Problem Set 0
Introductory Macaulay2 problems

Problem 1.

a) Install Macaulay2.[1] Hardcore version: install emacs and run Macaulay2 through emacs.

b) Make an .m2 file setting up a field k, a polynomial ring R over k, a nontrivial ideal I in R, the R-module $M = R/I$ and the ring $S = R/I$.

Problem 2 (Subalgebras)**.** Use Macaulay2 to find:

a) A presentation for the $\mathbb{Q}$-algebra $\mathbb{Q}[xy, xu, yv, uv] \subseteq \mathbb{Q}[x, y, u, v]$.

b) A presentation for the k-algebra U, where $k = \mathbb{Z}/101$ and

$$k\begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{k[u, v, x, y, z]}{(x^3 + y^3 + z^3)}.$$

Problem 3 (Graded rings)**.**

a) In Macaulay2, set up $A = \mathbb{Q}[s^2, st, t^2]$ as an $\mathbb{N}^2$-graded ring with the grading induced by setting s^2, st, t^2 as homogeneous elements of degrees

$$\deg(s^2) = (2, 0) \quad \deg(st) = (1, 1) \quad \deg(t^2) = (0, 2).$$

b) The ring $R = k[t^3, t^{13}, t^{42}]$ is a graded subring of $k[t]$ with the standard grading, meaning that the graded structure on $k[t]$ induces a grading on R. Set up R (with this grading) in Macaulay2.

Problem 4 (Modules)**.** Consider the domain $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$. Set up the following R-modules, making sure Macaulay2 actually sees them as modules over R:

a) The ideal $I = (x, a)$ viewed as an R-module.

b) The R-module $N = \mathbb{Q}$.

c) The 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

d) The submodule of R^3 generated by (a, b, c) and (x, y, z).

Problem 5 (Complexes in Macaulay2)**.** Let $R = \mathbb{Q}[x, y, z]/(x^2, xy)$.

a) Consider the bounded complex

$$C = \underset{3}{R} \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} \underset{2}{R^3} \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} \underset{1}{R^3} \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} \underset{0}{R}$$

Set C up in Macaulay2 and compute its homology. For which n is $\mathrm{H}_n(C) = 0$?

[1] If you don't have access to a computer, or if your computer runs only Windows, come talk to me about it.

b) Check that f below is a map of complexes, and compute its kernel, cokernel, and homology.

$$\begin{tikzcd}
D = \arrow[from=d, "f"] & R \arrow[r, "\begin{pmatrix} z \\ -y \\ x \end{pmatrix}"] & R^3 \arrow[r, "\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}"] & R^3 \arrow[r, "\begin{pmatrix} x & y & z \end{pmatrix}"] & R \\
C = & 0 \arrow[u, "0"] \arrow[r, "0"'] & R \arrow[u, "\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} -z \\ y \end{pmatrix}"'] & R^2 \arrow[u, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} y & z \end{pmatrix}"'] & R. \arrow[u, equal] \\
& {\scriptstyle 3} & {\scriptstyle 2} & {\scriptstyle 1} & {\scriptstyle 0}
\end{tikzcd}$$