Math 412. Adventure sheet on Quotient Groups

Fix an arbitrary group (G, \circ) .

DEFINITION: A subgroup N of G is **normal** if for all $g \in G$, the left and right N-cosets gN and Ng are the *same* subsets of G.

NOTATION: If $H \subseteq G$ is any subgroup, then G/H denotes the set of left cosets of H in G. Its elements are sets denoted gH where $g \in G$. The cardinality of G/H is called the **index** of H in G.

DEFINITION/THEOREM 8.13: Let N be a *normal subgroup* of G. Then there is a well-defined binary operation on the set G/N defined as follows:

$$G/N \times G/N \to G/N$$
 $g_1N \star g_2N = (g_1 \circ g_2)N$

making G/N into a group. We call this the **quotient group** "G modulo N".

A. WARMUP: Define the *sign map*:

$$S_n \to \{\pm 1\}$$
 $\sigma \mapsto 1$ if σ is even; $\sigma \mapsto -1$ if σ is odd.

- (1) Prove that sign map is a group homomorphism.
- (2) Use the sign map to give a different proof that A_n is a normal subgroup of S_n for all n.
- (3) Describe the A_n -cosets of S_n . Make a table to describe the quotient group structure S_n/A_n . What is the identity element?

B. OPERATIONS ON COSETS: Let (G, \circ) be a group and let $N \subseteq G$ be a **normal** subgroup.

- (1) Take arbitrary $ng \in Ng$. Prove that there exists $n' \in N$ such that ng = gn'.
- (2) Take any $x \in g_1N$ and any $y \in g_2N$. Prove that $xy \in g_1g_2N$.
- (3) Define a binary operation \star on the set G/N of left N-cosets as follows:

$$G/N \times G/N \to G/N \qquad g_1N \star g_2N = (g_1 \circ g_2)N.$$

Think through the meaning: the elements of G/N are sets and the operation \star combines two of these sets into a third set: how? Explain why the binary operation \star is **well-defined**. Where are you using normality of N?

- (4) Prove that the operation \star in (4) is associative.
- (5) Prove that N is an identity for the operation \star in (4).
- (6) Prove that every coset $gN \in G/N$ has an inverse under the operation \star in (4).
- (7) Conclude that $(G/N, \star)$ is a group.
- (8) Does the set of **right cosets** also have a natural group structure? What is it? Does it differ from G/N?

C. EASY EXAMPLES OF QUOTIENT GROUPS:

- (1) In $(\mathbb{Z}, +)$, explain why $n\mathbb{Z}$ is a normal subgroup and describe the corresponding quotient group.
- (2) For any group G, explain why G is a normal subgroup of itself. What is the quotient G/G?
- (3) For any group G, explain why $\{e\}$ is a normal subgroup of G. What is the quotient $G/\{e\}$?

- D. ANOTHER EXAMPLE. Let $G = \mathbb{Z}_{25}^{\times}$. Let N be the subgroup generated by [7].
 - (1) Give a one-line proof that N is normal.
 - (2) List out the elements of G and of N. Compute the order of both. Compute the index of N in G.
 - (3) List out the elements of G/N; don't forget that each one is a *coset* (in particular, a set whose elements you should list).
 - (4) Give each coset in G/N a reasonable name. Now make a multiplication table for the group G/N, using these names. Is G/N abelian?
- E. THE CANONICAL QUOTIENT MAP: Prove that the map

$$G \to G/N \qquad g \mapsto gN$$

is a group homomorphism. What is its kernel?

F. INDEX TWO. Suppose that H is an index two subgroup of G. Last time, we proved the

THEOREM: Every subgroup of index two in G is normal.

- (1) Describe the quotient group G/H. What are its elements? What is the table?
- (2) Find an example of an index two subgroup of D_n and describe its two cosets explicitly. Make a table for this group and describe the canonical quotient map $G \to G/H$ explicitly.
- G. PRODUCTS AND QUOTIENT GROUPS: Let K and H be arbitrary groups and let $G = K \times H$.
 - (1) Find a natural homomorphism $G \to H$ whose kernel K' is $K \times e_H$.
 - (2) Prove that K' is a normal subgroup of G, whose cosets are all of the form $K \times h$ for $h \in H$.
 - (3) Prove that G/K' is isomorphic to H.
- H. What goes wrong if we try to define a group structure on the set of right cosets G/H where H is a non-normal subgroup of G? Try illustrating the problem with the non-normal subgroup $\langle (1 \ 2) \rangle$ in S_3 .
- I. THE FIRST ISOMORPHISM THEOREM. Conjecture and prove first isomorphism theorem for groups.