as symmetric as possible over  $\mathbb{Q}$ . On the other hand,  $q(x) = x^4 - 2 \in \mathbb{Q}[x]$  has four roots, and we will soon see that these four roots are not as symmetric as possible over  $\mathbb{Q}$ .

Before starting the chapter, you might want a reminder of group actions. Below we include some of the definitions we will need for your convenience, though it is highly recommended that you read through the relevant portion of the 817 notes.

**Definition 6.1.** For a group  $(G, \cdot)$  and a set  $S$ , an **action** of  $G$  on  $S$  is a function

$$G \times S \rightarrow S,$$

typically written as  $(g, s) \mapsto g \cdot s$ , such that

- $g \cdot (g' \cdot s) = (gg') \cdot s$  for all  $g, g' \in G$  and  $s \in S$ , and
- $e_G \cdot s = s$  for all  $s \in S$ .

Let  $\text{Aut}(S)$  denote the set of automorphisms of the set  $S$ , which is a group under composition of functions. A group action of  $G$  on  $S$  is a group homomorphism  $G \rightarrow \text{Aut}(S)$ .

**Definition 6.2.** An action of a group  $G$  on a set  $S$  is called **faithful** if the associated group homomorphism is injective. Equivalently, an action is faithful if and only if for a given  $g \in G$ , whenever  $g \cdot s = s$  for all  $s \in S$ , it must be that  $g = e_G$ .

**Definition 6.3.** A group action of  $(G, \cdot)$  on  $S$  is **transitive** if for all  $p, q \in S$  there is a  $g \in G$  such that  $q = g \cdot p$ . Equivalently, an action is transitive if  $\text{Orb}_G(p) = S$  for any  $p \in S$ .

**Definition 6.4.** Let  $G$  be a group acting on a set  $S$ . The equivalence relation on  $S$  induced by the action of  $G$ , written  $\sim_G$ , is defined by  $s \sim_G s'$  if and only if there is a  $g \in G$  such that  $s' = g \cdot s$ . The equivalence classes of  $\sim_G$  are called **orbits**, specifically the equivalence class

$$\text{Orb}_G(s) = \{g \cdot s \mid g \in G\}$$

is the orbit of  $S$ . The set of equivalence classes with respect to  $\sim_G$  is written  $S/G$ .

## 6.1 Group actions on field extensions

**Definition 6.5.** Let  $K$  be a field. The **automorphism group** of  $K$ , denoted  $\text{Aut}(K)$ , is the collection of field automorphisms of  $K$ , with the binary operation of composition.

**Definition 6.6.** Let  $K/F$  be a field extension. The **automorphism group** of  $K/F$ , denoted  $\text{Aut}(K/F)$ , is the collection of field automorphisms of  $K$  that restrict to the identity on  $F$ , with the binary operation of composition.

**Exercise 16.** Let  $K/F$  be a field extension. Then  $\text{Aut}(K)$  is a group under composition of maps, and  $\text{Aut}(K/F)$  is a subgroup of  $\text{Aut}(K)$ .

Some books write  $\text{Gal}(L/F)$  for  $\text{Aut}(L/F)$ , and call it the Galois group of  $L$  over  $F$ . We will reserve that notation for a special type of finite extensions – those that are Galois – and use only  $\text{Aut}(L/F)$  for the general case.

**Example 6.7.** The automorphism group  $\text{Aut}(\mathbb{C}/\mathbb{R})$  has two elements: the identity map and the map given by complex conjugation. The fact that each of these is an element of  $\text{Aut}(\mathbb{C}/\mathbb{R})$  amounts to the fact that complex conjugation commutes with addition and multiplication of complex numbers. To see these are all the automorphisms, suppose  $\tau \in \text{Aut}(\mathbb{C}/\mathbb{R})$ . Since  $\tau|_{\mathbb{R}} = \text{id}_{\mathbb{R}}$ , then for any  $z = a + ib \in \mathbb{C}$  we have  $\tau(z) = a + b\tau(i)$ . Moreover,

$$-1 = \tau(-1) = \tau(i \cdot i) = \tau(i) \cdot \tau(i),$$

and so  $\tau(i) = \pm i$ .

**Example 6.8.** For any squarefree integer  $d$ , the group  $\text{Aut}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$  also has two elements: the identity and the map defined by  $a + b\sqrt{d} \mapsto a - b\sqrt{d}$ . The details are similar to the previous example, so we leave them as an exercise.

**Remark 6.9.** Let  $L$  be a field and let  $\sigma \in \text{Aut}(L)$ . Then the UMP of polynomial rings gives that there is an induced ring homomorphism  $(-)^{\sigma} : L[x] \rightarrow L[x]$  such that for each  $q = a_n x^n + \cdots + a_0 \in K[x]$ , we have

$$q^{\sigma}(x) = \sigma(a_n)x^n + \cdots + \sigma(a_0).$$

If  $\sigma \in \text{Aut}(L/K)$  and  $q \in K[x]$ , then  $q^{\sigma} = q$ .

**Lemma 6.10.** Let  $K/F$  be a field extension,  $\sigma \in \text{Aut}(K/F)$ , and  $q \in F[x]$ .

- a) For all  $c \in K$ , we have  $\sigma(q(c)) = q(\sigma(c))$ .
- b) If  $\alpha \in K$  is a root of  $q$ , then  $\sigma(\alpha)$  also is a root of  $q$ .

*Proof.*

- a) By assumption,  $\sigma$  is a homomorphism and it restricts to the identity on  $F$ . Thus for any polynomial  $q = a_n x^n + \cdots + a_0 \in F[x]$ , we have

$$\sigma(q(c)) = \sigma(a_n c^n + \cdots + a_0) = \sigma(a_n)\sigma(c)^n + \cdots + \sigma(a_0) = a_n \sigma(c)^n + \cdots + a_0 = q(\sigma(c))$$

b) If  $\alpha$  is a root of  $f$ , then

$$\begin{aligned} 0 &= \sigma(0) = \sigma(q(\alpha)) \\ &= q(\sigma(\alpha)) \quad \text{by a)} \end{aligned}$$

showing that  $\sigma(\alpha)$  is also a root of  $q$ .  $\square$

We now come to the main idea connecting field extensions and groups. It concerns the *action* of the group of automorphisms of a splitting field of a polynomial on the set of roots of that polynomial.

**Theorem 6.11.** *Let  $L$  be the splitting field of a polynomial  $f \in F[x]$ . Let  $S$  be the set of distinct roots of  $f$  in  $K$ , and let  $n := |S|$ .*

a) *The group  $\text{Aut}(L/F)$  acts faithfully on  $S$ , via*

$$\sigma \cdot b := \sigma(b)$$

*for all  $\sigma \in \text{Aut}(L/F)$  and  $b \in S$ , and hence  $\text{Aut}(L/F)$  is isomorphic to a subgroup of  $S_n$ .*

b) *If  $f \in F[x]$  is an irreducible polynomial, then  $\text{Aut}(L/F)$  acts transitively on  $S$ .*

*Proof.*

a) Let  $G = \text{Aut}(L/F)$ . To see that the action above is well-defined, notice that if  $b \in S$  then  $\sigma(b) \in S$  by Lemma 6.10. Now we have

$$\sigma \cdot (\sigma' \cdot b) = \sigma(\sigma'(b)) = (\sigma \circ \sigma')(b) \quad \text{for all } \sigma, \sigma' \in G, b \in S,$$

$$1_G \cdot b = \text{id}_K(b) = b \quad \text{for all } \sigma \in G \text{ and } b \in S,$$

so the given formula does indeed define an action of  $G$  on  $S$ .

This action is faithful: if  $\sigma$  fixes all the roots  $\alpha_1, \dots, \alpha_n$  of  $f$ , then it fixes every element of  $F(\alpha_1, \dots, \alpha_n) = L$ . Thus the corresponding group homomorphism  $\text{Aut}(L/F) \rightarrow \text{Aut}(S)$  is injective. On the other hand, the group of automorphisms on a set of  $n$  elements is isomorphic to  $S_n$ , so we have an inclusion of  $\text{Aut}(L/F)$  into  $S_n$ , and thus  $\text{Aut}(L/F)$  is isomorphic to a subgroup of  $S_n$ .

b) Let  $\alpha, \beta$  be any two roots of  $f$ . By Theorem 5.69, there is an isomorphism  $\theta: F(\alpha) \rightarrow F(\beta)$  that fixes  $F$ .

Our polynomial factors both as  $f = (x - \alpha)g$  and  $f = (x - \beta)h$ . Since  $f^\theta = f$  and  $(x - \alpha)^\theta = x - \beta$ , we must have  $g^\theta = h$ . Theorem 5.69 applies, showing there is an automorphism  $\sigma: L \rightarrow L$  that extends  $\theta$ . In particular,  $\sigma$  fixes  $F$ , since  $\sigma$  extends  $\theta$  and  $\theta|_F = \text{id}_F$ , so  $\sigma \in \text{Aut}(L/F)$ . Moreover, since  $\sigma$  extends  $\theta$  we have  $\sigma(\alpha) = \theta(\alpha) = \beta$ . This proves the action is transitive on the set of roots of any irreducible polynomial.  $\square$

Soon we will show that if  $f \in F[x]$  is separable but not necessarily irreducible, and  $L$  is the splitting field of  $f$ , then the orbits of the action of  $\text{Aut}(L/F)$  on the set of roots of  $f$  are precisely the sets of roots of the same irreducible factor of  $f$ . But to do so, we will need a little bit of Galois theory.

**Corollary 6.12.** *Let  $L$  be the splitting field of a polynomial  $f \in F[x]$  with  $n$  distinct roots. Then  $|\operatorname{Aut}(L/F)| \leq n!$ .*

*Proof.* We showed in Theorem 6.11 that  $\operatorname{Aut}(L/F)$  is isomorphic to a subgroup of  $S_n$ , and thus it has at most  $|S_n| = n!$  elements.  $\square$

We will give an improved version of this result soon.

**Exercise 17.** Let  $F$  be a field and  $L = F(a_1, \dots, a_n)$ , where  $a_1, \dots, a_n$  are elements in some extension of  $F$  that are algebraic over  $F$ . Each element  $\sigma \in \operatorname{Aut}(L/F)$  is uniquely determined by  $\sigma(a_1), \dots, \sigma(a_n)$ .

A typical question that arises from Theorem 6.11 is to explicitly identify the automorphisms of a splitting field extension as a subgroup of the symmetric group.

**Example 6.13.** Let  $L$  be the splitting field of  $f = x^3 - 2 \in \mathbb{Q}[x]$  and  $G := \operatorname{Aut}(L/\mathbb{Q})$ . Recall from Example 5.75 that  $L = \mathbb{Q}(\sqrt[3]{2}, \zeta)$ , where  $\zeta = e^{2\pi i/3}$ , and that  $[L : \mathbb{Q}] = 6$ . Let us write the roots of  $f$  as

$$\alpha_1 = \sqrt[3]{2}, \alpha_2 = \zeta\alpha_1, \alpha_3 = \zeta^2\alpha_1.$$

From Theorem 6.11,  $G$  acts transitively on  $\{\alpha_1, \alpha_2, \alpha_3\}$ , and hence  $G$  is isomorphic to a subgroup of  $S_3$ .

The restriction of complex conjugation to  $L$  determines an element  $s$  of  $G$  of order 2, since  $L$  is closed under complex conjugation. We have

$$s(\alpha_1) = \alpha_1, s(\alpha_2) = \alpha_3, s(\alpha_3) = \alpha_2$$

and so  $s$  corresponds to the permutation  $(23) \in S_3$ .

Since the action of  $G$  on the roots of  $f$  is transitive, there is also an element  $\tau \in G$  such that  $\tau(\alpha_1) = \alpha_2$ . Such a  $\tau$  corresponds to either  $(12)$ ,  $(123)$  of  $S_3$ . Either way,  $\tau$  and  $s$  generate all of  $S_3$ .

We conclude that  $|G| = 6$ , the maximum possible, and  $G$  is isomorphic to  $S_3$ . You should think of this as saying that the roots of  $x^3 - 2$  are as interchangeable as possible, since  $\operatorname{Aut}(L/\mathbb{Q})$  is as large as possible.

**Example 6.14.** Let  $L$  be the splitting field of  $f = x^4 - 2$  over  $\mathbb{Q}$ . The roots of  $f$  are

$$\alpha_1 = \sqrt[4]{2}, \quad \alpha_2 = i\alpha_1, \quad \alpha_3 = -\alpha_1, \quad \alpha_4 = -i\alpha_1,$$

and  $L = \mathbb{Q}(\alpha_1, i)$ . Let us start by computing  $[L : \mathbb{Q}]$ . Consider the chain of extensions

$$\mathbb{Q} \subseteq \mathbb{Q}(\alpha_1) \subseteq L = \mathbb{Q}(\alpha_1)(i).$$

The extension  $\mathbb{Q} \subseteq \mathbb{Q}(\alpha_1)$  has degree 4, since  $x^4 - 2$  is irreducible,<sup>1</sup> and the extension  $\mathbb{Q}(\alpha_1) \subseteq L$  has degree at most 2, since  $i$  is a root of the degree 2 polynomial  $x^2 + 1$ . Since  $\mathbb{Q}(\alpha_1) \subseteq \mathbb{R}$  and  $L$  contains elements that are not real, the extension  $\mathbb{Q}(\alpha_1) \subseteq L$  cannot be trivial, and thus it must have degree exactly 2. We conclude that  $[L : \mathbb{Q}] = 8$ .

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<sup>1</sup>By using Eisenstein's Criterion with the prime 2 to show  $f$  is irreducible over  $\mathbb{Z}$ , and Gauss' Lemma to show that  $f$  must then also be irreducible over  $\mathbb{Q}$ .

Set  $G := \text{Aut}(L/\mathbb{Q})$ . We know  $G$  is isomorphic to a subgroup of  $S_4$ . Since  $L = \mathbb{Q}(\alpha_1, i)$ , by Exercise 17 any  $\tau \in G$  is uniquely determined by what it does to  $\alpha_1$  and  $i$ . Such a  $\tau$  must send  $\alpha_1$  to a root of  $f$ , and thus to one of  $\alpha_1, \dots, \alpha_4$ . Moreover, since  $i$  is a root of  $x^2 + 1$ , so is  $\tau(i)$ , by Lemma 6.10, and thus  $\tau$  must send  $i$  to  $\pm i$ . By combining the possibilities for  $\tau(\alpha_1)$  and  $\tau(i)$ , we have at most 8 possibilities, so  $|G| \leq 8$ . In particular,  $G$  corresponds to a *proper* subgroup of  $S_4$ , and so the roots of  $x^4 - 2$  do not have as many symmetries as are conceivable.

*Claim:*  $|G| = 8$  and  $G$  is isomorphic to the subgroup of  $S_4$  generated by  $(24)$  and  $(1234)$ .

Let  $s$  be the map obtained by restricting complex conjugation to  $L$ , and note that indeed  $s \in \text{Aut}(L/\mathbb{Q})$ . This map  $s$  corresponds to  $(24) \in S_4$ .

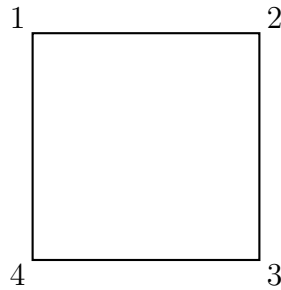
Now consider the field extension  $\mathbb{Q}(i) \subseteq L$ . Since  $[L : \mathbb{Q}] = 8$  and  $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ , by the Degree Formula we must have  $[L : \mathbb{Q}(i)] = 4$ . Since  $L = \mathbb{Q}(i)(\alpha_1)$ , the degree of  $m_{\alpha_1, \mathbb{Q}(i)}$  must be 4. In particular, this shows that  $x^4 - 2$  remains irreducible as a polynomial in  $\mathbb{Q}(i)[x]$ . So  $L$  is the splitting field of the irreducible polynomial  $x^4 - 2$  over  $\mathbb{Q}(i)$ , and we may thus apply Theorem 6.11 to get that there is an element  $\tau \in \text{Aut}(L/\mathbb{Q}(i))$  such that  $\tau(\alpha_1) = \alpha_2$ . We may regard  $\tau$  as an element of  $\text{Aut}(L/\mathbb{Q})$  too. Such a  $\tau$  satisfies  $\tau(i) = i$ , so

$$\tau(\alpha_2) = \tau(i\alpha_1) = i\tau(\alpha_1) = i\alpha_2 = \alpha_3.$$

A key point here is that if we had merely taken  $\tau$  to be an element of  $\text{Aut}(L/\mathbb{Q})$  sending  $\alpha_1$  to  $\alpha_2$ , we would have no idea what  $\tau$  does to  $\alpha_2$ : it was key to define  $\tau \in \text{Aut}(L/\mathbb{Q}(i))$  as we did. We also get  $\tau(\alpha_3) = \alpha_4$  and  $\tau(\alpha_4) = \alpha_1$ , so  $\tau$  corresponds to the permutation  $(1234)$ .

This proves that  $G$  is isomorphic to a subgroup of  $S_4$  that contains  $(24)$  and  $(1234)$ . Since the subgroup generated by these two elements has order 8 and  $|G| \leq 8$ , then  $G$  must be the subgroup  $\langle (24), (1234) \rangle$  of  $S_n$ , and  $|G| = 8$ .

Finally, we claim that this subgroup of  $S_4$  is isomorphic to  $D_8$ . Indeed, consider a square with



Let  $\rho \in D_8$  be the clockwise rotation by  $\frac{\pi}{2}$  and  $\tau \in D_8$  is the reflection across the line determined by the vertices 1 and 3. On the vertices of the square, the element  $\rho$  sends  $1 \mapsto 2$ ,  $2 \mapsto 3$ ,  $3 \mapsto 4$ , and  $4 \mapsto 1$ . Similarly,  $\tau$  switches the vertices 2 and 4. One can check that the map

$$\begin{aligned} G &\rightarrow D_8 \\ (1234) &\mapsto \rho \\ (24) &\mapsto \tau \end{aligned}$$

determines an isomorphism.



## 6.2 Automorphism groups of finite field extensions

We will now focus on finite field extensions and their automorphism groups. We start by giving a much better upper bound on the order of the automorphism group of a finite field extension.

**Theorem 6.15.** *Let  $L/F$  be a finite field extension. Then*

$$|\operatorname{Aut}(L/F)| \leq [L : F].$$

*If  $L$  is the splitting field of a separable polynomial in  $F[x]$ , then*

$$|\operatorname{Aut}(L/F)| = [L : F].$$

*Proof.* We proceed by induction on  $[L : F]$ . In the base case,  $[L : F] = 1$ , and thus  $L = F$ , so  $\operatorname{Aut}(L/F)$  is the trivial group, and both statements hold.

Now let  $n \geq 1$  and suppose that  $|\operatorname{Aut}(L/F)| \leq [L : F]$  holds for all  $L$  and  $F$  such that  $[L : F] < n$ . Let  $[L : F] = n$ . Pick  $\alpha \in L \setminus F$  and let  $m = m_{\alpha, F}$ , and consider  $F(\alpha)/F$ .

Note that  $H = \operatorname{Aut}(L/F(\alpha))$  is a proper subgroup of  $G = \operatorname{Aut}(L/F)$ . By induction, we have  $|H| \leq [L : F(\alpha)]$ . We claim that it suffices to prove  $[G : H] \leq [F(\alpha) : F]$ . Indeed, using the [Degree Formula](#) and the fact that  $|G| = |H| \cdot [G : H]$ , if  $[G : H] \leq [F(\alpha) : F]$  then

$$|G| = |H| \cdot [G : H] \leq [L : F(\alpha)][F(\alpha) : F] = [L : F].$$

To show that  $[G : H] \leq [F(\alpha) : F]$ , consider the function

$$\begin{aligned} G/H &= \{\text{cosets of } H \text{ in } G\} \xrightarrow{\Psi} \{\text{roots of } m \text{ in } L\} \\ ghH &\mapsto g(\alpha). \end{aligned}$$

By Lemma 6.10, for any  $g \in G$  the element  $g(\alpha)$  is also a root of  $m$ . For any  $h \in H$ , we have

$$gh(\alpha) = g(h(\alpha)) = g(\alpha).$$

Thus  $\Psi$  is well-defined. Moreover, for any  $g_1, g_2 \in G$  we have

$$\Psi(g_1H) = \Psi(g_2H) \iff g_1(\alpha) = g_2(\alpha) \iff g_2^{-1}g_1(\alpha) = \alpha$$

which is equivalent to saying that  $g_2^{-1}g_1$  fixes  $F(\alpha)$ , and equivalently

$$g_2^{-1}g_1 \in H \iff g_1H = g_2H.$$

This proves that the function  $\Psi$  is injective.

By Theorem 5.32,  $\deg(m) = [F(\alpha) : F]$ . Thus  $\Psi$  is an inclusion of  $G/H$  into a set with at most  $[F(\alpha) : F]$  many elements. Therefore,

$$[G : H] = |G/H| \leq [F(\alpha) : F].$$

Now suppose that  $f$  is separable, so that

$$f = c \prod_{i=1}^n (x - \alpha_i) \in L[x]$$

with  $\alpha_i \neq \alpha_j$  for  $i \neq j$  and  $L = F(\alpha_1, \dots, \alpha_n)$ .

Set  $\alpha = \alpha_1$  and let  $m$  be the irreducible factor of  $f$  that has  $\alpha$  as a root. Notice  $m = m_{\alpha, F}$ . As before, we consider  $F(\alpha)$  and set  $H = \text{Aut}(L/F(\alpha)) \leq \text{Aut}(L/F) = G$ . Note that  $L$  is also the splitting field of

$$g = \prod_{i=2}^n (x - \alpha_i) \in F(\alpha)[x]$$

over  $F(\alpha)$ , and  $g$  is also separable. By induction  $|H| = [L : F(\alpha)]$ , and it remains to show that

$$[G : H] = [F(\alpha) : F] = \deg(m).$$

Since  $f$  is separable, so is  $m$ , so  $\deg(m)$  is exactly the number of distinct roots of  $m$ . Showing that  $[G : H] = \deg(m)$  amounts to the assertion that the injective map  $\Psi$  is also surjective. This holds since  $G$  acts transitively on the roots of  $m$ , as shown in Theorem 6.11.  $\square$

The finite field extensions whose automorphism group is as large as possible are very important, and are the main subject of this final chapter.

**Definition 6.16.** A finite field extension  $F \subseteq L$  is **Galois** if  $|\text{Aut}(L/F)| = [L : F]$ . In this case we write  $\text{Gal}(L/F)$  for  $\text{Aut}(L/F)$ , and say  $\text{Gal}(L/F)$  is the **Galois group** of  $L$  over  $F$ .

**Example 6.17** (a nonexample). We claim that field extension  $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2})$  is not Galois. Indeed, suppose  $s \in \text{Aut}(L/\mathbb{Q})$ . Then  $s$  is entirely determined by where it sends  $\sqrt[3]{2}$  and it must send this element to another root of  $x^3 - 2$ . But the other two roots of this polynomial are not real, and hence not in  $L$ . So  $s(\sqrt[3]{2}) = \sqrt[3]{2}$  and  $s = \text{id}$ .

This shows  $\text{Aut}(L/\mathbb{Q})$  is the trivial group, so  $\text{Aut}(L/\mathbb{Q}) = 1 < 3 = [L : \mathbb{Q}]$ . In particular, the extension is not Galois.

Theorem 6.15 tells us how to construct Galois extensions:

**Corollary 6.18** (First construction of Galois extensions from splitting fields). *If  $L$  is the splitting field of a separable polynomial  $f \in F[x]$ , then  $L/F$  is Galois.*

**Definition 6.19.** Let  $f \in F[x]$  be a separable polynomial with splitting field  $L$ . The **Galois group** of  $f$  is  $\text{Gal}(L/F)$ .

We will need the following notation.

**Definition 6.20.** If  $G$  is subgroup of  $\text{Aut}(L)$ , the **subfield of  $L$  fixed by  $G$** , denoted  $L^G$ , is by definition

$$L^G := \{\alpha \in L \mid s(\alpha) = \alpha, \text{ for all } s \in G\}.$$

Note that the textbook writes this as  $L_G$ .

**Exercise 18.** If  $G$  is subgroup of  $\text{Aut}(L)$ , show that  $L^G$  is a subfield of  $L$ .

**Example 6.21.** Let  $G = \text{Aut}(\mathbb{C}/\mathbb{R})$ . Then  $\mathbb{C}^G$  is the subgroup of complex numbers fixed by all the elements in  $\text{Aut}(\mathbb{C}/\mathbb{R})$ , which we saw in Example 6.7 has only two elements, the identity and the conjugation map  $s$ . Therefore,  $\mathbb{C}^G$  is the set of complex numbers fixed by conjugation, and thus  $\mathbb{C}^{\text{Aut}(\mathbb{C}/\mathbb{R})} = \mathbb{R}$ .

## 6.3 The Fundamental Theorem of Galois Theory

The following is an important theorem with many corollaries. In fact, the Fundamental Theorem of Galois Theory, which we will state shortly, will follow from this result.

**Theorem 6.22** (Artin's Theorem). *Let  $L$  be any field. If  $G$  is a finite subgroup of  $\text{Aut}(L)$ , then  $L^G$  is a subfield of  $L$ , the extension  $L/L^G$  is finite and Galois, and  $\text{Gal}(L/L^G) = G$ .*

Note that we really do mean equality here: both  $G$  and  $\text{Gal}(L/L^G)$  are subgroups of  $\text{Aut}(L)$ , and the theorem states that they coincide. The containment  $G \subseteq \text{Gal}(L/L^G)$  is clear: if  $\sigma \in G$ , then by construction  $\sigma$  fixes every element of  $L^G$  and hence  $\sigma \in \text{Gal}(L/L^G)$ . The point of the theorem is that the extension  $L^G \subseteq L$  is always Galois and that if  $\sigma \in \text{Aut}(L)$  fixes every element of  $L^G$  then  $\sigma$  must belong to  $G$ .

We will not prove [Artin's Theorem](#) right away. Instead, we will first deduce some of its consequences, including the Fundamental Theorem of Galois Theory. We will then illustrate the Fundamental Theorem with many examples and give some consequences of it too. Only then will we circle back to prove [Artin's Theorem](#).

**Example 6.23.** The group  $G = \{\text{id}_{\mathbb{C}}, \sigma\}$ , where  $\sigma$  is complex conjugation, is a finite subgroup of  $\text{Aut}(\mathbb{C})$ . [Artin's Theorem](#) tells us that  $\mathbb{C}^G \subseteq \mathbb{C}$  is finite and Galois with Galois group  $G$ . It follows that  $[\mathbb{C} : \mathbb{C}^G] = |G| = 2$ . We already knew all this, since  $\mathbb{C}^G = \mathbb{R}$ .

As we head towards the Fundamental Theorem of Galois Theory, we start by stating a few helpful corollaries of [Artin's Theorem](#). These will also allow us to show that finite Galois extensions are precisely the splitting fields of separable polynomials.

**Corollary 6.24.** *Let  $L/F$  be any Galois extension. Then  $F = L^{\text{Gal}(L/F)}$ .*

*Proof.* Note that  $F \subseteq L^{\text{Gal}(L/F)}$  holds by definition, and so

$$[L : F] = [L : L^{\text{Gal}(L/F)}][L^{\text{Gal}(L/F)} : F]$$

by the [Degree Formula](#). But [Artin's Theorem](#) gives

$$[L : L^{\text{Gal}(L/F)}] = |\text{Gal}(L/F)|,$$

and we also know that  $[L : F] = |\text{Gal}(L/F)|$ . Therefore,  $[L^{\text{Gal}(L/F)} : F] = 1$  and thus  $F = L^{\text{Gal}(L/F)}$ .  $\square$

**Example 6.25.** We know from [Example 6.14](#) that  $L = \mathbb{Q}(\sqrt[4]{2}, i)$  is Galois over  $\mathbb{Q}$  with Galois group  $D_8$ . More precisely, this identification is given by writing

$$\alpha_1 = \sqrt[4]{2}, \quad \alpha_2 = i\sqrt[4]{2}, \quad \alpha_3 = -\sqrt[4]{2}, \quad \alpha_4 = -i\sqrt[4]{2}$$

and labelling the four corners of a square with  $\alpha_1, \dots, \alpha_4$ , counter-clockwise. Consider

$$\beta := \alpha_1 + \dots + \alpha_4$$

and  $\gamma = \alpha_1 \cdots \alpha_4$ . Then each of  $\beta$  and  $\gamma$  are fixed by every Galois automorphism and hence by [Corollary 6.24](#)  $\beta$  and  $\gamma$  must be rational. In fact, one can easily see that  $\beta = 0$  and  $\gamma = 2$ , but notice that the exact same reasoning would apply in general to the sum of roots and the product of roots in the splitting field of any separable polynomial.

**Corollary 6.26.** *Let  $F \subseteq L$  be a Galois extension. For every  $\alpha \in L$ ,  $m_{\alpha,F}$  is separable and all of its roots belong to  $L$ . Moreover,  $\text{Gal}(L/F)$  acts transitively on the set of roots of  $m_{\alpha,F}$ .*

*Proof.* Let  $\alpha \in L$  and consider the orbit  $\alpha_1 = \alpha, \dots, \alpha_m$  of  $\alpha$  under the action of  $\text{Gal}(L/F)$ . Set

$$f := (x - \alpha_1) \cdots (x - \alpha_m).$$

For any  $\tau \in \text{Gal}(L/F)$ , since  $\tau$  permutes the elements of any orbit then

$$f^\tau = (x - \tau(\alpha_1)) \cdots (x - \tau(\alpha_m)) = f.$$

This proves that  $f$  has all its coefficients in the field  $F^{\text{Gal}(L/F)}$ , which by Corollary 6.24 coincides with the field  $F$ . Thus  $f \in F[x]$ . Moreover, by construction  $f$  is separable. Since  $\alpha$  is a root of  $f$ , the minimal polynomial  $m_{\alpha,F}$  must divide  $f$ , and thus  $m_{\alpha,F}$  is also separable and has all its roots in  $L$ .

Finally, this also shows that all the roots of  $m_{\alpha,F}$  are on the orbit of  $\alpha$  with respect to the action of  $\text{Gal}(L/F)$ , and thus  $\text{Gal}(L/F)$  acts transitively on the set of roots of  $m_{\alpha,F}$ .  $\square$

**Remark 6.27.** Note that any irreducible polynomial over  $F$  with a root in  $L$  is the minimal polynomial of some element in  $L$ . So Corollary 6.26 says in particular that if  $F/L$  is Galois and  $f \in F$  is any irreducible polynomial, then  $\text{Gal}(F/L)$  acts transitively on the set of roots of  $f$ .

**Corollary 6.28.** *A finite field extension  $L/F$  is Galois if and only if  $L$  is the splitting field of some separable polynomial with coefficients in  $F$ .*

*Proof.* We already proved before in Theorem 6.15 that if  $L$  is the splitting field of some separable polynomial  $f \in F[x]$ , then  $F \subseteq L$  is Galois.

For the reverse direction, suppose that  $F \subseteq L$  is a Galois extension. In particular, it is a finite extension, so  $L = F(\beta_1, \dots, \beta_n)$  for some  $\beta_1, \dots, \beta_n \in L$ ; for example, the  $\beta_i$  could be chosen to be an  $F$ -basis of  $L$ .

By Corollary 6.26, each  $m_{\beta_i,F}$  is separable and all of its roots belong to  $L$ . Moreover, if  $\gamma_i$  is a root of  $m_{\beta_i,F}$  and  $i \neq j$ , then we claim that  $\gamma_i$  is not a root of  $m_{\beta_j,F}$ . Indeed, if  $m_{\beta_j,F}(\gamma_i) = 0$ , then  $m_{\alpha_i,F} | m_{\alpha_j,F}$ , but they are both monic irreducible polynomials over  $F$ , so we must have  $m_{\alpha_i,F} = m_{\alpha_j,F}$ .

Let  $m_1, \dots, m_s$  be the distinct polynomials among the  $m_{\beta_i,F}$ , and set

$$g := \prod_{i=1}^s m_i.$$

Since distinct  $m_i$  do not share any common roots and all the  $m_i$  are separable, their product  $g$  is also separable. Moreover, all of the roots of  $g$  belong to  $L$ , and hence the splitting field of  $g$  is contained in  $L$ . Since  $\beta_i$  is a root of  $g$  for all  $i$ ,  $L = F(\beta_1, \dots, \beta_n)$  must be precisely the splitting field of  $g$ .  $\square$

**Theorem 6.29.** *Let  $L$  be the splitting field of a separable polynomial  $f \in F[x]$ . The orbits of the action of  $\text{Aut}(L/F)$  on the set  $S$  of roots of  $f$  are the subsets of  $S$  that are the roots of the same irreducible factor of  $f$ .*

*Proof.* For each  $b \in S$ , the orbit of  $b$  is

$$\text{Orb}_{\text{Aut}(L/F)}(b) = \{\sigma(b) \mid \sigma \in \text{Aut}(L/F)\}.$$

Since  $b$  is a root of  $f$ , there exists an irreducible factor  $p \in F[x]$  of  $f$  such that  $b$  is a root of  $p$ . Since  $p \in F[x]$ , by Lemma 6.10 we know that  $\sigma(b)$  is a root of  $p$  for any  $\sigma \in \text{Aut}(L/F)$ . Thus the orbit of  $b$  is contained in the set of roots of  $p$  in  $L$ .

By Corollary 6.28,  $F \subseteq L$  is a Galois extension. By Corollary 6.26,  $\text{Gal}(L/F)$  acts transitively on the set of roots of  $p$ . Thus every root of  $p$  is in the orbit of  $b$  under the action of  $\text{Aut}(L/F)$ . We conclude that the orbit of  $b$  is precisely the set of roots of  $p$ .  $\square$

**Definition 6.30.** Given a field extension  $F \subseteq L$ , an **intermediate field** is a subfield  $E$  of  $L$  that contains  $F$ , so that  $F \subseteq E \subseteq L$ .

**Corollary 6.31.** *If  $F \subseteq L$  is Galois, then for any intermediate field  $E$  the extension  $E \subseteq L$  is Galois.*

*Proof.* This follows from Corollary 6.28: if  $L$  is the splitting field over  $F$  of a separable polynomial  $f \in F[x]$ , then  $L$  is also the splitting field over  $E$  of the same polynomial  $f$ , but now viewed as a polynomial in  $E$ .  $\square$

**Remark 6.32 (Warning!).** In the setting of Corollary 6.31, note that  $E$  need not be Galois over the original field  $F$ . For example,  $L = \mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$  is Galois over  $F = \mathbb{Q}$  but we saw in Example 6.17 that  $E = \mathbb{Q}(\sqrt[3]{2})$  is not Galois over  $\mathbb{Q}$ . Nevertheless, Corollary 6.31 says that  $L$  is Galois over  $E$ .

**Definition 6.33.** Let  $E_1$  and  $E_2$  be two subfields of  $K$ . The **composite** of  $E_1$  and  $E_2$ , denoted  $E_1E_2$ , is the smallest subfield of  $K$  containing both  $E_1$  and  $E_2$ ; more precisely, it is the intersection of all the subfields of  $K$  that contain both  $E_1$  and  $E_2$ .

**Example 6.34.** Let  $E = \mathbb{Q}(\sqrt{2})$  and  $F = \mathbb{Q}(\sqrt[3]{2})$ . We claim that the composite of  $E$  and  $F$  is the field  $L = \mathbb{Q}(\sqrt[6]{2})$ . On the one hand,  $\sqrt{2} = \sqrt[6]{2}^3 \in L$  and  $\sqrt[3]{2} = \sqrt[6]{2}^2 \in L$ , so  $E, F \subseteq L$ . On the other hand, any subfield of  $L$  containing both  $E$  and  $F$  must contain

$$\frac{\sqrt{2}}{\sqrt[3]{2}} = 2^{\frac{1}{2} - \frac{1}{3}} = 2^{\frac{3-2}{6}} = 2^{\frac{1}{6}} = \sqrt[6]{2}.$$

Thus  $L = EF$ .

**Remark 6.35.** Let  $F \subseteq L$  be a field extension and consider two intermediary fields  $E_1$  and  $E_2$ . If  $E_1 = F(\alpha_1, \dots, \alpha_n)$  and  $E_2 = F(\beta_1, \dots, \beta_m)$ , then  $E_1E_2 = F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ . If  $\alpha_1, \dots, \alpha_n$  is a basis for  $E_1/F$  and  $\beta_1, \dots, \beta_m$  is a basis for  $E_2/F$ , then  $\alpha_i\beta_j$  is a generating set for  $E_1E_2$  over  $F$ , so

$$[E_1E_2 : F] \leq [E_1 : F][E_2 : F].$$

Notice, however, that the inequality might be strict.

We are finally ready for the Fundamental Theorem of Galois Theory:

**Theorem 6.36** (Fundamental Theorem of Galois Theory). *Suppose  $L/F$  is a finite Galois extension. There is a bijection*

$$\begin{aligned} \{ \text{intermediate fields } E, \text{ with } F \subseteq E \subseteq L \} &\xleftrightarrow{\Psi} \{ \text{subgroups } H \text{ of } \text{Gal}(L/F) \} \\ E &\mapsto \Psi(E) = \text{Gal}(L/E) \\ \Psi^{-1}(H) = L^H &\longleftarrow H. \end{aligned}$$

Moreover, this bijective correspondence enjoys the following properties:

(a)  $\Psi$  and  $\Psi^{-1}$  each reverse the order of inclusion.

(b)  $\Psi$  and  $\Psi^{-1}$  convert between degrees of extensions and indices of subgroups:

$$[\text{Gal}(L/F) : H] = [L^H : F] \iff [\text{Gal}(L/F) : \text{Gal}(L/E)] = [E : F].$$

(c) Normal subgroups correspond to intermediate fields that are Galois over  $F$ :

- If  $N \trianglelefteq G$  then  $L^N/F$  is Galois.
- If  $E/F$  is Galois, then  $\text{Gal}(L/E)$  is a normal subgroup of  $\text{Gal}(L/F)$ .

(d) If  $E = L^N$  for a normal subgroup  $N \trianglelefteq \text{Gal}(L/F)$ , then  $\text{Gal}(E/F) \cong \text{Gal}(L/F)/N$ .

(e) If  $H_1, H_2$  are subgroups of  $G$  with fixed subfields  $E_1 = L^{H_1}$  and  $E_2 = L^{H_2}$ , then

- $E_1 \cap E_2 = L^{\langle H_1, H_2 \rangle}$  and  $\text{Gal}(L/E_1 \cap E_2) = \langle H_1, H_2 \rangle$ .
- $E_1 E_2 = L^{H_1 \cap H_2}$  and  $\text{Gal}(L/E_1 E_2) = H_1 \cap H_2$ .

*Proof.* First, we need to check that both functions are well-defined. For each intermediary field  $E$ , we know from Corollary 6.31 that  $L/E$  is also Galois, and hence it makes sense to write  $\text{Gal}(L/E)$ ; moreover, any  $\sigma \in \text{Gal}(L/E)$  is an automorphism of  $L$  that fixes  $E$ , and thus  $F \subseteq E$ , so  $\sigma \in \text{Gal}(L/F)$ . This shows that  $\Psi$  is well-defined. Conversely, given a subgroup  $H$  of  $\text{Gal}(L/F)$ ,  $L^H$  is a subfield of  $L$  by Exercise 18.

Next, we need to check that  $\Psi$  and  $\Psi^{-1}$  are indeed inverse functions. Given a subgroup  $H$  of  $\text{Gal}(L/F)$ , we have  $\text{Gal}(L/L^H) = H$  by Artin's Theorem. Thus

$$\Psi \circ \Psi^{-1}(H) = \Psi(L^H) = \text{Gal}(L/L^H) = H.$$

Conversely, given an intermediate field  $E$ ,  $L/E$  is Galois by Corollary 6.31, and hence  $L^{\text{Gal}(L/E)} = E$  by Corollary 6.24. Thus

$$\Psi^{-1} \circ \Psi(E) = \Psi(\text{Gal}(L/E)) = L^{\text{Gal}(L/E)} = E.$$

This establishes the fact that  $\Psi$  is indeed a bijective correspondence.

Now we check that  $\Psi$  satisfies the given list of properties. For brevity, set  $G := \text{Gal}(L/F)$ .

- (a) The fact that the correspondence is order reversing follows from the definitions. Given intermediate fields  $E_1 \subseteq E_2$ , any automorphism of  $L$  that preserves  $E_2$  must also preserve  $E_1$ , thus  $\text{Gal}(L/E_2) \supseteq \text{Gal}(L/E_1)$ . Conversely, if  $H_1 \leq H_2 \leq \text{Gal}(L/E)$ , then every  $x \in L$  that is fixed by every  $\sigma \in H_2$  must also be fixed in particular by every element of  $H_1$ , so  $L^{H_2} \supseteq L^{H_1}$ .
- (b) By definition of Galois extension,  $[L : F] = |G|$ . By [Artin's Theorem](#), for any subgroup  $H \leq G$  the extension  $L^H \subseteq L$  is also Galois, and thus by definition  $[L : L^H] = |H|$ . Using the [Degree Formula](#), we have

$$[L^H : F] = \frac{[L : F]}{[L : L^H]} = \frac{|G|}{|H|} = [G : H].$$

So if  $H = \Psi(E) = \text{Gal}(E/F)$ , then  $L^H = E$  and the formula above can be rewritten as

$$[\text{Gal}(L/F) : \text{Gal}(L/E)] = [E : F].$$

- (c) Suppose  $E$  is an intermediate field that is Galois over  $F$ . Fix  $\sigma \in G$  and  $\alpha \in E$ . Since  $E/F$  is Galois, by [Corollary 6.26](#) the polynomial  $m_{\alpha,F}$  is separable and all of its roots are in  $E$ . By [Lemma 6.10](#),  $\sigma(\alpha)$  is also a root of  $m_{\alpha,F}$ , and thus  $\sigma(\alpha) \in E$ .

Suppose  $\tau \in \text{Gal}(L/E)$ . For any  $\alpha \in E$  we have  $\sigma(\alpha) \in E$ , so  $\tau(\sigma(\alpha)) = \sigma(\alpha)$ . Thus

$$\sigma^{-1}(\tau(\sigma(\alpha))) = \sigma^{-1}(\sigma(\alpha)) = \alpha.$$

This proves that  $\sigma^{-1}\tau\sigma \in \text{Gal}(L/E)$  and hence that  $\text{Gal}(L/E) \trianglelefteq G$ . We have shown that if  $E$  is Galois over  $F$ , then the corresponding subgroup  $\text{Gal}(L/E)$  of  $G$  is normal.

For the converse, consider a normal subgroup  $N \trianglelefteq G$  and the corresponding intermediate field  $E = L^N$ , so that  $N = \text{Gal}(L/E)$ . We will show that  $E$  is the splitting field over  $F$  of a separable polynomial in  $F[x]$ , and hence is Galois over  $F$  by [Corollary 6.18](#).

Pick any  $\alpha \in E$  and set  $m := m_{\alpha,F}$ . By [Corollary 6.26](#),  $m$  is separable and all of its roots belong to  $L$ . We claim that all the roots must in fact belong to  $E$ . Since  $m$  is irreducible and  $L/F$  is Galois, by [Corollary 6.26](#)  $G$  acts transitively on the set of roots of  $m$ . Thus, given any other root  $\beta \in L$  of  $m$ , there is a  $\sigma \in G$  with  $\sigma(\alpha) = \beta$ . Since  $N$  is normal, for any  $\tau \in N$  we have  $\sigma\tau' = \tau\sigma$  for some  $\tau' \in N$ . But  $\tau' \in N$  fixes  $E$ , so  $\tau'(\alpha) = \alpha$ . Therefore,

$$\beta = \sigma(\alpha) = \sigma\tau'(\alpha) = \tau\sigma(\alpha) = \tau(\beta)$$

which shows that  $\beta$  is also fixed by  $N$ . But then  $\beta \in L^N = E$ . Therefore,  $E$  contains all the roots of  $m_{\alpha,F}$ , and thus  $E$  must contain the splitting field of  $m_{\alpha,F}$ .

We have  $E = F(\alpha_1, \dots, \alpha_l)$  for some  $\alpha_1, \dots, \alpha_l \in E$ . If  $m_1, \dots, m_n$  are the distinct polynomials among  $m_{\alpha_1,F}, \dots, m_{\alpha_l,F}$ , then  $E$  is the splitting field of the separable polynomial  $m_1 \cdots m_n$ . By [Corollary 6.28](#),  $E$  is Galois over  $F$ .

If  $E = L^N$  for a normal subgroup  $N \trianglelefteq \text{Gal}(L/F)$ , then  $\text{Gal}(E/F) \cong \text{Gal}(L/F)/N$ .



- (d) Let  $E = L^N$  for a normal subgroup  $N$  of  $G$ . We want to show that  $\text{Gal}(E/F)$  is isomorphic to  $G/N$ .

For each  $\sigma \in G$ , we claim that  $\sigma(E) \subseteq E$ . By Lemma 6.10, for all  $\alpha \in E$  the element  $\sigma(\alpha)$  is also a root of  $m_{\alpha,F}$ . But since  $E/F$  is Galois, it must contain all of the roots of  $m_{\alpha,F}$ , by Corollary 6.26, so  $\sigma(\alpha) \in E$ . Thus  $\sigma(E) \subseteq E$ , so the restriction of  $\sigma$  to  $E$  determines an injective field homomorphism  $\sigma|_E : E \rightarrow E$ . Since  $\sigma|_F = \text{id}_F$ , this map is also a linear transformation of vector spaces over  $F$ . But  $E$  is a finite vector space over  $F$ , and any injective linear transformation  $E \rightarrow E$  must be bijective. We conclude that  $\sigma|_E$  is an automorphism of  $E$ . We thus have a well-defined function

$$\begin{aligned} \phi : G &\rightarrow \text{Gal}(E/F) \\ \sigma &\mapsto \phi(\sigma) = \sigma|_E. \end{aligned}$$

Moreover, this map is a group homomorphism by construction. The kernel is the subgroup of  $G$  of automorphisms that restrict to the identity on  $E$ , which is precisely  $N$ . Hence we have an induced injective group homomorphism

$$\bar{\phi} : G/N \rightarrow \text{Gal}(E/F).$$

But  $|N| = |\text{Gal}(E/F)| < \infty$ , so this map  $\bar{\phi}$  must be an isomorphism.

- (e) Let  $H_1$  and  $H_2$  be subgroups of  $G$  with fixed fields  $E_1 = L^{H_1}$  and  $E_2 = L^{H_2}$ .

First, we will show that  $E_1 \cap E_2 = L^{\langle H_1, H_2 \rangle}$ . Given any  $\alpha \in E_1 \cap E_2$ ,  $\sigma(\alpha) = \alpha$  for all  $\sigma \in H_1$  and all  $\sigma \in H_2$ , since  $\alpha \in E_1$  and  $\alpha \in E_2$ , so  $\alpha \in L^{\langle H_1, H_2 \rangle}$ , and  $E_1 \cap E_2 \subseteq L^{\langle H_1, H_2 \rangle}$ . Conversely, if  $\alpha \in L^{\langle H_1, H_2 \rangle}$ , then  $\sigma(\alpha) = \alpha$  for all  $\sigma \in \langle H_1, H_2 \rangle$ , so in particular  $\sigma(\alpha) = \alpha$  for all  $\sigma \in H_i$  and thus  $\alpha \in L^{H_i} = E_i$ . We conclude that  $E_1 \cap E_2 = L^{\langle H_1, H_2 \rangle}$ .

Now let us show that  $E_1 E_2 = L^{H_1 \cap H_2}$ . Since  $L/F$  is a finite extension, then by the Degree Formula both of the extensions  $E_1/F$  and  $E_2/F$  are finite. Let  $E_1 = F(\alpha_1, \dots, \alpha_n)$  and  $E_2 = F(\beta_1, \dots, \beta_m)$ , so that  $E_1 E_2 = F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ . For any  $\sigma \in H_1 \cap H_2$  we have  $\sigma(\alpha_i) = \alpha_i$  and  $\sigma(\beta_j) = \beta_j$  for each  $i$ . Since  $\sigma|_E$  is completely determined by its values on the  $\alpha_i$  and  $\beta_j$ , by Exercise 17, but since  $\sigma$  restricted to  $E_1 E_2$  agrees with the identity map on the generators,  $\sigma|_{E_1 E_2} = \text{id}_{E_1 E_2}$ . We conclude that  $E_1 E_2 \subseteq L^{H_1 \cap H_2}$ .

Moreover, given any  $\sigma \in \text{Gal}(L/E_1 E_2)$ , its restriction to  $E_1 E_2$  is by definition the identity. Thus its restriction to the subfields  $E_1$  and  $E_2$  of  $E_1 E_2$  must also be the identity, and therefore we have  $\sigma \in \text{Gal}(L/E_1) = H_1$  and  $\sigma \in \text{Gal}(L/E_2) = H_2$ . We conclude that  $\sigma \in H_1 \cap H_2$ . Thus  $\text{Gal}(L/E_1 E_2) \leq H_1 \cap H_2$ . Since  $\Psi$  is order reversing, we conclude that  $E_1 E_2 \supseteq L^{H_1 \cap H_2}$ , giving us the desired equality.  $\square$

This bijection  $\Psi$  in Theorem 6.36 is sometimes called the **Galois correspondence**.

**Corollary 6.37.** *The Galois correspondence induces a lattice isomorphism between the lattice of intermediate fields of a Galois extension  $L/F$  and the dual of the lattice of subgroups of  $\text{Gal}(L/F)$ .*

This is just a fancy way to rephrase the fact that intermediate fields correspond to subgroups in an order-reversing bijection.



**Example 6.38.** Let  $L$  be the splitting field of  $x^4 - 2$  over  $\mathbb{Q}$ . Let us use the [Fundamental Theorem of Galois Theory](#) to list all intermediate fields for  $L/\mathbb{Q}$  and to determine which are Galois over  $\mathbb{Q}$ . By Example 6.14, we know  $G := \text{Gal}(L/\mathbb{Q})$  corresponds to the 8 element subgroup of  $S_4$  generated by  $\sigma = (2\ 4)$  and  $\tau = (1\ 2\ 3\ 4)$ , where we number the roots of  $x^4 - 2$  as

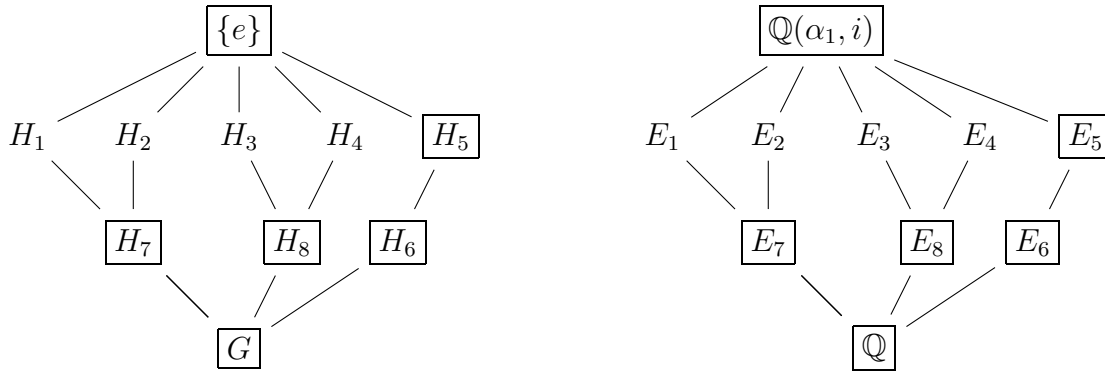
$$\alpha_1 = \sqrt[4]{2}, \quad \alpha_2 = i\alpha_1, \quad \alpha_3 = -\alpha_1, \quad \alpha_4 = -i\alpha_1.$$

We saw in Example 6.14 that this group is isomorphic to  $D_8$ , and we can make this isomorphism explicit by labeling the four corners of a square 1, 2, 3, 4 counterclockwise, so that  $\tau$  is rotation by 90 degrees and  $\sigma$  is reflection about the line joining vertices 1 and 3.

The subgroup lattice and intermediate field lattice are represented below, with normal subgroups and Galois extensions highlighted by boxes. The subgroups are

$$\begin{array}{ll} \{e\} & G = \langle (2\ 4), (1\ 2\ 3\ 4) \rangle \\ H_1 = \langle (2\ 4) \rangle & H_5 = \langle (1\ 3)(2\ 4) \rangle \\ H_2 = \langle (1\ 3) \rangle & H_6 = \langle (1\ 2\ 3\ 4) \rangle \\ H_3 = \langle (1\ 2)(3\ 4) \rangle & H_7 = \langle (1\ 3), (2\ 4) \rangle \\ H_4 = \langle (1\ 4)(2\ 3) \rangle & H_8 = \langle (1\ 2)(3\ 4), (1\ 4)(2\ 3) \rangle \end{array}$$

and the lattices are of subgroups of  $G$  and intermediate fields of  $\mathbb{Q} \subseteq \mathbb{Q}(\alpha_1, i)$  are



The intermediate fields are the fixed subfields of  $L$  associated to each of these subgroups. The group  $G$  corresponds to  $\mathbb{Q}$  and  $e$  corresponds to  $L = \mathbb{Q}(\alpha_1, i)$ . Set  $E_i = L^{H_i}$ .

The field  $E_1$  has degree  $4 = [G : H_1]$  over  $\mathbb{Q}$ . Since  $\alpha_1$  and  $\alpha_3$  belong to  $E_1$  and  $[\mathbb{Q}(\alpha_1) : \mathbb{Q}] = 4$ , we must have  $E_1 = \mathbb{Q}(\alpha_1)$ . Likewise,  $E_2 = \mathbb{Q}(\alpha_2)$ .

The field  $E_3$  also has degree 4 over  $\mathbb{Q}$ . Let

$$\beta := \alpha_1 + \alpha_2 = (1 + i)\sqrt[4]{2} \in E_3.$$

If  $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$ , then  $\beta$  would be fixed by a subgroup of index 2 that contains  $(1\ 2)(3\ 4)$ , and the only possibility is  $H_8$ . But  $(1\ 4)(2\ 3)$  sends  $\beta$  to  $\alpha_4 + \alpha_3 = -\beta \neq \beta$ . So we must have  $[\mathbb{Q}(\beta) : \mathbb{Q}] = 4$  and hence  $E_3 = \mathbb{Q}(\beta)$ . We claim that  $E_4 = \mathbb{Q}((1 - i)\alpha_1)$ ,  $E_5 = \mathbb{Q}(\sqrt{2}, i)$ , and  $E_7 = \mathbb{Q}(\sqrt{2})$ , and leave it as an exercise.

The field  $E_6$  has degree  $[G : H_6] = 2$  over  $\mathbb{Q}$ , and so we merely need to find a single nonrational element of  $L$  fixed by  $\tau$ . Since  $\tau(i) = i$ , we get  $E_6 = \mathbb{Q}(i)$ . Similarly, the field

$E_8$  also has degree 2 over  $\mathbb{Q}$  and so we just need to find a single nonrational element fixed by the two generators of  $H_8$ . Note that

$$\alpha_1\alpha_2 = \alpha_3\alpha_4 = i\sqrt{2},$$

and so  $i\sqrt{2}$  is fixed by both  $(1\ 2)(3\ 4)$  and  $(1\ 4)(2\ 3)$ . Thus  $E_8 = \mathbb{Q}(i\sqrt{2})$ .

Finally, we note that  $G, \{e\}, H_5, H_6, H_7, H_8$  are normal subgroups of  $D_8$ , since  $H_5$  is the center of  $D_8$  and each of  $H_6, H_7$ , and  $H_8$  has index two. Some messy checking reveals these to be the only normal subgroups of  $G$ . It follows from the [Fundamental Theorem](#) that  $\mathbb{Q}, L, E_5, E_6, E_7, E_8$  are the only intermediate fields that are Galois over  $\mathbb{Q}$ .

As an example, to see directly that  $E_3$  is not Galois over  $\mathbb{Q}$ , note that  $(1+i)\sqrt[4]{2}$  is a root of  $x^4 + 4$ , which is irreducible; but  $(1-i)\sqrt[4]{2}$  is also a root of this polynomial and it is not in  $E_3$ .

**Remark 6.39.** Let  $F \subseteq L$  be a Galois extension and consider an intermediate field  $E$  such that  $E/F$  is Galois, with corresponding normal subgroup  $N := \text{Gal}(E/F) \trianglelefteq \text{Gal}(L/F)$ . Part (d) of the [Fundamental Theorem](#) says there is an isomorphism  $\text{Gal}(E/F) \cong \text{Gal}(L/F)/N$ . In fact, the proof shows that the map

$$\begin{aligned} \phi : G &\rightarrow \text{Gal}(E/F) \\ \sigma &\mapsto \phi(\sigma) = \sigma|_E. \end{aligned}$$

is surjective. This says that every  $\tau \in \text{Gal}(E/F)$  can be lifted to some  $\sigma \in \text{Gal}(L/F)$ , so that  $\tau = \sigma|_E$ .

Notice that while the proof shows that for every  $\tau \in \text{Gal}(E/F)$  there exists  $\sigma \in \text{Gal}(L/F)$  such that  $\tau = \sigma|_E$ , the proof is very much nonconstructive. In specific examples, one can sometimes determine  $\sigma$  explicitly by using some of the other tricks we have discussed.

## 6.4 Solvable polynomials and solvable groups

Galois' theory has many fun applications. There are some famous questions in geometry whose impossibility is shown via Galois theory methods:

- Trisecting an angle: is it possible to trisect any given angle using only a compass and a straightedge?
- Doubling the cube: using only a compass and straightedge, and given a cube, is it possible to construct a cube whose volume is exactly twice the volume of the original cube?
- Squaring the circle: Using only a compass and straightedge, is it possible to construct a square with the same area of a given circle?

All these challenges turn out to be impossible, which one can show using Galois theory. Unfortunately, we will not be proving these here. Instead, we will focus on another famous classical question which is behind the origins of Galois theory:

**Question 6.40.** Does there exist a formula for the roots of a polynomial of degree  $n$  with rational coefficients using only addition, subtraction, multiplication, division, and taking radicals?

The formulas for the roots of polynomials of degree 2, 3, and 4 have been known for hundreds of years, and they involve only sums, products, quotients, and square roots, cube roots, and fourth roots. It turns out that there are polynomials for which no such formula exists; in fact, there is no general formula for polynomials of degree 5.

The first ingredient we need is to better understand the process of taking roots. With that in mind, we will now discuss the Galois groups of the splitting fields of polynomials of the form  $x^n - a$ . These calculations will be used to prove what Galois himself sort of proved: if the roots of a polynomial can be expressed using iterated radicals, then the Galois group of its splitting field must be a solvable group.

**Definition 6.41.** A **primitive  $n$ th root of unity** over an arbitrary field  $F$  is an element  $\zeta$  in the splitting field  $K$  of  $x^n - 1$  over  $F$  (or in the algebraic closure  $\overline{F}$ ) such that  $\zeta$  generates the (multiplicative) subgroup of  $K$ :

$$\mu_n(K) := \{\alpha \in K \mid \alpha^n = 1\} \leq (K^\times, \cdot).$$

**Exercise 19.** Show that for every field  $K$ , every finite subgroup of  $K^\times$  is cyclic.

**Remark 6.42.** In particular, the subgroup  $\mu_n(K)$  of  $K^\times$  of roots of  $x^n - 1$  is a cyclic group. As a consequence, a primitive  $n$ th root of unity always exists.

**Remark 6.43.** Let  $F$  be any field and let  $K$  be the splitting field of  $x^n - 1$  over  $F$ . Note that if  $\zeta$  is a generator of  $\mu_n(K)$ , then in particular  $F(\zeta) \subseteq K$  contains all the roots of  $x^n - 1$ , and thus  $F(\zeta)$  must be the splitting field of  $x^n - 1$ .

Note also that  $\mu_n(K)$  is a cyclic group of finite order, say  $\mu_n(K) \cong \mathbb{Z}/d$ . Moreover,  $\zeta$  is a generator of  $\mu_n(K)$ . By a result from Math 817, the complete list of primitive  $n$ th roots of unity is

$$\zeta^i \text{ such that } \gcd(i, d) = 1.$$

**Example 6.44.** When  $F = \mathbb{Q}$ , the element  $e^{2\pi i/n} \in \overline{\mathbb{Q}}$  is a primitive  $n$ th root of unity. Moreover, the primitive  $n$ th roots of unity over  $\mathbb{Q}$  are precisely the elements of the form  $e^{2\pi i j/n}$  with for any  $j$  with  $\gcd(n, j) = 1$ .

**Remark 6.45.** Note that if  $\text{char}(F) \nmid n$ , the polynomial  $x^n - 1 \in F[x]$  is separable by Lemma 5.90, since its derivative is  $nx^{n-1} \neq 0$  and hence  $\gcd(nx^{n-1}, x^n - 1) = 1$ . In this case,  $|\mu_n(K)| = n$  and so

$$\mu_n(K) = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}.$$

However, if  $\text{char}(F) \mid n$  then  $\mu_n(K)$  can have fewer than  $n$  elements. For example, if  $F$  is any field of characteristic 2, the polynomial  $x^2 - 1 = (x - 1)^2$  has a unique root over  $F$ : the unique second root of unity is 1.

**Theorem 6.46.** Let  $F$  be a field and  $n$  a positive integer with  $\text{char}(F) \nmid n$ , and let  $\zeta \in \overline{F}$  be a primitive  $n$ th root of unity. The extension  $F \subseteq F(\zeta)$  is Galois and  $\text{Gal}(F(\zeta)/F)$  is isomorphic to a subgroup of  $(\mathbb{Z}/n)^\times$ . In particular,  $\text{Gal}(F(\zeta)/F)$  is an abelian group.

*Proof.* By Remark 6.45,  $F(\zeta)$  is the splitting field of  $x^n - 1$  over  $F$ . As observed above, the polynomial  $x^n - 1$  is separable, and thus  $F(\zeta)/F$  is Galois by Corollary 6.28.

Given any  $\sigma \in \text{Gal}(F(\zeta)/F)$ , by Lemma 6.10  $\sigma(\zeta)$  is also an  $n$ th root of unity. One can check this explicitly by noting that

$$\sigma(\zeta)^n = \sigma(\zeta^n) = \sigma(1) = 1.$$

Moreover, we claim that  $\sigma(\zeta)$  must also be a primitive  $n$ th root of unity. Notice that since  $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$  are distinct, then the elements  $1, \sigma(\zeta), \sigma(\zeta)^2, \dots, \sigma(\zeta)^{n-1}$  must also be distinct, since  $\sigma(\zeta^l) = \sigma(\zeta)^l$  for all  $l$ .

This proves that  $\sigma(\zeta) = \zeta^j$  for an integer  $j$  (unique modulo  $n$ ) such that  $\gcd(j, n) = 1$ . Thus we have a well-defined function

$$\begin{aligned} \Phi: \text{Gal}(F(\zeta)/F) &\rightarrow (\mathbb{Z}/n)^\times \\ \sigma &\mapsto \Phi(\sigma) = j \quad \text{where } \sigma(\zeta) = \zeta^j. \end{aligned}$$

Given any other  $\tau \in \text{Gal}(F(\zeta)/F)$ , let  $\tau(\zeta) = \zeta^i$ . Then

$$(\tau \circ \sigma)(\zeta) = \tau(\zeta^j) = \tau(\zeta)^j = \zeta^{ij}.$$

This proves that  $\Phi(\tau \circ \sigma) = \Phi(\tau) \cdot \Phi(\sigma)$ , so  $\Phi$  is a group homomorphism.

If  $\Phi(\sigma) = 1$ , then  $\sigma$  fixes  $\zeta$  and hence must be the trivial automorphism. This shows that  $\Phi$  is injective.  $\square$

**Corollary 6.47.** *For any  $n > 1$ ,  $\text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q}) \cong (\mathbb{Z}/n)^\times$ .*

*Sketch of proof.* Consider the injective group homomorphism

$$\begin{aligned} \Phi: \text{Gal}(F(\zeta)/F) &\rightarrow (\mathbb{Z}/n)^\times \\ \sigma &\mapsto \Phi(\sigma) = j \quad \text{where } j \text{ satisfies } \sigma(\zeta) = \zeta^j. \end{aligned}$$

we constructed in the proof of Theorem 6.46. We claim that  $\Phi$  is an isomorphism.

To show that homomorphism must be surjective, one shows that the degree of the minimal polynomial of  $e^{2\pi i/n}$  is precisely the number of elements of  $(\mathbb{Z}/n)^\times$ , and thus  $|\text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})| = |(\mathbb{Z}/n)^\times|$ . We skip this detail.  $\square$

We now cover the Galois groups of polynomials of the form  $x^n - a$  in the case where the base field contains all the  $n$ th roots of unity.

**Theorem 6.48.** *Let  $F$  be a field,  $a \in F$ , and consider a positive integer  $n$  such that  $F$  contains a primitive  $n$ th root of unity and  $\text{char}(F) \nmid n$ . Let  $L$  be the splitting field of  $x^n - a$  over  $F$ . Then  $L/F$  is Galois and  $\text{Gal}(L/F)$  is isomorphic to a subgroup of  $\mathbb{Z}/n$ , and hence it is cyclic.*

*Proof.* If  $a = 0$ , then  $L = F$  and  $\text{Gal}(L/F)$  is the trivial group, so the result is trivially true. If  $a \neq 0$ , then

$$\gcd(x^n - a, nx^{n-1}) = 1,$$

and hence  $x^n - a$  is separable by Lemma 5.90. By Corollary 6.28, we conclude that  $F \subseteq L$  is Galois.

Let  $\alpha$  be a root of  $x^n - a$  in  $L$ , and let  $\zeta \in F$  be a primitive  $n$ th root of unity. Then the roots of  $x^n - a$  are  $\zeta^j \alpha$  for  $j = 0, \dots, n-1$ , and  $L = F(\alpha)$ . Also, the elements  $\zeta^j \alpha$  for  $j = 0, \dots, n-1$  are all distinct, and thus for each  $\sigma \in \text{Gal}(L/F)$  we have  $\sigma(\alpha) = \zeta^j \alpha$  with  $j$  well-defined modulo  $n$ . Define

$$\begin{aligned} \Phi: \text{Gal}(L/F) &\rightarrow \mathbb{Z}/n \\ \sigma &\mapsto \Phi(\sigma) = j \quad \text{where } \sigma(\alpha) = \zeta^j \alpha. \end{aligned}$$

Notice that such integer  $j$  is unique modulo  $n$ . Let  $\tau(\alpha) = \zeta^i \alpha$ . Note that  $\zeta \in F$  and hence that it is fixed by  $\tau$ . Then

$$(\tau \circ \sigma)(\alpha) = \tau(\zeta^j \alpha) = \zeta^j \zeta^i \alpha = \zeta^{j+i} \alpha,$$

so  $\Psi$  is a group homomorphism. It is injective since  $\Psi(\sigma) = 0$  implies that  $\sigma(\alpha) = \zeta^0 \alpha = \alpha$ , so  $\sigma$  fixes  $\alpha$  and hence all of  $L$ .  $\square$

We are interested in understanding when we can write a formula for all the roots of a given polynomial using only the usual elementary operations and radicals. We formalize this idea as follows:

**Definition 6.49.** For a field  $F$  of characteristic 0, we say  $f \in F[x]$  is **solvable by radicals** over  $F$  if there exists a finite chain of field extensions

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_m$$

such that  $f$  splits completely in  $F_m$ , and for each  $i$  the field extension  $F_i \subseteq F_{i+1}$  is the splitting field of a polynomial of the form  $x^{n_i} - a_i$  for some positive integer  $n_i$  and some element  $a_i \in F_i$ .

Note that  $a_i = 1$  is allowed here, so that some of the steps may involve adjoining  $n$ th roots of unity. Roughly speaking,  $f$  is solvable by radicals if each of its roots can be written by an expression involving sums, products, and iterated  $n$ th roots of elements of  $F$ . Granted, such an expression may perhaps be extremely complicated.

**Example 6.50.** The polynomial  $f = ax^2 + bx + c \in \mathbb{Q}[x]$  is solvable by radicals over  $\mathbb{Q}$  since its roots are

$$\frac{-b \pm \sqrt{b^2 - 4ca}}{2a}.$$

Explicitly, take  $F_1$  to be the splitting field of  $x^2 - (b^2 - 4ac)$ ; indeed,  $f$  splits completely in  $F_1$ .

**Example 6.51.** The polynomial  $f = x^4 + bx^2 + c \in \mathbb{Q}[x]$  is solvable by radicals over  $\mathbb{Q}$  since its roots are

$$\pm \sqrt{\frac{-b \pm \sqrt{b^2 - 4c}}{2}}.$$

Explicitly, we could set  $F_1$  to be the splitting field of  $x^2 - (b^2 - 4c)$  over  $\mathbb{Q}$ ,  $F_2$  to be the splitting field of  $x^2 - \left(\frac{-b + \sqrt{b^2 - 4c}}{2}\right)$  over  $F_1$ , and  $F_3$  to be the splitting field of  $x^2 - \left(\frac{-b - \sqrt{b^2 - 4c}}{2}\right)$  over  $F_2$ . It's not clear if  $F_3 = F_2$  or  $F_3 \subsetneq F_2$ , but either way the tower given shows that  $f$  is solvable by radicals.

The notion of solvable polynomial has a group theoretic counterpart.

**Definition 6.52.** A group  $G$  is **solvable** if there is a sequence of subgroups

$$\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_k = G$$

such that for all  $0 \leq i \leq k-1$  we have  $N_i \trianglelefteq N_{i+1} \leq G$  and the quotient groups  $N_{i+1}/N_i$  are abelian.

**Example 6.53.** One can show that every group  $G$  with  $|G| < 60$  is solvable.

**Remark 6.54.** Suppose that  $G$  is a finite simple group. Recall that this means that  $G$  has no nontrivial normal subgroups. Then the only sequence of normal subgroups of  $G$  is

$$\{e\} \trianglelefteq G,$$

and thus  $G$  is solvable if and only if  $G$  is abelian. But the only simple abelian groups are  $\mathbb{Z}/(p)$  for  $p$  a prime, so  $G \cong \mathbb{Z}/(p)$  for some prime  $p$ .

**Example 6.55.** The groups  $A_5$  and  $S_5$  are not solvable. To see why, recall that  $A_5$  is a finite simple group and it is not abelian, and thus by Remark 6.54 we conclude that  $A_5$  is not solvable. As for  $S_5$ , its only nontrivial normal subgroup is  $A_5$ , but both  $A_5$  and  $S_5$  are not abelian, so  $S_5$  is not solvable.

**Example 6.56.** We claim that the group  $S_4$  is solvable. To see that, consider the subgroup

$$V = \{e, (12)(34), (14)(23), (13)(24)\}$$

and the following sequence of subgroups:

$$\{e\} \trianglelefteq V \trianglelefteq A_4 \trianglelefteq S_4.$$

Since  $V$  has order 4 and any group of order 4 is abelian, then  $V$  is abelian. In fact, one can show that  $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ . Moreover, the quotients  $S_4/A_4$  and  $A_4/V$  have order 2, and thus are abelian.

It turns out that there is a close relationship between solvable groups and solvable polynomials. In what follows,  $\text{char}(F) = 0$  is not a necessary assumption, but we included it to make both the statement and the proof simpler.

**Theorem 6.57.** Assume  $F$  is a field of characteristic 0. If  $f \in F[x]$  is solvable by radicals, then the Galois group of the splitting field of  $f \in F[x]$  is a solvable group.

*Sketch of proof.* For a suitable  $n$ , we may assume there is a tower

$$F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m$$

such that

- The splitting field  $L$  of  $f$  satisfies  $L \subseteq F_m$ .
- The splitting field of  $x^n - 1$  over  $F$  is  $F_1$ .
- For each  $i \geq 1$ ,  $F_{i+1}$  is the splitting field of  $x^{d_i} - a_i \in F_i[x]$ , where  $a_i \in F_i$  and  $d_i \mid n$ .

Note that  $d_i \mid n$  means that  $F_i$  contains all the  $d_i$ th roots of 1, and thus Theorem 6.48 applies to the extension  $F_{i+1}/F_i$  for each  $i \geq 1$ .

It turns out that there is an extension  $E$  such that

$$F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m \subseteq E$$

and  $E/F$  is Galois with a chain of normal subgroup inclusions

$$\text{Gal}(E/F_m) \trianglelefteq \text{Gal}(E/F_{m-1}) \trianglelefteq \text{Gal}(E/F_{m-2}) \trianglelefteq \cdots \trianglelefteq \text{Gal}(E/F_1) \trianglelefteq \text{Gal}(E/F).$$

The key point is that by Theorem 6.46 and Theorem 6.48, the groups

$$\text{Gal}(F_{i+1}/F_i) \cong \text{Gal}(E/F_i) / \text{Gal}(E/F_{i+1}) \text{ for } i = 0, \dots, m-1$$

are all abelian. These properties imply that  $\text{Gal}(E/F)$  is a solvable group, and in turn this implies that  $\text{Gal}(L/F)$  is solvable.  $\square$

In characteristic 0, the converse of Theorem 6.57 is also true: if the Galois group of  $f$  is solvable, then  $f$  is solvable by radicals.

**Theorem 6.58.** *Every polynomial  $f \in \mathbb{Q}[x]$  of degree at most 4 is solvable by radicals.*

The main point is that if  $L$  is the splitting field of a polynomial of degree at most 4, then  $\text{Gal}(L/\mathbb{Q}) \trianglelefteq S_4$ , and that every subgroup of  $S_4$  is solvable. Indeed, formulas for the roots of polynomials of degree 2, 3, and 4 have been known for hundreds of years, and they involve only sums, products, quotients, and square roots, cube roots, and fourth roots.

We can now prove a theorem that has fascinated mathematicians and nonmathematicians alike for many centuries: the fact that the general quintic cannot be solved. More precisely, there is no formula involving only radicals, sums, and products for the zeroes of a general polynomial of degree 5 with rational coefficients. The key point turns out to lie in group theory:  $S_5$  is not a solvable group, and there are polynomials  $f \in \mathbb{Q}[x]$  of degree 5 such that the Galois group of the splitting field of  $f$  over  $\mathbb{Q}$  is  $S_5$ .

**Theorem 6.59.** *If  $f \in \mathbb{Q}[x]$  is any degree 5 irreducible polynomial with exactly 3 real roots, then  $f$  is not solvable by radicals.*

*Proof.* Let  $L$  be the splitting field of  $f$ . By Theorem 6.57, it suffices to prove  $\text{Gal}(L/\mathbb{Q})$  is not a solvable group. In fact, we show it is isomorphic to  $S_5$ .

Let  $\alpha_1, \alpha_2, \alpha_3$  be the three real roots of  $f$  and let  $\alpha_4, \alpha_5$  the two complex ones. Note that  $\overline{\alpha_4} = \alpha_5$ . Using this ordering of the roots, we identify  $\text{Gal}(L/\mathbb{Q})$  as a subgroup of  $S_5$ , and will identify an element of  $\text{Gal}(L/\mathbb{Q})$  sending  $\alpha_i$  to  $\alpha_j$  with a permutation sending  $i$  to  $j$ .

Let  $\sigma$  denote complex conjugation. Note that  $\sigma \in \text{Gal}(L/\mathbb{Q})$ , since  $\sigma$  preserves  $\mathbb{Q}$  and  $\alpha_1, \alpha_2$ , and  $\alpha_3$ , and it switches  $\alpha_2$  and  $\alpha_3$ . This element of  $\text{Gal}(L/\mathbb{Q})$  corresponds to the transposition  $(4, 5) \in S_5$ .

Since  $[\mathbb{Q}(\alpha_1) : \mathbb{Q}] = 5$ , by the Degree Formula 5  $|[L : \mathbb{Q}]$ . But the extension is Galois, so  $5 \mid |\text{Gal}(L/\mathbb{Q})|$ . Since 5 is prime, there is an element  $\tau \in \text{Gal}(L/K)$  of order 5 by Cauchy's Theorem. Such an element is necessarily a 5-cycle. The result follows since any 5-cycle and any transposition necessarily generate all of  $S_5$  (exercise).

Finally, we claim that  $S_5$  is not solvable. First,  $S_5$  is not abelian, so the series

$$H_0 = \{e\} \leq S_5$$

does not work since the unique quotient is not abelian. Moreover, as proven in Math 817, the only nontrivial normal subgroup of  $S_5$  is  $A_5$  and  $A_5$  has no nontrivial normal subgroups. Hence the only possible composition series for  $S_5$  would be

$$H_0 = \{e\} \leq A_5 \leq S_5,$$

but in this series the quotient  $A_5/\{e\} \cong A_5$  is not abelian. □

**Example 6.60.** The polynomial  $f(x) = x^5 - 4x + 2$  is not solvable by radicals over  $\mathbb{Q}$ . One can show it is irreducible in  $\mathbb{Q}[x]$  by the usual combination of Eisenstein's Criterion and Gauss' Lemma. Moreover, we claim that this polynomial has exactly 3 distinct real roots. Unfortunately, we cannot show this directly by *presenting* such roots, exactly because  $f$  is *not* solvable by radicals. One could check this informally by graphing  $f$  and checking that indeed it crosses the  $x$ -axis three times, and noting that all irreducible polynomials over a field of characteristic zero (thus perfect) are separable. If we wanted to check this more carefully, we could use some elementary calculus:  $f'(x) = 5x^4 - 4$  has precisely two roots and changes signs at these roots. It follows that  $f$  must have exactly 3 real roots. By Theorem 6.59,  $f$  is not solvable by radicals.

However, there *are* irreducible polynomials of degree 5 that are solvable by radicals.

**Example 6.61.** The polynomial  $x^5 - 1 \in \mathbb{Q}[x]$  is solvable by radicals; indeed, its roots are all 5th roots of unity over  $\mathbb{Q}$ .

We can check this using Galois Theory. By Corollary 6.47, the splitting field  $L$  of  $x^5 - 1$  over  $\mathbb{Q}$  satisfies  $\text{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/5)^\times$ . In particular,  $\text{Gal}(L/\mathbb{Q})$  is abelian and thus solvable.

More details on the other applications of Galois Theory we mentioned in the beginning of this section can often be found in any standard book on the subject.



## 6.5 The primitive element theorem

Let  $F \subseteq L$  be a field extension. Recall that an element  $\theta$  so that  $L = F(\theta)$  is called a **primitive element** for the a **simple extension**  $F \subseteq L$ .

**Lemma 6.62.** *If  $L/F$  is a finite extension with  $F$  infinite, then  $L = F(\theta)$  if and only if there are only finitely many subfields of  $L$  containing  $F$ .*

*Proof.* First we show if there are only finitely many subfields of  $L$  containing  $F$  then  $L$  is simple. It's sufficient to show  $F(\alpha, \beta)$  is simple for any  $\alpha, \beta \in L$  and then the statement about  $L$  will follow by induction on the dimension of  $L$ . Consider the intermediate fields  $E_c = F(\alpha + c\beta)$  for  $c \in F$ . Since there are only finitely many intermediate subfields, but infinitely many  $c \in F$  we have

$$F(\alpha + c\beta) = F(\alpha + c'\beta) =: E \text{ for some } c \neq c'.$$

Then

$$\alpha + c\beta - (\alpha + c'\beta) = (c - c')\beta \in E,$$

so  $\beta \in E$  and similarly  $\alpha \in E$ , thus  $E = F(\alpha + c\beta) = F(\alpha, \beta)$ .

For the converse, suppose  $L = F(\theta)$  is simple and let  $f = m_{\theta, F}$ . Let  $E$  be an intermediate field and  $g(x) = m_{\theta, E}$ . Then  $g \mid f$  in  $E[x]$ , so  $g$  is an irreducible factor of  $f$ . Consider  $E'$  to be the field obtained by adjoining the coefficients of  $g$  to  $F$ . Since  $g = m_{\theta, E} = m_{\theta, E'}$ , we have

$$[F(\theta) : E] = [F(\theta) : E'] = \deg(g).$$

Since  $E' \subseteq E$ , the [Degree Formula](#) gives  $E = E'$ . So all intermediate fields are generated by the coefficients of the irreducible factors of  $f$ .  $\square$

**Definition 6.63.** Let  $L/F$  be a finite separable extension. The **Galois closure** of  $L$  over  $F$  is the smallest (with respect to containment) Galois extension of  $F$  containing  $L$ , meaning

$$L^{\text{Gal}} = \bigcap_{\substack{F \subseteq K \text{ Galois} \\ F \subseteq L \subseteq K}} K.$$

**Remark 6.64.** We should check that every finite field extension has a Galois closure. For example, one can pick a basis  $\{\beta_1, \dots, \beta_n\}$  for  $L$  over  $F$  and take  $K$  to be the splitting field of the product of the minimal polynomials of  $\beta_1, \dots, \beta_n$ . Then  $L \subseteq K$  will be the splitting field of a separable polynomial, hence Galois. This shows that the set indexing the intersection above is not empty, so the Galois closure exists as defined.

**Theorem 6.65** (Primitive Element Theorem). *If  $F \subseteq L$  is a finite and separable extension, then  $L$  is simple over  $F$ , meaning  $L = F(\theta)$  for some  $\theta \in L$ .*

*Proof.* If  $F$  is a finite field then so is  $L$ . Since  $L$  is finite, then  $(L^\times, \cdot)$  is a cyclic group by a homework problem. Let  $\theta$  be a generator for this multiplicative group. Then  $L = F(\theta)$ .

Now suppose  $F$  is infinite. Let  $K$  be the Galois closure of  $L$  over  $F$ . Then  $G = \text{Gal}(K/F)$  is finite and has finitely many subgroups, thus by the [Galois correspondence](#) there are finitely many subfields of  $K$ , hence also of  $L$ , containing  $F$ . By Lemma 6.62 it follows that  $F \subseteq L$  is simple.  $\square$

**Corollary 6.66.** *If  $F \subseteq L$  is a finite extension with  $F$  perfect, then  $L = F(\theta)$  for some  $\theta \in L$ .*

*Proof.* By definition, if  $F$  is a perfect field then every finite extension of  $F$  is separable. By the [Primitive Element Theorem](#),  $F$  is simple over  $F$ . Also, recall that every field of characteristic zero is perfect, by [Corollary 5.94](#).  $\square$

In particular, if  $L/F$  is a finite extension of fields of characteristic zero, then  $L$  is simple over  $F$ . The most important special case of this, or at least the one we keep encountering, is that every finite extension of  $\mathbb{Q}$  is simple over  $\mathbb{Q}$ . On the other hand, our proof of the [Primitive Element Theorem](#) is not constructive, and it doesn't tell us how to find a primitive element for a given finite extension. We saw in [Example 5.21](#) that  $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ ; nevertheless, it is *not* always true that  $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha + \beta)$ , as we saw in [Example 5.77](#).

**Theorem 6.67.** *Every finite Galois extension is simple.*

*Proof.* Let  $F \subseteq L$  be a Galois extension. By [Corollary 6.26](#), for every  $\alpha \in L$  the minimal polynomial  $m_{\alpha, F}$  is separable, so  $F \subseteq L$  is separable. By the [Primitive Element Theorem](#),  $L$  is simple over  $F$ .  $\square$

For extensions of perfect fields of characteristic zero, this is just a very special case of [Corollary 6.66](#). In this case, we can also prove that the extension is simple directly from [Lemma 6.62](#): if  $F \subseteq L$  is Galois, then  $\text{Gal}(L/F)$  is finite, and thus it has finitely many subgroups, which by the [Galois correspondence](#) says that  $F \subseteq L$  has finitely many intermediate fields. By [Lemma 6.62](#), this implies that  $F \subseteq L$  is simple.

To construct an example of a finite field extension that is not simple, we need an infinite field of prime characteristic that is not perfect.

**Example 6.68.** Let  $L = \mathbb{Z}/p(s, t)$  be the fraction field of the polynomial ring in two variables  $\mathbb{Z}/p[s, t]$ , and consider the subfield  $K = \mathbb{Z}/p(s, t)$ . We claim that this is a finite extension that is not simple. First, note that

$$\{s^i t^j \mid 0 \leq i, j \leq p-1\}$$

is a basis for  $L$  over  $K$ , so  $[L : K] = p^2$ . Now let  $\alpha \in L$ , meaning a rational function

$$\alpha = \frac{f(s, t)}{g(s, t)}$$

for some polynomials  $f, g \in \mathbb{Z}/p[s, t]$ . For any  $a \in \mathbb{Z}/p$  we have  $a^p = a$  by Fermat's Little Theorem, so by the [Freshman's Dream](#) we have  $f(s, t)^p = f(s^p, t^p)$  and  $g(s, t)^p = g(s^p, t^p)$ . Therefore,

$$\alpha^p = \frac{f(s, t)^p}{g(s, t)^p} = \frac{f(s^p, t^p)}{g(s^p, t^p)} \in K.$$

Thus  $x^p - \alpha^p \in K[x]$ , and since  $\alpha$  is a root of this polynomial we conclude that

$$[K(\alpha) : K] \leq \deg(x^p - \alpha^p) = p.$$

Thus  $K(\alpha) \neq L$  for all  $\alpha \in L$ .

However, not every finite separable extension is Galois.

**Example 6.69.** We showed in [Example 6.17](#) that  $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2})$  is not Galois. However, the minimal polynomial of  $\sqrt[3]{2}$  over  $\mathbb{Q}$  is  $x^3 - 2$ , which is separable, and thus by [Exercise 15](#) the extension  $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2})$  is separable.

## 6.6 The proof of Artin's Theorem

We now embark on a proof of Artin's Theorem. A key ingredient is the “linear independence of characters”, which is useful in other contexts as well, such as representation theory.

**Definition 6.70.** For a group  $G$  and field  $F$ , a **character** of  $G$  with values in  $F$  is a group homomorphism of the form

$$\chi : G \rightarrow F^\times.$$

**Example 6.71.**

- 1) If  $G = C_n$ , cyclic of order  $n$ , with generator  $x$ , then the UMP for cyclic groups says there is a unique group homomorphism  $G \rightarrow \mathbb{C}^\times$  sending  $x \mapsto \zeta_n$ , and hence  $x^i \mapsto \zeta_n^i$ . This is an example of a character.
- 2) If  $K$  and  $F$  are two fields and  $\phi : K \rightarrow F$  is a field map, then  $\phi$  restricts to a character  $\phi' : K^\times \rightarrow F^\times$ .

Note that the set  $\text{Fun}(G, F)$  of all functions from  $G$  to  $F$  is an  $F$ -vector space and that the characters of  $G$  are elements of this vector space. Therefore it makes sense to talk about linear independence for sets of characters. A point to observe here is that arbitrary linear combinations  $\sum_i l_i \chi_i$  are not, in general, group homomorphisms.

**Definition 6.72.** For  $G$  and  $F$  and characters  $\chi_1, \dots, \chi_n$ , we say these characters are *linear independent* if whenever  $\sum_{i=1}^n l_i \chi_i = 0$  (the constant map 0), we must have  $l_i = 0$  for all  $i$ . Making this even more explicit:  $\chi_1, \dots, \chi_n$  are linear independent if given  $l_i \in F$  such that  $\sum_{i=1}^n l_i \chi_i(g) = 0$  for all  $g \in G$ , we must have  $l_i = 0$  for all  $i$ .

**Theorem 6.73** (Linear Independence of Characters). *Let  $G$  be a group,  $F$  be a field, and let  $\chi_j : G \rightarrow F^\times$  for  $j = 1, \dots, m$  be a finite list of distinct characters, meaning that for all  $i \neq j$  we have  $\chi_i(g) \neq \chi_j(g)$  for at least one  $g \in G$ . Then  $\chi_1, \dots, \chi_m$  are linearly independent.*

The Theorem is sort of a *Sophomore's dream*, since it is saying that if a list of a certain sort of vectors in a certain vector space has no repetitions, then the vectors are linearly independent.

*Proof.* We proceed by induction on  $m$ .

Base case: When  $m = 1$ , since  $\chi_1(g) \neq 0$  for all  $g$  then  $l_1 \chi_1 = 0$  iff  $l_1 = 0$ .

Induction Step: Suppose  $m > 1$  and that  $\sum_{i=1}^m l_i \chi_i(g) = 0$  for all  $g \in G$  for some  $l_i \in F$ . Suppose

$$\sum_{i=1}^m l_i \chi_i = 0. \tag{6.6.1}$$

Evaluating (6.6.1) at  $hg$  for  $g, h \in G$  and using that  $\chi_i$  are group homomorphisms gives

$$0 = \sum_{i=1}^m l_i \chi_i(hg) = \sum_{i=1}^m l_i \chi_i(h) \chi_i(g) \quad \text{for all } g, h \in G. \tag{6.6.2}$$

Multiplying (6.6.1) by  $\chi_1(h)$  gives

$$0 = \chi_1(h) \left( \sum_{i=1}^m l_i \chi_i(g) \right) \quad \text{for all } g, h \in G. \quad (6.6.3)$$

Subtracting (6.6.2) from (6.6.3) we get we get

$$0 = \chi_1(h) \left( \sum_{i=1}^m l_i \chi_i(g) \right) - \sum_{i=1}^m l_i \chi_i(h) \chi_i(g) = \sum_{i=2}^m (\chi_1(h) l_i - \chi_i(h) l_i) \chi_i(g) \quad \text{for all } g, h \in G.$$

Fixing  $h$ , the equation above gives a linear dependence between  $\chi_2, \dots, \chi_m$ . Using the induction hypothesis we conclude that

$$\chi_1(h) l_i - \chi_i(h) l_i = 0 \quad \text{for all } h \in G$$

for all  $i$ , including  $i = m$ . Since  $\chi_1(h) \neq \chi_m(h)$ , we get  $l_m = 0$ , and hence (6.6.1) reduces to

$$\sum_{i=1}^{m-1} l_i \chi_i(g) = 0, \quad \text{for all } g \in G.$$

Using the induction hypothesis again it follows that  $l_i = 0$  for all  $i$ . □

**Example 6.74.** Let  $G = C_n$ , generated by  $x$ , and define

$$\begin{aligned} \chi_j : G &\rightarrow \mathbb{C} \\ x &\mapsto \chi_j(x) = \zeta_n^j = e^{2\pi j/n} \end{aligned}$$

for  $j = 0, \dots, n-1$  by  $\chi_j(x) = \zeta_n^j = e^{2\pi j/n}$ . These are distinct, and hence by Theorem 6.73 they must be linearly independent.

We now restate Artin's theorem:

**Theorem** (Artin's Theorem). *Let  $L$  be any field and  $G$  any finite subgroup of  $\text{Aut}(L)$ . Then  $L^G$  is a subfield of  $L$ ,  $L/L^G$  is a finite Galois extension and  $\text{Gal}(L/L^G) = G$ .*

*Proof of Artin's Theorem.* Let  $G$  be a finite subgroup of  $\text{Aut}(L)$  for a field  $L$ . In Exercise 18, we left proving that  $L^G$  is a subfield of  $L$  as an exercise. We will prove the remaining statements. We need to prove  $L/L^G$  is a finite extension and that  $[L : L^G] = |\text{Aut}(L/L^G)|$ .

We start by showing that it suffices to show that  $[L : L^G] = |G|$ . If  $[L : L^G] = |G|$  does indeed hold, then in particular  $L/L^G$  is a finite extension. By Theorem 6.15, since  $L^G \subseteq L$  is finite then  $|\text{Aut}(L/L^G)| \leq [L : L^G]$ . Since  $[L : L^G] = |G|$ , then we obtain  $|\text{Aut}(L/L^G)| \leq |G|$ .

On the other hand, any element in  $G$  fixes  $L^G$  by definition, so  $G \leq \text{Aut}(L/L^G)$ . But  $|\text{Aut}(L/L^G)| \leq |G|$  and these are both finite groups, so  $G = \text{Aut}(L/L^G)$ . From the inequality before we now conclude that  $|\text{Aut}(L/L^G)| = [L : L^G]$ , and thus the extension is Galois. Finally, this gives  $\text{Gal}(L/L^G) = \text{Aut}(L/L^G) = G$ .

It remains to prove that  $[L : L^G] = |G|$ . Let  $n = |G|$  and  $G = \{\sigma_1, \dots, \sigma_n\}$  with  $\sigma_1 = \text{id}_L$ . By Theorem 6.15, we know that  $[L : L^G] \geq n$ . We want to show that equality holds. If  $[L : L^G] > n$ , then we can find  $n + 1$  many  $L^G$ -linearly independent elements  $\omega_1, \dots, \omega_{n+1}$  in  $L$ . Consider the system of  $n$  equations with  $n + 1$  unknowns

$$\begin{cases} \sigma_1(\omega_1)x_1 + \dots + \sigma_1(\omega_{n+1})x_{n+1} = 0 \\ \sigma_2(\omega_1)x_1 + \dots + \sigma_2(\omega_{n+1})x_{n+1} = 0 \\ \vdots \\ \sigma_n(\omega_1)x_1 + \dots + \sigma_n(\omega_{n+1})x_{n+1} = 0. \end{cases}$$

Since there are fewer equations than unknowns, this system has a nontrivial solution. Among these, choose the solution that has the least number  $r$  of nonzero components; by reordering the  $\omega_i$  we may assume this solution has the form  $(a_1, \dots, a_r, 0, \dots, 0)$  with  $a_i \neq 0$  for all  $i$ . By scaling, we may assume  $a_r = 1$ . Since  $\sigma_1 = \text{id}_G$ , the first equation says that

$$a_1\omega_1 + \dots + a_{r-1}\omega_{r-1} + \omega_r = 0.$$

If all the  $a_i$  belong to  $L^G$  then this equation of linear dependence would contradict the linear independence of  $\omega_1, \dots, \omega_{n+1}$ . Thus  $a_i \notin L^G$  for some  $i$ . Reordering again, we may assume  $a_1 \notin L^G$ . Since  $a_r = 1$ , note in particular that this shows  $r > 1$ . We thus have

$$\begin{cases} \sigma_1(\omega_1)a_1 + \dots + \sigma_1(\omega_{r-1})a_{r-1} + \sigma_1(\omega_r) = 0 \\ \sigma_2(\omega_1)a_1 + \dots + \sigma_2(\omega_{r-1})a_{r-1} + \sigma_2(\omega_r) = 0 \\ \vdots \\ \sigma_n(\omega_1)a_1 + \dots + \sigma_n(\omega_{r-1})a_{r-1} + \sigma_n(\omega_r) = 0 \end{cases}$$

Now, since  $a_1 \notin L^G$ , there is a  $k$  with  $\sigma_k(a_1) \neq a_1$ . Apply  $\sigma_k$  to the  $j$ th row to obtain

$$\sigma_k\sigma_j(\omega_1)\sigma_k(a_1) + \dots + \sigma_k\sigma_j(\omega_{r-1})\sigma_k(a_{r-1}) + \sigma_k\sigma_j(\omega_r) = 0$$

Since  $G$  is a group, as  $j$  ranges over all possibilities,  $\sigma_k\sigma_j$  ranges over all elements of  $G$ . Thus

$$\sigma_i(\omega_1)\sigma_k(a_1) + \dots + \sigma_i(\omega_{r-1})\sigma_k(a_{r-1}) + \sigma_i(\omega_r) = 0 \quad \text{for all } 1 \leq i \leq n.$$

For each  $i$ , subtracting this equation from the  $i$ th equation in the previous system yields

$$\sigma_i(\omega_1)(a_1 - \sigma_k(a_1)) + \dots + \sigma_i(\omega_{r-1})(a_{r-1} - \sigma_k(a_{r-1})) = 0 \quad \text{for all } 1 \leq i \leq n.$$

Since  $a_1 - \sigma_k(a_1) \neq 0$ , this is a nontrivial solution to original system of equations with fewer than  $r$  nonzero components, which is a contradiction.  $\square$

# Index

- $(-)^{\sigma}$ , 84
- $[f]_B^C$ , 28
- $[L : F]$ , 62
- $\text{Aut}(K)$ , 84
- $\text{Aut}(K/F)$ , 84
- $\text{char}(R)$ , 79
- $\text{Gal}(L/F)$ , 89
- $\text{im}(f)$ , 10
- $\text{ker}(f)$ , 10
- $\mu_n(K)$ , 98
- $\text{Orb}_G(s)$ , 83
- $\sim_G$ , 83
- $A + B$ , 8
- $EF$ , 92
- $F(A)$ , 64
- $F(a)$ , 63
- $F(a_1, \dots, a_n)$ , 64
- $IM$ , 8
- $L^{\text{Gal}}$ , 104
- $M \cong N$ , 10
- $R$ -algebra, 12
- $R$ -module, 5
- $R$ -module homomorphism, 10
- $R$ -module isomorphism, 10
- $R$ -module presented by  $A$ , 36
- $R$ -submodule, 8
- absorption, 4
- action, 83
- algebraic, 67
- algebraic closure, 71
- algebraic element, 67
- algebraic extension, 69
- algebraically closed, 71
- automorphism group of a field, 84
- automorphism group of a field extension, 84
- basis, 18
- canonical map, 14
- canonical quotient map, 14
- chain, 23
- change of basis matrix, 31
- character, 106
- characteristic, 79
- characteristic polynomial, 53
- commutative ring, 3
- companion matrix, 52
- composite of two fields, 92
- cyclic, 9
- degree of a field extension, 62
- derivative, 80
- diagonalizable, 59
- dimension, 25
- direct product, 20
- direct sum, 20
- direct sum of matrices, 52
- division ring, 4
- domain, 4
- eigenvalue, 55
- eigenvector, 55
- elementary basis change operation, 32
- elementary divisors, 45
- elementary matrix, 32
- elementary row operation, 32
- endomorphism ring, 11
- endomorphisms, 11
- faithful action, 83

- field, 4
- field extension, 62
- field of algebraic numbers, 74
- finite dimensional, 24
- finite field extension, 62, 69
- finitely generated, 17
- free, 18
- free module, 7
- free rank, 44
- Freshman's Dream, 79
  
- Galois closure, 104
- Galois correspondence, 95
- Galois extension, 89
- Galois group, 89
- Galois group of a Galois extension, 89
- Galois group of a polynomial, 89
- gcd, 45
- generated by, 8, 17
- greatest common divisor, 45
- group action, 83
  
- ideal, 4
- image, 10
- image of a homomorphism, 10
- inseparable, 81
- inseparable extension, 81
- integral domain, 4
- intermediate field, 92
- invariant factors, 44, 51
- isomorphic, 10
- isomorphic modules, 10
  
- Jordan block, 57
- Jordan canonical form, 59
  
- kernel, 10
- kernel of a homomorphism, 10
  
- left  $R$ -module, 5
- left ideal, 4
- linear combination, 17
- linear transformation, 10
- linearly dependent, 18
- linearly independent, 18
- matrix of the linear transformation, 28
  
- maximal element, 23
- minimal polynomial, 53, 54, 68
- module of relations, 36
- multiplicity, 80
- multiplicity of a root, 80
  
- noetherian ring, 40
- nontrivial ideal, 4
- nullspace, 27
  
- orbits of a group action, 83
  
- perfect field, 81
- PID, 40
- poset, 22, 23
- prime field, 79
- primitive  $n$ th root of unity, 98
- primitive element, 63, 104
- proper ideal, 4
  
- rank, 20, 27
- rational canonical form, 53
- relation, 35
- represent, 28
- restriction of scalars, 9
- right  $R$ -module, 5
- right ideal, 4
- ring, 3
- ring homomorphism, 4
- ring of scalars, 9
  
- separable extension, 81
- separable polynomial, 80
- similar, 31
- simple extension, 104
- simple field extension, 63
- solvable by radicals, 100
- solvable group, 101
- span, 22
- spanned by, 17
- splitting field, 75
- subfield generated by  $A$  over  $F$ , 64
- subfield of  $L$  fixed by  $G$ , 89
- submodule generated by, 8
- subring, 4
- sum of modules, 8

totally ordered, [23](#)

transcendental, [67](#)

transcendental element, [67](#)

transitive group action, [83](#)

trivial ideals, [4](#)

trivial submodules, [8](#)

upper bound, [23](#)

vector space, [6](#)

zero module, [8](#)

zerodivisors, [4](#)