The Zariski-Nagata theorem in mixed characteristic

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Let Q be a prime ideal in a noetherian ring R.

Symbolic Powers

The n-th symbolic power of Q is the ideal

$$\begin{aligned} Q^{(n)} &= Q^n R_Q \cap R. \\ &= \text{ smallest } Q\text{-primary ideal containing } Q^n \\ &= \text{ } Q\text{-primary component in a decomposition of } Q^n \end{aligned}$$

Theorem (Zariski–Nagata: order of vanishing)

Let Q be a prime ideal in a polynomial ring $R = K[x_1, \ldots, x_d]$ over a field K. Then

$$Q^{(n)} = \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \mathfrak{m} \supseteq Q}} \mathfrak{m}^n.$$

GEOMETRICALLY: $Q^{(n)}$ consists of the functions that vanish to order at least n on each point in $\mathcal{V}(Q)$.

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A closely related statement characterizes symbolic powers in terms of differential operators. This is the version of Zariski–Nagata that we are interested in.

Theorem (Zariski–Nagata: characteristic zero)

Let Q be a prime ideal in a polynomial ring $R = K[x_1, \ldots, x_d]$ over a field K of characteristic zero. Then

$$Q^{(n)} = \{f \in R \mid \forall \partial \in D^{n-1}_{R|K},\ \partial(f) \in Q\},$$

where $D^i_{R|K}$ is the set of differential operators of order at most i:

$$D^i_{R|K} \quad = \bigoplus_{0 \leqslant a_1 + \cdots + a_d \leqslant i} R \, \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_d}}{\partial x_d^{a_d}}.$$

Theorem (Zariski–Nagata: characteristic zero)

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Example

Let X be a 3×3 matrix of indeterminates, $R = K[X]$, and $P = I_2(X)$ be the ideal of 2×2 minors of X. Then, $\frac{\partial}{\partial x_{11}} \det(X) = x_{22}x_{33} - x_{23}x_{32} \in P$. By symmetry, $D^1_{R|K} \cdot \det(X) \subseteq P$, so $\det(X) \in P^{(2)}$.

$$Q^{(n)} = \{f \in R \mid \forall \partial \in D^{n-1}_{R|K},\ \partial(f) \in Q\}.$$

The fails in characteristic $p > 0$:

$$\frac{\partial}{\partial x_1}(x_1^p) = p x_1^{p-1} = 0 \qquad \frac{\partial}{\partial x_i}(x_1^p) = 0 \text{ for } i > 1$$

so every operator $\partial \in \bigoplus_{a_1,\ldots,a_d \geqslant 0} R \, \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_d}}{\partial x_d^{a_d}}$ satisfies $\partial(x_1^p) \in (x_1^p)$.

The previous version of Zariski–Nagata would say that

$$x_1^p \in (x_1)^{(n)} = (x_1^n) \quad \text{for all } n > 0, \text{ which is false.}$$

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Not enough differential operators (in this sense)!

If $R = A[x_1, \ldots, x_d]$, we define

$$D^i_{R|A} = \bigoplus_{0 \leqslant a_1 + \cdots + a_d \leqslant i} R \, \frac{1}{a_1!} \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{1}{a_d!} \frac{\partial^{a_d}}{\partial x_d^{a_d}}.$$

This makes sense even if the a_i's are not units, since if we differentiate a monomial by a variable a_i times, we either pick up a_i consecutive integers as coefficients, or run out of that variable.

This is a special case of Grothendieck's notion of differential operators.

Let R be an A-algebra.

Definition (Differential operators)

The A-linear differential operators of order 0 on R are given by

$$D^0_{R|A} = \operatorname{Hom}_R(R, R).$$

The A-linear differential operators of order n on R are given by

$$D^n_{R|A} = \{\partial \in \operatorname{Hom}_A(R, R) \,|\, \forall r \in D^0_{R|A},\ \ \partial \circ r - r \circ \partial \in D^{n-1}_{R|A}\}.$$

Theorem (Zariski–Nagata: characteristic zero or $p > 0$)

Let Q be a prime ideal in a polynomial ring $R = K[x_1, \ldots, x_d]$ over a field K (of characteristic 0 or $p > 0$). Assume that the field extension $K \hookrightarrow R_Q/QR_Q$ is separable. Then

$$Q^{(n)} = \{f \in R \mid \forall \partial \in D^{n-1}_{R|K},\ \partial(f) \in Q\}.$$

where $D^i_{R|K}$ is the set of differential operators of order at most i:

$$D^i_{R|K} = \bigoplus_{0 \leqslant a_1 + \cdots + a_d \leqslant i} R \, \frac{1}{a_1!} \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{1}{a_d!} \frac{\partial^{a_d}}{\partial x_d^{a_d}}.$$

Even with the correct notion of differential operators, the version of Zariski–Nagata we have seen fails in mixed characteristic.

Example

Let $R = \mathbb{Z}[x]$, p be a prime integer, and $Q = (p)$.
Every $\mathbb{Z}$-linear differential operator $\partial \in D^i_{R|\mathbb{Z}}$ on R satisfies

$$\partial(p) = p\partial(1) \in (p) = Q.$$

The previous version of Zariski–Nagata would say that $p \in Q^{(n)} = (p^n)$ for all n, which is false.

First version of Zariski–Nagata in mixed characteristic:

Theorem (De Stefani – G – Jeffries)

Let A be either $\mathbb{Z}$ or a DVR of mixed characteristic. Let $R = A[x_1, \ldots, x_d]$. If Q is a prime ideal in R such that $Q \cap A = (0)$, then for all $n \geqslant 1$,

$$Q^{(n)} = \{f \in R \mid \forall \partial \in D^{n-1}_{R|A},\ \partial(f) \in Q\}.$$

So what is the problem?

If our prime ideal $Q \ni p$, we need "differential operators" that decrease p-adic order.

- $p^n \in (p^n) = (p)^{(n)}$, so for a "differential operator" δ of order $n-1$ we should have $\delta(p^n) \in (p)$.
- $p^{n-1} \notin (p^n) = (p)^{(n)}$, so $\delta(p^{n-1}) \notin (p)$ for some "differential operator" δ of order $n-1$.

In particular, a "differential operator" of order 1 should decrease p-adic order by 1.

Fix a prime $p \in \mathbb{Z}$, and let S be a ring on which p is a nonzerodivisor.

Definition (Buium, Joyal)

We say that a set-theoretic map $\delta_p : S \to S$ is a *p-derivation* if $\delta_p(1) = 0$ and δ_p satisfies the following identities for all $x, y \in S$:

$$\delta_p(xy) = x^p \delta_p(y) + y^p \delta_p(x) + p\delta_p(x)\delta_p(y)$$

and

$$\delta_p(x + y) = \delta_p(x) + \delta_p(y) + \mathcal{C}_p(x, y),$$

where

$$\mathcal{C}_p(X, Y) = \frac{X^p + Y^p - (X + Y)^p}{p} \in \mathbb{Z}[X, Y].$$

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Given a lift ϕ_p of the Frobenius map on S/pS,

$$\delta_p(x) = \frac{\phi_p(x) - x^p}{p}$$

is a p-derivation.

Warning

Not all rings have a p-derivation!

But the following rings S do have p-derivations:

- $S = \mathbb{Z}$,
- $S = \mathbb{Z}_p$, the p-adic integers,
- S is a polynomial ring over a ring B that admits a p-derivation, or
- S is p-adically complete and formally smooth over a ring B that admits a p-derivation.

There is only one p-derivation over $\mathbb{Z}$

$$\delta_p(n) = \frac{n - n^p}{p}.$$

Every p-derivation on a ring S of characteristic 0 extends this one.

These are the maps we are looking for

A p-derivation δ does indeed decrease p-adic order:

$$\delta_p(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1} - p^{np-1} \in (p^{n-1}) \setminus (p^n).$$

Definition (De Stefani – G – Jeffries)

Let $A = \mathbb{Z}$ or a DVR of mixed characteristic, and $R = A[x_1, \ldots, x_d]$. Let ∂_p be a p-derivation on R. The *mixed differential operators* of order i are

$$D^{i,\mathrm{mix}}_{R|A} = \{\underbrace{\delta_p \circ \cdots \circ \delta_p}_{a \text{ times}} \circ \partial \mid \partial \in D^b_{R|A},\ a + b \leqslant i\}$$

Theorem (De Stefani – G – Jeffries)

Let $A = \mathbb{Z}$ or a DVR of mixed characteristic with a p-derivation, and $R = A[x_1, \ldots, x_d]$. Let Q be a prime ideal of R that contains p, and assume that the field extension $A/pA \hookrightarrow R_Q/QR_Q$ is separable. Then for all $n \geqslant 1$,

$$Q^{(n)} = \{f \in R \mid \forall \delta \in D_{R|A}^{n-1,\mathrm{mix}},\ \delta(f) \in Q\}.$$

$$D_{R|A}^{i,\mathrm{mix}} = \{\underbrace{\delta_p \circ \cdots \circ \delta_p}_{a \text{ times}} \circ \partial \mid \partial \in D_{R|A}^{b},\ a + b \leqslant i\};$$

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  $\Rightarrow$ It is enough to show that equality holds after localizing at Q.
- Finally, show $\mathfrak{m}^{\langle n \rangle_{\mathrm{mix}}} = \mathfrak{m}^n$ over a local ring $(R, \mathfrak{m})$.

(same proof sketch appears in Dao–De Stefani–G–Huneke–Núñez Betancourt and Brenner–Jeffries–Núñez Betancourt)

Obrigada!

Example (The order matters!)

Take $R = \mathbb{Z}_p[x]$ and $Q = (p, x)$. The lift of Frobenius that satisfies $\phi(x) = x^p$ induces a p-derivation δ on R such that $\delta(x) = 0$, and $D^1_{R|\mathbb{Z}_p} = R \oplus R\frac{d}{dx}$. Note that $px \notin Q^{(3)} = Q^3$. And in fact

$$px \notin Q^{\langle 3 \rangle_{\mathrm{mix}}} \text{ since } \left(\delta \circ \frac{d}{dx}\right)(px) \notin Q.$$

However,

$$\frac{d^2}{dx^2}(px), \left(\frac{d}{dx} \circ \delta\right)(px), \delta^2(px) \in Q.$$

Example (Not every ring has a p-derivation)

Let $S = \mathbb{Z}_p[x_1, \ldots, x_n]$, and $R = S/(p - f)$, where $f \in (x_1, \ldots, x_n)^2$. Suppose that there is some p-derivation δ on R. Considering $p = f$ in R, $\delta(p) = \delta(f) \in (x_1, \ldots, x_n, p)R$. However, $\delta(p) = 1 - p^{p-1}$, which yields a contradiction.

Warning!

The conclusion of Zariski-Nagata for fields fails if the extension $A/pA \hookrightarrow R_Q/QR_Q$ is not separable.

Example

Let $K = \mathbb{F}_p(t)$, $R = K[x]$, and $Q = (x^p - t)$.

Since $\frac{d}{dx}(x^p - t) = 0 \in Q$, and $D^1_{R|K} = R \oplus R\frac{d}{dx}$,

$$Q^{\langle 2 \rangle_K} = Q.$$

Then $Q^{(2)} = Q^2 \neq Q = Q^{\langle 2 \rangle_K}$.