## Problem Set 5

**Problem 1.** Let  $R = \mathbb{Z}[\sqrt{-5}]$ . While  $6 \in R$  cannot be written as a unique product of irreducibles, we are going to show that the ideal I = (6) does have a unique primary decomposition. Unfortunately, Macaulay2 cannot take primary decompositions over  $\mathbb{Z}$ , but this one we can do the old fashioned way.

- a) Prove that (2) is a primary ideal.
- b) Prove that (3) is *not* a primary ideal.
- c) Prove that  $(3, 1 + \sqrt{-5})$  and  $(3, 1 \sqrt{-5})$  are both primary.
- d) Show that  $(6) = (2) \cap (3, 1 + \sqrt{-5}) \cap (3, 1 \sqrt{-5}).$
- e) Why is this primary decomposition unique?

**Problem 2.** Let R be a Noetherian ring. Let I be an ideal in R, and  $x \in R$ . The saturation of I with respect to x is the ideal

$$(I:x^{\infty}) := \bigcup_{n=1}^{\infty} (I:x^n).$$

a) Let Q be a primary ideal. Show that

$$(Q:x^{\infty}) = \left\{ \begin{array}{ll} Q & \text{if } x \notin P \\ R & \text{if } x \in P \end{array} \right..$$

- b) Show that  $(I:x^{\infty})=(I:x^n)$  for some n.
- c) Show that  $(I \cap J : x^{\infty}) = (I : x^{\infty}) \cap (J : x^{\infty})$  for any ideals I and J.
- d) Let  $I = Q_1 \cap \cdots \cap Q_k$  be a primary decomposition, and  $x \in R$ . Show that

$$(I:x^{\infty}) = \bigcap_{x \notin P_i} Q_i.$$

**Problem 3.** Let I be a radical ideal in a Noetherian ring R. A beautiful theorem of Brodmann says that the set

$$\bigcup_{n\geqslant 1} \mathrm{Ass}\ (R/I^n)$$

is finite. Show that there exists an element x such that:

- x is contained in every embedded prime of  $I^n$  for every n, and
- $x \notin P$  for all  $P \in \text{Min}(I)$ .

Conclude that there exists  $x \in R$  such that  $I^{(n)} = (I^n : x^{\infty})$  for all  $n \ge 1$ .

**Problem 4.** Consider s points  $P_1 = (a_{11}, \ldots, a_{1d}), \ldots, P_s = (a_{s1}, \ldots, a_{sd})$  in  $\mathbb{A}^d$ , and let I be the corresponding radical ideal in  $\mathbb{C}[x_1, \ldots, x_d]$ . Show that for all  $n \geq 1$ ,

$$I^{(n)} = \bigcap_{i=1}^{s} (x_1 - a_{i1}, \dots, x_d - a_{id})^n.$$

**Problem 5.** Let R be a Noetherian ring.

- a) Show that if  $\mathfrak{m}$  is any maximal ideal in R, then  $\mathfrak{m}^n$  is  $\mathfrak{m}$ -primary for any  $n \geq 1$ .
- b) If R is a domain, then

$$\bigcap_{n\geqslant 1}I^n=0.$$

for any proper ideal I in R.

**Problem 6.** If  $(R, \mathfrak{m})$  is a Noetherian local ring, show that M has finite length if and only if M is finitely generated and  $\mathfrak{m}^n M = 0$  for some n.

**Problem 7.** Let k be a field, and R = k[a, b, c, d]/(ad - bc). Find prime ideals P and Q in R such that ht(P) + ht(Q) < ht(P+Q).

## Problem 8.

- a) Find the height of J=(ab,bc,cd,ad) in k[a,b,c,d] over any field k, and the dimension of k[a,b,c,d]/J.
- b) Find the dimension of the ring S, where  $S = \mathbb{Q}[x^3y^3, x^3y^2z, x^2z^3] \subseteq \mathbb{Q}[x, y, z]$ .
- c) Let I be the defining ideal of the curve parametrized by  $(t^{13}, t^{42}, t^{73})$  over  $\mathbb{Q}$ . Find the height of I, and notice that height  $(I) < \mu(I)$ .
- d) Let  $R = \mathbb{Q}[x, y, z]$ , and  $I = (x^3, x^2y, x^2z, xyz)$ . Find the dimension of R/I and the height of I.
- e) Find the dimension of the module  $I/I^2$ , where I=(xz) in  $R=\mathbb{C}[x,y,z]/(xy,yz)$ .

<sup>&</sup>lt;sup>1</sup>Hint: first, do the case where I is a maximal ideal in R. Be wise, localize!