Lech's Inequality for Generalized Multiplicities

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Introduction

Joint work with Kelsey Walters.

Throughout this talk all rings are Noetherian and all modules are finitely generated.

(R, m, k) Noetherian local, with k infinite.

If k is not infinite, pass from $R \to R[X]_{mR[X]}$ where X is an indeterminate.

Hilbert Samuel Multiplicity

If I is m-primary, then $\lambda(R/I) < \infty$ and also $\lambda(R/I^n) < \infty$.

For $n >> 0$, $\lambda(R/I^n)$ behaves as a polynomial in n of degree $d = dim\ R$.

$$\lim_{n\to\infty} d! \frac{\lambda(R/I^n)}{n^d} = e(I)$$

$$e(R) := e(m), \quad e(I^k) = k^d e(I).$$

Multiplicity detects integral dependence.

The multiplicity of ideals in R, in some sense, measures how badly behaved the ring is.

For instance (need to assume R is formally equidimensional for some of the following results):

- R is regular iff $e(R) = 1 = \lambda(R/m)$ [Nagata '62].
  - R is regular iff $e(I) = \lambda(R/I)$ for any (all) integrally closed ideal I [Ma-Quy-Smirnov '19].
- R is Cohen-Macaulay iff $e(I) = \lambda(R/I)$ for any (all) parameter ideal I.
  - Assume R is Cohen-Macaulay. Then I is a complete intersection iff $e(I) = \lambda(R/I)$.
- R is Buchsbaum iff $e(I) - \lambda(R/I)$ is the same for any parameter ideal I (Definition).
- R is generalized Cohen-Macaulay iff $sup\{e(I) - \lambda(R/I)\} < \infty$ taken over all parameter ideals I [Cuong-Schenzel-Trung '78].

Generalizations of Hilbert-Samuel Multiplicity

We can measure the colengths of multiple m-primary ideals $I_1, ..., I_r$.

$\lambda(R/{I_1}^{n_1} \cdots {I_r}^{n_r})$ behaves like a polynomial in r variables of total degree d when $(n_1, ..., n_r) >> 0$.

Look at a particular leading term. Choose positive integers $a_1 + ... + a_r = d$

$$c_{a_1,..,a_r} {x_1}^{a_1} \cdots {x_r}^{a_r}$$

Then the mixed multiplicity of type $(a_1, .., a_r)$ of the ideals $I_1, ..., I_r$ is

$$e({I_1}^{[a_1]}, ..., {I_r}^{[a_r]}) = a_1! \cdots a_r! c_{a_1,...,a_r}$$

Alternative notations: $e(\underbrace{I_1, ..., I_1}_{a_1\text{-times}}, ..., \underbrace{I_r, ..., I_r}_{a_r\text{-times}})$ or $e(I_1, ..., I_d)$.

Recover Hilbert-Samuel multiplicity when $r = 1$.

Also when $r > 1$, look at a single ideal I_1

$$e({I_1}^{[d]}, ..., {I_r}^{[0]}) = e(I_1, ..., I_1) = e(I_1)$$

$$\lambda(R/{I_1}^{n_1} \cdots {I_r}^{n_r}) = P(n_1, ..., n_r)$$

Fix $n_2, ..., n_r$. When $n_1 >> 0$ the term with x_1^d in $P(x_1, ..., x_r)$ dominates.

$P(x_1, n_2, ..., n_r) = P(x_1)$ agrees with the Hilbert polynomial for I_1 (up to a constant).

Buchsbaum-Rim Multiplicity

Can we replace I with a module M?

$M \subset F = R^r$. How to measure the growth of "powers" of M?

$\mathrm{Sym}(M) \to \mathrm{Sym}(F) = R[Y_1, ..., Y_r]$.

Image of $\mathrm{Sym}(M)$ is $\mathcal{R}[M]$ the Rees algebra of M.

$$M^n := \mathcal{R}[M]_n \subset F^n := \mathrm{Sym}(F)_n$$

If $\lambda(F/M) < \infty$ then $\lambda(F^n/M^n) < \infty$.

Definition [Buchsbaum-Rim '64]

If $M \subsetneq F$ with $\lambda(F/M) < \infty$ then $\lambda(F^n/M^n)$ is a polynomial of degree $d + r - 1$ for $n >> 0$. Then we define the Buchsbaum-Rim multiplicity of M to be

$$br(M) := \lim_{n \to \infty} (d + r - 1)! \frac{\lambda(F^n/M^n)}{n^{d+r-1}}.$$

When $r = 1$, M is an ideal in R, and M^n is the nth power of M as an ideal. $br(M) = e(M)$.

$$M = \bigoplus_{i=1}^{r} I_i \subset F = R^r$$

$$\frac{F^n}{M^n} \cong \bigoplus_{\substack{n_1+\ldots+n_r=n \\ n_1,\ldots,n_r\geq 0}} \frac{R}{{I_1}^{n_1}\cdots {I_r}^{n_r}}$$

Theorem [Kirby-Rees '96 , Bivia-Ausina '04]

Let $I_1, \ldots, I_r$ be m-primary ideals, and set $M = \bigoplus_{i=1}^{r} I_i$. Then

$$br(M) = \sum_{\substack{a_1+\ldots+a_r=d \\ a_1,\ldots,a_r\geq 0}} e({I_1}^{[a_1]}, \ldots, {I_r}^{[a_r]}).$$

Quick Mentions - More generalizations

For ideals $I \subset R$ not neccessary m-primary!

ϵ-**multiplicity [Ulrich-Validashti '11]**

Measure length of $H^0_m(R/I^n)$, length function does not necessarily have polynomial behavior.

ϵ-multiplicity can be an irrational number [Cutcosky].

j-**multiplicity [Achilles-Manaresi '93, Ulrich-Validashti '11]**

Measure length of $H^0_m(I^n/I^{n+1})$.

Lech's Inequality

There are interactions between $e(I)$ and $\lambda(R/I)$. Equality for various ideals implies certain properties about R.

Theorem [Lech '60]

For R a Noetherian local ring of dimension d and I an m-primary ideal,

$$e(I) \leq d!\lambda(R/I)e(R).$$

In general, the bound is is not sharp.

When $I = m$, $e(m) = e(R)$, we get $1 \leq d!$.

Huneke-Smirnov-Validashti's Bound

R regular local, $e(m) = e(R) = 1$. $e(m^n) = n^d e(m) = n^d$

$d!\lambda(R/m^n) = d!\binom{n+d-1}{d} = n(n+1)\cdots(n+d-1) = O(n^d)$.

Question [Huneke-Smirnov-Validashti '17]

Let R be a Noetherian local ring of dimension d and I an m-primary ideal. Define a numerical function $P(n) = n(n+1)\cdots(n+d-1)$. Is it true that

$$P(e(I)^{1/d}) \leq d!\lambda(R/I)e(R)?$$

R regular local, $e(m) = e(R) = 1$, take $I = m^n$

$$P(e(m^n)^{1/d}) = P((n^d e(m))^{1/d}) = P((n^d)^{1/d}) = P(n) = d!\lambda(R/m^n).$$

Theorem [Huneke-Smirnov-Validashti '17]

Let (R, m) be a Noetherian local ring of dimension $d \geq 4$ and I an m-primary ideal.

$$e(mI) \leq d!\lambda(R/I)e(R).$$

False if $1 \leq d < 4$. For instance if $I = m$ then

$$e(m^2) = 2^d e(m) = 2^d e(R) > d!e(R)$$

Main Results

Theorem [—, Walters]

Let R be a Noetherian local ring of dimension $d \geq 4$. Let $I_1, ..., I_d$ be m-primary ideals then

$$e(mI_1, ..., mI_d) < (d-1)! \sum_{i=1}^{d} \lambda(R/I_i) e(R).$$

Theorem [—, Walters]

Let R be a Noetherian local ring of dimension $d \geq 4$, and let $E \subset F = R^r$ such that $E \subset mF$ and $\lambda(F/E) < \infty$. Then

$$br(mE) < \frac{(d+r-1)!}{r!} \lambda(F/E) e(R).$$

Proof Sketch

1. Reduce to the polynomial ring case.
2. Induct on colength and dimension.
3. Reduce to low dimensional cases and prove bounds for them.

Reduction to Polynomial Rings

Lech type bound for monomial ideals in polynomial rings $\Longrightarrow$ Lech type bound for any Noetherian local ring.

Reduction steps: [Lech '60, Huneke-Smirnov-Validashti '17]

- Pass from R to $gr_m(R)$. (Reduction to graded rings)
- Pass to a Noether normalization $S \subset R$. (Reduction to polynomial rings)
- Pass from I to in I. (Reduction to monomial ideals)

The above steps can be extended to modules and mixed multiplicity [—, Walters].

Polynomial Ring Case

Next step: Induct on colength and dimension.

Key Lemma [Huneke-Smirnov-Validashti '17]

Let R be a Noetherian local ring, I an m-primary ideal, and $x \notin I$ a non-zero divisor. Set $J = I : x$, and denote by $-'$ images in $R' = R/(x)$ then

1. $\lambda(R/I) = \lambda(R/J) + \lambda(R'/I')$
2. $e(I) \leq e(J) + de(I')$

Strong tool for induction on both colength and dimension!

When R is a polynomial ring, we can take $x \notin I$ to be a general linear element. Then R' is still a polynomial ring in one less dimension and $\lambda(R/J) < \lambda(R/I)$.

Proof of Lech using Key Lemma

Key Lemma [Huneke-Smirnov-Validashti '17]

Let R be a Noetherian local ring, I an m-primary ideal, and $x \notin I$ a non-zero divisor. Set $J = I : x$, and denote by $-'$ images in $R' = R/(x)$ then

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Reduce to the polynomial ring case. Proceed by double induction on dimension and colength. Base cases are easy.

For the induction step take $x \notin I$ to be a general linear element, $J = I : x$.

$$e(I) \leq e(J) + de(I') \leq d!\lambda(R/J) + d(d-1)!\lambda(R'/I') = d!\lambda(R/I)$$

Sketch of Proof in Polynomial Ring Case

Main tools:

Lemma [Rees '84]

Suppose $I_1, ..., I_d, J$ are m-primary ideals then

$$e(JI_1, I_2, ..., I_d) = e(J, I_2, ..., I_d) + e(I_1, I_2, ..., I_d).$$

Theorem [Teissier '73]

Let $x_1 \in I_1$ be a general element (that is part of a joint reduction). Denote by $-'$ images modulo x_1, then

$$e(I_1, ..., I_d) = e(I_2', ..., I_d').$$

$$e(mI_1, ..., mI_d) < (d-1)! \sum_{i=1}^{d} \lambda(R/I_i) := B(I_1, ..., I_d)$$

We may assume our ideals are integrally closed. First induct on dimension. Base case: $d = 4$, assume that it holds for now.

Let $d > 4$. Now induct on $\sum_{i=1}^{d} \lambda(R/I_i)$. The base case is when $I_i = m$ and holds because $2^d < d!$.

The key lemma completes the induction steps for both colength and dimension.

Remains to show $d = 4$ case. Again induct on colength. Let x, y, z be general linear elements (that are part of joint reductions). $-', -''$, and $-'''$ denote images in quotient rings.

We will use $e(JI_1, I_2, ..., I_d) = e(J, I_2, ..., I_d) + e(I_1, I_2, ..., I_d)$.

$$
\begin{aligned}
e(mI_1, ..., mI_4) &= e(I_1, ..., I_4) + \sum e(m, I_i, I_j, I_k) \\
&\quad + \sum e(m, m, I_i, I_j) + \sum e(m, m, m, I_i) + 1 \\
&= e(I_1, ..., I_4) + \sum e(I_i', I_j', I_k') \\
&\quad + \sum e(I_i'', I_j'') + \sum e(I_i''') + 1 \\
&= e(I_1, ..., I_4) + \text{Error Terms in lower dimension}
\end{aligned}
$$

$$J_i = I_i : x, \quad mJ_i \subset I_i, \quad e(I_1, ..., I_4) \leq e(mJ_1, ..., mJ_4).$$

$$\begin{aligned} e(mI_1, ..., mI_4) &= e(I_1, ..., I_4) + \text{Error Terms} \\ &\leq e(mJ_1, ..., mJ_4) + \text{Error Terms} \\ &< B(J_1, ..., J_4) + \text{Error Terms} \end{aligned}$$

Remains to show

$$\text{Error Terms} \leq B(I_1, ..., I_4) - B(J_1, ..., J_4) = 6 \sum_{i=1}^{4} \lambda(R'/I_i').$$

Error Terms $= \sum e(I_i', I_j', I_k') + \sum e(I_i'', I_j'') + \sum e(I_i''') + 1$
This bound is a technical result in dimension 3.

Thank you!