Symbolic powers: the Containment Problem and Harbourne's Conjecture

Eloísa Grifo

University of Virginia

Joint Math Meetings 2018

Symbolic Power

The n-th **symbolic power** of a radical ideal I in a regular ring R is

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}(R/I)} I^n R_P \cap R.$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) If I is generated by a regular sequence, then $I^n = I^{(n)}$.

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) If I is generated by a regular sequence, then $I^n = I^{(n)}$.

(3) In general, $I^n \neq I^{(n)}$.

Containment Problem

When is $I^{(b)} \subseteq I^a$?

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a regular ring R and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a regular ring R and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a regular ring R and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

$$\text{In fact, } P^{(3)} \subseteq P^2.$$

Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Harbourne's Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Counterexample (Dumnicki, Szemberg, Tutaj-Gasińska, 2015)

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \not\subseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

Harbourne's Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

When does Harbourne's Conjecture hold?

- For squarefree monomial ideals.
- For general points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).
- For star configurations (Harbourne–Huneke).

Harbourne's Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

When does Harbourne's Conjecture hold?

- For squarefree monomial ideals.
- For general points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).
- For star configurations (Harbourne–Huneke).

Let R be a regular ring of characteristic p, I a radical ideal in R and h the maximal height of a minimal prime of I.

Theorem (G–Huneke)

If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Let R be a regular ring of characteristic p, I a radical ideal in R and h the maximal height of a minimal prime of I.

Theorem (G–Huneke)

If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Eg, determinantal rings, Veronese rings, nice rings of invariants.

Let R be a regular ring of characteristic p, I a radical ideal in R and h the maximal height of a minimal prime of I.

Theorem (G–Huneke)

If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Eg, determinantal rings, Veronese rings, nice rings of invariants.

Theorem (G–Huneke)

If R/I is strongly F-regular, then $I^{((h-1)n-(h-1)+1)} \subseteq I^n$ for all n.

Let R be a regular ring of characteristic p, I a radical ideal in R and h the maximal height of a minimal prime of I.

Theorem (G–Huneke)

If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Eg, determinantal rings, Veronese rings, nice rings of invariants.

Theorem (G–Huneke)

If R/I is strongly F-regular, then $I^{((h-1)n-(h-1)+1)} \subseteq I^n$ for all n.

Corollary

If R/I is strongly F-regular and $h = 2$, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

Evidence for the Stable Harbourne Conjecture

Let $a \geqslant 3$, k be a field, $R = k[x, y, z]$, and

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) .$$

This is a well-known counterexample to $I^{(3)} \subseteq I^2$. However, work of Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska implies that

$$I^{(2n-1)} \subseteq I^n$$

for all $n \geqslant 3$.

Stable Harbourne Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \gg 0$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Question

If there exists a value of n such that

$$I^{(hn-h+1)} \subseteq I^n,$$

does that imply that

$$I^{(hm-h+1)} \subseteq I^m$$

for all $m \gg 0$?

Theorem (–)

Let I be a radical ideal in a regular ring containing a field, and let h be the maximal height of a minimal prime of I. If there exists a value of n such that

$$I^{(hn-h)} \subseteq I^n,$$

then

$$I^{(hm-h)} \subseteq I^m$$

for all $m \gg 0$.

Theorem (–)

Let I be a radical ideal in a regular ring containing a field, and let h be the maximal height of a minimal prime of I. If there exists a value of n such that

$$I^{(hn-h)} \subseteq I^n,$$

then

$$I^{(hm-h)} \subseteq I^m,$$

for all $m \gg 0$.

Example

The defining ideal of $k[t^3, t^4, t^5]$ in $k[x, y, z]$ verifies $P^{(2\times 3-2=4)} \subseteq P^3$, and thus $P^{(2m-2)} \subseteq P^m$ for all $m \gg 0$.

Definition (Bocci-Harbourne)

The resurgence of an ideal I is given by

$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^{b}\right\}.$$

$$1 \leqslant \rho(I) \leqslant h.$$

If $\frac{a}{b} > \rho(I)$, then $I^{(a)} \subseteq I^{b}$.

Definition (Bocci-Harbourne)

The resurgence of an ideal I is given by

$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^{b}\right\}.$$

$$1 \leqslant \rho(I) \leqslant h.$$

If $\frac{a}{b} > \rho(I)$, then $I^{(a)} \subseteq I^{b}$.

Definition (Bocci-Harbourne)

The resurgence of an ideal I is given by

$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\right\}.$$

$$1 \leqslant \rho(I) \leqslant h.$$

If $\frac{a}{b} > \rho(I)$, then $I^{(a)} \subseteq I^b$.

Observation

Let I is a radical ideal, and h be the maximal height of a minimal prime of I. If $\rho(I) < h$, then for every constant $C > 0$,

$$I^{(hn-C)} \subseteq I^n$$

for all $n \gg 0$.

Question

Is there an ideal I with $\rho(I) = h$?

Observation

Let I is a radical ideal, and h be the maximal height of a minimal prime of I. If $\rho(I) < h$, then for every constant $C > 0$,

$$I^{(hn-C)} \subseteq I^n$$

for all $n \gg 0$.

Question

Is there an ideal I with $\rho(I) = h$?

Harbourne's Conjecture for primes

Let P be a prime ideal in a regular ring of height h. For all $n \geqslant 1$,

$$P^{(hn-h+1)} \subseteq P^n.$$

Harbourne's Conjecture for primes of height 2

Let P be a prime ideal of height 2 in a regular ring. For all $n \geqslant 1$,

$$P^{(2n-1)} \subseteq P^n.$$

Huneke's Question

If P is a prime of height 2, is $P^{(3)} \subseteq P^2$?

Theorem (–)

Let k be a field of characteristic not 3, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2.$$

Monomial space curves

Let k be a field. The kernel of the map

$$k[x, y, z] \longrightarrow k[t^a, t^b, t^c] \subseteq k[t]$$

is a prime ideal of height 2, generated by the maximal minors of

$$\begin{bmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{bmatrix}.$$

Theorem

Let k be a field of characteristic not 3, and $I \subseteq k[x,y,z]$ be the height 2 ideal generated by the maximal minors of

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

If $a_1 | b_2$, then $I^{(3)} \subseteq I^2$.

Theorem (–)

Let k be a field of characteristic not 2, 3 or 5, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2 \text{ and } P^{(5)} \subseteq P^3.$$

Ingredients in the proof (following work of Seceleanu)

- $I^{(a)} \subseteq I^b$ if and only if the map induced by $I^a \subseteq I^b$ on *Ext* $\operatorname{Ext}^2(R/I^b, R) \longrightarrow \operatorname{Ext}^2(R/I^a, R)$, is the 0 map.
- Use Rees Algebra techniques to find the resolutions of I^n.

Theorem (–)

Let k be a field of characteristic not 2 nor 3, let $a = 3$ or 4, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(4)} \subseteq P^3.$$

As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The defining ideal P of $k[t^9, t^{11}, t^{14}]$ fails the containment $P^{(4)} \subseteq P^3$, but Macaulay2 computations show that

$$P^{(2\times 4-2=6)} \subseteq P^4,$$

so $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Thank you!