

## Problem Set 3

**Problem 1.** Let  $R$  be a ring,  $I$  and  $J$  ideals in  $R$ , and  $M$  be an  $R$ -module.

- a) Show that  $R/I \otimes_R R/J \cong R/(I + J)$ .
- b) Show that  $R/I \otimes_R M \cong M/IM$ .
- c) There is an  $R$ -module map  $I \otimes_R M \rightarrow IM$  induced by the  $R$ -bilinear map  $(a, m) \mapsto am$ . This map is always clearly surjective; must it be injective?

**Problem 2.** Show that over a field  $k$ , the Hom functors and the tensor functor are always exact.

**Problem 3.**

- a) Consider  $R$ -module homomorphisms  $A \xrightarrow{f} B$  and  $B \xrightarrow{g} C$ . If

$$\mathrm{Hom}_R(M, A) \xrightarrow{f_*} \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C)$$

is exact for all  $M$ , then  $A \xrightarrow{f} B \xrightarrow{g} C$  is an exact sequence.<sup>1</sup>

- b) Suppose that  $(F, G)$  is an adjoint pair of covariant functors  $R\text{-}\mathbf{mod} \rightarrow R\text{-}\mathbf{mod}$ . Show that  $F$  is left exact and  $G$  is right exact.

Let  $R$  be a domain and  $M$  be an  $R$ -module. The **torsion** of  $M$  is the submodule

$$T(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

The elements of  $T(M)$  are called **torsion elements**, and we say that  $M$  is **torsion** if  $T(M) = M$ . Finally,  $M$  is **torsion free** if  $T(M) = 0$ .

**Problem 4.** Let  $R$  be a domain and  $M$  be an  $R$ -module.

- a) The  $R$ -module  $M/T(M)$  is torsion free.
- b) If  $f: M \rightarrow N$  is an  $R$ -module homomorphism,  $f(T(M)) \subseteq T(N)$ .
- c) Torsion is a left exact covariant functor  $R\text{-}\mathbf{mod} \rightarrow R\text{-}\mathbf{mod}$ .

**Problem 5.** Let  $R$  be a domain with fraction field  $Q$ .

- a) Show that for every  $Q$ -vector space  $V$  and every  $R$ -module  $M$ ,  $V \otimes_R M \cong V \otimes_R (M/T(M))$ .
- b) The kernel of the map  $M \rightarrow Q \otimes_R M$  given by  $m \mapsto 1 \otimes m$  is  $T(M)$ .
- c) Show that for every  $Q$ -vector space  $V$  and  $R$ -module  $M$ ,  $V \otimes_R M = 0$  if and only if  $M$  is torsion.
- d) Show that  $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/(\pi\mathbb{Z})) \neq 0$ .

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<sup>1</sup>In particular, we are not assuming  $A \xrightarrow{f} B \xrightarrow{g} C$  is a complex!

**Problem 6.** Consider the domain  $R = \mathbb{Q}[x, y, z, a, b, c]/(xb-ac, yc-bz, xc-az)$ , the ideal  $I = (x, a)$  in  $R$ , the  $R$ -module  $N = \mathbb{Q}$ ,  $I = (x, a)$ , and consider the 2-generated  $R$ -module  $M = Rf + Rg$ , where the generators  $f, g$  satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

- a) Are there nontrivial  $R$ -module homomorphisms  $M \rightarrow N$ ? How about  $N \rightarrow M$ ?
- b) Does  $- \otimes_R M$  preserve the injectivity of the inclusion  $I \subseteq R$ ?
- c) Apply  $\text{Hom}_R(-, R)$  to the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Is  $R$  an injective  $R$ -module?

Let  $R$  be a ring and  $I$  be an ideal in  $R$ . The functor  $\Gamma_I : R\text{-mod} \rightarrow R\text{-mod}$  that sends each  $R$ -module  $M$  to the  $R$ -module

$$\Gamma_I(M) := \bigcup_{n \geq 1} (0 :_M I^n) = \{m \in M \mid I^n m = 0 \text{ for some } n \geq 1\}$$

and that sends each  $R$ -module homomorphism  $M \xrightarrow{f} N$  to the  $R$ -module homomorphism  $\Gamma_I(M) \xrightarrow{\Gamma_I(f)} \Gamma_I(N)$  given by restricting the domain and codomain of  $f$  is called the  **$I$ -torsion functor**.

**Problem 7.**

- a) Check that any  $R$ -module homomorphism  $M \xrightarrow{f} N$  must send  $\Gamma_I(M)$  into  $\Gamma_I(N)$ , so that our definition of the functor  $\Gamma_I$  makes sense.
- b) Show that  $\Gamma_I$  is an additive covariant functor.
- c) Show that  $\Gamma_I$  is left exact.
- d) Show that  $\Gamma_I$  is not necessarily right exact.