

Commutative Algebra

Math 225 Winter 2021

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Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or incorrections, I will be most grateful to you for letting me know. If you are looking for a place where to learn commutative algebra, I strongly recommend the following excellent resources:

- [Mel Hochster's Lecture notes](#)
- Jack Jeffries' Lecture notes (either his [UMich 614 notes](#) or his [CIMAT notes](#))
- Atiyah and MacDonald's *Commutative Algebra* [[AM69](#)]
- Matsumura's *Commutative Ring Theory* [[Mat89](#)], or his other less known book *Commutative Algebra* [[Mat80](#)]
- Eisenbud's *Commutative Algebra with a view towards algebraic geometry* [[Eis95](#)]

The notes you see here are adapted from Jack Jeffries' notes, Alexandra Seceleanu's Spring 2020 Commutative Algebra notes, and inspired by all the other resources above.

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Chapter 0

Setting the stage

In this chapter we set the stage for what's to come in the rest of the class. The definitions and facts we collect here should be somewhat familiar to you already, and so we present them in rapid fire succession. You can learn more about the basic theory of (commutative) rings and R -modules in any introductory algebra book, such as [DF04].

0.1 Basic definitions: rings and ideals

Roughly speaking, Commutative Algebra is the branch of algebra that studies commutative rings and modules over such rings. For a commutative algebraist, every ring is commutative and has a $1 \neq 0$.

Definition 0.1 (Ring). A **ring** is a set R equipped with two binary operations $+$ and \cdot satisfying the following properties:

- 1) R is an abelian group under the addition operation $+$, with additive identity 0 .¹ Explicitly, this means that

- $a + (b + c) = (a + b) + c$ for all $a, b, c \in R$,
- $a + b = b + a$ for all $a, b \in R$,
- there is an element $0 \in R$ such that $0 + a = a$ for all $a \in R$, and
- for each $a \in R$ there exists an element $-a \in R$ such that $a + (-a) = 0$.

- 2) R is a commutative monoid under the multiplication operation \cdot , with multiplicative identity 1 .² Explicitly, this means that

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$,
- $a \cdot b = b \cdot a$ for all $a, b \in R$, and

¹Or 0_R if we need to specify which ring we are talking about.

²If we need to specify the corresponding ring, we may write 1_R .

- there exists an element $1 \in R$ such that $1 \cdot a = a \cdot 1$ for all $a \in R$.

3) multiplication is distributive with respect to addition, meaning that

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

for all $a, b, c \in R$.

4) $1 \neq 0$.

We typically write ab for $a \cdot b$.

While in some branches of algebra rings might fail to be commutative, we will explicitly say we have a *noncommutative ring* if that is the case, and otherwise all rings are assumed to be commutative. There also branches of algebra where rings might be assumed to not necessarily have a multiplicative identity; we recommend [Poo19] for an excellent read on the topic of *Why rings should have a 1*.

Example 0.2. Here are some examples of the kinds of rings we will be talking about.

- The integers \mathbb{Z} .
- Any quotient of \mathbb{Z} , which we write compactly as \mathbb{Z}/n .
- A polynomial ring. When we say polynomial ring, we typically mean $R = k[x_1, \dots, x_n]$, a polynomial ring in finitely many variables over a field k .
- A quotient of a polynomial ring by an ideal I , say $R = k[x_1, \dots, x_n]/I$.
- Rings of polynomials in infinitely many variables, $R = k[x_1, x_2, \dots]$.
- Power series rings $R = k[[x_1, \dots, x_n]]$. The elements are (formal) power series
$$\sum_{a_1, \dots, a_n \geq 0} c_{a_1, \dots, a_n} x_1^{a_1} \cdots x_n^{a_n}.$$
- While any field k is a ring, we will see that fields on their own are not very exciting from the perspective of the kinds of things we will be discussing in this class.

Definition 0.3 (ring homomorphism). A map $R \xrightarrow{f} S$ between rings is a **ring homomorphism** if f preserves the operations and the multiplicative identity, meaning

- $f(a + b) = f(a) + f(b)$ for all $a, b \in R$,
- $f(ab) = f(a)f(b)$ for all $a, b \in R$, and
- $f(1) = 1$.

A bijective ring homomorphism is an **isomorphism**. We should think about a ring isomorphism as a relabelling of the elements in our ring.

Definition 0.4. A subset $R \subseteq S$ of a ring S is a **subring** if R is also a ring with the structure induced by S , meaning that the each operation on R is the restrictions of the corresponding operation on S to R , and the 0 and 1 in R are the 0 and 1 in S , respectively.

Often, we care about the ideals in a ring more than we care about individual elements.

Definition 0.5 (ideal). A nonempty subset I of a ring R is an **ideal** if it is closed for the addition and for multiplication by any element in R : for any $a, b \in I$ and $r \in R$, we must have $a + b \in I$ and $ra \in I$.

The **ideal generated by** f_1, \dots, f_n , denoted (f_1, \dots, f_n) , is the smallest ideal containing f_1, \dots, f_n , or equivalently,

$$(f_1, \dots, f_n) = \{r_1 f_1 + \dots + r_n f_n \mid r_i \in R\}.$$

Example 0.6. Every ring has always at least 2 ideals, the zero ideal $(0) = \{0\}$ and the unit ideal $(1) = R$.

We will follow the convention that when we say *ideal* we actually mean every ideal $I \neq R$.

Exercise 1. The ideals in \mathbb{Z} are the sets of multiples of a fixed integer, meaning every ideal has the form (n) . In particular, every ideal in \mathbb{Z} can be generated by one element.

This makes \mathbb{Z} the canonical example of a **principal ideal domain**.

A **domain** is a ring with no zerodivisors, meaning that $rs = 0$ implies that $r = 0$ or $s = 0$. A **principal ideal** is an ideal generated by one element. A **principal ideal domain** or **PID** is a domain where every ideal is principal.

Exercise 2. Given a field k , $R = k[x]$ is a principal ideal domain, so every ideal in R is of the form $(f) = \{fg \mid g \in R\}$.

Exercise 3. While $R = k[x, y]$ is a domain, it is **not** a PID. We will see later that every ideal in R is finitely generated, and yet we can construct ideals in R with arbitrarily many generators!

Example 0.7. While $\mathbb{Z}[x]$ is a domain, it is also **not** a PID. For example, $(2, x)$ is not a principal ideal.

0.2 Basic definitions: modules

Similarly to how linear algebra is the study of vector spaces over fields, commutative algebra often focuses on the structure of modules over a given commutative ring R . While in other branches of algebra modules might be left- or right-modules, all our modules are two sided, and we refer to them simply as modules.

Definition 0.8 (Module). Given a ring R , an R -**module** $(M, +)$ is an abelian group equipped with an R -action that is compatible with the group structure. More precisely, there is an operation $\cdot : R \times M \longrightarrow M$ such that

- $r \cdot (a + b) = r \cdot a + r \cdot b$ for all $r \in R$ and $a, b \in M$,
- $(r + s) \cdot a = r \cdot a + s \cdot a$ for all $r, s \in R$ and $a \in M$,
- $(rs) \cdot a = r \cdot (s \cdot a)$ for all $r, s \in R$ and $a \in M$, and
- $1 \cdot a = a$ for all $a \in M$.

We typically write ra for $r \cdot a$. We denote the additive identity in M by 0 , or 0_M if we need to distinguish it from 0_R .

The definitions of submodule, quotient of modules, and homomorphism of modules are very natural and easy to guess, but here they are.

Definition 0.9. If $N \subseteq M$ are R -modules with compatible structures, we say that N is a **submodule** of M .

A map $M \xrightarrow{f} N$ between R -modules is a **homomorphism of R -modules** if it is a homomorphism of abelian groups that preserves the R -action, meaning $f(ra) = rf(a)$ for all $r \in R$ and all $a \in M$. We sometimes refer to R -module homomorphisms as **R -module maps**, or **maps of R -modules**. An isomorphism of R -modules is a bijective homomorphism, which we really should think about as a relabeling of the elements in our module. If two modules M and N are isomorphic, we write $M \cong N$.

Given an R -module M and a submodule $N \subseteq M$, the **quotient** M/N is an R -module whose elements are the equivalence classes determined by the relation on M given by $a \sim b \Leftrightarrow a - b \in N$. One can check that this set naturally inherits an R -module structure from the R -module structure on M , and it comes equipped with a natural **canonical map** $M \longrightarrow M/N$ induced by sending 1 to its equivalence class.

Example 0.10. The modules over a field k are precisely all the k -vector spaces. Linear transformations are precisely all the k -module maps.

While vector spaces make for a great first example, be warned that many of the basic facts we are used to from linear algebra are often a little more subtle in commutative algebra. These differences are features, not bugs.

Example 0.11. The \mathbb{Z} -modules are precisely all the abelian groups.

Example 0.12. When we think of the ring R as a module over itself, the submodules of R are precisely the ideals of R .

Exercise 4. The kernel $\ker f$ and image $\operatorname{im} f$ of an R -module homomorphism $M \xrightarrow{f} N$ are submodules of M and N , respectively.

Theorem 0.13 (First Isomorphism Theorem). *Given a homomorphism of R -modules $M \xrightarrow{f} N$, $M/\ker f \cong \operatorname{im} f$.*

The first big noticeable difference between vector spaces and more general R -modules is that while every vector space has a basis, most R -modules do not.

Definition 0.14. A subset $\Gamma \subseteq M$ of an R -module M is a **generating set**, or a **set of generators**, if every element in M can be written as a finite linear combination of elements in M with coefficients in R . A **basis** for an R -module M is a generating set Γ for M such that $\sum_i a_i \gamma_i = 0$ implies $a_i = 0$ for all i . An R -module is **free** if it has a basis.

Remark 0.15. Every vector space is a free module.

Remark 0.16. Every free R -module is isomorphic to a direct sum of copies of R . Indeed, let's construct such an isomorphism for a given free R -module M . Given a basis $\Gamma = \{\gamma_i\}_{i \in I}$ for M , let

$$\begin{aligned} \bigoplus_{i \in I} R &\xrightarrow{\pi} M \\ (r_i)_{i \in I} &\longrightarrow \sum_i r_i \gamma_i \end{aligned}$$

The condition that Γ is a basis for M can be restated into the statement that π is an isomorphism of R -modules.

One of the key things that makes commutative algebra so rich and beautiful is that most modules are in fact *not* free. In general, every R -module has a generating set — for example, M itself. Given some generating set Γ for M , we can always repeat the idea above and write a **presentation** $\bigoplus_{i \in I} R \xrightarrow{\pi} M$ for M , but in general the resulting map π will have a nontrivial kernel. A nonzero kernel element $(r_i)_{i \in I} \in \ker \pi$ corresponds to a **relation** between the generators of M .

Remark 0.17. Given a set of generators for an R -module M , any homomorphism of R -modules $M \rightarrow N$ is determined by the images of the generators.

We say that a module is **finitely generated** if we can find a finite generating set for M . The simplest finitely generated modules are the cyclic modules.

Example 0.18. An R -module is **cyclic** if it can be generated by one element. Equivalently, we can write M as a quotient of R by some ideal I . Indeed, given a generator m for M , the kernel of the map $R \xrightarrow{\pi} M$ induced by $1 \mapsto m$ is some ideal I . Since we assumed that m generates M , π is automatically surjective, and thus induces an isomorphism $R/I \cong M$.

Similarly, if an R -module has n generators, we can naturally think about it as a quotient of R^n by the submodule of relations among those n generators.

0.3 Why study commutative algebra?

There are many reasons why one would want to study commutative algebra. For starters, it's fun! Also, modern commutative algebra has connections with many fields of mathematics, including:

- Algebra Geometry
- Algebraic Topology
- Homological Algebra
- Category Theory
- Number Theory
- Arithmetic Geometry
- Combinatorics
- Invariant Theory
- Representation Theory
- Differential Algebra
- Lie Algebras
- Cluster Algebras

Chapter 1

Finiteness conditions

1.1 Noetherian rings and modules

The most common assumption in commutative algebra is to require that our rings be Noetherian. Noetherian rings are named after Emmy Noether, who is in many ways the mother of modern commutative algebra. Many rings that one would naturally want to study are noetherian.

Definition 1.1 (Noetherian ring). A ring R is *Noetherian* if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes: there is some N for which $I_n = I_{n+1}$ for all $n \geq N$.

This condition can be restated in various equivalent forms.

Proposition 1.2. *Let R be a ring. The following are equivalent:*

- 1) R is a Noetherian ring.
- 2) Every nonempty family of ideals has a maximal element (under \subseteq).
- 3) Every ascending chain of finitely generated ideals of R stabilizes.
- 4) Given any generating set S for an ideal I , I is generated by a finite subset of S .
- 5) Every ideal of R is finitely generated.

Proof.

(1) \Rightarrow (2): We prove the contrapositive. Suppose there is a nonempty family of ideals with no maximal element. This means that we can keep inductively choosing larger ideals from this family to obtain an infinite properly ascending chain.

(2) \Rightarrow (1): An ascending chain of ideals is a family of ideals, and the maximal ideal in the family indicates where our chain stabilizes.

(1) \Rightarrow (3): Clear.

(3) \Rightarrow (4): Let's prove the contrapositive. Suppose that there is an ideal I and a generating set S for I such that no finite subset of S generates I . So for any finite $S' \subseteq S$ we have $(S') \subsetneq (S) = I$, so there is some $s \in S \setminus (S')$. Thus, $(S') \subsetneq (S' \cup \{s\})$. Inductively, we can continue this process to obtain an infinite proper chain of finitely generated ideals, contradicting (3).

(4) \Rightarrow (5): Clear.

(5) \Rightarrow (1): Given an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

let $I = \bigcup_{n \in \mathbb{N}} I_n$. In general, the union of two ideals might fail to be an ideal, but the union of a chain of ideals is an ideal (exercise). By assumption, the ideal I is finitely generated, say $I = (a_1, \dots, a_t)$, and since each a_i is in some I_{n_i} , there is an N such that every a_i is in I_N . But then $I_N = I$, and thus $I_n = I_{n+1}$ for all $n \geq N$. \square

Remark 1.3. When we say that every non-empty family of ideals has a maximal element, that maximal element does not have to be unique in any way. An ideal I is maximal in the family \mathcal{F} if $I \subseteq J$ for some $J \in \mathcal{F}$ implies $I = J$; we might have many incomparable maximal elements in \mathcal{F} . For example, every element in the family of ideals in \mathbb{Z} given by

$$\mathcal{F} = \{(p) \mid p \text{ is a prime integer}\}$$

is maximal.

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Example 1.4.

- 1) If $R = k$ is a field, the only ideals in k are (0) and $(1) = k$, so k is a Noetherian ring.
- 2) \mathbb{Z} is a Noetherian ring. More generally, if R is a PID, then R is Noetherian. Indeed, every ideal is finitely generated!
- 3) As a special case of the previous example, consider the ring of germs of complex analytic functions near 0,

$$\mathbb{C}\{z\} := \{f(z) \in \mathbb{C}[[z]] \mid f \text{ is analytic on a neighborhood of } z = 0\}.$$

This ring is a PID: every ideal is of the form (z^n) , since any $f \in \mathbb{C}\{z\}$ can be written as $z^n g(z)$ for some $g(z) \neq 0$, and any such $g(z)$ is a unit in $\mathbb{C}\{z\}$.

- 4) A ring that is *not* Noetherian is a polynomial ring in infinitely many variables over a field k , $R = k[x_1, x_2, \dots]$: the ascending chain of ideals

$$(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \cdots$$

does *not* stabilize.

- 5) The ring $R = K[x, x^{1/2}, x^{1/3}, x^{1/4}, x^{1/5}, \dots]$ is also *not* Noetherian. A nice ascending chain of ideals is

$$(x) \subsetneq (x^{1/2}) \subsetneq (x^{1/3}) \subsetneq (x^{1/4}) \subsetneq \dots$$

- 6) The ring of continuous real-valued functions $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is *not* Noetherian: the chain of ideals

$$I_n = \{f(x) \mid f|_{[-1/n, 1/n]} \equiv 0\}$$

is increasing and proper. The same construction shows that the ring of infinitely differentiable real functions $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is not Noetherian: properness of the chain follows from, e.g., Urysohn's lemma (though it's not too hard to find functions distinguishing the ideals in the chain). Note that if we asked for analytic functions instead of infinitely-differentiable functions, every element of the chain would be the zero ideal!

Remark 1.5. If R is Noetherian, and I is an ideal of R , then R/I is Noetherian as well, since there is an order-preserving bijection

$$\{\text{ideals of } R \text{ that contain } I\} \longleftrightarrow \{\text{ideals of } R/I\}.$$

This gives us many more examples, by simply taking quotients of the examples above. We will also see huge classes of easy examples once we learn about localization.

Similarly, we can define noetherian modules.

Definition 1.6 (Noetherian module). An R -module M is *Noetherian* if every ascending chain of submodules of M eventually stabilizes.

There are analogous equivalent definitions for modules as we had above for rings, so we leave the proof as an exercise.

Proposition 1.7 (Equivalence definitions for Noetherian module). *Let M be an R -module. The following are equivalent:*

- 1) M is a Noetherian module.
- 2) Every nonempty family of submodules has a maximal element.
- 3) Every ascending chain of finitely generated submodules of M eventually stabilizes.
- 4) Given any generating set S for a submodule N , the submodule N is generated by a finite subset of S .
- 5) Every submodule of M is finitely generated.

In particular, a Noetherian module must be finitely generated.

Remark 1.8. A ring R is a Noetherian ring if and only if R is Noetherian as a module over itself. However, a Noetherian ring need not be a Noetherian module over a subring. For example, consider $\mathbb{Z} \subseteq \mathbb{Q}$. These are both Noetherian *rings*, but \mathbb{Q} is not a noetherian \mathbb{Z} -module; for example, the following is an ascending chain of submodules which does not stabilize:

$$0 \subsetneq \frac{1}{2}\mathbb{Z} \subsetneq \frac{1}{2}\mathbb{Z} + \frac{1}{3}\mathbb{Z} \subsetneq \frac{1}{2}\mathbb{Z} + \frac{1}{3}\mathbb{Z} + \frac{1}{5}\mathbb{Z} \subsetneq \cdots$$

Definition 1.9. An **exact sequence** of R -modules is a sequence

$$\cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \cdots$$

of R -modules and R -module homomorphisms such that $\text{im } f_n = \ker f_{n+1}$ for all n . An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a **short exact sequence**.

Remark 1.10. The sequence

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact if and only if f is injective. Similarly,

$$M \xrightarrow{f} N \longrightarrow 0$$

is exact if and only if f is surjective. So

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence if and only if

- f is injective
- g is surjective
- $\text{im } f = \ker g$.

So when this is indeed a short exact sequence, we can identify A with its image $f(A)$, and $A = \ker g$. Moreover, since g is surjective, by the First Isomorphism Theorem we conclude that $C \cong B/A$, so we might abuse notation and identify C with B/A .

Lemma 1.11 (Noetherianity in exact sequences). *In an exact sequence of modules*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

B is Noetherian if and only if A and C are Noetherian.

Proof. Assume B is Noetherian. Since A is a submodule of B , and its submodules are also submodules of B , A is Noetherian. Moreover, any submodule of B/A is of the form D/A for some submodule $D \supseteq A$ of B . Since every submodule of B is finitely generated, every submodule of C is also finitely generated. Therefore, C is Noetherian.

Conversely, assume that A and C are Noetherian, and let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

be a chain of submodules of B . First, note that

$$M_1 \cap A \subseteq M_2 \cap A \subseteq \cdots$$

is an ascending chain of submodules of A , and thus it stabilizes. Moreover,

$$g(M_1) \subseteq g(M_2) \subseteq g(M_3) \subseteq \cdots$$

is a chain of submodules of C , and thus it also stabilizes. Pick a large enough index n such that both of these chains stabilize. We claim that $M_n = M_{n+1}$, so that the original chain stabilizes as well. To show that, take $x \in M_{n+1}$. Then

$$g(x) \in g(M_{n+1}) = g(M_n)$$

so we can choose some $y \in M_n$ such that $g(x) = g(y)$. Then

$$x - y \in \ker g = \operatorname{im} f,$$

and $x - y \in A$. Now note that $x - y \in M_{n+1}$, so

$$x - y \in M_{n+1} \cap A = M_n \cap A.$$

Then $x - y \in M_n$, and since $y \in M_n$, we must have $x \in M_n$ as well. □

Corollary 1.12. *If A and B are Noetherian R -modules, then $A \oplus B$ is a Noetherian R -module.*

Proof. Apply the previous lemma to the short exact sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0.$$

□

Corollary 1.13. *A module M is Noetherian if and only if M^n is Noetherian for some n . In particular, if R is a Noetherian ring then R^n is a Noetherian module.*

Proof. By induction on n :

- The case $n = 1$ is a tautology.

- For $n > 1$, consider the short exact sequence

$$0 \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

Lemma 1.11 and the inductive hypothesis give the desired conclusion. \square

Proposition 1.14. *Let R be a Noetherian ring. Given an R -module M , M is a Noetherian R -module if and only if M is finitely generated.*

Consequently, if R is Noetherian, then any submodule of a finitely generated R -module is also finitely generated.

Proof. If M is Noetherian, M is finitely generated by the equivalent definitions above, and so are all of its submodules.

Now let R be Noetherian and M be a finitely generated R -module. Then M is isomorphic to a quotient of R^n for some n , which is Noetherian. \square

Remark 1.15. The Noetherianity hypothesis is important: if M is a finitely generated R -module over a non-Noetherian ring, M might not be Noetherian. For a dramatic example, note that R itself is a finitely generated R -module, but not Noetherian.

David Hilbert had a big influence in the early years of commutative algebra, in many different ways. Emmy Noether's early work in algebra was in part inspired by some of his work, and he later invited Emmy Noether to join the Göttingen Math Department — many of her amazing contributions to algebra happened during her time in Göttingen. Unfortunately, some of the faculty was opposed to having a woman joining the department, and for her first two years in Göttingen Noether did not have an official position nor was she paid. Hilbert's contributions also include three of the most fundamental results in commutative algebra — Hilbert's Basis Theorem, the Hilbert Syzygy Theorem, and Hilbert's Nullstellensatz. We can now prove the first.

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Theorem 1.16 (Hilbert's Basis Theorem). *Let R be a Noetherian ring. Then the rings $R[x_1, \dots, x_d]$ and $R[[x_1, \dots, x_d]]$ are Noetherian.*

Remark 1.17. We can rephrase this theorem in a way that can be understood by anyone with a basic high school algebra (as opposed to abstract algebra) knowledge:

Any system of polynomial equations in finitely many variables can be written in terms of finitely many equations.

Proof. We give the proof for polynomial rings, and indicate the difference in the power series argument.

Using induction on d , we can reduce to the case $d = 1$. Let $I \subseteq R[x]$, and let

$$J = \{a \in R \mid \exists ax^n + \text{lower order terms (wrt } x) \in I\}.$$

So $J \subseteq R$ consists of all the leading coefficients of polynomials in I . We can check (exercise) that this is an ideal of R . By our hypothesis, J is finitely generated, so let $J = (a_1, \dots, a_t)$. Pick $f_1, \dots, f_t \in R[x]$ such that the leading coefficient of f_i is a_i , and set $N = \max_i \{\deg f_i\}$.

Given any $f \in I$ of degree greater than N , we can cancel off the leading term of f by subtracting a suitable combination of the f_i , so any $f \in I$ can be written as $f = g + h$ where $h \in (f_1, \dots, f_t)$ and $g \in I$ has degree at most N , so $g \in I \cap (R + Rx + \dots + Rx^N)$. Note that since $I \cap (R + Rx + \dots + Rx^N)$ is a submodule of the finitely generated free R -module $R + Rx + \dots + Rx^N$, it is also finitely generated as an R -module. Given such a generating set, say $I \cap (R + Rx + \dots + Rx^N) = (f_{t+1}, \dots, f_s)$, we can write any such $f \in I$ as an $R[x]$ -linear combination of these generators and the f_i 's. Therefore, $I = (f_1, \dots, f_t, f_{t+1}, \dots, f_s)$ is finitely generated, and $R[x]$ is a Noetherian ring.

In the power series case, take J to be the coefficients of *lowest degree* terms. \square

1.2 Algebra finite-extensions

If R is a subring of S , then S is an **algebra** over R , meaning that S is a ring with a (natural) structure of an R -module that also satisfies

$$r(s_1 s_2) = (rs_1)s_2 \text{ for all } r \in R \text{ and } s_1, s_2 \in S.$$

More generally, given any ring homomorphism $\varphi : R \rightarrow S$, we can view S as an algebra over R via φ by setting $r \cdot s = \varphi(r)s$. We may abuse notation and write $r \in S$ for its image $\varphi(r) \in S$. We will see that in a lot of situations we want to study, it is enough to consider the case when φ is injective, so this abuse of notation makes sense.

Giving a ring homomorphism $R \rightarrow S$ is the same as giving an R -algebra structure to S . In particular, a ring S can have different R -algebra structures given by different homomorphisms $R \rightarrow S$.

A set of elements $\Lambda \subseteq S$ **generates** S as an R -algebra if the following equivalent conditions hold:

- The only subring of S containing $\varphi(R)$ and Λ is S itself.
- Every element of S admits a polynomial expression in Λ with coefficients in $\varphi(R)$.
- Given a polynomial ring $R[X]$ on $|\Lambda|$ indeterminates, the ring homomorphism

$$\begin{aligned} R[X] &\xrightarrow{\psi} S \\ x_i &\longmapsto \lambda_i \end{aligned}$$

is surjective.

Let S be an R -algebra and $\Lambda \subseteq S$ be a set of algebra generators for S over R . The ideal of **relations** on the elements Λ over R is the kernel of the map

$$\begin{aligned} R[X] &\xrightarrow{\psi} S \\ x_i &\longmapsto \lambda_i \end{aligned}$$

This consists of the polynomial functions with R -coefficients that the elements of Λ satisfy. Given an R -algebra S with generators Λ and ideal of relations I , we have a ring isomorphism $S \cong R[X]/I$ by the First Isomorphism Theorem. If we understand the ring R and generators and relations for S over R , we can get a pretty concrete understanding of S . If a sequence of elements has no nonzero relations, we say they are *algebraically independent* over R .

Remark 1.18. If $s_1, \dots, s_n \in S$ are algebraically independent over R , then $R[s_1, \dots, s_n]$ is isomorphic to the polynomial ring in n variables over R .

We say that $\varphi : R \rightarrow S$ is **algebra-finite**, or S is a **finitely generated R -algebra**, or S is of **finite type** over R , if there exists a *finite* set of elements $f_1, \dots, f_t \in S$ that generates S as an R -algebra. A better name might be *finitely generatable*, since to say that an algebra is finitely generated does not require knowing any actual finite set of generators. From the discussion above, we conclude that S is a finitely generated R -algebra if and only if S is a quotient of some polynomial ring $R[x_1, \dots, x_d]$ over R in finitely many variables. If S is generated over R by f_1, \dots, f_d , we will use the notation $R[f_1, \dots, f_d]$ to denote S . Of course, for this notation to properly specify a ring, we need to understand how these generators behave under the operations. This is no problem if A and \underline{f} are understood to be contained in some larger ring.

Remark 1.19. Any surjective ring homomorphism $\varphi : R \rightarrow S$ is algebra-finite, since then S is generated over R by 1. Moreover, we can always factor φ as the surjection $R \twoheadrightarrow R/\ker(\varphi)$ followed by the inclusion $R/\ker(\varphi) \hookrightarrow S$, so to understand algebra-finiteness it suffices to restrict our attention to injective homomorphisms.

Example 1.20. Every ring is a \mathbb{Z} -algebra, but generally not a finitely generated one.

Remark 1.21. Let $A \subseteq B \subseteq C$ be rings. It follows from the definitions that

- $\begin{array}{ccc} A \subseteq B \text{ algebra-finite} & & \\ \text{and} & \implies & A \subseteq C \text{ algebra-finite} \\ B \subseteq C \text{ algebra-finite} & & \end{array}$
- $A \subseteq C \text{ algebra-finite} \implies B \subseteq C \text{ algebra-finite}.$

However, $A \subseteq C \text{ algebra-finite} \not\Rightarrow A \subseteq B \text{ algebra-finite}.$

Example 1.22. Let k be a field and

$$B = k[x, xy, xy^2, xy^3, \dots] \subseteq C = k[x, y],$$

where x and y are indeterminates. While B and C are both k -algebras, C is a finitely generated k -algebra, while B is not. Indeed, any finitely generated subalgebra of B is contained in $k[x, xy, \dots, xy^m]$ for some m , since we can write the elements in any finite generating set as polynomial expressions in finitely many of the specified generators of B . However, note that every element of $k[x, xy, \dots, xy^m]$ is a k -linear combination of monomials with the property that the y exponent is no more than m times the x exponent, so this ring does not contain xy^{m+1} . Thus, B is not a finitely generated A -algebra.

There are many basic questions about algebra generators that are surprisingly difficult. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and $f_1, \dots, f_n \in R$. When do f_1, \dots, f_n generate R over \mathbb{C} ? It isn't too hard to show that the Jacobian determinant

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

must be a nonzero constant. It is a big open question whether this is in fact a sufficient condition!

Finally, note that an easy corollary of the Hilbert Basis Theorem is that finitely generated algebras over noetherian rings are also noetherian.

Corollary 1.23. *If R is a Noetherian ring, then any finitely generated R -algebra is Noetherian. In particular, any finitely generated algebra over a field is Noetherian.*

Proof. By our discussion above, a finitely generated R -algebra is isomorphic to a quotient of a polynomial ring over R in finitely many variables; polynomial rings over noetherian rings are Noetherian, by Hilbert's Basis Theorem, and quotients of Noetherian rings are Noetherian. \square

The converse to this statement is false: there are lots of Noetherian rings that are not finitely generated algebras over a field. For example, $\mathbb{C}\{z\}$ is not algebra-finite over \mathbb{C} .

1.3 Module-finite extensions

Given a ring homomorphism $\varphi : R \rightarrow S$, saying that S acquires an R -module structure via φ by $a \cdot r = \varphi(a)r$ is a particular case of *restriction of scalars*. By restriction of scalars, we mean that any S -module M also gains a new R -module structure given by

$r \cdot m = \varphi(r)m$.¹ We may write ${}_{\varphi}M$ for this R -module if we need to emphasize which map we are talking about.

Given an R -algebra S , we can consider the *algebra* structure of S over R , or its *module* structure over R . So instead of asking about how S is generated as an *algebra* over R , we can ask how it is generated as a *module* over R . Recall that an A -module M is generated by a set of elements $\Gamma \subseteq M$ if the following equivalent conditions hold:

- The smallest submodule of M that contains Γ is M itself.
- Every element of M can be written as an A -linear combination of elements in Γ .
- Given a free R -module on $|\Gamma|$ basis elements $R^{\oplus Y}$, the homomorphism

$$\begin{array}{ccc} R^{\oplus Y} & \xrightarrow{\theta} & M \\ y_i & \longmapsto & \gamma_i \end{array}$$

is surjective.

We use the notation $M = \sum_{\gamma \in \Gamma} A\gamma$ to indicate that M is generated by Γ as a module. We say that $\varphi : A \rightarrow R$ is *module-finite* if R is a finitely-generated A -module. This is also called simply *finite* in the literature, but we'll stick with the unambiguous “module-finite.”

As with algebra-finiteness, surjective maps are always module-finite in a trivial way, and it suffices to understand this notion for ring inclusions.

The notion of module-finite is much stronger than algebra-finite, since a linear combination is a very special type of polynomial expression.

Example 1.24.

- If $K \subseteq L$ are fields, saying L is module-finite over K just means that L is a finite field extension of K .
- The Gaussian integers $\mathbb{Z}[i]$ satisfy the well-known property (or definition, depending on your source) that any element $z \in \mathbb{Z}[i]$ admits a unique expression $z = a + bi$ with $a, b \in \mathbb{Z}$. That is, $\mathbb{Z}[i]$ is generated as a \mathbb{Z} -module by $\{1, i\}$; moreover, they form a free module basis!
- If R is a ring and x an indeterminate, $R \subseteq R[x]$ is not module-finite. Indeed, $R[x]$ is a free R -module on the basis $\{1, x, x^2, x^3, \dots\}$.
- Another map that is *not* module-finite is the inclusion $k[x] \subseteq k[x, 1/x]$. First, note that any element of $k[x, 1/x]$ can be written in the form $f(x)/x^n$ for some $f \in k[x]$ and some $n \geq 0$. Since $k[x]$ is a Noetherian ring, $k[x, 1/x]$ is a finitely-generated $k[x]$ -module if and only if it is a Noetherian $k[x]$ -module. But here is an infinite chain of submodules of $k[x, \frac{1}{x}]$:

$$k[x] \cdot \frac{1}{x} \subseteq k[x] \cdot \frac{1}{x^2} \subseteq k[x] \cdot \frac{1}{x^3} \subseteq \dots$$

¹This gives a functor from the category of S -modules to the category of R -modules.

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Remark 1.25. If R is an A -algebra,

- $A \subseteq R$ is algebra-finite if $R = A[f_1, \dots, f_n]$ for some $f_1, \dots, f_n \in R$.
- $A \subseteq R$ is module-finite if $R = Af_1 + \dots + Af_n$ for some $f_1, \dots, f_n \in R$.

Lemma 1.26. *If $R \subseteq S$ is module-finite and N is a finitely generated S -module, then N is a finitely generated R -module by restriction of scalars. In particular, the composition of two module-finite ring maps is module-finite.*

Proof. Let $S = Ra_1 + \dots + Ra_r$ and $N = Sb_1 + \dots + Sb_s$. Then we claim that

$$N = \sum_{i=1}^r \sum_{j=1}^s Ra_i b_j.$$

Indeed, given $n = \sum_{j=1}^s s_j b_j$, rewrite each $s_j = \sum_{i=1}^r r_{ij} a_i$ and substitute to get

$$n = \sum_{i=1}^r \sum_{j=1}^s r_{ij} a_i b_j$$

as an R -linear combination of the $a_i b_j$. □

Remark 1.27. Let $A \subseteq B \subseteq C$ be rings. It follows from the definitions that

- $$\begin{array}{ccc} A \subseteq B \text{ module-finite} & & \\ \text{and} & \implies & A \subseteq C \text{ module-finite} \\ B \subseteq C \text{ module-finite} & & \end{array}$$
- $A \subseteq C \text{ module-finite} \implies B \subseteq C \text{ module-finite}.$

However, $A \subseteq C \text{ module-finite} \not\Rightarrow A \subseteq B \text{ module-finite}$. We will see examples in the next problem set.

1.4 Integral extensions

In field theory, there is a close relationship between (vector space-)finite field extensions and algebraic equations. The situation for rings is similar.

Definition 1.28 (Integral element/extension). Let R be an A -algebra. The element $r \in R$ is **integral** over A if there are elements $a_0, \dots, a_{n-1} \in A$ such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0;$$

i.e., r satisfies an **equation of integral dependence** over A .

We say that R is **integral over** A if every $r \in R$ is integral over A .

Integral automatically implies algebraic, but the condition that there exists an equation of algebraic dependence that is *monic* is stronger in the setting of rings.

Again, we can restrict our focus to inclusion maps $A \subseteq R$.

Remark 1.29. An element $r \in R$ is integral over A if and only if r is integral over the subring $\varphi(A) \subseteq R$, so we might as well assume that φ is injective.

Definition 1.30. Given an inclusion of rings $A \subseteq R$, the **integral closure** of A in R is the set of elements in R that are integral over A . The integral closure of a domain R in its field of fractions is usually denoted by \overline{R} . We say A is **integrally closed** in R if A is its own integral closure in R ; a **normal domain** is a domain R that is integrally closed in its field of fractions, meaning $R = \overline{R}$.

Example 1.31. The ring $\mathbb{Z}[\sqrt{d}]$, where $d \in \mathbb{Z}$ is not a perfect square, is integral over \mathbb{Z} . Indeed, \sqrt{d} satisfies the monic polynomial $r^2 - d$, and since the integral closure of \mathbb{Z} is a ring containing \mathbb{Z} and \sqrt{d} , every element in $\mathbb{Z}[\sqrt{d}]$ is integral over \mathbb{Z} .

Proposition 1.32. Let $A \subseteq R$ be rings.

- 1) If $r \in R$ is integral over A then $A[r]$ is module-finite over A .
- 2) If $r_1, \dots, r_t \in R$ are integral over A then $A[r_1, \dots, r_t]$ is module-finite over A .

Proof.

- 1) Suppose r is integral over A , and $r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$. Then we claim that $A[r] = A + Ar + \dots + Ar^{n-1}$. First, note that to show that any polynomial $p(r) \in A[r]$ is in $A + Ar + \dots + Ar^{n-1}$, it is enough to show that $r^m \in A + Ar + \dots + Ar^{n-1}$ for all m . Using induction on m , the base cases $1, r, \dots, r^{n-1} \in A + Ar + \dots + Ar^{n-1}$ are obvious. On the other hand, we can use induction to conclude that $r^m \in A + Ar + \dots + Ar^{n-1}$ for all $m \geq n$, since we can use the equation above to rewrite r^m as

$$r^m = r^{m-n}(a_{n-1}r^{n-1} + \dots + a_1r + a_0),$$

which has degree $m - 1$ in r .

- 2) Write

$$A_0 := A \subseteq A_1 := A[r_1] \subseteq A_2 := A[r_1, r_2] \subseteq \dots \subseteq A_t := A[r_1, \dots, r_t].$$

Note that r_i is integral over A_{i-1} , via the same monic equation of r_i over A . Then, the inclusion $A \subseteq A[r_1, \dots, r_t]$ is a composition of module-finite maps, and thus it is also module-finite. \square

The name “ring” is roughly based on this idea: in an extension as above, the powers wrap around (like a ring).

We will need a linear algebra fact. The **classical adjoint** of an $n \times n$ matrix $B = [b_{ij}]$ is the matrix $\text{adj}(B)$ with entries $\text{adj}(B)_{ij} = (-1)^{i+j} \det(\widehat{B}_{ji})$, where \widehat{B}_{ji} is the matrix obtained from B by deleting its j th row and i th column. You may remember this matrix from linear algebra.

Lemma 1.33 (Determinantal trick). *Let R be a ring, $B \in M_{n \times n}(R)$, $v \in R^{\oplus n}$, and $r \in R$.*

- 1) $\text{adj}(B)B = \det(B)I_{n \times n}$.
- 2) If $Bv = rv$, then $\det(rI_{n \times n} - B)v = 0$.

Proof.

- 1) When R is a field, this is a basic linear algebra fact. We deduce the case of a general ring from the field case.

The ring R is a \mathbb{Z} -algebra, so we can write R as a quotient of some polynomial ring $\mathbb{Z}[X]$. Let $\psi : \mathbb{Z}[X] \twoheadrightarrow R$ be a surjection, $a_{ij} \in \mathbb{Z}[X]$ be such that $\psi(a_{ij}) = b_{ij}$, and let $A = [a_{ij}]$. Note that

$$\psi(\text{adj}(A)_{ij}) = \text{adj}(B)_{ij} \quad \text{and} \quad \psi((\text{adj}(A)A)_{ij}) = (\text{adj}(B)B)_{ij},$$

since ψ is a homomorphism, and the entries are the same polynomial functions of the entries of the matrices A and B , respectively. Thus, it suffices to establish

$$\text{adj}(B)B = \det(B)I_{n \times n}$$

in the case when $R = \mathbb{Z}[X]$, and we can do this entry by entry. Now, $R = \mathbb{Z}[X]$ is an integral domain, hence a subring of a field (its fraction field). Since both sides of the equation

$$(\text{adj}(B)B)_{ij} = (\det(B)I_{n \times n})_{ij}$$

live in R and are equal in the fraction field (by linear algebra) they are equal in R . This holds for all i, j , and thus 1) holds.

- 2) We have $(rI_{n \times n} - B)v = 0$, so by part 1)

$$\det(rI_{n \times n} - B)v = \text{adj}(rI_{n \times n} - B)(rI_{n \times n} - B)v = 0. \quad \square$$

Theorem 1.34 (Module finite implies integral). *Let $A \subseteq R$ be module-finite. Then R is integral over A .*

Proof. Given $r \in R$, we want to show that r is integral over A . The idea is to show that multiplication by r , realized as a linear transformation over A , satisfies the characteristic polynomial of that linear transformation.

Write $R = Ar_1 + \cdots Ar_t$. We may assume that $r_1 = 1$, perhaps by adding module generators. By assumption, we can find $a_{ij} \in A$ such that

$$rr_i = \sum_{j=1}^t a_{ij}r_j$$

for each i . Let $C = [a_{ij}]$, and v be the column vector (r_1, \dots, r_t) . We have $rv = Cv$, so by the determinant trick, $\det(rI_{n \times n} - C)v = 0$. Since we chose one of the entries of v to be 1, we have in particular that $\det(rI_{n \times n} - C) = 0$. Expanding this determinant as a polynomial in r , this is a monic equation with coefficients in A . \square

Collecting the previous results, we now have a useful characterization of module-finite extensions:

Corollary 1.35 (Characterization of module-finite extensions). *Let $A \subseteq R$ be rings. R is module-finite over A if and only if R is integral and algebra-finite over A .*

Proof. (\Rightarrow): A generating set for R as an A -module serves as a generating set as an A -algebra. The remainder of this direction comes from the previous theorem. (\Leftarrow): If $R = A[r_1, \dots, r_t]$ is integral over A , so that each r_i is integral over A , then R is module-finite over A by Proposition 1.32. \square

Corollary 1.36. *If R is generated over A by integral elements, then R is integral. Thus, if $A \subseteq S$, the set of elements of S that are integral over A form a subring of S .*

Proof. Let $R = A[\Lambda]$, with λ integral over A for all $\lambda \in \Lambda$. Given $r \in R$, there is a finite subset $L \subseteq \Lambda$ such that $r \in A[L]$. By the theorem, $A[L]$ is module-finite over A , and $r \in A[L]$ is integral over A .

For the latter statement, the first statement implies that

$$\{\text{integral elements}\} \subseteq A[\{\text{integral elements}\}] \subseteq \{\text{integral elements}\},$$

so equality holds throughout, and $\{\text{integral elements}\}$ is a ring. \square

Definition 1.37. If $A \subseteq R$, the **integral closure of A in R** is the set of elements of R that are integral over A .

So the previous result says that the integral closure of A in R is a subring of R (containing A).

Example 1.38.

- 1) Let $R = \mathbb{C}[x, y] \subseteq S = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2)$. Then S is module-finite over R : indeed, S is generated over R as an algebra by one element, z , and z satisfies the monic equation $z^2 + x^2 + y^2 = 0$, so it is integral over R .
- 2) Not all integral extensions are module-finite. Consider

$$A = k[x] \subseteq R = k[x, x^{1/2}, x^{1/3}, x^{1/4}, x^{1/5}, \dots].$$

R is generated by integral elements over $k[x]$, but it is not algebra-finite over $k[x]$.

Finally, we can prove a technical sounding result that puts together all our finiteness conditions in a useful way. We will then be able to answer a classical question using this result.

Theorem 1.39 (Artin-Tate Lemma). *Let $A \subseteq B \subseteq C$ be rings. Assume that*

- *A is Noetherian,*
- *C is module-finite over B , and*
- *C is algebra-finite over A .*

Then, B is algebra-finite over A .

Proof. Let $C = A[f_1, \dots, f_r]$ and $C = Bg_1 + \dots + Bg_s$. Then,

$$f_i = \sum_j b_{ij}g_j \quad \text{and} \quad g_i g_j = \sum_k b_{ijk}g_k$$

for some $b_{ij}, b_{ijk} \in B$. Let $B_0 = A[\{b_{ij}, b_{ijk}\}] \subseteq B$. Since A is Noetherian, so is B_0 .

We claim that $C = B_0 g_1 + \dots + B_0 g_s$. Given an element $c \in C$, write c as a polynomial expression in f_1, \dots, f_r , and since the f_i are linearly combinations of the g_i , we can rewrite $c \in A[\{b_{ij}\}][g_1, \dots, g_s]$. Then using the equations for $g_i g_j$ we can write c in the form required.

Now, since B_0 is Noetherian, C is a finitely generated B_0 -module, and $B \subseteq C$, then B is a finitely generated B_0 -module, too. In particular, $B_0 \subseteq B$ is algebra-finite. We conclude that $A \subseteq B$ is algebra-finite, as required. \square

1.5 An application to invariant rings

Historically, commutative algebra has roots in classical questions of algebraic and geometric flavors, including the following natural question:

Question 1.40. Given a (finite) set of symmetries, consider the collection of polynomial functions that are fixed by all of those symmetries. Can we describe all the fixed polynomials in terms of finitely many of them?

To make this precise, let G be a group acting on a ring R , or just as well, a group of automorphisms of R . The main case we have in mind is when $R = k[x_1, \dots, x_d]$ and k is a field. We are interested in the set of elements that are *invariant* under the action,

$$R^G := \{r \in R \mid g(r) = r \text{ for all } g \in G\}.$$

Note that R^G is a subring of R . Indeed, given $r, s \in R^G$, then

$$r + s = g(r) + g(s) = g(r + s) \quad \text{and} \quad rs = g(r)g(s) = g(rs) \quad \text{for all } g \in G,$$

since each g is a homomorphism. Note also that if $G = \langle g_1, \dots, g_t \rangle$, then $r \in R^G$ if and only if $g_i(r) = r$ for $i = 1, \dots, t$. The question above can now be rephrased as follows:

Question 1.41. Given a finite group G acting on $R = k[x_1, \dots, x_d]$, is R^G a finitely generated k -algebra?

Proposition 1.42. *Let k be a field, R be a finitely-generated k -algebra, and G a finite group of automorphisms of R that fix k . Then $R^G \subseteq R$ is module-finite.*

Proof. Since integral implies module-finite, we will show that R is algebra-finite and integral over R^G .

First, since R is generated by a finite set as a k -algebra, and $k \subseteq R^G$, it is generated by the same finite set as an R^G -algebra as well. Extend the action of G on R to $R[t]$ with G fixing t . Now, for $r \in R$, consider the polynomial $F_r(t) = \prod_{g \in G} (t - g(r)) \in R[t]$. Then G fixes $F_r(t)$, since for each $h \in G$,

$$h(F_r(t)) = h \prod_{g \in G} (t - g(r)) = \prod_{g \in G} (h \cdot t - hg(r)) = F_r(t)$$

Thus, $F_r(t) \in (R[t])^G$. Notice that $(R[t])^G = R^G[t]$, since

$g(a_n t^n + \cdots + a_0) = a_n t^n + \cdots + a_0 \implies (g \cdot a_n) t^n + \cdots + (g \cdot a_0) = a_n t^n + \cdots + a_0$. Therefore, $F_r(t) \in R^G[t]$. The leading term (with respect to t) of $F_r(t)$ is $t^{|G|}$, so $F_r(t)$ is monic, and r is integral over R^G . Therefore, R is integral over R^G . \square

Theorem 1.43 (Noether's finiteness theorem for invariants of finite groups). *Let k be a field, R be a polynomial ring over k , and G be a finite group acting k -linearly on R . Then R^G is a finitely generated k -algebra.*

Proof. Observe that $k \subseteq R^G \subseteq R$, that k is Noetherian, $k \subseteq R$ is algebra-finite, and $R^G \subseteq R$ is module-finite. Thus, by the Artin-Tate Lemma, we are done! \square

Chapter summary

- R is a Noetherian ring \iff every ideal I in R is Noetherian
- M is a Noetherian R -module $\xrightleftharpoons[R \text{ Noeth}]{\text{general}}$ M is a finitely generated R -module

$A \subseteq R$ extension of rings:

- $A \subseteq R$ module-finite $\iff R = Af_1 + \cdots + Af_n$ for some $f_i \in R \iff R \cong A^n/N$ $N \subseteq A^n$ submod
- $A \subseteq R$ algebra-finite $\iff R = A[f_1, \dots, f_n]$ for some $f_i \in R \iff R \cong A[x_1, \dots, x_i]/I$ x_i indeterminates
- $A \subseteq R$ algebra-finite $\iff R = A[f_1, \dots, f_n], f_i \in R$
- $A \subseteq R$ algebra-finite, A Noetherian $\implies R$ Noetherian ring
- $A \subseteq R$ module-finite $\iff \begin{cases} \text{algebra-finite} \\ \text{and integral} \end{cases} \not\iff \text{module-finite}$

$$\begin{array}{c} \text{Artin-Tate} \\ \text{Lemma:} \end{array} \quad \underbrace{\begin{array}{c} A \subseteq B \subseteq C \\ \text{Noeth} \quad \text{mod-fin} \end{array}}_{\text{alg-fin}}$$

Chapter 2

Graded rings

January 20

2.1 Graded rings

When we think of a polynomial ring R , we often think of R with its graded structure, even if we have never formalized what that means. Other rings we have seen also have a graded structure, and this structure is actually very powerful.

Definition 2.1. A ring R is **\mathbb{N} -graded** if we can write a direct sum decomposition of R as an abelian group indexed by \mathbb{N} ¹

$$R = \bigoplus_{a \geq 0} R_a,$$

where $R_a R_b \subseteq R_{a+b}$ for every $a, b \in \mathbb{N}$, meaning that for any $r \in R_a$ and $s \in R_b$, we have $rs \in R_{a+b}$. More generally, given a monoid T . The ring R is **T -graded** if there exists a direct sum decomposition of R as an abelian group indexed by T :

$$R = \bigoplus_{a \in T} R_a$$

satisfying $R_a R_b \subseteq R_{a+b}$.

An element that lies in one of the summands R_a is said to be **homogeneous of degree a** ; we write $|r|$ or $\deg(r)$ to denote the degree of a homogeneous element r .

By definition, an element in a graded ring is a *unique* sum of homogeneous elements, which we call its **homogeneous components** or **graded components**. One nice thing about graded rings is that many properties can usually be sufficiently checked on homogeneous elements, and these are often easier to deal with.

Remark 2.2. Note that whenever R is a graded ring, the multiplicative identity 1 must be a homogeneous element whose degree is the identity in T . In particular, if R is \mathbb{N} or \mathbb{Z} -graded, then $1 \in R_0$ and R_0 is a subring of R .

¹We follow the convention that 0 is a natural number.

Example 2.3.

- a) Any ring R is trivially an \mathbb{N} -graded ring, by setting $R_0 = R$ and $R_n = 0$ for $n \neq 0$.
- b) If k is a field and $R = k[x_1, \dots, x_n]$ is a polynomial ring, there is an \mathbb{N} -grading on R called the *standard grading* where R_d is the k -vector space with basis given by the monomials of total degree d , meaning those of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\sum_i \alpha_i = d$. Of course, this is the notion of degree familiar from middle school. So $x_1^2 + x_2x_3$ is homogeneous in the standard grading, while $x_1^2 + x_2$ is not.
- c) If k is a field, and $R = k[x_1, \dots, x_n]$ is a polynomial ring, we can give different \mathbb{N} -gradings on R by fixing some tuple $(\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ and letting x_i be a homogeneous element of degree β_i ; we call this a grading with *weights* $(\beta_1, \dots, \beta_n)$. For example, in $k[x_1, x_2]$, $x_1^2 + x_2^3$ is not homogeneous in the standard grading, but it is homogeneous of degree 6 under the \mathbb{N} -grading with weights $(3, 2)$.
- d) A polynomial ring $R = k[x_1, \dots, x_n]$ also admits a natural \mathbb{N}^n -grading, with $R_{(d_1, \dots, d_n)} = k \cdot x_1^{d_1} \cdots x_n^{d_n}$. This is called the *fine grading*.
- e) Let $\Gamma \subseteq \mathbb{N}^n$ be a subsemigroup of \mathbb{N}^n . Then

$$\bigoplus_{\gamma \in \Gamma} k \cdot \underline{x}^\gamma \subseteq k[\underline{x}] = k[x_1, \dots, x_n]$$

is an \mathbb{N}^n -graded subring of $k[x_1, \dots, x_n]$. Conversely, every \mathbb{N}^n -graded subring of $k[x_1, \dots, x_n]$ is of this form.

- f) Polynomial rings in Macaulay2 are graded with the standard grading by default. To define a different grading, we give Macaulay2 a list with the grading of each of the variables:

```
i1 : R = ZZ/101[a,b,c,Degrees=>{{1,2},{2,1},{1,0}}];
```

We can check whether an element of R is homogeneous, and the function `degree` applied to an element of R returns the least upper bound of the degrees of its monomials:

```
i2 : degree (a+b)
```

```
o2 = {2, 2}
```

```
o2 : List
```

```
i3 : isHomogeneous(a+b)
```

```
o3 = false
```

Remark 2.4. You may have seen the term *homogeneous polynomial* used to refer to a polynomial $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ that satisfies

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n)$$

for some d . This is equivalent to saying that all the terms in f have the same total degree, or that f is homogeneous with respect to the standard grading.

Similarly, a polynomial is *quasi-homogeneous*, or *weighted homogeneous*, if there exist integers w_1, \dots, w_n such that the sum $w = a_1 w_1 + \dots + a_n w_n$ is the same for all monomials $x_1^{a_1} \dots x_n^{a_n}$ appearing in f . So f satisfies

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^w f(x_1, \dots, x_n),$$

and $f(x_1^{w_1}, \dots, x_n^{w_n})$ is homogeneous (in the previous sense, so with respect to the standard grading). This condition is equivalent to asking that f be homogeneous with respect to some weighted grading on $k[x_1, \dots, x_n]$.

Definition 2.5. An ideal I in a graded ring R is called *homogeneous* if it can be generated by homogeneous elements.

Remark 2.6. Observe that an ideal is homogeneous if and only if I has the following property: for any element $f \in R$ we have $f \in I$ if and only if every homogeneous component of f lies in I . We can repackage this by saying that I is homogeneous if

$$I = \bigoplus_{a \in T} I_a,$$

where $I_a = I \cap R_a$.

Indeed, if I has this property, take a generating set $\{f_\lambda\}_\Lambda$ for I ; by assumption, all of the homogeneous components of each f_λ lie in I , and since each f_λ lies in the ideal generated by these components, the set of all the components generates I , and I is homogeneous. On the other hand, if all the components of f lie in I then so does f , whether or not I is homogeneous. If I is homogeneous and $f \in I$, write f as a combination of the (homogeneous) generators of I , say f_1, \dots, f_n :

$$f = r_1 f_1 + \dots + r_n f_n.$$

Now by writing each r_i as a sum of its components, say $r_i = r_{i,1} + \dots + r_{i,n_i}$, each $r_{i,j} f_i \in I$, and these contain all the components of f (and potentially some redundant terms).

Example 2.7. Given an \mathbb{N} -graded ring R , then $R_+ = \bigoplus_{d>0} R_d$ is a homogeneous ideal.

We now observe the following:

Lemma 2.8. *Let R be an T -graded ring, and I be a homogeneous ideal. Then R/I has a natural T -graded structure induced by the T -graded structure on R .*

Proof. The ideal I decomposes as the direct sum of its graded components, so we can write

$$R/I = \frac{\bigoplus R_a}{\bigoplus I_a} \cong \bigoplus \frac{R_a}{I_a}. \quad \square$$

Example 2.9.

- a) The ideal $I = (w^2x + wyz + z^3, x^2 + 3xy + 5xz + 7yz + 11z^2)$ in $R = k[w, x, y, z]$ is homogeneous with respect to the standard grading on R , and thus the ring R/I admits an \mathbb{N} -grading with $|w| = |x| = |y| = |z| = 1$.
- b) In contrast, the ring $R = k[x, y, z]/(x^2 + y^3 + z^5)$ does not admit a grading with $|x| = |y| = |z| = 1$, but does admit a grading with $|x| = 15, |y| = 10, |z| = 6$.

Definition 2.10. Let R be a T -graded ring, and M an R -module. The module M is **T -graded** if there exists a direct sum decomposition of M as an abelian group indexed by T :

$$M = \bigoplus_{a \in T} M_a \text{ such that } R_a M_b \subseteq M_{a+b}$$

for all $a, b \in T$.

The notions of homogeneous element of a module and degree of a homogeneous element of a module take the obvious meanings. A notable abuse of notation: we will often talk about \mathbb{Z} -graded modules over \mathbb{N} -graded rings, and likewise.

We can also talk about graded homomorphisms.

Definition 2.11. Let R and S be T -graded rings with the same grading monoid T . A ring homomorphism $\varphi : R \rightarrow S$ is **graded** or **degree-preserving** if $\varphi(R_a) \subseteq S_a$ for all $a \in T$.

Note that our definition of ring homomorphism requires $1_R \mapsto 1_S$, and thus it does not make sense to talk about graded ring homomorphisms of degree $d \neq 0$. But we can have graded module homomorphisms of any degree.

Definition 2.12. Let M and N be \mathbb{Z} -graded modules over the \mathbb{N} -graded ring R . A homomorphism of R -modules $\varphi : M \rightarrow N$ is **graded** if $\varphi(M_a) \subseteq N_{a+d}$ for all $a \in \mathbb{Z}$ and some fixed $d \in \mathbb{Z}$, called the **degree** of φ . A graded homomorphism of degree 0 is also called **degree-preserving**.

Example 2.13.

- a) Consider the ring map $k[x, y, z] \rightarrow k[s, t]$ given by $x \mapsto s^2, y \mapsto st, z \mapsto t^2$. If $k[s, t]$ has the fine grading, meaning $|s| = (1, 0)$ and $|t| = (0, 1)$, then the given map is degree preserving if and only if $k[x, y, z]$ is graded by

$$|x| = (2, 0), |y| = (1, 1), |z| = (0, 2).$$

- b) Let k be a field, and let $R = k[x_1, \dots, x_n]$ be a polynomial ring with the standard grading. Given $c \in k = R_0$, the homomorphism of R -modules $R \rightarrow R$ given by $f \mapsto cf$ is degree preserving. However, if instead we take $g \in k = R_d$ for some $d > 0$, then the map

$$\begin{aligned} R &\longrightarrow R \\ f &\longmapsto gf \end{aligned}$$

is not degree preserving, although it is a graded map of degree d . We can make this a degree-preserving map if we shift the grading on R by defining $R(-d)$ to be the R -module R but with the \mathbb{Z} -grading given by $R(-d)_t = R_{t-d}$. With this grading, the component of degree d of $R(-d)$ is $R(-d)_d = R_0 = k$. Now the map

$$\begin{aligned} R(-d) &\longrightarrow R \\ f &\longmapsto gf \end{aligned}$$

is degree preserving.

We observed earlier an important relationship between algebra-finiteness and Noetherianity that followed from the Hilbert basis theorem: if R is Noetherian, then any algebra-finite extension of R is also Noetherian. There isn't a converse to this in general: there are lots of algebras over fields K that are Noetherian but not algebra-finite over K . However, for graded rings, this converse relation holds.

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Proposition 2.14. *Let R be an \mathbb{N} -graded ring, and consider homogeneous elements $f_1, \dots, f_n \in R$ of positive degree. Then f_1, \dots, f_n generate the ideal $R_+ := \bigoplus_{d>0} R_d$ if and only if f_1, \dots, f_n generate R as an R_0 -algebra.*

Therefore, an \mathbb{N} -graded ring R is Noetherian if and only if R_0 is Noetherian and R is algebra-finite over R_0 .

Proof. If $R = R_0[f_1, \dots, f_n]$, then any element $r \in R_+$ can be written as a polynomial expression $r = P(f_1, \dots, f_n)$ for some $P \in R_0[x]$ with no constant term. Each monomial of P is a multiple of some x_i , and thus $r \in (f_1, \dots, f_n)$.

To show that $R_+ = (f_1, \dots, f_n)$ implies $R = R_0[f_1, \dots, f_n]$, it suffices to show that any homogeneous element $r \in R$ can be written as a polynomial expression in the f 's with coefficients in R_0 . We induce on the degree of r , with degree 0 as a trivial base case. For r homogeneous of positive degree, we must have $r \in R_+$, so by assumption we can write $r = a_1 f_1 + \dots + a_n f_n$; moreover, since r and f_1, \dots, f_n are all homogeneous, we can choose each coefficient a_i to be homogeneous of degree $|r| - |f_i|$. By the induction hypothesis, each a_i is a polynomial expression in the f 's, so we are done.

For the final statement, if R_0 is Noetherian and R algebra-finite over R_0 , then R is Noetherian by the Hilbert Basis Theorem. If R is Noetherian, then $R_0 \cong R/R_+$ is Noetherian. Moreover, R is algebra-finite over R_0 since R_+ is generated as an ideal by finitely many homogeneous elements by Noetherianity, so by the first statement, we get a finite algebra generating set for R over R_0 . \square

There are many interesting examples of \mathbb{N} -graded algebras with $R_0 = k$; in that case, R_+ is the largest homogeneous ideal in R . In fact, R_0 is the only maximal ideal of R that is also homogeneous, so we can call it *the homogeneous maximal ideal*; it is sometimes also called the **irrelevant maximal ideal** of R . This ideal plays a very

important role — in many ways, R and R_+ behave similarly to a local ring R and its unique maximal ideal. We will discuss this further when we learn about local rings.

2.2 Another application to invariant rings

If R is a graded ring, and G is a group acting on R by degree-preserving automorphisms, then R^G is a graded subring of R , meaning R^G is graded with respect to the same grading monoid. In particular, if G acts k -linearly on a polynomial ring over k , the invariant ring is \mathbb{N} -graded.

Using this perspective, we can now give a different proof of the finite generation of invariant rings that works under different hypotheses. The proof we will discuss now is essentially Hilbert's proof. To do that, we need another notion that is very useful in commutative algebra.

Definition 2.15. Let S be an R -algebra corresponding to the ring homomorphism $\varphi : R \rightarrow S$. We say that R is a **direct summand** of S if the map φ **splits** as a map of R -modules, meaning there is an R -module homomorphism

$$\begin{array}{ccc} & \pi & \\ \swarrow & & \searrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

such that $\pi\varphi$ is the identity on R .

First, observe that the condition on π implies that φ must be injective, so we can assume that $R \subseteq S$, perhaps after renaming elements. Then the condition on π is that $\pi(rs) = r\pi(s)$ for all $r \in R$ and $s \in S$ and that $\pi|_R$ is the identity. We call the map π the *splitting* of the inclusion. Note that given any R -linear map $\pi : S \rightarrow R$, if $\pi(1) = 1$ then π is a splitting: indeed, $\pi(R) = \pi(r \cdot 1) = r\pi(1) = r$ for all $r \in R$.

Being a direct summand is really nice, since many good properties of S pass onto its direct summands.

Lemma 2.16. *Let R be a direct summand of S . Then, for any ideal $I \subseteq R$, we have $IS \cap R = I$.*

Proof. Let π be the corresponding splitting. Clearly, $I \subseteq IS \cap R$. Conversely, if $r \in IS \cap R$, we can write $r = s_1 f_1 + \cdots + s_t f_t$ for some $f_i \in I$, $s_i \in S$. Applying π , we have

$$r = \pi(r) = \pi\left(\sum_{i=1}^t s_i f_i\right) = \sum_{i=1}^t \pi(s_i f_i) = \sum_{i=1}^t \pi(s_i) f_i \in I.$$

□

Proposition 2.17. *Let R be a direct summand of S . If S is Noetherian, then so is R .*

Proof. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be a chain of ideals in R . The chain of ideals in S

$$I_1 S \subseteq I_2 S \subseteq I_3 S \subseteq \cdots$$

stabilizes, so there exist J, N such that $I_n R = J$ for $n \geq N$. Contracting to R , we get that $I_n = I_n S \cap R = J \cap R$ for $n \geq N$, so the original chain also stabilizes. \square

Proposition 2.18. *Let k be a field, and R be a polynomial ring over k . Let G be a finite group acting k -linearly on R . Assume that the characteristic of k does not divide $|G|$. Then R^G is a direct summand of R .*

Remark 2.19. The condition that the characteristic of k does not divide the order of G is trivially satisfied if k has characteristic zero.

Proof. We consider the map $\rho : R \rightarrow R^G$ given by

$$\rho(r) = \frac{1}{|G|} \sum_{g \in G} g \cdot r.$$

First, note that the image of this map lies in R^G , since acting by g just permutes the elements in the sum, so the sum itself remains the same. We claim that this map ρ is a splitting for the inclusion $R^G \subseteq R$. To see that, let $s \in R^G$ and $r \in R$. We have

$$\rho(sr) = \frac{1}{|G|} \sum_{g \in G} g \cdot (sr) = \frac{1}{|G|} \sum_{g \in G} (g \cdot s)(g \cdot r) = \frac{1}{|G|} \sum_{g \in G} s(g \cdot r) = s \frac{1}{|G|} \sum_{g \in G} (g \cdot r) = s\rho(r),$$

so ρ is R^G -linear, and for $s \in R^G$,

$$\rho(s) = \frac{1}{|G|} \sum_{g \in G} g \cdot s = s.$$

\square

Theorem 2.20 (Hilbert's finiteness theorem for invariants). *Let k be a field, and R be a polynomial ring over k . Let G be a group acting k -linearly on R . Assume that G is finite and $|G|$ does not divide the characteristic of k , or more generally, that R^G is a direct summand of R . Then R^G is a finitely generated k -algebra.*

Proof. Since G acts linearly, R^G is an \mathbb{N} -graded subring of R with $R_0 = k$. Since R^G is a direct summand of R , R^G is Noetherian by Proposition 2.17. By our characterization of Noetherian graded rings, R^G is finitely generated over $R_0 = k$. \square

One important thing about this proof is that it applies to many infinite groups. In particular, for any *linearly reductive group*, including $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, and $(\mathbb{C}^\times)^n$, we can construct a splitting map ρ .

Chapter 3

Algebraic Geometry

Colloquially, we often identify systems of equations with their solution sets. We will make this correspondence more precise for systems of polynomial equations, and develop the beginning of a rich dictionary between algebraic and geometric objects.

Question 3.1. Let k be a field. To what extent is a system of polynomial equations

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_t = 0 \end{cases}$$

where polynomials $f_1, \dots, f_t \in k[x_1, \dots, x_d]$, determined by its solution set?

Let's consider one polynomial equation in one variable. Over \mathbb{R}, \mathbb{Q} , or other fields that are not algebraically closed, there are many polynomials with an empty solution set; for example, $z^2 + 1$ has an empty solution set over \mathbb{R} . On the other hand, over \mathbb{C} , or any algebraically closed field, if a_1, \dots, a_d are the solutions to $f(z) = 0$, we know that we can write f in the form $f(z) = \alpha(z - a_1)^{n_1} \cdots (z - a_d)^{n_d}$, so f is completely determined up to scalar multiple and repeated factors. If we insist that f have no repeated factors, then (f) is uniquely determined.

More generally, given any system of polynomial equations

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_t = 0 \end{cases}$$

where $f_i \in k[z]$ for some field k , notice that that $z = a$ is a solution to the system if and only if it is a solution for any polynomial $g \in (f_1, \dots, f_t)$. But since $k[z]$ is a PID, we have $(f_1, \dots, f_t) = (f)$, where f is a greatest common divisor of f_1, \dots, f_t . Therefore, $z = a$ is a solution to the system if and only if $f(a) = 0$.

3.1 Varieties

Definition 3.2. Given a field k , the *affine d -space over k* , denoted \mathbb{A}_k^d , is the set

$$\mathbb{A}_k^d = \{(a_1, \dots, a_d) \mid a_i \in k\}.$$

Definition 3.3. For a subset T of $k[x_1, \dots, x_d]$, we define $\mathcal{Z}(T) \subseteq \mathbb{A}_k^d$ to be the set of common zeros or the *zero set* of the polynomials (equations) in T :

$$\mathcal{Z}(T) = \{(a_1, \dots, a_d) \in \mathbb{A}_k^d \mid f(a_1, \dots, a_d) = 0 \text{ for all } f \in T\}.$$

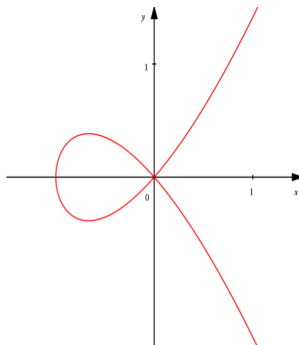
Sometimes, in order to emphasize the role of k , we will write this as $\mathcal{Z}_k(T)$.

A subset of \mathbb{A}_k^d of the form $\mathcal{Z}(T)$ for some subset T is called an **algebraic subset** of \mathbb{A}_k^d , or an **affine algebraic variety**. So a variety \mathbb{A}_k^d is the set of common solutions of some (possibly infinite) collection of polynomial equations. A variety is **irreducible** if it cannot be written as the union of two proper varieties.

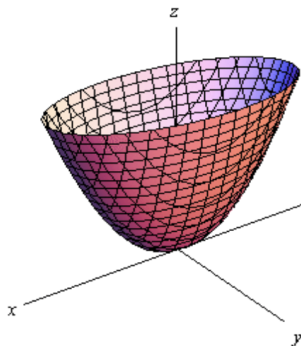
Note that some authors use the word *variety* to refer only to irreducible algebraic sets. Note also that the definitions given here are only completely standard when k is algebraically closed.

Example 3.4. Here are some simple examples of algebraic varieties:

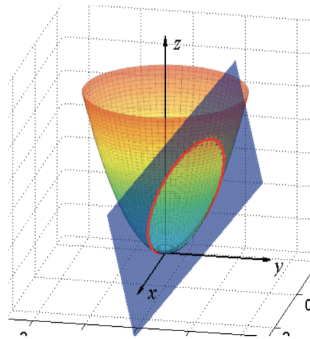
- a) For $k = \mathbb{R}$ and $n = 2$, $\mathcal{Z}(y^2 + x^2(x - 1))$ is a “nodal curve” in $\mathbb{A}_{\mathbb{R}}^2$, the real plane. Note that we’ve written x for x_1 and y for x_2 here.



- b) For $k = \mathbb{R}$ and $n = 3$, $\mathcal{Z}(z - x^2 - y^2)$ is a paraboloid in $\mathbb{A}_{\mathbb{R}}^3$, real three space.



- c) For $k = \mathbb{R}$ and $n = 3$, $\mathcal{Z}(z - x^2 - y^2, 3x - 2y + 7z - 7)$ is circle in $\mathbb{A}_{\mathbb{R}}^3$.



- d) For $k = \mathbb{R}$, $\mathcal{Z}_{\mathbb{R}}(x^2 + y^2 + 1) = \emptyset$. Note that $\mathcal{Z}_{\mathbb{C}}(x^2 + y^2 + 1) \neq \emptyset$.
 e) The subset $\mathbb{A}_k^2 \setminus \{(0, 0)\}$ is not an algebraic subset of \mathbb{A}_k^2 if k is infinite. Why?
 f) The graph of the sine function is not an algebraic subset of $\mathbb{A}_{\mathbb{R}}^2$. Why not?
 g) For $k = \mathbb{R}$, $\mathcal{Z}(y - x^2, xz - y^2, z - xy)$ is the so-called **twisted cubic (affine) curve**. It is the curve parametrized by (t, t^2, t^3) , meaning it is the image of the map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (t, t^2, t^3) \end{aligned}$$

We can check this with Macaulay2:

```
i1 : k = RR;

i2 : R = k[x,y,z];

i3 : f = map(k[t], R, {t, t^2, t^3});

i4 : ker f
      2          2
o4 = ideal (y  - x*z, x*y - z, x  - y)

o4 : Ideal of R
```

So in our computation above, f sets $x = t$, $y = t^2$, and $z = t^3$, and its kernel consists precisely of the polynomials that vanish at every point of this form. Note that computations over the reals in Macaulay2 are experimental, and yet we obtain the correct answer; we can also run the same computation over $k = \mathbb{Q}$.

- h) For any field k and elements $a_1, \dots, a_d \in k$, we have

$$\mathcal{Z}(x_1 - a_1, \dots, x_d - a_d) = \{(a_1, \dots, a_d)\}.$$

So, all one element subsets of \mathbb{A}_k^d are algebraic subsets.

We can consider the equations that a subset of affine space satisfies.

Definition 3.5. Given any subset X of \mathbb{A}_k^d for a field k , define

$$\mathcal{I}(X) = \{g(x_1, \dots, x_d) \in k[x_1, \dots, x_d] \mid g(a_1, \dots, a_d) = 0 \text{ for all } (a_1, \dots, a_d) \in X\}.$$

Exercise 5. $\mathcal{I}(X)$ is an ideal in $k[x_1, \dots, x_d]$ for any $X \subseteq \mathbb{A}_k^d$.

Example 3.6.

- a) $\mathcal{I}(\{(a_1, \dots, a_d)\}) = (x_1 - a_1, \dots, x_d - a_d)$, for any field k .
- b) $\mathcal{I}(\text{graph of the sine function in } \mathbb{A}_{\mathbb{R}}^2) = (0)$.

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Exercise 6. Here are some properties of the functions \mathcal{Z} and \mathcal{I} :

- a) For any field, we have $\mathcal{Z}(0) = \mathbb{A}_k^n$ and $\mathcal{Z}(1) = \emptyset$.
- b) $\mathcal{I}(\emptyset) = (1) = k[x_1, \dots, x_d]$ (the improper ideal).
- c) $\mathcal{I}(\mathbb{A}_k^d) = (0)$ if and only if k is infinite.
- d) If $I \subseteq J \subseteq k[x_1, \dots, x_d]$ then $\mathcal{Z}(I) \supseteq \mathcal{Z}(J)$.
- e) If $S \subseteq T$ are subsets of \mathbb{A}_k^n then $\mathcal{I}(S) \supseteq \mathcal{I}(T)$.
- f) If $I = (T)$ is the ideal generated by the elements of $T \subseteq k[x_1, \dots, x_d]$, then $\mathcal{Z}(T) = \mathcal{Z}(I)$.

So we will talk about the solution set of an ideal, rather than of an arbitrary set. Hilbert's Basis Theorem implies that every ideal in $k[x_1, \dots, x_d]$ is finitely generated, so any system of equations in $k[x_1, \dots, x_d]$ can be replaced with a system of *finitely many* equations.

Example 3.7. Let

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

be a 2×3 matrix of variables — we usually call these *generic* matrices — and let

$$R = k[X] = k \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Let $\Delta_1, \Delta_2, \Delta_3$ the 2×2 -minors of X . Consider the ideal $I = (\Delta_1, \Delta_2, \Delta_3)$. Thinking of these generators as equations, a solution to the system corresponds to a choice of 2×3 matrix whose 2×2 minors all vanish — that is, a matrix of rank at most one. So $\mathcal{Z}(I)$ is the set of rank at most one matrices. Note that $I \subseteq (x_1, x_2, x_3) =: J$, and $\mathcal{Z}(J)$ is the set of 2×3 matrices with top row zero. The containment $\mathcal{Z}(J) \subseteq \mathcal{Z}(I)$ we obtain from $I \subseteq J$ translates to the fact that a 2×3 matrix with a zero row has rank at most 1.

Finally, the union and intersection of varieties is also a variety.

Exercise 7. Suppose that I and J are ideals in $k[x_1, \dots, x_d]$.

- a) $\mathcal{Z}(I) \cap \mathcal{Z}(J) = \mathcal{Z}(I + J)$.
- b) $\mathcal{Z}(I) \cup \mathcal{Z}(J) = \mathcal{Z}(I \cap J) = \mathcal{Z}(IJ)$.

However, note that in general $IJ \neq I \cap J$.

3.2 Prime and maximal ideals

Before we talk more about geometry, let's recall some basic facts about prime and maximal ideals. As we will discover through the rest of the course, prime ideals play a very prominent role in commutative algebra.

Definition 3.8. An ideal $P \neq R$ is **prime** if $ab \in P$ implies $a \in P$ or $b \in P$.

Exercise 8. An ideal P in a ring R is prime if and only if R/P is a domain.

Example 3.9. The prime ideals in \mathbb{Z} are those of the form (p) for p a prime integer, and (0) .

Example 3.10. When k is a field, in $k[x]$ are easy to describe: $k[x]$ is a principal ideal domain, and $(f) \neq 0$ is prime if and only if f is an irreducible polynomial. Moreover, (0) is also a prime ideal, since $k[x]$ is a domain.

The prime ideals in $k[x_1, \dots, x_d]$ are, however, not so easy to describe. We will see many examples throughout the course; here are some.

Example 3.11. The ideal $P = (x^3 - y^2)$ in $R = k[x, y]$ is prime; one can show that $R/P \cong k[t^2, t^3] \subseteq k[t]$, which is a domain.

Example 3.12. The k -algebra $R = k[s^3, s^2t, st^2, t^2] \subseteq k[s, t]$ is a domain, so its defining ideal I in $k[x_1, x_2, x_3, x_4]$ is prime. This is the kernel of the presentation of R sending x_1, x_2, x_3, x_4 to each of our 4 algebra generators, which we can compute with Macaulay2:

```

1 : k = QQ;

i2 : f = map(k[s,t], k[x_1 .. x_4], {s^3, s^2*t, s*t^2, t^3})
           3      2      2      3
o2 = map(QQ[s..t], QQ[x .. x ], {s , s t, s*t , t })
           1      4

o2 : RingMap QQ[s..t] <--- QQ[x .. x ]
           1      4

i3 : I = ker f

```

```

      2                2
o3 = ideal (x  - x x , x x  - x x , x  - x x )
           3      2 4    2 3      1 4    2      1 3

o3 : Ideal of QQ[x ..x]
      1      4

```

Later we will show that prime ideals correspond to irreducible varieties; more precisely, that X is irreducible if and only if $\mathcal{I}(X)$ is prime.

Definition 3.13 (maximal ideal). An ideal \mathfrak{m} in R is **maximal** if for any ideal I

$$I \supseteq \mathfrak{m} \implies I = \mathfrak{m} \text{ or } I = R.$$

Exercise 9. An ideal \mathfrak{m} in R is maximal if and only if R/\mathfrak{m} is a field.

Given a maximal ideal \mathfrak{m} in R , the **residue field** of \mathfrak{m} is the field R/\mathfrak{m} . A field k is a residue field of R if $k \cong R/\mathfrak{m}$ for some maximal ideal \mathfrak{m} .

Remark 3.14. A ring may have many different residue fields. For example, the residue fields of \mathbb{Z} are all the finite fields with prime many elements, $\mathbb{F}_p \cong \mathbb{Z}/p$.

Exercise 10. Every maximal ideal is prime.

However, not every prime ideal is maximal. For example, in \mathbb{Z} , (0) is a prime ideal that is not maximal.

Theorem 3.15. *Given a ring R , every proper ideal $I \neq R$ is contained in some maximal ideal.*

Fun fact: this is actually *equivalent* to the Axiom of Choice. We will prove it (but not its equivalence to the Axiom of Choice!) using Zorn's Lemma, another equivalent version of the Axiom of Choice. Zorn's Lemma says that

Every non-empty partially ordered set in which every chain (i.e., totally ordered subset) has an upper bound contains at least one maximal element.

So let's prove that every ideal is contained in some maximal ideal.

Proof. First, we will show that Zorn's Lemma applies to proper ideals in any ring R . The statement will then follow by applying Zorn's Lemma to the non-empty set of ideals $J \supseteq I$, which is partially ordered by inclusion.

So consider a chain of proper ideals in R , say $\{I_i\}_i$. Now $I = \bigcup_i I_i$ is an ideal as well, and $I \neq R$ since $1 \notin I_i$ for all i . Note that unions of ideals are not ideals in general, but a union of totally ordered ideals *is* an ideal. Then I is an upper bound for our chain $\{I_i\}_i$, and Zorn's Lemma applies to the set of proper ideals in R with inclusion \subseteq . □

3.3 Nullstellensatz

Lemma 3.16. *Let k be a field, and $R = k[x_1, \dots, x_d]$ be a polynomial ring. There is a bijection*

$$\begin{aligned} \mathbb{A}_k^d &\longrightarrow \left\{ \begin{array}{l} \text{maximal ideals } \mathfrak{m} \text{ of } R \\ \text{with } R/\mathfrak{m} \cong k \end{array} \right\} \\ (a_1, \dots, a_d) &\longmapsto (x_1 - a_1, \dots, x_d - a_d) \end{aligned}$$

Proof. Each $\mathfrak{m} = (x_1 - a_1, \dots, x_d - a_d)$ is a maximal ideal satisfying $R/\mathfrak{m} \cong k$. Moreover, these ideals are distinct: if $x_i - a_i, x_i - a'_i$ are in the same ideal for $a_i \neq a'_i$, then the unit $a_i - a'_i$ is in the ideal, so it is not proper. Therefore, our map is injective. To see that it is surjective, let \mathfrak{m} be a maximal ideal with $R/\mathfrak{m} \cong k$. Each class in R/\mathfrak{m} corresponds to a unique $a \in k$, so in particular each x_i is in the class of a unique $a_i \in k$. This means that $x_i - a_i \in \mathfrak{m}$, and thus $(x_1 - a_1, \dots, x_d - a_d) \subseteq \mathfrak{m}$. Since $(x_1 - a_1, \dots, x_d - a_d)$ is a maximal ideal, we must have $(x_1 - a_1, \dots, x_d - a_d) = \mathfrak{m}$. \square

Example 3.17. Not all maximal ideals in $k[x_1, \dots, x_d]$ are necessarily of this form. For example, if $k = \mathbb{R}$ and $d = 1$, the ideal $(x^2 + 1)$ is maximal, but

$$k[x]/(x^2 + 1) \cong \mathbb{C} \not\cong k.$$

But this won't happen if k is algebraically closed.

Theorem 3.18 (Zariski's Lemma). *Consider an extension of fields $k \subseteq L$. If L is a finitely generated k -algebra, then L is a finite dimensional k -vector space. In particular, if k is algebraically closed then $L = k$.*

This is a nice application of the Artin-Tate Lemma, together with some facts about transcendent elements. We will skip the proof, but you can find it in [Jeffries' notes](#).

Corollary 3.19 (Nullstellensatz). *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over an algebraically closed field k . There is a bijection*

$$\begin{aligned} \mathbb{A}_k^d &\longrightarrow \{ \text{maximal ideals } \mathfrak{m} \text{ of } S \} \\ (a_1, \dots, a_d) &\longmapsto (x_1 - a_1, \dots, x_d - a_d) \end{aligned}$$

If R is a finitely generated k -algebra, we can write $R = S/I$ for a polynomial ring S , and there is an induced bijection

$$\mathcal{Z}_k(I) \subseteq \mathbb{A}_k^d \longleftrightarrow \{ \text{maximal ideals } \mathfrak{m} \text{ of } R \}.$$

Proof. The first part follows immediately from Lemma 3.16 and Lemma 3.18.

To show the second statement, fix an ideal I in S , and $R = S/I$. The maximal ideal ideals in R are in bijection with the maximal ideals \mathfrak{m} in S that contain I ; those are the ideals of the form $(x_1 - a_1, \dots, x_d - a_d)$ with $I \subseteq (x_1 - a_1, \dots, x_d - a_d)$. These are in bijection with the points $(a_1, \dots, a_d) \in \mathbb{A}_k^d$ satisfying $(a_1, \dots, a_d) \in \mathcal{Z}_k(I)$. \square

Theorem 3.20 (Weak Nullstellensatz). *Let k be an algebraically closed field. If I is a proper ideal in $R = k[x_1, \dots, x_d]$, then $\mathcal{Z}_k(I) \neq \emptyset$.*

Proof. If $I \subseteq R$ is a proper ideal, there is a maximal ideal $\mathfrak{m} \supseteq I$, so $\mathcal{Z}(\mathfrak{m}) \subseteq \mathcal{Z}(I)$. Since $\mathfrak{m} = (x_1 - a_1, \dots, x_d - a_d)$ for some $a_i \in k$, $\mathcal{Z}(\mathfrak{m})$ is a point, and thus nonempty. \square

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Over an algebraically closed field, maximal ideals in $k[x_1, \dots, x_d]$ correspond to points in \mathbb{A}^d . So we can start from the solution set — a point — and recover an ideal that corresponds to it. What if we start with some non-maximal ideal I , and consider its solution set $\mathcal{Z}_k(I)$ — can we recover I in some way?

Example 3.21. Many ideals define the same solution set. For example, in $R = k[x]$, the ideals $I_n = (x^n)$, for any $n \geq 1$, all define the same solution set $\mathcal{Z}_k(I_n) = \{0\}$.

To attack this question, we will need an observation on inequations.

Remark 3.22 (Rabinowitz's trick). Observe that, if $f(\underline{x})$ is a polynomial and $\underline{a} \in \mathbb{A}^d$, $f(\underline{a}) \neq 0$ if and only if $f(\underline{a}) \in k$ is invertible; equivalently, if there is a solution $y = b \in k$ to $yf(\underline{a}) - 1 = 0$. In particular, a system of polynomial equations and inequations

$$\begin{cases} f_1(\underline{x}) = 0 \\ \vdots \\ f_m(\underline{x}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} g_1(\underline{x}) \neq 0 \\ \vdots \\ g_n(\underline{x}) \neq 0 \end{cases}$$

has a solution $\underline{x} = \underline{a}$ if and only if the system

$$\begin{cases} f_1(\underline{x}) = 0 \\ \vdots \\ f_m(\underline{x}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} y_1 g_1(\underline{x}) - 1 = 0 \\ \vdots \\ y_n g_n(\underline{x}) - 1 = 0 \end{cases}$$

has a solution $(\underline{x}, y) = (\underline{a}, b)$. In fact, this is equivalent to a system in one extra variable:

$$\begin{cases} f_1(\underline{x}) = 0 \\ \vdots \\ f_m(\underline{x}) = 0 \\ y g_1(\underline{x}) \cdots g_n(\underline{x}) - 1 = 0 \end{cases}$$

Theorem 3.23 (Strong Nullstellensatz). *Let k be an algebraically closed field, and $R = k[x_1, \dots, x_d]$ be a polynomial ring. Let $I \subseteq R$ be an ideal. The polynomial f vanishes on $\mathcal{Z}_k(I)$ if and only if $f^n \in I$ for some $n \in \mathbb{N}$.*

Proof. Suppose that $f^n \in I$. For each $\underline{a} \in \mathcal{Z}_k(I)$, $f(\underline{a}) \in k$ satisfies $f(\underline{a})^n = 0 \in k$. Since k is a field, $f(\underline{a}) = 0$. Thus, $f \in \mathcal{Z}_k(I)$ as well.

Suppose that f vanishes along $\mathcal{Z}_k(I)$. This means that given any solution $\underline{a} \in \mathbb{A}^d$ to the system determined by I , $f(\underline{a}) = 0$. In other words, the system

$$\begin{cases} g(\underline{x}) = 0 \text{ for all } g \in I \\ f \neq 0 \end{cases}$$

has no solutions. By the discussion above, $\mathcal{Z}_k(I + (yf - 1)) = \emptyset$ in a polynomial ring in one more variable. By the Weak Nullstellensatz, we have $IR[y] + (yf - 1) = R[y]$, and equivalently $1 \in IR[y] + (yf - 1)$. Write $I = (g_1(\underline{x}), \dots, g_m(\underline{x}))$, and

$$1 = r_0(\underline{x}, y)(1 - yf(\underline{x})) + r_1(\underline{x}, y)g_1(\underline{x}) + \dots + r_m(\underline{x}, y)g_m(\underline{x}).$$

We can map y to $1/f$ to get

$$1 = r_1(\underline{x}, 1/f)g_1(\underline{x}) + \dots + r_m(\underline{x}, 1/f)g_m(\underline{x})$$

in the fraction field of $R[y]$. Since each r_i is polynomial, there is a largest negative power of f occurring; say that f^n serves as a common denominator. We can multiply by f^n to obtain f^n as a polynomial combination of the g 's. \square

Definition 3.24. The **radical** of an ideal I in a ring R is the ideal

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n\}.$$

An ideal is a **radical ideal** if $I = \sqrt{I}$.

To see that \sqrt{I} is an ideal, note that if $f^m, g^n \in I$, then

$$\begin{aligned} (f + g)^{m+n-1} &= \sum_{i=0}^{m+n-1} \binom{m+n-1}{i} f^i g^{m+n-1-i} \\ &= f^m \left(f^{n-1} + \binom{m+n-1}{1} f^{n-2} g + \dots + \binom{m+n-1}{n-1} g^{n-1} \right) \\ &\quad + g^n \left(\binom{m+n-1}{n} f^{m-1} + \binom{m+n-1}{n+1} f^{m-2} g + \dots + g^{m-1} \right) \in I, \end{aligned}$$

and $(rf)^m = r^m f^m \in I$.

Example 3.25. Prime ideals are radical.

Exercise 11. A nonzero element $f \in R$ is **nilpotent** if $f^n = 0$ for some $n > 1$; a ring R is **reduced** if it has no nilpotent elements. If R is a ring and I an ideal, then R/I is reduced if and only if I is a radical ideal.

Using this terminology, we can rephrase the Strong Nullstellensatz: if $k = \bar{k}$, then $f \in \mathcal{I}(\mathcal{Z}_k(I))$ if and only if $f \in \sqrt{I}$. Given any ideal I in $k[x_1, \dots, x_d]$, $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$.

We can now associate a ring to each subvariety of \mathbb{A}^d .

Definition 3.26. Let k be an algebraically closed field, and $X = \mathcal{Z}_k(I) \subseteq \mathbb{A}^d$ be a subvariety of \mathbb{A}^d . The **coordinate ring** of X is the ring $k[X] := k[x_1, \dots, x_d]/\mathcal{I}(X)$.

Since $k[X]$ is obtained from the polynomial ring on the ambient \mathbb{A}^d by quotienting out by exactly those polynomials that are zero on X , we interpret $k[X]$ as the ring of polynomial functions on X . Note that every reduced finitely generated k -algebra is a coordinate ring of some zero set X .

Remark 3.27. We showed before that $\mathcal{Z}(IJ) = \mathcal{Z}(I \cap J)$, despite the fact that we often have $IJ \neq I \cap J$. The Strong Nullstellensatz implies that $\sqrt{IJ} \neq \sqrt{I + J}$.

Remark 3.28. Observe that $\mathcal{Z}_k(\sqrt{J}) = \mathcal{Z}_k(J)$ whether or not k is algebraically closed, by the same proof we used above. The containment \subseteq is immediate since $J \subseteq \sqrt{J}$ from the definition. Moreover, if $f^n(\underline{a}) = 0$ then $f(\underline{a}) = 0$, so if $\underline{a} \in \mathcal{Z}_k(J)$ and $f \in \sqrt{J}$ then $f(\underline{a}) = 0$, and the equality of sets follows.

What might fail when the field is not algebraically closed is that $\mathcal{I}(\mathcal{Z}(I))$ is not necessarily \sqrt{I} . For example, $\mathcal{Z}_{\mathbb{R}}(x^2 + 1) = \emptyset$, so

$$\mathcal{I}(\mathcal{Z}_{\mathbb{R}}(x^2 + 1)) = \mathcal{I}(\emptyset) = \mathbb{R}[x] \neq \sqrt{(x^2 + 1)} = (x^2 + 1).$$

In fact, the ingredient that is missing is precisely the fact that the Weak Nullstellensatz is not satisfied over non-algebraically closed fields. If k is not algebraically closed, there exists some irreducible polynomial $f \in k[x]$ with no roots, so $\mathcal{Z}(f) = \emptyset$.

Remark 3.29. Note that if I is a radical ideal and $I \subsetneq J$, then $\mathcal{Z}(J) \subsetneq \mathcal{Z}(I)$. Indeed, there is some $f \in J$ such that $f \notin \sqrt{I} = I$, and thus $\mathcal{Z}(I) \not\subseteq \mathcal{Z}(f)$. Since $\mathcal{Z}(J) \subseteq \mathcal{Z}(f)$, we conclude that $\mathcal{Z}(I) \not\subseteq \mathcal{Z}(J)$.

Each variety corresponds to a unique radical ideal.

Corollary 3.30. Let k be an algebraically closed field and $R = k[x_1, \dots, x_d]$ a polynomial ring. There is an order-reversing bijection between the collection of subvarieties of \mathbb{A}_k^d and the collection of radical ideals of R :

$$\begin{array}{ccc} \{\text{subvarieties of } \mathbb{A}_k^d\} & \longleftrightarrow & \{\text{radical ideals } I \subseteq R\} \\ X & \xrightarrow{\mathcal{I}} & \{f \in R \mid X \subseteq \mathcal{Z}_k(f)\} \\ \mathcal{Z}_k(I) & \xleftarrow{\mathcal{Z}} & I \end{array}$$

In particular, given ideals I and J , we have $\mathcal{Z}_k(I) = \mathcal{Z}_k(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Proof. The Strong Nullstellensatz says that $\mathcal{I}(\mathcal{Z}(J)) = \sqrt{J}$ for any ideal J , hence $\mathcal{I}(\mathcal{Z}(J)) = J$ for a radical ideal J . Conversely, given X we can write $X = \mathcal{Z}_k(J)$ for some ideal J , and we without loss of generality we can assume J is radical, since $\mathcal{Z}_k(J) = \mathcal{Z}_k(\sqrt{J})$. Then $\mathcal{Z}(\mathcal{I}(X)) = \mathcal{Z}(\mathcal{I}(\mathcal{Z}(J))) = \mathcal{Z}(J) = X$.

This shows that \mathcal{I} and \mathcal{Z} are inverse operations, and we are done. \square

Under this bijection, irreducible varieties correspond to prime ideals.

Lemma 3.31. *A variety $X \subseteq \mathbb{A}_k^d$ is irreducible if and only if $\mathcal{I}(X)$ is prime.*

Proof. Suppose that X is reducible, say $X = V_1 \cup V_2$ for two varieties V_1 and V_2 such that $V_1, V_2 \subsetneq X$. Note that this implies that $\mathcal{I}(X) \subsetneq \mathcal{I}(V_1)$, $\mathcal{I}(X) \subsetneq \mathcal{I}(V_2)$, and $\mathcal{I}(X) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$. Then we can find $f \in \mathcal{I}(V_1)$ such that $f \notin \mathcal{I}(V_2)$, and $g \in \mathcal{I}(V_2)$ such that $g \notin \mathcal{I}(V_1)$. Notice that by construction $fg \in \mathcal{I}(V_1) \cap \mathcal{I}(V_2) = \mathcal{I}(X)$, while $f \notin \mathcal{I}(X)$ and $g \notin \mathcal{I}(X)$. Therefore, $\mathcal{I}(X)$ is not prime.

Now assume that $\mathcal{I}(X)$ is not prime, and fix $f, g \notin \mathcal{I}(X)$ with $fg \in \mathcal{I}(X)$. Then

$$X \subseteq \mathcal{Z}(fg) = \mathcal{Z}(f) \cup \mathcal{Z}(g).$$

The intersections

$$V_f = \mathcal{Z}(f) \cap X = \mathcal{Z}(\mathcal{I}(X) + (f))$$

and

$$V_g = \mathcal{Z}(g) \cap X = \mathcal{Z}(\mathcal{I}(X) + (g))$$

are varieties, and $X = V_f \cup V_g$. Finally, since $f \notin \mathcal{I}(X)$, then $X \not\subseteq V_f$. Similarly, $X \not\subseteq V_g$. Thus X is reducible. \square

Given a variety X , we can decompose it in irreducible components by writing it as a union $X = V_1 \cup \dots \cup V_n$. We can do this decomposition algebraically, by considering the radical ideal $I = \mathcal{I}(X)$ and writing it as an intersection of its minimal primes.

Definition 3.32. A prime P is a **minimal prime** of an ideal I if the only prime Q with $I \subseteq Q \subseteq P$ is $Q = P$. The set of minimal primes over I is denoted $\text{Min}(I)$.

Soon we will show that

$$\sqrt{I} = \bigcap_{\substack{P \text{ prime} \\ P \supseteq I}} P = \bigcap_{P \in \text{Min}(I)} P.$$

Later, we will prove that the set of minimal primes of an ideal in a Noetherian ring is finite, so in particular we can write $\mathcal{I}(X)$ as a finite intersection of prime ideals, say

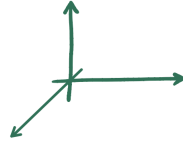
$$\mathcal{I}(X) = P_1 \cap \dots \cap P_k.$$

Then

$$X = \mathcal{Z}(P_1) \cup \dots \cup \mathcal{Z}(P_k)$$

is a decomposition of X into irreducible components.

Example 3.33. In $k[x, y, z]$, the radical ideal $I = (xy, xz, yz)$ corresponds to the variety X given by the union of the three coordinate axes.



Each of these axes is a variety in its own right, corresponding to the ideals (x, y) , (x, z) and (y, z) . The three axes are the irreducible components of X . And indeed, (x, y) , (x, z) and (y, z) are the three minimal primes over I , and

$$(xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z).$$

We will come back to this decomposition when we discuss primary decomposition.

In summary, Nullstellensatz gives us a dictionary between varieties and ideals:

<u>Algebra</u>	\longleftrightarrow	<u>Geometry</u>
algebra of ideals	\longleftrightarrow	geometry of varieties
algebra of $R = k[x_1, \dots, x_d]$	\longleftrightarrow	geometry of \mathbb{A}^d
radical ideals	\longleftrightarrow	varieties
prime ideals	\longleftrightarrow	irreducible varieties
maximal ideals	\longleftrightarrow	points
(0)	\longleftrightarrow	variety \mathbb{A}^d
$k[x_1, \dots, x_d]$	\longleftrightarrow	variety \emptyset
$(x_1 - a_1, \dots, x_d - a_d)$	\longleftrightarrow	point $\{(a_1, \dots, a_d)\}$
smaller ideals	\longleftrightarrow	larger varieties
larger ideals	\longleftrightarrow	smaller varieties

We now know that the subvarieties $\mathcal{Z}(I)$ of \mathbb{A}^d satisfy the following properties:

- The sets \emptyset and \mathbb{A}^d are varieties.
- The finite union of varieties is a variety.
- The arbitrary intersection of varieties is a variety.

These are the axioms of closed sets in a topology. So there is a topology on \mathbb{A}^d whose closed sets are precisely all the subvarieties $\mathcal{Z}(I)$ of \mathbb{A}^d . This topology is called the **Zariski topology**.

As a consequence, every variety inherits the Zariski topology, and this is the topology that algebraic geometers usually consider.

Exercise 12. A topological space X is **Noetherian** if it satisfies the descending chain condition for closed subsets: any descending chain of closed subsets

$$X_1 \supseteq X_2 \supseteq \cdots$$

stabilizes. Show that a variety with the Zariski topology is Noetherian.

Exercise 13. Show that if X is a Noetherian topological space, every open subset of X is quasicompact.

This topology is a little weird. In particular, it is *never* Hausdorff, unless the space we are considering is finite. The word *compact* is usually taken to include *Hausdorff*, so algebraic geometers say **quasicompact** to mean compact but maybe not Hausdorff.

Exercise 14. Show that $\mathbb{A}_{\mathbb{C}}^d$ with the Zariski topology is T_1 but not Hausdorff.

3.4 The prime spectrum of a ring

In modern algebraic geometry, one often studies schemes instead of varieties. We will now introduce the simplest scheme: the spectrum $\text{Spec}(R)$ of a ring R . The spectra of rings are to schemes as \mathbb{R}^n is to manifolds: while a manifold is a topological space that locally looks like \mathbb{R}^n , a scheme is, roughly speaking, a topological space that locally looks like $\text{Spec}(R)$ for some ring R .

The *maximal spectrum* of a ring R , denoted $\text{mSpec}(R)$, is the set of maximal ideals of R endowed with the topology with closed sets given by

$$V_{\text{Max}}(I) := \{\mathfrak{m} \in \text{Max}(R) \mid \mathfrak{m} \supseteq I\}$$

as I varies over all the ideals in R . By the Nullstellensatz, for polynomial rings S over an algebraically closed field k , this space $\text{mSpec}(S)$ has a natural homeomorphism to \mathbb{A}^n with its Zariski topology, and for an ideal I in $S = k[x_1, \dots, x_d]$, $\text{mSpec}(S/I)$ has a natural homeomorphism to $\mathcal{Z}_k(I) \subseteq \mathbb{A}^n$ with the subspace topology coming from the Zariski topology. Moreover, this is functorial: for any map of finitely generated k -algebras, there is an induced map on maximal ideals.

This is not quite the right notion to deal with general rings, for at least two reasons. First, there are many many interesting rings with only one maximal ideal! The topological space with one element, in contrast, is not that exciting. Second, we would like to have a geometric space that is assigned *functorially* to a ring, meaning that ring homomorphisms induce continuous maps of spaces (in the other direction). For the inclusion $A = k[x, y] = k[x - 1, y]$ into $B = k(x)[y] = k(x - 1)[y]$, what maximal ideal in A would we assign to $(y) \subseteq B$? How could one of (x, y) or $(x - 1, y)$ have a better claim than the other?

Definition 3.34. Let R be a ring. The **prime spectrum**, or **spectrum** of R is the set of prime ideals of R , denoted $\text{Spec}(R)$. This is naturally a poset, partially ordered by inclusion. We also endow it with the topology with closed sets

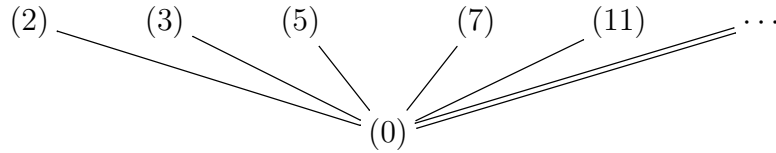
$$V(I) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq I\}$$

for (not necessarily proper) ideals $I \subseteq R$. In particular, $\emptyset = V(R)$ is closed.

Soon we will justify that this indeed forms a topology.

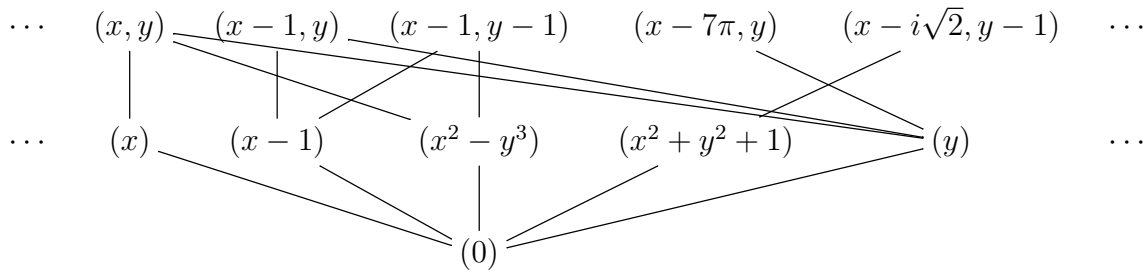
We will illustrate posets with Hasse diagrams: if an element is below something with a line connecting them, the higher element is \geq the lower one.

Example 3.35. The spectrum of \mathbb{Z} is the following poset:



The closed sets are of the form $V((n))$, which are the whole space when $n = 0$, the empty set with $n = 1$, and any finite union of things in the top row.

Example 3.36. Here are a few elements in $\mathbb{C}[x, y]$:



Note that $\text{Max}(R)$ is a subspace of $\text{Spec}(R)$ (that may be neither closed nor open).

Proposition 3.37. Let R be a ring, and let I , J , and I_λ be ideals (possibly improper).

- a) If $I \subseteq J$, then $V(J) \subseteq V(I)$.
- b) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.
- c) $\bigcap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$.
- d) $\text{Spec}(R)$ has a basis given by open sets of the form

$$D(f) := \text{Spec}(R) \setminus V(f) = \{\mathfrak{p} \in \text{Spec}(R) \mid f \notin \mathfrak{p}\}.$$

- e) $\text{Spec}(R)$ is quasicompact.

Proof.

- a) Clear from the definition.
- b) To see $V(I) \cup V(J) \subseteq V(I \cap J)$, just observe that if $\mathfrak{p} \supseteq I$ or $\mathfrak{p} \supseteq J$, then $\mathfrak{p} \supseteq I \cap J$. Since $IJ \subseteq I \cap J$, we have $V(I \cap J) \subseteq V(IJ)$. To show $V(IJ) \subseteq V(I) \cup V(J)$, if $\mathfrak{p} \not\supseteq I, J$, let $f \in I \setminus \mathfrak{p}$, and $g \in J \setminus \mathfrak{p}$. Then $fg \in IJ \setminus \mathfrak{p}$ since \mathfrak{p} is prime.
- c) Ideals are closed for sums, so if $\mathfrak{p} \supseteq I_\lambda$ for all λ , then $\mathfrak{p} \supseteq \sum_\lambda I_\lambda$. Moreover, if $\mathfrak{p} \supseteq \sum_\lambda I_\lambda$, then in particular $\mathfrak{p} \supseteq I_\lambda$.
- d) We can write any open set as the complement of $V(\{f_\lambda\}_\lambda) = \bigcap_\lambda V(f_\lambda)$, which is the union of $D(f_\lambda)$.
- e) Given a sequence of ideals I_λ , if $\sum_\lambda I_\lambda = R$, then 1 is in the sum on the left, and thus 1 can be realized in such a sum over finitely many indices, so

$$R = \sum_\lambda I_\lambda = I_{\lambda_1} + \cdots + I_{\lambda_t}.$$

Thus, if we have a family of closed sets with empty intersection,

$$\emptyset = \bigcap_\lambda V(I_\lambda) = V\left(\sum_\lambda I_\lambda\right) = V(I_{\lambda_1} + \cdots + I_{\lambda_t}) = V(I_{\lambda_1}) \cap \cdots \cap V(I_{\lambda_t}),$$

so some finite subcollection has an empty intersection. \square

Definition 3.38 (Induced map on Spec). Given a homomorphism of rings $R \xrightarrow{\varphi} S$, we obtain a map on spectra

$$\begin{aligned} \text{Spec}(S) &\xrightarrow{\varphi^*} \text{Spec}(R) . \\ \mathfrak{p} &\longrightarrow \varphi^{-1}(\mathfrak{p}) \end{aligned}$$

The key point here is that the preimage of a prime ideal is also prime. We will often write $\mathfrak{p} \cap R$ for $\varphi^{-1}(\mathfrak{p})$, even if the map is not necessarily an inclusion.

This is not only an order-preserving map, but also continuous: if $U \subseteq \text{Spec}(R)$ is open, say U is the complement of $V(I)$ for some ideal I , then for a prime \mathfrak{q} of S ,

$$\mathfrak{q} \in (\varphi^*)^{-1}(U) \iff \mathfrak{q} \cap R \not\supseteq I \iff \mathfrak{q} \not\supseteq IS \iff \mathfrak{q} \notin V(IS).$$

So $(\varphi^*)^{-1}(U)$ is the complement of $V(IS)$, and thus open.

Example 3.39. Let $R \xrightarrow{\pi} R/I$ be the canonical projection. Then

$$\text{Spec}(R/I) \xrightarrow{\pi^*} \text{Spec}(R)$$

corresponds to the inclusion of $V(I)$ into $\text{Spec}(R)$, since primes of R/I correspond to primes of R containing I .

We can use the spectrum of a ring to give an analogue of the strong Nullstellensatz that is valid for any ring. To prepare for this, we need a notion that we will use later.

Definition 3.40. A subset $W \subseteq R$ of a ring R is **multiplicatively closed** if $1 \in W$ and $a, b \in W \Rightarrow ab \in W$.

Lemma 3.41. Let R be a ring, I an ideal, and W a multiplicatively closed subset. If $W \cap I = \emptyset$, then there is a prime ideal \mathfrak{p} with $\mathfrak{p} \supseteq I$ and $\mathfrak{p} \cap W = \emptyset$.

Proof. Consider the family of ideals $\mathcal{F} := \{J \mid J \supseteq I, J \cap W = \emptyset\}$. This is nonempty, since it contains I , and any chain $J_1 \supseteq J_2 \supseteq \cdots \supseteq$ has an upper bound $\bigcap_i J_i$. Therefore, \mathcal{F} has some maximal element \mathbb{A} by a basic application of Zorn's Lemma. We claim \mathbb{A} is prime. Suppose $f, g \notin \mathbb{A}$. By maximality, $\mathbb{A} + (f)$ and $\mathbb{A} + (g)$ both have nonempty intersection with W , so there exist $r_1f + a_1, r_2g + a_2 \in W$, with $a_1, a_2 \in \mathbb{A}$. If $fg \in \mathbb{A}$, then $(r_1f + a_1)(r_2g + a_2) = r_1r_2fg + r_1fa_2 + r_2ga_1 + a_1a_2 \in W \cap \mathbb{A}$, a contradiction. \square

Proposition 3.42 (Spectrum analogue of strong Nullstellensatz). Let R be a ring, and I be an ideal. For $f \in R$,

$$V(I) \subseteq V(f) \iff f \in \sqrt{I}.$$

Equivalently,

$$\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \sqrt{I}.$$

Proof. First to justify the equivalence of the two statements we observe:

$$V(I) \subseteq V(f) \iff f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in V(I) \iff f \in \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

We prove that $\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \sqrt{I}$.

(\supseteq): It suffices to show that $\mathfrak{p} \supseteq I$ implies $\mathfrak{p} \supseteq \sqrt{I}$, and indeed

$$f^n \in I \subseteq \mathfrak{p} \implies f \in \mathfrak{p}.$$

(\subseteq): If $f \notin \sqrt{I}$, consider the multiplicatively closed set $W = \{1, f, f^2, f^3, \dots\}$. We have $W \cap I = \emptyset$ by hypothesis. By the previous lemma, there is a prime \mathfrak{p} in $V(I)$ that does not intersect W , and hence does not contain f . \square

The following corollary follows in exactly the same way as the analogous statement for subvarieties of \mathbb{A}^n , Corollary 3.30.

Corollary 3.43. Let R a ring. There is an order-reversing bijection

$$\{\text{closed subsets of } \text{Spec}(R)\} \longleftrightarrow \{\text{radical ideals } I \subseteq R\}$$

In particular, for two ideals I, J , $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Chapter 4

Local Rings

The study of local rings is central to commutative algebra. As we will see, life is easier in a local ring, so much so that we often want to *localize* so we can be in a local ring. A lot of the things we will say in this chapter also apply to \mathbb{N} -graded k -algebras and their homogenous maximal ideal — with some appropriate changes, such as considering only homogeneous ideals.

4.1 Local rings

Definition 4.1. A ring R is a **local ring** if it has exactly one maximal ideal. We often use the notation (R, \mathfrak{m}) to denote R and its maximal ideal, or (R, \mathfrak{m}, k) to also specify the residue field $k = R/\mathfrak{m}$. Some people reserve the term *local ring* for a Noetherian local ring, and call what we have defined a **quasilocal ring**; we will not follow this convention here.

Lemma 4.2. *A ring R is local if and only if the set of nonunits of R forms an ideal.*

Proof. If the set of nonunits is an ideal, that must be the only maximal ideal. □

Example 4.3.

- a) The ring $\mathbb{Z}/(p^n)$ is local with maximal ideal (p) .
- b) The ring $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b \text{ when in lowest terms}\}$ is a local ring with maximal ideal (p) .
- c) The ring of power series $k[[x]]$ over a field k is local. Indeed, a power series has an inverse if and only if its constant term is nonzero. The complement of this set of units is the ideal (x) .
- d) More generally, $k[[x_1, \dots, x_d]]$ is local with maximal ideal (x_1, \dots, x_d) .
- e) The ring of complex power series holomorphic at the origin, $\mathbb{C}\{\underline{x}\}$, is local. In the above setting, one proves that the series inverse of a holomorphic function at the origin is convergent on a neighborhood of 0.

- f) A polynomial ring over a field is certainly not local; we have seen it has so many maximal ideals!

We start with a comment about the characteristic of local rings.

Definition 4.4. The **characteristic** of a ring R is, if it exists, the smallest positive integer n such that

$$\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0.$$

If no such n exists, we say that R has characteristic 0. Equivalently, the characteristic of R is the integer $n \geq 0$ such that

$$(n) = \ker \begin{pmatrix} \mathbb{Z} \longrightarrow R \\ a \longmapsto a \cdot 1_R \end{pmatrix}.$$

Proposition 4.5. *Let (R, \mathfrak{m}, k) be a local ring. Then one of the following holds:*

- a) $\text{char}(R) = \text{char}(k) = 0$. We say that R has **equal characteristic zero**.
- b) $\text{char}(R) = 0$, $\text{char}(k) = p$ for a prime p , so R has **mixed characteristic** $(0, p)$.
- c) $\text{char}(R) = \text{char}(k) = p$ for a prime p , so R has **equal characteristic p** .
- d) $\text{char}(R) = p^n$, $\text{char}(k) = p$ for a prime p and an integer $n > 1$.

If R is reduced, then one of the first three cases holds.

Proof. Since k is a quotient of R , the characteristic of R must be a multiple of the characteristic of k , since the map $\mathbb{Z} \longrightarrow k$ factors through R . We must think of 0 as a multiple of any integer for this to make sense. Now k is a field, so its characteristic is 0 or p for a prime p . If $\text{char}(k) = 0$, then necessarily $\text{char}(R) = 0$. If $\text{char}(k) = p$, we claim that $\text{char}(R)$ must be either 0 or a power of p . Indeed, if we write $\text{char}(R) = p^n \cdot a$ with a coprime to p , note that $p \in \mathfrak{m}$, so if $a \in \mathfrak{m}$, we have $1 \in (p, a) \subseteq \mathfrak{m}$, which is a contradiction. Since R is local, this means that a is a unit. But then, $p^n a = 0$ implies $p^n = 0$, so the characteristic must be p^n . \square

Remark 4.6. If R is an \mathbb{N} -graded k -algebra with $R_0 = k$, and $\mathfrak{m} = \bigoplus_{n>0} R_n$ is the homogeneous maximal ideal, R and \mathfrak{m} behave a lot like a local ring and its maximal ideal, and we sometimes use the suggestive notation (R, \mathfrak{m}) to refer to it. Many properties of local rings also apply to the graded setting, so given a statement about local rings, you might take it as a suggestion that there might be a corresponding statement about graded rings — a statement that, nevertheless, still needs to be proved. There are usually some changes one needs to make to the statement; for example, if a theorem makes assertions about the ideals in a local ring, the corresponding graded statement will likely only apply to homogeneous ideals, and a theorem about finitely generated modules over a local ring will probably translate into a theorem about graded modules in the graded setting.

4.2 Localization

Recall that a multiplicative subset of a ring R is a set $W \ni 1$ that is closed for products. The three most important classes of multiplicative sets are the following:

Example 4.7. Let R be a ring.

- a) For any $f \in R$, the set $W = \{1, f, f^2, f^3, \dots\}$ is a multiplicative set.
- b) If $\mathfrak{p} \subseteq R$ is a prime ideal, the set $W = R \setminus \mathfrak{p}$ is multiplicative: this is an immediate translation of the definition.
- c) The set of *nonzerodivisors* in R — elements that are not zerodivisors — forms a multiplicatively closed subset.

Remark 4.8. An arbitrary intersection of multiplicatively closed subsets is multiplicatively closed. In particular, for any family of primes $\{\mathfrak{p}_\lambda\}$, the complement of $\bigcup_\lambda \mathfrak{p}_\lambda$ is multiplicatively closed.

Definition 4.9 (Localization of a ring). Let R be a ring, and W be a multiplicative set with $0 \notin W$. The **localization** of R at W is the ring

$$W^{-1}R := \left\{ \frac{r}{w} \mid r \in R, w \in W \right\} / \sim$$

where \sim is the equivalence relation

$$\frac{r}{w} \sim \frac{r'}{w'} \text{ if there exists } u \in W : u(rw' - r'w) = 0.$$

The operations are given by

$$\frac{r}{v} + \frac{s}{w} = \frac{rw + sv}{vw} \quad \text{and} \quad \frac{r}{v} \frac{s}{w} = \frac{rs}{vw}.$$

The zero in $W^{-1}R$ is $\frac{0}{1}$ and the identity is $\frac{1}{1}$. There is a canonical ring homomorphism

$$\begin{aligned} R &\longrightarrow W^{-1}R. \\ r &\longmapsto \frac{r}{1} \end{aligned}$$

Note that we write elements in $W^{-1}R$ in the form $\frac{r}{w}$ even though they are equivalence classes of such expressions.

Remark 4.10. Observe that if R is a domain, the equivalence relation simplifies to $rw' = r'w$, so $R \subseteq W^{-1}R \subseteq \text{Frac}(R)$, and in particular $W^{-1}R$ is a domain too. In particular, $\text{Frac}(R)$ is a localization of R .

In the localization of R at W , every element of W becomes a unit. The following universal property says roughly that $W^{-1}R$ is the smallest R -algebra in which every element of W is a unit.

Proposition 4.11. *Let R be a ring, and W a multiplicative set with $0 \notin W$. Let S be an R -algebra in which every element of W is a unit. Then there is a unique homomorphism α such that the following diagram commutes:*

$$\begin{array}{ccc} R & \longrightarrow & W^{-1}R \\ \downarrow & \nearrow \alpha & \\ S & & \end{array}$$

where the vertical map is the structure homomorphism and the horizontal map is the canonical homomorphism.

Example 4.12 (Most important localizations). Let R be a ring.

- a) For $f \in R$ and $W = \{1, f, f^2, f^3, \dots\}$, we usually write R_f for $W^{-1}R$.
- b) For a prime ideal \mathfrak{p} in R , we generally write $R_{\mathfrak{p}}$ for $(R \setminus \mathfrak{p})^{-1}R$, and call it **the localization of R at \mathfrak{p}** . Given an ideal I in R , we sometimes write $I_{\mathfrak{p}}$ to refer to $IR_{\mathfrak{p}}$, the image of I via the canonical map $R \rightarrow R_{\mathfrak{p}}$.
- c) When W is the set of nonzerodivisors on R , we call $W^{-1}R$ the **total ring of fractions** of R . When R is a domain, this is just the fraction field of R , and in this case this coincides with the localization at the prime (0) .

Remark 4.13. Notice that when we localize at a prime \mathfrak{p} , the resulting ring is a local ring $(R_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$. We can think of the process of localization at \mathfrak{p} as *zooming in* at the prime \mathfrak{p} . Many properties of an ideal I can be checked *locally*, by checking them for $IR_{\mathfrak{p}}$ for each prime $\mathfrak{p} \in V(I)$.

We can now add some more local rings to our list of examples.

Example 4.14.

- a) A local ring one often encounters is $k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$. We can consider this as the ring of rational functions that in lowest terms have a denominator with nonzero constant term. Note that we can talk about lowest terms since the polynomial ring is a UFD.
- b) Extending the following example, we have local rings like $(k[x_1, \dots, x_d]/I)_{(x_1, \dots, x_d)}$. If k is algebraically closed and I is a radical ideal, then $k[x_1, \dots, x_d]/I = k[X]$ is the coordinate ring of some affine variety, and $(x_1, \dots, x_d) = \mathfrak{m}_{\underline{0}}$ is the ideal defining the origin (as a point in $X \subseteq \mathbb{A}^d$). Then we call

$$k[X]_{\mathfrak{m}_{\underline{0}}} := (k[x_1, \dots, x_d]/I)_{(x_1, \dots, x_d)}$$

the **local ring of the point $\underline{0} \in X$** ; some people write $\mathcal{O}_{X, \underline{0}}$. The radical ideals of this ring consist of radical ideals of $k[X]$ that are contained in $\mathfrak{m}_{\underline{0}}$, which by the Nullstellensatz correspond to subvarieties of X that contain $\underline{0}$.

We state an analogous definition for modules, and for module homomorphisms.

Definition 4.15. Let R be a ring, W be a multiplicative set, and M an R -module. The **localization** of M at W is the $W^{-1}R$ -module

$$W^{-1}M := \left\{ \frac{m}{w} \mid m \in M, w \in W \right\} / \sim$$

where \sim is the equivalence relation $\frac{m}{w} \sim \frac{m'}{w'}$ if $u(mw' - m'w) = 0$ for some $u \in W$. The operations are given by

$$\frac{m}{v} + \frac{n}{w} = \frac{mw + nv}{vw} \quad \text{and} \quad \frac{r}{v} \frac{m}{w} = \frac{rm}{vw}.$$

If $M \xrightarrow{\alpha} N$ is an R -module homomorphism, then there is a $W^{-1}R$ -module homomorphism $W^{-1}M \xrightarrow{W^{-1}\alpha} W^{-1}N$ given by the rule $W^{-1}\alpha(\frac{m}{w}) = \frac{\alpha(m)}{w}$.

We will use the notations M_f and $M_{\mathfrak{p}}$ analogously to R_f and $R_{\mathfrak{p}}$.

To understand localizations of rings and modules, we will want to understand better how they are built from R . First, we take a small detour to talk about colons and annihilators.

Definition 4.16. The **annihilator** of a module M is the ideal

$$\text{ann}(M) := \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

Definition 4.17. Let I and J be ideals in a ring R . The **colon** of I and J is the ideal

$$(J : I) := \{r \in R \mid rI \subseteq J\}.$$

More generally, if M and N are submodules of some R -modules A , the colon of N and M is

$$(N :_R M) := \{r \in R \mid rM \subseteq N\}.$$

Exercise 15. The annihilator of M is an ideal in R , and

$$\text{ann}(M) = (0 :_R M).$$

Moreover, any colon $(N :_R M)$ is an ideal in R .

Lemma 4.18. Let M be an R -module, and W a multiplicative set. The class

$$\frac{m}{w} \in W^{-1}M \text{ is zero} \iff vm = 0 \text{ for some } v \in W \iff \text{ann}_R(m) \cap W \neq \emptyset.$$

Note in particular that this holds for $w = 1$.

Proof. For the first equivalence, we simply use the equivalence relation defining $W^{-1}R$ to note that $\frac{m}{w} = \frac{0}{1}$ in $W^{-1}M$ if and only if there exists some $v \in W$ such that $0 = v(1m - 0w) = vm$. The second equivalence just comes from the definition of the annihilator. \square

Remark 4.19. It follows from this lemma that if $N \xrightarrow{\alpha} M$ is injective, then $W^{-1}\alpha$ is also injective, since

$$0 = W^{-1}\alpha\left(\frac{n}{w}\right) = \frac{\alpha(n)}{w} \Rightarrow 0 = u\alpha(n) = \alpha(un) \text{ for some } u \in W \Rightarrow un = 0 \Rightarrow \frac{n}{w} = 0.$$

We want to collect one more lemma for later.

Lemma 4.20. *Let M be a module, and N_1, \dots, N_t be a finite collection of submodules. Let W be a multiplicative set. Then,*

$$W^{-1}(N_1 \cap \dots \cap N_t) = W^{-1}N_1 \cap \dots \cap W^{-1}N_t \subseteq W^{-1}M.$$

Proof. The containment $W^{-1}(N_1 \cap \dots \cap N_t) \subseteq W^{-1}N_1 \cap \dots \cap W^{-1}N_t$ is clear. Elements of $W^{-1}N_1 \cap \dots \cap W^{-1}N_t$ are of the form $\frac{n_1}{w_1} = \dots = \frac{n_t}{w_t}$; we can find a common denominator to realize this in $W^{-1}(N_1 \cap \dots \cap N_t)$. \square

Later we will show that localization has good homological properties: it's an exact functor.

Theorem 4.21. *Given a short exact sequence of R -modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and a multiplicative set W , the sequence

$$0 \longrightarrow W^{-1}A \longrightarrow W^{-1}B \longrightarrow W^{-1}C \longrightarrow 0$$

is also exact.

Remark 4.22. Given a submodule N of M , we can apply the statement above to the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

and conclude that $W^{-1}(M/N) \cong W^{-1}M/W^{-1}N$.

Proposition 4.23. *Let W be multiplicatively closed in R .*

- a) *If I is an ideal, then $W^{-1}I \cap R = \{r \in R \mid wr \in I \text{ for some } w \in W\}$.*
- b) *If \mathfrak{p} is prime and $W \cap \mathfrak{p} = \emptyset$, then $W^{-1}\mathfrak{p} = \mathfrak{p}(W^{-1}R)$ is prime.*

c) The map $\text{Spec}(W^{-1}R) \rightarrow \text{Spec}(R)$ is injective, with image

$$\{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \cap W = \emptyset\}.$$

Proof.

- a) Since $W^{-1}(R/I) \cong W^{-1}R/W^{-1}I$, we have $\ker(R \rightarrow W^{-1}(R/I)) = R \cap W^{-1}I$. The equality is then clear.
- b) First, since $W \cap \mathfrak{p} = \emptyset$, and \mathfrak{p} is prime, no element of W kills $\bar{1} = 1 + \mathfrak{p}$ in R/\mathfrak{p} , so $\bar{1}/1$ is nonzero in $W^{-1}(R/\mathfrak{p})$. Thus, $W^{-1}R/W^{-1}\mathfrak{p} \cong W^{-1}(R/\mathfrak{p})$ is nonzero, and a localization of a domain, hence is a domain. Thus, $W^{-1}\mathfrak{p}$ is prime.
- c) First, by part b), the map $\mathfrak{p} \mapsto W^{-1}\mathfrak{p}$, for $S = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \cap W = \emptyset\}$ sends primes to primes. We claim that

$$\begin{array}{ccc} \text{Spec}(W^{-1}R) & & S \\ \mathfrak{q} & \xrightarrow{\quad} & \mathfrak{q} \cap R \\ W^{-1}\mathfrak{p} & \xleftarrow{\quad} & \mathfrak{p} \end{array}$$

are inverse maps.

First take an ideal J in $W^{-1}R$, say $J = (\{\frac{a_i}{w_i}\})$. Then $J = (\{\frac{a_i}{1}\})$, since each generator was replaced by a unit multiple. Then $J \cap R = (\{a_i\})$, and $J = (J \cap R)W^{-1}R$.

If $W \cap \mathfrak{p} = \emptyset$, then using part a) and the definition of prime, we have that

$$W^{-1}\mathfrak{p} \cap R = \{r \in R \mid rw \in \mathfrak{p} \text{ for some } w \in W\} = \{r \in R \mid r \in \mathfrak{p}\} = \mathfrak{p}. \quad \square$$

Corollary 4.24. *Let R be a ring and \mathfrak{p} be a prime ideal in R . The map on Spectra induced by the canonical map $R \rightarrow R_{\mathfrak{p}}$ corresponds to the inclusion*

$$\{\mathfrak{q} \in \text{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\} \hookrightarrow \text{Spec}(R).$$

Appendix A

Macaulay2

There are several computer algebra systems dedicated to algebraic geometry and commutative algebra computations, such as [Singular](#) (more popular among algebraic geometers), [CoCoA](#) (which is more popular with european commutative algebraists, having originated in Genova, Italy), and [Macaulay2](#). There are many computations you could run on any of these systems (and others), but we will focus on Macaulay2 since it's the most popular computer algebra system among US based commutative algebraists.

Macaulay2, as the name suggests, is a successor of a previous computer algebra system named Macaulay. Macaulay was first developed in 1983 by Dave Bayer and Mike Stillman, and while some still use it today, the system has not been updated since its final release in 2000. In 1993, Daniel Grayson and Mike Stillman released the first version of Macaulay2, and the current stable version is Macaulay2 1.16.

Macaulay2, or M2 for short, is an open-source project, with many contributors writing packages that are then released with the newest Macaulay2 version. Journals like the *Journal of Software for Algebra and Geometry* publish peer-refereed short articles that describe and explain the functionality of new packages, with the package source code being peer reviewed as well.

The National Science Foundation has funded Macaulay2 since 1992. Besides funding the project through direct grants, the NSF has also funded several Macaulay2 workshops — conferences where Macaulay2 package developers gather to work on new packages, and to share updates to the Macaulay2 core code and recent packages.

A.1 Getting started

A Macaulay2 session often starts with defining some ambient ring we will be doing computations over. Common rings such as the rationals and the integers can be defined using the commands `QQ` and `ZZ`; one can easily take quotients or build polynomial rings (in finitely many variables) over these. For example,

```
i1 : R = ZZ/101[x,y]
```



```

o1 = R

o1 : PolynomialRing

and

i1 : k = ZZ/101;

i2 : R = k[x,y];

```

both store the ring $\mathbb{Z}/101$ as R , with the small difference that in the second example Macaulay2 has named the coefficient field k . One quirk that might make a difference later is that if we use the first option and later set k to be the field $\mathbb{Z}/101$, our ring R is *not* a polynomial ring over k . Also, in the second example we ended each line with a `;`, which tells Macaulay2 to run the command but not display the result of the computation — which is in this case was simply an assignment, so the result is not relevant. Lines indicated with `i` as in, where n is some integer, are input lines, whereas lines with an `o` indicate output lines.

We can now do all sorts of computations over our ring R . We can define ideals in R , and use them to either define a quotient ring S of R or an R -module M , as follows:

```

i3 : I = ideal(x^2,y^2,x*y)

o3 = ideal (x2, y2, x*y)

o3 : Ideal of R

i4 : M = R^1/I

o4 = cokernel | x2 y2 xy |

o4 : R-module, quotient of R1

i5 : S = R/I

o5 = S

o5 : QuotientRing

```

It's important to note that while R is a ring, R^1 is the R -module R — this is a very important difference for Macaulay2, since these two objects have different types.

So S defined above is a ring, while M is a module. Notice that Macaulay2 stored the module M as the cokernel of the map

$$R^3 \xrightarrow{\begin{bmatrix} x^2 & y^2 & xy \end{bmatrix}} R.$$

Note also that there is an alternative syntax to write our ideal I from above, as follows:

```
i15 : I = ideal"x2,xy,y2"
```

```
o15 = ideal (x2 , x*y, y2)
```

```
o15 : Ideal of R
```

When you make a new definition in Macaulay2, you might want to pay attention to what ring your new object is defined over. For example, now that we defined this ring S , Macaulay2 has automatically taken S to be our current ambient ring, and any calculation or definition we run next will be considered over S and not R . If you want to return to the original ring R , you must first run the command `use R`.

If you want to work over a finitely generated algebra over one of the basic rings you can define in Macaulay2, and your ring is not a quotient of a polynomial ring, you want to rewrite this algebra as a quotient of a polynomial ring. For example, suppose you want to work over the 2nd Veronese in 2 variables over our field k from before, meaning the algebra $k[x^2, xy, y^2]$. We need 3 algebra generators, which we will call a, b, c , corresponding to x^2 , xy , and y^2 :

```
i11 : U = k[a,b,c]
```

```
o11 = U
```

```
o11 : PolynomialRing
```

```
i12 : f = map(R,U,{x2,x*y,y2})
```

```
o12 = map(R,U,{x2 , x*y, y2 })
```

```
o12 : RingMap R <--- U
```

```
i13 : J = ker f
```

```
o13 = ideal(b2 - a*c)
```

```

o13 : Ideal of U

i14 : T = U/J

o14 = T

o14 : QuotientRing

```

Our ring T at the end is isomorphic to the 2nd Veronese of R , which is the ring we wanted.

A.2 Basic commands

Many Macaulay2 commands are easy to guess, and named exactly what you would expect them to be named. If you are not sure how to use a certain command, you can run `viewHelp` followed by the command you want to ask about; this will open an html file with the documentation for the method you asked about. Often, googling “Macaulay2” followed by descriptive words will easily land you on the documentation for whatever you are trying to do.

Here are some basic commands you will likely use:

- `ideal(f_1, \dots, f_n)` will return the ideal generated by f_1, \dots, f_n . Here products should be indicated by `*`, and powers with `^`. If you’d rather not use `^` (this might be nice if you have lots of powers), you can write `ideal(f_1, \dots, f_n)` instead.
- `map(S, R, f_1, \dots, f_n)` gives a ring map $R \rightarrow S$ if R and S are rings, and R is a quotient of $k[x_1, \dots, x_n]$. The resulting ring map will send $x_i \mapsto f_i$. There are many variations of `map` — for example, you can use it to define R -module homomorphisms — but you should carefully input the information in the required format. Try `viewHelp map` in Macaulay2 for more.
- `ker(f)` returns the kernel of the map f .
- `I + J` and `I * J` return the sum and product of the ideals I and J , respectively.
- `A = matrix{{ $a_{1,1}, \dots, a_{1,n}$ }, ..., { $a_{m,1}, \dots, a_{m,n}$ }}` returns the matrix

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}$$

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