Cohen-Macaulay property of the fiber cone of modules

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(R, m, k) Noetherian local ring

$E = Ra_1 + \dots + Ra_n$ a finite R-module with $\operatorname{rank} E = e > 0$

$$R^s \xrightarrow{\varphi} R^n \twoheadrightarrow E$$

$$S(E) \cong R[T_1, \dots, T_n] / (\ell_1, \dots, \ell_s) \qquad [\ell_1, \dots, \ell_s] = [\underline{T} \cdot \varphi]$$

symmetric algebra

$\downarrow$ mod $\tau_R(S(E))$

$\mathcal{R}(E)$ Rees algebra

$\downarrow$ mod $m(\mathcal{R}(E))$

$\mathcal{R}(E) \otimes_R k \cong \mathcal{F}(E)$ fiber cone.

Examples:

- $E =$ free module $= R^n$

  $\operatorname{Sym}(E) \cong \mathcal{R}(E) = R[T_1, \dots, T_n]$

  $\mathcal{F}(E) \cong k[T_1, \dots, T_n]$

- E torsion-free module of rank 1 $\cong$ ideal I of positive grade

  $\mathcal{R}(I) \cong R[It] = \bigoplus_{i \geq 0} I^i t^i \subseteq R[t]$

  $\mathcal{F}(I) \cong \mathcal{R}(I) \otimes_R k \cong \bigoplus_{i \geq 0} I^i / mI^i$

  $G(I) \cong \bigoplus_{i \geq 0} I^i / I^{i+1}$

Geometric Motivation

① Blow up constructions:

$$\mathrm{Proj}(\mathcal{R}(I)) = \text{blow up along } V(I)$$
$$\mathrm{Proj}(\mathcal{G}(I)) = \text{exceptional set of blow up}$$
$$\mathrm{Proj}(\mathcal{F}(I)) = \text{special fiber of blow up at the unique closed point}$$

disjoint ideals

$$\mathrm{Proj}(\mathcal{R}(I_1 \oplus \cdots \oplus I_s)) = \text{seq. of blow ups along } V(I_1), \ldots, V(I_s)$$
$$\mathrm{Proj}(\mathcal{F}(I_1 \oplus \cdots \oplus I_s)) = \text{special fiber of } \cdots$$

② $X \subseteq \mathbb{P}^n_k$, $k = \bar{k}$, $R = A(X)$

- rational maps: $X \overset{\Phi = [f_1 : \cdots : f_s]}{\dashrightarrow} \mathbb{P}^{s-1}_k$, $\deg f_i = \delta \ \forall i$

$$\mathrm{graph}(\Phi) = \mathrm{BiProj}(\mathcal{R}(I)) \qquad I = (f_1, \ldots, f_s)$$
$$\mathrm{im}(\Phi) = \mathrm{Proj}(\mathcal{F}(I))$$

- $\mathrm{Sec}\, X = \mathrm{Proj}(\mathcal{F}(\mathbb{D}))$ $\quad \mathbb{D}$ = diagonal ideal = $\ker(R \otimes_k R \longrightarrow R)$
- $\mathrm{Tan}\, X = \mathrm{Proj}(\mathcal{F}(\Omega_k(R)))$

③ $X \subseteq \mathbb{A}^n_k$, $k = \bar{k}$, $R = A(X)$, $\dim R = d$

Gauss map: $\mathbb{A}^n \supseteq X \overset{\mathcal{N}}{\dashrightarrow} G(d, \mathbb{A}^n_k) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$

$$p \longmapsto T_{X,p}$$

$$\mathrm{graph}(\mathcal{N}) = \mathrm{Proj}(\mathcal{R}(\Lambda^d(\Omega_k(R))))$$
$$\mathrm{im}(\mathcal{N}) = \mathrm{Proj}(\mathcal{F}(\Lambda^d(\Omega_k(R))))$$

Cohen-Macaulay property of blow up algebras of ideals

Assume that R is CM

[Trung-Ikeda, Johnston-Katz

if additional numerical conditions

$\mathcal{R}(I) \cong \bigoplus_{i\geq 0} I^i$ is CM $\quad \Longleftarrow \quad$ (if additional numerical conditions) $\quad \Longrightarrow$ (Huneke, 1982) $\quad \mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{G}(I) = \bigoplus_{i\geq 0} I^i/I^{i+1}$ is CM

[D'Anna, A. Guerrieri, Heinzer]

$\mathcal{F}(I) \cong \bigoplus_{i\geq 0} I^i/mI^i$ is CM

Modules:

$$E^i = [\mathcal{R}(E)]_i \qquad\qquad E^{i+1} \not\subseteq E^i$$

- Multi Rees rings $\mathcal{R}(I_1 \oplus \dots \oplus I_m)$
- little for modules $E \neq I_1 \oplus \dots \oplus I_m$ (small proj dim)
- Very little for fiber cones $\longrightarrow$ $\mathcal{F}(m^{n_1} \oplus \dots m^{n_k})$ — Balkrishnan Jayantan 2018

  $\mathcal{F}(E)$ $\quad$ E finite length or finete colength — Miranda-Neto 2020

Generic Bourbaki ideals and Rees algebras

[Simis-Ulrich-Vasconcelos, 2003]

(R, m, k) Noetherian local ring

E a finite R-module with $\operatorname{rank} E = e > 0$

$U = Ra_1 + \dots + Ra_n \subseteq E$ a submodule

$Z = \{ Z_{ij} \mid 1 \le i \le n,\ 1 \le j \le e-1 \}$

$R' = R[Z]$

$$0 \to \underset{\text{free}}{F''} \to E'' \to \underset{\text{ideal}}{I} \to 0$$

$R'' = R[Z]_{mR[Z]}$ local (ff extension of R)

$E'' = E \otimes_R R''$ $\qquad x_j = \sum_{i=1}^{n} Z_{ij} a_i$ $\qquad F'' = \sum_{j=1}^{e} R'' x_j$

<u>Thm</u>: If E is torsion free and locally free at primes p: $\operatorname{depth} R_p \le 1$,

then $E''/F'' \cong I$ ideal in R''

$\hookrightarrow$ generic Bourbaki ideal of E wrt U

<u>Theorem 1</u> [SUV, 2003]: R Noetherian local, E a finite R-module with rank

$U \subseteq E$ a reduction of E, $I \cong E''/F''$ a generic Bourbaki ideal of E wrt U

Then $\mathcal{R}(E)$ is CM $\iff$ $\mathcal{R}(I)$ is CM

Moreover, in this case $\mathcal{R}(I) \cong \mathcal{R}(E'')/(F'')\mathcal{R}(E'')$ $\rightsquigarrow$ $\mathcal{R}(I)$ is a deformation of $\mathcal{R}(E'')$

and $x_1, \dots, x_{e-1}$ form a regular sequence in $\mathcal{R}(E'')$

Generic Bourbaki ideals and fiber cones

Question: When is $\mathcal{F}(E'')/(F'') \cong \mathcal{F}(I)$?

Theorem 2 [–, 2020]: R Noeth. local, E a finite R-module, $\operatorname{rank} E = e > 0$

$U \subseteq E$ a reduction, $I \cong E''/F''$ a generic Bourbaki ideal of E wrt U.

Assume that ONE of the following conditions hold:

(i) $\operatorname{depth} \mathcal{F}(E) \geq e$

$\operatorname{Sym}(I) \cong \mathcal{R}(I)$

(ii) $\mathcal{R}(I)$ is S_2

(iii) $\operatorname{depth} \mathcal{R}(I_q) \geq 2$ for any $q \in \operatorname{Spec} R''$: I_q not of linear type

Then $\mathcal{F}(E'')/(F'') \cong \mathcal{F}(I)$

Theorem 3 [–, 2020]: R Noeth. local, E a finite R-module, $\operatorname{rank} E = e > 0$

$U \subseteq E$ a reduction, $I \cong E''/F''$ a generic Bourbaki ideal of E wrt U.

(a) $\mathcal{F}(E)$ is CM $\Rightarrow$ $\mathcal{F}(I)$ is CM

(b) If either (ii) or (iii) from Thm 2 are satisfied, then

$\mathcal{F}(I)$ CM $\Rightarrow$ $\mathcal{F}(E)$ CM. ///

Theorem 4 [–, 2020]: (R,m,k) local Gorenstein, $|k|=\infty$, $\dim R = d$

E a finite, torsion-free, orientable R-module, $\operatorname{rank} E = e > 0$, $\ell(E)=\ell$

$g=$ height of a generic Bourbaki ideal of E wrt $U=E$

Assume that:

$e=1$ (i),(ii),(iii) [Johnson-Ulrich, Goto-Nakamura-Nishida] 1996

(i) E is $G_{\ell-e+1}$ $\mu(E_p)\leq \dim R_p$ for all p $\dim R_p \leq \ell - e$

(ii) $r(E)\leq k$ for some integer k with $1\leq k\leq \ell-e$

(iii) $\operatorname{depth} E^j \geq \begin{cases} d-g-j+2 & \text{for } 1\leq j\leq \ell-e-k-g+1 \\ d-\ell+e+k-j & \text{for } \ell-e-k-g+2\leq j\leq k \end{cases}$

(iv) If $g=2$, $\operatorname{Ext}^{j+1}_{R_p}(E^j_p, R_p)=0$ for $\ell-e-k\leq j\leq \ell-e-3$ and for all $p\in \operatorname{Spec} R$ with $\dim R_p = \ell - e$ so that E_p is not free

Then, $\mathcal{R}(E)$ is CM. (Jacobian module of a normal c.i.) $\to$ ideal module

Moreover, $\mathcal{F}(E)$ is CM if also ONE of the following conditions holds:

(a) If $\mu(E)\geq \ell+2$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell-e-g+1\}$.

(b) If $\mu(E)=\ell+1$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \ell-e-g+1$.

$e=1$ Corso, Ghezzi, Polini, Ulrich 2003

Where to go from here

Theorem 5 [Montaño, 2015]: $(R, \mathfrak{m}, k)$ CM local, $|k| = \infty$, $\dim R = d$

I an ideal with $\operatorname{ht} I = g$, $\ell(I) = \ell$, $r(I) = r$.

Assume:

- I satisfies G_ℓ and $AN^-_{\ell-2}$ → good res. int. properties
- $I\mathfrak{m} = J\mathfrak{m}$ for a minimal reduction of I → condition core(I) = int. of all min. red.

Then:

$$\begin{array}{ccc} \mathcal{G}(I) \text{ is CM} & \Longrightarrow & \mathcal{F}(I) \text{ is CM and } a(\mathcal{F}(I)) \leq -g+1 \\ \Updownarrow \text{ if } g \geq 2 & & \Downarrow \\ \mathcal{R}(I) \text{ is CM} & \Longleftarrow & r \leq \ell - g + 1 \end{array}$$

if in addition

depth $I^j \geq d - g - j + 2$ for $1 \leq j \leq k$

Theorem 6 [–, 2019]: $(R, \mathfrak{m}, k)$ local Gorenstein, $|k| = \infty$, $\dim R = d$

E a finite, torsion-free, orientable R-module, $\operatorname{rank} E = e > 0$, $\ell(E) = \ell$

g = height of a generic Bourbaki ideal of E wrt $U = E$

Assume that conditions (i) – (iv) from Thm 4 hold and that

$U\mathfrak{m} = E\mathfrak{m}$ for a minimal reduction U of E.

Then, $\mathcal{F}(E)$ is CM and $a(\mathcal{F}(E)) \leq -e - g + 2$

What's missing: We don't understand core(E). (ongoing work Louiza Fouli Jooyoun Hong)