Homological Algebra

Math 915 Fall 2023

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September 29, 2023

Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, I will be most grateful to you for letting me know. If you are looking for a place where to learn homological algebra or category theory, I strongly recommend the following excellent resources:

- Rotman's *An introduction to homological algebra*, second edition. [Rot09]
- Weibel's *Homological Algebra* [Wei94].
- Mac Lane's *Categories for the working mathematician* [ML98].
- Emily Riehl's *Category Theory in context*

Acknowledgements

These notes are partially based on notes I wrote in Spring 2021 for a homological algebra class at the University of California, Riverside, and are heavily influenced by the references above. These notes also owe a big debt to all the students in that class for their comments and questions that lead to multiple improvements, especially Brandon Massaro, Rahul Rajkumar, Adam Richardson, Khoa Ta, Ryan Watson, and Noble Williamson, who found typos and errors. Thank you also to the students who found typos in this new version: Stephen Stern, Danny Anderson, Taylor Murray, Kesavan M, and Sara Mueller.

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Chapter 0

Where are we going?

Homological algebra first appeared in the study of topological spaces. Roughly speaking, homology is a way of associating a sequence of abelian groups (or modules, or other more sophisticated algebraic objects) to another object, for example a topological space. The homology of a topological space encodes topological information about the space in algebraic language — this is what algebraic topology is all about.

More formally, we will study *complexes* and their homology from a more abstract perspective. While algebraic topologists are often concerned with complexes of abelian groups, we will work a bit more generally with complexes of R-modules. The basic assumptions and notation about rings and modules we will use in this class can be found in Appendix A. As an appetizer, we begin with some basic homological algebra definitions.

Definition 0.1. A **chain complex** of R-modules $(C_\bullet, \partial_\bullet)$, also referred to simply as a **complex**, is a sequence of R-modules C_i and R-module homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

such that $\partial_n \partial_{n+1} = 0$ for all n. We refer to C_n as the module in **homological degree** n. The maps ∂_n are the **differentials** of our complex. We may sometimes omit the differentials ∂_n and simply refer to the complex $C_\bullet$ or even C; we may also sometimes refer to $\partial_\bullet$ as *the* differential of $C_\bullet$.

In some contexts, it is important to make a distinction between chain complexes and co-chain complexes, where the arrows go the opposite way: a co-chain complex would look like

$$\cdots \longrightarrow C_{n-1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n+1}} C_{n+1} \longrightarrow \cdots.$$

We will not need to make such a distinction, so we will call both of these complexes and most often follow the convention in the definition above. We will say a complex C is **bounded above** if $C_n = 0$ for all $n \gg 0$, and **bounded below** if $C_n = 0$ for all $n \ll 0$. A **bounded complex** is one that is both bounded above and below. If a complex is bounded, we may sometimes simply write it as a finite complex, say

$$C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_m.$$

Remark 0.2. The condition that $\partial_n \partial_{n+1} = 0$ for all n implies that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$.

Definition 0.3. The complex $(C_\bullet, \partial_\bullet)$ is **exact** at n if $\operatorname{im} \partial_{n+1} = \ker \partial_n$. An **exact sequence** is a complex that is exact everywhere. More precisely, an **exact sequence** of R-modules is a sequence

$$\cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \cdots$$

of R-modules and R-module homomorphisms such that $\operatorname{im} f_n = \ker f_{n+1}$ for all n. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a **short exact sequence**, sometimes written **ses**.

Remark 0.4. The sequence

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact if and only if f is injective. Similarly,

$$M \xrightarrow{f} N \longrightarrow 0$$

is exact if and only if f is surjective. So

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence if and only if

- f is injective
- g is surjective
- $\operatorname{im} f = \ker g$.

When this is indeed a short exact sequence, we can identify A with its image $f(A)$, and $A = \ker g$. Moreover, since g is surjective, by the First Isomorphism Theorem we conclude that $C \cong B/f(A)$, so we might abuse notation and identify C with B/A.

Notation 0.5. We write $A \twoheadrightarrow B$ to denote a surjective map, and $A \hookrightarrow B$ to denote an injective map.

Definition 0.6. The **cokernel** of a map of R-modules $A \xrightarrow{f} B$ is the module

$$\operatorname{coker} f := B / \operatorname{im}(f).$$

Remark 0.7. We can rephrase Remark 0.4 in a fancier language: if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence, then $A = \ker g$ and $C = \operatorname{coker} f$.

Example 0.8. Let π be the canonical projection $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. The following is a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0\,.$$

We will most often be interested in **complexes of R-modules**, where the abelian groups that show up are all modules over the same ring R.

Example 0.9. Let $R = k[x]$ be a polynomial ring over the field k. The following is a short exact sequence:

$$0 \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\pi} R/(x) \longrightarrow 0 \,.$$

The first map is multiplication by x, and the second map is the canonical projection.

Example 0.10. Given an ideal I in a ring R, the inclusion map $\iota : I \to R$ and the canonical projection $\pi : R \to R/I$ give us the following short exact sequence:

$$0 \longrightarrow I \xrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0.$$

Example 0.11. Let $R = k[x]/(x^2)$. The following complex is exact:

$$\cdots \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \longrightarrow \cdots \,.$$

Indeed, the image and the kernel of multiplication by x are both (x).

Sometimes we can show that certain modules vanish or compute them explicitly when they do not vanish by seeing that they fit in some naturally constructed exact sequence involving other modules we understand better. We will discuss this in more detail when we talk about long exact sequences.

Remark 0.12. The complex $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an isomorphism.

Remark 0.13. The complex $0 \longrightarrow M \longrightarrow 0$ is exact if and only if $M = 0$.

Historically, chain complexes first appeared in topology. To study a topological space, one constructs a particular chain complex that arises naturally from information from the space, and then calculates its homology, which ends up encoding important topological information in the form of a sequence of abelian groups.

Definition 0.14 (Homology)**.** The **homology** of the complex $(C_\bullet, \partial_\bullet)$ is the sequence of R-modules

$$\mathrm{H}_n(C_\bullet) = \mathrm{H}_n(C) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

The n**th homology** of $(C_\bullet, \partial_\bullet)$ is $\mathrm{H}_n(C)$. The submodules $Z_n(C_\bullet) = Z_n(C) := \ker \partial_n \subseteq C_n$ are called **cycles**, while the submodules $B_n(C_\bullet) = B_n(C) := \operatorname{im} \partial_{n+1} \subseteq C_n$ are called **boundaries**. One sometimes uses the word boundary to refer an element of $B_n(C)$ (an n-boundary), and the word cycle to refer to an element of $Z_n(C)$ (an n-cycle).

The homology of a complex measures how far our complex is from being exact at each point. Again, we can talk about the **cohomology** of a cochain complex instead, which we write as $\mathrm{H}^n(C)$; we will for now not worry about the distinction.

Remark 0.15. Note that $(C_\bullet, \partial_\bullet)$ is exact at n if and only if $\mathrm{H}_n(C_\bullet) = 0$.

Example 0.16. Let $R = k[x]/(x^3)$. Consider the following complex:

$$F_\bullet = \cdots \longrightarrow R \xrightarrow{\cdot x^2} R \xrightarrow{\cdot x^2} R \longrightarrow \cdots .$$

The image of multiplication by x^2 is (x^2), while the the kernel of multiplication by x^2 is $(x) \supseteq (x^2)$. For all n,

$$\mathrm{H}_n(F_\bullet) = (x)/(x^2) \cong R/(x).$$

Example 0.17. Let $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$ be the canonical projection map. Then

$$C = \underset{2}{\mathbb{Z}} \xrightarrow{4} \underset{1}{\mathbb{Z}} \xrightarrow{\pi} \underset{0}{\mathbb{Z}/2\mathbb{Z}}$$

is a complex of abelian groups, since the image of multiplication by 4 is $4\mathbb{Z}$, and that is certainly contained in $\ker \pi = 2\mathbb{Z}$. The homology of C is

$$\begin{aligned}
&\mathrm{H}_n(C) = 0 && \text{for } n \geqslant 3 \\
&\mathrm{H}_2(C) = \frac{\ker(\mathbb{Z} \xrightarrow{4} \mathbb{Z})}{\mathrm{im}(0 \longrightarrow \mathbb{Z})} = \frac{0}{0} = 0 && \\
&\mathrm{H}_1(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z})}{\mathrm{im}(\mathbb{Z} \xrightarrow{4} \mathbb{Z})} = \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} && \\
&\mathrm{H}_0(C) = \frac{\ker(\mathbb{Z}/2\mathbb{Z} \longrightarrow 0)}{\mathrm{im}(\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z})} = \frac{\mathbb{Z}/2\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} = 0 && \\
&\mathrm{H}_n(C) = 0 && \text{for } n < 0
\end{aligned}$$

Notice that our complex is exact at 2 and 0. The exactness at 2 says that the map $\mathbb{Z} \xrightarrow{4} \mathbb{Z}$ is injective, while exactness at 0 says that π is surjective.

Before we can continue any further into the world of homological algebra, we will need some categorical language. We will take a short break to introduce category theory, and then armed with that knowledge we will be ready to study homological algebra.

Chapter 1

Categories for the working homological algebraist

Most fields in modern mathematics follow the same basic recipe: there is a main type of object one wants to study – groups, rings, modules, topological spaces, etc – and a natural notion of arrows between these – group homomorphisms, ring homomorphisms, module homomorphisms, continuous maps, etc. The objects are often sets with some extra structure, and the arrows are often maps between the objects that preserve whatever that extra structure is. Category theory is born of this realization, by abstracting the basic notions that make math and studying them all at the same time. How many times have we felt a sense of déjà vu when learning about a new field of math? Category theory unifies all those ideas we have seen over and over in different contexts.

Category theory is an entire field of mathematics in its own right. As such, there is a lot to say about category theory, and unfortunately it doesn't all fit in the little time we have to cover it in this course. You are strongly encouraged to learn more about category theory, for example from [ML98] or [Rie17].

Before we go any further, note that there is a long and fun story about why we use the word *collection* when describing the objects in a category. Not all collections are allowed to be sets, an issue that was first discovered by Russel with his famous Russel's Paradox.[1] Russel exposed the fact that one has to be careful with how we formalize set theory. We follow the ZFC (Zermelo–Fraenkel with choice, short for the Zermelo–Fraenkel axioms plus the Axiom of Choice) axiomatization of set theory, and while we will not discuss the details of this formalization here, you are encouraged to read more on the subject.

1.1 Categories

A category consists of a collection of objects and arrows or morphisms between those objects. While these are often sets and some kind of functions between them, beware that this will not always be the case. We will use the words morphism and arrows interchangeably, though *arrow* has the advantage of reminding us we are not necessarily talking about functions.

[1] The collection of all sets that don't contain themselves cannot be a set. Do you see why?

Definition 1.1. A **category** $\mathscr{C}$ consists of three different pieces of data:

- a collection of **objects**, $\mathbf{ob}(\mathscr{C})$,
- for each two objects, say A and B, a collection $\mathrm{Hom}_{\mathscr{C}}(A,B)$ of **arrows** or **morphisms** from A to B, and
- for each three objects A, B, and C, a composition

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,C) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A,C)\ . \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

We will often drop the $\circ$ and write simply gf for $g \circ f$.

These ingredients satisfy the following axioms:

1) The $\mathrm{Hom}_{\mathscr{C}}(A,B)$ are all disjoint. In particular, if f is an arrow in $\mathscr{C}$, we can talk about its **source** A and its **target** B as the objects such that $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$.

2) For each object A, there is an **identity arrow** $1_A \in \mathrm{Hom}_{\mathscr{C}}(A,A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B,A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A,B)$.

3) Composition is **associative**: $f \circ (g \circ h) = (f \circ g) \circ h$ for all appropriately chosen arrows.

Notation 1.2. We sometimes write $f\colon A \to B$ or $A \xrightarrow{f} B$ for an arrow $f \in \mathrm{Hom}(A,B)$.

Exercise 1. Prove that every element in a category has a unique identity morphism.

Here are some categories you have likely encountered before:

Example 1.3.

1) The category **Set** with objects all sets and arrows all functions between sets.

2) The category **Grp** whose objects are the collection of all groups, and whose arrows are all the homomorphisms of groups. The identity arrows are the identity homomorphisms.

3) The category **Ab** with objects all abelian groups, and arrows the homomorphisms of abelian groups. The identity arrows are the identity homomorphisms.

4) The category **Ring** of rings and ring homomorphisms. Contrary to what you may expect, this is not nearly as important as the next one.

5) The category R-**mod** of left modules over a fixed ring R and with R-module homomorphisms. Sometimes one writes R-**Mod** for this category, and reserve R-**mod** for the category of finitely generated R-modules with R-module homomorphisms. When $R = k$ is a field, the objects in the category k-**Mod** are k-vector spaces, and the arrows are linear transformations; we may instead refer to this category as **Vect**-k.

6) The category **Top** of topological spaces and continuous functions.

One may consider many variations of the categories above. Here are some variations on vector spaces:

Example 1.4. Let k be a field.

1) The collection of finite dimensional k-vector spaces with all linear transformations is a category.

2) The collection of all n-dimensional k-vector spaces with all linear transformations is a category.

3) The collection of all k-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.

4) The collection of all k-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it is not closed under composition.

5) The collection of all n-dimensional vector spaces with linear transformations of determinant 0 is not a category, since it does not have identity maps.

Here is an important variation of **Set**:

Example 1.5. The category $\mathbf{Set}^*$ of pointed sets has objects all pairs (X, x) of sets X and points $x \in X$, and for two pointed sets (X, x) and (Y, y), the morphisms from (X, x) to (Y, y) are functions $f\colon X \to Y$ such that $f(x) = y$, with the usual composition of functions.

Example 1.6. The empty category has no objects and no arrows.

While the collections of objects and arrows might not actually be sets, sometimes they are.

Definition 1.7. A category $\mathscr{C}$ is **locally small** if for all objects A and B in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set. A category $\mathscr{C}$ is **small** if it is locally small and the collection of all objects in $\mathscr{C}$ is a set.

In fact, one can define a small category as one where the collection of all arrows is a set. It follows immediately that the collection of all objects is also a set, since it must be a subset of the set of arrows – for each object, there is an identity arrow.

Many important categories are at least locally small. For example, **Set** is locally small but not small. In a locally small category, we can now refer to its Hom-sets.

Categories where the objects are sets with some extra structure and the arrows are some kind of functions between the objects are called **concrete**. Not all categories are concrete.

Example 1.8. Given a partially ordered set $(X, \leqslant)$, we can regard X itself as a category: the objects are the elements of X, and for each x and y in X, $\mathrm{Hom}_X(x, y)$ is either a singleton if $x \leqslant y$ or empty if $x \nleqslant y$. There is only one possible way to define composition, and the transitive property of $\leqslant$ guarantees that the composition of arrows is indeed well-defined: if there is an arrow $i \to j$ and an arrow $j \to k$, then $i \leqslant j$ and $j \leqslant k$, so $i \leqslant k$ and thus there is a unique arrow $i \to k$. This category is clearly locally small, since all nonempty Hom-sets are in fact singletons. It is in fact small, since the objects are by construction the set X. We will denote this poset category by $\mathbf{PO}(X)$.

Example 1.9. For each positive integer n, the category $\mathbf{n}$ has n objects $0, 1, \ldots, n-1$ and $\text{Hom}(i, j)$ is either empty if $i > j$ or a singleton if $i \leqslant j$. As Example 1.8, composition is defined in the only way possible, and things work out. This is the poset category for the poset $(\{0, 1, \ldots, n-1\}, \leqslant)$ with the usual $\leqslant$.

Example 1.10. Fix a field k. We define a category **Mat-k** with objects all positive integers, and given two positive integers a and b, the Hom-set $\text{Hom}(a, b)$ consists of all $b \times a$ matrices with entries in k. The composition rule is given by product of matrices: given $A \in \text{Hom}(a, b)$ and $B \in \text{Hom}(b, c)$, the composition $B \circ A$ is the matrix $BA \in \text{Hom}(a, c)$. For each object a, its identity arrow is given by the $a \times a$ identity matrix.

Example 1.11. Let G be a directed graph. We can construct a category from G as follows: the objects are the vertices of G, and the arrows are directed paths in the graph G. In this category, composition of arrows corresponds to concatenation of paths. For each object A, the identity arrow corresponds to the empty path from A to A.

Remark 1.12. A locally small category with just one element is completely determined by its unique Hom-set; it thus consists of a set S with an associative operation that has an identity element, which in this class is what we call a **semigroup**.[2]

A key insight we get from category theory is that many important concepts can be understood through diagrams. Homological algebra is in many ways the study of commutative diagrams. One way to formalize what a diagram is involves talking about functors, which we will discuss in Section 1.2; here is a more down to earth definition.

Definition 1.13. A **diagram** in a category $\mathscr{C}$ is a directed multigraph whose vertices are objects in C and whose arrows/edges are morphisms in $\mathscr{C}$. A commutative diagram in $\mathscr{C}$ is a diagram in which for each pair of vertices A and B, any two paths from A to B compose to the same morphism.

Example 1.14. The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

commutes if and only if $gf = vu$.

There are some special types of arrows we will want to consider.

Definition 1.15. Let $\mathscr{C}$ be any category.

- An arrow $f \in \text{Hom}_{\mathscr{C}}(A, B)$ is **left invertible** if there exists $g \in \text{Hom}_{\mathscr{C}}(B, A)$ such that $gf = 1_A$. In this case, we say that g is the **left inverse** of f. So g is a left inverse of f if the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd}$$

  commutes.

[2]Some authors prefer the term monoid.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is **right invertible** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $fg = 1_B$. In this case, we say that g is the **right inverse** of f. So g is a right inverse of f if the diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is an **isomorphism** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$. Unsurprisingly, such an arrow g is called the **inverse** of f. We say two objects A and B are **isomorphic** if there exists an isomorphism $A \to B$.

- An arrow $f \in \mathrm{Hom}(B, C)$ is **monic**, a **monomorphism**, or a **mono** if for all arrows

$$\begin{tikzcd} A \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

if $fg_1 = fg_2$ then $g_1 = g_2$.

- Similarly, an arrow $f \in \mathrm{Hom}(A, B)$ is an **epi** or an **epimorphism** if for all arrows

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

if $g_1 f = g_2 f$ then $g_1 = g_2$.

Here are some examples:

Exercise 2. Show that in **Set**, the monos coincide with the injective functions and the epis coincide with the surjective functions.

Example 1.16.

a) In **Grp**, **Ring**, and R-**Mod** the isomorphisms are the morphisms that are bijective functions.

b) In contrast, in **Top** the isomorphisms are the homeomorphisms, which are the bijective continuous functions with continuous inverses. These are *not* the same thing as just the bijective continuous functions.

Exercise 3. Show that in any category, every isomorphism is both epi and mono.

Exercise 4. Show that the usual inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epi in the category **Ring**.

This *should* feel weird: it says being epi and being surjective are *not* the same thing. Similarly, being monic and being injective are *not* the same thing.

Exercise 5. Show that the canonical projection $\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a mono in the category of divisible abelian groups.[3]

[3] An abelian group A is divisible if for every $a \in A$ and every positive integer n there exists $b \in A$ such that $nb = a$.

Exercise 6. Show that given any poset P, in the poset category of P every morphism is both monic and epic, but no nonidentity morphism has a left or right inverse.

There are some special types of objects we will want to consider.

Definition 1.17. Let $\mathscr{C}$ be a category. An **initial object** in $\mathscr{C}$ is an object i such that for every object x in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(i, x)$ is a singleton, meaning there exists a unique arrow $i \longrightarrow x$. A **terminal object** in $\mathscr{C}$ is an object t such that for every object x in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(x, t)$ is a singleton, meaning there exists a unique arrow $x \longrightarrow t$. A **zero object** in $\mathscr{C}$ is an object that is both initial and terminal.

Exercise 7. Initial objects are unique up to unique isomorphism. Terminal objects are unique up to unique isomorphism.

So we can talk about *the* initial object, *the* terminal object, and *the* zero object, if they exist.

Example 1.18.

a) The empty set is initial in **Set**. Any singleton is terminal. Since the empty set and a singleton are not isomorphic in **Set**, there is no zero object in **Set**.

b) The 0 module is the zero object in R-**Mod**.

c) The trivial group $\{e\}$ is the zero object in **Grp**.

d) In the category of rings, $\mathbb{Z}$ is the initial object, but there is no terminal object unless we allow the 0 ring.

e) There are no initial nor terminal objects in the category of fields.

We will now continue to follow a familiar pattern and define the related concepts one can guess should be defined.

Definition 1.19. A **subcategory** $\mathscr{C}$ of a category $\mathscr{D}$ consists of a subcollection of the objects of $\mathscr{D}$ and a subcollection of the morphisms of $\mathscr{D}$ such that the following hold:

- For every object C in $\mathscr{C}$, the arrow $1_C \in \mathrm{Hom}_{\mathscr{D}}(C, C)$ is an arrow in $\mathscr{C}$.
- For every arrow in $\mathscr{C}$, its source and target in $\mathscr{D}$ are objects in $\mathscr{C}$.
- For every pair of arrows f and g in $\mathscr{C}$ such that fg is an arrow that makes sense in $\mathscr{D}$, fg is an arrow in $\mathscr{C}$.

In particular, $\mathscr{C}$ is a category in its own right.

Example 1.20. The category of finitely generated R-modules with R-module homomorphisms is a subcategory of R-**Mod**.

Definition 1.21. A subcategory $\mathscr{C}$ of $\mathscr{D}$ is a **full subcategory** if $\mathscr{C}$ includes *all* of the arrows in $\mathscr{D}$ between any two objects in $\mathscr{C}$.

Example 1.22.

a) The category **Ab** of abelian groups is a full subcategory of **Grp**.

b) Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, not every function between two groups is a group homomorphism, so **Grp** is not a full subcategory of **Set**.

c) The category whose objects are all sets and with arrows all bijections is a subcategory of **Set** that is not full.

Here is another way of constructing a new category out of an old one.

Definition 1.23. Let $\mathscr{C}$ be a category. The **opposite category** of $\mathscr{C}$, denoted $\mathscr{C}^{\mathrm{op}}$, is a category whose objects are the objects of $\mathscr{C}$, and such that each arrow $f \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B)$ is the same as some arrow in $\mathrm{Hom}_{\mathscr{C}}(B, A)$. The composition fg of two morphisms f and g in $\mathscr{C}^{\mathrm{op}}$ is defined as the composition gf in $\mathscr{C}$.

Many objects and concepts one might want to describe are obtained from existing ones by flipping the arrows. Opposite categories give us the formal framework to talk about such things. We will often want to refer to **dual** notions, which will essentially mean considering the same notion in a category $\mathscr{C}$ and in the opposite category $\mathscr{C}^{\mathrm{op}}$; in practice, this means we should flip all the arrows involved. We will see examples of this later on.

The dual category construction gives us a formal framework to talk about **dual notions**. We will often make a statement in a category $\mathscr{C}$ and make comments about the **dual statement**; in practice, this corresponds to simply switching the way all arrows go. Here are some examples of dual notions and statements:

source	target
epi	mono
g is a right inverse for f	g is a left inverse for f
f is invertible	f is invertible
initial objects	terminal objects
homology	cohomology

The prefix co- is often used to denote the dual of something, such as in *co*homology. Note that the dual of the dual is the original statement; formally, $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$. Sometimes we can easily prove a statement by dualizing; however, this is not always straightforward, and one needs to carefully dualize all portions of the statement in question. Nevertheless, Sanders MacLane, one of the fathers of category theory, wrote that "If any statement about a category is deducible from the axioms for a category, the dual statement is likely deducible" [Mac50]. One of the upshots of duality is that any theorem in category theory must simultaneously prove two theorems: the original statement and its dual. But for this to hold, we need proofs that use the abstraction of a purely categorical proof.

Opposite categories are more interesting than they might appear at first; there is more than just flipping all the arrows. For example, consider the opposite category of **Set**. For any nonempty set X, there is a unique morphism in **Set** (a function) $i : \emptyset \to X$, but there are no functions $X \to \emptyset$, so $i^{\mathrm{op}}\colon \emptyset \to X$ is not a function. Thus thinking about $\mathbf{Set^{op}}$ is a bit difficult. One can show that this is the category of complete atomic Boolean algebras – but we won't concern ourselves with what that means.

1.2 Functors

Many mathematical constructions are *functorial*, in the sense that they behave well with respect to morphisms. In the formalism of category theory, this means that we can think of a functorial construction as a functor.

Definition 1.24. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a mapping that assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$, such that

- F preserves the composition of maps, meaning $F(fg) = F(f)F(g)$ for all arrows f and g in $\mathscr{C}$, and
- F preserves the identity arrows, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

A **contravariant functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a mapping that assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(B), F(A))$, such that

- F preserves the composition of maps, meaning $F(fg) = F(g)F(f)$ for all composable arrows f and g in $\mathscr{C}$, and
- F preserves the identity arrows, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

So a contravariant functor is a functor that flips all the arrows. We can also describe a contravariant functor as a covariant functor from $\mathscr{C}$ to the opposite category of $\mathscr{D}$, $\mathscr{D}^{\mathrm{op}}$.

Remark 1.25. A contravariant functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ can be thought of as a covariant functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$, or also as a covariant functor $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$. If using one of these conventions, one needs to be careful, however, when composing functors, so that the respective sources and targets match up correctly. While we haven't specially discussed how one composes functors, it should be clear that applying a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$ is the same as applying a functor $\mathscr{C} \longrightarrow \mathscr{D}$, which we can write as GF.

For example, if $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$ are both contravariant functors, the composition $GF\colon \mathscr{C} \longrightarrow \mathscr{E}$ is a covariant functor, since

$$\begin{tikzcd} A \arrow[d, "f"'] & \rightsquigarrow & F(A) & \rightsquigarrow & GF(A) \arrow[d, "GF(f)"'] \\ B & & F(B) \arrow[u, "F(f)"] & & GF(B) \end{tikzcd}$$

So we could think of F as a covariant functor $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$ and G as a covariant functor $\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{E}$. Similarly, if $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a covariant functor and $G : \mathscr{D} \longrightarrow \mathscr{E}$ is a contravariant functor, $GF\colon \mathscr{C} \longrightarrow \mathscr{E}$ is a contravariant functor. In this case, we can think of G as a covariant functor $\mathscr{D} \longrightarrow \mathscr{E}^{\mathrm{op}}$, so that GF is now a covariant functor $\mathscr{C} \longrightarrow \mathscr{E}^{\mathrm{op}}$.

Exercise 8. Show that functors preserve isomorphisms.

Remark 1.26. Any functor sends isos to isos, since it preserves compositions and identities.

Example 1.27. Here are some examples of functors you may have encountered before.

a) Many categories one may think about are concrete categories, where the objects are sets with some extra structure, and the arrows are functions between those sets that preserved that extra structure. The **forgetful functor** from such a category to **Set** is the functor that, just as the name says, *forgets* that extra structure, and sees only the underlying sets and functions of sets. For example, the forgetful functor $\mathbf{Gr} \longrightarrow \mathbf{Set}$ sends each group to its underlying set, and each group homomorphism to the corresponding function of sets.

b) The identity functor $1_{\mathscr{C}}$ on any category $\mathscr{C}$ does what the name suggests: it sends each object to itself and each arrow to itself.

c) Given an object C in a category $\mathscr{C}$, the **constant functor** at C is the functor $\Delta C : \mathscr{C} \to \mathscr{C}$ that sends every object to C every arrow to 1_C.

d) Given a group G, the subgroup $[G, G]$ of G generated by the set of commutators
$$\{ghg^{-1}h^{-1} \mid g, h \in G\}$$
is a normal subgroup, and the quotient $G^{\text{ab}} := G/[G, G]$ is called the **abelianization** of G. The group G^{ab} is abelian. Given a group homomorphism $f\colon G \to H$, f automatically takes commutators to commutators, so it induces a homomorphism $\tilde{f}\colon G^{\text{ab}} \to H^{\text{ab}}$. More precisely, abelianization gives a covariant functor from **Grp** to **Ab**.

e) The unit group functor $-^*\colon \mathbf{Ring} \to \mathbf{Grp}$ sends a ring R to its group of units R^*. To see this is indeed a functor, we should check it behaves well on morphisms; and indeed if $f\colon R \to S$ is a ring homomorphism, and $u \in R^*$ is a unit in R, then
$$f(u)f(u^{-1}) = f(uu^{-1}) = f(1_R) = 1_S,$$
so $f(u)$ is a unit in S. Thus f induces a function $R^* \to S^*$ given by restriction of f to R^*, which must therefore be a group homomorphism since f preserves products.

f) Fix a field k. Given a vector space V, the set V^* of linear transformations from V to k is a k-vector space, the **dual vector space** of V. If $\varphi\colon W \to V$ is a linear transformation and $\ell : V \to k$ is an element of V^*, then $\ell \circ \varphi : W \to k$ is in W^*. Doing this for all elements $\ell \in V^*$ gives a function $\varphi^* : V^* \to W^*$, and one can show that φ^* is a linear transformation. The assignment that sends each vector space V to its dual vector space V^* and each linear transformation φ to φ^* is a contravariant functor $\mathbf{Vect}\text{-}k \to \mathbf{Vect}\text{-}k$.

g) Localization is a functor. Let R be a ring and W be a multiplicatively closed set in R. The localization at W induces a a functor $R\text{-}\mathbf{mod} \longrightarrow W^{-1}R\text{-}\mathbf{mod}$: this functor sends each R-module M to $W^{-1}M$, and each R-module homomorphism $\alpha : M \to N$ to the R-module homomorphism $W^{-1}\alpha : W^{-1}M \to W^{-1}N$.

Remark 1.28. If we apply a covariant functor to a diagram, then we get a diagram of the same shape:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] & & F(A) \arrow[r, "F(f)"] \arrow[d, "F(u)"'] & F(B) \arrow[d, "F(g)"] \\ C \arrow[r, "v"'] & D & \arrow[r, squiggly, "F"] & & F(C) \arrow[r, "F(v)"'] & F(D) \end{tikzcd}$$

However, if we apply a contravariant functor to the same diagram, we get a similar diagram but with the arrows reversed:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] & & F(A) & F(B) \arrow[l, "F(f)"'] \\ C \arrow[r, "v"'] & D & {} \arrow[r, phantom, "\overset{F}{\rightsquigarrow}"] & F(C) \arrow[u, "F(u)"] & F(D) \arrow[l, "F(v)"] \arrow[u, "F(g)"'] \end{tikzcd}$$

Definition 1.29. The category **Cat** has objects all small categories and arrows all functors between them.

If we think about functors as functions between categories, it's natural to consider what would be the appropriate versions of the notions of injective or surjective.

Definition 1.30. A covariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ between locally small categories is

- **faithful** if all the functions of sets
$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A, B) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)) \\ f &\longmapsto F(f) \end{aligned}$$
are injective.
- **full** if all the functions of sets
$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A, B) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)) \\ f &\longmapsto F(f) \end{aligned}$$
are surjective.
- **fully faithful** if it is full and faithful.
- **essentially surjective** if every object d in $\mathscr{D}$ is isomorphic to $F(c)$ for some c in $\mathscr{C}$.
- an **embedding** if it is fully faithful and injective on objects.

Example 1.31. The forgetful functor R-**Mod** $\longrightarrow$ **Set** is faithful since any two maps of R-modules with the same source and target coincide if and only if they are the same function of sets. This functor is not full, since not every function between the underlying sets of two R-modules is an R-module homomorphism.

Remark 1.32. A fully faithful functor is not necessarily injective on objects, but it is injective on objects up to isomorphism.

Remark 1.33. A subcategory $\mathscr{C}$ of $\mathscr{D}$ is full if the inclusion functor $\mathscr{C} \longrightarrow \mathscr{D}$ is full.

Exercise 9. Show that every fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ reflects isos:

a) If f is an arrow in $\mathscr{C}$ such that $F(f)$ is an iso, then f is an iso.

b) If $F(X)$ and $F(Y)$ are isomorphic, then the objects X and Y are isomorphic in $\mathscr{C}$.

Note that the converses of these statements hold for any functor.

To close this section, here are the two of the most important functors we will discuss this semester:

Definition 1.34. Let $\mathscr{C}$ be a locally small category. An object A in $\mathscr{C}$ induces two Hom functors:

- The covariant functor $\operatorname{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} \longrightarrow \mathbf{Set}$ is defined as follows:

$$\begin{array}{lccc} & \mathscr{C} \longrightarrow \mathbf{Set} & & \\ \text{on objects:} & X \longmapsto \operatorname{Hom}_{\mathscr{C}}(A,X) & & \\ \text{on arrows:} & \begin{array}{c} B \\ {\scriptstyle f}\downarrow \\ C \end{array} \quad \rightsquigarrow & \begin{array}{c} \operatorname{Hom}_{\mathscr{C}}(A,B) \\ \downarrow \\ \operatorname{Hom}_{\mathscr{C}}(A,C) \end{array} & \begin{array}{c} \ni g \\ \downarrow \\ \ni f \circ g \end{array} \end{array}$$

  We read $\operatorname{Hom}_{\mathscr{C}}(A,-)$ as *Hom from* A, and may refer to this functor as the covariant functor **represented by** A. Given an arrow f in $\mathscr{C}$, we write $f_* := \operatorname{Hom}_{\mathscr{C}}(A,f)$. It is easier to see what f_* does through the following commutative diagram:

$$f_* = \operatorname{Hom}_{\mathscr{C}}(A,f) : \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed, "f_*(g)=fg"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

- The contravariant functor $\operatorname{Hom}_{\mathscr{C}}(-,B) : \mathscr{C} \longrightarrow \mathbf{Set}$ is defined as follows:

$$\begin{array}{lccc} & \mathscr{C} \longrightarrow \mathbf{Set} & & \\ \text{on objects:} & X \longmapsto \operatorname{Hom}_{\mathscr{C}}(X,B) & & \\ \text{on arrows:} & \begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ C \end{array} \quad \rightsquigarrow & \begin{array}{c} \operatorname{Hom}_{\mathscr{C}}(A,B) \\ \uparrow \\ \operatorname{Hom}_{\mathscr{C}}(C,B) \end{array} & \begin{array}{c} \ni g \circ f \\ \uparrow \\ \ni g \end{array} \end{array}$$

  We read $\operatorname{Hom}_{\mathscr{C}}(-,B)$ as *Hom to* B, and we may refer to this functor as the contravariant functor **represented by** B. Given an arrow f in $\mathscr{C}$, we write $f^* := \operatorname{Hom}_{\mathscr{C}}(-,B)$. It is easier to see what f^* does through the following commutative diagram:

$$f^* = \operatorname{Hom}_{\mathscr{C}}(f,B) : \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "f^*(g)=gf"'] & C \arrow[d, "g"] \\ & B \end{tikzcd}$$

Exercise 10. Check that $\operatorname{Hom}(A,-)$ and $\operatorname{Hom}(-,B)$ are indeed functors.

We will be particularly interested in the Hom-functors in the category R**-mod**, which we will study in detail in a later chapter.

1.3 Natural transformations

Definition 1.35. Let F and G be covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A **natural transformation** between F and G is a mapping that to each object A in $\mathscr{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes. We sometimes write

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathscr{D} \end{tikzcd}$$

or simply $\eta \colon F \implies G$ to indicate that η is a natural transformation from F to G.

Definition 1.36. Let F and G be contravariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A **natural transformation** between F and G is a mapping that to each object A in $\mathscr{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] & G(A) \\ F(B) \arrow[r, "\eta_B"'] \arrow[u, "F(f)"] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

commutes.

Often, when studying a particular topic, we sometimes say a certain map is *natural* to mean that there is actually a natural transformation behind it.

Example 1.37. Recall the abelianization functor we discussed in Example 1.27. The abelianization comes equipped with a natural projection map $\pi_G : G \longrightarrow G^{\mathrm{ab}}$, the usual quotient map from G to a normal subgroup. Here we mean natural in two different ways: both that this is the common sense map to consider, and that this is in fact coming from a natural transformation. What's happening behind the scenes is that abelianization is a functor $\mathrm{ab} \colon \mathbf{Grp} \longrightarrow \mathbf{Grp}$. On objects, the abelianizations functor is defined as $G \mapsto G^{\mathrm{ab}}$. Given an arrow, meaning a group homomorphism $G \xrightarrow{f} H$, one can check that $[G, G]$ is contained in the kernel of $\pi_H f$, so $\pi_H f$ factors through G^{ab}, and there exists a group homomorphism f^{ab} making the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d, "f"'] & G^{\mathrm{ab}} \arrow[d, dashed, "f^{\mathrm{ab}}"] \\ H \arrow[r, "\pi_H"'] & H^{\mathrm{ab}} \end{tikzcd}.$$

So the abelianization functor takes the arrow f to f^{ab}. The commutativity of the diagram above says that π_{-} is a natural transformation π between the identity functor on **Grp** and the abelianization functor, which we can write more compactly as

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{ab}"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathbf{Grp} \end{tikzcd}.$$

Example 1.38. The determinant gives rise to a natural transformation. Fix an integer $n \geqslant 1$, and consider the GL_n functor

$$\mathrm{GL}_n\ \mathbf{Ring} \to \mathbf{Grp}$$

that takes each ring R to the group GL_n of invertible $n \times n$ matrices with entries in R, and that takes each ring homomorphism $f\colon R \to S$ to the map

$$\mathrm{GL}_n(f)\colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$$

that applies f to all the entries of each matrix $A \in \mathrm{GL}_n(R)$, and which can be shown to be a group homomorphism. We claim that the determinant is a natural transformation from GL_n to the unit functor $(\ )^*$ we defined in Example 1.27. First, note that the determinant of an invertible matrix is a unit, so the determinant gives a map $\mathrm{GL}_n(R) \to R^*$. Moreover, given any ring homomorphism $f\colon R \to S$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\det"] \arrow[d, "f"'] & R^* \arrow[d, "f"] \\ \mathrm{GL}_n(S) \arrow[r, "\det"'] & S^*. \end{tikzcd}$$

Above we identified f with both the map $\mathrm{GL}_n(f)$ obtained by applying f to all coordinates of A and the restriction of f to the unit groups, meaning the image of f under the units functor. This commutative diagram just encodes the fact that taking determinants commutes with applying f: for any invertible $n \times n$ matrix A,

$$f(\det(A)) = \det(f(A)).$$

Definition 1.39. A **natural isomorphism** is a natural transformation η where each η_A is an isomorphism.

Exercise 11. Show that a natural transformation $\eta\colon F \implies G$ is a natural isomorphism if and only if there exists a natural transformation $\mu\colon G \implies F$ such that $\eta \circ \mu$ is the identity natural isomorphism on G and $\mu \circ \eta$ is the identity natural isomorphism on F.

Warning: there are many theorems that say that a particular isomorphism is natural; however, not all isomorphisms are natural! Whenever S is an infinite set, the sets $S \times S$ are in bijection S, but no such bijection can be natural. Details below.

Exercise 12. Let $\mathbf{Set}^\infty$ be the full subcategory of $\mathbf{Set}$ consisting of all infinite sets. Let

$$F\colon \mathbf{Set}^\infty \to \mathbf{Set}^\infty$$

be the functor that on objects is given by the rule $F(S) = S \times S$, and on morphisms is given by $F(f) = (f, f)$. Show that there is no natural isomorphism $\eta : F \Rightarrow 1_{\mathbf{Set}^\infty}$.

Definition 1.40. Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be two functors between the categories $\mathscr{C}$ and $\mathscr{D}$. We write

$$\mathrm{Nat}(F, G) = \{\text{natural transformations } F \implies G\}.$$

Given two categories $\mathscr{C}$ and $\mathscr{D}$, one can build a **functor category**[4] with objects all covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$, and arrows the corresponding natural transformations. This category is denoted $\mathscr{D}^{\mathscr{C}}$. Sometimes one writes $\mathrm{Hom}(F, G)$ for $\mathrm{Nat}(F, G)$, but we will avoid that, as it might make things even more confusing.

For the functor category to truly be a category, though, we need to know how to compose natural transformations.

Remark 1.41. Consider natural transformations

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{D} \arrow[from=U, to=D, phantom, "\Downarrow\varphi"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathscr{D}. \arrow[from=U, to=D, phantom, "\Downarrow\eta"] \end{tikzcd}$$

We can compose them for form a new natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathscr{D}. \arrow[from=U, to=D, phantom, "\Downarrow\eta\varphi"] \end{tikzcd}$$

For each object C in $\mathscr{C}$, $\eta\varphi$ sends C to the arrow

$$F(C) \xrightarrow{\varphi_C} G(C) \xrightarrow{\eta_C} H(C).$$

This makes the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\eta_A"] \arrow[d, "G(f)"'] & H(A) \arrow[d, "H(f)"] \\ F(B) \arrow[r, "\varphi_B"'] & G(B) \arrow[r, "\eta_B"'] & H(B) \end{tikzcd}$$

commute; replacing the horizontal arrows with the composition gives us the commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A\varphi_A"] \arrow[d, "F(f)"'] & H(A) \arrow[d, "H(f)"] \\ F(B) \arrow[r, "\eta_B\varphi_B"'] & H(B) \end{tikzcd}$$

which encodes the fact that $\eta\varphi$ is a natural transformation.

[4]Yes, the madness is neverending.

Definition 1.42. Two categories $\mathscr{C}$ and $\mathscr{D}$ are **equivalent** if there exist functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and two natural isomorphisms $\alpha : GF \implies 1_{\mathscr{C}}$ and $\beta : FG \implies 1_{\mathscr{D}}$. We say that a functor $F : \mathscr{C} \to \mathscr{D}$ is an **equivalence of categories** if there exists a functor G and natural isomorphisms α and β as above.

If one assumes the Axiom of Choice, this is the right notion of isomorphism of two categories (though not in the categorical sense!); better said, two categories that are equivalent are essentially the same. Note that this does not mean that there is a bijection between the objects of $\mathscr{C}$ and the objects of $\mathscr{D}$. In fact, one can show that a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective – though this fact requires the Axiom of Choice!

Exercise 13. Let $\mathscr{C}$ be the category with one object C and a unique arrow 1_C. Let $\mathscr{D}$ be the category with two objects D_1 and D_2 and four arrows: the identities 1_{D_i} and two isomorphisms $\alpha : D_1 \to D_2$ and $\beta : D_2 \to D_1$. Let $\mathscr{E}$ be the category with two objects E_1 and E_2 and only two arrows, 1_{E_1} and 1_{E_2}.

a) Show that $\mathscr{C}$ and $\mathscr{D}$ are equivalent categories.

b) Show that $\mathscr{C}$ and $\mathscr{E}$ are not equivalent categories.

The functors that are naturally isomorphic to some Hom functor are important.

Definition 1.43. A covariant functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists an object A in $\mathscr{C}$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(A, -)$. A contravariant functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists an object B in $\mathscr{C}$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, B)$.

Example 1.44. We claim that the identity functor $\mathbf{Set} \longrightarrow \mathbf{Set}$ is representable. Let $\mathbf{1}$ be a singleton set. Given any set X, there is a bijection between elements $x \in X$ and functions $\mathbf{1} \longrightarrow X$ sending the one element in $\mathbf{1}$ to each x. Moreover, given any other set Y, and a function $f : X \longrightarrow Y$, our bijections make the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"'] \\
\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, Y) \arrow[r, "\cong"] & Y.
\end{tikzcd}$$

This data gives a natural isomorphism between the identity functor and $\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, -)$.

Exercise 14. Show that the forgetful functor $\mathbf{Grp} \longrightarrow \mathbf{Set}$ is representable.

Exercise 15. Given a ring R, show that the forgetful functor R-**mod** $\longrightarrow \mathbf{Set}$ is representable.

The Yoneda Lemma tells us that in order to study a locally small category $\mathscr{C}$, it is in many ways sufficient to study the category of functors from $\mathscr{C}$ to $\mathbf{Set}$, and that representable functors are the most important functors of all.

1.4 The Yoneda Lemma

Even though this is only a short introduction to category theory, we would be remiss not to mention the Yoneda Lemma, arguably the most important statement in category theory.

Theorem 1.45 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a locally small category, and fix an object A in $\mathscr{C}$. Let $F\colon \mathscr{C} \longrightarrow$ **Set** be a covariant functor. Then there is a bijection*

$$\mathit{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \xrightarrow{\ \gamma\ } F(A)\ .$$

Moreover, this correspondence is natural in both A and F.

Proof. Let φ be a natural transformation in $\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F)$. The proof is essentially the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(A,A) \arrow[rrr, "\mathrm{Hom}_{\mathscr{C}}(A,f)"] \arrow[ddd, "\varphi_A"'] & & & \mathrm{Hom}_{\mathscr{C}}(A,X) \arrow[ddd, "\varphi_X"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & (F(f))u = \varphi_X(f) & \\
F(A) \arrow[rrr, "F(f)"'] & & & F(X)
\end{tikzcd}$$

Our bijection will be defined by

$$\gamma(\varphi) := \varphi_A(1_A).$$

We should first check that this makes sense: arrows in **Set** are just functions between sets, and so φ_A is a function of sets $\mathrm{Hom}_{\mathscr{C}}(A,A) \longrightarrow F(A)$. Also, $\mathrm{Hom}_{\mathscr{C}}(A,A)$ is a set that contains at least the element 1_A, and $\varphi_A(1_A)$ is some element in the set $F(A)$.

Given any fixed arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A,X)$, the fact that φ is a natural transformation translates into the outer commutative diagram. In particular, the functions of sets $F(f)\varphi_A$ and $\varphi_X \mathrm{Hom}_{\mathscr{C}}(A,f)$ coincide, and must in particular take 1_A to the same element in $F(X)$. This is the commutativity of the inner diagram, with $u := \varphi_A(1_A)$.

The commutativity of the diagram above says that φ is completely determined by $\varphi_A(1_A)$, since for any other object X in $\mathscr{C}$ and any arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A,X)$, we necessarily have $\varphi_X(f) = F(f)\varphi_A(1_A)$. Thus if φ and η are distinct natural transformations, then there exists some object X and some $f \in \mathrm{Hom}_{\mathscr{C}}(A,X)$ such that

$$\varphi_X(f) \neq \eta_X(f), \quad \text{so } F(f)\varphi_A(1_A) \neq F(f)\eta_A(1_A) \quad \text{and thus } \varphi_A(1_A) \neq \eta_A(1_A).$$

In particular, our map $\gamma(\varphi) = \varphi_A(1_A)$ is injective.

Moreover, note that each choice of $u \in F(A)$ gives rise to a different natural transformation φ by setting $\varphi_X(f) = F(f)u$. To check that this is in fact a natural transformation, one needs to check that for all arrows $g\colon X \to Y$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(A,X) \arrow[r, "\varphi_X"] \arrow[d, "g_*"'] & F(X) \arrow[d, "F(g)"] \\ \mathrm{Hom}_{\mathscr{C}}(A,Y) \arrow[r, "\varphi_Y"'] & F(Y) \end{tikzcd}$$

commutes. And indeed, given any $f \in \mathrm{Hom}_{\mathscr{C}}(A,X)$,

$$\begin{aligned} F(g) \circ \varphi_X(f) &= F(g)F(f)u && \text{by definition of } \varphi \\ &= F(gf)u && \text{since } F \text{ is a functor} \\ &= \varphi_Y(gf) && \text{by definition of } \varphi \\ &\varphi_Y \circ g_*(f) && \text{by definition of } g_*. \end{aligned}$$

This shows that the diagram above commutes, and we conclude that the assignment φ given by $\varphi_X(f) = F(f)u$ is indeed a natural transformation. We have shown that our proposed map γ is a bijection.

We now have two naturality statements to prove. Naturality in the functor means that given a natural isomorphism $\eta\colon F \longrightarrow G$, the following diagram must commute:

$$\begin{tikzcd} \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\gamma_F"] \arrow[d, "\eta_*"'] & F(A) \arrow[d, "\eta_A"] \\ \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),G) \arrow[r, "\gamma_G"'] & G(A) \end{tikzcd}$$

Given a natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F,

$$\begin{aligned} \eta_A \circ \gamma_F(\varphi) &= \eta_A(\varphi_A(1_A)) && \text{by definition of } \gamma \\ &= (\eta \circ \varphi)_A(1_A) && \text{by definition of composition of natural transformations} \\ &= \gamma_G(\eta \circ \varphi) && \text{by definition of } \gamma \\ &= \gamma_G \circ \eta_*(\varphi) && \text{by definition of } \eta_* \end{aligned}$$

so commutativity does hold. Naturality on the object means that given an arrow $f\colon A \longrightarrow B$, the diagram

$$\begin{tikzcd} \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\gamma_A"] \arrow[d, "(f^*)^*"'] & F(A) \arrow[d, "F(f)"] \\ \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(B,-),F) \arrow[r, "\gamma_B"'] & F(B) \end{tikzcd}$$

commutes. Given a natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F,

$$F(f) \circ \gamma_A(\varphi) = F(f)(\varphi_A(1_A)),$$

while

$$\gamma_B \circ (f^*)^*(\varphi) = \gamma_B(\varphi \circ f^*) = (\varphi \circ f^*)_B(1_B).$$

Now notice that

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathscr{C}}(B,B) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathscr{C}}(A,B) & \xrightarrow{\varphi_B} & F(B) \\ 1_B & \longmapsto & f & \longmapsto & \varphi_B(f) \end{array} .$$

Let's look back at the big commutative diagram we started our proof with: it says in particular that $\varphi_B(f) = F(f)(\varphi_A(1_A))$. So commutativity does hold, and we are done. $\square$

One can naturally (pun intended) define the notion of functor category of contravariant functors, and then prove the corresponding Yoneda Lemma, which will instead use the contravariant Hom functor.

Exercise 16 (Contravariant version of the Yoneda Lemma)**.** Let $\mathscr{C}$ be a locally small category, and fix an object B in $\mathscr{C}$. Let $F\colon \mathscr{C} \longrightarrow \mathbf{Set}$ be a contravariant functor. Then there is a bijection

$$\mathrm{Nat}\ (\mathrm{Hom}_{\mathscr{C}}(-,B),F) \xrightarrow{\gamma} F(B)$$

which is natural on both B and F.

The Yoneda Lemma says that to give a natural transformation between the functors $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F is choosing an element in the set $F(A)$.

Remark 1.46. Notice that the Yoneda Lemma says in particular that the collection of all natural transformations from $\mathrm{Hom}_{\mathscr{C}}(A,-)$ to F is a set. This wasn't clear a priori, since the collection of objects in $\mathscr{C}$ is not necessarily a set.

The Yoneda Lemma says that natural transformations between representable functors correspond to arrows between the representing objects.

Remark 1.47. If we apply the Yoneda Lemma to the case when F itself is also a Hom functor, say $F = \mathrm{Hom}_{\mathscr{C}}(B,-)$, the Yoneda Lemma says that there is a bijection between $\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),\mathrm{Hom}_{\mathscr{C}}(B,-))$ and $\mathrm{Hom}_{\mathscr{C}}(B,A)$. In particular, each arrow in $\mathscr{C}$ determines a natural transformation between Hom functors.

The Yoneda Embedding, which we will prove next, formalizes the remark above. It roughly says that every locally small category can be embedded into the category of contravariant functors from $\mathscr{C}$ to **Set**. It is common to refer to both Theorem 1.45 and Theorem 1.49 as the Yoneda Lemma.

Remark 1.48. Let $\mathscr{C}$ be a locally small category. Each arrow $f\colon A \to B$ in $\mathscr{C}$ gives rise to a natural transformation $\mathrm{Hom}_{\mathscr{C}}(-,A) \implies \mathrm{Hom}_{\mathscr{C}}(-,B)$ that sends each object X to the arrow (function)

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X,A) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathscr{C}}(X,B) \\ g & \longmapsto & fg. \end{array}$$

The fact that this is a natural transformation is encoded in the following commutative diagram; we have one such diagram for each arrow $g\colon X \to Y$.

$$\begin{tikzcd} X \arrow[d, "g"'] & \mathrm{Hom}_{\mathscr{C}}(X,A) \arrow[r, "f_*"] & \mathrm{Hom}_{\mathscr{C}}(X,B) \\ Y & \mathrm{Hom}_{\mathscr{C}}(Y,A) \arrow[r, "f_*"'] \arrow[u, "\mathrm{Hom}_{\mathscr{C}}(g,A)=g^*"] & \mathrm{Hom}_{\mathscr{C}}(Y,B) \arrow[u, "\mathrm{Hom}_{\mathscr{C}}(g,B)=g^*"'] \end{tikzcd}$$

This diagram commutes since

$$g^* f_*(h) = g^*(fh) = (fh)g = f(hg) = f_*(hg) = f_* g^*(h).$$

Conversely, f^* indicates the natural transformation $\mathrm{Hom}_{\mathscr{C}}(B,-) \implies \mathrm{Hom}_{\mathscr{C}}(A,-)$ sending each object X to the arrow (function)

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(B,X) &\xrightarrow{f^*} \mathrm{Hom}_{\mathscr{C}}(A,X) \\ g &\longmapsto gf. \end{aligned}$$

Theorem 1.49 (Yoneda Embedding)**.** *Let $\mathscr{C}$ be a locally small category. The covariant functor*

$$\begin{tikzcd} \mathscr{C} \arrow[rr] & & \boldsymbol{Set}^{\mathscr{C}^{op}} \\ A \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(-,A) \arrow[d, "f_*"] \\ B & & \mathrm{Hom}_{\mathscr{C}}(-,B) \end{tikzcd}$$

*from $\mathscr{C}$ to the category of contravariant functors $\mathscr{C} \longrightarrow$ **Set** is an embedding. Moreover, the contravariant functor*

$$\begin{tikzcd} \mathscr{C} \arrow[rr] & & \boldsymbol{Set}^{\mathscr{C}} \\ A \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(A,-) \\ B & & \mathrm{Hom}_{\mathscr{C}}(B,-) \arrow[u, "f^*"'] \end{tikzcd}$$

*from the category $\mathscr{C}$ to the category of covariant functors $\mathscr{C} \longrightarrow$ **Set** is also an embedding.*

Proof. First, note that our functors are injective on objects because the Hom-sets in our category are all disjoint. So all we need to check is that given objects A and B in $\mathscr{C}$, we have bijections

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(-,A), \mathrm{Hom}_{\mathscr{C}}(-,B))$$

and

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-), \mathrm{Hom}_{\mathscr{C}}(B,-)).$$

Note that the left hand side are the Hom-sets in $\mathscr{C}$, and the right hand side are Hom-sets in **Set**. We will do the details for the second one, and leave the first as an exercise.

This follows from Remark 1.47, but let's carefully check the details. First, in Remark 1.48 we have already checked that each arrow is indeed taken to a natural transformation, so we just need to check injectivity and surjectivy at the level of arrows.

The Yoneda Lemma applied here tells us that each natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(B,-)$ and $F = \mathrm{Hom}_{\mathscr{C}}(A,-)$ corresponds to an element $u \in F(B) = \mathrm{Hom}_{\mathscr{C}}(A,B)$, which we obtain by taking $u := \varphi_B(1_B)$. The Yoneda Lemma says this correspondence is bijective.

Indeed, we can recover φ from u by taking the natural transformation φ that for each object X in $\mathscr{C}$ has $\varphi_X \colon \mathrm{Hom}_{\mathscr{C}}(B,X) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A,X)$ given by

$$\varphi_X(f) = \mathrm{Hom}_{\mathscr{C}}(f,A)(u) = f_*(u).$$

This shows surjectivity on arrows. Finally, different arrows f give rise to different natural transformations by applying the resulting natural transformation f_* to the identity arrow 1_A, which takes it to f. This shows injectivity on arrows. □

Finally, the Yoneda Embedding says that you can essentially recover an object in a category by knowing the maps from it or into it.

Theorem 1.50. *Let X and Y be objects in a locally small category $\mathscr{C}$. If $\mathrm{Hom}_{\mathscr{C}}(-,X)$ and $\mathrm{Hom}_{\mathscr{C}}(-,Y)$ are naturally isomorphic, or if $\mathrm{Hom}_{\mathscr{C}}(X,-)$ and $\mathrm{Hom}_{\mathscr{C}}(Y,-)$ are naturally isomorphic, then X and Y are isomorphic objects.*

Proof. The Yoneda Embeddings from Theorem 1.49 are fully faithful, and thus by Exercise 9 they must reflect isomorphisms. A natural isomorphism between the functors $\mathrm{Hom}_{\mathscr{C}}(X,-)$ and $\mathrm{Hom}_{\mathscr{C}}(Y,-)$ (or the functors $\mathrm{Hom}_{\mathscr{C}}(-,X)$ and $\mathrm{Hom}_{\mathscr{C}}(-,Y)$) is an isomorphism in the target functor category, and it corresponds to f_* (respectively, f^*) for some arrow f from Y to X. By Exercise 9, f must be an isomorphism. In particular, X and Y are isomorphic. □

To summarize the content of this chapter, here is the Yoneda Lemma in slogans:

- To give a natural transformation from $\mathrm{Hom}(A,-)$ to F is the same as giving an element in the set $F(A)$.
- The collection of all natural transformations from $\mathrm{Hom}(A,-)$ to F is a set.
- To give a natural transformation between representable functors is to give an arrow between the corresponding representing objects.
- Every locally small category $\mathscr{C}$ can be embedded into the functor category of (covariant or contravariant) functors from $\mathscr{C}$ to **Set**. So rather than studying the category $\mathscr{C}$, we can study functor category to **Set**.
- We can recover an object in a category by knowing the maps from it or into it.

1.5 Products and coproducts

Definition 1.51. Let $\mathscr{C}$ be a locally small category, and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathscr{C}$. The **product** of the A_i is an object in $\mathscr{C}$, denoted by $\prod_i A_i$, together with arrows $\pi_j \in \operatorname{Hom}_{\mathscr{C}}(\prod_i A_i, A_j)$ for each j, called **projections**, satisfying the following universal property: given any object B in $\mathscr{C}$ and arrows $f_i \colon B \longrightarrow A_i$ for each i, there exists a unique arrow f such that

$$\begin{tikzcd}
B \arrow[d, dashed, "\exists! f"'] \arrow[dr, "f_j"] & \\
\prod_i A_i \arrow[r, "\pi_j"'] & A_j
\end{tikzcd}$$

commutes for all j. When I is finite, we may write $A_1 \times \cdots \times A_n$ for the product of $A_1, \ldots, A_n$.

Here is a larger diagram for the (first few) maps involved in a product when the indexing set $I = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & \prod_i A_i \arrow[dr, "\pi_1"] \arrow[ddr, "\pi_2"] \arrow[dddr, "\pi_3"] \arrow[ddddr] & \\
 & & A_1 \\
B \arrow[uur, dashed, "f"] \arrow[urr, "f_1"] \arrow[rr, "f_2"] \arrow[drr, "f_3"] \arrow[ddrr] & & A_2 \\
 & & A_3 \\
 & & \vdots
\end{tikzcd}$$

We can also take a "big picture" view of this universal property of the product:

$$\begin{tikzcd}
 & \prod_i A_i \arrow[dr, squiggly, "\{\pi_i\}"] & \\
B \arrow[ur, dashed, "f"] \arrow[rr, squiggly, "\{f_i\}"'] & & \{A_i\},
\end{tikzcd}$$

where the squiggly arrows are again collections of maps instead of maps.

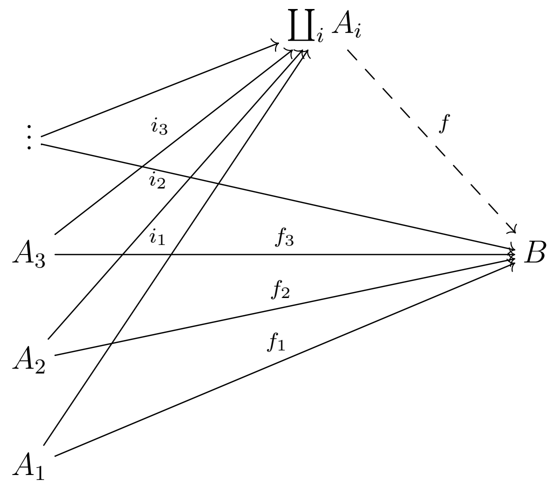
The dual notion is the coproduct.

Definition 1.52. Let $\mathscr{C}$ be a locally small category, and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathscr{C}$. The **coproduct** of the A_i is an object in $\mathscr{C}$, denoted by $\coprod_i A_i$, together with arrows $\iota_j \in \operatorname{Hom}_{\mathscr{C}}(A_j, \coprod_i A_i,)$ for each j, satisfying the following universal property: given any object B in $\mathscr{C}$ and arrows $f_i \colon A_i \longrightarrow B$ for each i, the following diagram commutes:

$$\begin{tikzcd}
B & \\
\coprod_i A_i \arrow[u, dashed, "\exists! f"] & A_j \arrow[l, "\iota_j"] \arrow[ul, "f_j"']
\end{tikzcd}$$

When I is finite, we may write $A_1 \amalg \cdots \amalg A_n$ for the coproduct of $A_1, \ldots, A_n$.

Here is a diagram for the (first few) maps involved in a coproduct when $\Lambda = \mathbb{N}$ is countable:



We can also take a "big picture" view of the universal property of the coproduct:

$$\begin{tikzcd} & \coprod_i A_i \arrow[dr, dashed, "f"] & \\ \{A_i\} \arrow[ur, squiggly, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_i\}"'] & & B, \end{tikzcd}$$

where the squiggly arrows are now collections of maps instead of maps.

Theorem 1.53. *If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda \in \Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda \in \Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda \in \Lambda}$ in a category $\mathscr{C}$, then there is a unique isomorphism $\alpha\colon P \xrightarrow{\sim} P'$ such that $p'_\lambda \circ \alpha = p_\lambda$ for all λ. The analogous statement holds for coproducts.*

Proof. We will just deal with products. The following picture is a rough guide:

$$\begin{tikzcd} P \arrow[r, dashed, "\alpha"] \arrow[drr, squiggly, "\{p_\lambda\}"'] & P' \arrow[r, dashed, "\beta"] \arrow[dr, squiggly, "\{p'_\lambda\}"] & P \arrow[d, squiggly, "\{p_\lambda\}"] \\ & & \{X_\lambda\} \end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta : P' \to P$ such that $p_\lambda \circ \beta = p'_\lambda$. Switching roles, we obtain a unique map $\alpha : P \to P'$ such that $p'_\lambda \circ \alpha = p_\lambda$.

Consider the composition $\beta \circ \alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ for all λ. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ for all λ, so by the uniqueness property of products, $\beta \circ \alpha = 1_P$. We can again switch roles to see that $\alpha \circ \beta = 1_{P'}$. Thus α is an isomorphism. The uniqueness of α in the statement is part of the universal property. $\square$

Exercise 17. Prove the analogous statement to Theorem 1.53 for coproducts.

This explains why the notations $\prod_i A_i$ and $\coprod_i A_i$ make sense: we can talk about *the* product and *the* coproduct of the A_i, if they exist.

The key thing to remember about these constructions and their universal properties is the following:

- Mapping *into* a product is completely determined by mapping into each of the factors.
- Mapping *out* of a coproduct is completely determined by mapping out of each factor.

Example 1.54. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The product of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the cartesian product of sets along with the canonical projection maps.

The familiar notion of Cartesian product or direct product serves as a product in many of our favorite categories. Let's note first that given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in any of the categories $\mathbf{Sgrp}, \mathbf{Grp}, \mathbf{Ring}, R\text{-}\mathbf{Mod}, \mathbf{Top}$, the usual direct product $\prod_{\lambda\in\Lambda} X_\lambda$ is an object of the same category:

- for semigroups, groups, and rings, take the operation coordinate by coordinate:

$$(x_\lambda)_{\lambda\in\Lambda} \cdot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda};$$

- for modules, addition is coordinate by coordinate, and the action is the same on each coordinate: $r \cdot (x_\lambda)_{\lambda\in\Lambda} = (r \cdot x_\lambda)_{\lambda\in\Lambda}$;
- for topological spaces, use the product topology.

Note that this is not true for fields! The usual product of fields is not a field. In fact, there is no product in this category.

Theorem 1.55. *In each of the categories* **Set**, **Grp**, **Ring**, R-**Mod**, *and* **Top**, *given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$, the object $\prod_{\lambda\in\Lambda} X_\lambda$ given by the usual direct product along with the usual projection maps $\pi_\lambda\colon \prod_{\gamma\in\Lambda} X_\gamma \to X_\lambda$ forms a product in the category.*

Proof. We observe that in each category, the direct product is an object, and the projection maps π_λ are morphisms in the category.

Let $\mathscr{C}$ be one of these categories, and suppose that we have morphisms $g_\lambda\colon Y \to X_\lambda$ for all λ in $\mathscr{C}$. We need to show there is a unique morphism $\phi\colon Y \to \prod_{\lambda\in\Lambda} X_\lambda$ such that $\pi_\lambda \circ \phi = g_\lambda$ for all λ. The last condition is equivalent to

$$(\phi(y))_\lambda = (\pi_\lambda \circ \phi)(y) = g_\lambda(y)$$

for all λ, which is equivalent to $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$, so if this is a valid morphism, it is unique. Thus, it suffices to show that the map $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$ is a morphism in $\mathscr{C}$; we leave the details as an exercise. $\square$

Example 1.56. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ in **Set** is given by the disjoint union with the various inclusion maps. By disjoint union, we simply mean union if the sets are disjoint; in general do something like replace X_λ with $X_\lambda \times \{\lambda\}$ to make them disjoint.

Theorem 1.57. *Let R be a ring, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left R-modules. A coproduct for the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is given by the direct sum of modules*

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \text{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda$$

together with the inclusion maps

$$M_\lambda \xrightarrow{\iota_\lambda} \bigoplus_{\lambda\in\Lambda} M_\lambda$$

that send each $m \in M_\lambda$ to the tuple that has m in coordinate λ and zeroes elsewhere.

Proof. Given R-module homomorphisms $g_\lambda : M_\lambda \to N$ for each λ, we need to show that there is a unique R-module homomorphism $\alpha\colon \bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ such that $\alpha \circ \iota_\lambda = g_\lambda$. We define

$$\alpha((m_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} g_\lambda(m_\lambda).$$

Note that since $(m_\lambda)_{\lambda\in\Lambda}$ is in the direct sum, at most finitely many m_λ are nonzero, so the sum on the right hand side is finite, and hence makes sense in N. We need to check that α is R-linear; indeed,

$$\begin{aligned}
\alpha((m_\lambda) + (n_\lambda)) &= \alpha((m_\lambda + n_\lambda)) \\
&= \sum g_\lambda(m_\lambda + n_\lambda) \\
&= \sum g_\lambda(m_\lambda) + \sum g_\lambda(n_\lambda) \\
&= \alpha((m_\lambda)) + \alpha((n_\lambda)),
\end{aligned}$$

and the check for scalar multiplication is similar. For uniqueness of α, note that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is generated by the elements $\iota_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha' \circ \iota_\lambda = g_\lambda$ for all λ, then $\alpha(\iota_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(\iota_\lambda(m_\lambda))$ so the maps must be equal. □

Remark 1.58. If the index set Λ is finite, then the objects $\prod_{\lambda\in\Lambda} M_\lambda$ and $\bigoplus_{\lambda\in\Lambda} M_\lambda$ are identical, but the product and coproduct are not the same since one involves projection maps and the other involves inclusion maps. When Λ is infinite, the two objects are truly distinct, and in fact the direct sum is a submodule of the product.

Remark 1.59. For any indexing set Λ, $\oplus_{\lambda\in\Lambda} R$ is a free R-module. If $R = k$ happens to be a field, then $\prod_{\lambda\in\Lambda} k$ is free, since all vector spaces are free modules, but in general, $\prod_{\lambda\in\Lambda} R$ is not free for an infinite set Λ.

Example 1.60.

1) In **Top**, disjoint unions serve as coproducts.

2) In **Sgrp** and **Grp**, coproducts exist, and are given as free products. You may see or have seen them in topology in the context of Van Kampen's theorem.

3) In **Ring**, the story is more complicated. Let's note first that disjoint unions won't work, since they are not rings. Direct sums of infinitely many rings do not have 1, so they are not rings in this class, but even finite direct sums or products will not work, since the inclusion maps does not send 1 to 1. We will later on construct coproducts in the full subcategory of **Ring** consisting of commutative rings.

1.6 Limits and colimits

Definition 1.61. Let $(I, \geqslant)$ be a partially ordered set and let $\mathscr{C}$ be a category. An **inverse system** in $\mathscr{C}$ indexed by I is a contravariant functor $\mathbf{PO}(I) \to \mathscr{C}$.

Remark 1.62. Let's unwrap the definition of inverse system a bit. For each $i \in I$, we get an object M_i in $\mathscr{C}$. Moreover, in the category $\mathbf{PO}(I)$, there is exactly one arrow $i \to j$ for each $i \leqslant j$, and the image of this arrow under any contravariant functor $\mathbf{PO}(I) \to \mathscr{C}$ is an arrow $M_j \to M_i$. Finally, our functor must preserve compositions of arrows, so whenever $k \geqslant j \geqslant i$, the arrow $M_k \to M_i$ should match the composition of arrows through j. Thus an inverse system in $\mathscr{C}$ indexed by I consists of the following data:

- for each $i \in I$, an object M_i in $\mathscr{C}$, and
- for each $i \leqslant j$, an arrow $\varphi_i^j \colon M_j \to M_i$ in $\mathscr{C}$

such that whenever $i \leqslant j \leqslant k$, the following diagram must commute:

$$\begin{tikzcd} M_k \arrow[rr, "\varphi_i^k"] \arrow[dr, "\varphi_j^k"'] & & M_i \\ & M_j \arrow[ur, "\varphi_i^j"'] & \end{tikzcd}$$

Note moreover that $\varphi_i^i = \mathrm{id}_{M_i}$, since functors preserve identities. To indicate all this data in a compact way, we say that $\{M_i, \varphi_i^j\}$ is an inverse system.

Example 1.63.

a) An inverse system in a category $\mathscr{C}$ indexed by $\mathbb{N}$ is determined by a diagram of the form

$$X_0 \xleftarrow{a_0} X_1 \xleftarrow{a_1} X_2 \xleftarrow{a_2} X_3 \xleftarrow{a_3} X_4 \xleftarrow{a_4} X_5 \leftarrow \cdots .$$

All the other arrows $X_j \to X_i$ for $i < j$ are given by composition.

b) Let I be a family of submodules of an R-module M. Then we can think of I as a partially ordered set with the reverse inclusion $\supseteq$, so that $L \leqslant N$ if and only if $L \supseteq N$. Whenever $N \subseteq L$, we have an inclusion map $N \to L$, and the family of submodules I together with the inclusion maps forms an inverse system of R-modules.

A special case of this is when we have a descending chain of submodules of M

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

which is also a special case of an inverse system indexed by $\mathbb{N}$.

c) If I is a poset with the **discrete partial order**, meaning $i \leqslant j$ if and only if $i = j$, then an inverse system indexed by I is just a family of objects indexed by I.

d) If $I = \{1, 2, 3\}$ is a poset with $1 \leqslant 2$ and $1 \leqslant 3$, then an inverse system indexed by I is just a diagram of the form

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A. \end{tikzcd}$$

Exercise 18. Let J be an ideal in a commutative ring R, and consider its nth power, which is the ideal

$$J^n := (f_1 \cdots f_n \mid f_i \in J)$$

generated by all n-fold products of elements in R. For each $m \geqslant n$, consider the maps

$$\begin{aligned} R/J^m &\xrightarrow{\varphi_n^m} R/J^n \\ r + J^m &\longmapsto r + J^n. \end{aligned}$$

Show that these form an inverse system in R-**Mod** indexed by $\mathbb{N}_{>0}$. Note that this can be represented as

$$R/J \xleftarrow{\varphi_1^2} R/J^2 \xleftarrow{\varphi_2^3} R/J^3 \xleftarrow{\varphi_3^4} R/J^4 \xleftarrow{\varphi_4^5} R/J^5 \longleftarrow \cdots.$$

Definition 1.64. Let $\mathscr{C}$ be a category and let $\{M_i, \varphi_i^j\}_i$ be an inverse system on $\mathscr{C}$ indexed by I. The **limit** or **inverse limit** of $\{M_i, \varphi_i^j\}$ consists of an object

$$\varprojlim M_i$$

and arrows

$$\pi_i \colon \varprojlim M_i \to M_i$$

called **projections** such that for all $j \leqslant k$ in I, the diagram

$$\begin{tikzcd} M_k & \varprojlim M_i \arrow[l, "\pi_k"'] \arrow[dl, "\pi_j"] \\ M_j \arrow[u, "\varphi_k^j"] & \end{tikzcd}$$

commutes, and that satisfy the following universal property: for all arrows $f_i \colon X \to M_i$ such that $\varphi_i^j f_j = f_i$ for all i, j, meaning that the diagram

$$\begin{tikzcd} M_i & X \arrow[l, "f_i"'] \arrow[dl, "f_j"] \\ M_j \arrow[u, "\varphi_i^j"] & \end{tikzcd}$$

commutes, there exists a unique arrow $f \colon X \to \varprojlim M_i$ such that

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_j"'] & & X \arrow[ll, dashed, "\exists! f"'] \arrow[dl, "f_j"] \\ & M_j & \end{tikzcd}$$

commute for all j.

One can show that if it exists, the object $\varprojlim M_i$ is unique up to isomorphism; in fact, this is the terminal object in some appropriate (and technical) category. So we can refer to *the* limit of an inverse system. The notation $\varprojlim M_i$ is sometimes replaced by $\lim_i M_i$.

Remark 1.65. Given an inverse system $\{M_i, \varphi_i^j\}$ indexed by I in a category $\mathscr{C}$, say corresponding to the contravariant functor $\varphi : I \to \mathscr{C}$, suppose that its limit exists, and let $L = \varprojlim M_i$. The projections π_i give us commutative diagrams

$$\begin{tikzcd} L \arrow[r, "1_L"] \arrow[d, "\pi_i"'] & L \arrow[d, "\pi_j"] \\ M_i \arrow[r, "\varphi_i^j"'] & M_j \end{tikzcd}$$

This is the same data as a natural transformation

$$\begin{tikzcd} \mathbf{PO}(I) \arrow[r, bend left, "\Delta L"] \arrow[r, bend right, "\varphi"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C} \end{tikzcd} .$$

In other words, a limit for α consists of an object and a natural transformation from the constant functor on that object to the functor α.

Example 1.66. A terminal object can be viewed as a limit of the empty diagram: since there are no objects in an inverse limit from the empty category, the limit is just an object L that must satisfy the condition that for every object X, there is a unique arrow $X \to L$.

Exercise 19. Show that if I is a partially ordered set with the discrete order, then the limit of any inverse system indexed by I is the product on the corresponding set of objects.

Theorem 1.67. *Let R be any ring. Every inverse system of left R-modules over any partially ordered set has a limit.*

Proof. Let I be a partially ordered set and consider an inverse system of R-modules indexed by I, say with modules M_i and homomorphisms $\varphi_i^j \colon M_j \to M_i$. Let

$$L := \{(m_i) \in \prod_i M_i \mid \varphi_i^j(m_j) = m_i \text{ for all } i \leqslant j\}.$$

One can show (exercise!) that this is a submodule of the product of the M_i. For each i, let $\pi_i \colon L \to M_i$ be the restriction of the projection maps $\prod M_i \to M_i$ to L. We claim that L is a limit for the inverse system, together with the projection maps π_i.

First, note that

$$\varphi_i^j \pi_j((m_k)_k) = \varphi_i^j(m_j) = m_i = \pi_i((m_k)_k),$$

by construction, so $\varphi_i^j \pi_j = \pi_i$.

Moreover, suppose that we are given an R-module X and R-module homomorphisms $f_i \colon X \to M_i$ such that $\varphi_i^j f_j = f_i$ for all $i \leqslant j$. Define

$$\begin{aligned} X &\xrightarrow{\ g\ } \textstyle\prod_i M_i \\ x &\longmapsto (f_i(x))_i. \end{aligned}$$

First, note that $\pi_i(g(x)) = f_i(x)$ for all i by construction. Moreover, this is an R-module homomorphism; it is induced by the universal property of the product. We claim that the image of g is contained in L, and thus that we can restrict g to an R-module homomorphism $f\colon X \to L$. Indeed, given any $x \in X$,

$$\varphi_i^j(\pi_j(g(x))) = \varphi_i^j(f_j(x)) = f_i(x) = \pi_i(g(x)).$$

This says that $g(x) \in L$, so we get an R-module homomorphism $f\colon X \to L$ given by

$$f(x) = (f_i(x))_i.$$

Finally, we claim that L and f satisfy the desired universal property, and for that, we need first to check that

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"'] & & X \arrow[ll, dashed, "f"'] \arrow[dl, "f_i"] \\ & M_i & \end{tikzcd}$$

commutes, and we need to check that such f is unique. The commutativity is immediate, since as noted above $\pi(f(x)) = f_i(x)$ for all $x \in X$ by construction. For uniqueness, suppose that h is any other R-module homomorphism $X \to L$ such that

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"'] & & X \arrow[ll, dashed, "h"'] \arrow[dl, "f_i"] \\ & M_i & \end{tikzcd}$$

also commutes. Given any $x \in X$, let $h(x) = (m_i)$. Then

$$m_i = \pi_i(h(x)) = f_i(x)$$

for all i, so

$$h(x) = (m_i)_i = (f_i(x))_i = f(x),$$

and thus $h = f$. This completes the proof that L is a limit for the given inverse system. $\square$

Remark 1.68. One can adapt the proof of Theorem 1.67 to show that all limits in **Set** exist, and can be constructed explicitly as a subset of the product of the sets forming the inverse system: the limit of an inverse system $\{M_i, \varphi_i^j\}$ is the subset of the product given by

$$L := \{(m_i) \in \prod_i M_i \mid \varphi_i^j(m_j) = m_i \text{ for all } i \leqslant j\}$$

together with the canonical projections from the product restricted to the subset L.

Example 1.69.

a) If I is a partially ordered set with the discrete order, then the limit of any inverse system just the product.

b) Given a ring R and an ideal J, the limit of the inverse system

$$R/J \longleftarrow R/J^2 \longleftarrow R/J^3 \longleftarrow R/J^4 \longleftarrow R/J^5 \longleftarrow \cdots$$

is the J-adic completion of R.

The dual construction to limits is the notion of a colimit.

Definition 1.70. Let $(I, \geqslant)$ be a partially ordered set and let $\mathscr{C}$ be a category. A **direct system** in $\mathscr{C}$ indexed by I is a covariant functor $\mathbf{PO}(I) \to \mathscr{C}$.

Remark 1.71. An inverse system in $\mathscr{C}$ indexed by I consists of the following data:

- for each $i \in I$, an object M_i in $\mathscr{C}$, and
- for each $i \leqslant j$, an arrow $\varphi^i_j \colon M_i \to M_j$ in $\mathscr{C}$

such that whenever $i \leqslant j \leqslant k$, the following diagram must commute:

$$\begin{tikzcd} M_i \arrow[rr, "\varphi^k_i"] \arrow[dr, "\varphi^i_k"'] & & M_k \\ & M_j \arrow[ur, "\varphi^j_k"'] & \end{tikzcd}$$

Note moreover that $\varphi^i_i = \mathrm{id}_{M_i}$, since functors preserve identities. To indicate all this data in a compact way, we say that $\{M_i, \varphi^j_i\}$ is an inverse system.

Example 1.72.

a) A direct system in a category $\mathscr{C}$ indexed by $\mathbb{N}$ is determined by a diagram of the form

$$X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3 \xrightarrow{a_3} X_4 \xrightarrow{a_4} X_5 \to \cdots .$$

All the other arrows $X_i \to X_j$ for $i < j$ are given by composition.

b) Let I be a family of submodules of an R-module M. Then we can think of I as a partially ordered set with $\subseteq$. Whenever $N \subseteq L$, we have an inclusion map $N \to L$, and the family of submodules I together with the inclusion maps forms a direct system of R-modules.

A special case of this is when we have an ascending chain of submodules of M

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

which is also a special case of a direct system indexed by $\mathbb{N}$.

c) If I is a poset with the discrete partial order, then an inverse system indexed by I is just a family of objects indexed by I.

d) If $I = \{1, 2, 3\}$ is a poset with $1 \leqslant 2$ and $1 \leqslant 3$, then a direct system indexed by I is just a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C. & \end{tikzcd}$$

Definition 1.73. Let $\mathscr{C}$ be a category and let $\{M_i, \varphi^i_j\}_i$ be a direct system on $\mathscr{C}$ indexed by I. The **colimit** or **direct limit** of $\{M_i, \varphi^i_j\}$ consists of an object

$$\varinjlim M_i$$

and arrows

$$\alpha_i \colon M_i \to \varinjlim M_i$$

called **insertion arrows** such that

$$\alpha_j \varphi^i_j = \alpha_i \quad \text{for all } i, j \in I$$

satisfying the following universal property: for all arrows $f_i \colon M_i \to X$ such that $f_j \varphi^i_j = f_i$ for all i, j, meaning that the diagram

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "\varphi^i_j"'] & X \\ M_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

commutes, there exists a unique arrow $f \colon \varinjlim M_i \to X$ such that

$$\begin{tikzcd} \varinjlim M_i \arrow[rr, dashed, "\exists! f"] & & X \\ & M_j \arrow[ul, "\alpha_j"] \arrow[ur, "f_j"'] & \end{tikzcd}$$

commutes.

One can show that if it exists, the object $\varinjlim M_i$ is unique up to isomorphism; in fact, this is the initial object in some appropriate (and technical) category. So we can refer to *the* colimit of a direct system. The notation $\varinjlim M_i$ is sometimes replaced by $\operatorname{colim}_i M_i$.

Remark 1.74. Given a direct system $\{M_i, \varphi^i_j\}$ indexed by I in a category $\mathscr{C}$, say corresponding to the covariant functor $\varphi \colon I \to \mathscr{C}$, suppose that its colimit exists, and let $L = \varinjlim M_i$. The α_i give us commutative diagrams

$$\begin{tikzcd} L \arrow[r, "1_L"] & L \\ M_i \arrow[u, "\alpha_i"] \arrow[r, "\varphi^i_j"'] & M_j \arrow[u, "\alpha_j"'] \end{tikzcd}$$

This is the same data as a natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\Delta L"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C} \end{tikzcd} .$$

In other words, a limit for α consists of an object and a natural transformation from α to the constant functor on that object.

Example 1.75. An initial object can be viewed as a colimit of the empty diagram: since there are no objects in a direct limit from the empty category, the colimit is an object C that must satisfy the condition that for every object X, there is a unique arrow $C \to X$.

Exercise 20. Show that if I is a poset with the discrete order, then the colimit of any inverse system indexed by I is the same as the coproduct of the corresponding set of objects.

Theorem 1.76. *Let R be any ring. Every direct system of left R-modules over any partially ordered has a colimit.*

Proof. Let I be a partially ordered set and consider a direct system of R-modules indexed by I, say with modules M_i and homomorphisms $\varphi^i_j \colon M_j \to M_i$. Let $\iota_i \colon M_i \to \oplus_j M_j$ be the inclusions into the direct sum, let S be the submodule of $\oplus M_i$ generated by all elements of the form

$$\iota_i(\varphi^i_j(m_i)) - \iota_i(m_i),$$

and define

$$C := \bigoplus_i M_i / S.$$

For each i, let

$$\begin{aligned} M_i &\xrightarrow{\alpha_i} C \\ m &\longmapsto \iota_i(m) + S. \end{aligned}$$

We claim that C together with the maps α_i is a colimit for the direct system; we leave the details as an exercise. $\square$

Remark 1.77. One can adapt the proof of Theorem 1.67 to show that all colimits in **Set** exist, and can be constructed explicitly as the set of equivalence classes of an appropriate equivalence relation on the coproduct.

There are many other important constructions that arise as special cases of limits and colimits, some of which we will study later in the class. Here is one more example:

Definition 1.78. Let $\mathscr{C}$ be a category. A **pullback** of the arrows f and g consists of an object P and arrows p_1 and p_2 such that

$$\begin{tikzcd} P \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes, and satisfying the following universal property: for all objects Q and arrows q_1 and q_2 such that

$$\begin{tikzcd} Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes, there exists a unique u such that

$$
\begin{tikzcd}
Q \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] \arrow[dr, dashed, "u"] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}
$$

commutes. One sometimes refers to the following diagram as a **pullback diagram**:

$$
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
$$

The dual construction is the pushout.

Definition 1.79. Let $\mathscr{C}$ be a category. A **pushout** of the arrows f and g consists of an object P and arrows p_1 and p_2 such that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dashed, "p_1"] \\
C \arrow[r, dashed, "p_2"'] & P
\end{tikzcd}
$$

commutes, and satisfying the following universal property: for all objects Q and arrows q_1 and q_2 such that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q_1"] \\
C \arrow[r, "q_2"'] & Q
\end{tikzcd}
$$

commutes, there exists a unique u such that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "p_1"] \arrow[ddr, bend left, "q_1"] & \\
B \arrow[r, "p_2"'] \arrow[drr, bend right, "q_2"'] & P \arrow[dr, dashed, "u"] & \\
& & Q
\end{tikzcd}
$$

commutes. One sometimes refers to the following diagram as a **pushout diagram**:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_1"] \\
C \arrow[r, "p_2"'] & P
\end{tikzcd}
$$

Exercise 21. Interpret the notion of pullback as a limit and a pushout as a colimit. More precisely, describe a partially ordered set and corresponding inverse system or direct system whose limit or colimit is the same as a pushout or pullback.

We showed in Theorem 1.67 and Theorem 1.76 that R-**Mod** has all limits and colimits. In the case of pullbacks and pushouts, one can describe the corresponding module in a more manageable way.

Exercise 22. Explicitly describe pullbacks and pushouts in R-**Mod**.

We have defined the limit of an inverse system, and the colimit of a direct system. One can define limits and colimits even more generally – every functor may have a limit, even if the source is not a poset category.

Definition 1.80. Let $\mathscr{C}$ be a category and let J be a small category. A **diagram** in $\mathscr{C}$ of **shape** J is a functor $J \to \mathscr{C}$. We may call J the **index category**.

Remark 1.81. Let $\mathscr{C}$ be a category and let J be a small category. Let I be the set of objects in J. To give a covariant functor $J \to \mathscr{C}$ is to give

- a set $\{X_i\}_{i \in I}$ in $\mathscr{C}$ indexed by I,
- or every pair (i, j) of objects $i, j \in I$, a set of arrows $A_{i,j} := \{f_\alpha\}$ in $\operatorname{Hom}_{\mathscr{C}}(X_i, X_j)$ indexed by the set $\operatorname{Hom}_I(i, j)$

satisfying the necessary properties to guarantee that that 1_i gets sent to 1_{X_i} and that composition of arrows is preserved. One can give a diagram by forgetting the underlying indexing category J and just presenting the set of objects, sets of arrows, and corresponding composition rules.

One advantage of giving this data, as opposed to the functor $F\colon J \to \mathscr{C}$, is that we do not need to distinguish between covariant and contravariant functors – we are simply giving a set of objects and various sets of arrows.

Definition 1.82. Consider a diagram in $\mathscr{C}$ with objects $\{X_i\}_{i \in I}$ in $\mathscr{C}$ and arrows $A_{i,j} = \{f_\alpha\}$. A **cone** over this diagram consists of

- an object C in $\mathscr{C}$, and
- for each $i \in I$, an arrow $p_i\colon C \to X_i$

such that for every pair (i, j) and every arrow $f : X_i \to X_j$ in the diagram, the following triangle commutes:

$$\begin{tikzcd} & C \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ X_i \arrow[rr, "f"'] & & X_j \end{tikzcd}$$

Dually, a **cocone** over this diagram consists of

- an object C in $\mathscr{C}$, and

- for each $i \in I$, an arrow $p_i \colon X_i \to C$

such that for every pair (i, j) and every arrow $f : X_i \to X_j$ in the diagram, the following triangle commutes:

$$\begin{tikzcd}
X_i \arrow[rr, "f"] \arrow[dr, "p_i"'] & & X_j \arrow[dl, "p_j"] \\
& C &
\end{tikzcd}$$

Definition 1.83. Consider a diagram in $\mathscr{C}$ with objects $\{X_i\}_{i\in I}$ in $\mathscr{C}$ and arrows $A_{i,j} = \{f_\alpha\}$. The **limit** of this diagram is, if it exists, a cone

$$\begin{tikzcd}
& \lim X_i \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\
X_i \arrow[rr] & & X_j
\end{tikzcd}$$

which is terminal with respect to all other cones, meaning that for every other cone

$$\begin{tikzcd}
& C \arrow[dl, "q_i"'] \arrow[dr, "q_j"] & \\
X_i \arrow[rr] & & X_j
\end{tikzcd}$$

there exists a unique arrow $u \colon C \to \lim X_i$ such that

$$\begin{tikzcd}
& \lim X_i \arrow[dr, "p_j"] & \\
C \arrow[ur, "u"] \arrow[rr, "q_j"'] & & X_j
\end{tikzcd}$$

commutes.

The **colimit** of this diagram is, if it exists, a cocone

$$\begin{tikzcd}
X_i \arrow[rr] \arrow[dr, "p_i"'] & & X_j \arrow[dl, "p_j"] \\
& \operatorname{colim} X_i &
\end{tikzcd}$$

which is initial with respect to all other cones, meaning that for every other cone

$$\begin{tikzcd}
X_i \arrow[rr] \arrow[dr, "q_i"'] & & X_j \arrow[dl, "q_j"] \\
& C &
\end{tikzcd}$$

there exists a unique arrow $u \colon \operatorname{colim} X_i \to C$ such that

$$\begin{tikzcd}
X_j \arrow[rr, "q_j"] \arrow[dr, "p_j"'] & & C \\
& \operatorname{colim} X_i \arrow[ur, "u"'] &
\end{tikzcd}$$

commutes.

One can check that if we take a limit of a contravariant diagram indexed by a poset category, we recover the limit of an inverse system, and analogously the colimit of a covariant diagram indexed by a poset category is the colimit of a direct system.

Definition 1.84. A covariant functor $F\colon \mathscr{C} \to \mathscr{D}$

- **preserves colimits** if
$$F(\operatorname{colim} M_i) = \operatorname{colim} F(M_i).$$
More precisely, if the object $\operatorname{colim} M_i$ and the arrows $\alpha_i : M_i \to \operatorname{colim} M_i$ form the colimit of diagram D, then $F(\operatorname{colim} M_i)$ is the colimit of the diagram $F \circ D$ with insertion arrows $F(\alpha_j)\colon F(M_j) \to F(\operatorname{colim} M_i)$.

- **preserves limits** if
$$F(\lim M_i) = \lim F(M_i).$$
More precisely, if $\lim M_i$ is the limit of a diagram D with projections $\pi_j\colon \lim M_i \to M_j$, then the object $F(\lim M_i)$ and the projection arrows $F(\pi_j)\colon F(\lim M_i) \to F(M_j)$ form a limit of the diagram $F \circ D$.

Definition 1.85. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ **converts limits to colimits** or **sends limits to colimits** if

$$F(\lim M_i) = \operatorname{colim} F(M_i).$$

Similarly, a contravariant functor $F : \mathscr{C} \to \mathscr{D}$ **converts colimits to limits** or **sends colimits to limits** if
$$F(\operatorname{colim} M_i) = \lim F(M_i).$$

The Hom functors preserve limits and colimits.

Theorem 1.86. *Let $\mathscr{C}$ be any category and let A be an object in $\mathscr{C}$.*

a) If the limit $\lim_i M_i$ exists, then there is a natural isomorphism

$$\operatorname{Hom}_{\mathscr{C}}(A, \lim_i M_i) \cong \lim_i \operatorname{Hom}_{\mathscr{C}}(A, M_i).$$

In particular, the limit of $\operatorname{Hom}_{\mathscr{C}}(A, M_i)$ exists.

b) If the limit $\lim_i M_i$ exists, then there is a natural isomorphism

$$\operatorname{Hom}_{\mathscr{C}}(\operatorname{colim}_i M_i, A) \cong \lim_i \operatorname{Hom}_{\mathscr{C}}(M_i, B).$$

In particular, the limit of $\operatorname{Hom}_{\mathscr{C}}(M_i, A)$ exists.

1.7 Universal properties

We have all seen constructions that are at first a bit messy but that end up satisfying some nice universal property that makes everything work out. At the end of the day, a universal property allows us to ignore the messy details and focus on the universal property, which usually says everything we need to know about the construction.

Universal properties are *everywhere*. Limits and colimits are a big example; products and coproducts are a special case of limits and colimits. A representable functor encodes a *universal property* of the object that represents it: for example, in Example 1.44, mapping out of the singleton set is the same as choosing an element x in a set X.

In this section, we will briefly describe how one can formalize the idea of a universal property in categorical language. This is not necessary to understand what comes afterwards; this section is here for our own amusement. The most interesting observation in this section is perhaps that any universal property can be phrased in terms of representable functors. There are a few different equivalent frameworks in the literature, and we will briefly try to reconcile two of them. We note, however, that understanding this formalism is not necessarily for what we will do next; this level of abstraction can be confusing at first, and this is a section that can be better understood once the reader has had some time to get comfortable with categorical language.

Definition 1.87. Let $\mathscr{C}$ be a locally small category. A **universal property** of an object C in $\mathscr{C}$ consists of a representable functor $F\colon \mathscr{C} \longrightarrow \mathbf{Set}$ together with a **universal element** $X \in F(C)$ such that F is naturally isomorphic to either $\mathrm{Hom}_{\mathscr{C}}(C, -)$ (if F is covariant) or $\mathrm{Hom}_{\mathscr{C}}(-, C)$ (if F is contravariant), via the natural isomorphism that corresponds to X via the bijection in the Yoneda Lemma.

We can rephrase this in terms of universal arrows.

Definition 1.88. Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be covariant functor and let D be and object in $\mathscr{D}$. A **universal arrow from D to F** is a pair (U, u) where U is an object in $\mathscr{C}$ and an arrow $u \in \mathrm{Hom}_{\mathscr{D}}(D, F(U))$ with the following **universal property**: for any arrow $f \in \mathrm{Hom}_{\mathscr{D}}(D, F(Y))$, there exists a unique arrow $h \in \mathrm{Hom}_{\mathscr{C}}(U, Y)$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, dashed, "h"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "u"] \arrow[dr, "f"'] & F(U) \arrow[d, dashed, "F(h)"] \\ & F(Y). \end{tikzcd}$$

There is a dual to this definition. A **universal arrow from F to D** is a pair (U, u), where C is an object in $\mathscr{C}$ and $u \in \mathrm{Hom}_{\mathscr{D}}(F(U), D)$ that satisfy the following **universal property**: for any arrow $f \in \mathrm{Hom}_{\mathscr{D}}(F(Y), D)$, there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(Y, U)$ such that the following diagram commutes:

$$\begin{tikzcd} U \\ Y \arrow[u, dashed, "h"] \end{tikzcd} \qquad \begin{tikzcd} D & F(U) \arrow[l, "u"'] \\ & F(Y). \arrow[ul, "f"] \arrow[u, dashed, "F(h)"'] \end{tikzcd}$$

Let's see in detail why it is that giving a universal arrow is equivalent to giving a universal property as defined above.

Remark 1.89. Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a covariant functor, and fix an object U in $\mathscr{C}$, an object D in $\mathscr{D}$, and an arrow $u \in \mathrm{Hom}_{\mathscr{D}}(D, F(U))$. Notice that $\mathrm{Hom}_{\mathscr{D}}(D, F(-))$ determines a covariant functor $\mathscr{C} \longrightarrow \mathbf{Set}$. By the Yoneda Lemma, the following is a recipe for a natural transformation between $\mathrm{Hom}_{\mathscr{C}}(U, -)$ and $\mathrm{Hom}_{\mathscr{D}}(D, F(-))$: for each object Y in $\mathscr{C}$ and each arrow $h \in \mathrm{Hom}_{\mathscr{C}}(U, Y)$, set

$$\varphi_Y(h) := \mathrm{Hom}_{\mathscr{D}}(D, F(h))(u).$$

Notice that

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(D, F(U)) \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(D,F(h))"] & \mathrm{Hom}_{\mathscr{D}}(D, F(Y)) \ , \\ f \arrow[r, mapsto] & F(h) \circ u \end{tikzcd}$$

so $\varphi_Y(h)(f) = F(h) \circ u$.

We get the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(U, U) \arrow[rrr, "\mathrm{Hom}_{\mathscr{C}}(U,h)"] \arrow[ddd, "\varphi_U"'] & & & \mathrm{Hom}_{\mathscr{C}}(U, Y) \arrow[ddd, "\varphi_Y"] \\ & 1_U \arrow[r, mapsto] \arrow[d, mapsto] & h \arrow[d, mapsto] & \\ & u \arrow[r, mapsto] & F(h) \circ u =: \varphi_Y(h) & \\ \mathrm{Hom}_{\mathscr{D}}(D, F(U)) \arrow[rrr, "\mathrm{Hom}_{\mathscr{D}}(D,F(h))"'] & & & \mathrm{Hom}_{\mathscr{D}}(D, F(Y)) \end{tikzcd}$$

Given an arrow $f \in \mathrm{Hom}_{\mathscr{D}}(D, F(Y))$, $\varphi_Y(h) = f$ for some $h \in \mathrm{Hom}_{\mathscr{C}}(U, Y)$ if and only if $F(h) \circ u = f$.

On the one hand, φ is a natural isomorphism if and only if for every object Y in $\mathscr{C}$ and every $f \in \mathrm{Hom}_{\mathscr{D}}(D, F(Y))$ there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(U, Y)$ such that $F(h) \circ u = f$. On the other hand, that is exactly the condition required for (U, u) to be a universal arrow from D to F. So we have shown that the following are equivalent:

- (U, u) is a universal arrow from D to F.
- U represents the functor $\mathrm{Hom}_{\mathscr{D}}(D, F(-))\colon \mathscr{C} \longrightarrow \mathbf{Set}$, via $u \in \mathrm{Hom}_{\mathscr{D}}(D, F(U))$.

Similarly, one can prove the dual equivalence:

- (U, u) is a universal arrow from F to D.
- U represents the functor $\mathrm{Hom}_{\mathscr{D}}(F(-), D)\colon \mathscr{C} \longrightarrow \mathbf{Set}$, via $u \in \mathrm{Hom}_{\mathscr{D}}(F(U), D)$.

Conversely, suppose that we are given a representable functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ together with an element $X \in F(C)$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(C, -)$ via the natural isomorphism that corresponds to X via the bijection in the Yoneda Lemma. First, let $\{\star\}$ be a singleton. Recall that we saw in Example 1.44 that the functor $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, -)$ is naturally isomorphic to the identity functor on **Set**; by composing natural isomorphisms, this implies that $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, F(-)) = \mathrm{Hom}_{\mathbf{Set}}(\{\star\}, -) \circ F$ is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(C, -)$. So the object C represents the functor $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, F(-))$; this is half the recipe for a universal arrow.

Now if we actually want to keep track of the arrow $u \in \mathrm{Hom}_{\mathbf{Set}}(\{\star\}, F(C))$ that corresponds to this natural isomorphism, we need to keep track of what happens when we compose with $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, -)$. We started with a natural isomorphism corresponding to $X \in F(C)$, and composed with the functor $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, -)$, so our original $X \in F(C)$ will now correspond to some element in $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, F(C))$; this set is in natural bijection with the original set $F(C)$, and the element $X \in F(C)$ corresponds to the function $u \in \mathrm{Hom}_{\mathbf{Set}}(\{\star\}, F(C))$ given by $\star \mapsto X$. This is the arrow u we are searching for.

In conclusion: we have an equivalence between the following pieces of data:

- A representable functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ together with an element $X \in F(C)$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(C, -)$ via the natural isomorphism that corresponds to X via the bijection in the Yoneda Lemma.

- A universal arrow (C, u) from $\{\star\}$ to F, where $u \in \mathrm{Hom}_{\mathbf{Set}}(\{\star\}, F(C))$ is given by $\star \mapsto X$.

Let's take some of the universal properties we have encountered before and try to rephrase them via this formal lens.

Example 1.90. A singleton set $\{\star\}$ (or *the* singleton set, if we think about sets up to isomorphism) has the following simple universal property: to give a function out of $\{\star\}$ is the same as choosing an element in the target set. We saw in Example 1.44 that this is encoded in the fact that the identity functor on **Set** is representable, with representing object $\{\star\}$. Now here is a fun fact: the natural isomorphism between the identity on **Set** and the functor $\mathrm{Hom}_{\mathbf{Set}}(\{\star\}, -)$ used is the *only* natural transformation between them: indeed, the Yoneda Lemma says that each natural transformation corresponds to an element in $1_{\mathbf{Set}}(\{\star\}) = \{\star\}$; but there is only one such element!

Example 1.91. Let's phrase the universal property of products as a universal property in this formal sense, at least in the case of the product of two object C_1 and C_2 in $\mathscr{C}$. To do that, we need to consider the **product category** $\mathscr{C} \times \mathscr{C}$ whose objects are pairs (C_1, C_2) of objects in $\mathscr{C}$, and an arrow $(C_1, C_2) \longrightarrow (C_3, C_4)$ is given by a pair (f_1, f_2) with $f_1 \in \mathrm{Hom}_{\mathscr{C}}(C_1, C_3)$ and $f_2 \in \mathrm{Hom}_{\mathscr{C}}(C_2, C_4)$. The diagonal functor $\Delta : \mathscr{C} \longrightarrow \mathscr{C} \times \mathscr{C}$ is exactly what it sounds like: $\Delta(C) = (C, C)$ for every object C in $\mathscr{C}$ and $\Delta(f) = (f, f)$ for every arrow f in $\mathscr{C}$.

Given objects X and Y in $\mathscr{C}$, consider the projection arrows $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$. We claim that the object $X \times Y$ together with the arrow (π_1, π_2) in $\mathscr{C} \times \mathscr{C}$ form a universal arrow from Δ to (X, Y) in $\mathscr{C} \times \mathscr{C}$. Why? If true, this would mean that given any object Z in $\mathscr{C}$ and any arrow $(f_1, f_2) \in \mathrm{Hom}_{\mathscr{C} \times \mathscr{C}}(\Delta(Z), (X, Y))$, there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(Z, X \times Y)$ such that

$$\begin{tikzcd} X \times Y & & (X,Y) & \Delta(X \times Y) \arrow[l, "(\pi_1,\pi_2)"'] \\ Z \arrow[u, dashed, "h"] & & & \Delta(Z) \arrow[u, dashed, "\Delta(h)"'] \arrow[ull, "(f_1,f_2)"] \end{tikzcd}$$

commutes. This is indeed the universal property of products we described less formally when we first defined products: given $f_1 : Z \longrightarrow X$ and $f_2 : Z \longrightarrow Y$, there is a unique $h \colon Y \longrightarrow X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \times Y & & (X,Y) & (X \times Y, X \times Y) \arrow[l, "(\pi_1,\pi_2)"'] \\ Z \arrow[u, dashed, "h"] & & & (Z,Z). \arrow[u, dashed, "\Delta(h)"'] \arrow[ull, "(f_1,f_2)"] \end{tikzcd}$$

The diagram in $\mathscr{C} \times \mathscr{C}$ translates into two commutative diagrams in $\mathscr{C}$:

$$\begin{tikzcd} X & X \times Y \arrow[l, "\pi_1"'] \\ & Z \arrow[u, dashed, "h"'] \arrow[ul, "f_1"] \end{tikzcd} \qquad \begin{tikzcd} Y & X \times Y \arrow[l, "\pi_2"'] \\ & Z. \arrow[u, dashed, "h"'] \arrow[ul, "f_2"] \end{tikzcd}$$

This is precisely the universal property of the product that we described before.

Equivalently, following the recipe we described in Remark 1.89, the universal property of the product is encoded in the representable functor $\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(-),(X,Y))$, which is represented by $X \times Y$ via (π_1, π_2). So there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(-, X \times Y) \cong \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(-),(X,Y)),$$

which means that to give an arrow to $X \times Y$ is the same as giving an arrow to X and an arrow to Y. In fact, this natural iso is the natural transformation that the Yoneda bijection we constructed in Theorem 1.45 takes to $(\pi_1, \pi_2) \in \mathrm{Hom}_{\mathscr{C}}(\Delta(X \times Y), (X,Y))$. If we follow that bijection, our natural isomorphism φ sends an object Z in $\mathscr{C}$ to the arrow

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(Z, X \times Y) & \xrightarrow{\quad \varphi_Z \quad} & \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(Z),(X,Y)) \\ f & \longmapsto & \left(\Delta(Z) \xrightarrow{(f,f)} \Delta(X \times Y) \xrightarrow{(\pi_1,\pi_2)} (X,Y)\right). \end{array}$$

Since φ_Z is a bijection, every arrow $(f_1, f_2) \in \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(Z),(X,Y))$ is $\varphi_Z(f)$ for some $(f_1, f_2) \in \mathrm{Hom}_{\mathscr{C}}(Z, X \times Y)$. In particular, there exists (f_1, f_2) such that $f_1 = \pi_1 f$ and $f_2 = \pi_2 f$. And surprise surprise: we just rediscovered the universal property of the product!

Exercise 23. Rephrase the universal property of the coproduct in this formal sense.

1.8 Adjoint functors

Universal properties are closely related to adjoint functors.

Definition 1.92. Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Two covariant functors

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

form an **adjoint** pair (F, G) if given any objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$, there is a bijection between the Hom-sets

$$\mathrm{Hom}_{\mathscr{D}}(F(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(C, G(D))$$

which is natural on both objects, meaning that for all $f \in \mathrm{Hom}_{\mathscr{C}}(C_1, C_2)$ and $g \in \mathrm{Hom}_{\mathscr{D}}(D_1, D_2)$, the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(C_1), D) \arrow[r, "\cong"] \arrow[d, "F(f)^*"'] & \mathrm{Hom}_{\mathscr{C}}(C_1, G(D)) \arrow[d, "f^*"] \\
\mathrm{Hom}_{\mathscr{D}}(F(C_2), D) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(C_2, G(D))
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(C), D_1) \arrow[r, "\cong"] \arrow[d, "g_*"'] & \mathrm{Hom}_{\mathscr{C}}(C, G(D_1)) \arrow[d, "G(g)_*"] \\
\mathrm{Hom}_{\mathscr{D}}(F(C), D_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(C, G(D_2))
\end{tikzcd}$$

commute for all $C \in \mathscr{C}$ and all $D \in \mathscr{D}$. We say that F is the **left adjoint** of G, or that F has a **right adjoint**, and that G is the **right adjoint** of F, or that G has a **left adjoint**.

We can think of adjoint functors as solutions to optimization problems. A particular adjoint functor gives the most efficient functorial solution to some problem.

Example 1.93. Fix a ring R. Given a set I, what is the most efficient way to assign an R-module to I in a functorial way? The solution to this problem is the construction of free modules. Formally, the free functor is the functor $\mathbf{Free} \colon \mathbf{Set} \longrightarrow R\text{-}\mathbf{Mod}$ that sends each set I to the free R-module on I

$$R^I = \bigoplus_I R.$$

The free functor is precisely a left adjoint to the forgetful functor $R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$. That is, there is a natural bijection

$$\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(\bigoplus_I R, M) \cong \mathrm{Hom}_{\mathbf{Set}}(I, M).$$

(On the right side we identified the image of M by the forgetful functor with M, since it's simply the underlying set.) Even without any category theory, one often describes the free R-module on a set I by the following universal property: given a function f from a set I to an R-module M, there exists a unique R-module homomorphism $\oplus_I R \to M$ that agrees with f on the basis elements. And indeed, this is what is encoded in the bijection above.

This type of *free* construction is quite common, and often gives rise to adjunctions. We can think about the free functor from **Set** to R-**Mod** as the most efficient way of defining an R-module from a given set. It's efficient because it comes with a nice universal property.

Quoting Mac Lane [ML98], one of the fathers of category theory, "the slogan is *adjoint functors arise everywhere*". We will see a very important example of adjunction later on – the Hom-tensor adjunction.

Remark 1.94. We can rephrase the condition that $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ as follows: for every object C in $\mathscr{C}$, there is a universal arrow from C to G, and for every object D in $\mathscr{D}$ there exists a universal arrow from F to D. To see that, let $\eta_D \in \mathrm{Hom}_{\mathscr{D}}(F(G(D)), D)$ be the image of the identity on $\mathrm{Hom}_{\mathscr{D}}(G(D), G(D))$ via the bijection

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(G(D), G(D)) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{D}}(F(G(D)), D) \\
\mathrm{id}_{G(D)} \arrow[r, mapsto] & \eta_D
\end{tikzcd}$$

given by the definition of adjoint functors, and let $\varepsilon_C \in \mathrm{Hom}_{\mathscr{C}}(C, GF(C))$ be be the image of the identity on $\mathrm{Hom}_{\mathscr{C}}(F(C), F(C))$ via the bijection

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(C), F(C)) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{D}}(C, GF(C))\ . \\
\mathrm{id}_{F(C)} \arrow[r, mapsto] & \varepsilon_C
\end{tikzcd}$$

We claim that $(F(C), \varepsilon_C)$ is a universal arrow from C to G. That would mean that given arrow $f \in \mathrm{Hom}_{\mathscr{C}}(C, G(Y))$, there must exist a unique arrow $h \in \mathrm{Hom}_{\mathscr{D}}(F(C), Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
F(C) \arrow[d, dashed, "h"'] & & D \arrow[r, "\varepsilon_C"] \arrow[dr, "f"'] & G(F(C)) \arrow[d, dashed, "G(h)"] \\
Y & & & G(Y).
\end{tikzcd}$$

This says that $G(h)_*(\varepsilon_C) = G(h) \circ \varepsilon_C = f$, which means that

$$\begin{tikzcd}
1_{F(C)} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \varepsilon_C \arrow[ddd, mapsto] \\
& \mathrm{Hom}_{\mathscr{D}}(F(C), F(C)) \arrow[r, "\cong"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathscr{C}}(C, GF(C)) \arrow[d, "G(h)_*"] & \\
& \mathrm{Hom}_{\mathscr{D}}(F(C), Y) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathscr{C}}(C, G(Y)) & \\
h \arrow[rrr, mapsto] & & & f
\end{tikzcd}$$

On the one hand, such an h does exist: just take $h \in \mathrm{Hom}_{\mathscr{C}}(F(C), Y)$ that is sent to f via the bijection between $\mathrm{Hom}_{\mathscr{D}}(F(C), Y)$ and $\mathrm{Hom}_{\mathscr{C}}(C, G(Y))$. Since this map is a bijection, such an h is unique.

Similarly, we claim that $(G(D), \eta_D)$ is a universal arrow from F to D. That would mean that for any arrow $f \in \mathrm{Hom}_{\mathscr{D}}(F(Y), D)$, there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(Y, G(D))$ such that the following diagram commutes:

$$\begin{tikzcd}
G(D) & & D & F(G(D)) \arrow[l, "\eta_D"'] \\
Y \arrow[u, dashed, "h"] & & & F(Y) \arrow[ul, "f"] \arrow[u, dashed, "F(h)"']
\end{tikzcd}$$

This means that $(F(h))^*(\eta_D) = \eta_D \circ F(h) = f$, so

$$\begin{tikzcd}
1_{G(D)} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \eta_D \arrow[ddd, mapsto] \\
& \operatorname{Hom}_{\mathscr{C}}(G(D), G(D)) \arrow[r, "\cong"] \arrow[d, "h^*"'] & \operatorname{Hom}_{\mathscr{D}}(D, FG(D)) \arrow[d, "F(h)^*"] & \\
& \operatorname{Hom}_{\mathscr{C}}(G(D), Y) \arrow[r, "\cong"'] & \operatorname{Hom}_{\mathscr{D}}(D, F(Y)) & \\
h \arrow[rrr, mapsto] & & & f
\end{tikzcd}$$

Again, such an h exists and it is unique because it must correspond to f via the bijection between $\operatorname{Hom}_{\mathscr{D}}(D, F(Y))$ and $\operatorname{Hom}_{\mathscr{C}}(G(D), Y)$.

We can talk about *the* left or right adjoint to a given functor.

Exercise 24. Left and right adjoints are unique up to natural isomorphism. More precisely, given an adjoint pair of functors (F, G), show that if G' is also a right adjoint to F, then G' and G are naturally isomorphic. Similarly, show that if F' is also a left adjoint to G, then F and F' are naturally isomorphic.

We close this short detour into the wonderful world of category theory to point out that if we wanted to sound really obscure, we could have defined chain complexes in this categorical language.

Remark 1.95. First, we view $\mathbb{Z}$ as a partially ordered set under $\geqslant$. As in Example 1.8, $\mathbb{Z}$ now gives us a category whose objects are the integers, and where we have an arrow in $\operatorname{Hom}_{\mathbb{Z}}(n, m)$ if $n \geqslant m$. If we ignore the identity maps $\operatorname{Hom}_{\mathbb{Z}}(n, n)$ and composite maps, we can represent this category in the following diagram:

$$\cdots \longrightarrow n+1 \longrightarrow n \longrightarrow n-1 \longrightarrow \cdots.$$

From this perspective, a chain complex is a functor $F \colon \mathbb{Z} \longrightarrow R\text{-}\mathbf{Mod}$: for each $n \in \mathbb{Z}$, we get an R-module F_n, and we also get an R-module homomorphisms $F_{n+1} \longrightarrow F_n$ for each n. Indeed, this can all be represented as a sequence

$$\cdots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots.$$

For our functor to truly be a complex, though, we must require that all compositions $F_{n+1} \longrightarrow F_n \longrightarrow F_{n-1}$ be 0. A map of complexes, also known as a chain map, is a natural transformation between two such functors.

Chapter 2

The category of chain complexes

We are finally ready to introduce the category of chain complexes, and to talk more about exact sequences and homology.

2.1 Maps of complexes

Unsurprisingly, we can form a category of complexes, but to do that we need the right definition of maps between complexes. We also take this section as a chance to set up some definitions we will need later. One thing to keep in mind as we build our basic definitions: we also want homology to be functorial.

Definition 2.1. Let $(F_\bullet, \partial_\bullet^F)$ and $(G_\bullet, \partial_\bullet^G)$ be complexes. A **map of complexes** or a **chain map**, which we write as $h\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ or simply $h\colon F \longrightarrow G$, is a sequence of homomorphisms of R-modules $h_n\colon F_n \longrightarrow G_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, "h_{n+1}"'] & F_n \arrow[r] \arrow[d, "h_n"'] & F_{n-1} \arrow[r] \arrow[d, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r] & G_n \arrow[r] & G_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

This means that $h_n \partial_{n+1}^F = \partial_{n+1}^G h_{n+1}$ for all n.

Note that throughout, whenever we call a function $f\colon M \to N$ between R-modules M and N a *map*, we really mean to say it is a homomorphism of R-modules.

Example 2.2. The zero and the identity maps of complexes $(F_\bullet, \partial_\bullet) \longrightarrow (F_\bullet, \partial_\bullet)$ are exactly what they sound like: the zero map $0_{F_\bullet}$ is 0 in every homological degree, and the identity map $1_{F_\bullet}$ is the identity in every homological degree.

This is the notion of morphism we would want to form a category of chain complexes.

Definition 2.3. Let R be a ring. The **category of chain complexes** of R-modules, denoted $\mathrm{Ch}(R\text{-}\mathbf{mod})$ or simply $\mathrm{Ch}(R)$, is the category with objects all chain complexes of R-modules and arrows all maps of complexes of R-modules. When $R = \mathbb{Z}$, we write $\mathrm{Ch}(\mathbf{Ab})$ for $\mathrm{Ch}(\mathbb{Z})$, the category of chain complexes of abelian groups.

Note that the identity maps defined above are precisely the identity arrows in the category of chain complexes.

Exercise 25. Show that the isomorphisms in the category $\mathrm{Ch}(R)$ are precisely the maps of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, "h_{n+1}"'] & F_n \arrow[r] \arrow[d, "h_n"'] & F_{n-1} \arrow[r] \arrow[d, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r] & G_n \arrow[r] & G_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that h_n is an isomorphism for all n.

This is a good notion of map of complexes: it induces homomorphisms in homology, which in particular allows us to say that homology is a functor.

Lemma 2.4. *Let $h\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ be a map of complexes. For all n, h_n restricts to homomorphisms $B_n(h) : B_n(F_\bullet) \longrightarrow B_n(G_\bullet)$ and $Z_n(h) : Z_n(F_\bullet) \longrightarrow Z_n(G_\bullet)$. As a consequence, h induces homomorphisms on homology $\mathrm{H}_n(h) : \mathrm{H}_n(F_\bullet) \longrightarrow \mathrm{H}_n(G_\bullet)$.*

Proof. Since $h_n\partial_{n+1}^F = \partial_{n+1}^G h_{n+1}$, any element $a \in B_n(F_\bullet)$, say $a = \partial_{n+1}^F(b)$, is taken to

$$h_n(a) = h_n\partial_{n+1}^F(b) = \partial_{n+1}^G h_{n+1}(b) \in \operatorname{im} \partial_{n+1}^G = B_n(G_\bullet).$$

Similarly, if $a \in Z_n(F_\bullet) = \ker \partial_n^F$, then

$$\partial_n h_n(a) = h_{n-1}\partial_n^F(a) = 0,$$

so $h_n(a) \in \ker \partial_n^G = Z_n(G_\bullet)$. Finally, the restriction of h_n to $Z_n(F_\bullet) \longrightarrow Z_n(G_\bullet)$ sends $B_n(F_\bullet)$ into $B_n(G_\bullet)$, and thus it induces a well-defined homomorphism on the quotients $\mathrm{H}_n(F_\bullet) \longrightarrow \mathrm{H}_n(G_\bullet)$. $\square$

Definition 2.5. Let $h\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ be a map of complexes. We call the map

$$\begin{aligned}
\mathrm{H}_n(h)\colon \quad \mathrm{H}_n(F_\bullet) &\longrightarrow \mathrm{H}_n(G_\bullet) \\
a + B_n(F) &\longmapsto h_n(a) + B_n(G)
\end{aligned}$$

the **induced map in homology**, and sometimes denote it by h_*.

One can show that H_n preserves compositions, and that moreover, the map in homology induced by the identity is the identity. Thus taking nth homology is a functor

$$\mathrm{H}_n\colon \mathrm{Ch}(R) \longrightarrow R\text{-}\mathbf{Mod}$$

which takes each map of complexes $h\colon F_\bullet, \longrightarrow G_\bullet$ to the R-module homomorphism

$$\mathrm{H}_n(h) : \mathrm{H}_n(F_\bullet) \longrightarrow \mathrm{H}_n(G_\bullet).$$

Definition 2.6. A map of chain complexes h is a **quasi-isomorphism** if it induces an isomorphism in homology, meaning $\mathrm{H}_n(h)$ is an isomorphism of R-modules for all n. If there exists a quasi-isomorphism between two complexes C and D, we say that C and D are **quasi-isomorphic**, and write $C \simeq D$.

Remark 2.7. Note that saying that if f is a quasi-isomorphism between F and G is a stronger statement that the fact that $\mathrm{H}_n(F) \cong \mathrm{H}_n(G)$ for all n: it also says that there are such isomorphisms that are all induced by f.

Not all quasi-isomorphisms are isomorphisms, as the following example shows:

Exercise 26. Let π denote the projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. The chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism.

Definition 2.8. Let $f, g\colon F \longrightarrow G$ be maps complexes. A **homotopy**, sometimes referred to as a **chain homotopy**, between f and g is a sequence of maps $h_n\colon F_n \longrightarrow G_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & F_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "g_{n+1}"'] \arrow[d, shift left, "f_{n+1}"] & F_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"] \arrow[d, shift right, "g_n"'] \arrow[d, shift left, "f_n"] & F_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "h_{n-1}"] \arrow[d, shift right, "g_{n-1}"'] \arrow[d, shift left, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial_{n+2}"'] & G_{n+1} \arrow[r, "\partial_{n+1}"'] & G_n \arrow[r, "\partial_n"'] & G_{n-1} \arrow[r, "\partial_{n-1}"'] & \cdots
\end{tikzcd}$$

such that

$$\partial_{n+1} h_n + h_{n-1} \partial_n = f_n - g_n$$

for all n. If there exists a homotopy between f and g, we say that f and g are **homotopic** or that they **have the same homotopy type**. We write $f \simeq g$ to say that f and g are homotopic. If f is homotopic to the zero map, we say f is **nullhomotopic**, and write $f \simeq 0$. This should not be confused with the notation $C \simeq D$ on complexes.

Exercise 27. Homotopy is an equivalence relation.

The equivalence classes under homotopy are called **homotopy classes**. Homotopy is an interesting equivalence relation because homotopic maps induce the same map on homology.

Lemma 2.9. *Let $f, g\colon (F_\bullet, \partial^F_\bullet) \longrightarrow (G_\bullet, \partial^G_\bullet)$ be maps of complexes. If f is homotopic to g, then $\mathrm{H}_n(f) = \mathrm{H}_n(g)$ for all n. In particular, every nullhomotopic map induces the zero map in homology.*

Proof. Let $f, g : (F_\bullet, \partial^F_\bullet) \longrightarrow (G_\bullet, \partial^G_\bullet)$ be homotopic maps of complexes, and let h be a homotopy between f and g. We claim that the map of complexes $f - g$ (defined in the obvious way) sends cycles to boundaries. If $a \in Z_n(F_\bullet)$, then

$$(f - g)_n(a) = \partial_{n+1} h_n + h_{n-1} \underbrace{\partial_n(a)}_{0} = \partial_{n+1}(h_n(a)) \in B_n(G_\bullet).$$

The map on homology induced by $f - g$ must then be the 0 map, so f and g induce the same map on homology. Here we are implicitly using the fact that $\mathrm{H}_n(f + h) = \mathrm{H}_n(f) + \mathrm{H}_n(g)$, which we leave as an exercise to be further explored in Remark 3.4. $\square$

Notice, however, that the converse is fall: the induced map in homology can be the zero map (for all homological degrees) even if the original map of complexes is not nullhomotopic.

Exercise 28. Consider the following map of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, "2"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "2"'] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Show that this map is not nullhomotopic, but that the induced map in homology is zero.

Definition 2.10. If $f : (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ and $g : (G_\bullet, \partial_\bullet^G) \longrightarrow (F_\bullet, \partial_\bullet^F)$ are maps of complexes such that fg is homotopic to the identity map on $(G_\bullet, \partial_\bullet^G)$ and gf is homotopic to the identity chain map on $(F_\bullet, \partial_\bullet^F)$, we say that f and g are **homotopy equivalences** and $(F_\bullet, \partial_\bullet^F)$ and $(G_\bullet, \partial_\bullet^G)$ are **homotopy equivalent**.

Corollary 2.11. *Homotopy equivalences are quasi-isomorphisms.*

Proof. If $f\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ and $g\colon (G_\bullet, \partial_\bullet^G) \longrightarrow (F_\bullet, \partial_\bullet^F)$ are such that fg is homotopic to $1_{G_\bullet}$ and gf is homotopic to $1_{F_\bullet}$, then by Lemma 2.9 the map fg induces the identity map on homology. So for all n we have

$$\mathrm{H}_n(f)\,\mathrm{H}_n(g) = \mathrm{H}_n(fg) = \mathrm{H}_n(1) = 1.$$

Therefore, $\mathrm{H}_n(f)$ and $\mathrm{H}_n(g)$ must both be isomorphisms. ☐

The converse is false.

Exercise 29. Let π denote the projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. The chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism but not a homotopy equivalence.

Remark 2.12. The relation $F \simeq G$, meaning "there is a quasi-isomorphism from F to G", is not symmetric: in Exercise 29, there is no quasi-isomorphism going in the opposite direction of the one given.

Now that we know about maps between complexes, it's time to point out that we can also talk about complexes and exact sequences of complexes. While we will later formalize this a little better when we discover that $\mathrm{Ch}(R)$ is an abelian category, let's for now give quick definitions that we can use.

Definition 2.13. Given complexes B and C, B is a **subcomplex** of C if B_n is a submodule of C_n for all n, and the inclusion maps $\iota_n\colon B_n \subseteq C_n$ define a map of complexes $\iota : B \longrightarrow C$. Given a subcomplex B of C, the **quotient** of C by B is the complex C/B that has C_n/B_n in homological degree n, with differential induced by the differential on C_n.

Exercise 30. If B is a subcomplex of C, then the differential d on C satisfies $d_n(B_n) \subseteq B_{n-1}$. Therefore, d_n induces a map of R-modules $C_n/B_n \longrightarrow C_{n-1}/B_{n-1}$ for all n, so that our definition of the differential on C/B actually makes sense.

We can also talk about kernels and cokernels of maps of complexes.

Definition 2.14. Given any map of complexes $f\colon B_\bullet \longrightarrow C_\bullet$, the **kernel** of f is the subcomplex $\ker f$ of $B_\bullet$ that we can assemble from the the kernels $\ker f_n$. More precisely, $\ker f$ is the complex

$$\cdots \longrightarrow \ker f_{n+1} \longrightarrow \ker f_n \longrightarrow \ker f_{n-1} \longrightarrow \cdots$$

where the differentials are simply the corresponding restrictions of the differentials on $B_\bullet$. Similarly, the **image** of f is the subcomplex of $C_\bullet$

$$\cdots \longrightarrow \operatorname{im} f_{n+1} \longrightarrow \operatorname{im} f_n \longrightarrow \operatorname{im} f_{n-1} \longrightarrow \cdots$$

where the differentials are given by restriction of the corresponding differentials in $C_\bullet$. The **cokernel** of f is the quotient complex $C_\bullet/\operatorname{im} f$.

Again, there are some details to check.

Exercise 31. Show that the kernel, image, and cokernel of a complex map are indeed complexes.

Definition 2.15. A **complex** in $\mathrm{Ch}(R)$ is a sequence of complexes of R-modules C^n and chain maps $d_n\colon C^n \longrightarrow C^{n-1}$ between them

$$\cdots \longrightarrow C^{n+1} \xrightarrow{d_{n+1}} C^n \xrightarrow{d_n} C^{n-1} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all n. A complex of complexes is a diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n+1}_{i+1} \arrow[r, "d_{n+1}"] \arrow[d, "\partial_{i+1}"'] & C^n_{i+1} \arrow[r, "d_n"] \arrow[d, "\partial_{i+1}"'] & C^{n-1}_{i+1} \arrow[r] \arrow[d, "\partial_{i+1}"'] & \cdots \\
t\cdots \arrow[r] & C^{n+1}_{i} \arrow[r, "d_{n+1}"] \arrow[d, "\partial_{i}"'] & C^n_{i} \arrow[r, "d_n"] \arrow[d, "\partial_{i}"'] & C^{n-1}_{i} \arrow[r] \arrow[d, "\partial_{i}"'] & \cdots \\
\cdots \arrow[r] & C^{n+1}_{i-1} \arrow[r, "d_{n+1}"] & C^n_{i-1} \arrow[r, "d_n"] & C^{n-1}_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

where the nth column corresponds to the complex C^n, and every row is also a complex.

Given a complex C in $\mathrm{Ch}(R)$, we can talk about cycles and boundaries, which are a sequence of subcomplexes of the complexes in C, and thus its homology. Such a complex is exact if $\operatorname{im} d_{n+1} = \ker d_n$ for all n.

Definition 2.16. A **short exact sequence** of complexes is an exact complex in $\mathrm{Ch}(R)$ of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Equivalently, a short exact sequence of complexes is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\partial_{i+1}"'] & B_{i+1} \arrow[r, "g_{i+1}"] \arrow[d, "\partial_{i+1}"'] & C_{i+1} \arrow[r] \arrow[d, "\partial_{i+1}"'] & 0 \\
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "\partial_i"'] & B_i \arrow[r, "g_i"] \arrow[d, "\partial_i"'] & C_i \arrow[r] \arrow[d, "\partial_i"'] & \cdots \\
\cdots \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] \arrow[d] & B_{i-1} \arrow[r, "g_{i-1}"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the rows are exact and the columns are complexes.

2.2 Short exact sequences

In this section, we will discuss short exact sequences of modules in a bit more detail. We note, however, that everything we will discuss here can be extended for short exact sequences of complexes, and that the generalization is not too difficult: one just needs to replace modules with complexes and maps of modules by maps of complexes.

Example 2.17. Fix a ring R, and let A and C be R-modules. Consider the inclusion $i\colon A \to A \oplus C$ of A into the first component of the direct sum, and the projection map $\pi\colon A \oplus C \to C$ onto the second component of the product. These two maps fit into a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0.$$

These are sometimes called **trivial short exact sequences**.

On the one hand, the short exact sequences that look like this one are very important; on the other hand, not all short exact sequences are of this type.

Definition 2.18. We say that a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

splits or is a **split short exact sequence** if it is isomorphic to

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0$$

where i is the inclusion of the first component and p is the projection onto the second component.

Lemma 2.19 (Splitting Lemma)**.** *Consider the short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules. The following are equivalent:

a) There exists a homomorphism of R-modules $q\colon B \longrightarrow A$ such that $qf = \mathrm{id}_A$.

b) There exists a homomorphism of R-modules $r\colon C \longrightarrow B$ such that $gr = \mathrm{id}_C$.

c) The short exact sequence splits.

Definition 2.20. Given a split short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

maps q and r satisfying the conditions of the Splitting Lemma are called **splittings**.

Remark 2.21. In the split short exact sequence

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0,$$

the canonical projection $q\colon A \oplus C \to A$ and the usual inclusion $r\colon C \to A \oplus C$ are splittings.

Proof. First, we will show that c implies a and b. If the sequence splits, then consider an isomorphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[r, "p"] \arrow[l, bend left, "\pi"] & C \arrow[r] \arrow[l, bend left, "j"] & 0,
\end{tikzcd}$$

meaning that the diagram commutes and a, b, and c are isomorphisms of R-modules, i is the inclusion in the first component, and p is the projection onto the second component. Let $\pi : A \oplus C \longrightarrow A$ be the projection onto the first component, and $j\colon C \longrightarrow A \oplus C$ be the inclusion onto the first component. Now consider the maps $q := a^{-1}\pi b$ and $r := b^{-1}jc$. Then

$$\begin{aligned}
qf &= a^{-1}\pi b f \\
&= a^{-1}\pi i a && \text{by commutativity} \\
&= a^{-1}a && \text{because } \pi i = \mathrm{id}_A \\
&= 1_A
\end{aligned}$$

and

$$\begin{aligned}
gr &= g b^{-1} j c \\
&= c^{-1}(cg) b^{-1} j c && \text{multiplying by } c^{-1}c = 1_C \\
&= c^{-1}(pb) b^{-1} j c && \text{by commutativity} \\
&= c^{-1} p j c && \text{because } bb^{-1} = 1_B \\
&= c^{-1} c && \text{because } pj = \mathrm{id}_C \\
&= 1_C.
\end{aligned}$$

Therefore, c implies a and b.

Now suppose that a holds, and let's show that the sequence splits. First, we need to show that $B \cong A \oplus C$. Every $b \in B$ can be written as

$$b = (b - fq(b)) + fq(b),$$

where $fq(b) \in \operatorname{im} f \cong A$, and

$$q(b - fq(b)) = q(b) - \underbrace{qf}_{\mathrm{id}_A}(q(b)) = q(b) - q(b) = 0,$$

so $b - fq(b) \in \ker q$. This shows that $B = \operatorname{im} f + \ker q$. Moreover, if $f(a) \in \ker q$, then $a = qf(a) = 0$, so $\operatorname{im} f \cap \ker q = 0$, and $B = \operatorname{im} f \oplus \ker q$. Now when we restrict g to $\ker q$, g becomes injective. We claim it is also surjective, and thus an isomorphism. Indeed, for any $c \in C$ we can pick $b \in B$ such that $g(b) = c$, since g is surjective, and we showed that we can write $b = f(a) + k$ for some $k \in \ker q$. Then

$$g(k) = \underbrace{gf}_{0}(a) + g(k) = g(b) = c.$$

Finally, note that $\operatorname{im} f \cong A$, so we conclude that $B \cong A \oplus C$, via the isomorphism φ given by

$$\begin{array}{ccccc} B & \longrightarrow & \operatorname{im} f \oplus \ker q & \longrightarrow & A \oplus C \\ b & \longmapsto & (fq(b), b - fq(b)) & \longmapsto & (q(b), g(b)). \end{array}$$

Since $gf = 0$ and $qf = \mathrm{id}_A$, $\varphi f(a) = (qf(a), 0) = (a, 0)$, so $\varphi f = i$, where $i \colon A \longrightarrow A \oplus C$ is the inclusion on the first factor. If $p \colon A \oplus C \longrightarrow C$ denotes the projection onto the second factor, $p\varphi = g$. Together, these two facts say that the following is a map of complexes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\varphi"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"'] & A \oplus C \arrow[r, "p"'] & C \arrow[r] & 0. \end{tikzcd}$$

Since φ is an isomorphism, so is our map of complexes, and thus our original sequence is a split exact sequence. This shows that a implies c.

Now assume b holds. Every $b \in B$ can be written as

$$b = (b - rg(b)) + rg(b),$$

where $rg(b) \in \operatorname{im} r$ and

$$g(b - rg(b)) = g(b) - \underbrace{gr}_{\mathrm{id}_C}(g(b)) = g(b) - g(b) = 0,$$

so $b - rg(b) \in \ker g$. This shows that $B = \ker g + \operatorname{im} r$. Moreover, if $r(c) \in \ker g$, then

$$c = \mathrm{id}_C(c) = gr(c) = 0.$$

Therefore, $B = \ker g \oplus \operatorname{im} r$. Now r is injective, since $r(c) = 0 \implies c = gr(c) = 0$, and thus $\operatorname{im} r \cong C$. Since $\ker g = \operatorname{im} f \cong A$, we conclude that $B \cong A \oplus C$, via the isomorphism

$$\begin{aligned} A \oplus C &\xrightarrow{\quad\psi\quad} B \\ (a, c) &\longmapsto f(a) + r(c). \end{aligned}$$

Finally, let $i \colon A \longrightarrow A \oplus C$ denote the inclusion of the first factor, and $p \colon A \oplus C \longrightarrow C$ denote the projection onto the second factor. By construction, $\psi i = f$. Moreover,

$$g\psi(a, c) = \underbrace{gf}_{0}(a) + \underbrace{gr}_{\mathrm{id}_C}(c) = c,$$

so $g\psi = p$. Together, these say that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & A \oplus C \arrow[r, "p"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

commutes, and must then be an isomorphism of short exact sequences. $\square$

Exercise 32. Let k be a field. Show that every short exact sequence of k-vector spaces splits.

The Rank-Nulity Theorem can be recast in this setting as a consequence of the fact that every short exact sequence of k-vector spaces splits.

Exercise 33. Prove the Rank-Nulity Theorem using Exercise 32: show that given any linear transformation $T \colon V \to W$ of k-vector spaces,

$$\dim(\operatorname{im} T) + \dim(\ker T) = \dim V.$$

But over a general ring, not every short exact sequence splits.

Example 2.22. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

is not split. Indeed, $\mathbb{Z}$ does not have any 2-torsion elements, so it is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$.

An alternative explanation is that there is no splitting to the inclusion $\mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z}$. On the one hand, every $\mathbb{Z}$-module map is given by multiplication by a fixed integer n, so a splitting $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ would be of the form $f(a) = na$ for some fixed n. On the other hand, our proposed splitting f must send 2 to 1, but there is no integer solution n to $2n = f(2) = 1$.

More surprisingly, a short exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\ f\ } A \oplus C \xrightarrow{\ g\ } C \longrightarrow 0$$

is not necessarily split, not unless f is the inclusion of the first component and g is the projection onto the second component, as the next example will show.

Example 2.23. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{f} \mathbb{Z}/(4) \xrightarrow{g} \mathbb{Z}/(2) \longrightarrow 0$$

where f is the inclusion of the subgroup generated by 2, so $f(1+(2)) = 2+(4)$, and g is the quotient onto that subgroup, meaning $g(1) = 1$. This is not a split short exact sequence, because $\mathbb{Z}/(4) \ncong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. Now let

$$M := \bigoplus_{\mathbb{N}} (\mathbb{Z}/(2) \oplus \mathbb{Z}/(4))$$

be the direct sum of infinitely many copies of $\mathbb{Z}/(2) \oplus \mathbb{Z}/(4)$. Then

$$\mathbb{Z}/(2) \oplus M \cong M \cong M \oplus \mathbb{Z}/(4),$$

and the sequence

$$0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{h} \mathbb{Z}/(4) \oplus M \xrightarrow{t} \mathbb{Z}/(2) \oplus M \longrightarrow 0$$

with $h(a) = (f(a), 0)$ and $t(a, m) = (g(a), m)$ is still exact. The middle term is indeed isomorphic to the direct sum of the other two:

$$\mathbb{Z}/(4) \oplus M \cong M \cong (M \oplus \mathbb{Z}/(2)) \oplus \mathbb{Z}/(2).$$

And yet this is not a split exact sequence: if we had a splitting $q\colon \mathbb{Z}/(4) \oplus M \longrightarrow \mathbb{Z}/(2)$ of h, then its restriction to the first factor would give us a splitting $\mathbb{Z}/(4) \longrightarrow \mathbb{Z}/(2)$ of f, which we know cannot exist, since

$$0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{f} \mathbb{Z}/(4) \xrightarrow{g} \mathbb{Z}/(2) \longrightarrow 0$$

does not split.

Given splittings q and r for a short exact sequence as in Lemma 2.19, we can quickly show that our short exact sequence splits using the Five Lemma. To prove the Five Lemma, one needs to use diagram chasing. Diagram chasing is a common technique in homological algebra, which essentially consists of tracing elements around in the diagram. We will see some examples of diagram chasing in the next section.

Exercise 34 (The Five Lemma)**.** Consider the following commutative diagram of R-modules with exact rows:

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"'] & B' \arrow[r] \arrow[d, "b"'] & C' \arrow[r] \arrow[d, "c"'] & D' \arrow[r] \arrow[d, "d"'] & E' \arrow[d, "e"'] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

Show that if a, b, d, and e are isomorphisms, then c is an isomorphism.

Remark 2.24. Given a short exact sequence, suppose we have R-module homomorphisms q and r

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, bend left, "q"] & C \arrow[r] \arrow[l, bend left, "r"] & 0 \end{tikzcd}$$

such that $qf = \mathrm{id}_A$ and $rg = \mathrm{id}_C$. Then we get an induced map

$$\begin{aligned} B &\xrightarrow{\varphi} A \oplus C \\ b &\longmapsto (q(b), g(b)) \end{aligned}$$

such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, dashed, "\varphi"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"'] & A \oplus C \arrow[r, "p"'] & C \arrow[r] & 0. \end{tikzcd}$$

commutes. The Five Lemma guarantees that φ must be an isomorphism, so our diagram is an isomorphism of short exact sequences.

There are many ways in which R-**Mod** behaves better than the category of groups, and this is one of them.

Remark 2.25. The Splitting Lemma does not hold if we replace R-modules with the category **Grp** of groups. For example, consider the symmetric group on 3 elements S_3 and the inclusion $A_3 \hookrightarrow S_3$ of the alternating group in S_3. Notice that A_3 is precisely the kernel of the sign map

$$\mathrm{sign}\colon S_3 \longrightarrow \mathbb{Z}/2,$$

which sends even permutations to 0 and odd permutations to 1. Therefore,

$$0 \longrightarrow A_3 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

is a short exact sequence. Moreover, this exact sequence is not split, since S_3 is not abelian but $A_3 \oplus \mathbb{Z}/2$ is, and thus $S_3 \ncong A_3 \oplus \mathbb{Z}/2$. However, any group homomorphism $u\colon \mathbb{Z}/2 \longrightarrow S_3$ defined by sending the generator to any two cycle is a splitting for our short exact sequence, meaning $\mathrm{sign} \circ u = \mathrm{id}_{\mathbb{Z}/2}$.

Funny enough, there is no splitting for the inclusion $A_3 \subseteq S_3$, since there are no nontrivial homomorphisms $S_3 \longrightarrow A_3$: A_3 has no elements of order 2, so a group homomorphism $S_3 \longrightarrow A_3$ must send every 2-cycle in S_3 must be sent to the identity, but 2-cycles generate S_3.

We will return to the topic of split short exact sequences when we talk about projective and injective modules.

Exercise 35. Fix a ring R. Show that if F is a free R-module, then every short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow R \longrightarrow 0$$

splits.

2.3 Long exact sequences

A long exact sequence is just what it sounds like: an exact sequence that is, well, long. Usually, we use the term long exact sequence to refer to any exact sequence, especially if it is not a short exact sequence. So in particular, a long exact sequence does not literally have to be that long.

Long exact sequences arise naturally in various ways, and are often induced by some short exact sequence. The first long exact sequence one encounters is the long exact sequence on homology. All other long exact sequences are, in some way, a special case of this one. The main tool we need to build it is the Snake Lemma.

Theorem 2.26 (Snake Lemma)**.** *Consider the commutative diagram of R-modules*

$$\begin{tikzcd} & A' \arrow[r, "i'"] \arrow[d, "f"'] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C & \end{tikzcd} .$$

If the rows of the diagram are exact, then there exists an exact sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

Given $c' \in \ker h$, pick $b' \in B'$ such that $p'(b') = c'$, and $a \in A$ such that $i(a) = g(b')$. Then

$$\partial(c') = a + \operatorname{im} f \in \operatorname{coker} f.$$

The picture to keep in mind (and which explains the name of the lemma) is the following:

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\partial"] \\ A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r] \arrow[d, "g"'] & C' \arrow[d, "h"'] \\ A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h \end{tikzcd}$$

Definition 2.27. The map ∂ in the Snake Lemma is the **connecting homomorphism**.

Proof. If $a' \in \ker f$, then

$$g(i'(a')) = if(a') = 0,$$

by commutativity, so $i'(a') \in \ker g$. Similarly, if $b' \in \ker g$ then $p'(b') \in \ker(g)$. So

$$A' \xrightarrow{i'} B' \xrightarrow{p'} C' \qquad \text{restrict to maps} \qquad \ker f \xrightarrow{i'} \ker g \xrightarrow{p'} \ker h\,.$$

We claim that the sequence obtained by restriction

$$\ker f \xrightarrow{i'} \ker g \xrightarrow{p'} \ker h$$

is exact. On the one hand, we already know that the original maps satisfy $p'i' = 0$, so their restrictions must satisfy this as well, guaranteeing that

$$i'(\ker f) \subseteq \ker(\ker g \xrightarrow{p'} \ker h).$$

On the other and, if $b' \in \ker g$ is such that $p'(b') = 0$, then by exactness of the original sequence there exists $a' \in A'$ such that $i'(a') = b'$; we only need to check that we can choose such a' satisfying $a' \in \ker f$. An indeed, by commutativity, any a' with $i'(a') = b'$ satisfies

$$if(a') = gi'(a') = g(b') = 0,$$

and since i is injective, we must have $f(a') = 0$. So we have shown that the following is an exact sequence:

$$\ker f \xrightarrow{i'} \ker g \xrightarrow{p'} \ker h.$$

Similarly, if $a \in \operatorname{im} f$, the commutativity of the diagram guarantees that $i(a) \in \operatorname{im} g$, and if $b \in \operatorname{im} g$, then $p(b) \in \operatorname{im} h$. So the maps $A \xrightarrow{i} B \xrightarrow{p} C$ restrict to maps

$$\operatorname{im} f \xrightarrow{i} \operatorname{im} g \xrightarrow{p} \operatorname{im} h\ ,$$

which then induce maps

$$\operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

To make the notation less heavy, we denote the induced maps on the quotients by i and p. Again, the fact that $pi = 0$ automatically gives us that the restrictions satisfy

$$\operatorname{im}(\operatorname{coker} f \to \operatorname{coker} g) \subseteq \ker(\operatorname{coker} g \to \operatorname{coker} h),$$

so we only need to check equality. Consider $b + \operatorname{im} g$ such that $p(b + \operatorname{im} g) = 0$, meaning that $p(b) = 0$, meaning that $p(b) \in \operatorname{im} h$. Let $c' \in C$ be such that $h(c') = p(b)$. Since p' is surjective, there exists $b' \in B'$ such that $p'(b') = c'$, and by commutativity,

$$pg(b') = hp'(b') = h(c') = p(b).$$

Then $b - g(b') \in \ker p = \operatorname{im} i$. Let $a \in A$ be such that $i(a) = b - g(b')$. Now in $\operatorname{coker} g$ we have

$$\begin{aligned} b + \operatorname{im} g &= b - g(b') + \operatorname{im} g \\ &= i(a) + \operatorname{im} g \\ &= i(a + \operatorname{im} f). \end{aligned}$$

This concludes the proof of exactness of

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \qquad \text{and} \qquad \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

We still need to show the parts of the statement related to the connecting homomorphism ∂. Our definition of ∂ can be visualized as follows:

$$\begin{tikzcd}
 & & c' \in \ker h \arrow[d] & \\
A' \arrow[r, "i'"] \arrow[d, "f"'] & b' \in B' \arrow[r, dashed, mapsto, "p'"] \arrow[d, mapsto, "g"] & c' \in C' \arrow[d, "h"] & \\
a \in A \arrow[r, mapsto, "i"'] \arrow[d, mapsto] & g(b') \in B \arrow[r, "p"'] & 0 \in C \arrow[r, Rightarrow] & g(b') \in \ker p = \operatorname{im} i \\
a + \operatorname{im} f \in \operatorname{coker} f & & &
\end{tikzcd}$$

Let's recap the process in words. First, we fix $c' \in \ker h \subseteq C'$. Since p' is surjective, we can always pick $b' \in B'$ such that $p'(b') = c'$. Since $c' \in \ker h$, by commutativity we have

$$pg(b') = hp'(b') = h(c') = 0,$$

so $g(b') \in \ker p = \operatorname{im} i$. Therefore, there exists $a \in A$ such that $i(a) = g(b')$. In fact, since i is injective, there exists a unique $a \in A$ such that $i(a) = g(b')$. Our definition of $\partial(c')$ sets

$$\partial(c') = a + \operatorname{im} f \in \operatorname{coker} f.$$

The fact that ∂ is a homomorphism of R-modules follows from the fact that all the maps involved are homomorphisms of R-modules: given $c'_1, c'_2 \in \ker h$, and $b'_1, b'_2 \in B'$, $a_1, a_2 \in A$ such that

$$p'(b'_1) = c'_1, \quad p'(b'_2) = c'_2, \quad i(a_1) = g(b'_1), \quad i(a_2) = g(b'_2),$$

we have

$$i(a_1 + a_2) = i(a_1) + i(a_2) = g(b'_1) + g(b'_2) = g(b'_1 + b'_2),$$

so

$$\partial(c'_1) = a_1 + \operatorname{im} f, \quad \partial(c'_2) = a_2 + \operatorname{im} f, \quad \text{and} \quad \partial(c'_1 + c'_2) = (a_1 + a_2) + \operatorname{im} f.$$

Therefore, $\partial(c'_1) + \partial(c'_2) = \partial(c'_1 + c'_2)$. Similarly, given any $r \in R$,

$$r(a_1 + \operatorname{im} f) = ra_1 + \operatorname{im} f, \quad i(ra_1) = ri(a_1) = rg(b'_1) = g(rb'_1), \quad \text{and} \quad p'(rb_1) = rp'(b_1) = rc_1,$$

so $\partial(rc_1) = r(a_1 + \operatorname{im} f) = r\partial(c_1)$. We now need to show the following:

1) ∂ is well-defined. 2) $p'(\ker g) = \ker \partial$. 3) $\operatorname{im} \partial = \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$.

Points 2) and 3) together say that the sequence

$$\ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g$$

is exact, and this will complete the proof.

First, let's show that $\partial(0)$ is well-defined. Ultimately, our definition of ∂ only involves one choice, when we pick $b' \in B'$ such that $p'(b') = 0$; we need to show that $\partial(0)$ does not depend on the choice of b'. Given $b' \in B'$ such that $p'(b') = 0$, by exactness we have $b' \in \ker p' = \operatorname{im} i'$. Therefore, there exists $a' \in A'$ such that $i'(a') = b'$. Notice that $a := f(a') \in A$ is such that
$$i(a) = if(a') = gi'(a') = g(b').$$
Thus our definition says that $\partial(0) = a + \operatorname{im} f \in \operatorname{coker} f$. Since $a = f(a') \in \operatorname{im} f$, we conclude that $a + \operatorname{im} f = 0$, so $\partial(0) = 0$ for any choice of b'.

Now consider any $c' \in \ker h$. Again, to show ∂ is well-defined, we need only to show it does not depend on the choice of b' such that $p'(b') = c'$. Consider $b'_1, b'_2 \in B'$ such that
$$p'(b'_1) = p'(b'_2) = c',$$
and $a_1, a_2 \in A$ such that
$$i(a_1) = g(b'_1) \qquad \text{and} \qquad i(a_2) = g(b'_2).$$
Note that
$$i(a_1 - a_2) = g(b'_1 - b'_2),$$
and since
$$p'(b'_1 - b'_2) = c' - c' = 0,$$
we must have
$$a_1 - a_2 + \operatorname{im} f = \partial(0) = 0.$$
Thus
$$a_1 + \operatorname{im} f = a_2 + \operatorname{im} f,$$
and this concludes our proof that ∂ is well-defined.

Now we show 1): that $p'(\ker g) = \ker \partial$.

If $b' \in \ker g$, then the only $a \in A$ such that $i(a) = g(b') = 0$ is $a = 0$. Therefore, $\partial(p'(b')) = 0$, so $p'(\ker g) \subseteq \ker \partial$. On the other hand, let $c' \in \ker h$ be such that $\partial(c') = 0$. That means that for any $b' \in B'$ such that $p'(b') = c'$ we must have $g(b') = i(a)$ for some $a \in \operatorname{im} f$. Let $a' \in A'$ be such that $f(a') = a$. Then
$$gi'(a') = if(a') = i(a) = g(b')$$
so $b' - i'(a') \in \ker g$. Since $p'i' = 0$,
$$c' = p'(b') = p'(b' - i'(a')) \in p'(\ker g).$$
We conclude that $\ker \partial = p'(\ker g)$, and this shows 2).

Now we show 3), that is, $\operatorname{im} \partial = \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$.

Let $a \in A$ be such that $i(a + \operatorname{im} f) = 0$. In B, this says that $i(a) \in \operatorname{im} g$, so we can choose $b' \in B'$ such that $g(b') = i(a)$. Using commutativity and the fact that $pi = 0$, we have
$$hp'(b') = pg(b') = pi(a) = 0 \quad \text{so} \quad p'(b') \in \ker h.$$
This shows that $a + \operatorname{im} f = \partial(p'(b'))$, and thus $\ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g) \subseteq \operatorname{im} \partial$. Finally, if $p'(b') = c'$ and $i(a) = g(b')$, then
$$i\partial(c') = i(a + \operatorname{im} f) = g(b') + \operatorname{im} g = 0, \quad \text{so} \quad \operatorname{im} \partial \subseteq \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g). \qquad \square$$

The proof of the Snake Lemma is what we call a *diagram chase*, for reasons that may be obvious by now: we followed the diagram in the natural way, and everything worked out in the end. The Five Lemma is another classical example of a diagram chase.

Now that we have the Snake Lemma, we can construct the long exact sequence in homology:

Theorem 2.28 (Long exact sequence in homology)**.** *Given a short exact sequence in* $\mathrm{Ch}(R)$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

there are connecting homomorphisms $\partial : \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A)$ *such that*

$$\cdots \longrightarrow \mathrm{H}_{n+1}(C) \xrightarrow{\partial} \mathrm{H}_n(A) \xrightarrow{f} \mathrm{H}_n(B) \xrightarrow{g} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \longrightarrow \cdots$$

is an exact sequence.

Proof. For each n, we have short exact sequences

$$0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0.$$

The condition that f and g are maps of complexes implies, by Lemma 2.4, that f and g take cycles to cycles, so we get exact sequences

$$0 \longrightarrow Z_n(A) \longrightarrow Z_n(B) \longrightarrow Z_n(C)\ .$$

Again by Lemma 2.4, the condition that f and g are maps of complexes also implies that f and g both take boundaries to boundaries, so that we get exact sequences

$$A_n/\operatorname{im} d^A_{n+1} \longrightarrow B_n/\operatorname{im} d^B_{n+1} \longrightarrow C_n/\operatorname{im} d^C_{n+1} \longrightarrow 0\ .$$

Let F be any complex. The boundary maps on F induce maps $F_n \longrightarrow Z_{n-1}(F)$ that send $\operatorname{im} d_{n+1}$ to 0, so we get induced maps $F_n/\operatorname{im} d_{n+1} \longrightarrow Z_{n-1}(F)$. Applying this general fact to A, B, and C, and putting all this together, we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & A_n/\operatorname{im} d^A_{n+1} \arrow[r] \arrow[d, "d^A_n"'] & B_n/\operatorname{im} d^B_{n+1} \arrow[r] \arrow[d, "d^B_n"'] & C_n/\operatorname{im} d^C_{n+1} \arrow[r] \arrow[d, "d^C_n"'] & 0\ . \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) &
\end{tikzcd}$$

For any complex F,

$$\ker(F_n/\operatorname{im} d^F_{n+1} \xrightarrow{d^F_n} Z_{n-1}(F)) = \mathrm{H}_n(F)$$

and

$$\operatorname{coker}(F_n/\operatorname{im} d^F_{n+1} \xrightarrow{d^F_n} Z_{n-1}(F)) = Z_{n-1}(F)/\operatorname{im} d^F_n = \mathrm{H}_{n-1}(F).$$

The Snake Lemma now gives us exact sequences

$$\mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \longrightarrow \mathrm{H}_{n-1}(B) \longrightarrow \mathrm{H}_{n-1}(C).$$

Finally, we glue all these together to obtain the long exact sequence in homology. □

Remark 2.29. It's helpful to carefully consider how to compute the connecting homomorphisms in the long exact sequence in homology, which we can easily put together from the proof of the Snake Lemma. Suppose that $c \in Z_{n+1}(C) = \ker d^C_{n+1}$. When we view c as an element in C_{n+1}, we can find $b \in B_{n+1}$ such that $g_{n+1}(b) = c$, since g_{n+1} is surjective by assumption. Since g is a map of complexes, we have

$$g_n d^B_{n+1}(b) = d^C_{n+1} g_{n+1}(b) = d^C_{n+1}(c) = 0,$$

so $d^B_{n+1}(b) \in \ker g_n$. In fact, note that $d^B_{n+1}(b) \in \mathbb{Z}_n(B)$, so

$$b \in \ker(Z_n(B) \xrightarrow{g_n} Z_n(C)) = \operatorname{im}(Z_n(A) \to Z_n(B)).$$

Thus there exists $a \in Z_n(A)$ such that $f_n(a) = d^B_{n+1}(b)$. Finally,

$$\partial(c + \operatorname{im} d_{n+2}) = a + \operatorname{im} d^A_{n+1}.$$

So in summary, the recipe goes as follows: given $c + \operatorname{im} d_{n+2} \in H_{n+1}(C)$, we find $b \in B_{n+1}$ such that $g_{n+1}(b) = c$ and $a \in Z_n(A)$ such that $f_n(a) = d^B_{n+1}(b)$, and

$$\partial(c) = a + \operatorname{im} d^A_{n+1}.$$

We will soon see that long exact sequences appear everywhere, and that they are very helpful. Before we see more examples, we want to highlight a connection between long and short exact sequences.

Remark 2.30. Suppose that

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots$$

is a long exact sequence. This long exact sequence breaks into the short exact sequences

$$0 \longrightarrow \ker f_n \xrightarrow{i} C_n \xrightarrow{\pi} \operatorname{coker} f_{n+1} \longrightarrow 0\,.$$

The first map i is simply the inclusion of the submodule $\ker f_n$ into C_n, while the second map π is the canonical projection onto the quotient. While it is clear that i is injective and π is surjective, exactness at the middle is less obvious. This follows from the exactness of the original complex, which gives $\operatorname{im} i = \ker f_n = \operatorname{im} f_{n+1} = \ker \pi$.

The long exact sequence in homology is natural.

Theorem 2.31 (Naturality of the long exact sequence in homology)**.** *Any commutative diagram in* $\mathrm{Ch}(R)$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "p'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows induces a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{n+1}(C) \arrow[r, "\partial"] \arrow[d, "h"'] & \mathrm{H}_n(A) \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{H}_n(B) \arrow[r, "p"] \arrow[d, "g"'] & \mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "h"'] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d, "f"'] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_{n+1}(C') \arrow[r, "\partial'"'] & \mathrm{H}_n(A') \arrow[r, "i'"'] & \mathrm{H}_n(B') \arrow[r, "p'"'] & \mathrm{H}_n(C') \arrow[r, "\partial'"'] & \mathrm{H}_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The rows of the resulting diagram are the long exact sequences in homology induced by each row of the original diagram, as in Theorem 2.28. So the content of the theorem is that the maps induced in homology by f, g, and h make the diagram commute. The commutativity of

$$\begin{tikzcd}
\mathrm{H}_n(A) \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{H}_n(B) \arrow[r, "p"] \arrow[d, "g"'] & \mathrm{H}_n(C) \arrow[d, "h"'] \\
\mathrm{H}_n(A') \arrow[r, "i'"'] & \mathrm{H}_n(B') \arrow[r, "p'"'] & \mathrm{H}_n(C')
\end{tikzcd}$$

follows from the fact that H_n is a functor, so we only need to check commutativity of the square

$$\begin{tikzcd}
\mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "h"'] & \mathrm{H}_{n-1}(A) \arrow[d, "f"] \\
\mathrm{H}_n(C') \arrow[r, "\partial'"'] & \mathrm{H}_{n-1}(A')
\end{tikzcd}$$

that involves the connecting homomorphisms ∂ and ∂'. Consider the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[rr] & & A_n \arrow[rr, "i"] \arrow[dl, "d"'] \arrow[dd, dashed] & & B_n \arrow[rr, "p"] \arrow[dl, "d"'] \arrow[dd, dashed] & & C_n \arrow[rr] \arrow[dl, "d"'] \arrow[dd, dashed] & & 0 \\
0 \arrow[rr] & & A_{n-1} \arrow[rr] \arrow[dd] & & B_{n-1} \arrow[rr] \arrow[dd] & & C_{n-1} \arrow[rr] \arrow[dd] & & 0 & \\
& 0 \arrow[rr] & & A'_n \arrow[rr] \arrow[dl, "d"'] & & B'_n \arrow[rr] \arrow[dl, "d"'] & & C'_n \arrow[rr] \arrow[dl, "d"'] & & 0 \\
0 \arrow[rr] & & A'_{n-1} \arrow[rr, "i'"'] & & B'_{n-1} \arrow[rr, "p'"'] & & C'_{n-1} \arrow[rr] & & 0 &
\end{tikzcd}$$

Given $c \in \ker(d_n \colon C_n \longrightarrow C_{n-1})$, we need to check that $f_{n-1}(\partial(c)) = \partial' h_n(c)$ in $\mathrm{H}_{n-1}(A)$. To compute $\partial(c)$, we find a lift $b \in B_n$ such that $p_n(b) = c$, and $a \in A_{n-1}$ with $i_{n-1}(a) = d_n(b)$, and set $\partial(c) = a + \operatorname{im} d_n \in \mathrm{H}_{n-1}(A)$. So $f_{n-1}\partial(c) = f_{n-1}(a) + \operatorname{im} d_n$. On the other hand, to compute $\partial' h_n(c)$, we start by finding $b' \in B'_n$ such that $p'_n(b') = h_n(c)$. By commutativity of the top square

$$\begin{tikzcd}
B_n \arrow[r, "p_n"] \arrow[d, "g_n"'] & C_n \arrow[d, "h_n"] \\
B'_n \arrow[r, "p'_n"'] & C'_n
\end{tikzcd}$$

we can choose $b' = g_n(b)$, since

$$p'_n(b') = p'_n g_n(b) = h_n p_n(b) = h_n(c).$$

Next we take $a' \in A'_{n-1}$ such that $i'_{n-1}(a') = d_n(b')$, and set $\partial'(h(c)) = a' + \operatorname{im} d_n \in \mathrm{H}_{n-1}(A')$.

By commutativity of the middle square

$$\begin{tikzcd} B_n \arrow[r, "d_n"] \arrow[d, "g_n"'] & B_{n-1} \arrow[d, "g_{n-1}"] \\ B'_n \arrow[r, "d_n"'] & B'_{n-1} \end{tikzcd}$$

we have

$$d_n(b') = d_n g_n(b) = g_{n-1} d_n(b).$$

By our choice of a, we have

$$d_n(b') = g_{n-1} d_n(b) = g_{n-1} i_{n-1}(a),$$

and by commutativity of the front left square

$$\begin{tikzcd} A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"'] & B_{n-1} \arrow[d, "g_{n-1}"] \\ A'_{n-1} \arrow[r, "i'_{n-1}"'] & B'_{n-1} \end{tikzcd}$$

we have

$$i'_{n-1} f_{n-1}(a) = g_{n-1} i_{n-1}(a) = d_n(b').$$

So we can take $a' = f_{n-1}(a)$. Finally, this means $\partial'(h_n(c)) = f_{n-1}(a) + \operatorname{im} d_{n-1}$, as we wanted to prove. □

Remark 2.32. Let

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

be a short exact sequence in $\mathrm{Ch}(R)$. We can think of Theorem 2.31 as saying that the induced maps on homology $i_* : \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B)$ and $p_* : \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C)$ and the connecting homomorphism $\partial : \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A)$ are all natural transformations. More precisely, consider the category **SES** of short exact sequences of R-modules, which is a full subcategory of $\mathrm{Ch}(R)$. Homology gives us functors **SES** $\longrightarrow$ R-**Mod** that given a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

return the R-modules $\mathrm{H}_n(A)$, $\mathrm{H}_n(B)$, or $\mathrm{H}_n(C)$). A map between two short exact sequences then induces R-module homomorphisms between the corresponding homologies. With this framework, Theorem 2.31 says that $i_* : \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B)$, and $p_* : \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C)$ and the connecting homomorphism $\partial : \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A)$ are all natural transformations between the corresponding homology functors.

Chapter 3

R-Mod

Before we study abelian categories in general, we want to understand our best prototype for what an abelian category looks like: the category R-**Mod** of R-modules and R-module homomorphisms.

3.1 Hom

From now on, let's fix a ring R. Recall that whenever we say an R-module M, we mean a *left* R-module; any general facts about left modules can be naturally converted into statements about right R-modules, under small appropriate corrections. When M is commutative, left and right module structures agree, so the distinction is not relevant.

Our goal is to get to know the category R-**Mod**, which as we are about to discover is a very nice category. One of the many nice things about R-**Mod** is that the Hom-sets have an extra structure. (Roughly speaking, a locally small category where the Hom-sets are objects in some other category is called an *enriched category*).

To make the notation less heavy, we write $\mathrm{Hom}_R(M, N)$ instead of $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$ for the Hom-set between M and N in R-**Mod**. The arrows in $\mathrm{Hom}_R(M, N)$ are all the R-module homomorphisms from M to N. This is a locally small category, meaning that the Hom-sets are actual sets, but more even is true: the Hom-sets are actually abelian groups, and when R is commutative, they are even R-modules.

Given $f, g \in \mathrm{Hom}_R(M, N)$, $f + g$ is the R-module homomorphism defined by

$$(f + g)(m) := f(m) + g(m).$$

When R is a commutative ring, given $r \in R$ and $f \in \mathrm{Hom}_R(M, N)$, $r \cdot f$ is the R-module homomorphism defined by

$$(r \cdot f)(m) := f(rm).$$

Exercise 36. Let M and N be R-modules. Then $\mathrm{Hom}_R(M, N)$ is an abelian group under the sum defined above.

Exercise 37. Let M and N be R-modules over a commutative ring R. Then $\mathrm{Hom}_R(M, N)$ is an R-module.

Remark 3.1. The main reason we need commutativity for $\operatorname{Hom}_R(M, N)$ to be a module is that given any $r \in R$ and $f \in \operatorname{Hom}_R(M, N)$, we need rf to be an R-module homomorphism, so in particular for any $a \in M$ and any $s \in R$ we need

$$(rf)(sa) = s(rf)(a),$$

so

$$(rs)f(a) = rf(sa) = (rf)(sa) = s(rf)(a) = s(rf(a)) = (sr)f(a).$$

This holds whenever $rs = sr$, but not in general.

Some Hom-sets can easily be identified with other well-understood modules.

Exercise 38. Let R be a commutative ring. Let M be an R-module, and I an ideal in R. Then we have the following isomorphisms of R-modules:

a) $\operatorname{Hom}_R(R, M) \cong M$.

b) $\operatorname{Hom}_R(R^n, M) \cong M^n$ for any $n \geqslant 1$.

c) $\operatorname{Hom}_R(R/I, M) \cong (0 :_M I) := \{m \in M \mid Im = 0\}$.

Since R-**Mod** is a locally small category, we saw in Definition 1.34 that there are two Hom-functors from R-**Mod** to **Set**, the covariant functor $\operatorname{Hom}_R(M, -)\colon R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$ and the contravariant functor $\operatorname{Hom}_R(-, N)\colon R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$. In light of Exercise 37, we can upgrade these functors to land in **Ab**, or in R-**Mod** when R is commutative, not just in **Set**. Note that while there are two Hom-functors, we will sometimes refer to *the* Hom functor when talking about properties that are common to both of them.

A functor that lands in $\mathbb{R} - mod$, or **Ab** in particular, can have some additional good properties.

Definition 3.2. Let R and S be rings. A functor $T\colon R\text{-}\mathbf{Mod} \longrightarrow S\text{-}\mathbf{Mod}$ is an **additive functor** if

$$T(f + g) = T(f) + T(g)$$

for all $f, g \in \operatorname{Hom}_R(M, N)$.

Note that to say that T is a covariant additive functor is to say that for all A and B, the map

$$\begin{aligned} \operatorname{Hom}(A, B) &\longrightarrow \operatorname{Hom}(T(A), T(B)) \\ f &\longmapsto T(f) \end{aligned}$$

induced by T is a homomorphism of abelian groups. Similarly, a contravariant additive functor T is one such that

$$\begin{aligned} \operatorname{Hom}(A, B) &\longrightarrow \operatorname{Hom}(T(B), T(A)) \\ f &\longmapsto T(f) \end{aligned}$$

is a homomorphism of abelian groups. Notice moreover that this definition makes sense more generally in any category $\mathscr{C}$ whose objects have an abelian group structure.

Exercise 39. Show that $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$ are both additive functors.

Note that in the previous exercise we were purposely vague about where $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$ land: these are additive functors whether we consider them as functors with target **Ab** or target R-**Mod**, when appropriate.

Lemma 3.3. *Let $T\colon R$-**Mod** $\longrightarrow S$-**Mod** be an additive functor.*

a) Let 0 denote the 0-map between any two R-modules M and N. Then $T(0) = 0$ is the 0-map $T(M) \to T(N)$.

b) Let 0 denote the zero R-module. Then $T(0) = 0$ is the zero S-module.

Proof.

a) As a function defined on each fixed $\mathrm{Hom}_R(M, N)$, T is a group homomorphism, so it must send 0 to 0.

b) An R-module M is the zero module if and only if the zero and identity maps on M coincide. Let N be the image of the zero R-module via T. On the one hand, any functor must send identity maps to identity maps, so the identity map on the zero module must be sent to the identity on N. On the other hand, we have shown that the zero map must be sent to the zero map on N, so the zero and identity maps on N must coincide, so $N = 0$. □

Remark 3.4. Note that the category of chain complexes also has a similar structure to R-**Mod**: given two maps of complexes $f, g\colon C \to D$, we define a map of complexes $f + g\colon C \to D$ given by

$$(f + g)_n := f_n + g_n.$$

It is routine to check that this again gives a map of complexes, and that this operation gives the Hom-sets in $\mathrm{Ch}(R)$ the structure of an abelian group. In fact, this abelian group structure can be upgraded to an R-module structure when R is commutative, by setting

$$(rf)_n := rf_n$$

for all $r \in R$. This allows us to talk about additive functors to and from the category $\mathrm{Ch}(R)$, and there is a version of Lemma 3.3 in $\mathrm{Ch}(R)$.

Exercise 40. Show that homology is an additive functor.

Most functors between categories or modules or chain complexes are additive. In fact, we will spend the rest of this chapter studying three very important additive functors: the two Hom functors, and a third functor we have yet to define.

Additive functors have many nice properties. For example, they preserve (finite) coproducts.

Exercise 41. Let R and S be rings and let $T\colon R$-**Mod** $\longrightarrow S$-**Mod** be an additive functor. Show that for all R-modules A and B,

$$T(A \oplus B) \cong T(A) \oplus T(B).$$

Hom satisfies a stronger version of this property.

Theorem 3.5. *For all R-modules M, N, M_i, N_i, there are isomorphisms of abelian groups*

$$\operatorname{Hom}_R(M, \prod_i N_i) \cong \prod_i \operatorname{Hom}_R(M, N_i) \text{ and } \operatorname{Hom}_R(\bigoplus_i M_i, N) \cong \prod_i \operatorname{Hom}_R(M_i, N).$$

Moreover, when R is commutative, these are in fact isomorphisms of R-modules.

In particular,

$$\operatorname{Hom}_R(A \oplus B, C) \cong \operatorname{Hom}_R(A, C) \oplus \operatorname{Hom}_R(B, C)$$

and

$$\operatorname{Hom}_R(A, B \oplus C) \cong \operatorname{Hom}_R(A, B) \oplus \operatorname{Hom}_R(A, C).$$

These two properties, however, are consequences of Exercise 39 and Exercise 41: Hom is additive, and additive functors preserve finite direct sums.

Proof. For each i, let $\pi_i : \prod_j N_j \longrightarrow N_i$ be the canonical projection map. Consider the map

$$\begin{aligned} \operatorname{Hom}_R(M, \textstyle\prod_i N_i) &\xrightarrow{\alpha} \textstyle\prod_i \operatorname{Hom}_R(M, N_i) \ . \\ f &\longmapsto (\pi_i f) \end{aligned}$$

We claim this map is the desired isomorphism. We leave it as an exercise to show that α is a homomorphism of abelian groups, and a homomorphism of R-modules when R is commutative; we focus on proving that α is a bijection. First, take $(f_i)_i \in \prod_i \operatorname{Hom}_R(M, N_i)$. Define a map

$$\begin{aligned} M &\xrightarrow{\psi} \textstyle\prod_i N_i \ . \\ m &\longmapsto (f_i(m)) \end{aligned}$$

This makes the diagram

$$\begin{tikzcd} & N_i & \\ \prod_j N_j \arrow[ur, "\pi_i"] & & M \arrow[ul, "f_i"'] \arrow[ll, dashed, "\psi"] \end{tikzcd}$$

commute, so that $\alpha(\psi) = (\pi_i \psi)_i = (f_i)$. This shows that α us surjective.

Now let us show that α is injective. Suppose $f \in \operatorname{Hom}_R(M, \prod_i N_i)$ is such that $\alpha(f) = 0$. For each $m \in M$, let $f(m) = (n_i)_i$, so $\pi_i f(m) = n_i$. By assumption, $(\pi_i f(m)) = 0$, which means that $\pi_i \alpha = 0$ for all i, and thus $n_i = 0$ for all i. So $f = 0$. We conclude that α is an isomorphism.

Now consider the map

$$\begin{aligned} \operatorname{Hom}_R(\bigoplus_i M_i, N) &\xrightarrow{\beta} \prod_i \operatorname{Hom}_R(M_i, N) \\ f &\longmapsto (f \iota_i) \end{aligned}$$

where $\iota_j \colon M_j \longrightarrow \bigoplus_i M_i$ is the inclusion of the jth factor. We leave it as an exercise to prove that β is a homomorphism of abelian groups, and that whenever R is commutative, β is in fact a homomorphism of R-modules.

Given $(f_i)_i \in \prod\limits_i \operatorname{Hom}_R(M_i, N)$, let

$$\begin{array}{rcl} \bigoplus\limits_i M_i & \xrightarrow{\psi} & N \\ (m_i) & \longmapsto & \sum\limits_i f_i(m_i) \end{array}$$

Then $\beta(\psi) = (\psi\iota_i)_i$, so for each i and each $m_i \in M_i$, $\psi\iota_i(m_i) = f_i(m_i)$, and $\beta(\psi) = (f_i)_i$. This shows that β is surjective.

Now assume $\beta(f) = 0$, which implies that $f\iota_i$ is the zero map for each i. Consider any $(m_i)_i \in \bigoplus_i M_i$. For each i, $f\iota_i(m_i) = 0$. On the other hand, $(m_i)_i = \sum_i \iota_i(m_i)$, so $f((m_i)_i) = \sum_i \iota_i(m_i) = 0$. We conclude that $f = 0$, and β is injective. $\square$

Exercise 42. Show that the isomorphisms in Theorem 3.5 are natural on both components. More precisely, given any other family of R-modules L_i such that for each i there exists j, a map σ_{ij} there exist R-module maps making the following diagrams commute:

$$\begin{tikzcd} \operatorname{Hom}_R(M, \prod_i N_i) \arrow[r, "\cong"] \arrow[d] & \prod_i \operatorname{Hom}_R(M, N_i) \arrow[d] \\ \operatorname{Hom}_R(M, \prod_i L_i) \arrow[r, "\cong"'] & \prod_i \operatorname{Hom}_R(M, L_i) \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Hom}_R(\bigoplus_i M_i, N) \arrow[r, "\cong"] \arrow[d] & \bigoplus_i \operatorname{Hom}_R(M_i, N) \arrow[d] \\ \operatorname{Hom}_R(\bigoplus_i L_i, N) \arrow[r, "\cong"'] & \bigoplus_i \operatorname{Hom}_R(L_i, N) \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Hom}_R(M, \prod_i N_i) \arrow[r, "\cong"] \arrow[d] & \prod_i \operatorname{Hom}_R(M, N_i) \arrow[d] \\ \operatorname{Hom}_R(M, \prod_i L_i) \arrow[r, "\cong"'] & \prod_i \operatorname{Hom}_R(M, L_i) \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Hom}_R(\bigoplus_i M_i, N) \arrow[r, "\cong"] \arrow[d] & \bigoplus_i \operatorname{Hom}_R(M_i, N) \arrow[d] \\ \operatorname{Hom}_R(\bigoplus_i L_i, N) \arrow[r, "\cong"'] & \bigoplus_i \operatorname{Hom}_R(L_i, N) \end{tikzcd}$$

In fact, one can show that more generally, Hom behaves well with limits and colimits.

Exercise 43. Let R be any ring and consider R-modules A and $\{M_i\}$.

a) For any inverse system $\{M_i\}$, there is a natural isomorphism

$$\operatorname{Hom}_R(A, \lim_i M_i) \cong \lim_i \operatorname{Hom}_R(A, M_i).$$

b) For any direct system $\{M_i\}$ or R-modules, there is a natural isomorphism

$$\operatorname{Hom}_R(\operatorname{colim}_i M_i, A) \cong \lim_i \operatorname{Hom}_R(M_i, B).$$

Moreover, when R is commutative, these are isomorphisms of modules.

Another important property of Hom is how it interacts with exact sequences.

Definition 3.6. A covariant additive functor $T: R$-**Mod** $\longrightarrow S$-**Mod** is **left exact** if it takes every exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

of R-modules to the exact sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$$

of S-modules, and **right exact** if it takes every exact sequence of R-modules

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

to the exact sequence of S-modules

$$T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0.$$

Finally, T is an **exact functor** if it preserves short exact sequences, meaning every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is taken to the short exact sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0.$$

A contravariant additive functor $T: R$-**Mod** $\longrightarrow S$-**Mod** is **left exact** if it takes every exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules to the exact sequence

$$0 \longrightarrow T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A)$$

of S-modules, and **right exact** if it takes every exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

to the exact sequence of S-modules

$$T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A) \longrightarrow 0.$$

Finally, T is an **exact functor** if it preserves short exact sequences, meaning every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is taken to the short exact sequence

$$0 \longrightarrow T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A) \longrightarrow 0.$$

Exercise 44. The definitions above all stay unchanged if for each condition we start with a short exact sequence. For example, a covariant additive functor T is left exact if and only if for every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules,

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$$

is exact.

Remark 3.7. Left exact covariant functors take kernels to kernels, while right exact covariant functors take cokernels to cokernels: the kernel of f fits in an exact sequence

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B$$

and applying a left exact functor F gives us an exact sequence

$$0 \longrightarrow F(\ker f) \longrightarrow F(A) \xrightarrow{F(f)} F(B).$$

Exactness tells us that $F(\ker f)$ is the kernel of $F(f)$. Similarly, the cokernel of f fits into an exact sequence

$$A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0,$$

which any right exact functor G will take to an exact sequence

$$G(A) \xrightarrow{G(f)} G(B) \longrightarrow G(\operatorname{coker} f) \longrightarrow 0.$$

Exactness says that $G(\operatorname{coker} f)$ is the cokernel of $G(f)$.

Similarly, left exact contravariant functors take cokernels to kernels, and right exact contravariant functors take kernels to cokernels. A left exact contravariant functor F will take the exact sequence

$$A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

to an exact sequence

$$0 \longrightarrow F(\operatorname{coker} f) \longrightarrow F(B) \xrightarrow{F(f)} F(A),$$

and exactness tells us that $F(\operatorname{coker} f)$ is the kernel of $F(f)$.

A right exact contravariant functor G will take the exact sequence

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B$$

to the exact sequence

$$G(B) \xrightarrow{G(f)} G(A) \longrightarrow G(\ker f) \longrightarrow 0,$$

and exactness says that $G(\ker f)$ is the cokernel of $G(f)$.

Exactness is preserved by natural isomorphisms

Remark 3.8. Suppose that $F, G\colon R\text{-}\mathbf{Mod} \longrightarrow S\text{-}\mathbf{Mod}$ are naturally isomorphic additive functors. We claim that F is exact if and only if G is exact. Let's prove it in the case when F and G are covariant. Given any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

applying each of our functors yields complexes of R-modules which may or may not be exact. Our natural isomorphism gives us an isomorphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] \arrow[d] & F(B) \arrow[r] \arrow[d] & F(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G(A) \arrow[r] & G(B) \arrow[r] & G(C) \arrow[r] & 0.
\end{tikzcd}$$

Isomorphisms of complexes induce isomorphisms in homology, so the top sequence is exact if and only if the bottom sequence is exact. Thus F preserves the short exact sequence if and only if G does.

A similar argument shows that F is left (respectively, right) exact if and only if G is left (respectively, right) exact; we leave the details as an exercise.

Notice that every additive functor sends complexes to complexes; but not every additive functor preserves exactness.

Lemma 3.9. *Let $T\colon R\text{-}\boldsymbol{Mod} \longrightarrow S\text{-}\boldsymbol{Mod}$ be an additive functor and let C be a complex of R-modules. Then $T(C)$ is a complex.*

Proof. Given two maps of R-modules f and g such that $gf = 0$, $T(g)T(f) = T(gf) = T(0)$, but by Lemma 3.3 $T(0) = 0$, so $T(g)T(f) = 0$. Therefore, T sends complexes to complexes. ☐

However, an additive functor does not have to be left exact nor right exact.

Example 3.10. We claim that the homology functor is neither left exact nor right exact. On the one hand, note that the homology functor is exact *in the middle*: given a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0,$$

the exactness of the long exact sequence in homology says in particular that

$$\mathrm{H}_n(A) \xrightarrow{\mathrm{H}_n(f)} \mathrm{H}_n(B) \xrightarrow{\mathrm{H}_n(g)} \mathrm{H}_n(C)$$

is exact for all n. On the other hand, $\mathrm{H}_n(f)$ might fail to be injective and $\mathrm{H}_n(g)$ might fail to be surjective. To find a counterexample amounts to finding a short exact sequence of complexes such that the connecting homomorphism in the long exact sequence in homology is not the zero map.

For example, consider the following complexes and maps of complexes:

$$\begin{tikzcd}[row sep=small]
 & & 2 & 1 & 0 & -1 & \\
A = \arrow[d, "f"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
B = \arrow[d, "g"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
C = & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Applying H_0 gives us

$$\begin{array}{c} \mathrm{H}_0(A) \xrightarrow{\mathrm{H}_0(f)} \mathrm{H}_0(B) \\ \mathbb{Z} \xrightarrow{\quad 0 \quad} 0, \end{array}$$

which is not injective, so

$$0 \longrightarrow \mathrm{H}_0(A) \xrightarrow{\mathrm{H}_0(f)} \mathrm{H}_0(B) \xrightarrow{\mathrm{H}_0(g)} \mathrm{H}_0(C)$$

is not exact. Similarly, applying H_1 gives

$$\begin{array}{c} \mathrm{H}_1(B) \xrightarrow{\mathrm{H}_1(g)} \mathrm{H}_1(C) \\ 0 \xrightarrow{\quad 0 \quad} \mathbb{Z}, \end{array}$$

which is not surjective, so

$$\mathrm{H}_1(A) \xrightarrow{\mathrm{H}_1(f)} \mathrm{H}_1(B) \xrightarrow{\mathrm{H}_1(g)} \mathrm{H}_1(C) \longrightarrow 0$$

is not exact.

In fact, an additive functor might fail to preserve exactness even *in the middle*.

Example 3.11. Fix a prime p and consider the functor $F\colon \mathbf{Ab} \to \mathbf{Ab}$ which on objects is defined by

$$F(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, M/p^2M);$$

given a homomorphism of abelian groups $M \xrightarrow{f} N$, we get an induced homomorphism of abelian groups

$$\begin{aligned} M/p^2M &\xrightarrow{\overline{f}} N/p^2N \\ m + p^2M &\longmapsto f(m) + p^2N, \end{aligned}$$

and $F(f) = \overline{f} \circ -$ is postcomposition with $\overline{f}$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^2 \xrightarrow{f} \mathbb{Z}/p^3 \xrightarrow{g} \mathbb{Z}/p \longrightarrow 0,$$

where f is the multiplication by p map, which sends $1 \mapsto p$, and g is the canonical quotient map by the subgroup generated by p.

Note that $F(\mathbb{Z}/p^2) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p^2)$ is the submodule of $\mathbb{Z}/p^2$ of elements killed by p, which is generated by the class of p, so $F(\mathbb{Z}/p^2) = \mathbb{Z}/p$. Moreover,

$$\frac{\mathbb{Z}/p^3}{p^2\mathbb{Z}/p^3} \cong \mathbb{Z}/p^2,$$

so $F(\mathbb{Z}/p^3)$ is the the submodule of $\mathbb{Z}/p^2$ of elements killed by p, which is generated by p and isomorphic to $\mathbb{Z}/p$, so $F(\mathbb{Z}/p^3) = \mathbb{Z}/p$. Now $F(f) \colon \mathbb{Z}/p \to \mathbb{Z}/p$ is the map induced by multiplication by p, so it is the zero map. The map $\overline{g} \colon \mathbb{Z}/p^2 \to \mathbb{Z}/p$ is the canonical quotient by the subgroup generated by p; any element in $F(Z/p^3) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p^2)$ corresponds to choosing an element of order p, and thus in the subgroup generated by p, so applying $\overline{g}$ always results in 0. We conclude that $F(g) = 0$. Finally, this shows that applying F to the original short exact sequence gives us

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \longrightarrow 0,$$

which is not exact anywhere.

Hom is left exact.

Theorem 3.12. *Let M be an R-module.*

a) The covariant functor $\mathrm{Hom}_R(M, -)$ is left exact: for every exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

of R-modules, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{\mathrm{Hom}_R(M,f)} \mathrm{Hom}_R(M, B) \xrightarrow{\mathrm{Hom}_R(M,g)} \mathrm{Hom}_R(M, C)$$

is exact.

b) The contravariant functor $\mathrm{Hom}_R(-, M)$ is left exact: for every exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{\mathrm{Hom}_R(g,M)} \mathrm{Hom}_R(B, M) \xrightarrow{\mathrm{Hom}_R(f,M)} \mathrm{Hom}_R(A, M)$$

is exact.

Proof. To make the notation less heavy, we will write $f_* := \mathrm{Hom}_R(M, f)$, $g_* := \mathrm{Hom}_R(M, g)$, $f^* := \mathrm{Hom}_R(f, M)$, and $g^* := \mathrm{Hom}_R(g, M)$.

a) We have three things to show:

<u>f_* is injective</u>: Suppose that $h \in \mathrm{Hom}_R(M, A)$ is such that $f_*(h) = 0$. By definition, this means that $fh = 0$. But f is injective, so for any $m \in M$

$$fh(m) = 0 \implies h(m) = 0.$$

We conclude that $h = 0$, and f_* is injective.

<u>$\mathrm{im}\, f_* \subseteq \ker g_*$</u>: For any $h \in \mathrm{Hom}_R(M, A)$, we have

$$g_* f_*(h) = \underbrace{gf}_{0} h = 0.$$

<u>$\ker g_* \subseteq \mathrm{im}\, f_*$</u>: Let $h \in \mathrm{Hom}_R(M, B)$ be in the kernel of g_*. Then $gh = g_*(h) = 0$, so for each $m \in M$, $gh(m) = 0$. Then $h(m) \in \ker g = \mathrm{im}\, f$, so there exists $a \in A$ such that $f(a) = h(m)$. Since f is injective, this element a is unique for each $m \in M$. So setting $k(m) := a$ gives us a well-defined function $k \colon M \longrightarrow A$. We claim that k is in fact an R-module homomorphism. To see that, notice that if $k(m_1) = a_1$ and $k(m_2) = a_2$, then

$$f(a_1 + a_2) = f(a_1) + f(a_2) = h(m_1) + h(m_2) = h(m_1 + m_2),$$

so that $k(m_1 + m_2) = a_1 + a_2 = k(m_1) + k(m_2)$. Similarly, given any $r \in R$,

$$f(ra_1) = rf(a_1) = rh(m_1) = h(rm_1),$$

so $k(rm_1) = ra_1 = rk(m_1)$. Finally, this element $k \in \mathrm{Hom}_R(M, A)$ satisfies

$$f_*(k)(m) = f(k(m)) = h(m)$$

for all $m \in M$, so $f_*(k) = h$ and $h \in \mathrm{im}\, f_*$.

b) Again, we have three things to show:

<u>g^* is injective</u>: If $g^*(h) = 0$ for some $h \in \mathrm{Hom}_R(C, M)$, then $hg = g^*(h) = 0$. Consider any $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Then $h(c) = hg(b) = 0$, so $h = 0$.

<u>$\mathrm{im}\, g^* \subseteq \ker f^*$</u>: Let $h \in \mathrm{Hom}_R(B, M)$ be in $\mathrm{im}\, g^*$, so that there exists $k \in \mathrm{Hom}_R(C, M)$ such that $kg = g^*(k) = h$. Then

$$f^*(h) = hf = k \underbrace{gf}_{0} = 0, \qquad \text{so } h \in \ker f^*.$$

<u>$\ker f^* \subseteq \mathrm{im}\, g^*$</u>: Let $h \in \mathrm{Hom}_R(B, M)$ be in $\ker f^*$, so that $hf = 0$. Given any $c \in C$, there exists $b \in B$ such that $g(b) = c$, since g is surjective. Let $k : C \longrightarrow M$ be the function defined by $k(c) := h(b)$ for some b with $g(b) = c$. This function is well-defined, since whenever $g(b') = g(b) = c$, $b - b' \in \ker g = \mathrm{im}\, f$, say $b - b' = f(a)$, and thus $h(b - b') = h(f(a)) = 0$. Moreover, we claim that k is indeed a homomorphism of R-modules. If $c_1, c_2 \in C$, and $g(b_1) = c_1$, $g(b_2) = c_2$, then $g(b_1 + b_2) = c_1 + c_2$, so

$$k(c_1 + c_2) = h(b_1 + b_2) = h(b_1) + h(b_2) = k(b_1) + k(b_2).$$

Finally, this element $k \in \mathrm{Hom}_R(C, M)$ is such that $g^*(k)$ satisfies

$$(g_*(k))(b) = k(g(b)) = h(b)$$

for all $b \in B$, so $g^*(k) = h$, and $h \in \operatorname{im} g^*$. $\square$

So $\mathrm{Hom}_R(M, -)$ preserves kernels, and $\mathrm{Hom}_R(-, N)$ sends cokernels to kernels. However, Hom is *not* right exact in general.

Example 3.13. Consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where the first map is the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$, and the second map is the canonical projection. The elements in the abelian group $\mathbb{Q}/\mathbb{Z}$ are cosets of the form $\frac{p}{q} + \mathbb{Z}$, where $\frac{p}{q} \in \mathbb{Q}$, and whenever $\frac{p}{q} \in \mathbb{Z}$, $\frac{p}{q} + \mathbb{Z} = 0$. While Theorem 3.12 says that

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$$

is exact, we claim that this cannot be extended to a short exact sequence, since the map $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$ is not surjective.

On the one hand, there are no nontrivial homomorphisms from $\mathbb{Z}/2$ to either $\mathbb{Z}$ nor $\mathbb{Q}$, since there are no elements in $\mathbb{Z}$ nor $\mathbb{Q}$ of order 2. This shows that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}) \cong 0.$$

On the other hand, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$ is nonzero: to give a homomorphism of abelian groups $\mathbb{Z}/2 \to \mathbb{Q}/ZZ$ is to choose an element in $\mathbb{Q}/\mathbb{Z}$ of order 2. Since $\frac{1}{2} + \mathbb{Z}$ is an element of order 2 in $\mathbb{Q}/\mathbb{Z}$, the map sending 1 in $\mathbb{Z}/2$ to $\frac{1}{2} + \mathbb{Z}$ in $\mathbb{Z}/\mathbb{Q}$ is nonzero. So after applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$, we get the exact sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z}).$$

So this shows that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ is not an exact functor, only left exact.

Similarly, we can show that $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is not exact:

Example 3.14. Let's apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

This time, Theorem 3.12 says that

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}).$$

is exact. We claim that the last map is not surjective.

First, we claim that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Indeed, if $f : \mathbb{Q} \longrightarrow \mathbb{Z}$ is a homomorphism of abelian groups, then for all $n \geqslant 1$ we have

$$f(1) = nf(\frac{1}{n}).$$

So $f(1)$ is an integer that is divisible by every integer, which is impossible unless $f(1) = 0$. We conclude that $f = 0$, and thus $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong 0$. So our exact sequence above is actually

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \longrightarrow 0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}).$$

By Exercise 38, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \neq 0$, so the last map in our sequence can't possibly be surjective, so our sequence is not a short exact sequence.

The other fun consequence is that since $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ and we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0,$$

we can now conclude that

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = 0.$$

The last observation is a common trick: once we know we have an exact sequence involving certain modules we do not know, we can sometimes calculate them exactly by studying the other modules and maps in the exact sequence.

We can use the left exactness of Hom to compute some modules of interest:

Example 3.15. Let R be a commutative ring and M be a finitely presented R-module. This means that M has a presentation with finitely many generators and relations, which translates into an exact sequence of the form

$$R^m \xrightarrow{f} R^n \longrightarrow M \longrightarrow 0.$$

Since R^m and R^n are free modules, we can think of the map f as multiplication by a matrix A with n rows and m columns, after we fix a basis for R^n and R^m. Applying $\operatorname{Hom}_R(-, R)$ to the exact sequence above, we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, R) \longrightarrow \operatorname{Hom}_R(R^n, R) \xrightarrow{f^*} \operatorname{Hom}(R^m, R).$$

By Exercise 38, $\operatorname{Hom}_R(R^n, R) \cong R^n$ and $\operatorname{Hom}_R(R^m, R) \cong R^m$. Moreover, we claim that f^* is multiplication by the transpose of A.

First, note that given a basis $\{e_1, \ldots, e_n\}$ for R^n, we get a dual basis $\{e_1^*, \ldots, e_n^*\}$ for $\operatorname{Hom}_R(R^n, R)$, where

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have a dual basis $\{e_1^*, \ldots, e_m^*\}$ for $\operatorname{Hom}_R(R^m, R) \cong R^m$; we might as well assume that we picked the canonical basis in both cases, so that we can use similar notation on both.

Now the map f^* is also given by multiplication by a matrix, now having m rows and n columns. To calculate its jth column, we need to calculate $f^*(e_j^*)$, which is given by precomposition with f, so $f^*(e_j^*) = e_j^* A$; this reads off the jth row of A. Thus f^* is indeed multiplication by A^T, and we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, R) \longrightarrow R^n \xrightarrow{A^T} R^m.$$

In particular, we have shown that $\operatorname{Hom}_R(M, R)$ is the kernel of multiplication by A^T.

3.2 Tensor products

Definition 3.16. Fix a ring R, and consider:

- a right R-module M,
- a left R-module N,
- an abelian group L.

A function $f\colon M \times N \longrightarrow L$ is R-**biadditive** if for all $m, m' \in M$, all $n, n' \in N$, and all $r \in R$ we have

- $f(m + m', n) = f(m, n) + f(m', n)$
- $f(m, n + n') = f(m, n) + f(m, n')$
- $f(mr, n) = f(m, rn)$.

When R is a commutative ring, suppose that L is also an R-module. We say that a function $f : M \times N \longrightarrow L$ is R-**bilinear** if for all $m, m' \in M$, all $n, n' \in N$, and all $r \in R$ we have

- $f(m + m', n) = f(m, n) + f(m', n)$
- $f(m, n + n') = f(m, n) + f(m, n')$
- $f(rm, n) = f(m, rn) = rf(m, n)$.

Note that an R-bilinear function is an R-biadditive function that satisfies

$$f(rm, n) = f(m, rn) = rf(m, n).$$

Example 3.17. The product on R is an R-biadditive function $R \times R \longrightarrow R$. The first two rules follow from distributivity of multiplication over the sum; the final rule is a consequence of the associativity of multiplication.

When R is commutative, this is an R-bilinear function.

Definition 3.18. Let M be a right R-module and let N be a left R-module. The **tensor product** of M and N is an abelian group $M \otimes_R N$ together with an R-biadditive function $\tau : M \times N \longrightarrow M \otimes_R N$ with the following universal property: for every abelian group A and every R-bidditive map $f : M \times N \longrightarrow A$, there exists a unique group homomorphism $\tilde{f}\colon M \otimes_R N \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd}
M \otimes_R N \arrow[dr, dashed, "\tilde{f}"] & \\
M \times N \arrow[u, "\tau"] \arrow[r, "f"'] & A
\end{tikzcd}$$

We will now show that tensor products exist and are unique up to isomorphism; in particular, we can talk about *the* tensor product of M and N.

Lemma 3.19. *Let R be any ring, M be a right R-module, and N a left R-module. The tensor product of M and N is unique up to unique isomorphism. More precisely, if $M \times N \xrightarrow{\tau_1} T_1$ and $M \times N \xrightarrow{\tau_2} T_2$ are two tensor products, then there exists a unique isomorphism $T_1 \xrightarrow{i} T_2$ such that*

$$\begin{tikzcd}
& T_1 \arrow[dd, dashed, "i"] \\
M \times N \arrow[ur, "\tau_1"] \arrow[dr, "\tau_2"'] & \\
& T_2
\end{tikzcd}$$

Proof. First, note that the universal property of the tensor product implies that there exists a unique φ such that

$$\begin{tikzcd}
T_i \arrow[dr, dashed, "\varphi"] & \\
M \times N \arrow[u, "\tau_i"] \arrow[r, "\tau_i"'] & T_i
\end{tikzcd}$$

commutes. Since the identity map $T_i \longrightarrow T_i$ *is* such a map, it must be the *only* such map.

Similarly, there are unique maps $\varphi_1 \colon T_1 \longrightarrow T_2$ and $\varphi_2 \colon T_2 \longrightarrow T_1$ such that

$$\begin{tikzcd}
T_1 \arrow[dr, dashed, "\varphi_1"] & \\
M \times N \arrow[u, "\tau_1"] \arrow[r, "\tau_2"'] & T_2
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2 \arrow[dr, dashed, "\varphi_2"] & \\
M \times N \arrow[u, "\tau_2"] \arrow[r, "\tau_1"'] & T_1
\end{tikzcd}$$

both commute. Stacking these up, we get commutative diagrams

$$\begin{tikzcd}
T_1 \arrow[dr, dashed, "\varphi_1"] & & \\
& T_2 \arrow[dr, dashed, "\varphi_2"] & \\
M \times N \arrow[uu, "\tau_1"] \arrow[ur, "\tau_2"] \arrow[rr, "\tau_1"'] & & T_1
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2 \arrow[dr, dashed, "\varphi_2"] & & \\
& T_1 \arrow[dr, dashed, "\varphi_1"] & \\
M \times N \arrow[uu, "\tau_2"] \arrow[ur, "\tau_1"] \arrow[rr, "\tau_2"'] & & T_2.
\end{tikzcd}$$

Note that the identity maps on T_1 and T_2 *are* homomorphisms $T_1 \to T_1$ and $T_2 \to T_2$ that would make each of these triangles commute:

$$\begin{tikzcd}
T_1 \arrow[dr, dashed, "\mathrm{id}_1"] & \\
M \times N \arrow[u, "\tau_1"] \arrow[r, "\tau_2"'] & T_2
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2 \arrow[dr, dashed, "\mathrm{id}_2"] & \\
M \times N \arrow[u, "\tau_2"] \arrow[r, "\tau_1"'] & T_2.
\end{tikzcd}$$

By uniqueness, $\varphi_2\varphi_1$ must be the identity on T_1 and $\varphi_1\varphi_2$ must be the identity on T_2. In particular, T_1 and T_2 are isomorphic, and the isomorphisms φ_1 and φ_2 are unique. $\square$

Theorem 3.20. *Given any right R-modules M and any left R-module N, their tensor product $M \otimes_R N$ exists, and it is given by the abelian group $M \otimes_R N$ defined as follows:*

- *Generators:* *For each pair of elements $m \in M$ and $n \in N$, we have a generator $m \otimes n$.*
- *Relations:* *the generators of $m \otimes n$ satisfy the following relations, where $m, m' \in M$, $n, n' \in N$, and $r \in R$:*

$$\begin{aligned} m \otimes (n + n') &= m \otimes n + m \otimes n' \\ (m + m') \otimes n &= m \otimes n + m \otimes n' \\ (mr) \otimes n &= m \otimes (rn). \end{aligned}$$

Proof. Let F be the free abelian group on the set $M \times N$. In what follows, we identify a pair $(m, n) \in M \times N$ with the corresponding basis element for F. Let S be the subgroup of F generated by

$$S = \left(\left\{ \begin{array}{c|c} (m, n + n') - (m, n) - (m, n') & m, m' \in M \\ (m + m', n) - (m, n) - (m', n) & n, n' \in N \\ (mr, n) - (m, rn) & r \in R \end{array} \right\} \right).$$

Let $M \otimes_R N := F/S$, and let $m \otimes n$ denote the class of (m, n) in the quotient. We claim that this abelian group $M \otimes_R N$ is a tensor product for M and N, together with the map

$$\begin{aligned} M \times N &\xrightarrow{\tau} M \otimes N \\ (m, n) &\longmapsto m \otimes n \end{aligned}$$

Notice τ is the restriction of the quotient map $F \longrightarrow F/S$ to the basis elements of F. Moreover, by construction of $M \otimes_R N$, the following identities hold:

$$\begin{aligned} m \otimes (n + n') &= m \otimes n + m \otimes n' \\ (m + m') \otimes n &= m \otimes n + m \otimes n' \\ (mr) \otimes n &= m \otimes (rn) \end{aligned}$$

Together, these make τ an R-biadditive map. The map $M \times N \longrightarrow F$ that sends each pair (m, n) to the corresponding basis element is R-bilinear by construction. Moreover, there is a natural quotient map $F \longrightarrow M \otimes_R N$, and these maps make the diagram

$$\begin{tikzcd} M \times N \arrow[rr, "\tau"] \arrow[dr, "i"'] & & M \otimes_R N \\ & F \arrow[ur, two heads] & \end{tikzcd}$$

commute.

Now suppose that A is any other abelian group, and let $M \times N \xrightarrow{f} A$ by any R-biadditive map. Since F is the free R-module on $M \times N$, f induces a homomorphism of abelian groups $\varphi \colon F \longrightarrow A$ such that $fi = \varphi$, meaning $f(m, n) = \varphi(m, n)$ for all $m \in M$ and all $n \in N$.

Finally, the fact that f is bilinear implies that $S \subseteq \ker \varphi$. Therefore, φ induces a group homomorphism on $F/S = M \otimes_R N$. All this fits in the following commutative diagram:

$$\begin{tikzcd}
M \times N \arrow[rr, "\tau"] \arrow[dr, "i"'] \arrow[ddr, "f"'] & & M \otimes_R N \arrow[ddl, dashed, "\tilde{f}"] \\
& F \arrow[ur, two heads] \arrow[d, dashed, "\varphi"'] & \\
& A &
\end{tikzcd}$$

Finally, this map $\tilde{f}$ we constructed satisfies $\tilde{f}(n \otimes n) = f(m, n)$, and since $M \otimes_R N$ is generated by such elements, $\tilde{f}$ is completely determined by the images of $m \otimes n$, and thus unique. $\square$

The construction in Theorem 3.20 gives us generators $m \otimes n$ for $M \otimes_R N$. These are usually called **simple tensors**. So any element in $M \otimes_R N$ is of the form

$$\sum_{i=1}^{k} m_i \otimes n_i.$$

Such expressions are *not* unique. For a cheap example, consider the relations we used to construct $M \otimes_R N$ from the abelian group on $M \times N$, which gives us nontrivial ways to write the 0 element in $M \otimes_R N$:

$$\begin{aligned}
0 &= m \otimes (n + n') - m \otimes n - m \otimes n' \\
0 &= (m + m') \otimes n - m \otimes n - m \otimes n' \\
0 &= (mr) \otimes n - m \otimes (rn).
\end{aligned}$$

This makes things unexpectedly tricky. For example, a particular tensor product might unexpectedly be the zero module. Also, whenever we try to define some R-module homomorphism from $M \otimes_R N$ into some other R-module, we must carefully check that our map is well-defined, which is in principle not an easy task. Therefore, the easiest way to define some R-module homomorphism from $M \otimes_R N$ is to give some R-bilinear map from $M \times N$ into our desired R-module.

In summary: the tensor product $M \otimes_R N$ of M and N is generated by the simple tensors $m \otimes n$, but it's important to remember (though we're all bound to forget once or twice) that *not* all elements in $M \otimes_R N$ are simple tensors. Moreover, even if M and N are nonzero, $M \otimes_R N$ could very well be zero.

Remark 3.21. Two group homomorphisms $M \otimes_R N \longrightarrow L$ coincide if and only if they agree on simple tensors, since these are generators for $M \otimes_R N$.

Remark 3.22. In any tensor product $M \otimes_R N$, the simple tensor $0 \otimes 0$ is the zero element, and

$$m \otimes 0 = 0 = 0 \otimes n$$

for all $m \in M$ and $n \in N$.

Let's see some examples of how tensor products can be zero.

Example 3.23. We claim that $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, despite the fact that both of these $\mathbb{Z}$-modules are nonzero. To see that, simply note that given any $a \in \mathbb{Z}/2$ and any $p \in \mathbb{Q}$,

$$a \otimes p = a \otimes \frac{2p}{2} = (2a) \otimes \frac{p}{2} = 0 \otimes \frac{p}{2} = 0.$$

Since $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by simple tensors, which are all 0, we conclude that $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Example 3.24. Consider the abelian group $\mathbb{Q}/\mathbb{Z}$. Again, this is very much nonzero, and yet we claim that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$. For any simple tensor,

$$\begin{aligned} \left(\frac{p}{q} + \mathbb{Z}\right) \otimes \left(\frac{a}{b} + \mathbb{Z}\right) = \left(\frac{bp}{bq} + \mathbb{Z}\right) \otimes \left(\frac{a}{b} + \mathbb{Z}\right) &= \left(\frac{p}{bq} + \mathbb{Z}\right) \otimes b\left(\frac{a}{b} + \mathbb{Z}\right) \\ &= \left(\frac{p}{bq} + \mathbb{Z}\right) \otimes 0 = 0 \otimes 0 = 0. \end{aligned}$$

Example 3.25. Let p and q be distinct prime integers. Then p has inverse modulo q, say $ap \equiv 1 \mod q$, and q has an inverse modulo p, say $bq \equiv 1 \mod p$. Given any simple tensor $n \otimes m$ in $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q$,

$$n \otimes m = ((bq)n) \otimes ((ap)m) = (pbn) \otimes (qam) = 0 \otimes 0.$$

Since all simple tensors are 0 and $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q$ is generated by simple tensors, we conclude that $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q = 0$.

More generally, the following holds:

Exercise 45. Show that if $d = \gcd(m, n)$, then $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/d$.

Of course not all tensor products are 0. But showing an element in a tensor product is nonzero is somehow harder than showing an element is zero. A good method for showing that a particular element m in a module M is nonzero is to give a homomorphism from M sending m to some nonzero element. We apply this technique to tensor products: to show that a particular element in $M \otimes_R N$ is nonzero, one shows there is a homomorphism from $M \otimes_R N$ that takes that particular element no some nonzero element in L. This is typically easier for simple tensors, since we can give an R-biadditive or R-bilinear map out of $M \times N$ that sends the corresponding pair to a nonzero element.

Example 3.26. Consider the abelian group $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2$. The map

$$\begin{aligned} 2\mathbb{Z} \times \mathbb{Z}/2 &\longrightarrow \mathbb{Z}/2 \\ (a, b) &\longmapsto \tfrac{ab}{2} \end{aligned}$$

is $\mathbb{Z}$-bilinear, and thus it induces a homomorphism $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$. Via this map, $2 \otimes 1 \mapsto 1 \neq 0$, so $2 \otimes 1$ is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2$, and $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 \neq 0$.

Moreover, not all elements in a tensor product are simple tensors.

Exercise 46. Let $R = \mathbb{Z}[x]$ and consider the ideal $I = (2, x)$. Show that in $I \otimes_R I$, the element $2 \otimes 2 + x \otimes x$ is not a simple tensor.

We can sometimes give $M \otimes_R N$ the structure of an R-module.

Remark 3.27. Let R be a commutative ring, and let M and N be R-modules. We can give $M \otimes_R N$ the structure of an R-module, as follows: given $r \in R$ and a simple tensor $m \otimes n$,

$$r(m \otimes n) = (rm) \otimes n = m \otimes (rm).$$

We can then extend this linearly to all other elements of $M \otimes_R N$. We leave it as an exercise to check that this does indeed make the abelian group $M \otimes_R N$ into an R-module.

Alternatively, over a commutative ring we can define the tensor product as follows:

Definition 3.28. Let M and N be R-modules. The **tensor product** of M and N is an R-module $M \otimes_R N$ together with an R-bilinear map $\tau \colon M \times N \longrightarrow M \otimes_R N$ with the following universal property: for every R-module A and every R-bilinear map $f \colon M \times N \longrightarrow A$ there exists a unique R-module homomorphism $\tilde{f} \colon M \otimes_R N \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} M \otimes_R N \arrow[dr, dashed, "\tilde{f}"] & \\ M \times N \arrow[u, "\tau"] \arrow[r, "f"'] & A \end{tikzcd}$$

One can now check that if we take the abelian group $M \otimes_R N$ we defined above, which is the unique abelian group which satisfies the universal property of the tensor product (as defined for a general ring R), and endow it with the R-module structure defined in Remark 3.27, the resulting R-module satisfies the universal property in Definition 3.28, and the argument we gave in Lemma 3.19 can be repurposed to show that this is the unique R-module satisfying this universal property.

Remark 3.29. We can express the universal property of the tensor product in the framework of Definition 1.87. For simplicity, assume that R is a commutative ring. Consider the functor $\mathrm{Bilin}(M \times N, -) \colon R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$ that sends an R-module A to the set of R-bilinear maps $M \times N \longrightarrow A$, and a map of R-modules $f\ A \longrightarrow B$ to the function of sets induced by post-composition of functions. The universal property of the tensor product is encoded in the representable functor $\mathrm{Bilin}(M \times N, -) : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$ together with the bilinear map $\tau \in \mathrm{Bilin}(M \times N, M \otimes_R N)$. Indeed, this says that τ induces a natural isomorphism between $\mathrm{Hom}_R(M \otimes_R N, -)$ and $\mathrm{Bilin}(M \times N, -)$ by sending each R-module A to the bijection

$$\begin{aligned} \mathrm{Hom}_R(M \otimes_R N, A) &\longrightarrow \mathrm{Bilin}(M \times N, A) \\ f &\longmapsto \mathrm{Bilin}(M \times N, f)\tau = f_*(\tau) = f\tau. \end{aligned}$$

The fact that this is a bijection says that for every R-bilinear map g there exists a unique R-module homomorphism f such that

$$\begin{tikzcd} M \otimes_R N \arrow[dr, dashed, "f"] & \\ M \times N \arrow[u, "\tau"] \arrow[r, "g"'] & A \end{tikzcd}$$

commutes. So this is indeed the universal property we described before.

More generally, $M \otimes_R N$ has a module structure when one of M or N is a bimodule.

Definition 3.30. Fix rings R and S. An (R, S)**-bimodule** is an abelian group M together with a left R-module structure and a right S-module structure such that for all $r \in R$, $s \in S$, and $m \in M$,

$$(rm)s = r(ms).$$

One sometimes writes ${}_R M_S$ to indicate M is an (R, S)-bimodule. An R**-bimodule** is an (R, R)-bimodule.

Example 3.31.

a) Let $\mathrm{M}_{m,n}(R)$ denote the ring of $m \times n$ matrices with entries in a ring R. We can also view $\mathrm{M}_{m,n}(R)$ as an $(\mathrm{M}_{m,m}, \mathrm{M}_{n,n})$-bimodule via left and right multiplication of matrices.

b) Any two-sided ideal I of a ring R is an R-bimodule.

c) Let R be a commutative ring and let M be any left R-module. Then M is also a right R-module under the same module structure, by setting

$$m \cdot r := rm.$$

Moreover, M is also an R-bimodule using both of these structures at once.

d) Let $f\colon R \to S$ be a ring homomorphism. We can view S as an (R, S)-bimodule via

$$t \cdot s \cdot r := tsf(r)$$

for $t, s \in S$ and $r \in R$, where the right hand side is just multiplication in s. Similarly, S can be viewed as an (S, R)-bimodule and as an (R, R)-bimodule.

e) Let R be a commutative ring of prime characteristic $p > 0$, meaning that R contains a copy of $\mathbb{F}_p$, or equivalently, that

$$\underbrace{1 + \cdots + 1}_{p \text{ times}} = 0.$$

Then R is an R-bimodule with the left module structure given by the Frobenius map

$$\begin{aligned} R &\xrightarrow{F} R \\ r &\longmapsto r^p \end{aligned}$$

and right module structure given by the usual multiplication on R. More precisely, given $r, s, t \in R$,

$$r \cdot s \cdot t := r^p st$$

where the right hand side is just multiplication in R.

Exercise 47. Let M be an (S, R)-bimodule and N an R-module. Consider $M \times N$ as a left S-module via

$$s(m, n) = (sm, n).$$

Then $M \otimes_R N$ is an R-module via

$$s\left(\sum_i m_i \otimes n_i\right) = (sm_i) \otimes n_i.$$

The map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_R N \\ (m, n) &\longrightarrow m \otimes n \end{aligned}$$

is left S-linear. Finally, for any left S-module A and any left S-linear R-biadditive map $b\colon M \times N \to A$, there is a unique left S-module homomorphism $\alpha\colon M \otimes_R N \to A$ such that $\alpha(m \otimes n) = b(m, n)$.

Similarly, for an R-module M and an (R, S)-bimodule N, $M \times N$ is a right S-module via

$$(m, n)s = (m, ns).$$

Then $M \otimes_R N$ is an R-bimodule via

$$r\left(\sum_i m_i \otimes n_i\right) = (rm_i) \otimes n_i,$$

and the map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_R N \\ (m, n) &\longrightarrow m \otimes n \end{aligned}$$

is right S-linear. Finally, for any S-module A and any right S-linear R-biadditive map $b\colon M \times N \to A$, there is a unique right S-module homomorphism $\alpha\colon M \otimes_R N \to A$ such that $\alpha(m \otimes n) = b(m, n)$.

We can also take tensor products of maps.

Lemma 3.32. *Let R be a ring, $f : A \to C$ be a homomorphism of right R-modules, and $g\colon B \to D$ be a homomorphism of left R-modules. There exists a unique homomorphism of abelian groups $f \otimes g\colon A \otimes_R B \longrightarrow C \otimes_R D$ such that*

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$$

for all $a \in A$ and $b \in B$. When R is commutative, $f \otimes g$ is a homomorphism of R-modules. Moreover, if M and N are (S, R)-bimodules and f is S-linear, then $f \otimes g$ is also a homomorphism of S-modules.

Proof sketch. The function

$$\begin{aligned} A \times B &\longrightarrow C \otimes_R D \\ (a, b) &\longmapsto f(a) \otimes g(b) \end{aligned}$$

is R-biadditive, and R-bilinear when R is commutative, so the universal property of tensor products in each case gives the desired homomorphism and its uniqueness. $\square$

Lemma 3.33. *Given R-module maps* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ *and* $B_1 \xrightarrow{b_1} B_2 \xrightarrow{g_2} B_3$ *, the composition of* $f_1 \otimes g_1$ *satisfies* $f_2 \otimes g_2$

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 f_1) \otimes (g_2 g_1).$$

Proof. It's sufficient to check that these maps agree on simple tensors, and indeed they both take $a \otimes b$ to $(f_2 f_1(a)) \otimes (g_2 g_1(b))$. □

We are particularly interested in tensor products because of the tensor functor.

Theorem 3.34. *Let M be a right R-module. There is an additive covariant functor*

$$M \otimes_R -\colon R\text{-}\boldsymbol{Mod} \longrightarrow \mathbf{Ab}$$

that takes each R-module N to $M \otimes_R N$, and each R-module homomorphism $f\colon A \longrightarrow B$ to the homomorphism of abelian groups $1_M \otimes f\colon M \otimes_R A \longrightarrow M \otimes_R B$.

*When R is commutative, we can view $M \otimes_R -$ as an additive functor R-**Mod** $\to$ R-**Mod**.*

Proof. Let $T := M \otimes_R -$. First, note that T preserves identities, meaning $T(1_N) = 1_{T(N)}$, since the identity map on $M \otimes_R N$ agrees with $T(1_N) = 1_M \otimes 1_N$ on simple tensors. Moreover, T preserves compositions, since by Lemma 3.33 we have

$$T(f)T(g) = (1 \otimes f)(1 \otimes g) = 1 \otimes (fg) = T(fg).$$

Therefore, T is a functor. To check that it is an additive functor, we need to prove that $T(f+g) = T(f) + T(g)$ for all $f, g \in \mathrm{Hom}_R(A, B)$. It is sufficient to check that the maps $T(f+g) = 1 \otimes (f+g)$ and $T(f) + T(g) = 1 \otimes f + 1 \otimes g$ agree on simple tensors. Indeed,

$$\begin{aligned}
T(f+g)(a \otimes b) &= (1 \otimes (f+g))(a \otimes b) \\
&= a \otimes (f+g)(b) \\
&= a \otimes f(b) + g(b) \\
&= a \otimes f(b) + a \otimes g(b) \\
&= (1 \otimes f)(a \otimes b) + (1 \otimes g)(a \otimes b) \\
&= T(f)(a \otimes b) + T(g)(a \otimes b).
\end{aligned}$$

We conclude that $T(f+g) = T(f) + T(g)$. □

Definition 3.35. Given a ring R and a right R-module M, the functor $M \otimes_R -$ is the **tensor product functor**.

Note that we were purposely vague on the target of the tensor product functor: when R is commutative, we get both a functor R-**Mod** $\to$ **Ab** and a functor R-**Mod** $\to$ R-**Mod**. The two functors are essentially the same: the tensor product functor R-**Mod** $\to$ **Ab** is the composition of functor R-**Mod** $\to$ R-**Mod** followed by the forgetful functor R-**Mod** $\to$ **Ab**.

We can similarly define the tensor product functor $- \otimes_R N$; when R is commutative, it turns out that the two constructions are essentially the same.

Lemma 3.36 (Commutativity of tensor products)**.** *Let R be a commutative ring. There is a natural isomorphism $M \otimes_R - \cong - \otimes_R N$. In particular, for all R-modules M and N we have*

$$M \otimes_R N \cong N \otimes_R M.$$

Proof. One can check (exercise!) that the map $M \times N \longrightarrow N \otimes_R M$ given by $(m, n) \mapsto n \otimes m$ is R-biadditive, and R-bilinear if R is commutative. The universal property of the tensor product $M \otimes_R N$ gives us an R-module homomorphism φ such that the diagram

$$\begin{tikzcd}
& M \otimes_R N \arrow[dr, dashed, "\varphi"] & \\
M \times N \arrow[ur] \arrow[rr] & & N \otimes_R M \\
(m,n) \arrow[rr, mapsto] & & n \otimes m
\end{tikzcd} .$$

commutes. Similarly, we get a map ψ and a commutative diagram

$$\begin{tikzcd}
& N \otimes_R M \arrow[dr, dashed, "\psi"] & \\
N \times M \arrow[ur] \arrow[rr] & & M \otimes_R N \\
(m,n) \arrow[rr, mapsto] & & n \otimes m
\end{tikzcd} .$$

Then $\varphi\psi$ agrees with the identity on $N \otimes_R M$ on simple tensors, so it is the identity. Similarly, $\psi\varphi$ is the identity on $M \otimes_R N$, and these are the desired isomorphisms.

The statement about naturality is more precisely the following: for every R-module maps $f : M_1 \longrightarrow M_2$ and $g : N_1 \longrightarrow N_2$, our isomorphisms $M_1 \otimes_R N_1 \cong N_1 \otimes_R M_1$ and $M_2 \otimes_R N_2 \cong N_2 \otimes_R M_2$ make the diagram

$$\begin{tikzcd}
M_1 \otimes_R N_1 \arrow[r, "\cong"] \arrow[d, "f \otimes g"'] & N_1 \otimes_R M_1 \arrow[d, "g \otimes f"] \\
M_2 \otimes_R N_2 \arrow[r, "\cong"] & N_2 \otimes_R M_2
\end{tikzcd}$$

commute. To check this, it's sufficient to check commutativity on simple tensors, and indeed

$$\begin{tikzcd}
m \otimes n \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & n \otimes m \arrow[ddd, mapsto] \\
& M_1 \otimes_R N_1 \arrow[r, "\cong"] \arrow[d, "f \otimes g"'] & N_1 \otimes_R M_1 \arrow[d, "g \otimes f"] & \\
& M_2 \otimes_R N_2 \arrow[r, "\cong"] & N_2 \otimes_R M_2 & \\
f(m) \otimes g(n) \arrow[rrr, mapsto] & & & g(n) \otimes f(m).
\end{tikzcd}$$

Lemma 3.37 (Associativity of tensors)**.** *Given a right R-module A, an (R,S)-bimodule B, and an S-module C,*

$$(A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C).$$

Proof. Fix $c \in C$. The map

$$\begin{aligned} A \times B &\longrightarrow A \otimes_R (B \otimes_R C) \\ (a,b) &\longmapsto a \otimes (b \otimes c) \end{aligned}$$

is R-biadditive, so it induces a homomorphism abelian groups $\varphi_c \colon A \otimes_R B \longrightarrow A \otimes_R (B \otimes_R C)$. This map is in fact a homomorphism of R-modules when R is commutative. Then

$$\begin{aligned} (A \otimes_R B) \times C &\longrightarrow A \otimes_R (B \otimes_R C) \\ (a \otimes b, c) &\longmapsto a \otimes (b \otimes c) \end{aligned}$$

is also R-bilinear, and it induces a homomorphism of R-modules that sends $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$. Similarly, we can define a homomorphism of R-modules $A \otimes_R (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_R C$ that sends $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$. The composition of these two homomorphisms of R-modules in either order is the identity on simple tensors, and thus they are both isomorphisms. □

Lemma 3.38. *Let R be any ring. There is a natural isomorphism between $R \otimes_R -$ and the identity functor on R-**Mod**. In particular, for every left R-module M there is an isomorphism of R-modules*

$$R \otimes_R M \cong M.$$

Proof. First, note that R is an R-bimodule, so $R \otimes_R M$ is an R-module. The map

$$\begin{aligned} R \times M &\longrightarrow M \\ (r,m) &\longmapsto rm \end{aligned}$$

is R-biadditive (by the distributive laws), R-bilinear (by associativity of the action on a module), and R-linear, so it induces a homomorphism of R-modules $R \otimes_R M \xrightarrow{\varphi_M} M$. By definition, φ_M is surjective. Moreover, the map

$$\begin{aligned} M &\xrightarrow{f_M} R \otimes_R M \\ m &\longmapsto 1 \otimes m \end{aligned}$$

is a homomorphism of R-modules, since

$$f_M(a+b) = 1 \otimes (a+b) = 1 \otimes a + 1 \otimes b \text{ and } f_M(ra) = 1 \otimes (ra) = r(1 \otimes a) = rf_M(a).$$

For every $m \in M$, $\varphi_M f_M(m) = \varphi_M(1 \otimes m) = 1m = m$, and for every simple tensor, $f_M \varphi_M(r \otimes m) = f_M(rm) = 1 \otimes (rm) = r \otimes m$. This shows that φ_M is an isomorphism.

Finally, given any $f \in \mathrm{Hom}_R(M,N)$, since f is R-linear we conclude that the diagram

$$\begin{tikzcd}
r \otimes m \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & rm \arrow[ddd, mapsto] \\
& R \otimes_R M \arrow[r, "\varphi_M"] \arrow[d, "1 \otimes f"'] & M \arrow[d, "f"] & \\
& R \otimes N \arrow[r, "\varphi_N"'] & N & \\
r \otimes f(m) \arrow[rrr, mapsto] & & & rf(m) = f(rm)
\end{tikzcd}$$

commutes, so our isomorphism is natural. $\square$

Similarly to the Hom functor, tensor behaves well with respect to arbitrary direct sums.

Theorem 3.39. *Let M be and R-module, and let $\{N_i\}_{i \in I}$ be an arbitrary family of left R-modules. Then the map*

$$\begin{aligned} M \otimes_R \left(\bigoplus_{i \in I} N_i \right) &\xrightarrow{\cong} \bigoplus_{i \in I} M \otimes_R N_i \\ m \otimes (a_i)_i &\longmapsto (m \otimes a_i) \end{aligned}$$

is a natural bijection, meaning that given two families of left R-modules $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$, and left R-module homomorphisms $\sigma_{ij} \colon A_i \longrightarrow B_j$, the R-module homomorphisms

$$\begin{aligned} \bigoplus_{i \in I} A_i &\xrightarrow{\sigma} \bigoplus_{j \in J} B_j \\ (a_i)_{i \in I} &\longmapsto (\sigma_{ij}(a_i))_{j \in J} \end{aligned} \qquad \textit{and} \qquad \tilde{\sigma} = \bigoplus_{i \in I} \sigma_{ij} \colon \bigoplus_{i \in I} M \otimes_R A_i \longrightarrow \bigoplus_{j \in J} M \otimes_R B_j$$

give a commutative diagram

$$\begin{tikzcd}
M \otimes_R \left(\bigoplus_{i \in I} A_i \right) \arrow[r, "\cong"] \arrow[d, "1 \otimes \sigma"'] & \bigoplus_{i \in I} M \otimes_R A_i \arrow[d, "\tilde{\sigma}"] \\
M \otimes_R \left(\bigoplus_{j \in J} B_j \right) \arrow[r, "\cong"'] & \bigoplus_{j \in J} M \otimes_R B_j.
\end{tikzcd}$$

This map is an isomorphism of abelian groups in general, of R-modules in the commutative case, of S-modules if each N_i is an (S,R)-bimodule, and of right S-modules if N is an (R,S)-bimodule.

Proof. First, note that the function

$$\begin{array}{ccc} M \times \left(\bigoplus_{i \in I} A_i \right) & \longrightarrow & \bigoplus_{i \in I} (M \otimes_R A_i) \\ (m, (a_i)_i) & \longmapsto & (m \otimes a_i) \end{array}$$

is R-bilinear, so it induces a homomorphism

$$M \otimes_R \left(\bigoplus_{i \in I} A_i \right) \xrightarrow{\tau} \bigoplus_{i \in I} (M \otimes_R A_i) .$$

For each $k \in I$, let ι_k denote the inclusion map $A_k \subseteq \bigoplus_i A_i$. The universal property of the coproduct (which in the case of R-modules, means the direct sum) gives an R-module homomorphism

$$\begin{array}{ccc} \bigoplus_{i \in I} (M \otimes_R A_i) & \xrightarrow{\lambda} & M \otimes_R \bigoplus_{i \in I} (A_i) \\ (m \otimes a_i)_i & \longmapsto & m \otimes \sum_i \iota_i(a_i) \end{array}$$

which we obtain by assembling the R-module homomorphisms $1 \otimes \iota_i$. It is routine to check that λ is the inverse of τ, which must then be an isomorphism. Finally, we can check naturality by checking commutativity of the square above, element by element:

$$\begin{tikzcd} m \otimes (a_i)_i \arrow[r, mapsto] \arrow[d, mapsto] & (m \otimes a_i)_i \arrow[d, mapsto] \\ m \otimes (\sigma_{ij}(a_i))_i \arrow[r, mapsto] & (m \otimes \sigma_{ij}(a_i)). \end{tikzcd}$$

Remark 3.40. By commutativity of the tensor product, we also get that

$$\left(\bigoplus_{i \in I} N_i \right) \otimes_R M \xrightarrow{\cong} \bigoplus_{i \in I} N_i \otimes_R M.$$

The following follows as a corollary of Lemma 3.38 and Theorem 3.39:

Exercise 48. If F and G are free R-modules on basis $\{e_\lambda\}_{\lambda \in \Lambda}$ and $\{e_\gamma\}_{\gamma \in \Gamma}$, respectively, then $F \otimes_R G$ is the free R-module on basis

$$\{e_\lambda \otimes e_\gamma \mid \lambda \in \Lambda, \gamma \in \Gamma\}.$$

In particular,

$$R^n \otimes R^m \cong R^{nm}.$$

Example 3.41. Let R be any ring and consider $R^2 \otimes_R R^2$. Let $e_1 = (1,0) \in R^2$ and $e_2 = (0,1) \in R^2$. We claim that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ is not a simple tensor. Suppose, by contradiction, that there exist $v, y \in R^2$ such that

$$e_1 \otimes e_2 + e_2 \otimes e_1 = v \otimes w.$$

Since $\{e_1, e_2\}$ is a basis for the free module R^2, we can write

$$v = v_1 e_1 + v_2 e_2 \qquad \text{and} \qquad w = w_1 e_1 + w_2 e_2.$$

Substituting above, we see that

$$\begin{aligned} v \otimes w &= (v_1 e_1 + v_2 e_2) \otimes (w_1 e_1 + w_2 e_2) \\ &= v_1 w_1 e_1 \otimes e_1 + v_1 w_2 e_1 \otimes e_2 + v_2 w_1 e_2 \otimes e_1 + v_2 w_2 e_2 \otimes e_2. \end{aligned}$$

But by Exercise 48, $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ is a basis for the free R-module $R^2 \otimes R^2 \cong R^4$, so we can now compare coefficients: since

$$e_1 \otimes e_2 + e_2 \otimes e_1 = v_1 w_1 e_1 \otimes e_1 + v_1 w_2 e_1 \otimes e_2 + v_2 w_1 e_2 \otimes e_1 + v_2 w_2 e_2 \otimes e_2,$$

we must have

$$\begin{cases} v_1 w_1 = 1 \\ v_1 w_2 = 0 \\ v_2 w_1 = 0 \\ v_2 w_2 = 1 \end{cases} \implies \begin{cases} v_1 \text{ and } w_1 \text{ are units} \\ v_1 w_2 = 0 \\ v_2 w_1 = 0 \\ v_2 \text{ and } w_2 \text{ are units} \end{cases}$$

But since v_1 is a unit and $v_1 w_2 = 0$, we must have $w_2 = 0$; similarly, since v_2 is a unit and $v_2 w_1 = 0$, we must have $w_1 = 0$. But we have both $w_1 = w_2 = 0$ and that w_1, w_2 are units, which is a contradiction. We conclude that $e_1 \otimes e_2 + e_2 \otimes e_1$ is not a simple tensor.

One of the reasons tensor products are useful is that we can use tensor products to extend module structures to ring extensions.

Lemma 3.42. *Let $f : R \to S$ be a ring homomorphism. Then $S \otimes_R -$ determines a functor from R-modules to S-modules.*

Remark 3.43. Let $f : R \to S$ be a ring homomorphism. Since S is an (S, R)-bimodule, the abelian group $S \otimes_R M$ has an S-module structure for every left R-module M. Thus $S \otimes_R -$ determines a functor from R-modules to S-modules.

Definition 3.44. Let $f : R \to S$ be a ring homomorphism. The **extension of scalars** from R to S is the functor $S \otimes_R - : R\text{-}\mathbf{Mod} \longrightarrow S\text{-}\mathbf{mod}$. For each R-module M, we get an S-module $S \otimes_R M$ with

$$s \cdot \left(\sum_i s_i \otimes m_i \right) := \sum_i (s s_i) \otimes m_i,$$

and for each R-module homomorphism $f : M \to N$ we get the S-module homomorphism $1 \otimes_R f : S \otimes_R M \longrightarrow S \otimes_R N$.

This functor is closely related to restriction of scalars: we will soon show that restriction and extension of scalars are adjoint functors.

Definition 3.45. Let $f : R \to S$ be a ring homomorphism. The **restriction of scalars functor** from S to R is the functor $f^* \colon S\text{-}\mathbf{mod} \longrightarrow R\text{-}\mathbf{Mod}$ that takes each S-module M to the R-module f^*M with underlying abelian group M and R-module structure

$$r \cdot m := f(r)m$$

induced by f. Moreover, for each S-module homomorphism $g \colon M \longrightarrow N$ we get the R-module homomorphism $f^*(g) \colon f^*(M) \longrightarrow f^*(N)$ defined by $f^*(g)(m) := g(n)$.

Exercise 49. Check that restriction of scalars as defined above is indeed a functor.

Tensor is right exact.

Theorem 3.46. *Let M be a right R-module. The functor $M \otimes_R -$ is right exact, meaning that for every exact sequence*

$$A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

the sequence

$$M \otimes_R A \xrightarrow{1\otimes i} M \otimes_R B \xrightarrow{1\otimes p} M \otimes_R C \longrightarrow 0$$

is exact.

Proof. Since additive functors send complexes to complexes, $(1 \otimes p)(1 \otimes i) = 0$. We have two more things to show:

$\underline{1 \otimes p \text{ is surjective}}$: Consider any $m_1 \otimes c_1 + \cdots + m_n \otimes c_n \in M \otimes_R C$. Since p is surjective, we can find $b_1, \ldots, b_n \in B$ such that $p(b_i) = c_i$. Therefore,

$$(1 \otimes p)(m_1 \otimes b_1 + \cdots + m_n \otimes b_n) = m_1 \otimes p(b_1) + \cdots + m_n \otimes p(b_n) = m_1 \otimes c_1 + \cdots + m_n \otimes c_n.$$

$\underline{\ker(1 \otimes p) = \operatorname{im}(1 \otimes i)}$: Let $I = \operatorname{im}(1 \otimes i)$. We have already shown that $I \subseteq \ker(1 \otimes p)$, so $1 \otimes p$ induces a map $q \colon (M \otimes_R B)/I \longrightarrow M \otimes_R C$. Let $\pi \colon M \otimes_R B \longrightarrow (M \otimes_R B)/I$ be the canonical projection. By definition, $q\pi = 1 \otimes p$.

Consider the map

$$\begin{aligned} M \times C &\xrightarrow{f} (M \otimes_R B)/I\,, \\ (m, c) &\longmapsto m \otimes b \end{aligned}$$

where b is such that $p(b) = c$. First, we should check this map f is well-defined. To see that, suppose that $b' \in B$ is another element with $p(b') = c$, so that $p(b - b') = 0$. Then $b - b' \in \ker p = \operatorname{im} i$, so $m \otimes (b - b') \in \operatorname{im}(1 \otimes i) \subseteq I$. Therefore, $m \otimes b = m \otimes b'$ modulo I, and f is well-defined.

Moreover, we can easily check that f is R-additive, so it induces a homomorphism of R-modules $M \otimes_R C \longrightarrow (M \otimes_R B)/I$, which we will denote by $\hat{f}$. We will show that $\hat{f}$ is a left inverse of q, so q is injective. And indeed, given $m_i \in M$ and $b_i \in B$, we have

$$\hat{f}q\left(\sum_{i=1}^{n} m_i \otimes b_i\right) = f\left(\sum_{i=1}^{n} m_i \otimes p(b_i)\right) = \sum_{i=1}^{n} f(m_i \otimes p(b_i)) = \sum_{i=1}^{n} m_i \otimes b_i.$$

We conclude that q is injective, and thus

$$\ker(1 \otimes p) = \ker(q\pi) = \ker \pi = I = \operatorname{im}(1 \otimes i). \qquad \square$$

However, tensor is not exact.

Example 3.47. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Applying the functor $\mathbb{Z}/2 \otimes_{\mathbb{Z}} -$, we get an exact sequence

$$\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

However, we claim that $1 \otimes i$ is not injective. On the one hand, $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/2$ by Lemma 3.38. On the other hand, we have seen in Example 3.23 that $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, so the map $1 \otimes i \colon \mathbb{Z}/2 \to 0$ cannot possibly be injective.

We can now show that extension of scalars turns an R-module into the S-module with the same presentation.

Remark 3.48. Let R be a ring, M be a right R-module, and N be a left R-module. We can compute $M \otimes_R N$ by taking a presentation of M

$$R^{\oplus\Gamma} \xrightarrow{\phi} R^{\oplus\Lambda} \longrightarrow M \longrightarrow 0$$

and tensoring with N to get

$$N^{\oplus\Gamma} \longrightarrow N^{\oplus\Lambda} \longrightarrow M \otimes_R Nar[r]0,$$

so $M \otimes_R N$ is isomorphic to the cokernel of the map $N^{\oplus\Gamma} \to N^{\oplus\Lambda}$ induced by ϕ. We can also compute $M \otimes_R N$ by taking a presentation of M

$$R^{\oplus\Xi} \xrightarrow{\psi} R^{\oplus\Omega} \longrightarrow N \longrightarrow 0$$

and tensoring with M to get

$$M^{\oplus\Xi} \longrightarrow M^{\oplus\Omega} \longrightarrow M \otimes_R N \longrightarrow 0,$$

so $M \otimes_R N$ is isomorphic to the cokernel of the map $M^{\oplus\Gamma} \to M^{\oplus\Lambda}$ induced by ψ.

Exercise 50. Let R and S be rings.

- If A is an (R, S)-bimodule and B is a left R-module, then $\operatorname{Hom}_R(A, B)$ is a left S-module via $(s \cdot f)(a) = f(as)$.
- If A is an (R, S)-bimodule and B is a right S-module, then $\operatorname{Hom}_R(A, B)$ is a right R-module via $(f \cdot r)(a) = f(ra)$.
- If B is an (S, R)-bimodule and A is a right R-module, then $\operatorname{Hom}_R(A, B)$ is a left S-module via $(s \cdot f)(a) = sf(a)$.
- If B is an (S, R)-bimodule and A is a left S-module, then $\operatorname{Hom}_R(A, B)$ is a right R-module via $(f \cdot r)(a) = f(a)r$.

3.3 Hom-tensor adjunction

The Hom and tensor functors are closely related.

Theorem 3.49 (Hom-tensor adjunction I)**.** *Let R be a commutative ring and let M, N, and P be R-modules. There is an isomorphism of R-modules*

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

that is natural on M, N, and P.

Proof. The universal property of the tensor product says that to give an R-module homomorphism $M \otimes_R N \longrightarrow P$ is the same as giving an R-bilinear map $M \times N \longrightarrow P$. Given such a bilinear map f, the map $n \mapsto f(m \otimes n)$ is R-linear for each $m \in M$, so it defines an R-module homomorphism $N \longrightarrow P$. Now the assignment

$$\begin{aligned} M &\longrightarrow \mathrm{Hom}_S(N, P) \\ m &\longrightarrow (n \mapsto f(m \otimes n)) \end{aligned}$$

is R-linear, f is an R-module homomorphism, and $m \mapsto m \otimes n$ is R-linear on m.

Conversely, given an R-module homomorphism $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$, one can easily check (exercise!) that $(m, n) \mapsto f(m)(n)$ is an R-bilinear map, so it induces an R-module homomorphism $M \otimes_R N \longrightarrow P$.

So we have constructed a bijection of Hom-sets

$$\begin{aligned} \mathrm{Hom}_R(M \otimes_R N, P) &\xrightarrow{\ \tau\ } \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \,. \\ f &\longmapsto (m \mapsto (n \mapsto f(m \otimes n))) \\ (m \otimes n \mapsto g(m)(n)) &\longleftarrow\!\shortmid\ g \end{aligned}$$

It's routine to check that both of these bijections are indeed homomorphisms of R-modules, so we leave it as an exercise.

Finally, we have the following commutative diagrams:

$$\begin{tikzcd} A \arrow[d, "f"'] & \rightsquigarrow & \mathrm{Hom}_R(A \otimes_R N, P) \arrow[r, "\cong"] \arrow[d, "(f\otimes 1_N)^*"'] & \mathrm{Hom}_R(A, \mathrm{Hom}_R(N, P)) \arrow[d, "f^*"] \\ B & & \mathrm{Hom}_R(B \otimes_R N, P) \arrow[r, "\cong"'] & \mathrm{Hom}_R(B, \mathrm{Hom}_R(N, P)) \end{tikzcd} ,$$

$$\begin{tikzcd} A \arrow[d, "f"'] & \rightsquigarrow & \mathrm{Hom}_R(M \otimes_R A, P) \arrow[r, "\cong"] \arrow[d, "(1_M\otimes f)^*"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(A, P)) \arrow[d, "(f^*)_*"] \\ B & & \mathrm{Hom}_R(M \otimes_R B, P) \arrow[r, "\cong"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(B, P)) \end{tikzcd} ,$$

and

$$\begin{tikzcd} A \arrow[d, "f"'] & \rightsquigarrow & \mathrm{Hom}_R(M \otimes_R N, A) \arrow[r, "\cong"] \arrow[d, "f_*"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, A)) \arrow[d, "(f^*)^*"] \\ B & & \mathrm{Hom}_R(M \otimes_R N, B) \arrow[r, "\cong"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, B)) \end{tikzcd} .$$

We leave checking these do indeed commute as an exercise. $\square$

Theorem 3.50. *Let M be a right R-module, N be a R-bimodule, and P a left R-module. Then there is a natural isomorphism*

$$\operatorname{Hom}_R(M \otimes_R N, P) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, P)).$$

Corollary 3.51 (Tensor and Hom are adjoint functors)**.** *Let R be a commutative ring, and M an R-module. The functor $- \otimes_R M : R\text{-}\boldsymbol{Mod} \longrightarrow R\text{-}\boldsymbol{Mod}$ is left adjoint to the functor $\operatorname{Hom}_R(M, -) \colon R\text{-}\boldsymbol{Mod} \longrightarrow R\text{-}\boldsymbol{Mod}$.*

Proof. The adjointness translates into the fact that for all R-modules N and P there is a bijection

$$\operatorname{Hom}_R(N \otimes_R M, P) \cong \operatorname{Hom}_R(N, \operatorname{Hom}_R(M, P))$$

which is natural on N and P, which is a corollary of Theorem 3.49. □

Later, when we talk about more general abelian categories, we will see that this adjunction *implies* that Hom is left exact and that tensor is right exact; in fact, this is a more general fact about adjoint pairs. For now, we want to discuss a more general version of this Hom-tensor adjunction.

Theorem 3.52 (Hom-tensor adjunction II)**.** *Let $f : R \to S$ be a ring homomorphism of commutative rings. Let M be an R-module, and P and N be S-modules. There is an isomorphism of abelian groups*

$$\operatorname{Hom}_S(M \otimes_R N, P) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_S(N, P)).$$

Moreover, this isomorphism is natural on M, N, and P, so it induces natural isomorphisms

- *between* $\operatorname{Hom}_S(- \otimes_R N, P)$ *and* $\operatorname{Hom}_R(-, \operatorname{Hom}_S(N, P))$.
- *between* $\operatorname{Hom}_S(M \otimes_R -, P)$ *and* $\operatorname{Hom}_R(M, \operatorname{Hom}_S(-, P))$.
- *between* $\operatorname{Hom}_S(M \otimes_R N, -)$ *and* $\operatorname{Hom}_R(M, \operatorname{Hom}_S(N, -))$.

Proof. Consider the map

$$\begin{array}{ccc} \operatorname{Hom}_S(M \otimes_R N, P) & \xrightarrow{\ \tau\ } & \operatorname{Hom}_R(M, \operatorname{Hom}_S(N, P)) \\ f & \longmapsto & m \mapsto (n \mapsto f(m \otimes n)) \end{array}.$$

Fix f. For each $m \in M$, let τ_m be the map $N \longrightarrow P$ defined by $\tau_m(n) := f(m \otimes n)$. Note that τ_m is indeed a homomorphism of S-modules, since it is the composition of two S-module maps, f and $m \otimes_R \mathrm{id}_N$, where m is the constant map $M \longrightarrow M$ equal to m.

We should check that our proposed map τ is indeed a map of abelian groups. It is immediate from the definition that τ sends the 0-map to the 0-map. Moreover, given S-module homomorphisms $f, g \colon M \otimes N \longrightarrow P$, and any $n \in N$, we have

$$\begin{aligned} \tau_m(f+g)(n) &= (f+g)(m \otimes n) && \text{by definition} \\ &= f(m \otimes n) + g(m \otimes n) && \text{since } f \text{ and } g \text{ are } S\text{-module maps} \\ &= \tau_m(f)(n) + \tau_m(g)(n) && \text{by definition} \end{aligned}$$

so $\tau_m(f+g) = \tau_m(f) + \tau_m(g)$ for all $m \in M$, and thus $\tau(f+g) = \tau(f) + \tau(g)$.

Suppose that $\tau(f) = 0$. Then for every $m \in M$ and every $n \in N$,

$$0 = \tau(f)(m)(n) = \tau_m(f)(n) = f(m \otimes n),$$

so f vanishes at every simple tensor, and we must have $f = 0$. On the other hand, if we are given $g \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$, consider the map $M \times N \longrightarrow P$ defined by $\tilde{f}(m, n) = g(m)(n)$. Since g is a homomorphism of R-modules, it is R-linear on m. Moreover, for each fixed m, $g(m)$ is a homomorphism of S-modules, so in particular $g(m)$ is R-linear. Together, these say that $\tilde{f}$ is an R-bilinear map. Let f be the homomorphism of R-modules $M \otimes_R N \longrightarrow P$ induced by $\tilde{f}$. By definition, $f(m \otimes n) = \tilde{f}(m, n) = g(m)(n)$, so $\tau(f) = g$. We conclude that τ is a bijection.

We leave the statements about naturality as exercises. □

Once more, there is a more general statement, but we leave the proof as an exercise:

Theorem 3.53. *Let $f: R \to S$ be a ring homomorphism. Let M be a left R-module, P be a left S-module, and N be an (S, R)-bimodule. There is an isomorphism of abelian groups*

$$\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$$

that induces natural isomorphisms

- *between* $\mathrm{Hom}_S(- \otimes_R N, P)$ *and* $\mathrm{Hom}_R(-, \mathrm{Hom}_S(N, P))$.
- *between* $\mathrm{Hom}_S(M \otimes_R -, P)$ *and* $\mathrm{Hom}_R(M, \mathrm{Hom}_S(-, P))$.
- *between* $\mathrm{Hom}_S(M \otimes_R N, -)$ *and* $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N, -))$.

Corollary 3.54 (Adjointness of restriction and extension of scalars)**.** *Let $f\ R \longrightarrow S$ be a ring homomorphism. The restriction of scalars functor $f^*: S\textbf{-mod} \longrightarrow R\textbf{-Mod}$ is the right adjoint of the extension of scalars functor $f_!: R\textbf{-Mod} \longrightarrow S\textbf{-mod}$.*

Proof. We need to show that for every R-module M and every S-module N there are bijections

$$\mathrm{Hom}_S(f_!(M), N) \cong \mathrm{Hom}_R(M, f^*(N))$$

which are natural on both M and N. By Theorem 3.52, we have natural bijections

$$\mathrm{Hom}_S(M \otimes_R S, N) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N)).$$

The module $M \otimes_R S$ is precisely $f_!(M)$. By Exercise 38, $\mathrm{Hom}_S(S, N) \cong N$ as an S-module. An isomorphism of S-modules $\mathrm{Hom}_S(S, N) \longrightarrow N$ is in particular an R-linear map, and thus also an isomorphism of R-modules. So $\mathrm{Hom}_S(S, N) \cong f^*(N)$ as R-modules. Therefore, the Hom-tensor adjuntion gives us the natural bijections we were looking for. □

The idea is that restriction of scalars and extension of scalars are the most efficient ways of making an R-module out of an S-module, and vice-versa.

Appendix A

Rings and modules

We will study complexes of R-modules; to make sure we are all speaking the same language, we record here our basic assumptions on rings and modules. You can learn more about the basic theory of rings and modules in any introductory algebra book, such as [DF04].

A.1 Rings and why they have 1

In this class, all rings have a multiplicative identity, written as 1 or 1_R is we want to emphasize that we are referring to the ring R. This is what some authors call *unital rings*; since for us all rings are unital, we will omit the adjective. Moreover, we will think of 1 as part of the structure of the ring, and thus require it be preserved by all natural constructions. As such, a subring S of R must share the same multiplicative identity with R, meaning $1_R = 1_S$. Moreover, any ring homomorphism must preserve the multiplicative identity. To clear any possible confusion, we include below the relevant definitions.

Definition A.1. A **ring** is a set R equipped with two binary operations, $+$ and $\cdot$, satisfying:

1) $(R,+)$ is an abelian group with identity element denoted 0 or 0_R.

2) The operation $\cdot$ is associative, so that $(R,\cdot)$ is a semigroup.

3) For all $a,b,c \in R$, we have

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad \text{and} \quad (a+b) \cdot c = a \cdot c + b \cdot c.$$

4) there is a multiplicative identity, written as 1 or 1_R, such that $1 \neq 0$ and $1 \cdot a = a = a \cdot 1$ for all $a \in R$.

To simplify notation, we will often drop the $\cdot$ when writing the multiplication of two elements, so that ab will mean $a \cdot b$.

Note that the requirement that $1 \neq 0$ makes it so that the *zero ring* is not a ring.

Definition A.2. A ring R is a **commutative ring** if for all $a,b \in R$ we have $a \cdot b = b \cdot a$.

Definition A.3. A ring R is a **division ring** if $1 \neq 0$ and $R \setminus \{0\}$ is a group under $\cdot$, so every nonzero $r \in R$ has a multiplicative inverse. A **field** is a commutative division ring.

Definition A.4. A commutative ring R is a **domain**, sometimes called an **integral domain** if it has no zerodivisors: $ab = 0 \Rightarrow a = 0$ or $b = 0$.

For some familiar examples, $\mathrm{M}_n(R)$ (the set of $n \times n$ matrices) is a ring with the usual addition and multiplication of matrices, $\mathbb{Z}$ and $\mathbb{Z}/n$ are commutative rings, $\mathbb{C}$ and $\mathbb{Q}$ are fields, and the real Hamiltonian quaternion ring $\mathbb{H}$ is a division ring.

Definition A.5. A **ring homomorphism** is a function $f\colon R \to S$ satisfying the following:

- $f(a+b) = f(a) + f(b)$ for all $a, b \in R$.
- $f(ab) = f(a)f(b)$ for all $a, b \in R$.
- $f(1_R) = 1_S$.

Under this definition, the map $f\colon \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$ sending $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ preserves addition and multiplication but not the multiplicative identities, and thus it is not a ring homomorphism.

Exercise 51. For any ring R, there exists a unique homomorphism $\mathbb{Z} \to R$.

Definition A.6. A subset S of a ring R is a **subring** of R if it is a ring under the same addition and multiplication operations and $1_R = 1_S$.

So under this definition, $2\mathbb{Z}$, the set of even integers, is not a subring of $\mathbb{Z}$; in fact, it is not even a ring, since it does not have a multiplicative identity!

Definition A.7. Let R be a ring. A subset I of R is an **ideal** if:

- I is nonempty.
- $(I, +)$ is a subgroup of $(R, +)$.
- For every $a \in I$ and every $r \in R$, we have $ra \in I$ and $ar \in I$.

The final property is often called **absorption**. A **left ideal** satisfies only absorption on the left, meaning that we require only that $ra \in I$ for all $r \in R$ and $a \in I$. Similarly, a **right ideal** satisfies only absorption on the right, meaning that $ar \in I$ for all $r \in R$ and $a \in I$.

When R is a commutative ring, the left ideals, right ideals, and ideals over R are all the same. However, if R is not commutative, then these can be very different classes.

One key distinction between unital rings and nonunital rings is that if one requires every ring to have a 1, as we do, then the ideals and subrings of a ring R are very different creatures. In fact, the *only* subring of R that is also an ideal is R itself. The change lies in what constitutes a subring; notice that nothing has changed in the definition of ideal.

Remark A.8. Every ring R has two **trivial ideals**: R itself and the zero ideal $(0) = \{0\}$.

A **nontrivial ideal** I of R is an ideal that $I \neq R$ and $I \neq (0)$. An ideal I of R is a **proper ideal** if $I \neq R$.

A.2 Modules

You can learn more about the basic theory of (commutative) rings and R-modules in any introductory algebra book, such as [DF04].

Definition A.9. Let R be a ring with $1 \neq 0$. A **left R-module** is an abelian group $(M, +)$ together with an action $R \times M \to M$ of R on M, written as $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$ we have the following:

- $(r + s)m = rm + sm$,
- $(rs)m = r(sm)$,
- $r(m + n) = rm + rn$, and
- $1m = m$.

A **right R-module** is an abelian group $(M, +)$ together with an action of R on M, written as $M \times R \to M, (m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$ we have

- $m(r + s) = mr + ms$,
- $m(rs) = (mr)s$,
- $(m + n)r = mr + nr$, and
- $m1 = m$.

By default, we will be studying left R-modules. To make the writing less heavy, we will sometimes say **R-module** rather than left R-module whenever there is no ambiguity.

Remark A.10. If R is a commutative ring, then any left R-module M may be regarded as a right R-module by setting $mr := rm$. Likewise, any right R-module may be regarded as a left R-module. Thus for commutative rings, we just refer to modules, and not left or right modules.

The definitions of submodule, quotient of modules, and homomorphism of modules are very natural and easy to guess, but here they are.

Definition A.11. If $N \subseteq M$ are R-modules with compatible structures, we say that N is a **submodule** of M.

A map $M \xrightarrow{f} N$ between R-modules is a **homomorphism of R-modules** if it is a homomorphism of abelian groups that preserves the R-action, meaning $f(ra) = rf(a)$ for all $r \in R$ and all $a \in M$. We sometimes refer to R-module homomorphisms as **R-module maps**, or **maps of R-modules**. An isomorphism of R-modules is a bijective homomorphism, which we really should think about as a relabeling of the elements in our module. If two modules M and N are isomorphic, we write $M \cong N$.

Given an R-module M and a submodule $N \subseteq M$, the **quotient** M/N is an R-module whose elements are the equivalence classes determined by the relation on M given by $a \sim b \Leftrightarrow a - b \in N$. One can check that this set naturally inherits an R-module structure from the R-module structure on M, and it comes equipped with a natural **canonical map** $M \longrightarrow M/N$ induced by sending 1 to its equivalence class.

Example A.12. The modules over a field k are precisely all the k-vector spaces. Linear transformations are precisely all the k-module maps.

While vector spaces make for a great first example, be warned that many of the basic facts we are used to from linear algebra are often a little more subtle in commutative algebra. These differences are features, not bugs.

Example A.13. The $\mathbb{Z}$-modules are precisely all the abelian groups.

Example A.14. When we think of the ring R as a module over itself, the submodules of R are precisely the ideals of R.

Theorem A.15 (First Isomorphism Theorem)**.** *Any R-module homomorphism $M \xrightarrow{f} N$ satisfies $M/\ker f \cong \operatorname{im} f$.*

The first big noticeable difference between vector spaces and more general R-modules is that while every vector space has a basis, most R-modules do not.

Definition A.16. A subset $\Gamma \subseteq M$ of an R-module M is a **generating set**, or a **set of generators**, if every element in M can be written as a finite linear combination of elements in M with coefficients in R. A **basis** for an R-module M is a generating set Γ for M such that $\sum_i a_i \gamma_i = 0$ implies $a_i = 0$ for all i. An R-module is **free** if it has a basis.

Remark A.17. Every vector space is a free module.

Remark A.18. Every free R-module is isomorphic to a direct sum of copies of R. Indeed, let's construct such an isomorphism for a given free R-module M. Given a basis $\Gamma = \{\gamma_i\}_{i\in I}$ for M, let

$$\begin{aligned} \textstyle\bigoplus_{i\in I} R &\xrightarrow{\quad\pi\quad} M \\ (r_i)_{i\in I} &\longrightarrow \sum_i r_i \gamma_i \end{aligned} \quad .$$

The condition that Γ is a basis for M can be restated into the statement that π is an isomorphism of R-modules.

One of the key things that makes commutative algebra so rich and beautiful is that most modules are in fact *not* free. In general, every R-module has a generating set — for example, M itself. Given some generating set Γ for M, we can always repeat the idea above and write a **presentation** $\oplus_{i\in I} R \xrightarrow{\pi} M$ for M, but in general the resulting map π will have a nontrivial kernel. A nonzero kernel element $(r_i)_{i\in I} \in \ker \pi$ corresponds to a **relation** between the generators of M.

Remark A.19. Given a set of generators for an R-module M, any homomorphism of R-modules $M \longrightarrow N$ is determined by the images of the generators.

We say that a module is **finitely generated** if we can find a finite generating set for M. The simplest finitely generated modules are the cyclic modules.

Example A.20. An R-module is **cyclic** if it can be generated by one element. Equivalently, we can write M as a quotient of R by some ideal I. Indeed, given a generator m for M, the kernel of the map $R \xrightarrow{\pi} M$ induced by $1 \mapsto m$ is some ideal I. Since we assumed that m generates M, π is automatically surjective, and thus induces an isomorphism $R/I \cong M$.

Similarly, if an R-module has n generators, we can naturally think about it as a quotient of R^n by the submodule of relations among those n generators.

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