

# Symbolic powers

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# Warning!

These are notes for a topics course on symbolic powers I am teaching at the University of Nebraska — Lincoln in Spring 2022. These are under construction and are guaranteed to contain typos. Please let me know if you find any typos or errors, no matter how small.

## Assumptions

Throughout, all rings are commutative with identity. Moreover, we will almost always assume our rings are Noetherian, so we settle on the convention that rings are Noetherian unless we say otherwise.

We will assume the reader has some knowledge of elementary commutative algebra, as in [AM69], [Mat80], [Mat89], [BH93], or [Eis95]. Any elementary commutative algebra details or proofs we skip can be found in the references above.

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# Chapter 1

## Primary Decomposition

One of the first theorems one learns in abstract algebra is the Fundamental Theorem of Arithmetic:

**Theorem 1.1.** *Every integer can be written as a product of primes, which is unique up to sign and the order of the factors. More precisely, for every  $n \in \mathbb{Z}$  there exist primes  $p_1, \dots, p_k \in \mathbb{Z}$  and integers  $a_1, \dots, a_k > 0$  such that*

$$n = \pm p_1^{a_1} \cdots p_k^{a_k}.$$

Under the philosophy that general rings are modeled after the integers, we might expect such a theorem in any ring. But what would a general version of this theorem look like? A first guess would perhaps be that every element in any reasonable ring should be a product of irreducible elements, unique up to the order of the factors and up to multiplication by a unit. But this fails easily.

**Example 1.2.** In  $\mathbb{Z}[\sqrt{-5}]$ ,

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two *different* ways to write 6 as a product of irreducible elements: there is no way to obtain 2 or 3 from  $1 + \sqrt{-5}$  or  $1 - \sqrt{-5}$  by multiplying by a unit.

The problem here is that we are focusing on *elements* when we ought to focus on *ideals*. Instead of writing *elements* as products of irreducibles, we will write *ideals* in terms of *primary ideals*.

But before we discuss primary decomposition, we will need to recall some elementary facts about minimal primes and associated primes; the details and proofs we skip can be found in any standard introduction to commutative algebra, such as [Eis95], [AM69], or [BH93].

### 1.1 Minimal primes

**Definition 1.3.** Given a ring  $R$ , the set of prime ideals in  $R$  is denoted  $\text{Spec}(R)$ , and called the **spectrum** of  $R$ . Given an ideal  $I$ , the set of all primes that contain  $I$  is denoted  $V(I)$ .

**Definition 1.4.** Let  $I$  be an ideal in a ring  $R$ . A **minimal prime** of  $I$  is a minimal element (with respect to containment) in  $V(I)$ . More precisely,  $P$  is a minimal prime of  $I$  if the following hold:

- $P$  is a prime ideal,
- $P \supseteq I$ , and
- if  $Q$  is also a prime ideal and  $I \subseteq Q \subseteq P$ , then  $Q = P$ .

The set of minimal primes of  $I$  is denoted  $\text{Min}(I)$ .

**Example 1.5.** Let  $k$  be a field and  $R = k[x, y]$ . Every prime containing  $I = (x^2, xy)$  must contain  $x^2$ , and thus  $x$ . On the other hand,  $(x)$  is a prime ideal containing  $I$ . Therefore,  $I$  has a unique minimal prime, and  $\text{Min}(I) = \{(x)\}$ .

**Remark 1.6.** If  $\mathfrak{p}$  is prime, then  $\text{Min}(\mathfrak{p}) = \{\mathfrak{p}\}$ .

Minimal primes are closely associated to radicals.

**Definition 1.7.** The **radical** of an ideal  $I$  in a ring  $R$  is the ideal

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n\}.$$

An ideal is a **radical ideal** if  $I = \sqrt{I}$ .

**Macaulay2.** Given an ideal  $I$ , `radical I` returns the radical of  $I$ .

**Example 1.8.** Prime ideals are radical.

**Example 1.9.** The ideal  $I = (x^2, xy)$  in the ring  $R = k[x, y]$  from Example 1.5 is not radical, since it contains  $x^2$  but not  $x$ .

**Theorem 1.10.** Let  $R$  be a ring, and  $I$  be an ideal. Then

$$\sqrt{I} = \bigcap_{P \in V(I)} P = \bigcap_{P \in \text{Min}(I)} P.$$

**Example 1.11.** The radical of the ideal  $I = (x^2, xy)$  in the ring  $R = k[x, y]$  from Example 1.5 is  $\sqrt{I} = (x)$ . We saw before that  $\text{Min}(I) = \{(x)\}$ .

**Example 1.12.** In  $R = k[x, y]$ , the ideal  $I = (xy, xz) = (x) \cap (y, z)$  is radical, and its minimal primes are  $(x)$  and  $(y, z)$ .

Over a noetherian ring, we can realize every radical ideal as the intersection of finitely many primes.

**Theorem 1.13.** Over any noetherian ring, any ideal  $I$  has finitely many minimal primes, and thus  $\sqrt{I}$  is a finite intersection of primes.

**Macaulay2.** The method `minimalPrimes` receives an ideal and returns a list of its minimal primes.

**Example 1.14.** The nilpotent elements of a ring  $R$  are exactly the elements in the radical of  $(0)$ . By Theorem 1.10,  $\sqrt{(0)}$  is the intersection of the minimal primes of  $(0)$ , or equivalently, the intersection of the minimal elements among all the primes in  $R$ . The radical of  $(0)$  is often called the **nilradical** of  $R$ , denoted  $\mathcal{N}(R)$ .

**Example 1.15.** Let  $k$  be a field. The nilradical of  $R = k[x, y]/(x^2, xy)$  corresponds to the radical of  $(x^2, xy)$  is  $k[x, y]$ , so it is the ideal  $(x)/(x^2, xy)$ .

**Remark 1.16.** Since  $I \subseteq \sqrt{I}$ , any prime containing  $\sqrt{I}$  must also contain  $I$ . On the other hand, if  $P$  is a prime containing  $I$ , and  $f \in \sqrt{I}$ , then for some  $n$  we have  $f^n \in I \subseteq P$ , and since  $P$  is prime we must have  $f \in P$ . This shows that  $V(I) = V(\sqrt{I})$ , and therefore  $\text{Min}(I) = \text{Min}(\sqrt{I})$ .

**Lemma 1.17.** If  $\text{Min}(I) = \{P_1, \dots, P_n\}$ , no  $P_i$  can be deleted in the intersection  $P_1 \cap \dots \cap P_n$ .

*Proof.* Suppose that we can delete  $P_i$ , meaning that

$$\bigcap_{j=1}^n P_j = \bigcap_{j \neq i} P_j.$$

Then

$$P_i \supseteq \bigcap_{j=1}^n P_j = \bigcap_{j \neq i} P_j \supseteq \prod_{j \neq i} P_j.$$

Since  $P_i$  is prime, this implies that  $P_i \supseteq P_j$  for some  $j \neq i$ , but this contradicts the assumption that the primes are incomparable.  $\square$

**Lemma 1.18.** Let  $I$  be an ideal in  $R$ . If  $I = P_1 \cap \dots \cap P_n$  where each  $P_i$  is prime and  $P_i \not\subseteq P_j$  for each  $i \neq j$ , then  $\text{Min}(I) = \{P_1, \dots, P_n\}$ . Moreover,  $I$  must be radical.

*Proof.* If  $Q$  is a prime containing  $I$ , then  $Q \supseteq (P_1 \cap \dots \cap P_n)$ . We claim that  $Q$  must contain one of the  $P_i$ . Indeed, if  $Q \not\supseteq P_i$  for all  $i$ , then there are elements  $f_i \in P_i$  such that  $f_i \notin Q$ , so their product satisfies  $f_1 \cdots f_n \in (P_1 \cap \dots \cap P_n)$  but  $f_1 \cdots f_n \notin Q$ . This is a contradiction, so indeed any prime containing  $I$  must contain some  $P_i$ . Therefore, any minimal prime of  $I$  must be one of the  $P_i$ . Since we assumed that the  $P_i$  are incomparable, these are exactly all the minimal primes of  $I$ .

By assumption,  $I$  coincides with the intersection of its minimal primes, and by Theorem 1.10 this intersection is  $\sqrt{I}$ . Therefore,  $I = \sqrt{I}$ .  $\square$

**Remark 1.19.** If  $I = P_1 \cap \dots \cap P_n$  for some primes  $P_i$ , we can always delete unnecessary components until no component can be deleted. Therefore,  $\text{Min}(I) \subseteq \{P_1, \dots, P_n\}$ .

As a consequence of Lemma 1.17 and Lemma 1.18, if  $I$  is a radical ideal, there is a unique way to write  $I$  as a finite intersection of incomparable prime ideals. Moreover, any ideal that can be written as a finite intersection of prime ideals is radical.

## 1.2 Localization

**Definition 1.20.** A **multiplicative closed** set  $W$  in a ring  $R$  is a subset  $W \subseteq R$  containing 1 and closed for products, meaning  $a, b \in W \implies ab \in W$ .

Given a multiplicative closed subset  $W$ , we can form a new ring, the **localization** of  $R$  at  $W$ , which is a ring with elements

$$W^{-1}R := \left\{ \frac{r}{w} \mid r \in R, w \in W \right\} / \sim$$

where  $\sim$  is the equivalence relation

$$\frac{r}{w} \sim \frac{r'}{w'} \text{ if there exists } u \in W : u(rw' - r'w) = 0.$$

The operations are given by

$$\frac{r}{v} + \frac{s}{w} = \frac{rw + sv}{vw} \quad \text{and} \quad \frac{r}{v} \frac{s}{w} = \frac{rs}{vw}.$$

We write elements in  $W^{-1}R$  in the form  $\frac{r}{w}$ , though they are equivalence classes of such expressions. The zero in  $W^{-1}R$  is  $\frac{0}{1}$  and the identity is  $\frac{1}{1}$ . There is a canonical ring homomorphism

$$\begin{aligned} R &\longrightarrow W^{-1}R. \\ r &\longmapsto \frac{r}{1} \end{aligned}$$

We can also localize modules. Given a module  $M$ , the **localization** of  $M$  at  $W$  is a module over the ring  $W^{-1}R$ , given by

$$W^{-1}M := \left\{ \frac{m}{w} \mid m \in M, w \in W \right\} / \sim$$

where  $\sim$  is the equivalence relation  $\frac{m}{w} \sim \frac{m'}{w'}$  if  $u(mw' - m'w) = 0$  for some  $u \in W$ . The operations are given by

$$\frac{m}{v} + \frac{n}{w} = \frac{mw + nv}{vw} \quad \text{and} \quad \frac{r}{v} \frac{m}{w} = \frac{rm}{vw}.$$

**Remark 1.21.** If  $\alpha : M \rightarrow N$  is an  $R$ -module homomorphism, then there is a  $W^{-1}R$ -module homomorphism  $W^{-1}\alpha : W^{-1}M \rightarrow W^{-1}N$  given by the rule  $W^{-1}\alpha(\frac{m}{w}) = \frac{\alpha(m)}{w}$ .

One thing to keep in mind is that things might unexpectedly become zero in a localization.

**Definition 1.22.** The **annihilator** of an  $R$ -module  $M$  is the ideal

$$\text{ann}(M) := \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

Given ideals  $I$  and  $J$  in  $R$ , the **colon** of  $I$  and  $J$  is the ideal

$$(J : I) := \{r \in R \mid rI \subseteq J\}.$$

More generally, if  $M$  and  $N$  are submodules of some  $R$ -module  $A$ , the colon of  $N$  and  $M$  is

$$(N :_R M) := \{r \in R \mid rM \subseteq N\}.$$



The annihilator of  $M$  is an ideal in  $R$ , and  $\text{ann}(M) = (0 :_R M)$ . Moreover, any colon  $(N :_R M)$  is an ideal in  $R$ .

**Macaulay2.** The methods `ann` and `annihilator` are equivalent, and both receive a module  $M$  or an element  $f$  in a module  $M$  and returns  $\text{ann}(M)$  or  $\text{ann}(f)$ . Given two  $R$ -modules  $M$  and  $N$  or two ideals  $I$  and  $J$  in  $R$ , `M : N` and `I : J` return the corresponding colon ideals.

**Remark 1.23.** Given ideals  $I$  and  $J$  in  $R$ ,  $(I : J) = R$  if and only if  $J \subseteq I$ . Moreover, note that we always have  $(I : J) \supseteq I$ .

**Lemma 1.24.** *Given an  $R$ -module  $M$ ,*

$$\frac{m}{w} \in W^{-1}M \text{ is zero} \iff vm = 0 \text{ for some } v \in W \iff \text{ann}_R(m) \cap W \neq \emptyset.$$

*Proof.* For the first equivalence, we use the equivalence relation defining  $W^{-1}R$  to note that  $\frac{m}{w} = \frac{0}{1}$  in  $W^{-1}M$  if and only if there exists some  $v \in W$  such that  $0 = v(1m - 0w) = vm$ . The second equivalence just comes from the definition of the annihilator.  $\square$

**Remark 1.25.** As a corollary to Lemma 1.24, we obtain that if  $W$  is a multiplicatively closed set in  $R$  and  $W$  does not contain any zerodivisors, then the localization map  $R \rightarrow W^{-1}R$  is injective, since no element in  $W$  can be in the annihilator of a nonzero element.

**Notation 1.26.** Let  $f : R \rightarrow S$  be a ring homomorphism. Given an ideal  $I$  in  $R$ , the **expansion** of  $I$  to  $S$  is the ideal of  $S$  generated by  $f(I)$ , and we denote it by  $IS$ . Given an ideal  $J$  of  $S$ , the **contraction** of  $J$  to  $R$  is the preimage of  $J$  via  $f$ , which we denote by  $J \cap R$ :

$$J \cap R := f^{-1}(J) = \{r \in R \mid f(r) \in J\}.$$

There is much to say about expansion and contraction of ideals, the details of which can be found in any standard commutative algebra reference such as [Mat89] or [AM69]. Later on, it will be helpful to know the following facts:

- For any ideals  $I$  in  $R$  and  $J$  in  $S$ , we have  $(IS \cap R)S = IS$  and  $(J \cap R)S \cap R = J \cap R$ .
- The contraction of a prime ideal is prime.
- The expansion of a prime ideal might fail to be prime.
- A prime ideal  $P$  in  $R$  is the contraction of a prime in  $S$  if and only if  $PS \cap R = P$ .

We will see contractions of ideals most often when talking about localizations. Given a multiplicatively closed set  $W$  in a ring  $R$  and an ideal  $I$  in  $W^{-1}R$ , we write  $I \cap R$  for the **contraction** of  $I$  into  $R$ , which is the preimage of  $I$  in  $R$  via the canonical map  $R \rightarrow W^{-1}R$ . More precisely,

$$I \cap R = \left\{ r \in R \mid \frac{r}{1} \in I \right\}.$$

**Theorem 1.27.** *Let  $R$  be a ring and  $W$  be a multiplicatively closed set in  $R$ .*

- a) *If  $I$  is an ideal in  $R$ , then  $W^{-1}I \cap R = \{r \in R \mid wr \in I \text{ for some } w \in W\}$ .*

- b) If  $J$  is an ideal in  $W^{-1}R$ , then  $W^{-1}(J \cap R) = J$ .
- c) If  $P$  is a prime ideal and  $W \cap P = \emptyset$ , then  $W^{-1}P = P(W^{-1}R)$  is prime.
- d) The set of prime ideals in  $W^{-1}R$  is in bijection with

$$\{P \in \text{Spec}(R) \mid P \cap W = \emptyset\}.$$

**Theorem 1.28.** *Localization is exact: given a short exact sequence of  $R$ -modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*and a multiplicative set  $W$ , the sequence*

$$0 \longrightarrow W^{-1}A \longrightarrow W^{-1}B \longrightarrow W^{-1}C \longrightarrow 0$$

*is also exact.*

The most important multiplicatively closed sets we will do localization on are the complements of prime ideals. As an immediate consequence of the definition of prime ideal, we get that  $W = R \setminus P$  is a multiplicatively closed set.

**Definition 1.29.** Let  $R$  be a ring and  $P$  be a prime ideal in  $R$ . We write  $R_P$  for the localization  $(R \setminus P)^{-1}R$ , and call it the **localization of  $R$  at  $P$** . Given an ideal  $I$  in  $R$ , we write  $I_P$  instead of  $(R \setminus P)^{-1}I$ , the ideal in  $R_P$  generated by all the images of elements in  $I$ . This ideal  $I_P$  is the same as  $IR_P$ , the expansion of  $I$  to  $R_P$ .

The ring  $R_P$  is a local ring with unique maximal ideal  $P_P$ . The proper ideals in  $R_P$  are of the form  $I_P$ , where  $I \subseteq P$  is an ideal in  $R$ , and the primes in  $R_P$  are precisely the ideals of the form  $Q_P$  with  $Q \subseteq P$  prime.

We can also localize at the complement of a union of primes.

**Remark 1.30.** Let  $A$  be a finite set of prime ideals in a noetherian ring  $R$ , and consider the set

$$W := \{r \in R \mid r \notin \bigcup_{P \in A} P\}.$$

If  $x, y \in W$ , then  $x, y \notin P$  for any of the primes  $P \in A$ , and thus  $xy \in W$ . Moreover,  $1 \notin P$  for any  $P \in A$ , so  $1 \in W$ . We conclude that  $W$  is a multiplicatively closed set. The localization  $W^{-1}R$  is now a semilocal ring, which means it has only finitely many maximal ideals — the maximal elements in  $A$ .

**Definition 1.31.** If  $M$  is an  $R$ -module, the **support** of  $M$  is

$$\text{Supp}(M) := \{\mathfrak{p} \in \text{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

**Example 1.32.** If  $M = R/I$ , then  $\text{Supp}(M) = V(I)$ . Indeed,  $M_{\mathfrak{p}}$  is generated by the image of 1, so  $M_{\mathfrak{p}} = 0$  iff the image of 1 is zero in the localization. But this happens if and only if

$$\exists w \notin \mathfrak{p} : w \cdot 1 = 0 \text{ in } R/I \Leftrightarrow \exists w \notin \mathfrak{p}, w \in I \Leftrightarrow \mathfrak{p} \not\supseteq I.$$

**Proposition 1.33.** *Given  $M$  a finitely generated  $R$ -module over a ring  $R$ ,*

$$\mathrm{Supp}(M) = V(\mathrm{ann}_R(M)).$$

*In particular,  $\mathrm{Supp}(R/I) = V(I)$ .*

*Proof.* Let  $M = Rm_1 + \cdots + Rm_n$ . We have

$$\mathrm{ann}_R(M) = \bigcap_{i=1}^n \mathrm{ann}_R(m_i),$$

so

$$V(\mathrm{ann}_R(M)) = \bigcup_{i=1}^n V(\mathrm{ann}_R(m_i)).$$

Notice that we need finiteness here. Also, we claim that

$$\mathrm{Supp}(M) = \bigcup_{i=1}^n \mathrm{Supp}(Rm_i).$$

To show  $(\supseteq)$ , notice that  $(Rm_i)_{\mathfrak{p}} \subseteq M_{\mathfrak{p}}$ , so

$$\mathfrak{p} \in \mathrm{Supp}(Rm_i) \implies 0 \neq (Rm_i)_{\mathfrak{p}} \subseteq M_{\mathfrak{p}} \implies \mathfrak{p} \in \mathrm{Supp}(M).$$

On the other hand, the images of  $m_1, \dots, m_n$  in  $M_{\mathfrak{p}}$  generate  $M_{\mathfrak{p}}$  for each  $\mathfrak{p}$ , so  $\mathfrak{p} \in \mathrm{Supp}(M)$  if and only if  $\mathfrak{p} \in \mathrm{Supp}(Rm_i)$  for some  $m_i$ . Thus, we can reduce to the case of a cyclic module  $Rm$ . Now  $\frac{m}{1} = 0$  in  $M_{\mathfrak{p}}$  if and only if  $(R \setminus \mathfrak{p}) \cap \mathrm{ann}_R(m) \neq \emptyset$ , which happens if and only if  $\mathrm{ann}_R(m) \not\subseteq \mathfrak{p}$ .  $\square$

## 1.3 Associated primes

**Remark 1.34.** Let  $R$  be a ring,  $I$  be an ideal in  $R$ , and  $M$  be an  $R$ -module. To give an  $R$ -module homomorphism  $R \rightarrow M$  is the same as choosing an element  $m$  of  $M$  (the image of 1 via our map) or equivalently, to choose a cyclic submodule of  $M$  (the submodule generated by  $m$ ).

To give an  $R$ -module homomorphism  $R/I \rightarrow M$  is the same as giving an  $R$ -module homomorphism  $R \rightarrow M$  whose image is killed by  $I$ . Thus giving an  $R$ -module homomorphism  $R/I \rightarrow M$  is to choose an element  $m \in M$  that is killed by  $I$ , meaning  $I \subseteq \mathrm{ann}(m)$ . The kernel of the map  $R \rightarrow M$  given by  $1 \mapsto m$  is precisely  $\mathrm{ann}(m)$ , so a well-defined map  $R/I \rightarrow M$  given by  $1 \mapsto m$  is injective if and only if  $I = \mathrm{ann}(m)$ .

**Definition 1.35.** Let  $R$  be a ring, and  $M$  a module. We say that  $P \in \mathrm{Spec}(R)$  is an **associated prime** of  $M$  if  $P = \mathrm{ann}_R(m)$  for some  $m \in M$ . Equivalently,  $P$  is associated to  $M$  if there is an injective homomorphism  $R/P \rightarrow M$ . We write  $\mathrm{Ass}_R(M)$  for the set of associated primes of  $M$ .

We will be primarily interested in the associated primes of  $R/I$ , where  $I$  is an ideal in  $R$ . We will abuse notation and call these the **associated primes of  $I$** , and even write  $\mathrm{Ass}_R(I)$ .

**Lemma 1.36.** *Let  $R$  be a noetherian ring and  $M$  be an  $R$ -module. A prime  $P$  is associated to  $M$  if and only if  $P_P \in \text{Ass}(M_P)$ .*

*Proof.* By Theorem 1.28, localization is exact, so any inclusion  $R/P \subseteq M$  gives an inclusion  $R_P/P_P \subseteq M_P$ . Conversely, let  $P_P = \text{ann}(\frac{m}{w})$  for some  $\frac{m}{w} \in M_P$ . Let  $P = (f_1, \dots, f_n)$ . Since  $\frac{f_i}{1} \frac{m}{w} = \frac{0}{1}$ , there exists  $u_i \notin P$  such that  $u_i f_i m = 0$ . Then  $u = u_1 \cdots u_n$  is not in  $P$ , since  $P$  is prime, and  $u f_i m = 0$  for all  $i$ . Since the  $f_i$  generate  $P$ , we have  $P(um) = 0$ . On the other hand, if  $r \in \text{ann}(um)$ , then  $\frac{ru}{1} \in \text{ann}(\frac{m}{w}) = P_P$ . We conclude that  $ru \in P_P \cap R = P$ . Since  $u \notin P$ , we conclude that  $r \in P$ .  $\square$

**Lemma 1.37.** *If  $P$  is prime, then  $\text{Ass}_R(R/P) = \{P\}$ .*

*Proof.* An element  $r + P \in R/P$  is nonzero if and only if  $r \notin P$ . Given any nonzero  $r + P \in R/P$  we have  $\text{ann}_R(r + P) = \{s \in R \mid rs \in P\} = P$  by definition of prime ideal.  $\square$

Soon we will see that the converse does not hold.

**Definition 1.38.** Let  $M$  be an  $R$ -module. An element  $r \in R$  is a **zerodivisor** on  $M$  if  $rm = 0$  for some  $m \in M$ . We sometimes write the set of zerodivisors of  $M$  as  $\mathcal{Z}(M)$ .

**Theorem 1.39.** *If  $R$  is Noetherian, and  $M$  is an arbitrary  $R$ -module, then the following hold:*

- 1) *For any nonzero  $m \in M$ ,  $\text{ann}_R(m)$  is contained in an associated prime of  $M$ .*
- 2)  $\text{Ass}(M) = \emptyset \iff M = 0$ .
- 3)  $\bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p} = \mathcal{Z}(M)$ .

*Proof.* Even if  $R$  is not Noetherian,  $M = 0$  implies  $\text{Ass}(M) = \emptyset$  by definition. So we focus on the case when  $M \neq 0$ .

First, we are going to show that 1) implies 2) and 3). To do that, let's suppose that we have shown 1). If  $M \neq 0$ , then  $M$  contains a nonzero element  $m$ , and  $\text{ann}(m)$  is contained in an associated prime of  $M$ . In particular,  $\text{Ass}(M) \neq \emptyset$ , and 2) holds. Now if  $r \in \mathcal{Z}(M)$ , then by definition we have  $r \in \text{ann}(m)$  for some nonzero  $m \in M$ . Since  $\text{ann}(m)$  is contained in some associated prime of  $M$ , so is  $r$ . On the other hand, if  $\mathfrak{p}$  is an associated prime of  $M$ , then by definition all elements in  $\mathfrak{p}$  are zerodivisors on  $M$ . This shows that 3) holds. So all that is left is to prove 1).

Now we show 1) for any  $M \neq 0$ . The set of ideals  $S := \{\text{ann}_R(m) \mid m \in M, m \neq 0\}$  is nonempty, and any element in  $S$  is contained in a maximal element, by Noetherianity. Note in fact that any element in  $S$  must be contained in a maximal element of  $S$ . Let  $I = \text{ann}(m)$  be any maximal element, and let  $rs \in I$ ,  $s \notin I$ . We always have  $\text{ann}(sm) \supseteq \text{ann}(m)$ , and equality holds by the maximality of  $\text{ann}(m)$  in  $S$ . Then  $r(sm) = (rs)m = 0$ , so  $r \in \text{ann}(sm) = \text{ann}(m) = I$ . We conclude that  $I$  is prime, and therefore it is an associated prime of  $M$ .  $\square$

**Lemma 1.40.** *If*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

*is an exact sequence of  $R$ -modules, then  $\text{Ass}(L) \subseteq \text{Ass}(M) \subseteq \text{Ass}(L) \cup \text{Ass}(N)$ .*

*Proof.* If  $R/P$  includes in  $L$ , then composition with the inclusion  $L \hookrightarrow M$  gives an inclusion  $R/P \hookrightarrow M$ . This shows that  $\text{Ass}(L) \subseteq \text{Ass}(M)$ .

Now let  $\mathfrak{p} \in \text{Ass}(M)$ , and let  $m \in M$  be such that  $\mathfrak{p} = \text{ann}(m)$ . First, note that  $\mathfrak{p} \subseteq \text{ann}(rm)$  for all  $r \in R$ .

Thinking of  $L$  as a submodule of  $N$ , suppose that there exists  $r \notin \mathfrak{p}$  such that  $rm \in L$ . Then

$$s(rm) = 0 \iff (sr)m = 0 \implies sr \in \mathfrak{p} \implies s \in \mathfrak{p}.$$

So  $\mathfrak{p} = \text{ann}(rm)$ , and thus  $\mathfrak{p} \in \text{Ass}(L)$ .

If  $rm \notin L$  for all  $r \notin \mathfrak{p}$ , let  $n$  be the image of  $m$  in  $N$ . Thinking of  $N$  as  $M/L$ , if  $rn = 0$ , then we must have  $rm \in L$ , and by assumption this implies  $r \in \mathfrak{p}$ . Since  $\mathfrak{p} = \text{ann}(m) \subseteq \text{ann}(n)$ , we conclude that  $\mathfrak{p} = \text{ann}(n)$ . Therefore,  $\mathfrak{p} \in \text{Ass}(N)$ .  $\square$

Note that the inclusions in Lemma 1.40 are not necessarily equalities.

**Example 1.41.** If  $M$  is a module with at least two associated primes, and  $P$  is an associated prime of  $M$ , then

$$0 \longrightarrow R/P \longrightarrow M$$

is exact, but  $\{P\} = \text{Ass}(R/P) \subsetneq \text{Ass}(M)$ .

**Example 1.42.** Let  $R = k[x]$ , where  $k$  is a field, and consider the short exact sequence of  $R$ -modules

$$0 \longrightarrow (x) \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

Then one can check that:

- $\text{Ass}(R/(x)) = \text{Ass}(k) = \{(x)\}$ .
- $\text{Ass}(R) = \text{Ass}((x)) = \{(0)\}$ .

In particular,  $\text{Ass}(R) \subsetneq \text{Ass}(R/(x)) \cup \text{Ass}((x))$ .

**Corollary 1.43.** Let  $A$  and  $B$  be  $R$ -modules. Then  $\text{Ass}(A \oplus B) = \text{Ass}(A) \cup \text{Ass}(B)$ .

*Proof.* Apply Lemma 1.40 to the short exact sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0.$$

This gives us  $\text{Ass}(A) \subseteq \text{Ass}(A \oplus B) \subseteq \text{Ass}(A) \cup \text{Ass}(B)$ . Repeating with

$$0 \longrightarrow B \longrightarrow A \oplus B \longrightarrow A \longrightarrow 0,$$

we conclude that  $\text{Ass}(B) \subseteq \text{Ass}(A \oplus B)$ . So we have shown both  $\text{Ass}(A) \subseteq \text{Ass}(A \oplus B)$  and  $\text{Ass}(B) \subseteq \text{Ass}(A \oplus B)$ , so  $\text{Ass}(A) \cup \text{Ass}(B) \subseteq \text{Ass}(A \oplus B)$ . Since we have also already shown  $\text{Ass}(A \oplus B) \subseteq \text{Ass}(A) \cup \text{Ass}(B)$ , we must have  $\text{Ass}(A \oplus B) = \text{Ass}(A) \cup \text{Ass}(B)$ .  $\square$

**Theorem 1.44.** If  $R$  is a noetherian ring, and  $M$  is a finitely generated  $R$ -module, then  $\text{Ass}_R(M)$  is finite. Moreover, if  $R$  and  $M$  are graded, then  $\text{Ass}_R(M)$  is a finite set of homogeneous primes.

In a Noetherian ring, associated primes localize.

**Theorem 1.45.** *Let  $R$  be a Noetherian ring,  $W$  a multiplicative set, and  $M$  a module. Then*

$$\text{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}P \mid P \in \text{Ass}_R(M), P \cap W = \emptyset\}.$$

*Proof.* Given  $P \in \text{Ass}_R(M)$  such that  $P \cap W = \emptyset$ , Theorem 1.27 says that  $W^{-1}P$  is a prime in  $W^{-1}R$ . Since  $P \in \text{Ass}(M)$ , we have an inclusion map  $R/P \subseteq M$ . Localization is exact, by Theorem 1.28, so we get an inclusion  $W^{-1}R/W^{-1}P \cong W^{-1}(R/P) \subseteq W^{-1}M$ , which says that  $W^{-1}P$  is an associated prime of  $W^{-1}M$ .

Now suppose that  $Q \in \text{Spec}(W^{-1}R)$  is associated to  $W^{-1}M$ . By Theorem 1.27, we know this is of the form  $W^{-1}P$  for some prime  $P$  in  $R$  such that  $P \cap W = \emptyset$ , so we just need to prove that  $P \in \text{Ass}(M)$ . Since  $R$  is noetherian,  $P$  is finitely generated, say  $P = (f_1, \dots, f_n)$  in  $R$ , and so  $Q = (\frac{f_1}{1}, \dots, \frac{f_n}{1})$ .

By assumption,  $Q = \text{ann}(\frac{m}{w})$  for some  $m \in M$ ,  $w \in W$ . Since  $w$  is a unit in  $W^{-1}R$ , we can also write  $Q = \text{ann}(\frac{m}{1})$ . By definition, this means that for each  $i$

$$\frac{f_i}{1} \frac{m}{1} = \frac{0}{1} \iff u_i f_i m = 0 \text{ for some } u_i \in W.$$

Let  $u = u_1 \cdots u_n \in W$ . Then  $u f_i m = 0$  for all  $i$ , and thus  $Pum = 0$ . We claim that in fact  $P = \text{ann}(um)$  in  $R$ . Given any  $v \in \text{ann}(um)$ , we have  $u(vm) = v(um) = 0$ , and since  $u \in W$ , this implies that  $\frac{vm}{1} = 0$ . Therefore,  $\frac{v}{1} \in \text{ann}(\frac{m}{1}) = W^{-1}\mathfrak{p}$ , and  $vw \in P$  for some  $w \in W$ . But  $P \cap W = \emptyset$ , and thus  $v \in P$ . Thus  $P \in \text{Ass}(M)$ .  $\square$

**Theorem 1.46.** *Let  $R$  be Noetherian, and  $M$  be an  $R$ -module.*

$$a) \text{ Supp}_R(M) = \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} V(\mathfrak{p}).$$

b) *If  $M$  is a finitely generated  $R$ -module, then  $\text{Min}(\text{ann}_R(M)) \subseteq \text{Ass}_R(M)$ . In particular,  $\text{Min}(I) \subseteq \text{Ass}_R(R/I)$ .*

*Proof.*

a) Let  $P \in \text{Ass}_R(M)$ , and fix  $m \in M$  such that  $P = \text{ann}_R(m)$ . Let  $Q \in V(P)$ . By Proposition 1.33,  $Q \in \text{Supp}(R/P)$ . Since  $0 \rightarrow R/\mathfrak{p} \xrightarrow{m} M$  is exact and localization is exact, by Theorem 1.28,  $0 \rightarrow (R/P)_Q \rightarrow M_Q$  is also exact. Since  $(R/P)_Q \neq 0$ , we must also have  $M_Q \neq 0$ , and thus  $Q \in \text{Supp}(M)$ . This shows  $\text{Supp}_R(M) \supseteq \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} V(\mathfrak{p})$ .

Now let  $Q$  be a prime ideal and suppose that

$$Q \not\subseteq \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} V(\mathfrak{p}).$$

In particular,  $Q$  does not contain any associated prime of  $M$ . Then there is no associated prime of  $M$  that does not intersect  $R \setminus Q$ , so by Theorem 1.45,  $\text{Ass}_{R_Q}(M_Q) = \emptyset$ . By Theorem 1.39,  $M_Q = 0$ .

b) We have shown that

$$V(\text{ann}_R(M)) = \text{Supp}_R(M) = \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} V(\mathfrak{p}),$$

so the minimal elements of both sets agree. In particular, the right hand side has the minimal primes of  $\text{ann}_R(M)$  as minimal elements, and they must be associated primes of  $M$ , or else this would contradict minimality.  $\square$

So the minimal primes of a module  $M$  are all associated to  $M$ , and they are precisely the minimal elements in the support of  $M$ .

**Definition 1.47.** If  $I$  is an ideal, then an associated prime of  $I$  that is not a minimal prime of  $I$  is called an **embedded prime** of  $I$ .

**Macaulay2.** Given an  $R$ -module  $M$ , `associatedPrimes M` will return a list, the list of all the primes in  $\text{Ass}_R(M)$ . If  $I$  is an ideal in  $R$ , Macaulay2 follows the same convention we do: `associatedPrimes I` will return  $\text{Ass}(I)$ , the associated primes of the module  $R/I$ .

**Example 1.48.** Let's get back to the ideal  $I = (x^2, xy)$  in  $R = k[x, y]$  from Example 1.5. We saw before that this ideal has only one minimal prime,  $(x)$ . We claim it also has an embedded prime:  $\mathfrak{m} = (x, y)$ . Indeed,  $x + I$  is a nonzero element in  $R/I$  killed by  $\mathfrak{m}$ , and since  $\mathfrak{m}$  is a maximal ideal and  $x + I \neq 0$ , we must have  $\text{ann}(x + I) = \mathfrak{m}$ . We can also rewrite all this as  $\mathfrak{m} = (I : x)$ . This shows that indeed  $\mathfrak{m}$  is an associated prime of  $I$ .

We can use Macaulay2 to check that these are all the associated primes of  $I$  for some choice of  $k$ .

```
i1 : R = QQ[x,y];
i2 : I = ideal"x2,xy";
i3 : associatedPrimes I
o3 = {ideal x, ideal (y, x)}
o3 : List
```

**Theorem 1.49.** Let  $f: R \rightarrow S$  be a ring homomorphism and let  $M$  be a finitely generated  $S$ -module. Then

$$\text{Ass}_R(M) = \{P \cap R \mid P \in \text{Ass}_S(M)\}.$$

*Proof.* First let's reduce to the case when  $R \subseteq S$  is an inclusion of noetherian rings. Setting  $I = \ker f$ , we can factor  $f$  as follows:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ & \searrow & \nearrow \\ & R/I & \end{array}$$

Given any  $S$ -module  $M$ , when we view it as an  $R$ -module via restriction of scalars we must have  $I \subseteq \text{ann}(M)$ , so  $M$  is an  $R/I$ -module with the exact same structure it has as an  $R$ -module. In particular, we see that a prime  $P$  in  $R$  is in  $\text{Ass}_R(M)$  if and only if  $P/I \in \text{Ass}_{R/I}(M)$ . Moreover, given any prime  $P$  in  $S$ , if its contraction to  $R$  or  $R/I$  is prime, and if  $Q$  is a prime ideal in  $R$  such that  $Q/I = P \cap R/I$ , then  $Q = P \cap R$ . This shows that it is sufficient to consider the case when  $f$  is injective.

Given an element  $m \in M$ , an element  $r \in R$  acts on  $m$  by  $r \cdot m = f(r)m$ , so

$$r \in (0 :_R m) \Leftrightarrow r \cdot m = 0 \Leftrightarrow f(r)m = 0 \Leftrightarrow f(r) \in (0 :_S m).$$

Therefore,  $\text{ann}_R(m) = \text{ann}_S(m) \cap R$ . In particular, if a prime  $P$  in  $S$  is associated to  $M$ , then  $P = \text{ann}_S(m)$  for some  $m$ , and  $P \cap R = \text{ann}_R(m)$ . Since the contraction of a prime is prime,  $P \cap R \in \text{Ass}_R(M)$ .

Now let  $Q \in \text{Ass}_R(M)$ , and let  $m \in M$  be such that  $Q = \text{ann}_R(m)$ . Then  $I := \text{ann}_S(m)$  satisfies  $I \cap R = Q$ . We want to find a prime ideal  $P$  in  $S$  such that  $P \cap R = Q$ . First, note that since  $I = \text{ann}_S(m)$ , the map  $S \rightarrow M$  determined by sending 1 to  $m$  has kernel  $I$ , so we have an inclusion  $S/I \hookrightarrow M$ . By Lemma 1.40,  $\text{Ass}_S(S/I) \subseteq \text{Ass}_S(M)$ . So if we find a prime  $P \in \text{Ass}_S(S/I)$  such that  $P \cap R = Q$ , this  $P$  will be an element of  $\text{Ass}_S(M)$ , and we will be done.

Since  $I \cap R = Q$ , we have an inclusion  $R/Q \rightarrow S/I$ . Write  $\bar{R} := R/Q$  and  $\bar{S} := S/I$ . Since  $\bar{R}$  is a domain, the zero ideal  $J$  in  $\bar{R}$  is prime. On the other hand,  $J\bar{S}$  is the zero ideal in  $\bar{S}$ , so  $J\bar{S} \cap \bar{R} = J$ . Any prime ideal  $J$  satisfying  $J\bar{S} \cap \bar{R} = J$  is the contraction of a prime, so let  $P$  be a prime ideal in  $S$  whose image in  $\bar{S}$  contracts to  $J = (0)$  in  $\bar{R}$ . Lifting to  $R$  and  $S$ , this prime ideal  $P$  in  $S$  must contract to  $Q$ . Moreover, we can take this  $P$  to be minimal over  $I$ : if  $P$  is not minimal over  $I$ , then it contains a minimal prime  $P'$  of  $I$ , and  $Q = P \cap R \supseteq P' \cap R \supseteq I \cap R = Q$ , so we can ultimately replace  $P$  with  $P'$ .

We found a prime ideal  $P$  in  $S$  with  $P \cap R = Q$ . By construction, this prime  $P$  is a minimal prime of  $I$ , and thus by Theorem 1.46,  $P \in \text{Ass}_S(R/I) \subseteq \text{Ass}_S(M)$ .  $\square$

Finally, we record a lemma we will use later about the associated primes over  $R$  and  $R/I$ .

**Lemma 1.50.** *Given an ideal  $I$  in a noetherian ring  $R$ , and let  $P$  be a prime ideal in  $R$ . If  $M$  is a finitely generated  $R$ -module with  $I \subseteq \text{ann}(M)$ , then*

$$P \in \text{Ass}_R(M) \text{ if and only if } P/I \in \text{Ass}_{R/I}(M).$$

*Proof.* Since  $I \subseteq \text{ann}(M)$ ,  $M$  has a structure of an  $R/I$ -module that is compatible with its  $R$ -module structure; these two structures are essentially the same.

If  $P \in \text{Ass}_R(M)$ , there exists  $a \in M$  such that

$$P = \text{ann}_R(m) = \{r \in R \mid ra = 0\}.$$

Then

$$P/I = \{r + I \mid ra = 0\} = \text{ann}_{R/I}(m).$$

On the other hand, all primes in  $R/I$  are of the form  $P/I$  for some prime ideal  $P \supseteq I$  in  $R$ , so if  $P/I = \text{ann}_{R/I}(m)$  for some  $m \in M$ , then

$$P/I = \{r + I \mid ra = 0\} = \{r + I \mid (r + I)a = 0\},$$

and thus  $P \subseteq \text{ann}_R(m)$ . Moreover, if  $r \in R$  satisfies  $ra = 0$ , then  $(r + I)a = ra = 0$ , so  $r + I \in \text{ann}_{R/I}(m) = P/I$ . Since  $P \supseteq I$ , we conclude that  $r \in P$ .  $\square$



## 1.4 Primary Ideals

**Definition 1.51.** We say that an ideal  $I$  is **primary** if

$$xy \in I \implies x \in I \text{ or } y \in \sqrt{I}.$$

We say that an ideal  $I$  is  **$P$ -primary** if  $I$  is primary and  $\sqrt{I} = P$ .

**Remark 1.52.** Suppose that  $Q$  is primary and  $xy \in \sqrt{Q}$ . Then  $x^n y^n \in Q$  for some  $n$ . If  $y \notin \sqrt{Q}$ , then  $y^n \notin \sqrt{Q}$ . Since  $Q$  is primary, we must have  $x^n \in Q$ , so  $x \in \sqrt{Q}$ . Therefore, the radical of a primary ideal is always prime, so every primary ideal  $Q$  is  $\sqrt{Q}$ -primary.

**Example 1.53.** a) Any prime ideal is primary.

- b) If  $R$  is a UFD, we claim that a principal ideal is primary if and only if it is generated by a power of an irreducible element. If  $f \in R$  is an irreducible element, then

$$xy \in (f^n) \iff f^n | xy.$$

Since  $f$  is irreducible and every element in  $R$  can be written as a unique product of irreducibles, if  $f^n$  divides  $xy$  but  $f^n$  does not divide  $x$ , then  $f$  must divide  $y$ . So

$$f^n | x \text{ or } f | y \iff x \in (f^n) \text{ or } y \in \sqrt{(f^n)} = (f).$$

We conclude that  $(f^n)$  is indeed primary. Conversely, if  $a$  is not a prime power, then  $a = gh$ , for some  $g, h$  nonunits with no common factor, so we can take  $gh \in (a)$  but  $g \notin (a)$  and  $h \notin \sqrt{(a)}$ . This shows that  $(a)$  is not primary.

- c) As a particular case of the previous example, the nonzero primary ideals in  $\mathbb{Z}$  are of the form  $(p^n)$  for some prime  $p$  and some  $n \geq 1$ . This example is a bit misleading, as it suggests that primary ideals are the same as powers of primes. We will soon see that it not the case.
- d) In  $R = k[x, y, z]$ , the ideal  $I = (y^2, yz, z^2)$  is primary. Give  $R$  the grading with weights  $|y| = |z| = 1$ , and  $|x| = 0$ . If  $g \notin \sqrt{I} = (y, z)$ , then  $g$  has a degree zero term. If  $f \notin I$ , then  $f$  has a term of degree zero or one. The product  $fg$  has a term of degree zero or one, so it is not in  $I$ .

If the radical of an ideal is prime, that does not imply that ideal is primary.

**Example 1.54.** In  $R = k[x, y, z]$ , the ideal  $Q = (x^2, xy)$  is not primary, even though  $\sqrt{Q} = (x)$  is prime. The offending product is  $xy$ :  $x \notin Q$  and  $y \notin Q$ .

The definition of primary can be reinterpreted in many ways.

**Proposition 1.55.** *If  $R$  is noetherian, the following are equivalent:*

- (1)  $Q$  is primary.
- (2) Every zerodivisor in  $R/Q$  is nilpotent on  $R/Q$ .

(3)  $\text{Ass}(R/Q)$  is a singleton.

(4)  $Q$  has exactly one minimal prime, and no embedded primes.

(5)  $\sqrt{Q} = P$  is prime and for all  $r, w \in R$  with  $w \notin P$ ,  $rw \in Q$  implies  $r \in Q$ .

(6)  $\sqrt{Q} = P$  is prime, and  $QR_P \cap R = Q$ .

*Proof.* (1)  $\iff$  (2):  $y$  is a zerodivisor modulo  $Q$  if there is some  $x \notin Q$  with  $xy \in Q$ ; the primary assumption translates to saying that a power of  $y$  is in  $Q$ .

(2)  $\iff$  (3): On the one hand, (2) says that the set of zerodivisors on  $R/Q$  coincides with the elements in the nilradical of  $R/Q$ . In general, the set of zerodivisors is the union of all the associated primes, while the nilradical is the intersection of all the minimal primes. Now notice that the associated primes of  $R/Q$  are the associated primes of the ideal  $Q$ , while the minimal primes of  $R/Q$  are the minimal primes of  $Q$ . So we always have

$$\bigcup_{\mathfrak{p} \in \text{Ass}(Q)} \mathfrak{p} = \mathcal{Z}(R/Q) \supseteq \{r \in R \mid r + Q \in \mathcal{N}(R/Q)\} = \bigcap_{\mathfrak{p} \in \text{Min}(Q)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \text{Ass}(Q)} \mathfrak{p}.$$

The rest of the proof is elementary set theory: the intersection and union of a collection of sets agree if and only if there is only one set. More precisely, we have equality above if and only if there is only one associated prime.

(3)  $\iff$  (4) is clear, since each statement is just a restatement of the other one.

(1)  $\iff$  (5): Given the observation that the radical of a primary ideal is prime, this is just a rewording of the definition.

(5)  $\iff$  (6): This is immediate from the following characterization:

$$\begin{aligned} QR_P \cap R &= \left\{ r \in R \mid \frac{r}{1} \in QR_P \right\} \\ &= \left\{ r \in R \mid \frac{r}{1} = \frac{a}{w} \text{ for some } a \in Q, w \notin P \right\} \end{aligned}$$

The equality  $\frac{r}{1} = \frac{a}{w}$  is equivalent to  $u(wr - a) = 0$  for some  $a \in Q$  and  $u, w \notin P$ , so equivalently  $s := uw \in W$  satisfies  $sr = ua \in Q$ . Therefore,

$$QR_P \cap R = \{r \in R \mid sr \in Q \text{ for some } s \notin P\}$$

□

If the radical of an ideal is maximal, that *does* imply the ideal is primary.

**Remark 1.56.** Let  $I$  be an ideal with  $\sqrt{I} = \mathfrak{m}$  a maximal ideal. If  $R$  is noetherian, then  $\text{Ass}_R(R/I)$  is nonempty and contained in  $\text{Supp}(R/I) = V(I) = \{\mathfrak{m}\}$ , so  $\text{Ass}_R(R/I) = \{\mathfrak{m}\}$ , and hence  $I$  is primary.

Note that the assumption that  $\mathfrak{m}$  is maximal was necessary here. Indeed, having a prime radical does not guarantee an ideal is primary, as we saw in Example 1.54. Moreover, even the powers of a prime ideal may fail to be primary.

**Example 1.57.** Let  $R = k[x, y, z]/(xy - z^n)$ , where  $k$  is a field and  $n \geq 2$  is an integer. Consider the prime ideal  $P = (x, z)$  in  $R$ , and note that  $y \notin P$ . We have  $xy = z^n \in P^n$ , while  $x \notin P^n$  and  $y \notin \sqrt{P^n} = P$ . Therefore,  $P^n$  is not a primary ideal, even though its radical is the prime  $P$ .

**Lemma 1.58.** *Let  $P$  be a prime ideal in a noetherian ring  $R$ . If  $Q \subseteq P$  is a primary ideal, then  $Q_P$  is a primary ideal in  $R_P$ , even if  $Q$  is not  $P$ -primary.*

*Proof.* Since  $Q \subseteq P$  and  $P$  is prime, we must have  $\sqrt{Q} \subseteq P$ , so  $Q_P = QR_P$  and  $\sqrt{Q}R_P$  are proper ideals of  $R_P$  with  $\sqrt{Q}R_P$  prime. Since  $\text{Ass}(Q) = \{\sqrt{Q}\}$ , by Theorem 1.45 we have  $\text{Ass}(Q_P) = \{\sqrt{Q}R_P\}$ . By Proposition 1.55,  $Q_P$  is a primary ideal.  $\square$

The contraction of primary ideals is always primary.

**Lemma 1.59.** *Let  $f: R \rightarrow S$  be a ring homomorphism. If  $Q$  is a primary ideal in  $S$ , then  $Q \cap R$  is a primary ideal in  $R$ . In particular, given any multiplicatively closed set  $W$  in  $R$ , and a primary ideal  $Q$  in  $W^{-1}R$ ,  $Q \cap R$  is a primary ideal in  $R$ .*

*Proof.* If  $xy \in Q \cap R$  and  $x \notin Q \cap R$ , then  $f(x) \notin Q$ . But  $f(x)f(y) = f(xy) \in Q$ , and since  $Q$  is primary, so we must have  $f(y^n) = f(y)^n \in Q$  for some  $n$ . Therefore,  $y^n \in Q \cap R$ , and  $Q \cap R$  is indeed primary.  $\square$

**Lemma 1.60.** *Let  $Q$  be a primary ideal in a noetherian ring  $R$ . If  $P$  is a prime ideal containing  $Q$ , then  $QR_P \cap R = Q$ .*

*Proof.* By Lemma 1.58,  $QR_P$  is a primary ideal in  $R_P$ . By Lemma 1.59,  $QR_P \cap R$  is also primary.

$$Q = QR_{\sqrt{Q}} \cap R = \left\{ r \in R \mid sr \in Q \text{ for some } s \notin \sqrt{Q} \right\}$$

Since  $P$  is prime and  $P \supseteq Q$ , then  $P \supseteq \sqrt{Q}$ . Therefore, the complement of  $\sqrt{Q}$  contains the complement of  $P$ , and

$$\left\{ r \in R \mid sr \in Q \text{ for some } s \notin \sqrt{Q} \right\} \supseteq \left\{ r \in R \mid sr \in Q \text{ for some } s \notin P \right\} = QR_P \cap R.$$

Therefore,  $Q \supseteq QR_P \cap R$ . But by elementary set theory,  $Q \subseteq QR_P \cap R$ , so we must have  $Q = QR_P \cap R$ .  $\square$

The intersection of primary ideals is also primary.

**Lemma 1.61.** *If  $I_1, \dots, I_t$  are ideals, then*

$$\text{Ass} \left( R / \bigcap_{j=1}^t I_j \right) \subseteq \bigcup_{j=1}^t \text{Ass}(R/I_j).$$

*In particular, a finite intersection of  $P$ -primary ideals is  $P$ -primary.*

*Proof.* The map  $R \rightarrow R/I_1 \oplus R/I_2$  given by  $r \mapsto (r + I_1, r + I_2)$  has kernel  $I_1 \cap I_2$ , so there is an inclusion  $R/(I_1 \cap I_2) \subseteq R/I_1 \oplus R/I_2$ . Hence, by Lemma 1.40 we have

$$\text{Ass}(R/(I_1 \cap I_2)) \subseteq \text{Ass}(R/I_1) \cup \text{Ass}(R/I_2).$$

For  $t \geq 3$ ,

$$\text{Ass} \left( R / \bigcap_{j=1}^t I_j \right) \subseteq \bigcup_{j=1}^t \text{Ass}(R/I_j)$$

can be shown by an easy induction.

Now suppose all the  $I_j$  are all  $P$ -primary. Then

$$\text{Ass}\left(R/\bigcap_{j=1}^t I_j\right) \subseteq \bigcup_{j=1}^t \text{Ass}(R/I_j) = \{P\}.$$

On the other hand,  $\bigcap_{j=1}^t I_j \subseteq I_1 \neq R$ , so  $R/(\bigcap_{j=1}^t I_j) \neq 0$ . Thus  $\text{Ass}(R/(\bigcap_{j=1}^t I_j))$  is nonempty, by Theorem 1.39, and therefore it must be the singleton  $\{P\}$ . Then  $\bigcap_{j=1}^t I_j$  is  $P$ -primary by the characterization of primary in Proposition 1.55 (3).  $\square$

## 1.5 Primary decomposition

**Definition 1.62** (Primary decomposition). A **primary decomposition** of an ideal  $I$  is an expression of the form

$$I = Q_1 \cap \cdots \cap Q_t,$$

with each  $Q_i$  primary. An **irredundant primary decomposition** of an ideal  $I$  is a primary decomposition as above in which  $\sqrt{Q_i} \neq \sqrt{Q_j}$  for  $i \neq j$ , and  $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$  for all  $i$ .

**Remark 1.63.** By Lemma 1.61, we can turn any primary decomposition into an irredundant one by combining the terms with the same radical, then removing redundant terms.

**Example 1.64** (Primary decomposition in  $\mathbb{Z}$ ). Given a decomposition of  $n \in \mathbb{Z}$  as a product of distinct primes, say  $n = p_1^{a_1} \cdots p_k^{a_k}$ , then the primary decomposition of the ideal  $(n)$  is  $(n) = (p_1^{a_1}) \cap \cdots \cap (p_k^{a_k})$ . However, this example can be deceiving, in that it suggests that primary ideals are just powers of primes; as we saw in Example 1.57, they are not!

The existence of primary decompositions was first shown by Emanuel Lasker (yes, the chess champion!) for polynomial rings and power series rings in 1905 [Las05], and then extended to noetherian rings (in the same paper where she introduced noetherian rings – though not by that name) by Emmy Noether in 1921 [Noe21].

**Theorem 1.65** (Existence of primary decompositions). *Every ideal in a noetherian ring has a primary decomposition.*

*Proof.* We will say that an ideal is irreducible if it cannot be written as a proper intersection of larger ideals. If  $R$  is Noetherian, we claim that any ideal of  $R$  can be expressed as a finite intersection of irreducible ideals. If the set of ideals that are not a finite intersection of irreducibles were nonempty, then by Noetherianity there would be an ideal maximal with the property of not being an intersection of irreducible ideals. Such a maximal element must be an intersection of two larger ideals, each of which are finite intersections of irreducibles, giving a contradiction.

Next, we claim that every irreducible ideal is primary. To prove the contrapositive, suppose that  $Q$  is not primary, and take  $xy \in Q$  with  $x \notin Q$ ,  $y \notin \sqrt{Q}$ . The ascending chain of ideals

$$(Q : y) \subseteq (Q : y^2) \subseteq (Q : y^3) \subseteq \cdots$$

stabilizes for some  $n$ , since  $R$  is Noetherian. It will soon be helpful to realize that this means that for any element  $f \in R$ ,  $y^{n+1}f \in Q \implies y^n f \in Q$ . Using this, we will show that

$$(Q + (y^n)) \cap (Q + (x)) = Q,$$

proving that  $Q$  is not irreducible.

The containment  $Q \subseteq (Q + (y^n)) \cap (Q + (x))$  is clear. On the other hand, if

$$a \in (Q + (y^n)) \cap (Q + (x)),$$

we can write  $a = q + by^n$  for some  $q \in Q$ , and

$$a \in Q + (x) \implies ay \in Q + (xy) = Q.$$

So

$$by^{n+1} = ay - aq \in Q \implies b \in (Q : y^{n+1}) = (Q : y^n).$$

By definition, this means that  $by^n \in Q$ , and thus  $a = q + by^n \in Q$ . This shows that  $Q$  is not irreducible, concluding the proof.  $\square$

Primary decompositions, even irredundant ones, are not unique.

**Example 1.66.** Let  $R = k[x, y]$ , where  $k$  is a field, and  $I = (x^2, xy)$ . We can write

$$I = (x) \cap (x^2, xy, y^2) = (x) \cap (x^2, y).$$

The ideals  $(x^2, xy, y^2)$  and  $(x^2, y)$  are primary, since each has radical  $\mathfrak{m} = (x, y)$ , which is maximal, and by Remark 1.56 any ideal whose radical is maximal must be primary. In fact, our ideal  $I$  has infinitely many irredundant primary decompositions: given any  $n \geq 1$ ,

$$I = (x) \cap (x^2, xy, y^n)$$

is a irredundant primary decomposition. One thing all of these have in common is the radicals of the primary components: they are always  $(x)$  and  $(x, y)$ . Indeed, we saw in Example 1.48 that these are the associated primes of  $I$ .

In the previous example, the fact that all our irredundant primary decompositions had primary components always with the same radical was not an accident. Indeed, there are some aspects of primary decompositions that are unique, and this is one of them.

**Theorem 1.67** (First uniqueness theorem for primary decompositions). *Suppose  $I$  is an ideal in a noetherian ring  $R$ . Given any irredundant primary decomposition of  $I$ , say*

$$I = Q_1 \cap \cdots \cap Q_t,$$

*we have*

$$\{\sqrt{Q_1}, \dots, \sqrt{Q_t}\} = \text{Ass}(R/I).$$

*In particular, this set is the same for all irredundant primary decompositions of  $I$ .*

*Proof.* For any primary decomposition, irredundant or not, we have

$$\text{Ass}(I) \subseteq \bigcup_i \text{Ass}(Q_i) = \{\sqrt{Q_1}, \dots, \sqrt{Q_t}\}$$

by Lemma 1.61. We just need to show that in an irredundant decomposition as above, every  $P_j := \sqrt{Q_j}$  is indeed an associated prime of  $I$ .

So fix  $j$ , and let

$$I_j = \bigcap_{i \neq j} Q_i \supseteq I.$$

Since the decomposition is irredundant, the module  $I_j/I$  is nonzero, hence by Theorem 1.39 it has an associated prime, say  $\mathfrak{a}$ . Fix  $x_j \in R$  such that  $\mathfrak{a}$  is the annihilator of  $\overline{x_j}$  in  $I_j/I$ . Since

$$Q_j x_j \subseteq Q_j \cdot \bigcap_{i \neq j} Q_i \subseteq Q_1 \cap \dots \cap Q_n = I,$$

we conclude that  $Q_j$  is contained in the annihilator of  $\overline{x_j}$ , meaning  $Q_j \subseteq \mathfrak{a}$ . Since  $P_j$  is the unique minimal prime of  $Q_j$  and  $\mathfrak{a}$  is a prime containing  $Q_j$ , we must have  $P_j \subseteq \mathfrak{a}$ . On the other hand, if  $r \in \mathfrak{a}$ , we have  $rx_j \in I \subseteq Q_j$ , and since  $x_j \notin Q_j$ , we must have  $r \in \sqrt{Q_j} = P_j$  by the definition of primary ideal. Thus  $\mathfrak{a} \subseteq P_j$ , so we can now conclude that  $\mathfrak{a} = P_j$ . This shows that  $P_j$  is an associated prime of  $R/I$ .  $\square$

If we do not assume that  $R$  is noetherian, we may or may not have a primary decomposition for a given ideal. It is true that if an ideal  $I$  in a general ring has a primary decomposition, then the primes occurring are the same in any irredundant decomposition. However, they are not the associated primes of  $I$  in general; rather, they are the primes that occur as radicals of annihilators of elements.

There is also a partial uniqueness result for the actual primary ideals that occur in an irredundant decomposition.

**Theorem 1.68** (Second uniqueness theorem for primary decompositions). *If  $I$  is an ideal in a noetherian ring  $R$ , then the minimal components in any irredundant primary decomposition of  $I$  are unique. More precisely, if*

$$I = Q_1 \cap \dots \cap Q_t$$

*is an irredundant primary decomposition, and  $\sqrt{Q_i} \in \text{Min}(I)$ , then  $Q_i$  is given by the formula*

$$Q_i = IR_{\sqrt{Q_i}} \cap R,$$

*which does not depend on our choice of irredundant decomposition.*

*Proof.* Let  $Q$  be a primary ideal, and let  $P$  be any prime. The localization  $Q_P$  is either:

- the unit ideal, if  $Q \not\subseteq P$ , or
- a  $P$ -primary ideal, if  $Q \subseteq P$ .

This follows from Theorem 1.45, the fact that the associated primes of  $Q$  localize, since  $Q \subseteq P$  implies  $\sqrt{Q} \subseteq P$ , and  $Q_P$  will still have a unique associated prime.

Finite intersections commute with localization, so for any prime  $P$ ,

$$I_P = (Q_1)_P \cap \cdots \cap (Q_t)_P$$

is a primary decomposition, although not necessarily irredundant. Fix a minimal prime  $P = P_i$  of  $I$ , and let  $Q = Q_i$ . When we localize at  $P$ , all the other components become the unit ideal, since their radicals are not contained in  $P$ , and thus  $I_P = Q_P$ . We can then contract to  $R$  to get  $I_P \cap R = (Q_i)_{P_i} \cap R = Q_i$ , since  $Q_i$  is  $P_i$ -primary and we can then apply Proposition 1.55 (6).  $\square$

We saw in Example 1.66 that the components associated to embedded primes are not necessarily unique. One might be tempted to think that the formula above,  $IR_P \cap R$ , still gives a  $P$ -primary component for  $I$  for any associated prime; unfortunately, that is *always* false when  $P$  is embedded.

**Remark 1.69.** If  $P$  is an embedded prime of  $I$ , we claim that  $IR_P \cap R$  is not primary. Let  $P_1, \dots, P_k$  be all the associated primes of  $I$  that are contained in  $P$ , and let  $Q_1, \dots, Q_k$  be the  $P_i$ -primary components and  $Q$  be the  $P$ -primary component in an irredundant primary decomposition of  $I$ . Since  $Q_i \subseteq P_i \subseteq P$ , by Lemma 1.60 we have  $Q_i R_P \cap R = Q_i$ . On the other hand, if  $\mathfrak{q}$  is another primary component of  $I$  but with  $\sqrt{\mathfrak{q}} \not\subseteq P$ , then  $\mathfrak{q} R_P = R_P$ , and  $\mathfrak{q} R_P \cap R = R$ .

Finally, localization commutes with intersections, so we conclude that

$$IR_P \cap R = (Q_1 R_P \cap R) \cap \cdots \cap (Q_k R_P \cap R) = Q_1 \cap \cdots \cap Q_k.$$

This says  $IR_P \cap R$  collects *all* the primary components of  $I$  corresponding to primes contained in  $P$ . If  $P$  is an embedded prime of  $I$ , there are at least two such components, and thus this ideal is not primary.

This does show, however, that if  $\text{mAss}(I)$  denotes the set of maximal elements in  $\text{Ass}(I)$ , then

$$I = \bigcap_{P \in \text{Ass}(I)} IR_P \cap R = \bigcap_{P \in \text{mAss}(I)} IR_P \cap R,$$

since each  $IR_P \cap R$  is the intersection of all the primary components associated to elements in  $\text{Ass}(I)$  that are contained in  $P$ .

**Remark 1.70.** Fix an ideal  $I$  and suppose that  $A$  is a finite set of primes that contains every associated prime of  $I$ . By Remark 1.30,  $W = R \setminus \bigcup_{P \in A} P$  is a multiplicatively closed set, so we can consider the localization ring  $W^{-1}R$ . For each associated prime  $P$  of  $I$ ,  $R \setminus P \supseteq W$ , so if

$$r \in W^{-1}I \cap R \implies sr \in I \text{ for some } s \in W \implies sr \in I \text{ for some } s \notin P \implies r \in IR_P \cap R.$$

Therefore,

$$I \subseteq W^{-1}I \cap R \subseteq \bigcap_{P \in \text{Ass}(I)} IR_P \cap R = I,$$

where the last equality was discussed in Remark 1.69. We conclude that  $W^{-1}I \cap R = I$ .

It is relatively easy to give a primary decomposition for a radical ideal:

**Example 1.71.** If  $R$  is noetherian, and  $I$  is a radical ideal, then we have seen that  $I$  coincides with the intersection of its minimal primes, say  $I = P_1 \cap \cdots \cap P_t$ . This is the *only* primary decomposition of a radical ideal.

For a more concrete example, take the ideal  $I = (xy, xz, yz)$  in  $k[x, y, z]$ . This ideal is radical, so we just need to find its minimal primes. And indeed, one can check that  $(xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$ .

**Example 1.72.** Let's get back to our motivating example in  $\mathbb{Z}[\sqrt{-5}]$ , where some elements can be written as products of irreducible elements in more than one way. For example, we saw that

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

In fact,  $(6) = (2) \cap (3)$ , but while  $(2)$  is primary,  $(3)$  is not. In fact,  $(3)$  has two distinct minimal primes, and the following is a minimal primary decomposition for  $(6)$ :

$$(6) = (2) \cap (3, 1 + \sqrt{-5}) \cap (3, 1 - \sqrt{-5}).$$

## 1.6 Computing primary decompositions

Providing efficient algorithms for primary decomposition of an ideal [...] is [...] still one of the big challenges for computational algebra and computational algebraic geometry.

(Decker, Greuel, and Pfister write in [DGP99])

Computing primary decompositions is a computationally difficult problem. This is true in a formal sense: even if one restricts to ideals monomial in a polynomial ring over a field, the problem of finding a primary decompositions is NP-complete [HS02].

The method `primaryDecomposition` in Macaulay2 computes primary decompositions using several strategies: there are special algorithms for monomial and binomial ideals, which we will discuss later, an algorithm of Shimoyama and Yokoyama [SY96] for ideals in polynomial rings, and an algorithm by Eisenbud, Huneke, and Vasconcelos [EHV92], and a hybrid of the last two. The user can force Macaulay2 to choose a particular strategy using the command `Strategy`. Here's an example from the Macaulay2 documentation:

```
i2 : Q = QQ[a..d];

i3 : I = ideal(a^2*b, a*c^2, b*d, c*d^2);

o3 : Ideal of Q

i4 : primaryDecomposition(I, Strategy => Monomial)
                                2          2    2
o4 = {ideal (d, a), ideal (c, b), ideal (b, a, d), ideal (d, c , a ),
```



---

```

      2      2
ideal (b, d , c )}

```

```
o4 : List
```

To compute a primary decomposition of  $I$ , we want to isolate the components with a particular radical, which can be done via saturations.

**Definition 1.73.** Let  $I$  and  $J$  be ideals in a ring  $R$ . The **saturation of  $I$  with respect to  $J$**  is the ideal

$$(I : J^\infty) := \bigcup_{n \geq 1} (I : J^n) = \{r \in R \mid rJ^n \subseteq I \text{ for some } n\}.$$

Given an element  $x \in R$ , we write  $(I : x^\infty)$  instead of  $(I : (x)^\infty)$ . Some authors drop the parenthesis altogether, and write simply  $I : J^\infty$ .

We leave the following basic properties of saturations as an exercise.

**Exercise 1.** Let  $I$ ,  $J$ , and  $L$  be ideals in a noetherian ring  $R$ .

- a) There exists  $n$  such that  $(I : J^\infty) = (I : J^n)$ .
- b) If  $Q$  is a  $P$ -primary ideal, then

$$(Q : J^\infty) = \begin{cases} Q & \text{if } J \not\subseteq P \\ R & \text{if } J \subseteq P \end{cases}.$$

- c)  $(I \cap J : L^\infty) = (I : L^\infty) \cap (J : L^\infty)$ .
- d) Given a primary decomposition  $I = Q_1 \cap \cdots \cap Q_k$ ,

$$(I : J^\infty) = \bigcap_{J \not\subseteq \sqrt{Q_i}} Q_i.$$

- e) If  $\text{Ass}(I) = \text{Min}(I) = \{P_1, \dots, P_k\}$ , then for each  $i$  there exists an element  $x_i \in R$  such that the  $P_i$ -primary component of  $I$  is given by  $(I : x_i^\infty)$ .

# Chapter 2

## Symbolic powers

Given an ideal  $I$ , its  **$n$ th power** is the ideal

$$I^n := (f_1 \cdots f_n \mid f_i \in I).$$

By convention, we set  $I^0 = R$ . While  $\sqrt{I^n} = \sqrt{I}$  and the minimal primes of  $I^n$  are the same as the minimal primes of  $R$ ,  $I^n$  might have new, exciting embedded primes. This is the beginning of a beautiful friendship.

### 2.1 What are symbolic powers?

**Definition 2.1.** Let  $I$  be an ideal in a noetherian ring  $R$  and fix an integer  $n \geq 1$ . The  **$n$ th symbolic power** of  $I$  is the ideal

$$I^{(n)} := \bigcap_{P \in \text{Ass}(I)} I^n R_P \cap R.$$

Our main focus will be on symbolic powers of ideals  $I$  with no embedded primes. In that case,  $\text{Ass}(I) = \text{Min}(I)$ , and since  $\text{Min}(I^n) = \text{Min}(I)$ , the  $n$ th symbolic power of  $I$  is the ideal

$$I^{(n)} = \bigcap_{P \in \text{Ass}(I)} I^n R_P \cap R = \bigcap_{P \in \text{Min}(I)} I^n R_P \cap R = \bigcap_{P \in \text{Min}(I^n)} I^n R_P \cap R.$$

For each minimal prime  $P$  of  $I^n$ ,  $I^n R_P \cap R$  is the  $P$ -primary component of  $I^n$ . The  $n$ th symbolic power of  $I$  is thus obtained by computing a primary decomposition for  $I^n$ , collecting the minimal components, and discarding the embedded ones.

When  $I$  does have embedded primes, the two ideals

$$\bigcap_{P \in \text{Ass}(I)} I^n R_P \cap R \quad \text{and} \quad \bigcap_{P \in \text{Min}(I)} I^n R_P \cap R$$

are necessarily distinct. Both appear in the literature as definitions for symbolic powers, and the ambiguity only arises when one considers ideals with embedded components. While we will briefly discuss the advantages and disadvantages of each definition, our main focus will be the situation where  $I$  has no embedded primes and there is no ambiguity to worry about. In fact, even the study of symbolic powers of primes is interesting enough to keep us busy all semester, and so focusing on ideals with no embedded primes is not a big concession.

**Lemma 2.2.** *Let  $P$  be a prime ideal in a noetherian ring  $R$ .*

- (1)  $P^{(n)} = P^n R_P \cap P$ .
- (2)  $P^{(n)} = \{r \in R \mid sr \in P^n \text{ for some } s \notin P\}$ .
- (3)  $P^{(n)}$  is the unique  $P$ -primary component in a primary decomposition of  $P^n$ .
- (4)  $P^{(n)}$  is the smallest  $P$ -primary ideal containing  $P^n$ .

Moreover,  $P^n = P^{(n)}$  if and only if  $P^n$  is primary.

*Proof.* Prime ideals are primary, and  $\text{Ass}(P) = \{P\}$ , so (1) is immediate. The characterization in (2) follows from (1) and the definition of  $R_P$ . Theorem 1.68 says that  $P^n R_P \cap R$  is the unique minimal ( $P$ -primary) component in a primary decomposition of  $P^n$ , which is (3).

So  $P^{(n)}$  is a  $P$ -primary ideal containing  $P^n$ . To show it is the smallest such ideal, consider any other  $P$ -primary ideal  $Q$  containing  $P^n$ . Since images and preimages preserve containments,  $P^n R_P \cap R \subseteq Q R_P \cap R$ . Since  $Q$  is  $P$ -primary, the characterization of primary ideals in Proposition 1.55 says that  $Q R_P \cap R = Q$ . We conclude that

$$P^{(n)} = P^n R_P \cap R \subseteq Q R_P \cap R = Q,$$

and  $P^{(n)}$  is the smallest  $P$ -primary ideal containing  $P^n$ , which is (4).

The final statement is now immediate from (4). □

The symbolic powers of a prime ideal do not necessarily coincide with its powers.

**Example 2.3.** Fix  $n \geq 2$ . Let  $k$  be a field,  $R = k[x, y, z]/(xy - z^n)$ , and consider the prime ideal  $P = (x, z)$  in  $R$ . While  $y \notin P$ ,  $xy = z^n \in P^n$ , so  $x \in P^{(n)}$ .

In fact, the symbolic powers of a prime ideal might not coincide with its powers even over a polynomial ring.

**Example 2.4.** Fix a field  $k$ , and let  $R = k[x, y, z]$ . Consider the ideal  $P$  given by

$$P = \left( \underbrace{x^3 - yz}_f, \underbrace{y^2 - xz}_g, \underbrace{z^2 - x^2 y}_h \right).$$

We will show that the ring homomorphism

$$\begin{aligned} \frac{k[x, y, z]}{P} &\xrightarrow{\pi} k[t^3, t^4, t^5] \\ (x, y, z) &\longmapsto (t^3, t^4, t^5) \end{aligned}$$

is an isomorphism. First, note that it is immediately surjective by construction, so we just need to prove it is injective. If we set  $\deg(x) = 3, \deg(y) = 4, \deg(z) = 5, \deg(t) = 1$ ,  $\pi$  is a graded homomorphism of graded rings, whose kernel is homogeneous. Since  $[k[t^3, t^4, t^5]]_n$  is a 1-dimensional vector space generated by  $t^n$  for all  $n \geq 3$  (and zero in degrees 1 and 2), it suffices to show that  $\dim([\frac{k[x, y, z]}{P}]_n) = 1$  for all  $n \leq 3$  (and zero in degrees 1 and 2).

Given any monomial in  $\frac{k[x,y,z]}{P}$ , we can use the relations  $y^2 - xz$ ,  $z^2 - x^2y$ , and  $yz - x^3$  to obtain an equivalent monomial where the sum of the  $y$  and  $z$  exponents is smaller until we get a monomial of the form  $x^a$ ,  $x^ay$ , or  $x^az$ . If  $n = 1, 2$ , there is no such monomial; if  $n \geq 3$ , there is exactly one, namely,

$$\begin{cases} x^{n/3} & \text{if } n \equiv 0 \pmod{3} \\ x^{(n-4)/3}y & \text{if } n \equiv 1 \pmod{3} \\ x^{(n-5)/3}y & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

This shows that  $P$  is the kernel of the map

$$\begin{aligned} k[x, y, z] &\longrightarrow k[t^3, t^4, t^5] \\ (x, y, z) &\longmapsto (t^3, t^4, t^5) \end{aligned}$$

Since  $k[t^3, t^4, t^5] \subseteq k[t]$  is a domain, we conclude that  $P$  is a prime ideal. In fact,  $P$  is a homogeneous ideal with the grading we considered above: our generators  $f$ ,  $g$ , and  $h$  are now homogeneous, with  $\deg(f) = 9$ ,  $\deg(g) = 8$ , and  $\deg(h) = 10$ . We claim that  $P^{(2)} \neq P^2$ .

Consider the homogeneous element  $fg - h^2 \in (x)$ , which has degree 18, and let  $q$  be such that  $fg = qx$ . Since  $x \notin P$  and  $xq = fg - h^2 \in P^2$ , we conclude that  $q \in P^{(2)}$ . However, since  $\deg(x) = 3$  and  $\deg(fg) = 18$ ,  $q$  must be a homogeneous element of degree 15, but the smallest degree of any element in  $P^2$  is  $2 \times 8 = 16$ , so  $q \notin P^2$ .

The symbolic powers of an ideal do sometimes coincide with its ordinary powers.

**Lemma 2.5.** *Let  $k$  be a field and consider a polynomial ring  $R = k[x_1, \dots, x_d]$ . If  $I$  is an ideal generated by some of the variables, then  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

*Proof.* Since  $I$  is a prime ideal, we want to show that  $I^n$  is primary for all  $n$ . Without loss of generality, assume  $I = (x_1, \dots, x_t)$  for some  $t \leq d$ . Give  $R$  the grading with weights  $|x_1| = \dots = |x_t| = 1$ , and  $|x_{t+1}| = \dots = |x_n| = 0$ . With this grading,  $I$  is a homogeneous ideal, and the nonzero homogeneous elements in  $I$  are precisely the homogeneous elements of positive degree. Similarly, the nonzero homogeneous elements in  $I^n$  are the homogeneous elements in  $R$  of degree at least  $n$ . If  $g \notin \sqrt{I^n} = I$ , then  $g$  has a term of degree zero. If  $f \notin I$ , then  $f$  has a term of degree strictly smaller than  $n$ . The product  $fg$  must then have a term of degree strictly smaller than  $n$ , so it is not in  $I^n$ . We conclude that  $I^n$  is primary.  $\square$

**Lemma 2.6.** *If  $\mathfrak{m}$  is a maximal ideal in a noetherian ring  $R$ , then  $\mathfrak{m}^n = \mathfrak{m}^{(n)}$  for all  $n$ .*

*Proof.* We just need to show that  $\mathfrak{m}^n$  is primary. But  $\mathfrak{m}$  is the only prime containing  $\mathfrak{m}^n$ , so we necessarily have  $\text{Ass}(\mathfrak{m}^n) \subseteq \{\mathfrak{m}\}$ . By Theorem 1.39 Item 3),  $\text{Ass}(\mathfrak{m}^n) \neq \emptyset$ , so  $\text{Ass}(\mathfrak{m}^n) = \{\mathfrak{m}\}$ .  $\square$

Now we are ready to discuss the general case. When  $I$  is an ideal with no embedded primes, the symbolic powers of  $I$  are given by

$$I^{(n)} = \bigcap_{P \in \text{Ass}(I)} I^n R_P \cap R = \bigcap_{P \in \text{Min}(I)} I^n R_P \cap R.$$

This is a primary decomposition of  $I^{(n)}$ : the associated primes of  $I^{(n)}$  are precisely the minimal primes of  $I$ , and for each  $P \in \text{Min}(I)$  the  $P$ -primary component of  $I^{(n)}$  is given by  $I^n R_P \cap R$ .

**Lemma 2.7.** *Let  $I$  be an ideal with no embedded primes in a noetherian ring  $R$ .*

- (1)  $I^n = I^{(n)}$  if and only if  $I^n$  has no embedded primes.
- (2)  $\text{Ass}(I^{(n)}) = \text{Ass}(I) = \text{Min}(I) = \text{Min}(I^n)$ .

*Proof.*

- (1) Since  $\sqrt{I^n} = \sqrt{I}$ , the minimal primes of  $I^n$  coincide with those of  $I$ . Therefore, an irredundant primary decomposition of  $I^n$  consists of

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_k,$$

where  $Q_1, \dots, Q_k$  are primary components corresponding to embedded primes of  $I^n$ . There are no such components precisely when  $I^n = I^{(n)}$ .

- (2) Saying  $I$  has no embedded primes is the same as saying that  $\text{Ass}(I) = \text{Min}(I)$ , and  $\text{Min}(I) = \text{Min}(I^n)$  always holds. For each  $P \in \text{Ass}(I) = \text{Min}(I) = \text{Min}(I^n)$ ,  $I^n R_P \cap R$  is a  $P$ -primary ideal, so

$$\bigcap_{P \in \text{Ass}(I)} I^n R_P \cap R$$

is an intersection of primary ideals with distinct, incomparable radicals. This must then be a primary decomposition of  $I^{(n)}$ , and the corresponding primes are the associated primes of  $I^{(n)}$ .  $\square$

We can write  $I^{(n)}$  more directly as the elements that live in  $I^n$  up to multiplication by an element not in any associated prime of  $I$ .

**Remark 2.8.** Let  $I$  be an ideal in a noetherian ring  $R$ . Setting

$$W := R \setminus \bigcup_{P \in \text{Ass}(I)} P$$

we claim that

$$\begin{aligned} I^{(n)} &= W^{-1} I^n \cap R = \{r \in R \mid sr \in I^n \text{ for some } s \in W\} \\ &= \{r \in R \mid sr \in I^n \text{ for some } s \notin P \text{ for all } P \in \text{Ass}(I)\} \end{aligned}$$

If  $a \in I^{(n)}$ , then  $a \in I^n R_P \cap R$  for each  $P \in \text{Min}(I)$ , so for each such  $P$  there exists an element  $s \notin P$  with  $sa \in I^n$ . Therefore,  $(I^n : a)$  is not contained in any  $P \in \text{Ass}(I)$ . There are finitely many such primes, by Theorem 1.44, so by [Prime Avoidance](#)

$$(I^n : a) \not\subseteq \bigcup_{P \in \text{Ass}(I)} P.$$

Therefore, there exists  $s \in W$  such that  $sa \in I^n$ . Now assume  $s \in W$  and  $a \in R$  satisfy  $sa \in I^n$ . For each  $P \in \text{Ass}(I)$ ,  $s \notin P$ , so  $sa \in I^n$  gives  $a \in I^n R_P \cap R$ . Therefore,  $a \in I^{(n)}$ .

**Lemma 2.9.** *Let  $I$  be an ideal in a noetherian ring  $R$ .*

- (1) *For all  $n \geq 1$ ,  $I^n \subseteq I^{(n)}$ .*
- (2) *If  $I$  has no embedded primes, then  $I^{(1)} = I$ .*
- (3) *If  $R$  is a domain and  $I$  is a nonzero ideal, then  $I^a \subseteq I^{(b)}$  implies  $a \geq b$ .*
- (4) *If  $a \geq b$ , then  $I^{(a)} \subseteq I^{(b)}$ .*
- (5) *For all  $a, b \geq 1$ ,  $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$ .*

*Proof.*

- (1) This is a set-theoretic statement: any set is contained in the preimage of its own image by any map. In particular, for all associated primes  $P$  of  $I$ ,  $I^n \subseteq I^n R_P \cap R$ .
- (2) By Theorem 1.68, the minimal primary components of  $I^n$  are unique, and the component associated to each  $P \in \text{Min}(I^n) = \text{Min}(I) = \text{Ass}(I)$  is given by  $I^n R_P \cap R$ . Since all the associated primes of  $I$  are minimal,

$$I = \bigcap_{P \in \text{Min}(I)} (I R_P \cap R) = \bigcap_{P \in \text{Ass}(I)} (I R_P \cap R) = I^{(1)}.$$

- (3) Suppose  $I^a \subseteq I^{(b)}$ , and let  $P$  be an associated prime of  $I$ . We have

$$(I_P)^a = (I^a)_P \subseteq (I^{(b)})_P = (I_P)^b.$$

Write  $J = I_P$ , and note that  $J$  is contained in the unique maximal ideal of  $R_P$ . If  $a < b$ , it would follow that  $J^a = J^b$ , which by NAK implies  $J = 0$ . Since  $R$  is a domain, the localization map is injective, and thus  $J = 0$  happens only if  $I = 0$ .

- (4) Since  $a \geq b$ ,  $I^a \subseteq I^b$ , so  $I^a R_P \subseteq I^b R_P$  for any prime ideal  $P$ . Since taking preimages preserves inclusions, we conclude that  $I^a R_P \cap R \subseteq I^b R_P \cap R$ , and thus  $I^{(a)} \subseteq I^{(b)}$ .
- (5) Let  $P \in \text{Ass}(I)$  and let  $\pi$  be the canonical homomorphism  $R \rightarrow R_P$ . Given  $x \in I^{(a)}$  and  $y \in I^{(b)}$ ,  $\pi(xy) = \pi(x)\pi(y) \in I^a I^b R_P$ . Now notice that  $I^a I^b = I^{a+b}$ , so  $\pi(xy) \in I^{a+b} R_P$ , and  $xy \in I^{a+b} R_P \cap R$ . Since this holds for all  $P \in \text{Ass}(I)$ , we conclude that  $xy \in I^{(a+b)}$ .  $\square$

**Remark 2.10.** Let  $I$  be a radical ideal with minimal primes  $P_1, \dots, P_s$ , so  $I = P_1 \cap \dots \cap P_s$ . As we discussed in the proof of Theorem 1.68,  $I R_{P_i} = P_i R_{P_i}$ . Since localization commutes with powers, we conclude that  $I^n R_{P_i} = P_i^n R_{P_i}$ , and thus

$$I^{(n)} = P_1^{(n)} \cap \dots \cap P_s^{(n)}.$$

**Example 2.11.** Consider the radical ideal  $I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$ . As in Remark 2.10, its symbolic powers are given by

$$I^{(n)} = (x, y)^{(n)} \cap (x, z)^{(n)} \cap (y, z)^{(n)},$$

which by Lemma 2.5 can be simplified to

$$I^{(n)} = (x, y)^n \cap (x, z)^n \cap (y, z)^n.$$

Now  $xyz \in I^{(2)}$  but  $xyz \notin I^2$ , since  $I^2$  is generated by homogeneous elements of degree 2.

Just because  $I^{(n)} = I^n$  for some  $n > 1$ , that does not mean that  $I^{(m)} = I^m$  for other values of  $m$ . Here is an example by Susan Morey [Mor96].

**Example 2.12** (Morey). Let  $k$  be a field,  $R = k[x_1, x_2, x_3, x_4]$ , and consider the matrix

$$M = \begin{pmatrix} 0 & -x_1 & -x_3 & x_2 & -x_1 & x_4 & -x_3 \\ x_1 & 0 & -x_3 & x_2 & x_1 & -x_4 & -x_1 \\ x_3 & x_3 & 0 & 0 & -x_3 & x_1 & x_4 \\ -x_2 & -x_2 & 0 & 0 & -x_4 & x_2 & 0 \\ x_1 & -x_1 & x_3 & x_4 & 0 & -x_3 & x_1 \\ -x_4 & x_4 & -x_1 & -x_2 & x_3 & 0 & x_2 \\ x_3 & x_1 & x_4 & 0 & -x_1 & x_2 & 0 \end{pmatrix}.$$

This is a skew-symmetric matrix, meaning that  $M_{ij} = -M_{ji}$  for all  $i, j$ . The even sized minors of such a matrix turn out to be squares, so we can consider polynomials  $f$  such that  $f^2$  is an even-sized minor of  $M$ . The ideal generated by the  $2t$ -sized minors of a skew-symmetric matrix is called the **ideal of  $2t \times 2t$  pfaffians** of  $M$ .

The ideal  $I$  of  $6 \times 6$  pfaffians of  $M$  is an example of an ideal that satisfies  $I^{(2)} = I^2$  but  $I^{(3)} \neq I^3$ , which one can check with Macaulay2.

We end this section with a discussion on the symbolic powers of a general ideal.

**Discussion 2.13.** Let  $I$  be any ideal, possibly with embedded primes. By definition, if  $Q \subseteq P$  are primes containing  $I$ , then  $I^n R_P \cap R \subseteq I^n R_Q \cap R$ . Thus to compute  $I^{(n)}$ , it is sufficient to consider the maximal primes in  $\text{Ass}(I)$ , denoted  $\text{mAss}(I)$ , and

$$I^{(n)} = \bigcap_{P \in \text{Ass}(I)} I^n R_P \cap R = \bigcap_{P \in \text{mAss}(I)} I^n R_P \cap R.$$

Now fix  $P \in \text{Ass}(I)$  and let  $J = IR_P \cap R$ . By Remark 1.69,  $J = IR_P \cap P$  is the intersection of all the primary components of  $I$  that are contained in  $P$ , which is by construction a primary decomposition of  $J$ . Therefore,  $\text{Ass}(J) = \text{Ass}(IR_P \cap R) = \{Q \in \text{Ass}(I) \mid Q \subseteq P\}$ . In particular,  $JR_P = IR_P$ , and since powers commute with localization,  $J^n R_P = I^n R_P$  for all  $n$ . Therefore,  $J^{(n)} = I^n R_P \cap R$ , and

$$I^{(n)} = \bigcap_{P \in \text{mAss}(I)} (IR_P \cap R)^{(n)}.$$

In words,  $I^{(n)}$  can be decomposed into the symbolic powers of the ideals obtained by collecting all the primary components of  $I^n$  contained in each maximal associated prime of  $I$ ; each one of these is an ideal with only one maximal associated prime.

Now suppose that  $I$  is an ideal with a unique maximal associated prime, say  $P$ . An element  $s$  is contained in some associated prime of  $I$  if and only if  $s \in P$ , so

$$I^{(n)} = \{r \in R \mid sr \in I^n \text{ for some } s \notin \bigcup_{Q \in \text{Ass}(I)} Q\} = \{r \in R \mid sr \in I^n \text{ for some } s \notin P\}.$$

Thus for a general ideal  $I$ ,

$$I^{(n)} = \left\{ r \in R \mid sr \in I^n \text{ for some } s \notin \bigcup_{P \in \text{mAss}(I)} P \right\}.$$

## 2.2 Computing symbolic powers

From what we have seen so far, it appears that to compute the symbolic powers of a given ideal we need only to find a primary decomposition of  $I^n$ , and then collect the appropriate components. In practice, that would be a terrible idea: as we briefly discussed in the previous chapter, computing primary decompositions is a computationally difficult problem, and to add to that, the number of calculations involved in computing powers of ideals grows very fast. One of the problems we will study is how to find effective and practical ways to compute the symbolic powers of an ideal — and most importantly, to avoid computing primary decompositions.

We saw in Exercise 1 that we can compute the minimal primary components of any ideal via saturation, so we can now put the same trick to use to compute symbolic powers. More interestingly, given an ideal  $I$  with no embedded primes, we can construct an ideal  $J$  such that  $I^{(n)} = (I^n : J^\infty)$  for all  $n$ . Saturations are computationally simple, so the difficult task is only to find the appropriate ideal to saturate with.

To show that we can compute the symbolic powers of  $I$  by saturating with a fixed  $J$ , we will use the beautiful and extraordinary fact that there are only finitely many primes that are associated to some power of  $I$ . This was shown by Ratliff in 1976 [Rat76]. In 1979, Brodmann [Bro79] showed that more is true: when  $n$  is sufficiently large,  $\text{Ass}(I^n)$  does not depend on  $n$ . Unfortunately, we will have to establish a bit more background before we can prove this.

**Theorem 2.14** (Ratliff, 1976 [Rat76], Brodmann, 1979 [Bro79]). *If  $I$  is an ideal in a noetherian ring  $R$ , then  $\text{Ass}(I^n)$  stabilizes, meaning that there exists  $N$  such that  $\text{Ass}(I^n)$  is independent of  $n \geq N$ . In particular,*

$$\bigcup_{n \geq 1} \text{Ass}(I^n)$$

*is finite.*

**Notation 2.15.** We denote the set of all primes that are associated to some power of  $I$  by

$$\mathcal{A}(I) := \bigcup_{n \geq 1} \text{Ass}(I^n).$$



This will allow us to show that to compute the symbolic powers of an ideal  $I$ , we can now take saturations with respect to a fixed ideal; in fact, we can take saturations with respect to a fixed principal ideal. To do that, however, we will need to use [Prime Avoidance](#).

**Exercise 2.** Let  $I$  be an ideal with no embedded primes in a noetherian ring  $R$ . There exists an ideal  $J$ , which we can take to be principal, such that  $I^{(n)} = (I^n : J^\infty)$  for all  $n \geq 1$ .

The hard part, of course, is constructing this ideal explicitly and in a computationally efficient way.

## 2.3 Homogeneous ideals

We have already seen a few examples of symbolic powers of homogeneous ideals where we took advantage of being able to look at degrees of homogeneous elements. And indeed, the symbolic powers of a homogeneous ideal are also homogeneous.

**Theorem 2.16.** *Let  $R$  be a noetherian graded ring. If  $I$  is a homogeneous ideal, then its symbolic powers  $I^{(n)}$  are homogeneous for all  $n \geq 1$ .*

*Proof.* By Exercise 2, there exists an element  $t$  not in any associated prime of  $I$ , but in every prime  $P$  such that  $P \in \text{Ass}(I^n)$  for some  $n$  but  $P \notin \text{Ass}(I)$ , and we can compute  $I^{(n)}$  by taking  $I^{(n)} = (I^n : t^\infty)$ . By Theorem 1.44, the associated primes of  $I^n$  are all homogeneous for all  $n$ , so we can choose such  $t$  to be homogeneous.

If  $f \in I^{(n)}$ , then for all  $m$  large enough,  $ft^m \in I^n$ . Since  $I$  is homogeneous, so is  $I^n$ , and thus every homogeneous component of  $ft^m$  must be in  $I^n$ . Decompose  $f = \sum_i f_i$  as a finite sum of homogeneous elements of degree  $i$ . Since  $t^m$  is homogeneous, the homogeneous components of  $ft^m$  are of the form  $f_i t^m$ , so each  $f_i t^m \in I^n$ . Therefore,  $f_i \in I^{(n)}$  for all  $i$ , and thus  $I^{(n)}$  is homogeneous.  $\square$

We have seen several examples of homogeneous ideals whose symbolic powers do not coincide with the powers because they have elements in *wrong* degrees. This is a very common phenomenon. In the examples we saw before, we had elements of degrees that were too small.

**Definition 2.17.** If  $I$  is a homogeneous ideal in an  $\mathbb{N}$ -graded ring  $R$ , then

$$\alpha(I) := \min \{ \deg(f) \mid 0 \neq f \in I \text{ is a homogeneous element} \}.$$

So  $\alpha(I)$  is the smallest degree of a homogeneous generator for  $I$ .

**Example 2.18.** Consider a  $3 \times 3$  matrix of variables,

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}.$$

Given a field  $k$ , we write  $R = k[X]$  for the polynomial ring  $R = k[x_1, \dots, x_9]$ , and  $I_t(X)$  for the ideal in  $R$  generated by the  $t$ -minors of  $X$ . Let  $I = I_2(X)$ , which is a homogeneous

ideal generated in degree 2. It is a nontrivial fact (which we won't prove) that this ideal is in fact prime. We claim  $I^{(2)} \neq I^2$ , and once more we will use a degree argument. To see this, consider the element  $\det(X)$ , which we can write for example using cofactor expansion on the first row:

$$\det(X) = x_1 \begin{vmatrix} x_5 & x_6 \\ x_8 & x_9 \end{vmatrix} - x_2 \begin{vmatrix} x_4 & x_6 \\ x_7 & x_9 \end{vmatrix} + x_3 \begin{vmatrix} x_4 & x_5 \\ x_7 & x_8 \end{vmatrix}.$$

This is clearly an element in  $I$ , since we wrote it as a linear combination of elements in  $I$ . On the other hand, this is *not* an element in  $I^2$ , since it has degree 3 and  $\alpha(I^2) = 4$ . On the other hand, with a few careful computations one can see that

$$x_1 \det(X) = \begin{vmatrix} x_1 & x_3 \\ x_7 & x_9 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ x_4 & x_5 \end{vmatrix} - \begin{vmatrix} x_1 & x_2 \\ x_7 & x_8 \end{vmatrix} \begin{vmatrix} x_1 & x_3 \\ x_4 & x_6 \end{vmatrix} \in I^2,$$

and since  $x_1 \notin I$ , we conclude that  $\det(X) \in I^{(2)}$ . So we have shown that  $I^2 \neq I^{(2)}$ .

In Example 2.18, Example 2.4 and Example 2.11, we found a homogeneous element in  $I^{(n)}$  of degree smaller than  $\alpha(I^n)$ . The existence of such elements is not necessary for  $I^n \neq I^{(n)}$ .

**Example 2.19.** Let  $R = \mathbb{Q}[x, y, z]$ , and let  $P$  be the defining ideal of the curve parametrized by  $(t^9, t^{11}, t^{14})$ . Here are some Macaulay2 computations.

```
i1 : k = QQ;

i2 : a = 9; b = 11; c = 14;

i6 : R = k[x,y,z, Degrees => {a,b,c}];

i7 : P = ker map(QQ[t],R,{t^a,t^b,t^c})

o7 = ideal (x4 - y2 z, x*y3 - z3, y5 - x3 z2)

o7 : Ideal of R

i8 : associatedPrimes(P^2)

o8 = {ideal (x4 - y2 z, x*y3 - z3, y5 - x3 z2), ideal (x, y, z)}

o8 : List

i9 : symbolic2 = (select(primaryDecomposition(P^2), Q -> radical(Q) == P))_0

o9 = ideal (x8 - 2x4 y2 z + y4 z2, x5 y3 - x*y5 z - x4 z3 + y2 z6, x3 y3 - 2x*y3 z + z6,
-----
y8 + x7 y*z - 3x3 y3 z + x2 z5, x4 y5 - y7 z - x7 z2 + x3 y2 z3)
```

```

o9 : Ideal of R

i10 : degrees symbolic2

o10 = {{72}, {78}, {84}, {88}, {91}}

o10 : List

i11 : degrees (P^2)

o11 = {{72}, {78}, {91}, {84}, {97}, {110}}

o11 : List

```

In line i8 we learned that  $P^2$  has one embedded prime, the homogeneous maximal ideal. This automatically tells us that  $P^2 \neq P^{(2)}$ . We computed  $P^{(2)}$  and called it `symbolic2` by simply taking a primary decomposition of  $P^2$  and selecting the  $P$ -primary component. Since both  $P^2$  and  $P^{(2)}$  are homogeneous ideals, we then asked Macaulay2 for the degrees of the generators using the command `degrees`, and found two interesting things:

- $\alpha(P^2) = \alpha(P^{(2)})$ , despite the fact that  $P^2 \neq P^{(2)}$ .
- $P^{(2)}$  has a minimal generator of degree 84, while  $P^2$  does not.

We can ask Macaulay2 to help us identify this generator of  $P^{(2)}$  of degree 84:

```

i14 : select(flatten entries mingens (symbolic2), f -> degree(f) == {88})
      8      7      3 3 2      2 5
o14 = {y  + x y*z - 3x y z  + x z }

o14 : List

```

We can do even more, and find all the generators of  $P^{(2)}$  in our generating set that are not in  $P^2$ .

```

i19 : select(flatten entries mingens (symbolic2), f -> f%(P^2) != 0)
      8      7      3 3 2      2 5
o19 = {y  + x y*z - 3x y z  + x z }

o19 : List

i20 : f = oo_0
      8      7      3 3 2      2 5
o20 = y  + x y*z - 3x y z  + x z

o20 : R

```

```
i21 : P^2 + ideal(f) == symbolic2
```

```
o21 = true
```

This last computation tells us that this element  $f = y^8 + x^7yz - 3x^3y^3z^2 + x^2z^5$  of degree 88 satisfies  $P^{(2)} = P^2 + (f)$ .

## 2.4 Where are we going?

In order to understand symbolic powers better, we will need to first take a short detour to develop a few more fundamental commutative algebra tools. Here are some of the main questions that will motivate our study:

### Computing symbolic powers

As we saw in the previous section, we need only to find an appropriate element  $x$  such that  $I^{(n)} = (I^n : J^\infty)$ . We have discussed how to find this element in theory, but our description appears to require knowledge of all the associated primes of all the powers of  $I$ . This is of course unreasonable, and our goal is to find more effective ways to find such an  $x$ .

We will also discuss how to find symbolic powers via other methods for some special classes of ideals. To do this, we will use all sorts of different tools. For example, we will discuss how to apply combinatorics to study the symbolic powers of monomial ideals.

### Equality

Computing symbolic powers would be an easy problem if  $I^{(n)} = I^n$ . So when does this hold? What are some sufficient or necessary conditions for this to hold for some fixed  $n$ , or for all  $n$ ? Can we test the equality  $I^{(n)} = I^n$  for all  $n$  by doing only finitely many tests?

### Minimal degrees

When  $I$  is a homogeneous ideal,  $I^{(n)}$  is also homogeneous, but its elements may have unexpected degrees. We have briefly discussed some of the behavior we may find, but we would like to explore this further. Will keep a special eye towards lower bounds on  $\alpha(I^{(n)})$ , since  $n\alpha(I) = \alpha(I^n)$  is the obvious upper bound.

### Finite generation of symbolic Rees algebras

In Lemma 2.9 we showed that  $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$ . So as we compute higher and higher symbolic powers of  $I$ , we can think of all the elements in

$$\sum_{a_1 + \dots + a_{n-1} = n} I^{a_1} \dots (I^{(n-1)})^{a_{n-1}} \subseteq I^{(n)}$$

as *expected*, and any other element in  $I^{(n)}$  as *unexpected*. Roughly speaking, we want to understand whether we will see unexpected elements in  $I^{(n)}$  for arbitrarily large  $n$ , or whether there is a finite set of symbolic powers which describe all the remaining ones. We will make this problem precise when we study the symbolic Rees algebra of  $I$ , which is a graded algebra we construct from the symbolic powers of  $I$ . In a surprising twist, these algebras will sometimes be NOT noetherian.

## Comparing powers and symbolic powers of ideals

While the symbolic and ordinary powers of an ideal are not necessarily equal, they are of course related. How different are  $I^n$  and  $I^{(n)}$ , really? And how can we formalize this question? One way is of course by studying  $\alpha(I^{(n)})$  and comparing it to  $\alpha(I^n)$ , but there are other ways to formalize this comparison. One in particular we will discuss is known as the Containment Problem, which is the question of when  $I^{(a)} \subseteq I^b$ . Another way we can formalize this question is to ask whether  $I^{(n+1)}$  contains any minimal generator of  $I^{(n)}$ , which is roughly speaking the content of an open problem known as the Eisenbud-Mazur conjecture.

## A geometric perspective

Over a polynomial ring over a perfect field, there is a geometric interpretation for the symbolic powers of a radical ideal. We will discuss this, and take it in part as a motivation to study symbolic powers.

But before we do any of this, it's time for a short detour through some classical commutative algebra topics that algebraists should keep in their back pocket.

# Chapter 3

## Sharpening our tools

### 3.1 Dimension and height

**Definition 3.1.** A chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

has **length**  $n$ . We say a chain of primes is **saturated** if for each  $i$  there is no prime  $\mathfrak{q}$  with  $\mathfrak{p}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_{i+1}$ . The **Krull dimension** of a ring  $R$  is the supremum of the lengths of chains of primes in  $R$ , and we denote it by  $\dim(R)$ . The **height** of a prime  $\mathfrak{p}$  is the supremum of the lengths of chains of primes in  $R$  that end in  $\mathfrak{p}$ , i.e., with  $\mathfrak{p} = \mathfrak{p}_n$  above, and we denote it by  $\text{ht}(\mathfrak{p})$ . The **height** of an ideal  $I$  is given by

$$\text{ht}(I) := \inf \{ \text{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \text{Min}(I) \}$$

**Macaulay2.** We can compute the dimension of a ring using `dim`. To compute the height of an ideal  $I$ , we use the method `codim`.

**Definition 3.2.** The **dimension** of an  $R$ -module  $M$  is defined as  $\dim(R/\text{ann}_R(M))$ .

Note that if  $M$  is finitely generated,  $\dim(M)$  is the same as the supremum of the lengths of chains of primes in  $\text{Supp}_R(M)$ .

**Remark 3.3.**

- 1) If  $I$  is an ideal, then  $\dim(R/I)$  is the supremum of the lengths of chains of primes in  $R$

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$$

with each  $\mathfrak{q}_i \in V(I)$ .

- 2) If  $W$  is a multiplicative set, then  $\dim(W^{-1}R) \leq \dim(R)$ .
- 3) If  $\mathfrak{p}$  is prime, then  $\dim(R_{\mathfrak{p}}) = \text{ht}(\mathfrak{p})$ .
- 4) If  $\mathfrak{q} \supseteq \mathfrak{p}$  are primes, then  $\dim(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}})$  is the supremum of the lengths of all chains of primes in  $R$  of the form

$$\mathfrak{p} = \mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots \subsetneq \mathfrak{a}_n = \mathfrak{q}.$$

- 5)  $\dim(R) = \sup\{\text{ht}(\mathfrak{m}) \mid \mathfrak{m} \in \text{mSpec}(R)\}.$
- 6)  $\dim(R) = \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Min}(R)\}.$
- 7) If  $I$  is an ideal,  $\dim(R/I) + \text{ht}(I) \leq \dim(R).$
- 8) The ideal  $(0)$  has height 0.
- 9) A prime ideal has height zero if and only if it is a minimal prime of  $R.$

We will need a few theorems before we compute the height and dimension of many examples, but we can handle a few basic cases.

**Example 3.4.**

- a) The dimension of a field is zero.
- b) A ring is zero-dimensional if and only if every minimal prime is maximal.
- c) The ring of integers  $\mathbb{Z}$  has dimension 1: there is one minimal prime  $(0)$  and every other prime is maximal. More generally, any principal ideal domain has dimension 1.
- d) In a UFD, we claim that  $I$  is a prime of height 1 if and only if  $I = (f)$  with  $f$  prime element.

To see this, note that if  $I = (f)$  with  $f$  irreducible, and  $0 \subsetneq \mathfrak{p} \subseteq I$ , then  $\mathfrak{p}$  contains some nonzero multiple of  $f$ , say  $af^n$  with  $a$  and  $f$  coprime. Since  $a \notin I$ ,  $a \notin \mathfrak{p}$ , so we must have  $f \in \mathfrak{p}$ , so  $\mathfrak{p} = (f)$ . Thus,  $I$  has height one. On the other hand, if  $I$  is a prime of height one, we claim  $I$  contains an irreducible element. Indeed,  $I$  is nonzero, so it contains some  $f \neq 0$ , and primeness implies one of the prime factors of  $f$  is contained in  $I$ . Thus, any nonzero prime contains a prime ideal of the form  $(f)$ , so a height one prime must be of this form.

- e) If  $k$  is a field, then  $\dim(k[x_1, \dots, x_d]) \geq d$ , since there is a saturated chain of primes  $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_d).$

**Definition 3.5.** A ring is **catenary** if for every pair of primes  $\mathfrak{q} \supseteq \mathfrak{p}$  in  $R$ , every saturated chain of primes

$$\mathfrak{p} = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = \mathfrak{q}$$

has the same length. A ring is **equidimensional** if every maximal ideal has the same finite height and  $\dim(R/P)$  is the same finite number for every minimal prime  $P$ .

Here are some examples of what can go wrong.

**Example 3.6.** We can find the minimal primes of the ring

$$R = \frac{k[x, y, z]}{(xy, xz)}$$

by computing  $\text{Min}((xy, xz))$  in  $k[x, y, z]$ . As we saw in Example 1.12, the primes  $(x)$  and  $(y, z)$  are incomparable, and  $(x) \cap (y, z) = (xy, xz)$ , so  $\text{Min}(R) = \{(x), (y, z)\}$ . We claim that

the height of  $(x-1, y, z)$  in  $R$  is one: it contains the minimal prime  $(y, z)$ , and any saturated chain from  $(y, z)$  to  $(x-1, y, z)$  corresponds to a saturated chain from  $(0)$  to  $(x-1)$  in  $k[x]$ , which must have length 1 since this is a PID. The height of  $(x, y-1, z)$  is at least 2, as witnessed by the chain  $(x) \subseteq (x, y-1) \subseteq (x, y-1, z)$ . So  $R$  is not equidimensional. One can also show that the minimal primes of  $(xy, xz)$  in  $k[x, y, z]$  have different heights: the prime  $(x)$  has height 1, while  $(y, z)$  has height at least 2, since  $(0) \subsetneq (y) \subsetneq (y, z)$ . In fact, we will soon see that any ideal generated by 2 elements must have height at most 2, and thus  $(y, z)$  has height exactly 2.

While the previous example is not a domain, even domains may fail to be equidimensional.

**Example 3.7.** The ring  $\mathbb{Z}_{(2)}[x]$  is a domain that is not equidimensional. On the one hand, the maximal ideal  $(2, x)$  has height at least two, which we see from the chain

$$(0) \subsetneq (x) \subseteq (x, 2).$$

On the other hand, the ideal  $(2x-1)$  has height 1, and it is maximal since the quotient is  $\mathbb{Q}$ .

**Remark 3.8.**

- a) If  $R$  is a finite dimensional domain, and  $f \neq 0$ , then  $\dim(R/(f)) < \dim(R)$ .
- b) If  $R$  is equidimensional, then  $\dim(R/(f)) < \dim(R)$  if and only if  $f \notin \bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}$ .
- c) In general,  $\dim(R/(f)) < \dim(R)$  if and only if  $f \notin \bigcup_{\substack{\mathfrak{p} \in \text{Min}(R) \\ \dim(R/\mathfrak{p}) = \dim(R)}} \mathfrak{p}$ .
- d)  $f \notin \bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}$  if and only if  $\dim(R/(\mathfrak{p} + (f))) < \dim(R/\mathfrak{p})$  for all  $\mathfrak{p} \in \text{Min}(R)$ .

Krull's Height Theorem, which we will now prove, gives us the most elementary and important connection between height and number of generators. This is also an excellent example of a theorem in elementary commutative algebra that we can prove using symbolic powers.

**Theorem 3.9** (Krull's Principal Ideal theorem). *Let  $R$  be a Noetherian ring, and  $f \in R$ . Then, every minimal prime of  $(f)$  has height at most one.*

Note that this is stronger than the statement that the height of  $(f)$  is at most one: that would only mean that some minimal prime of  $(f)$  has height at most one.

*Proof.* Suppose the theorem is false, so that there is some ring  $R$ , a prime  $\mathfrak{p}$ , and an element  $f$  such that  $\mathfrak{p}$  is minimal over  $(f)$  and  $\text{ht}(\mathfrak{p}) > 1$ . If we localize at  $\mathfrak{p}$  and then mod out by an appropriate minimal prime, we obtain a Noetherian local domain  $(R, \mathfrak{m})$  of dimension at least two in which  $\mathfrak{m}$  is the unique minimal prime of  $(f)$ , so let's work over that Noetherian local domain  $(R, \mathfrak{m})$ . Note that  $\overline{R} = R/(f)$  is zero-dimensional, since  $\mathfrak{m}$  is the only minimal prime over  $(f)$ . Back in  $R$ , let  $\mathfrak{q}$  be a prime strictly in between  $(0)$  and  $\mathfrak{m}$ , and notice that we necessarily have  $f \notin \mathfrak{q}$ .



Consider the symbolic powers  $\mathfrak{q}^{(n)}$  of  $\mathfrak{q}$ . We will show that these stabilize in  $R$ . Since  $\overline{R} = R/(f)$  is Artinian, the descending chain of ideals

$$\mathfrak{q}\overline{R} \supseteq \mathfrak{q}^{(2)}\overline{R} \supseteq \mathfrak{q}^{(3)}\overline{R} \supseteq \dots$$

stabilizes. We then have some  $n$  such that  $\mathfrak{q}^{(n)}\overline{R} = \mathfrak{q}^{(m)}\overline{R}$  for all  $m \geq n$ , and in particular,  $\mathfrak{q}^{(n)}\overline{R} = \mathfrak{q}^{(n+1)}\overline{R}$ . Pulling back to  $R$ , we get  $\mathfrak{q}^{(n)} \subseteq \mathfrak{q}^{(n+1)} + (f)$ . Then any element  $a \in \mathfrak{q}^{(n)}$  can be written as  $a = b + fr$ , where  $b \in \mathfrak{q}^{(n+1)} \subseteq \mathfrak{q}^{(n)}$  and  $r \in R$ . Notice that this implies that  $fr \in \mathfrak{q}^{(n)}$ . Since  $f \notin \mathfrak{q}$ , we must have  $r \in \mathfrak{q}^{(n)}$ . This yields  $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$ . Thus,  $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f(\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)})$ , so  $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = \mathfrak{m}(\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)})$ . By [NAK](#),  $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$  in  $R$ . Similarly, we obtain  $\mathfrak{q}^{(n)} = \mathfrak{q}^{(m)}$  for all  $m \geq n$ .

Now, if  $a \in \mathfrak{q}$  is nonzero, we have  $a^n \in \mathfrak{q}^n \subseteq \mathfrak{q}^{(n)} = \mathfrak{q}^{(m)}$  for all  $m$ , so

$$\bigcap_{m \geq 1} \mathfrak{q}^{(m)} = \bigcap_{m \geq n} \mathfrak{q}^{(m)} = \mathfrak{q}^{(n)}.$$

Notice that  $\mathfrak{q}^n \neq 0$  because  $R$  is a domain, and so  $\mathfrak{q}^{(n)} \supseteq \mathfrak{q}^n$  is also nonzero. So

$$\bigcap_{m \geq 1} \mathfrak{q}^{(m)} = \mathfrak{q}^{(n)} \neq 0.$$

On the other hand,  $\mathfrak{q}^{(m)} = \mathfrak{q}^m R_{\mathfrak{q}} \cap R$  for all  $m$ , and

$$\bigcap_{m \geq 1} \mathfrak{q}^{(m)} R_{\mathfrak{q}} \subseteq \bigcap_{m \geq 1} \mathfrak{q}^m R_{\mathfrak{q}} = \bigcap_{m \geq 1} (\mathfrak{q} R_{\mathfrak{q}})^m = 0$$

by [Krull's Intersection theorem](#). Since  $R$  is a domain, the contraction of  $(0)$  in  $R_{\mathfrak{q}}$  back in  $R$  is  $(0)$ . This is the contradiction we seek. So no such  $\mathfrak{q}$  exists, so that  $R$  has dimension 1, and in the original ring, all the minimal primes over  $f$  must have height at most 1.  $\square$

To generalize this to ideals generated by  $n$  elements, it is not so straightforward to run an induction. We will need a lemma that allows us to control the chains of primes we get.

**Lemma 3.10.** *Let  $R$  be Noetherian,  $\mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{a}$  be primes, and  $f \in \mathfrak{a}$ . Then there is some  $\mathfrak{q}'$  with  $\mathfrak{p} \subsetneq \mathfrak{q}' \subsetneq \mathfrak{a}$  and  $f \in \mathfrak{q}'$ .*

*Proof.* If  $f \in \mathfrak{p}$ , there is nothing to prove, since we can simply take  $\mathfrak{q}' = \mathfrak{q}$ . Suppose  $f \notin \mathfrak{p}$ . After we quotient out by  $\mathfrak{p}$  and localize at  $\mathfrak{a}$ , we may assume that  $\mathfrak{a}$  is the maximal ideal. We want to find a nonzero prime  $\mathfrak{q}' \subsetneq \mathfrak{a}$ . Our assumption implies that  $f \neq 0$ , and then by [Theorem 3.9](#), minimal primes of  $(f)$  have height one, hence are not  $\mathfrak{a}$  nor  $\mathfrak{p}$ . We can take  $\mathfrak{q}'$  to be one of the minimal primes of  $f$ .  $\square$

**Theorem 3.11** (Krull's Height Theorem). *Let  $R$  be a Noetherian ring. If  $I$  is an ideal generated by  $n$  elements, then every minimal prime of  $I$  has height at most  $n$ .*

*Proof.* By induction on  $n$ . The case  $n = 1$  is the [Principal Ideal Theorem](#).

Let  $I = (f_1, \dots, f_n)$  be an ideal,  $\mathfrak{p}$  a minimal prime of  $I$ , and  $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_h = \mathfrak{p}$  be a saturated chain of length  $h$  ending at  $\mathfrak{p}$ . If  $f_1 \in \mathfrak{p}_1$ , then we can apply the induction hypothesis to the ring  $\overline{R} = R/((f_1) + \mathfrak{p}_0)$  and the ideal  $(f_2, \dots, f_n)\overline{R}$ . Then by induction

hypothesis, the chain  $\mathfrak{p}_1 \overline{R} \subsetneq \cdots \subsetneq \mathfrak{p}_h \overline{R}$  has length at most  $n - 1$ , so  $h - 1 \leq n - 1$  and  $\mathfrak{p}$  has height at most  $n$ .

If  $f_1 \notin \mathfrak{p}_1$ , we use the previous lemma to replace our given chain with a chain of the same length but such that  $f_1 \in \mathfrak{p}_1$ . To do this, note that  $f_1 \in \mathfrak{p}_i$  for some  $i$ ; after all,  $f_1 \in I \subseteq \mathfrak{p}$ . So in the given chain, suppose that  $f_1 \in \mathfrak{p}_{i+1}$  but  $f_1 \notin \mathfrak{p}_i$ . If  $i > 0$ , apply the previous lemma with  $\mathfrak{a} = \mathfrak{p}_{i+1}$ ,  $\mathfrak{q} = \mathfrak{p}_i$ , and  $\mathfrak{p} = \mathfrak{p}_{i-1}$  to find  $\mathfrak{q}_i$  such that  $f_1 \in \mathfrak{q}_i$ . Replace the chain with

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{i-1} \subsetneq \mathfrak{q}_i \subsetneq \mathfrak{p}_i \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathfrak{p}.$$

Repeat until  $f_1 \in \mathfrak{p}_1$ . □

The bound given by [Krull's Height Theorem](#) is sharp:

**Example 3.12.** Any ideal generated by  $n$  variables in a polynomial ring has height  $n$ . The ideal  $(u^3 - xyz, x^2 + 2xz - 6y^5, vx + 7vy)$  in  $k[u, v, w, x, y, z]$  has height 3.

**Definition 3.13.** An ideal of height  $n$  generated by  $n$  elements is a **complete intersection**.

Here are some other examples.

**Example 3.14.**

- a) As we saw in [Example 3.6](#), the ideal  $(xy, xz)$  in  $k[x, y, z]$  has minimal primes of heights 1 and 2. Its height is 1, though its minimal number of generators is 2.
- b) It is possible to have associated primes of height greater than the number of generators. For a cheap example, in  $R = k[x, y]/(x^2, xy)$ , the ideal generated by zero elements (the zero ideal) has an associated prime of height two, namely  $(x, y)$ .
- c) The same phenomenon can happen even in a nice polynomial ring. For example, consider the ideal  $I = (x^3, y^3, x^2u + xyv + y^2w)$  in  $R = k[u, v, w, x, y]$ . The element  $x^2y^2 \notin I$  has  $(u, v, w, x, y) = (I : x^2y^2)$ , so  $I$  has an associated prime of height 5.
- d) Noetherianity is necessary. Let  $R = k[x, xy, xy^2, \dots] \subseteq k[x, y]$ . For all  $a \geq 1$ ,  $xy^a \notin (x)$ , since  $y^a \notin R$ , but  $(xy^a)^2 = x \cdot xy^{2a} \in (x)$ . Then  $(x)$  is not prime, and  $\mathfrak{m} = (x, xy, xy^2, \dots) \subseteq \sqrt{(x)}$ . Since  $\mathfrak{m}$  is a maximal ideal, we have equality, so  $\text{Min}(x) = \{\mathfrak{m}\}$ . However,  $\mathfrak{p} = (xy, xy^2, xy^3, \dots) = (y)k[x, y] \cap R$  is prime, and the chain  $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$  shows that  $\text{ht}(\mathfrak{m}) > 1$ .

**Corollary 3.15.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring. Then

$$\dim(R) \leq \mu(\mathfrak{m}).$$

*In particular, a Noetherian local ring has finite dimension.*

*Proof.* The dimension of a local ring is the height of its unique maximal ideal, so this is just [Krull's Height Theorem](#) applied to  $\mathfrak{m}$ . If  $R$  is noetherian, then  $\mu(\mathfrak{m})$  is finite, so  $\dim(R)$  must also be finite. □

**Definition 3.16.** The **embedding dimension** of a local ring  $(R, \mathfrak{m})$  is the minimal number of generators of  $\mathfrak{m}$ ,  $\mu(\mathfrak{m})$ . We write  $\text{embdim}(R) := \mu(\mathfrak{m})$  for the embedding dimension of  $R$ .

**Macaulay2.** The minimal number of generators of an ideal or module can be computed via the method `numgens`. We can also find a particular minimal generating set with `mingens`. Unfortunately, these computations are not reliable if the ideal is not homogeneous.

So Corollary 3.15 can be restated as  $\dim(R) \leq \text{embdim}(R)$ . Rings whose dimension and embedding dimension agree are very nicely behaved.

**Definition 3.17.** A Noetherian local ring  $(R, \mathfrak{m})$  is **regular** if  $\dim(R) = \text{embdim}(R)$ .

**Corollary 3.18.** *Let  $k$  be a field. The power series ring  $R = k[[x_1, \dots, x_d]]$  is a regular local ring. Then  $\dim(R) = d$  and  $R$  is a regular local ring.*

*Proof.* Let  $\mathfrak{m} = (x_1, \dots, x_d)$ . The images of  $x_1, \dots, x_d$  in  $\mathfrak{m}/\mathfrak{m}^2$  are linearly independent, so  $\mu(\mathfrak{m}) = d$ . To show that  $R$  is regular, we need to show that  $\dim(R) = d$ . The strict chain of primes

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_d)$$

shows that  $\dim(R) \geq d$ . By Corollary 3.15,  $\dim(R) = d$ . □

But before we can say more about regular rings, we will need to discuss regular sequences. We record here a few results one would cover in a first course in commutative algebra that we will not prove, but that we will want to use later.

**Theorem 3.19.** *If a domain  $R$  is a finitely generated algebra over a field  $k$ , or a quotient of a power series ring over a field, then all the maximal ideals of  $R$  have the same finite height. Moreover,  $\dim(k[x_1, \dots, x_d]) = d$ .*

**Theorem 3.20.** *Let  $R$  be a finitely generated algebra or a quotient of a power series ring over a field.*

1)  $R$  is catenary.

*If additionally  $R$  is a domain, then*

2)  $R$  is equidimensional, and

2)  $\text{ht}(I) = \dim(R) - \dim(R/I)$  for all ideals  $I$ .

**Definition 3.21.** Let  $K \subseteq L$  be an extension of fields. A **transcendence basis** for  $L$  over  $K$  is a maximal algebraically independent subset of  $L$ . The **transcendence degree** of a field extension  $L$  over  $K$  is the common size of any transcendence basis for the extension.

**Definition 3.22.** For a prime  $P$  in a ring  $R$ , we denote the residue field of  $R_P$  by  $\kappa(P)$ . Equivalently,  $\kappa(P)$  is the field of fractions of  $R/P$ .

**Theorem 3.23** (Dimension inequality). *Let  $R \subseteq S$  be an inclusion of domains with  $R$  noetherian. Let  $\mathfrak{q} \in \text{Spec}(S)$  and  $\mathfrak{p} = \mathfrak{q} \cap R \in \text{Spec}(R)$ . Then*

$$\text{height}(\mathfrak{q}) + \text{trdeg}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) \leq \text{height}(\mathfrak{p}) + \text{trdeg}(\text{frac}(S)/\text{frac}(R)).$$

## 3.2 The Koszul complex

The Koszul complex is arguably the most important complex in commutative algebra (and beyond). It appears everywhere, and it is a very powerful yet elementary tool any homological algebraist needs in their toolbox. Every sequence of elements  $x_1, \dots, x_n$  in any ring  $R$  gives rise to a Koszul complex.

**Definition 3.24.** The tensor product of two complexes of  $R$ -modules  $C_\bullet$  and  $D_\bullet$  is the complex  $C_\bullet \otimes_R D_\bullet$  with

$$(C_\bullet \otimes_R D_\bullet)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j,$$

and with differential  $\delta_n$  defined on simple tensors  $x \otimes y \in C_i \otimes_R D_j$  by

$$\delta_n(x \otimes y) = \delta_i^{C_\bullet}(x) \otimes y + (-1)^i x \otimes \delta_i^{D_\bullet}(y).$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 C_{i+1} \otimes D_{j+1} & \xrightarrow{1 \otimes \delta_{j+1}^{D_\bullet}} & C_{i+1} \otimes D_j & \xrightarrow{1 \otimes \delta_j^{D_\bullet}} & C_{i+1} \otimes D_{j-1} & \xrightarrow{1 \otimes \delta_{j-1}^{D_\bullet}} & C_{i+1} \otimes D_{j-2} \\
 & & \downarrow \delta_{i+1}^{C_\bullet} \otimes 1 & & \downarrow \delta_{i+1}^{C_\bullet} \otimes 1 & & \\
 C_i \otimes D_{j+1} & \xrightarrow{1 \otimes \delta_{j+1}^{D_\bullet}} & C_i \otimes D_j & \xrightarrow{1 \otimes \delta_j^{D_\bullet}} & C_i \otimes D_{j-1} & \xrightarrow{1 \otimes \delta_{j-1}^{D_\bullet}} & C_i \otimes D_{j-2} \\
 & & \downarrow \delta_i^{C_\bullet} \otimes 1 & & \downarrow \delta_i^{C_\bullet} \otimes 1 & & \\
 & & C_{i-1} \otimes_R D_j & \xrightarrow{1 \otimes \delta_j^{D_\bullet}} & C_{i-1} \otimes_R D_{j-1} & \xrightarrow{1 \otimes \delta_{j-1}^{D_\bullet}} & C_{i-1} \otimes_R D_{j-2} \\
 & & \downarrow \delta_{i-1}^{C_\bullet} \otimes 1 & & \downarrow \delta_{i-1}^{C_\bullet} \otimes 1 & & \\
 & & \vdots & & \vdots & & 
 \end{array}$$

**Definition 3.25.** The **Koszul complex** on  $r \in R$  is the complex

$$K(r) := 0 \longrightarrow R \xrightarrow{r} R \longrightarrow 0.$$

More generally, the **Koszul complex** on the  $R$ -module  $M$  with respect to  $r \in R$  is

$$K(r; M) = K(r) \otimes_R M = 0 \longrightarrow M \xrightarrow{r} M \longrightarrow 0.$$

Finally, given  $x_1, \dots, x_n \in R$ , the **Koszul complex** on  $M$  with respect to  $x_1, \dots, x_n$  is the complex  $K(x_1, \dots, x_n)$  defined inductively as

$$K(x_1, \dots, x_n; M) = K(x_1, \dots, x_{n-1}; M) \otimes_R K(x_n).$$



for any homogeneous elements  $a$  and  $b$ . We denote the homogeneous elements of degree  $n$  by  $\bigwedge^n M$ . Note also that this construction is functorial: a map  $M \xrightarrow{f} N$  of  $R$ -modules induces a map  $\bigwedge M \xrightarrow{\bigwedge f} \bigwedge N$  given by  $m_1 \wedge \cdots \wedge m_s \mapsto f(m_1) \wedge \cdots \wedge f(m_s)$ .

We will use this construction in the case of free modules. When  $M = R^n$  with basis  $e_1, \dots, e_n$ ,  $\bigwedge^k M \cong R^{\binom{n}{k}}$ , with basis  $e_{i_1} \wedge \cdots \wedge e_{i_s}$  ranging over  $i_1 < i_2 < \cdots < i_s$ ,  $s = \binom{n}{k}$ .

**Definition 3.30.** Let  $x_1, \dots, x_n$  be elements in  $R$ . The **Koszul complex** on  $x_1, \dots, x_n$  is the complex

$$K(x_1, \dots, x_n) := 0 \longrightarrow \bigwedge^n R^n \longrightarrow \bigwedge^{n-1} R^n \longrightarrow \cdots \longrightarrow \bigwedge^1 R^n \longrightarrow R \longrightarrow 0$$

with differential given by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{1 \leq p \leq s} (-1)^{p+1} x_{i_p} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}.$$

More generally, given an  $R$ -module  $M$ , the Koszul complex on  $M$  with respect to  $x_1, \dots, x_n$  is  $K(x_1, \dots, x_n; M) := K(x_1, \dots, x_n) \otimes_R M$ .

**Exercise 3.** Show that  $d$  as defined above is indeed a differential, meaning  $d^2 = 0$ .

**Exercise 4.** Check that our two definitions of the Koszul complex coincide.

**Example 3.31.** Let's compute the Koszul complex on 2 elements  $x_1, x_2$  via this second definition. The complex looks like

$$K_\bullet(x_1, x_2) := 0 \longrightarrow \bigwedge^2 R^2 \longrightarrow \bigwedge^1 R^2 \longrightarrow R \longrightarrow 0 = 0 \longrightarrow R^1 \longrightarrow R^2 \longrightarrow R \longrightarrow 0.$$

The differential in degree 1 is given by

$$d(e_1) = x_1 \text{ and } d(e_2) = x_2$$

while the differential in degree 2 is

$$d(e_1 \wedge e_2) = x_1 e_2 - x_2 e_1,$$

so the Koszul complex is

$$K_\bullet(x_1, x_2) = 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} R \longrightarrow 0.$$

**Remark 3.32.** The Koszul complex  $K(\underline{x}; M)$  looks like

$$0 \longrightarrow M \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

where the map in degree  $n$  is a column with entries  $\pm x_i$ , which after reordering looks like

$$\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \\ \vdots \\ (-1)^{n+1} x_n \end{pmatrix}$$

and the map in degree 0 is

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

The homology of the Koszul complex has nice properties.

**Definition 3.33.** Let  $M$  be an  $R$ -module and  $x_1, \dots, x_n \in R$ . The  $i$ th **Koszul homology** module of  $M$  with respect to  $x_1, \dots, x_d$  is

$$H_i(x_1, \dots, x_d; M) := H_i(K(x_1, \dots, x_d; M)).$$

**Example 3.34.** In  $R = k[x, y]$ ,

$$H_1(x, y; R) = \text{homology of } R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R = 0.$$

**Example 3.35.** In  $R = k[x, y, u, v]/(xu - yv)$ ,

$$H_1(x, y; R) = \frac{\langle \begin{pmatrix} -y \\ x \end{pmatrix}, \begin{pmatrix} u \\ -v \end{pmatrix} \rangle}{\langle \begin{pmatrix} -y \\ x \end{pmatrix} \rangle}.$$

Note that

$$x \begin{pmatrix} u \\ -v \end{pmatrix} = \begin{pmatrix} xu \\ -xv \end{pmatrix} = \begin{pmatrix} yv \\ -xv \end{pmatrix} = -v \begin{pmatrix} -y \\ x \end{pmatrix} = 0 \text{ in } H_1(x, y; R).$$

**Proposition 3.36.** Let  $R$  be a ring,  $\underline{x} = x_1, \dots, x_n \in R$ , and  $I = (x_1, \dots, x_n)$ .

- a)  $H_i(\underline{x}; M) = 0$  whenever  $i < 0$  or  $i > n$ .
- b)  $H_0(\underline{x}; M) = M/IM$ .
- c)  $H_n(\underline{x}; M) = (0 :_M I) = \text{ann}_M(I)$ .
- d) Every Koszul homology module  $H_i(\underline{x}; M)$  is killed by  $\text{ann}_R(M)$ .
- e) Every Koszul homology module  $H_i(\underline{x}; M)$  is killed by  $I$ .
- f) If  $M$  is a Noetherian  $R$ -module, so is  $H_i(\underline{x}; M)$  for every  $i$ .
- g) For every  $i$ ,  $H_i(\underline{x}; -)$  is a covariant additive functor  $R\text{-}\mathbf{mod} \rightarrow R\text{-}\mathbf{mod}$ .
- h) Every short exact sequence of  $R$ -modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives rise to a long exact sequence on Koszul homology,

$$\cdots \longrightarrow H_1(\underline{x}; C) \longrightarrow H_0(\underline{x}; A) \longrightarrow H_0(\underline{x}; B) \longrightarrow H_0(\underline{x}; C) \longrightarrow 0.$$

*Proof.*

- a) Immediate from the definition, since the Koszul complex is only nonzero in homological degrees 0 through  $n$ .

b) The comment above tells us that

$$H_n(\underline{x}; M) = \text{coker} \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix} = M/IM.$$

c) The comment above tells us that

$$\begin{aligned} H_n(\underline{x}; M) &= \ker \left( M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{n+1}x_n \end{pmatrix}^T} M \right) \\ &= \{m \in M \mid rx_1 = rx_2 = \cdots = rx_n = 0\} = (0 :_M I). \end{aligned}$$

d) In each homological degree, the Koszul complex is simply a direct sum of copies of  $M$ . So the modules in  $K(\underline{x}; M)$  in each degree are themselves already killed by  $\text{ann}_I(M)$ .

e) We are going to show something stronger: that for every  $a \in I$ , multiplication by  $a$  on  $K(\underline{x}; M)$  is nullhomotopic, which proves that  $a$  kills the homology of  $K(\underline{x}; M)$ . It is in fact sufficient to show that multiplication by  $a$  is nullhomotopic on  $K(\underline{x})$ , since additive functors preserve the homotopy relation. To do this, we will explicitly use the multiplicative structure of the Koszul complex, and think of our description of the Koszul complex via exterior powers. Given  $a \in I = (x_1, \dots, x_n)$ , write  $a = a_1x_1 + \cdots + a_nx_n$ . Consider the map  $s_a: K(\underline{x}) \rightarrow \Sigma^{-1}K(\underline{x})$  given by multiplication by  $a_1e_1 \wedge \cdots \wedge a_ne_n$ , meaning

$$s_a(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^n a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Now we claim this map  $s_a$  is a nullhomotopy for the map of complexes  $K(\underline{x}) \rightarrow K(\underline{x})$  given by multiplication by  $a$  in every component. To check that, it is sufficient to check that

$$s_a d(e_{i_1} \wedge \cdots \wedge e_{i_t}) + ds_a(e_{i_1} \wedge \cdots \wedge e_{i_t}) = ae_{i_1} \wedge \cdots \wedge e_{i_t}.$$

We have

$$\begin{aligned} s_a d(e_{i_1} \wedge \cdots \wedge e_{i_t}) &= s_a \left( \sum_{k=1}^t (-1)^{k+1} x_k e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \right) \\ &= \sum_{j=1}^n \sum_{k=1}^t (-1)^{k+1} a_j x_k e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and

$$\begin{aligned} ds_a(e_{i_1} \wedge \cdots \wedge e_{i_t}) &= d \left( \sum_{j=1}^n a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t} \right) \\ &= \sum_{j=1}^n \sum_{k=1}^t (-1)^{k+2} a_j e_j \wedge e_{j_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_s} + \sum_{j=1}^n a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \\ &= -s_a d(e_{i_1} \wedge \cdots \wedge e_{i_t}) + \sum_{j=1}^n a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \end{aligned}$$



and since  $\sum_{j=1}^n a_j x_j = a$ , we conclude that

$$(s_a d + ds_a)(e_{i_1} \wedge \cdots \wedge e_{i_t}) = a e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

- f) If  $M$  is Noetherian, then so is  $M^k$  for any  $k$ , as well as any submodules of  $M^k$  and any of their quotients. Each  $H_i(\underline{x}; M)$  is a subquotient of a direct sum of copies of  $M$ , so it must be Noetherian.
- g) An  $R$ -module homomorphism  $M \xrightarrow{f} N$  induces map  $K(f) : K(\underline{x}; M) \longrightarrow K(\underline{x}; N)$  given by  $K(\underline{x}) \otimes f$ , so  $H_i(\underline{x}; f) = H_i(K(\underline{x}) \otimes f)$ .
- h) In each homological degree,  $K(\underline{x})$  has a free module, so  $K(\underline{x}) \otimes_R -$  is exact. We conclude that

$$0 \longrightarrow K(\underline{x}) \otimes_R A \longrightarrow K(\underline{x}) \otimes_R B \longrightarrow K(\underline{x}) \otimes_R C \longrightarrow 0$$

is a short exact sequence of complexes, and the long exact sequence we want is precisely resulting the long exact sequence in homology.  $\square$

**Remark 3.37.** If  $C_\bullet$  is a complex, there exists a short exact sequence of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_\bullet & \longrightarrow & C_\bullet \otimes K_\bullet(f) & \longrightarrow & C_\bullet(-1) \longrightarrow 0 \\ & & & & \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) & & \\ 0 & \longrightarrow & C_n & \xrightarrow{\left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)} & C_n \oplus C'_{n-1} & \xrightarrow{\left( \begin{smallmatrix} 0 & (-1)^{n-1} \end{smallmatrix} \right)} & C'_{n-1} \longrightarrow 0 \end{array}$$

where  $C'_i \cong C_i$ , and the  $'$  indicates that this is the copy tensored with  $K_1(f)$ . Indeed, these are clearly exact, and we only need to check that these give maps of complexes; i.e., that the maps above commute with the differentials. An element  $\nu \in C_n$  maps to  $(\nu, 0)$  in  $C_n \oplus C'_{n-1}$ , which maps to  $(\delta(\nu), 0)$  by the differential on  $C_\bullet \otimes K_\bullet(f)$ , so the map  $C_\bullet \rightarrow C_\bullet \otimes K_\bullet(f)$  is a map of complexes. Likewise, an element  $(\nu, \mu)$  in  $C_n \oplus C'_{n-1}$  maps to an element with second component  $\delta(\mu)$  by the differential on  $C_\bullet \otimes K_\bullet(f)$ , so the map  $C_\bullet \otimes K_\bullet(f) \rightarrow C_\bullet(-1)$  is a map of complexes as well.

The corresponding long exact sequence in homology is

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(C_\bullet) & \longrightarrow & H_n(C_\bullet \otimes K_\bullet(f)) & \longrightarrow & H_n(C'_\bullet(-1)) \xrightarrow{\delta} \cdots \\ & & & & & & \parallel \\ & & & & & & H_{n-1}(C_\bullet) \end{array}$$

We claim that the connecting homomorphism  $\delta$  agrees with multiplication by  $f$ . Indeed, for  $[\eta] \in H_i(C'_\bullet)$ , one has that  $(0, [\eta]) \in H_i(C_\bullet \otimes K_\bullet(f)) \mapsto [\eta] \in H_i(C'_\bullet)$ . Applying the differential yields  $([f\eta], [\delta(g)]) = ([f\eta], 0) \in H_{i-1}(C_\bullet \otimes K_\bullet(f))$ , and  $[f\eta] \in H_{i-1}(C_\bullet) \mapsto ([f\eta], 0) \in H_{i-1}(C_\bullet \otimes K_\bullet(f))$ .

Thus, the long exact sequence breaks into short exact sequences

$$0 \longrightarrow \frac{H_n(C_\bullet)}{f H_n(C_\bullet)} \longrightarrow H_n(C_\bullet \otimes K_\bullet(f)) \longrightarrow \text{ann}_{H_{n-1}(C_\bullet)}(f) \longrightarrow 0.$$

In particular, if  $\underline{x}, y$  is a sequence of elements of  $R$ , and  $M$  is an  $R$ -module, then

$$0 \longrightarrow \frac{H_n(\underline{x}; M)}{y H_n(\underline{x}; M)} \longrightarrow H_n(\underline{x}, y; M) \longrightarrow \text{ann}_{H_{n-1}(\underline{x})}(y) \longrightarrow 0$$

### 3.3 Regular sequences

**Definition 3.38.** Let  $R$  be a ring and  $M$  be an  $R$ -module. An element  $r \in R$  is **regular** (or a nonzerodivisor) on an  $R$ -module  $M$  if

$$rm = 0 \implies m = 0$$

for any  $m \in M$ . More generally, a sequence of elements  $x_1, \dots, x_n$  is a **regular sequence** on  $M$  if

- $(x_1, \dots, x_n)M \neq M$ , and
- for each  $i$ ,  $x_i$  is regular on  $M/(x_1, \dots, x_{i-1})M$ .

**Remark 3.39.** Requiring that  $x_i$  is regular on  $M/(x_1, \dots, x_{i-1})M$  is equivalent to asking that  $((x_1, \dots, x_{i-1})M :_M x_i) = (x_1, \dots, x_{i-1})M$ .

**Example 3.40.**

- a) Consider the polynomial ring  $R = k[x_1, \dots, x_n]$  in  $n$  variables over a field  $k$ . The variables  $x_1, \dots, x_n$  form a regular sequence on  $R$ .
- b) Let  $k$  be a field and  $R = k[x, y, z]$ . The sequence  $xy, xz$  is not regular on  $R$ , since  $xz$  kills  $y$  on  $R/(xy)$ .

The order we write the elements in is important.

**Example 3.41.** Let  $k$  be a field and  $R = k[x, y, z]$ . The sequence  $x, (x-1)y, (x-1)z$  is regular, while  $(x-1)y, (x-1)z, x$  is not.

**Remark 3.42.** An element  $r$  is regular on  $M$  if and only if  $H_1(K(r; M)) = 0$ . Indeed,

$$H_1(K(r; M)) = \ker(M \xrightarrow{r} M) = (0 :_M r),$$

and by definition,  $r$  is regular on  $M$  if and only if  $(0 :_M r) = 0$ .

The Koszul complex on a regular sequence is exact in all positive degrees.

**Theorem 3.43.** If  $\underline{x} = x_1, \dots, x_n \in R$  is a regular sequence on the  $R$ -module  $M$ , then  $H_i(\underline{x}; M) = 0$  for all  $i > 0$ .

*Proof.* We proceed by induction on the length of the sequence, noting that the case  $n = 1$  is Remark 3.42. Now suppose that  $H_j(x_1, \dots, x_i; M) = 0$  for all  $j > 0$ . The long exact sequence

$$\cdots \longrightarrow H_n(x_1, \dots, x_{i+1}; M) \longrightarrow H_{n-1}(x_1, \dots, x_i; M) \xrightarrow{x_{i+1}} H_{n-1}(x_1, \dots, x_i; M) \longrightarrow \cdots$$

we discussed in Remark 3.37 forces  $H_j(x_1, \dots, x_{i+1}; M) = 0$  for all  $j > 1$ . Moreover, Remark 3.37 also gave us the short exact sequence

$$0 \longrightarrow \frac{H_1(x_1, \dots, x_i; M)}{x_{i+1} \cdot H_1(x_1, \dots, x_i; M)} \longrightarrow H_1(x_1, \dots, x_{i+1}; M) \longrightarrow \text{ann}_{H_0(x_1, \dots, x_i; M)}(x_{i+1}) \longrightarrow 0.$$

Since  $x_{i+1}$  is regular on  $M/(x_1, \dots, x_i)M = H_0(x_1, \dots, x_i; M)$ ,  $\text{ann}_{H_0(x_1, \dots, x_i; M)}(x_{i+1}) = 0$ . Moreover,  $H_1(x_1, \dots, x_i; M) = 0$  by hypothesis. Therefore, the short exact sequence above gives us  $H_1(x_1, \dots, x_{i+1}; M) = 0$ .  $\square$

It is natural to ask if Theorem 3.43 has a converse; over a nice enough ring, the answer is yes: the vanishing of Koszul homology does characterize regular sequences.

**Theorem 3.44.** *Let  $(R, \mathfrak{m}, k)$  is either a Noetherian local ring or an  $\mathbb{N}$ -graded algebra over a field  $k$  with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ . Let  $M \neq 0$  be a finitely generated  $R$ -module, and consider  $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$ . In the graded case, we assume that  $M$  is graded and  $x_1, \dots, x_n$  are all homogeneous. If  $H_i(\underline{x}; M) = 0$  for all  $i \geq 1$ , then  $x_1, \dots, x_n$  is a regular sequence on  $R$ .*

*Proof.* We proceed by induction on  $n$ , noting that the case  $n = 1$  is simply Remark 3.42. Now let  $n > 1$  and suppose that the statement holds for all sequences of  $n - 1$  elements. By Remark 3.37, we have short exact sequences

$$0 \longrightarrow \frac{H_i(x_1, \dots, x_{n-1}; M)}{x_n \cdot H_i(x_1, \dots, x_n; M)} \longrightarrow H_i(x_1, \dots, x_n; M) \longrightarrow \text{ann}_{H_{i-1}(x_1, \dots, x_{n-1}; M)}(x_n) \longrightarrow 0,$$

and since the middle term is 0 for all  $i \geq 1$ , we conclude that

- $\frac{H_i(x_1, \dots, x_{n-1}; M)}{x_n \cdot H_i(x_1, \dots, x_n; M)} = 0$  for all  $i \geq 1$ , and
- $\text{ann}_{H_0(x_1, \dots, x_{n-1}; M)}(x_n) = 0$ , so  $x_n$  is regular on  $H_0(x_1, \dots, x_{n-1}; M) = M/(x_1, \dots, x_{n-1})M$ .

By Proposition 3.36,  $H_i(x_1, \dots, x_{n-1}; M)$  is a finitely generated  $R$ -module for all  $i$ . Since  $x_n \in \mathfrak{m}$  and  $x_n H_i(x_1, \dots, x_{n-1}; M) = H_i(x_1, \dots, x_{n-1}; M)$ , NAK (or Proposition B.11 in the graded case) implies that  $H_i(x_1, \dots, x_{n-1}; M) = 0$  for all  $i \geq 1$ . By induction hypothesis,  $x_1, \dots, x_{n-1}$  is a regular sequence. We conclude that  $x_1, \dots, x_n$  is a regular sequence.  $\square$

A corollary of Theorem 3.44 is that in a regular ring, the order of the elements in a regular sequence does not matter.

**Corollary 3.45.** *Let  $(R, \mathfrak{m}, k)$  is either a noetherian local ring or a finitely generated  $\mathbb{N}$ -graded algebra over a field  $k$  with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ . Let  $M$  be a finitely generated  $R$ -module, and consider  $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$ . In the graded case, we assume that  $M$  is graded and  $x_1, \dots, x_n$  are all homogeneous. If the sequence  $\underline{x}$  is regular on  $M$ , then so is any of its permutations.*

*Proof.* If  $x_1, \dots, x_n$  is a regular sequence, then  $H_i(x_1, \dots, x_n; M) = 0$  for all  $i > 0$ , by Proposition 3.36. The Koszul homology on  $\underline{x}$  agrees with the Koszul homology on any permutation of  $\underline{x}$ , which must then also vanish. By Theorem 3.44, any permutation of  $\underline{x}$  is a regular sequence.  $\square$

In fact, we can extend this to any ring and any module under a reasonable assumption.

**Lemma 3.46.** *Let  $R$  be a ring and  $M$  an  $R$ -module. If  $x, y$  is a regular sequence on  $M$  and  $y$  is regular on  $M$ , then  $y, x$  is a regular sequence on  $M$ .*

*Proof.* Suppose that  $xm = yn$  for some  $m, n \in M$ . Since  $x, y$  is a regular sequence on  $M$  and  $yn \in (x)M$ , we must have  $n \in (x)M$ , so there exists some  $w \in M$  such that  $n = xw$ . But then  $xm = yn = xyw$ . Since  $x$  is regular on  $M$ , we conclude that  $m = yw$ , so  $m \in (y)M$ . In particular, this shows that  $x$  is regular on  $M/(y)M$ .  $\square$

**Lemma 3.47.** *Let  $(R, \mathfrak{m}, k)$  is either a Noetherian local ring or an  $\mathbb{N}$ -graded algebra over a field  $k$  with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ . Let  $M \neq 0$  be a finitely generated  $R$ -module, and consider  $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$ . In the graded case, we assume that  $M$  is graded and  $x_1, \dots, x_n$  are all homogeneous.*

*If  $x_1, \dots, x_n$  is a regular sequence on  $M$ , then so is  $x_1^{a_1}, \dots, x_n^{a_n}$  for any integers  $a_i > 0$ .*

*Proof.* By Corollary 3.45, we are allowed to permute the elements in our sequence. Let's use this fact to reduce our proof to the case  $n = 1$ . If the case  $n = 1$  holds, since  $x_n$  is a regular sequence on  $M/(x_1, \dots, x_{n-1})$  we can now say  $x_n^{a_n}$  is regular on  $M/(x_1, \dots, x_{n-1})$ . Therefore,  $x_1, \dots, x_{n-1}, x_n^{a_n}$  is regular on  $M$ . Now switch the order and repeat the argument with each  $x_i$ , until we conclude that  $x_1^{a_1}, \dots, x_n^{a_n}$  is also regular on  $M$ .

Finally, we give a proof when  $n = 1$ . Now if  $x$  is a regular element on  $M$ , if  $x^a \neq 0$ , then  $x^a m = 0 \implies x x^{a-1} m = 0$ , and since  $x$  is regular we must have  $x^{a-1} m = 0$ . Repeating this  $a - 1$  times, we conclude that  $xm = 0$ , and  $m = 0$ .  $\square$

There is a connection between regular sequences and height.

**Theorem 3.48.** *If  $x_1, \dots, x_n$  is a regular sequence on  $R$ , then  $\text{ht}(x_1, \dots, x_n) = n$ .*

*Proof.* We use induction on  $n$ . When  $n = 1$ ,  $x_1$  is regular if and only if  $x_1$  is not in the set of zero divisors of  $R$ . By Theorem 1.39, this means  $x_1$  is not in any associated prime of  $R$ , and in particular,  $x_1$  is not in any of the minimal primes of  $R$ . Therefore, any prime containing  $x_1$  must have height at least 1. By Theorem 3.11,  $\text{ht}(x_1) \leq 1$ , so  $\text{ht}(x_1) = 1$ . When  $n > 1$ ,  $x_n$  is regular on  $R/(x_1, \dots, x_{n-1})$ , so by case  $n = 1$ ,  $(x_1, \dots, x_n)/(x_1, \dots, x_{n-1})$  has height 1 on  $R/(x_1, \dots, x_{n-1})$ . By induction hypothesis,  $\text{ht}(x_1, \dots, x_{n-1}) = n - 1$ . We conclude that  $\text{ht}(x_1, \dots, x_n) = n$ .  $\square$

We now record some useful facts about regular sequences we will need when we get back to talking about symbolic powers.

**Lemma 3.49.** *Let  $x_1, \dots, x_t$  be a regular sequence on  $R$ . If  $r_1, \dots, r_t \in R$  are such that  $r_1 x_1 + \dots + r_t x_t = 0$ , then  $r_1, \dots, r_t \in (x_1, \dots, x_t)$ .*

*Proof.* We will do induction on  $t$ . When  $t = 1$ ,  $x_1$  is a regular element on  $R$ , so  $r_1 x_1 = 0$  implies  $r_1 = 0 \in (x_1)$ .

Now suppose the claim holds for all regular sequences of length  $t - 1$  for some  $t - 1 \geq 1$ . Since  $x_t$  is regular on  $R/(x_1, \dots, x_{t-1})$ , the fact that  $r_t x_t = -(r_1 x_1 + \dots + r_{t-1} x_{t-1}) \in (x_1, \dots, x_{t-1})$  implies that  $r_t \in (x_1, \dots, x_{t-1})$ . Rewriting  $r_t = s_1 x_1 + \dots + s_{t-1} x_{t-1}$ , our assumption is that

$$0 = r_1 x_1 + \dots + r_t x_t = (r_1 + s_1 x_t) x_1 + \dots + (r_{t-1} + s_{t-1} x_t) x_{t-1}.$$

By induction hypothesis, we must have  $r_1 + s_1 x_t, \dots, r_{t-1} + s_{t-1} x_t \in (x_1, \dots, x_{t-1})$ , and thus  $r_1, \dots, r_{t-1} \in (x_1, \dots, x_t)$ .  $\square$

**Theorem 3.50.** *Let  $R$  be a ring and consider  $I = (a_1, \dots, a_t)$ , where  $a_1, \dots, a_t$  is a regular sequence on  $R$ . Given a homogeneous polynomial  $F \in R[x_1, \dots, x_t]$  of degree  $n \geq 1$ , if  $F(a_1, \dots, a_t) \in I^{n+1}$ , then the coefficients of  $F$  must all be in  $I$ .*

*Proof.* Fix nonnegative integers  $c_1, \dots, c_t$  with  $c_1 + \dots + c_t = n$ , and let  $u$  be the coefficient of  $F$  in  $x_1^{c_1} \dots x_t^{c_t}$ . Since  $F$  is homogeneous, all the remaining monomials in  $F$  have a factor of at least one of the terms  $x_1^{c_1+1}, \dots, x_t^{c_t+1}$ . So we can write

$$F(a_1, \dots, a_t) = ua_1^{c_1} \dots a_t^{c_t} + \sum_{j=1}^t a_j^{c_j+1} v_j$$

for some  $v_j \in R$ . The ideal  $I^{n+1}$  is generated by all  $a_1^{d_1} \dots a_t^{d_t}$  with  $d_1 + \dots + d_t = n+1$ . For each of these generators, we must have  $d_i \geq c_i + 1$  for some  $i$ , since  $c_1 + \dots + c_t = n$ , and thus  $I^{n+1} \subseteq (a_1^{c_1+1}, \dots, a_t^{c_t+1})$ . Since  $F(a_1, \dots, a_t) \in I^{n+1}$ , we can now find  $b_j \in R$  such that

$$ua_1^{c_1} \dots a_t^{c_t} = \sum_{j=1}^t a_j^{c_j+1} b_j.$$

We will show that this implies that  $u \in I$ , and since we chose  $c_1, \dots, c_t$  to be arbitrary, we will then be able to conclude that all the coefficients of  $F$  are in  $I$ . To do that, we will do induction on the number of nonzero  $c_j$ . If all the  $c_j$  are 0, then the assumption is that  $u = \sum_{j=1}^t a_j b_j$ , so  $u \in I = (a_1, \dots, a_t)$ . For the induction step, set

$$y := \prod_{j \neq i} a_j^{c_j}.$$

Now the equality for  $ua_1^{c_1} \dots a_t^{c_t}$  above can be rewritten as

$$\sum_{j \neq i} a_j^{c_j+1} b_j - a_i^{c_i} (uy - a_i b_i) = 0.$$

By Lemma 3.47,  $a_1^{c_1+1}, \dots, a_{i-1}^{c_{i-1}+1}, a_i^{c_i}, a_{i+1}^{c_{i+1}+1}, \dots, a_t^{c_t+1}$  is also a regular sequence, so

$$\sum_{j \neq i} a_j^{c_j+1} b_j - a_i^{c_i} (uy - a_i b_i)$$

is a linear combination of the elements in a regular sequence. By Lemma 3.49, all the coefficients must be in the ideal generated by this regular sequence. Therefore,

$$uy - a_i b_i \in (a_i^{c_i}) + (a_j^{c_j+1} \mid j \neq i) \implies uy \in (a_i) + (a_j^{c_j+1} \mid j \neq i).$$

Now

$$uy = u \prod_{j \neq i} a_j^{c_j} = ua_1^{c_1} \dots a_{i-1}^{c_{i-1}} a_i^0 a_{i+1}^{c_{i+1}} \dots a_t^{c_t} = b'_i a_i^{0+1} + \sum_{j \neq i} d_j a_j^{c_j+1}$$

has one fewer nonzero exponent than our original  $ua_1^{c_1} \dots a_t^{c_t}$ , and it is of the form

$$uy = ua_1^{d_1} \dots a_t^{d_t} = \sum_{j=1}^t e_j a_j^{d_j+1}$$

with  $d_i = 0$  and  $d_j = c_j$  for  $j \neq i$ , and some  $e_j \in R$ . Therefore, the induction hypothesis applies, and thus  $u \in (a_1, \dots, a_t)$ .  $\square$

### 3.4 Free resolutions

**Definition 3.51.** Let  $M$  be an  $R$ -module. A **free resolution** is a complex

$$F_{\bullet} = \cdots \longrightarrow F_n \xrightarrow{\quad} \cdots \xrightarrow{\quad} F_1 \xrightarrow{\quad} F_0 \longrightarrow 0$$

$\qquad\qquad\qquad n \qquad\qquad\qquad 1 \qquad\qquad\qquad 0$

where all the  $F_i$  are free  $R$ -modules,  $H_0(F) = M$ , and  $H_i(F) = 0$  for all  $i \neq 0$ . We may also write a free resolution for  $M$  as an exact sequence free modules  $F_i$  of the form

$$\cdots \longrightarrow F_n \xrightarrow{\quad} \cdots \xrightarrow{\quad} F_1 \xrightarrow{\quad} F_0 \longrightarrow M \longrightarrow 0 .$$

$\qquad\qquad\qquad n \qquad\qquad\qquad 1 \qquad\qquad\qquad 0$

You will find both these definitions in the literature, often indicating the second option as an abuse of notation. We will be a bit sloppy and consider both equivalently, since at the end of the day they contain the same information.

**Theorem 3.52.** *Every  $R$ -module has a free resolution.*

*Proof.* Let  $M$  be an  $R$ -module. We are going to construct a free resolution quite explicitly. The first step is to find a free module surjecting onto  $M$ , which we can do by taking a free module on any set of generators for  $M$ . Now consider the kernel of that projection, say

$$0 \longrightarrow K_0 \xrightarrow{i_0} P_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

Set  $\partial_0 := \pi_0$ . There exists a free module  $P_1$  surjecting onto  $K_0$ . Now the map  $\partial_1 = i_0\pi_1$  satisfies  $\text{im } \partial_1 = K_0 = \ker \partial_0$ .

$$\begin{array}{ccccc}
 0 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & K_0 & & \\
 & \nearrow \pi_1 & & \searrow & \\
 P_1 & \xrightarrow{\quad \partial_1 \quad} & P_0 & \xrightarrow{\partial_0} & M.
 \end{array}$$

Now the process continues analogously. We find a free module  $P_2$  surjecting onto  $K_1 := \ker \partial_1$ , and set

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \searrow & & \nearrow \\
 & & & & & K_0 & \\
 & & & \nearrow \pi_1 & & \searrow i_0 & \\
 & & & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} M. \\
 P_2 & \xrightarrow{\partial_2} & P_1 & & & & \\
 & \searrow \pi_2 & & \nearrow i_1 & & & \\
 & & K_1 & & & & \\
 & \nearrow & & \searrow & & & \\
 0 & & & & 0 & & 
 \end{array}$$

At each stage,  $\pi_i: P_i \longrightarrow K_{i-1}$  is a surjective map,  $K_i := \ker \partial_i$ ,  $i_i$  is the inclusion of the kernel of  $\partial_i$  into  $P_i$ , and we get short exact sequences

$$0 \longrightarrow K_{n+1} \xrightarrow{i_{n+1}} P_{n+1} \xrightarrow{\pi_{n+1}} K_n \longrightarrow 0.$$

In fact,  $\text{im}(i_{n+1}) = \ker \partial_{n+1} = \ker(i_n \pi_{n+1}) = \ker \pi_{n+1}$ . We can continue this process indefinitely for as long as  $P_n \neq 0$ , and the resulting sequence will be a projective resolution for  $M$ .  $\square$

We can think of a free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$$

as giving a detailed description of our module  $M$ . The first free module,  $F_0$ , gives us generators for  $M$ . The second free module,  $F_1$ , gives us generators for all the relations among our generators for  $M$ . The next module describes the relations among the relations among our generators. And so on.

**Definition 3.53.** Let  $(R, \mathfrak{m}, k)$  is either a noetherian local ring or a finitely generated  $\mathbb{N}$ -graded algebra over a field  $k$  with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ . Let  $M$  be a finitely generated  $R$ -module, which we assume to be graded in the case when  $R$  is graded. The **projective dimension** of  $M$  is

$$\text{pdim}_R(M) := \inf \left\{ c \mid 0 \longrightarrow P_c \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0 \text{ is a free resolution for } M \right\}.$$

The projective dimension of a finitely generated module can be infinite.

**Example 3.54.** Let  $k$  be a field and  $R = k[x]/(x^2)$ , which is a local ring with maximal ideal  $\mathfrak{m} = (x)$ . The residue field  $k = R/\mathfrak{m}$  has infinite projective dimension:

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0.$$

So even cyclic modules can have infinite projective dimension.

When  $(R, \mathfrak{m})$  is either a local ring or an  $\mathbb{N}$ -graded graded  $k$ -algebra with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ , we can talk about **minimal** free resolutions. To construct a minimal free resolution of  $M$ , we simply take as few generators as possible in each step. This is equivalent to having a **minimal complex**  $F$ , meaning that the differential  $\partial$  satisfies  $\text{im}(\partial^F) \subseteq \mathfrak{m}F$ . If we find basis for each free module  $F_i$ , and write the differentials as matrices in those basis, the entries in every differential matrix are all in  $\mathfrak{m}$ . Ultimately, we can talk about *the* minimal free resolution of  $M$ , since all minimal free resolutions are isomorphic. While we will not discuss the details here, here are some examples.

**Definition 3.55.** Let  $(R, \mathfrak{m})$  be either a local ring or an  $\mathbb{N}$ -graded graded  $k$ -algebra with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ . Let  $F$  be a minimal free resolution for the finitely generated (graded)  $R$ -module  $M$ . The  $n$ th betti number of  $M$  is

$$\beta_i(M) := \text{rank } F_i = \mu(F_i).$$

In the graded case, the  $(i, j)$ th betti number of  $M$ ,  $\beta_{ij}(M)$ , counts the number of generators of  $F_i$  in degree  $j$ . We often collect the betti numbers of a module in its **betti table**:

| $\beta(M)$ | 0               | 1               | 2               | ...      |
|------------|-----------------|-----------------|-----------------|----------|
| 0          | $\beta_{00}(M)$ | $\beta_{11}(M)$ | $\beta_{22}(M)$ |          |
| 1          | $\beta_{01}(M)$ | $\beta_{12}(M)$ | $\beta_{23}(M)$ |          |
| 2          | $\beta_{02}(M)$ | $\beta_{13}(M)$ |                 |          |
| $\vdots$   |                 |                 |                 | $\ddots$ |

By convention, the entry corresponding to  $(i, j)$  in the betti table of  $M$  contains  $\beta_{i,i+j}(M)$ , and *not*  $\beta_{ij}(M)$ . This is how Macaulay2 displays betti tables as well, using the command **betti**.

**Example 3.56.** Suppose that  $R = k[x, y, z]$  and that  $M = R/(xy, xz, yz)$  corresponds to the variety defining the union of the three coordinate lines in  $\mathbb{A}_k^3$ . This variety has dimension 1 and degree 3. The minimal free resolution for  $M$  is

$$0 \longrightarrow R^{\textcolor{red}{2}} \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^{\textcolor{blue}{3}} \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M.$$

From this minimal resolution, we can read the betti numbers of  $M$ :

- $\beta_0(M) = 1$ , since  $M$  is a cyclic module;
- $\beta_1(M) = \textcolor{blue}{3}$ , and these three quadratic generators live in degree  $\textcolor{teal}{2}$ ;
- $\beta_2(M) = \textcolor{red}{2}$ , and these represent linear syzygies on quadrics, and thus live in degree  $\textcolor{blue}{3}$ .

Here is the graded free resolution of  $M$ :

$$0 \longrightarrow R(\textcolor{blue}{-3})^{\textcolor{red}{2}} \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R(\textcolor{teal}{-2})^{\textcolor{blue}{3}} \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M.$$

Notice that the graded shifts in lower homological degrees affect all the higher homological degrees as well. For example, when we write the map in degree 2, we only need to shift the degree of each generator by  $\textcolor{orange}{1}$ , but since our map now lands on  $R(\textcolor{teal}{-2})^{\textcolor{blue}{3}}$ , we have to bump up degrees from 2 to 3, and write  $R(\textcolor{blue}{-3})^{\textcolor{red}{2}}$ . The graded betti number  $\beta_{ij}(M)$  of  $M$  counts the number of copies of  $R(-j)$  in homological degree  $i$  in our resolution. So we have

$$\beta_{00} = 1, \beta_{12} = \textcolor{blue}{3}, \text{ and } \beta_{23} = \textcolor{red}{2}.$$

We can collect the graded betti numbers of  $M$  in what is called a *betti table*:

| $\beta(M)$ | 0 | 1                     | 2                    |
|------------|---|-----------------------|----------------------|
| 0          | 1 | —                     | —                    |
| 1          | — | $\textcolor{blue}{3}$ | $\textcolor{red}{2}$ |



**Example 3.57.** Let  $k$  be a field,  $R = k[x, y]$ , and consider the ideal

$$I = (x^2, xy, y^3)$$

which has two generators of degree 2 and one of degree 3, so there are graded betti numbers  $\beta_{12}$  and  $\beta_{13}$ . The minimal free resolution for  $R/I$  is

$$0 \longrightarrow \begin{array}{c} R(-3)^1 \\ \oplus \\ R(-4)^1 \end{array} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{pmatrix}} \begin{array}{c} R(-2)^2 \\ \oplus \\ R(-3)^1 \end{array} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R \longrightarrow R/I.$$

$$\begin{array}{ll} \beta_{23}(R/I) = 1 & \beta_{12}(R/I) = 2 \\ \beta_{24}(R/I) = 1 & \beta_{13}(R/I) = 1 \end{array}$$

So the betti table of  $R/I$  is

| $\beta(M)$ | 0 | 1 | 2 |
|------------|---|---|---|
| 0          | 1 | — | — |
| 1          | — | 2 | 1 |
| 2          | — | 1 | 1 |

These invariants can give us some information about Ext and Tor, and vice-versa.

**Remark 3.58.** If  $\text{pdim}_R(M) = n$  is finite, then we can take a free resolution of  $M$  with length  $n$  to compute  $\text{Tor}_i^R(M, -)$ , and thus  $\text{Tor}_i^R(M, -) = 0$  for all  $i > n$ . For the same reason,  $\text{Ext}_R^i(M, -) = 0$  for all  $i > n$ .

**Remark 3.59.** Let  $(R, \mathfrak{m}, k)$  be either a local ring or an  $\mathbb{N}$ -graded  $k$ -algebra, where  $k$  is a field, with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ , and  $M$  a finitely generated (graded)  $R$ -module. The homomorphisms of  $R$ -modules  $R \rightarrow R$  are precisely the multiplication maps by each fixed  $r \in R$ . The map  $(r \cdot -) \otimes_R k$  is simply multiplication by the image of  $r$  in  $k = R/\mathfrak{m}$  on  $k$ . More generally, a homomorphism of  $R$ -modules  $R^n \xrightarrow{f} R^m$  can be represented by an  $m \times n$  matrix  $A$  with entries in  $R$  after we fix bases for  $R^n$  and  $R^m$ , and the matrix representing  $f \otimes_R k$  in the corresponding bases for  $k^m$  and  $k^n$  is the matrix obtained from  $A$  by considering the images of the entries in  $k$ .

Given any  $g \in \text{Hom}_R(R, k)$ , the composition of  $f$  with multiplication by  $r$  is the map  $g(r \cdot -) = rg(-)$ . So  $\text{Hom}_R(r \cdot -, k)$  is multiplication by the image of  $r$  in  $k$ . More generally, if  $A$  is an  $m \times n$  matrix representing  $R^n \xrightarrow{f} R^m$ , the map  $\text{Hom}_R(f, k)$  is represented in the corresponding bases for  $k^n$  and  $k^m$  by the transpose of  $A$ , where the entries are now replaced by their images in  $k = R/\mathfrak{m}$ .

**Theorem 3.60.** Let  $(R, \mathfrak{m}, k)$  be either a Noetherian local ring or an  $\mathbb{N}$ -graded  $k$ -algebra, where  $k$  is a field, with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ , and  $M$  a finitely generated (graded)  $R$ -module. Then

$$\beta_i(M) = \dim_k(\text{Tor}_i^R(M, k)) = \dim_k(\text{Ext}_R^i(M, k)).$$

*Proof.* Let  $F$  be a minimal free resolution for  $M$ . The module in degree  $i$  in the complex  $F \otimes_R k$  is

$$F_i \otimes_R k = R^{\beta_i(M)} \otimes_R k = k^{\beta_i(M)}.$$

Minimal free resolutions are minimal complexes, so  $\text{im}(\partial^F) \subseteq \mathfrak{m}F$ , and thus  $\partial \otimes_R k = 0$ . So

$$F \otimes_R k = \cdots \longrightarrow k^{\beta_i(M)} \longrightarrow k^{\beta_{i-1}(M)} \longrightarrow \cdots \longrightarrow k^{\beta_1(M)} \longrightarrow k^{\beta_0} \longrightarrow 0.$$

Therefore,

$$\text{Tor}_i^R(M, k) = H_i(F \otimes_R k) = k^{\beta_i(M)}.$$

The module in degree  $i$  in the complex  $\text{Hom}_R(F, k)$  is

$$\text{Hom}_R(F_i, k) = \text{Hom}_R(R^{\beta_i(M)}, k) = k^{\beta_i(M)}.$$

Following the discussion in Remark 3.59, the fact that  $\partial(F) \subseteq \mathfrak{m}F$  implies  $\text{Hom}_R(\partial^F, k) = 0$ . Therefore,

$$\text{Hom}_R(F, k) = 0 \longrightarrow k^{\beta_0(M)} \xrightarrow{0} k^{\beta_1(M)} \longrightarrow \cdots \longrightarrow k^{\beta_i(M)} \longrightarrow \cdots$$

so

$$\text{Ext}_R^i(M, k) = H^i(\text{Hom}_R(F, k)) = k^{\beta_i(M)}. \quad \square$$

**Corollary 3.61.** *Let  $(R, \mathfrak{m}, k)$  be either a local ring or an  $\mathbb{N}$ -graded  $k$ -algebra, where  $k$  is a field, with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ . For every finitely generated (graded)  $R$ -module  $M$ ,  $\text{pdim}_R(M) \leq \text{pdim}_R(k)$ .*

*Proof.* When  $i > \text{pdim}_R(k)$ ,  $\text{Tor}_i^R(M, k) = 0$ , so  $\beta_i(M) = 0$  by Theorem 3.60.  $\square$

**Remark 3.62.** Also as a consequence of Theorem 3.60, we learn that

$$\text{pdim}_R(M) = \sup\{i \mid \beta_i(M) \neq 0\} = \sup\{i \mid \text{Ext}_R^i(M, k) \neq 0\} = \sup\{i \mid \text{Tor}_R^i(M, k) \neq 0\}.$$

We can extend this to graded betti numbers once we realize that  $\text{Tor}$  and  $\text{Ext}$  of graded modules can also be given graded structures.

**Exercise 5.** Let  $(R, \mathfrak{m}, k)$  be an  $\mathbb{N}$ -graded  $k$ -algebra, where  $k$  is a field, with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ , and  $M$  a finitely generated graded  $R$ -module. Fix a graded minimal free resolution  $F$  of  $M$ . Then

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i = \dim_k(\text{Tor}_i^R(M, k)_j) = \dim_k(\text{Ext}_R^i(M, k)_j).$$

If only we had an explicit minimal free resolution of  $k$ , maybe we could use it to say something about the minimal free resolutions of other finitely generated  $R$ -modules. With that goal in mind, we return to regular sequences.

**Corollary 3.63.** *If  $x_1, \dots, x_n$  is a regular sequence on  $R$ , then the Koszul complex on  $x_1, \dots, x_n$  is a free resolution for  $R/(x_1, \dots, x_n)$ . Moreover, if  $(R, \mathfrak{m}, k)$  is either a local ring or an  $\mathbb{N}$ -graded algebra over a field  $k$  with  $R_0 = k$  and homogeneous maximal ideal  $\mathfrak{m} = R_+$ , then the Koszul complex  $K_\bullet(x_1, \dots, x_n)$  is a minimal free resolution for  $R/(x_1, \dots, x_n)$ .*

*Proof.* By Theorem 3.43, the Koszul complex  $P = K_\bullet(x_1, \dots, x_n)$  has  $H_i(P) = 0$  for all  $i > 0$ . This is a complex of free modules, and thus a free resolution of  $H_0(P)$ , which by Proposition 3.36 is  $R/(x_1, \dots, x_n)$ .  $\square$

**Theorem 3.64** (Hilbert Syzygy Theorem). *Every finitely generated graded module  $M$  over a polynomial ring  $R = k[x_1, \dots, x_n]$  over a field  $k$  has finite projective dimension. In fact,  $\text{pdim}(M) \leq n$ .*

*Proof.* By Corollary 3.63, the Koszul complex on the regular sequence  $x_1, \dots, x_d$  is a minimal free resolution for  $k = R/(x_1, \dots, x_n)$ , so  $\text{pdim}_R k = n$ . But  $\beta_i(M) = \dim_k \text{Tor}_R^i(M, k)$  by Theorem 3.60, and since  $\text{Tor}_i^R(M, k) = 0$  for all  $i > n = \text{pdim}_R(k)$ ,  $\text{pdim}_R(M) \leq n$ .  $\square$

## 3.5 Regular rings

Regular rings are the nicest possible kinds of rings. A **regular local ring**  $(R, \mathfrak{m}, k)$  is a finite dimensional Noetherian ring of dimension  $d$  whose maximal ideal  $\mathfrak{m}$  is generated by  $d$  elements. When we discussed height and dimension, we saw that this is in fact the smallest possible value for the minimal number of generators of  $\mathfrak{m}$ ; in general,  $\mu(\mathfrak{m}) \geq d$ . We are now ready to give a completely homological characterization of regular local rings. This characterization, first proved by Auslander and Buchsbaum and independently by Serre, solved a famous open problem called the Localization Problem.

**Problem 3.65** (Localization Problem). *If  $R$  is a regular local ring, must  $R_P$  be regular for every prime  $P$  in  $R$ ?*

This is asking if being regular is a local property. A positive answer allows for a simple global definition of regularity:

**Definition 3.66.** A ring  $R$  is **regular** if  $R_P$  is a regular local ring for all prime ideals  $P$ .

Before we can get to this famous homological characterization of regular local rings, and the solution to the localization problem, we will need to sharpen our tools a bit.

**Theorem 3.67** (Serre). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring. Moreover, if  $\mu(\mathfrak{m}) = s$ , then*

$$\dim_k(\text{Tor}_i^R(k, k)) \geq \binom{s}{i}.$$

Replacing  $k$  by  $M$  in the previous result leads to a famous open question.

**Conjecture 3.68** (Buchsbaum—Eisenbud, Horrocks). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring of dimension  $d$  and  $M$  a finitely generated Artinian  $R$ -module of finite projective dimension. Then*

$$\beta_i(M) = \dim_k(\text{Tor}_i^R(M, k)) \geq \binom{d}{i}.$$

While this remains an open question, there is much evidence to support it. For example, the conjecture predicts that

$$\sum_i \beta_i(M) \geq \sum_i \binom{d}{i} = 2^d.$$

This is known as the Total Rank Conjecture, and it was recently shown by Walker in almost all cases.

**Theorem 3.69** (Walker, 2017). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring of dimension  $d$  and characteristic not 2,  $M \neq 0$  a finitely generated  $R$ -module of finite projective dimension, and  $c = \text{ht}(\text{ann}(M))$ . Then*

$$\sum_i \beta_i(M) \geq 2^c.$$

The famous homological characterization of regular rings that solved the localization problem is the following:

**Theorem 3.70** (Auslander–Buchsbaum, Serre). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring of dimension  $d$ . The following are equivalent:*

- a) *The residue field  $k$  has finite projective dimension.*
- b) *Every finitely generated  $R$ -module has finite projective dimension.*
- c) *The maximal ideal  $\mathfrak{m}$  is generated by a regular sequence.*
- d) *The maximal ideal  $\mathfrak{m}$  is generated by  $d$  elements.*

*Proof.* The implication **b**)  $\implies$  **a**) is obvious: just take  $M = k$ . The proof of **a**)  $\implies$  **b**) is essentially the same as Hilbert’s Syzygy Theorem:  $\beta_i(M) = \dim_k \text{Tor}_i^k(M, k)$  for all  $i$ , and  $\text{Tor}_i^k(M, k) = 0$  for all  $i > \text{pdim}_R(k)$ .

If  $\mathfrak{m}$  is generated by a regular sequence, then the Koszul complex on that regular sequence is a minimal free resolution of  $k$ , by Corollary 3.63, so  $k$  has projective dimension  $d$ . This is **c**)  $\implies$  **a**).

Let’s now show that **d**)  $\implies$  **c**). Set  $\mathfrak{m} = (x_1, \dots, x_d)$ . In fact, we will show something stronger: that  $(0), (x_1), (x_1, x_2), \dots, (x_1, \dots, x_d)$  are distinct prime ideals in  $R$ . Notice in particular that this implies that  $x_1, \dots, x_d$  form a regular sequence.

If  $d = 0$ , then  $\mathfrak{m} = (\{ \}) = (0)$ , and there is nothing to prove. We proceed by induction on  $d$ , assuming that  $d > 0$  and that we have shown that whenever  $\mathfrak{m}$  is generated by  $d - 1$  elements,  $x_1, \dots, x_{d-1}$ , the ideals  $(0), (x_1), (x_1, x_2), \dots, (x_1, \dots, x_{d-1})$  are distinct prime ideals in  $R$ .

When  $d > 0$ ,  $\mathfrak{m}$  is not a minimal prime. By **Prime Avoidance**,

$$\mathfrak{m} \not\subseteq \bigcup_{P \in \text{Min}(R)} P,$$

and using Theorem B.3 we can find an element

$$y_1 = x_1 + r_2 x_2 + \dots + r_d x_d \notin \bigcup_{P \in \text{Min}(R)} P.$$

Now we can replace  $x_1$  by  $y_1$ , since  $\mathfrak{m} = (y_1, x_2, \dots, x_d)$ , so we can assume that  $x_1$  is not in any minimal prime. By Krull's Height Theorem 3.11,  $\text{ht}(x_1) \leq 1$ , so  $\dim(R/(x_1)) = \text{ht}(\mathfrak{m}/(x_1)) \geq d - 1$ . By construction,  $\mathfrak{m}/(x_1)$  is generated by  $d - 1$  elements, so again by Krull's Height Theorem,  $\dim(R/(x_1)) = \text{ht}(\mathfrak{m}/(x_1)) \leq d - 1$ . We conclude that  $\dim(R/(x_1)) = d - 1$ . By induction hypothesis,  $(x_1)/(x_1), \dots, (x_1, \dots, x_d)/(x_1)$  are distinct prime ideals in  $R/(x_1)$ . Therefore,  $(x_1), (x_1, x_2), \dots, (x_1, \dots, x_d)$  are distinct prime ideals in  $R$ .

Now we claim that  $R$  is a domain, which will show that  $(0) \subsetneq (x_1)$  is also a prime ideal. First, note that  $x_1$  is not contained in any minimal prime, but  $(x_1)$  is a prime ideal, so there exists some minimal prime  $P \subsetneq (x_1)$ . Given any  $y \in P \subseteq (x_1)$ , we can write  $y = rx_1$  for some  $r$ . By construction,  $x_1 \notin P$ , so we must have  $r \in P$ . But we just showed that every element in  $P$  is of the form  $rx_1$ , so  $P = x_1P$ . By NAK,  $P = (0)$ . We conclude that  $R$  is a domain, and this finishes the proof of d)  $\implies$  c).

Finally, all that's left to show is a)  $\implies$  d). We claim that  $\text{pdim}_R(k) < \infty$  implies  $\text{pdim}_R(k) \leq \dim(R) = d$ . If the claim holds, then Theorem 3.67 and Theorem 3.60 say that

$$\beta_i(k) = \dim_k(\text{Tor}_i^R(k, k)) \geq \binom{\mu(\mathfrak{m})}{i}$$

for all  $i$ . Since  $\beta_i(k) = 0$  for all  $i > \text{pdim}_R(k)$ , we must have

$$\mu(\mathfrak{m}) \leq \text{pdim}_R(k) \leq \dim(R) = d.$$

But  $\text{ht}(\mathfrak{m}) = \dim(R) = d$ , so by Theorem 3.11,  $\mu(\mathfrak{m}) \geq d$ . We conclude that  $\mathfrak{m}$  is generated by exactly  $d$  elements, which is precisely d). So all we have left to do is to prove the claim that  $\text{pdim}_R(k) < \infty$  implies  $\text{pdim}_R(k) \leq \dim(R) = d$ .

By contradiction, suppose  $\text{pdim}_R(k) > d$  but  $\text{pdim}_R(k) < \infty$ . Choose a maximal regular sequence  $y_1, \dots, y_t \in \mathfrak{m}$ . By Theorem 3.48,  $t \leq d$ .

Since our regular sequence  $y_1, \dots, y_t$  was chosen to be maximal inside  $\mathfrak{m}$ , every element in  $\mathfrak{m}$  is a zerodivisor on  $R/(y_1, \dots, y_t)$ , or else we could increase our regular sequence. So  $\mathfrak{m}$  is contained in the union of the zerodivisors on  $R/(y_1, \dots, y_t)$ , which by Theorem 1.39 is the same as the union of the associated primes of  $R/(y_1, \dots, y_t)$ . By Prime Avoidance,  $\mathfrak{m}$  must be contained in some associated prime of  $R/(y_1, \dots, y_t)$ . But  $\mathfrak{m}$  is maximal, so  $\mathfrak{m}$  is an associated prime of  $R/(y_1, \dots, y_t)$ . Equivalently,  $k = R/\mathfrak{m}$  embeds into  $R/(y_1, \dots, y_t)$ . This gives us some short exact sequence

$$0 \longrightarrow k \longrightarrow R/(y_1, \dots, y_t) \longrightarrow M \longrightarrow 0.$$

The corresponding long exact sequence for Tor is

$$\dots \longrightarrow \text{Tor}_{i+1}^R(M, k) \longrightarrow \text{Tor}_i^R(k, k) \longrightarrow \text{Tor}_i^R(R/(y_1, \dots, y_t), k) \longrightarrow \dots$$

We know  $t = \text{pdim}_R(R/(y_1, \dots, y_t))$ , by Corollary 3.63, so  $\text{Tor}_i^R(R/(y_1, \dots, y_t), k) = 0$  for  $i > t$ . But  $t \leq d < \text{pdim}_R(k)$ , so in particular  $\text{Tor}_i^R(R/(y_1, \dots, y_t), k) = 0$  for  $i = \text{pdim}_R(k)$ .

Moreover, Corollary 3.61 says that  $\text{pdim}_R(M) \leq \text{pdim}_R(k)$  for any finitely generated  $R$ -module  $M$ , so in particular  $\text{Tor}_{i+1}^R(M, k) = 0$  for  $i = \text{pdim}_R(k)$ . But this is impossible: our long exact sequence would then have  $\text{Tor}_{\text{pdim}_R(k)}^R(k, k) \neq 0$  sandwiched between two zero modules.  $\square$

Our proof also showed the following:

**Corollary 3.71.** *Every regular local ring is a domain.*

**Corollary 3.72.** *Every regular local ring  $(R, \mathfrak{m}, k)$  has  $\text{pdim}_R(k) = \dim R$ .*

Now we can solve the localization problem very easily.

**Exercise 6.** If  $R$  is a regular local ring, then  $R_P$  is a regular local ring for every prime  $P$ .

**Remark 3.73.** If we want to show that a particular ring (not necessarily local) is regular, it is sufficient to show that  $R_{\mathfrak{m}}$  is a regular local ring for every maximal ideal  $\mathfrak{m}$  — this will imply that  $R_P$  is a localization of a regular local ring for every prime  $P$ .

**Exercise 7.** Show that every principal ideal domain is a regular ring.

We have shown that finitely generated *graded* modules over a polynomial ring  $k[x_1, \dots, x_d]$  have finite projective dimension, but this is not quite enough to conclude that polynomial rings are regular.

**Theorem 3.74.** *Every polynomial ring  $R = k[x_1, \dots, x_d]$  over a field  $k$  is a regular ring.*

We have seen that regular rings are very nice. Modulo some technical conditions, it turns out that *every* noetherian local ring is a quotient of a regular ring. More precisely, every *complete* local ring is a quotient of a regular local ring. If a local ring  $R$  is not complete, we can always take its *completion*, which is now a quotient of a regular local ring. This very important fact is the Cohen Structure Theorem.<sup>1</sup> When our local ring  $R$  contains a field  $k$ , the Cohen Structure Theorem actually says that  $R$  is a quotient of  $k[[x_1, \dots, x_d]]$  for some  $d$ .

The nice things we proved about regular local rings have analogues in any regular ring, not necessarily local. For example, when  $R$  is a regular ring of dimension  $d$ , then it is still true that every finitely generated  $R$ -module has projective dimension at most  $d$ , even if  $R$  is not local; if  $R$  is not regular, then it has finitely generated modules with infinite projective dimension.

## 3.6 Depth

We now get back to regular sequences to talk about their length.

**Definition 3.75.** Let  $I$  be an ideal in a ring  $R$  and  $M$  be an  $R$ -module. The  **$I$ -depth** of  $M$  is the maximal length of a regular sequence on  $M$  consisting of elements in  $I$ , denoted  $\text{depth}_I(M)$ . When  $(R, \mathfrak{m})$  is a local ring, we write  $\text{depth}(M)$  for  $\text{depth}_{\mathfrak{m}}(M)$ , and call it the **depth** of  $M$ .

While it is not yet clear that all maximal regular sequences on  $M$  inside an ideal  $I$  have the same length, we already have an upper bound for depth.

**Remark 3.76.** If  $x_1, \dots, x_n$  is a regular sequence on  $R$  inside  $I$ , we saw in Theorem 3.48 that  $\text{ht}(x_1, \dots, x_n) = n$ , so  $\text{depth}_I(R) \leq \text{ht}(I)$ . In particular,  $\text{depth}(R) \leq \dim(R)$ .

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<sup>1</sup>In fact, this amazing theorem was I. S. Cohen's PhD thesis!

We can construct maximal regular sequences explicitly.

**Construction 3.77.** Let  $R$  be a Noetherian ring,  $I$  be an ideal in  $R$ , and  $M \neq 0$  be a finitely generated  $R$ -module. To construct a regular sequence on  $M$  inside  $I$ , we start by finding a regular element on  $M$  inside  $I$ . Either  $I$  is contained in some associated prime of  $M$ , in which case every element in  $I$  is a zerodivisor on  $M$ , or there exists an element  $x_1$  in  $I$  not in any associated prime of  $M$ , by [Prime Avoidance](#). Such an element is regular on  $M$ , since the union of the associated primes of  $M$  is precisely the set of zerodivisors, by Theorem 1.39.

Now we repeat the process: either  $I$  is contained in some associated prime of  $M/(x_1)M$ , in which case there are no regular elements on  $M/(x_1)M$  inside  $I$ , or we can find  $x_2 \in I$  not in any associated prime of  $M/(x_1)M$ , which is necessarily regular on  $M/(x_1)M$ . At each step,  $(x_1, \dots, x_i)M \subsetneq (x_1, \dots, x_{i+1})M$ , and since  $M$  is Noetherian, the process must stop.

In fact, we get such an increasing sequence given any regular sequence on  $M$  inside of  $I$ , so all such sequences are finite. We will now show that all maximal regular sequences on  $M$  inside  $I$  have the same length, which proves that  $\text{depth}_I(M)$  is finite. First, we need a few lemmas.

**Lemma 3.78.** *Let  $R$  be a Noetherian ring,  $I$  be an ideal in  $R$ , and  $M \neq 0$  be a finitely generated  $R$ -module. There exists  $r \in I$  which is regular on  $M$  if and only if  $I \not\subseteq P$  for all  $P \in \text{Ass}(M)$ . In particular, if  $(R, \mathfrak{m})$  is a noetherian local ring,  $\mathfrak{m} \in \text{Ass}(M)$  if and only if there are no regular elements on  $M$ .*

*Proof.* If  $r \in I$  is regular on  $M$ , then  $r$  is not a zerodivisor on  $M$ , so  $r$  is not in the union of the associated primes of  $M$ , by Theorem 1.39. As a consequence,  $I$  cannot be contained in any associated prime of  $M$ .

Conversely, recall that  $M$  has finitely many associated primes, by Theorem 1.44. If  $I$  is not contained in any associated prime of  $M$ , then by Prime Avoidance B.1 it also cannot be contained in the union of the associated primes of  $M$ . Recall Theorem 1.39, which says that the union of the associated primes is the set of zero divisors. We conclude that  $I$  contains some regular element on  $M$ .

The final statement now follows once we note that the fact that  $\mathfrak{m}$  is a maximal ideal implies that  $\mathfrak{m} \in \text{Ass}(M)$  if and only if  $\mathfrak{m}$  is contained in some associated prime of  $M$ .  $\square$

**Lemma 3.79.** *Let  $R$  be a noetherian ring and  $M$  and  $N$  be finitely generated  $R$ -modules. If  $a \in \text{ann}_R(M)$ , then  $a \text{Ext}_R^i(M, N) = 0$  for all  $i$ . Moreover, if  $b \in \text{ann}_R(N)$ , then  $b \text{Ext}_R^i(M, N) = 0$  for all  $i$ .*

*Proof.* When  $i = 0$ , we want to show that  $a \in \text{ann}(\text{Hom}_R(M, N))$  and  $b \in \text{ann}(\text{Hom}_R(M, N))$ . Given any  $f \in \text{Hom}_R(M, N)$  and any  $m \in M$ ,

$$af(m) = f(am) = f(0) = 0,$$

so  $af = 0$ . Moreover,  $b$  kills every element in  $N$ , so  $bf = 0$ . Now let  $P \rightarrow M$  be a projective resolution on  $M$ , and  $N \rightarrow E$  be an injective resolution of  $N$ . Then

$$\begin{aligned} \text{Ext}_R^i(M, N) &= H^i \left( 0 \longrightarrow \text{Hom}_R(P_0, N) \longrightarrow \text{Hom}_R(P_1, N) \longrightarrow \text{Hom}_R(P_2, N) \longrightarrow \cdots \right) \\ &= H^i \left( 0 \longrightarrow \text{Hom}_R(M, E^0) \longrightarrow \text{Hom}_R(M, E^1) \longrightarrow \text{Hom}_R(M, E^2) \longrightarrow \cdots \right), \end{aligned}$$



so  $\text{Ext}_R^i(M, N)$  is a subquotient of both  $\text{Hom}_R(P_i, N)$  and  $\text{Hom}_R(M, E^i)$ . We conclude that  $a$  and  $b$  both kill  $\text{Ext}_R^i(M, N)$ .  $\square$

**Theorem 3.80.** *Let  $R$  be a Noetherian ring,  $I$  be an ideal in  $R$ , and  $M \neq 0$  be a finitely generated  $R$ -module with  $M \neq IM$ . Then all maximal regular sequences on  $M$  inside  $I$  have the same length.*

*Proof.* Let  $x_1, \dots, x_n \in I$  and  $y_1, \dots, y_\ell \in I$  both be maximal regular sequences on  $M$ , and assume  $n \leq \ell$ .

When  $n = 0$ , every element in  $I$  is a zerodivisor on  $M$ , so  $\ell = 0$ . When  $n = 1$ , every element in  $I$  is a zerodivisor on  $M/(x_1)M$ , so  $I \subseteq P$  for some  $P \in \text{Ass}(M/(x_1)M)$ . In particular, there exists some  $m \in M$ ,  $m \notin (x_1)M$ , such that  $I \subseteq ((x_1)M :_R m)$ , so  $Im \in (x_1)M$ . In particular,  $y_1 m = x_1 a$  for some  $a \in M$ . If  $a \in (y_1)M$ , then we would have  $y_1 m = x_1 a \in (x_1 y_1)M$ , and since  $y_1$  is regular on  $M$ , that would imply  $m \in (x_1)M$ , which is a contradiction. Thus  $a \notin (y_1)M$ . Moreover,

$$(x_1)Ia = Ix_1 a = Iy_1 m = y_1(Im) \subseteq (x_1)(y_1)M,$$

and since  $x_1$  is a regular element on  $M$ , we must have  $Ia \subseteq (y_1)M$ . Therefore,  $a \in M$  is an element that both satisfies  $a \notin (y_1)M$  and  $Ia \subseteq (y_1)M$ , so every element in  $I$  kills  $a$  in  $M/(y_1)M$ , and is thus a zerodivisor on  $M/(y_1)M$ . This proves that  $\ell = 1$ .

We proceed by induction on  $n$ . Now assume that  $n > 1$  and  $\ell > n$ . In particular,  $I$  contains a regular element on  $M/(x_1, \dots, x_i)$  for all  $i < n$  and a regular element on  $M/(y_1, \dots, y_j)$  for all  $j < \ell$ , so by [Prime Avoidance](#) we can pick  $c \in I$  that avoids both all the (finitely many) associated primes of  $M/(x_1, \dots, x_i)M$  for all  $i < n$  and  $M/(y_1, \dots, y_j)M$  for all  $j < \ell$ . In particular,  $x_1, \dots, x_{n-1}, c$  and  $y_1, \dots, y_n, c$  are both regular sequences on  $M$ . Now  $x_n$  and  $c$  are both regular sequences on  $M/(x_1, \dots, x_{n-1})$ , so the case  $n = 1$  says  $x_1, \dots, x_{n-1}, c$  is also a maximal regular sequence on  $M$ . Now by [Lemma 3.46](#),  $x_1, \dots, c, x_{n-1}$  is also a regular sequence on  $M$ , since  $c$  is also regular on  $M/(x_1, \dots, x_{n-2})$ , and so on, until we conclude that  $c, x_1, \dots, x_{n-1}$  is a regular sequence on  $M$ . Similarly,  $c, y_1, \dots, y_n$  is a regular sequence on  $M$ . Notice in fact that  $c, x_1, \dots, x_{n-1}$  is maximal inside  $I$ , or else we could increase its size, move  $c$  back to after  $x_{n-1}$ , and obtain a contradiction. Therefore,  $x_1, \dots, x_{n-1}$  and  $c, y_1, \dots, y_n$  are both regular sequences on  $M/(c)M$ , and  $x_1, \dots, x_{n-1}$  is maximal. But by induction hypothesis, all maximal regular sequences on  $M/(c)M$  inside  $I$  have the same length, which would say that the length of  $y_1, \dots, y_n, n$ , is at most  $n - 1$ . This is a contradiction, and we conclude that  $\ell = n$ .  $\square$

It turns out that depth can be described in a purely homological way.

**Theorem 3.81.** *Let  $R$  be a Noetherian ring and  $M$  a finitely generated  $R$ -module. Then*

$$\text{depth}_I(M) = \min\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}.$$

*Proof.* When  $\text{depth}_I(M) = 0$ , there is no regular sequence on  $M$  inside  $I$ . By [Lemma 3.78](#),  $I \subseteq P$  for some  $P \in \text{Ass}(M)$ . We have an inclusion  $R/P \hookrightarrow M$ , so consider the composition

$$R/I \twoheadrightarrow R/P \hookrightarrow M.$$



This is a nonzero map, so  $\text{Ext}^0(R/I, M) = \text{Hom}(R/I, M) \neq 0$ . On the other hand, if  $\text{Hom}_R(R/I, M) = \text{Ext}^0(R/I, M) \neq 0$ , there exists a nonzero  $R$ -module homomorphism  $R/I \rightarrow M$ . But to choose an  $R$ -module homomorphism  $R/I \rightarrow M$  is the same as choosing an element in  $M$  that is killed by  $I$ , so  $I$  contains no nonzero divisors on  $M$  and  $\text{depth}_I(M) = 0$ .

We proceed by induction on  $\text{depth}_I(M) = n$ , assuming we have proved the statement whenever  $\text{depth}_I(M) < n$ . Suppose that  $x_1, \dots, x_n$  is a maximal regular sequence on  $M$  inside  $I$ . Then  $x_2, \dots, x_n$  is a maximal regular sequence on  $M/x_1M$ , so by induction we know  $n-1 = \min\{i \mid \text{Ext}_R^i(R/I, M/x_1M) \neq 0\}$ . Applying  $\text{Hom}_R(R/I, -)$  to the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

we get a long exact sequence

$$\cdots \longrightarrow \text{Ext}_R^{i-1}(R/I, M/x_1M) \longrightarrow \text{Ext}_R^i(R/I, M) \xrightarrow{x_1} \text{Ext}_R^i(R/I, M) \longrightarrow \cdots$$

We know that  $\text{Ext}_R^{n-1}(R/I, M/x_1M) \neq 0$  and that  $\text{Ext}_R^i(R/I, M/x_1M) = 0$  for all  $i < n-1$ . Therefore, whenever  $i < n-1$ ,

$$\text{Ext}_R^i(R/I, M) \xrightarrow{x_1} \text{Ext}_R^i(R/I, M)$$

is an isomorphism. However,  $x_1 \in I = \text{ann}(R/I)$ , so  $\text{ann}(R/I) \subseteq \text{ann}(\text{Ext}_R^i(R/I, M))$  by Lemma 3.79. Therefore,  $\text{Ext}_R^i(R/I, M) = 0$  for all  $i < n-1$ . Moreover, multiplication by  $x_1$  is the zero map on  $\text{Ext}_R^i(R/I, M)$  for any  $i$ , also by Lemma 3.79. Finally, we have an exact sequence

$$\text{Ext}_R^{n-1}(R/I, M) \xrightarrow{x_1} \text{Ext}_R^{n-1}(R/I, M) \longrightarrow \text{Ext}_R^{n-1}(R/I, M/x_1M) \longrightarrow \cdots$$

where the multiplication by  $x_1$  maps are 0, so our exact sequence is

$$0 \longrightarrow \text{Ext}_R^{n-1}(R/I, M) \xrightarrow{0} \text{Ext}_R^{n-1}(R/I, M) \longrightarrow \underbrace{\text{Ext}_R^{n-1}(R/I, M/x_1M)}_{\neq 0} \longrightarrow \text{Ext}_R^n(R/I, M) \xrightarrow{0} \cdots$$

In particular,  $\text{Ext}_R^{n-1}(R/I, M) = 0$  and  $\text{Ext}_R^n(R/I, M) \neq 0$ . We conclude that

$$\text{depth}_I(M) = n = \min\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}. \quad \square$$

For yet another homological characterization of depth, we turn to Koszul homology.

**Theorem 3.82** (Depth sensitivity of the Koszul complex). *Let  $R$  be a Noetherian ring and  $M$  be finitely generated  $R$ -module. Given any  $\underline{x} = x_1, \dots, x_n$  such that  $(\underline{x})M \neq M$ ,*

$$\text{depth}_{(\underline{x})}(M) = \max\{r \mid H^i(\underline{x}; M) = 0 \text{ for all } i > n-r\}.$$

So we can measure  $\text{depth}_I(M)$  by looking at the first nonzero Koszul homology we see when we start counting *from the top*.

$$\begin{array}{ccccccc} K(\underline{x}; M) & 0 \rightarrow M \rightarrow \cdots \rightarrow M^{\binom{n}{n-r+1}} \rightarrow M^{\binom{n}{n-r}} \rightarrow \cdots \rightarrow M \rightarrow 0 \\ H(\underline{x}; M) & 0 & \cdots & 0 & \neq 0 & & \end{array}$$

*Proof.* We are going to show that if  $I = (\underline{x})$  contains a regular sequence  $y_1, \dots, y_m$  on  $M$ , then  $H_{n-i+1}(\underline{x}; M) = 0$  for all  $i = 1, \dots, m$ , and  $H_{n-m}(\underline{x}; M) \cong \text{Ext}_R^m(R/I, M)$ . This will prove the theorem, since depth is the largest possible  $m$  we could take, and Theorem 3.81 says  $\text{Ext}_R^{\text{depth}}(R/I, M) \neq 0$ .

We proceed by induction on  $m$ . When  $m = 0$ , Proposition 3.36 c) says that

$$H_n(\underline{x}; M) \cong (0 :_M I) \cong \text{Hom}_R(R/I, M),$$

and we are done. When  $m > 0$ , the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/(y_1)M \longrightarrow 0$$

induces a long exact sequence in koszul homology (see Proposition 3.36 h))

$$\cdots \longrightarrow H_{i+1}(\underline{x}; M/(y_1)M) \longrightarrow H_i(\underline{x}; M) \xrightarrow{y_1} H_i(\underline{x}; M) \longrightarrow H_i(\underline{x}; M/(y_1)M) \longrightarrow \cdots$$

Now by Proposition 3.36 e),  $I$  kills  $H_i(\underline{x}; M)$ , so the multiplication by  $y_1$  map is zero in the long exact sequence above, which must then break into short exact sequences

$$0 \longrightarrow H_i(\underline{x}; M) \longrightarrow H_i(\underline{x}; M/(y_1)M) \longrightarrow H_{i-1}(\underline{x}; M) \longrightarrow 0.$$

We have an exact sequence  $y_2, \dots, y_m$  of  $m - 1$  elements on  $M$  inside  $(\underline{x})$ , so by induction hypothesis

$$H_{n-i+1}(\underline{x}; M/(y_1)M) = 0 \text{ for all } i = 1, \dots, m - 1$$

and

$$H_{n-m+1}(\underline{x}; M/(y_1)M) \cong \text{Ext}_R^{m-1}(R/I, M/y_1M).$$

Therefore, for all  $i \leq m - 1$ ,

$$0 \longrightarrow H_{n-i+1}(\underline{x}; M) \longrightarrow \underbrace{H_{n-i+1}(\underline{x}; M/(y_1)M)}_0 \longrightarrow H_{n-i}(\underline{x}; M) \longrightarrow 0.$$

This implies  $H_{n-i+1}(\underline{x}; M) = 0$  for all  $i = 1, \dots, m$ . Moreover, we have a short exact sequence

$$0 \longrightarrow \underbrace{H_{n-m+1}(\underline{x}; M)}_0 \longrightarrow H_{n-m+1}(\underline{x}; M/(y_1)M) \longrightarrow H_{n-m}(\underline{x}; M) \longrightarrow 0.$$

so

$$H_{n-m}(\underline{x}; M) \cong H_{n-m+1}(\underline{x}; M/(y_1)M) \cong \text{Ext}_R^{m-1}(R/I, M/y_1M).$$

Finally, we claim that  $\text{Ext}_R^m(R/I, M) \cong \text{Ext}_R^{m-1}(R/I, M/(y_1)M)$ . For that purpose, consider the long exact sequence on  $\text{Ext}_R^i(R/I, -)$  induced by the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/(y_1)M \longrightarrow 0,$$

which looks like

$$\cdots \longrightarrow \text{Ext}_R^i(R/I, M) \xrightarrow{y_1} \text{Ext}_R^i(R/I, M) \longrightarrow \text{Ext}_R^i(R/I, M/(y_1)M) \longrightarrow \cdots$$

We do have a regular sequence on  $M$  of length  $m$  inside  $I$ , so  $\text{depth}_I(M) \geq m$ . Therefore,  $\text{Ext}_R^{m-1}(R/I, M) = 0$  by Theorem 3.81. By Lemma 3.79, multiplication by  $y_1$  on  $\text{Ext}_R^i(R/I, M)$  is the zero map, since  $y_1 \in I$ , so we get an exact sequence

$$0 \longrightarrow \text{Ext}_R^{m-1}(R/I, M/(y_1)M) \longrightarrow \text{Ext}_R^m(R/I, M) \longrightarrow 0.$$

Therefore,

$$\text{Ext}_R^m(R/I, M) \cong \text{Ext}_R^{m-1}(R/I, M/(y_1)M),$$

which finishes our proof.  $\square$

**Remark 3.83.** If  $I = (x_1, \dots, x_n)$  and  $\text{depth}_I(M) = n$ , then by Theorem 3.82 the Koszul complex  $K(\underline{x}; M)$  must be exact. This does not necessarily say that  $\underline{x}$  is a regular sequence, only that there exists some regular sequence on  $M$  of length  $n$  inside  $I$ . However, that implication does hold in the local or graded setting, by Theorem 3.44.

We now have all the tools we need to prove a very useful formula relating depth and projective dimension.

**Theorem 3.84** (Auslander—Buchsbaum Formula). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and  $M \neq 0$  a finitely generated  $R$ -module of finite projective dimension. Then*

$$\text{depth}(M) + \text{pdim}_R(M) = \text{depth}(R).$$

*Proof.* Suppose  $\text{depth}(R) = 0$ . In that case, the claim is that  $\text{pdim}_R(M) = \text{depth}(M) = 0$ . First, note that the fact that  $\text{depth}(R) = 0$  implies immediately that  $\mathfrak{m} \in \text{Ass}(R)$ , by Lemma 3.78, so  $\mathfrak{m}$  kills some nonzero  $r \in R$ . Consider a minimal free resolution for  $M$ , say

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0.$$

Suppose  $n > 0$ , so that  $\varphi_n \neq 0$ . By minimality,  $\varphi_n(F_n) \subseteq \mathfrak{m}F_{n-1}$ , so  $\varphi_n(r, 0, \dots, 0) = r\varphi_n(1, 0, \dots, 0) \in r\mathfrak{m} = 0$ , so  $\varphi_n$  is not injective. This is a contradiction, so we must have  $n = 0$ , and  $M$  is free, say  $M \cong R^n$ . Therefore,  $\text{pdim}_R(M) = 0$  and  $\text{depth}(M) = \text{depth}(R^n) = \text{depth}(R) = 0$ .

Now assume that  $\text{depth}(M) = 0$ . Set  $t := \text{depth}(R)$ , and fix a maximal regular sequence  $x_1, \dots, x_t \in \mathfrak{m}$ . By Corollary 3.63,  $\text{pdim}_R(R/(x_1, \dots, x_t)) = t$ . Our goal is to show that  $n := \text{pdim}_R(M) = \text{depth}(R) = t$ . Notice that  $\text{Tor}_i^R(R/(x_1, \dots, x_t), M)$  can be computed via minimal free resolutions for either  $M$  or  $R/(x_1, \dots, x_t)$ , so it vanishes for  $i > \min\{t, n\}$ . We are going to show that both  $\text{Tor}_t^R(R/(x_1, \dots, x_t), M) \neq 0$  and  $\text{Tor}_n^R(R/(x_1, \dots, x_t), M) \neq 0$ , which proves that  $\text{depth}(R) = t = n = \text{pdim}_R(M)$ .

The Koszul complex is a minimal free resolution for  $R/(x_1, \dots, x_t)$ , by Corollary 3.63, so the last map in the minimal free resolution looks like

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{n+1}x_n \end{pmatrix}^T} R^t$$

so applying  $- \otimes_R M$  gives

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{n+1}x_n \end{pmatrix}^T} M^t.$$

Therefore,

$$\mathrm{Tor}_t^R(R/(x_1, \dots, x_t), M) = \bigcap_{i=1}^t \ker(M \xrightarrow{\pm x_i} M).$$

Our assumption that  $\mathrm{depth}(M) = 0$  says that there are no regular elements on  $M$ , so by Lemma 3.78,  $\mathfrak{m} \in \mathrm{Ass}(M)$ . Therefore, there exists a nonzero element  $m \in M$  such that  $\mathrm{ann}(m) = \mathfrak{m} \supseteq (x_1, \dots, x_t)$ , so  $\mathrm{Tor}_t^R(R/(x_1, \dots, x_t), M) \neq 0$ .

On the other hand, to compute  $\mathrm{Tor}_n^R(R/(x_1, \dots, x_t), M)$  we can take a minimal free resolution of  $M$ , say

$$0 \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \xrightarrow{\varphi_{n-1}} \dots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0,$$

and apply  $-\otimes_R M$ , so that  $\mathrm{Tor}_n^R(R/(x_1, \dots, x_t), M)$  is the kernel of

$$(R/(x_1, \dots, x_t))^{\beta_n} \longrightarrow (R/(x_1, \dots, x_t))^{\beta_{n-1}}.$$

Our assumption that  $x_1, \dots, x_t$  is a maximal regular sequence on  $R$  implies that any other element in  $R$  is a zerodivisor on  $R/(x_1, \dots, x_t)$ , and  $\mathrm{depth}(R/(x_1, \dots, x_t)) = 0$ . In particular,  $\mathfrak{m} \in \mathrm{Ass}(R/(x_1, \dots, x_t))$ , so there exists some  $r \notin (x_1, \dots, x_t)$  such that  $\mathfrak{m}r \subseteq (x_1, \dots, x_t)$ . The map

$$(R/(x_1, \dots, x_t))^{\beta_n} \longrightarrow (R/(x_1, \dots, x_t))^{\beta_{n-1}}$$

is given by multiplication by a matrix whose entries are all in  $\mathfrak{m}$ , so its kernel is nonzero, meaning  $\mathrm{Tor}_n^R(R/(x_1, \dots, x_t), M) \neq 0$ .

So we have shown the theorem holds in two situations: when  $\mathrm{depth}(R) = 0$  and when  $\mathrm{depth}(M) = 0$ . So now we assume that both  $t := \mathrm{depth}(R) > 0$  and  $n := \mathrm{depth}(M) > 0$ , and assume we have shown the theorem holds when  $\mathrm{depth}(R) \leq t-1$  and  $\mathrm{depth}(M) \leq n-1$ .

By Prime Avoidance, Theorem B.1, we can find  $r \in \mathfrak{m}$  that avoids both the associated primes of  $M$  and  $R$ , so  $r$  is both regular on  $M$  and on  $R$ . In particular,  $\mathrm{pdim}_R(R/(x)) = 1$ , by Corollary 3.63, so  $\mathrm{Tor}_i^R(R/(x), M) = 0$  for all  $i \geq 2$ . Let

$$0 \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \xrightarrow{\varphi_{n-1}} \dots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0$$

be a minimal free resolution for  $M$ . In particular, if we choose basis for each  $R^{\beta_i}$ , the entries in the matrices representing  $\varphi_i$  are all in  $\mathfrak{m}$ . Applying  $-\otimes_R R/(x)$ , we get a complex

$$0 \longrightarrow (R/(x))^{\beta_n} \longrightarrow (R/(x))^{\beta_{n-1}} \longrightarrow \dots \longrightarrow (R/(x))^{\beta_0} \longrightarrow M/xM \longrightarrow 0$$

which is exact at  $M \otimes_R R/(x) \cong M/xM$ , since tensor is right exact, and whose homology is otherwise given by  $\mathrm{Tor}_i^R(R/(x), M)$ . In particular, our complex is exact for all  $i \geq 2$ , since we have seen that  $\mathrm{Tor}_i^R(R/(x), M) = 0$  for all  $i \geq 2$ . The only remaining possibly interesting homology is given by

$$\mathrm{Tor}_1^R(R/(x), M) = H_1(M \otimes (0 \rightarrow R \xrightarrow{x} R \rightarrow 0)) = H_1(0 \rightarrow M \xrightarrow{x} M \rightarrow 0) = (0 :_M x).$$

By assumption,  $x$  is regular on  $M$ , so  $\mathrm{Tor}_1^R(R/(x), M) = (0 :_M x) = 0$ . So the complex above is exact, and thus a free resolution for  $M/xM$  over  $R/(x)$ . In fact, the maps in this is free

resolution for  $M/xM$  were obtained by tensoring  $\varphi$  with  $R/(x)$ , so we can obtain matrices representing each map by taking the matrix representing  $\varphi_i$  and setting all the entries in  $(x)$  equal to 0. In particular, all the entries are still in  $\mathfrak{m}/(x)$ , and our resolution for  $M/xM$  over  $R/(x)$  is minimal. This shows that  $\text{pdim}_{R/(x)}(M/xM) = \text{pdim}_R(M)$ .

Now notice that we picked  $x$  to be regular on  $M$ , so that  $\text{depth}(M/xM) = \text{depth}(M) - 1$ . Similarly,  $x$  is regular on  $R$ , so  $\text{depth}(R/(x)) = \text{depth}(R) - 1$ . Using our assumption, we conclude that

$$\begin{aligned} \text{depth}(M/x) + \text{pdim}_{R/(x)}(M/xM) &= \text{depth}(R/(x)) \\ \Leftrightarrow \text{depth}(M) - 1 + \text{pdim}_R(M) &= \text{depth}(R) - 1 \\ \Leftrightarrow \text{depth}(M) + \text{pdim}_R(M) &= \text{depth}(R). \end{aligned}$$

□

This formula is very useful. For example, when doing explicit computations, it is often easier to compute a minimal free resolution for  $M$  than to compute its depth. If we happen to know  $\text{depth}(R)$ , one can deduce  $\text{depth}(M)$  by computing  $\text{pdim}_R(M)$ .

**Remark 3.85.** One of the consequences of Theorem 3.84 is that if a finitely generated  $R$ -module  $M$  has finite projective dimension, then  $\text{pdim}_R(M) \leq \text{depth}(R) \leq \dim R$ .

We close this section with an easy fact about depth.

**Lemma 3.86.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  a finitely generated  $R$ -module. Given any ideal  $I$  in  $R$ ,  $\text{depth}_I(M) = \text{depth}_{\sqrt{I}}(M)$ .*

*Proof.* On the one hand,  $I \subseteq \sqrt{I}$ , so  $\text{depth}_I(M) \leq \text{depth}_{\sqrt{I}}(M)$ . On the other hand, if  $x_1, \dots, x_n$  is a maximal regular sequence on  $M$  inside  $\sqrt{I}$ , then there exists  $a_1, \dots, a_n > 0$  such that  $x_1^{a_1}, \dots, x_n^{a_n} \in I$ , but by Lemma 3.47  $x_1^{a_1}, \dots, x_n^{a_n}$  is a regular sequence on  $M$ . □

## 3.7 Cohen-Macaulay rings

Life is really worth living in a Noetherian ring  $R$  when all the local rings have the property that every system of parameters is an  $R$ -sequence. Such a ring is called Cohen-Macaulay (C-M for short).

(Mel Hochster, page 887 of [Hoc78])

Cohen-Macaulay rings, named after Irvin Cohen and Francis Macaulay, two big influences in the early days of commutative algebra, are by some measure the largest class of nice rings commutative algebraists study. They are on the border of being just nice enough to make life is easier, and just broad enough to contain many interesting examples. One of the main reference books in any commutative algebraist's shelf is dedicated to Cohen-Macaulay rings specifically [BH93]. In this section, we will see some of the reasons why life really is worth living in a Cohen-Macaulay ring.

Given a local ring  $R$ ,

$$\text{depth}(R) \leq \dim(R) \leq \text{embdim}(R).$$

When the second inequality is an equality, we have a regular ring. When the first inequality is an equality, our ring is Cohen-Macaulay.

**Definition 3.87.** A Noetherian local ring  $R$  is **Cohen-Macaulay** if  $\text{depth}(R) = \dim(R)$ . More generally, an  $R$ -module  $M$  is **Cohen-Macaulay** if  $\text{depth}(M) = \dim(M)$ . A Noetherian ring  $R$  is **Cohen-Macaulay** if  $R_{\mathfrak{m}}$  is Cohen-Macaulay for every maximal ideal  $\mathfrak{m}$ .

**Example 3.88.**

- a) Every regular ring is Cohen-Macaulay, since our homological characterization of regular local rings, Theorem 3.70, says that the maximal ideal is generated by a regular sequence of dimension many elements.
- b) Every Artinian ring is Cohen-Macaulay: since we always have  $\dim(R) \geq \text{depth}(R)$ , having  $\dim(R) = 0$  automatically implies  $\text{depth}(R) = 0$ .
- c) Every 1-dimensional domain is Cohen-Macaulay, since any nonzero nonunit is a regular element.
- d) The ring  $k[x]/(x^2)$  is Cohen-Macaulay because it has dimension 0, but it is not regular, since 0-dimensional regular rings must be fields; in fact, the embedding dimension of  $k[x]/(x^2)$  is 1.
- e) We claim that the local ring  $R = k[[x, y, z]]/(xy, xz)$  is not Cohen-Macaulay. First, we compute its dimension: since  $(x)$  is the unique minimal prime over  $(xy, xz)$  in  $k[[x, y, z]]$ , by Theorem 3.20 we have  $\dim(R) = \dim(k[[x, y, z]]) - \text{ht}(xy, xz) = 3 - 1 = 2$ . So for  $R$  to be Cohen-Macaulay, we must be able to find a regular sequence of length 2 in  $R$ . To find one regular element, we need to find an element in  $R$  that is not a unit and not in any associated prime of  $R$ : the associated primes of  $(xy, xz)$  in  $k[[x, y, z]]$  are  $(x)$  and  $(y, z)$ , so we can take for example  $x - y$ . Now  $R/(x - y) \cong k[[x, z]]/(x^2, xz)$ , and the unique maximal ideal  $(x, y, z)$  of  $R$  is associated to  $R/(x - y)$ : it is the annihilator of  $x$ , since  $x^2 = 0$ ,  $xz = 0$ , and  $yx = x^2 = 0$ . As we saw in Lemma 3.78, this means that there are no elements on  $R$  that are regular on  $R/(x - y)$ , so we conclude that  $x - y$  is a maximal regular sequence on  $R$ , and  $\text{depth}(R) = 1$ . Since  $R$  is 2-dimensional,  $R$  is not Cohen-Macaulay.

We could have concluded this ring is not Cohen-Macaulay by noting that it is not equidimensional, which as we will see is a property shared by all Cohen-Macaulay rings.

Many rings with *nice* singularities are Cohen-Macaulay. For example, Hochster and Roberts famously showed [HR74] that the ring of invariants of any finite group  $G$  over a field  $k$  of characteristic not dividing  $|G|$  is Cohen-Macaulay. Their proof used prime characteristic techniques, introducing what is now a very important class of characteristic  $p$  singularities, and are essentially homological in nature.

**Remark 3.89.** Given a Noetherian local ring  $(R, \mathfrak{m})$ , we can decide whether  $R$  is Cohen-Macaulay by computing  $\dim(R)$  and  $\text{depth}(R)$ . By Theorem 3.81, we can compute  $\text{depth}(R)$  by finding the smallest  $i$  such that  $\text{Ext}_R^i(R/\mathfrak{m}, R) \neq 0$ . On the other hand, if  $R$  can easily be seen to be regular, then we can immediately conclude  $R$  is Cohen-Macaulay. The easiest way to decide whether a given regular local ring is regular is by finding  $\mu(\mathfrak{m})$  and to compare its size to  $\dim(R)$ .

To compute the depth of a finitely generated  $R$ -module  $M$  over a Cohen-Macaulay local ring  $(R, \mathfrak{m})$ , we can start by computing a minimal free resolution of  $M$ . If  $\text{pdim}_R(M) < \infty$ , then the [Auslander–Buchsbaum formula](#) tells us that

$$\text{depth}(M) = \text{depth}(R) - \text{pdim}_R(M) = \dim(R) - \text{pdim}_R(M).$$

If  $R$  is not Cohen-Macaulay, we need to also compute  $\text{depth}(R)$ , which we can do for example by finding the smallest  $i$  such that  $\text{Ext}_R^i(R/\mathfrak{m}, R) \neq 0$ , or by explicitly constructing a maximal regular sequence as in Construction 3.77. If  $\text{pdim}_R(M) = \infty$ , then Theorem 3.84 does not apply, and we need to explicitly find  $\text{depth}(M)$ , for example by finding the smallest  $i$  such that  $\text{Ext}_R^i(R/\mathfrak{m}, M) \neq 0$ , or by explicitly constructing a maximal regular sequence as in Construction 3.77.

Hochster’s quote in the beginning of this section pointed us to an equivalent characterization of Cohen-Macaulayness, for which we will need to recall the notion of a system of parameters.

**Definition 3.90.** A sequence of  $d$  elements  $x_1, \dots, x_d$  in a  $d$ -dimensional noetherian local ring  $(R, \mathfrak{m})$  is a **system of parameters** or **SOP** if  $\sqrt{(x_1, \dots, x_d)} = \mathfrak{m}$ . If  $k$  is a field, a sequence of  $d$  homogeneous elements  $x_1, \dots, x_d$  in a  $d$ -dimensional  $\mathbb{N}$ -graded finitely generated  $k$ -algebra  $R$ , with  $R_0 = k$ , is a *homogeneous system of parameters* if  $\sqrt{(x_1, \dots, x_d)} = R_+$ .

We say that elements  $x_1, \dots, x_t$  are **parameters** if they are part of a system of parameters; this is a property of the set, not just the elements.

**Lemma 3.91.** *Let  $R$  be a Noetherian ring, and  $I$  be an ideal. Let  $f_1, \dots, f_t \in I$ , and  $J_i = (f_1, \dots, f_i)$  for each  $i$ . Suppose that for each  $i$ ,*

$$f_i \notin \bigcup_{\substack{\mathfrak{a} \in \text{Min}(J_{i-1}) \\ \mathfrak{a} \notin V(I)}} \mathfrak{a}.$$

*Then any minimal prime of  $J_i$  either contains  $I$  or has height  $i$ .*

*Proof.* We use induction on  $i$ . For  $i = 0$ ,  $J_0 = (0)$ , and every minimal prime has height zero. Suppose now the statement holds for  $i = m$ , and consider a minimal prime  $\mathfrak{q}$  of  $J_{m+1}$ . Since  $J_m \subseteq J_{m+1}$ ,  $\mathfrak{q}$  must contain some minimal prime of  $J_m$ , say  $\mathfrak{p}$ . If  $\mathfrak{p} \supseteq I$ , then  $\mathfrak{q} \supseteq I$ . If  $\mathfrak{q}$  does not contain  $I$ , then neither does  $\mathfrak{p}$ . On the one hand,  $f_{m+1} \in J_{m+1} \subseteq \mathfrak{q}$ . On the other hand, since  $\mathfrak{p} \in \text{Min}(J_m)$  and  $\mathfrak{p} \notin V(I)$ , our assumption implies that  $f_{m+1} \notin \mathfrak{p}$ . In particular,  $\mathfrak{p} \subsetneq \mathfrak{q}$ . By the induction hypothesis,  $\mathfrak{p}$  has height  $m$ , and thus the height of  $\mathfrak{q}$  is at least  $m + 1$ . But  $J_{m+1}$  is generated by  $m + 1$  elements, so by the Krull Height Theorem 3.11, the height of  $\mathfrak{q}$  is then exactly  $m + 1$ .  $\square$



**Theorem 3.92.** *Let  $R$  be a Noetherian ring of dimension  $d$ .*

- a) *If  $\mathfrak{p}$  is a prime of height  $h$ , then there are  $h$  elements  $f_1, \dots, f_h \in \mathfrak{p}$  such that  $\mathfrak{p}$  is a minimal prime of  $(f_1, \dots, f_h)$ .*
- b) *Suppose that  $R$  is either a local ring or an  $\mathbb{N}$ -graded ring with  $R_0$  a field. Let  $I$  is an ideal in  $R$ , homogeneous in the graded case. There are  $d$  elements, which can be chosen to be homogeneous in the graded case, say  $f_1, \dots, f_d \in I$ , such that  $\sqrt{I} = \sqrt{(f_1, \dots, f_d)}$ .*

*Proof.* We will use the notation from the previous lemma.

- a) If  $\mathfrak{p}$  is a minimal prime in  $R$ , then  $\mathfrak{p}$  is minimal over the ideal generated by 0 elements,  $(0)$ . Otherwise, we will use the recipe from the lemma above with  $I = \mathfrak{p}$ . First, we need to show that we can choose  $h$  elements satisfying the hypotheses. So we will show that starting from  $J_0 = (0)$ , we can find elements  $f_1, \dots, f_h \in \mathfrak{p}$  such that  $J_i = (f_1, \dots, f_i)$

$$\mathfrak{p} \not\subseteq \bigcup_{\substack{\mathfrak{a} \in \text{Min}(J_i) \\ \mathfrak{a} \notin V(I)}} \mathfrak{a}$$

for  $i = 0, \dots, h-1$ . As long as the set on the right is nonempty,

$$(f_1, \dots, f_i) \subseteq \bigcup_{\substack{\mathfrak{a} \in \text{Min}(J_i) \\ \mathfrak{a} \notin V(I)}} \mathfrak{a},$$

so the previous statement allows us to choose  $f_{i+1}$  as in the Lemma. So fix any  $i \leq h-1$ , and suppose we have constructed  $J_i$ . The Krull Height Theorem 3.11 implies that all the elements in  $\text{Min}(J_i)$  have height strictly less than  $h$ . Since  $\mathfrak{p}$  has height  $h$ , that implies that the sets  $\text{Min}(J_i)$  and  $V(\mathfrak{p})$  are disjoint. So we want to show that

$$\mathfrak{p} \not\subseteq \bigcup_{\substack{\mathfrak{a} \in \text{Min}(f_1, \dots, f_i) \\ \mathfrak{a} \notin V(I)}} \mathfrak{a} = \bigcup_{\mathfrak{a} \in \text{Min}(f_1, \dots, f_i)} \mathfrak{a}$$

This is immediate by prime avoidance B.1, again because  $\mathfrak{p}$  is not contained in a minimal prime of  $(f_1, \dots, f_i)$ . Thus, we can choose  $(f_1, \dots, f_h) \subseteq \mathfrak{p}$  as in the lemma, and by the lemma its minimal primes either have height  $h$  or contain  $\mathfrak{p}$ . Since  $(f_1, \dots, f_h) \subseteq \mathfrak{p}$ , some minimal prime  $\mathfrak{q}$  of  $J_h$  is contained in  $\mathfrak{p}$ . We know that this  $\mathfrak{q}$  either contains  $\mathfrak{p}$ , and hence is  $\mathfrak{p}$ , or else is contained in and has the same height as  $\mathfrak{p}$ , so again must be equal to  $\mathfrak{p}$ . Therefore,  $\mathfrak{p}$  is a minimal prime of  $(f_1, \dots, f_h)$ .

- b) We again run the same argument, using homogeneous prime avoidance in the graded case. The point is that the only (homogeneous, in the graded case) ideal of height  $d$  already contains  $I$ .  $\square$

By Theorem 3.92, every local (or graded) ring admits a system of parameters, and these can be useful in characterizing the dimension of a local Noetherian ring, or the height of a prime in a Noetherian ring. Moreover, we can characterize Cohen-Macaulayness in terms of sops.



**Theorem 3.93.** *The following are equivalent for any Noetherian local ring  $(R, \mathfrak{m})$ :*

- a)  *$R$  is Cohen-Macaulay.*
- b) *Some system of parameters in  $R$  is a regular sequence on  $R$ .*
- c) *Every system of parameters in  $R$  is a regular sequence on  $R$ .*

*Proof.* Clearly, c)  $\Rightarrow$  b)  $\Rightarrow$  a). Suppose  $R$  is Cohen-Macaulay and let  $\underline{x} = x_1, \dots, x_d$  be a system of parameters on  $R$ , meaning  $\sqrt{(\underline{x})} = \mathfrak{m}$ . By Lemma 3.86,  $\text{depth}_{(\underline{x})}(R) = \text{depth}_{\mathfrak{m}}(R) = d$ . By Theorem 3.82,  $K(\underline{x})$  is exact, and by Theorem 3.44, this implies that  $\underline{x}$  is a regular sequence.  $\square$

To prove some other nice properties of Cohen-Macaulay rings, we will need the following technical looking result.

**Theorem 3.94.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  and  $N$  finitely generated  $R$ -modules. Then  $\text{Ext}_R^i(M, N) = 0$  for all  $i < \text{depth}(N) - \dim(M)$ .*

*Proof.* First, we reduce to the case when  $M = R/P$  for some prime ideal  $P$ . To do that, fix a prime filtration of  $M$ ; such a filtration is well-known to exist. More precisely, this is an ascending chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_n = M$$

such that  $M_i/M_{i-1} \cong R/P_i$  for some primes  $P_i$ . First, we claim that we can reduce the problem to showing  $\text{Ext}_R^j(R/P_i, N) = 0$  for all  $j < \text{depth}(N) - \dim(M)$  and all  $i$ .

Break this filtration into short exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow R/P_i \longrightarrow 0$$

and look at the long exact sequence we get when we apply  $\text{Hom}_R(-, N)$ :

$$\dots \longrightarrow \text{Ext}_R^j(R/P_i, N) \longrightarrow \text{Ext}_R^j(M_i, N) \longrightarrow \text{Ext}_R^j(M_{i-1}, N) \longrightarrow \dots$$

If  $\text{Ext}_R^j(R/P_i, N) = 0$  for all  $i$ , then we must have  $\text{Ext}_R^j(M_i, N) = 0$  for all  $i$ , and therefore  $\text{Ext}_R^j(M, N) = 0$ .

So we have reduced the problem to showing that  $\text{Ext}_R^j(R/P_i, N) = 0$  for all  $i$  and all  $j < \text{depth}(N) - \dim(M)$ . Notice also that by the construction of the prime filtration, all the  $P_i$  contain  $\text{ann}(M)$ , so  $\dim(R/P_i) \leq \dim(M)$ . Therefore, it is sufficient to show that if  $P$  is prime, then  $\text{Ext}_R^i(R/P, N) = 0$  for all  $i < \text{depth}(N) - \dim(R/P)$ .

We proceed by induction on  $\dim(R/P)$ . If  $\dim(R/P) = 0$ , then  $P = \mathfrak{m}$ , so by Theorem 3.81 we have  $\text{Ext}_R^i(R/\mathfrak{m}, N) = 0$  for all  $i < \text{depth}(N) = \text{depth}(N) - \dim(R/P)$ . If  $\dim(R/P) > 0$ , the  $P \neq \mathfrak{m}$ , so pick  $x \in \mathfrak{m}$  but  $x \notin P$ . The short exact sequence

$$0 \longrightarrow R/P \xrightarrow{x} R/P \longrightarrow R/P + (x) \longrightarrow 0$$

gives rise to the long exact sequence

$$\dots \rightarrow \text{Ext}_R^i(R/P + (x), N) \rightarrow \text{Ext}_R^i(R/P, N) \xrightarrow{x} \text{Ext}_R^i(R/P, N) \rightarrow \text{Ext}_R^i(R/P + (x), N) \rightarrow \dots$$

Since  $x \notin P$ , we necessarily have  $\dim(R/P + (x)) < \dim(R/P)$ , so by induction hypothesis we have  $\text{Ext}_R^i(R/P + (x), N) = 0$  for all

$$i < \text{depth}(N) - \dim(R/P + (x)) = \text{depth}(N) - \dim(R/P) + 1.$$

Therefore, for all  $i < \text{depth}(N) - \dim(R/P)$ ,

$$\text{Ext}_R^i(R/P, N) \xrightarrow{x} \text{Ext}_R^i(R/P, N)$$

is an isomorphism. In particular,  $\text{Ext}_R^i(R/P, N) = x \cdot \text{Ext}_R^i(R/P, N)$ , so  $\text{Ext}_R^i(R/P, N) = 0$  by [NAK](#), where we used that  $\text{Ext}_R^i(R/P, N)$  is finitely generated.

It remains to show that  $\text{Ext}_R^i(R/P, N) = 0$  when  $i < \text{depth}(N) - \dim(R/P)$ .  $\square$

**Corollary 3.95.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $M$  be a finitely generated  $R$ -module. For every associated prime of  $M$ ,  $\text{depth}(M) \leq \dim(R/P)$ .*

*Proof.* By Theorem [3.94](#),  $\text{Ext}_R^i(R/P, M) = 0$  for all  $i < \text{depth}(M) - \dim(R/P)$ . But every element in  $P$  is a zerodivisor on  $M$ , so  $\text{depth}_P(M) = 0$ , and by Theorem [3.81](#),  $\text{Ext}_R^0(R/P, M) \neq 0$ . We conclude that  $\text{depth}(M) \leq \dim(R/P)$ .  $\square$

One of the nicest properties all Cohen-Macaulay rings have is called unmixedness.

**Theorem 3.96** (Unmixedness theorem). *Let  $R$  be a Noetherian local ring. If  $M$  is a Cohen-Macaulay  $R$ -module, then*

$$\text{depth}(M) = \dim(R/\mathfrak{p})$$

*for every  $\mathfrak{p} \in \text{Ass}(M)$ . In particular, if  $R$  is a Cohen-Macaulay ring, then  $R$  has no embedded primes, and  $\dim(R/\mathfrak{p}) = \dim(R)$  for each  $\mathfrak{p} \in \text{Min}(R)$ .*

*Proof.* This follows from the inequality on depth and dimension of associated primes from Corollary [3.95](#):

$$\dim(M) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Ass}(M)\} \geq \min\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Ass}(M)\} \geq \text{depth}(M).$$

Since  $M$  is Cohen-Macaulay, equality holds throughout, and that implies in fact that equality holds for each  $\dim(R/\mathfrak{p})$  with  $\mathfrak{p} \in \text{Ass}(M)$ .  $\square$

The Cohen-Macaulay property localizes.

**Theorem 3.97.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring and  $P$  be a prime ideal in  $R$ . Then  $R_P$  is a Cohen-Macaulay local ring.*

*Proof.* We claim that there is a regular sequence contained in  $R$  of length equal to the height of  $P$ . If  $P$  is minimal, there is nothing to show. If  $P$  is not minimal, it is not contained in the union of the minimal primes, hence not in the union of the associated primes of  $R$  by Theorem [3.96](#). Thus, there is a nonzerodivisor in  $P$ . We can mod out by this to get a Cohen-Macaulay ring of lower dimension, and inductively, the claim follows. Now localizing at  $P$  this stays a regular sequence that is a system of parameters for  $R_P$ .  $\square$

**Theorem 3.98** (Dimension formula). *Let  $R$  be a Cohen-Macaulay ring, and  $\mathfrak{p} \subseteq \mathfrak{q}$  be primes. Then*

$$\text{ht}(\mathfrak{q}) - \text{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}).$$

*In particular, if  $(R, \mathfrak{m})$  is a Cohen-Macaulay local ring, then  $\dim(R) - \text{height}(\mathfrak{p}) = \dim(R/\mathfrak{p})$ .*

*Proof.* Cohen-Macaulayness localizes, by Theorem 3.97, so  $R_{\mathfrak{q}}$  is a Cohen-Macaulay local ring of dimension  $\text{ht}(\mathfrak{q})$ . Pick  $h = \text{ht}(\mathfrak{p})$  elements  $r_1, \dots, r_h \in \mathfrak{p}$  that form a regular sequence. Now  $R_{\mathfrak{q}}/(r_1, \dots, r_h)R_{\mathfrak{q}}$  is Cohen-Macaulay, and  $\dim(R_{\mathfrak{q}}/(r_1, \dots, r_h)R_{\mathfrak{q}}) = \dim(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}})$ , since  $r_1, \dots, r_h$  is a maximal regular sequence in  $\mathfrak{p}$ , so  $\mathfrak{p}R_{\mathfrak{q}} \in \text{Ass}(R_{\mathfrak{q}}/(r_1, \dots, r_h)R_{\mathfrak{q}})$ . On the other hand,  $\dim(R_{\mathfrak{q}}/(r_1, \dots, r_h)R_{\mathfrak{q}}) = \dim(R_{\mathfrak{q}}) - h$ . The equality follows.  $\square$

**Remark 3.99.** If  $(R, \mathfrak{m})$  is Cohen-Macaulay and  $x_1, \dots, x_a$  is a regular sequence on  $R$ , then we claim that  $R/(x_1, \dots, x_a)$  is also Cohen-Macaulay. On the one hand,  $\text{ht}(x_1, \dots, x_a) = a$  by Theorem 3.48. On the other hand, by Theorem 3.11 all the minimal primes of  $R/(x_1, \dots, x_a)$  have height at most  $a$ , so we conclude that all the minimal primes of  $R/(x_1, \dots, x_a)$  have the same height  $a$ . By Theorem 3.98,  $\dim(R/(x_1, \dots, x_a)) = \dim(R) - a$ . On the other hand, going modulo each  $x_i$  decreases the depth by 1, so  $\text{depth}(R/(x_1, \dots, x_a)) = \text{depth}(R) - a = \dim(R) - a$ .

To help characterize systems of parameters, we will use the following definition:

**Definition 3.100.** Let  $R$  be a Noetherian ring. A prime  $\mathfrak{p}$  of  $R$  is *absolutely minimal* if  $\dim(R) = \dim(R/\mathfrak{p})$ .

An absolutely minimal prime is minimal, since  $\dim(R) \geq \dim(R/\mathfrak{p}) + \text{height}(\mathfrak{p})$ .

**Theorem 3.101.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and  $x_1, \dots, x_t \in \mathfrak{m}$ .*

- 1)  $\dim(R/(x_1, \dots, x_t)) \geq \dim(R) - t$ .
- 2)  $x_1, \dots, x_t$  are parameters if and only if  $\dim(R/(x_1, \dots, x_t)) = \dim(R) - t$ .
- 3)  $x_1, \dots, x_t$  are parameters if and only if  $x_1$  is not in any absolutely minimal prime of  $R$  and  $x_i$  is not contained in any absolutely minimal prime of  $R/(x_1, \dots, x_{i-1})$  for each  $i = 2, \dots, t$ .

*Proof.*

- 1) If  $\dim(R/(x_1, \dots, x_t)) = s$ , then take a system of parameters  $y_1, \dots, y_s$  for  $R/(x_1, \dots, x_t)$ , and pull back to  $R$  to get  $x_1, \dots, x_t, y'_1, \dots, y'_s$  in  $R$  such that the quotient of  $R$  modulo the ideal generated by these elements has dimension zero. By Krull's Height Theorem, we get that  $t + s \geq \dim(R)$ .
- 2) Let  $d = \dim(R)$ . Suppose first that  $\dim(R/(x_1, \dots, x_t)) = d - t$ . Then, there is a SOP  $y_1, \dots, y_{d-t}$  for  $R/(x_1, \dots, x_t)$ ; lift back to  $R$  to get a sequence of  $d$  elements  $x_1, \dots, x_t, y_1, \dots, y_{d-t}$  that generate an  $\mathfrak{m}$ -primary ideal. This is a SOP, so  $x_1, \dots, x_t$  are parameters.

On the other hand, if  $x_1, \dots, x_t$  are parameters, extend to a SOP  $x_1, \dots, x_d$ . If  $I$  is the image of  $(x_{t+1}, \dots, x_d)$  in  $R' = R/(x_1, \dots, x_t)$ , we have  $R'/I$  is zero-dimensional, hence

has finite length, so  $\text{Ass}_{R'}(R'/I) = \{\mathfrak{m}\}$ , and  $I$  is  $\mathfrak{m}$ -primary in  $R'$ . Thus,  $\dim(R')$  is equal to the height of  $I$ , which is then  $\leq d - t$  by Krull height. That is,  $\dim(R') \leq d - t$ , and using the first statement, we have equality.

- 3) This follows from the previous statement and the observation that  $\dim(S/(f)) \leq \dim(S)$  if and only if  $f$  is not in any absolutely minimal prime of  $S$ .  $\square$

**Lemma 3.102.** *Given a Cohen-Macaulay ring  $R$  and a regular sequence  $x_1, \dots, x_n$ , every associated prime of  $(x_1, \dots, x_n)$  must have height  $n$ .*

*Proof.* We can reduce to the local case, since associated primes localize,  $\frac{x_1}{1}, \dots, \frac{x_n}{1}$  must also be regular on  $R_P$  for each  $P \in \text{Ass}(x_1, \dots, x_n)$ , and  $\text{ht}(P) = \text{ht}(P_P)$ .

Now note that  $\dim(R/(x_1, \dots, x_n)) = \dim(R) - n$ , by Remark 3.99, so  $x_1, \dots, x_n$  form part of a system of parameters by Theorem 3.101. Let  $P$  be an associated prime of  $(x_1, \dots, x_n)$ . Then  $P_P$  is an associated prime of  $(x_1, \dots, x_n)_P$ , by Theorem 1.45, and  $R_P$  is Cohen-Macaulay by Theorem 3.97. The images of  $x_1, \dots, x_n$  can be extended to a system of parameters, which by Theorem 3.93 must be a maximal regular sequence. However, every element in  $P_P$  is a zerodivisor on  $R_P/(x_1, \dots, x_n)_P$ , since  $P_P$  is associated to  $x_1, \dots, x_n$ , so  $x_1, \dots, x_n$  cannot be extended inside  $P_P$ . Thus  $\text{depth}(R_P/(x_1, \dots, x_n)_P) = 0$ . By Remark 3.99,  $R_P/(x_1, \dots, x_n)_P$  is Cohen-Macaulay, so it must have dimension 0. Therefore,  $P$  is minimal over  $(x_1, \dots, x_n)$ .

By Theorem 3.11,  $\text{ht } P \leq n$ . By Theorem 3.48,  $\text{ht}(P) \leq n$ , so  $\text{ht}(P) = n$ .  $\square$

**Theorem 3.103.** *Let  $R$  be a Noetherian ring. The following are equivalent:*

- 1)  $R$  is Cohen-Macaulay.
- 2) Every ideal  $I$  in  $R$  contains a regular sequence of length  $\text{ht}(I)$ .
- 3) Every maximal regular sequence inside  $I$  has length  $\text{ht}(I)$ .

*Proof.* Notice that 3)  $\Rightarrow$  2) is obvious, and 2)  $\Rightarrow$  1) is clear once we take  $I$  to be equal to each maximal ideal in  $R$ . To show 1)  $\Rightarrow$  2), let  $x_1, \dots, x_n$  be a maximal regular sequence inside  $I$ . The elements of  $I$  must all be zerodivisors on  $R/(x_1, \dots, x_n)$ , by maximality, so by Lemma 3.78 we must have  $I$  contained in some associated prime  $P$  of  $R/(x_1, \dots, x_n)$ . By Lemma 3.102,  $P$  has height  $n$ , so  $\text{ht}(I) \leq \text{ht}(P) = n$ . But  $I \supseteq (x_1, \dots, x_n)$  and  $\text{ht}(x_1, \dots, x_n) = n$  by Theorem 3.48, so  $\text{ht}(I) = n$ .  $\square$

**Remark 3.104.** Let  $I$  be an ideal in a Cohen-Macaulay ring  $R$ . If  $I$  is generated by a regular sequence, then that regular sequence must have length  $\text{ht}(I)$ , so  $I$  must be generated by exactly  $\text{ht}(I)$  elements. If  $R$  is a local ring and  $\mu(I) > \text{ht}(I)$ , then  $I$  is not generated by a regular sequence.

The most important classes of rings we will consider are

$$\text{Regular rings} \subseteq \text{complete intersections} \subseteq \text{Cohen-Macaulay rings}.$$

We will unfortunately not have a chance to discuss Gorenstein rings, which are a special subclass of Cohen-Macaulay rings that contain all complete intersections.

### 3.8 A few direct applications to symbolic powers

We now collect some immediate applications of the tools we developed in the remainder of the chapter to associated primes and symbolic powers.

**Theorem 3.105.** *Let  $I$  be an ideal in a noetherian ring  $R$ . A prime  $P$  is associated to  $I$  if and only if  $\text{depth}(R_P/I_P) = 0$ . In particular, if  $(R, \mathfrak{m})$  is a local ring, then  $\mathfrak{m} \in \text{Ass}(I)$  if and only if  $\text{depth}(R/I) = 0$ .*

*Proof.* By Lemma 1.36,  $P \in \text{Ass}(I)$  if and only if  $P_P \in \text{Ass}_{R_P}(I_P)$ . This means we can reduce to the local case, and that it is enough to show that in a local ring  $(R, \mathfrak{m})$ ,  $\mathfrak{m} \in \text{Ass}(I)$  if and only if  $\text{depth}(R/I) = 0$ . If  $\mathfrak{m}$  is associated to  $I$ , then every element in  $\mathfrak{m}$  is a zerodivisor on  $R/I$ , so there are no regular elements on  $R/I$ , and  $\text{depth}(R/I) = 0$ . If  $\text{depth}(R/I) = 0$ , then every element in  $\mathfrak{m}$  must be a zerodivisor on  $R/I$ , so  $\mathfrak{m}$  is contained in the zerodivisors of  $R/I$ . By Theorem 1.39, this means that

$$\mathfrak{m} \subseteq \bigcup_{P \in \text{Ass}(I)} P.$$

By Theorem 1.44, this is a union of finitely many primes, so by [Prime Avoidance](#)  $\mathfrak{m}$  must be contained in some associated prime of  $I$ . But  $\mathfrak{m}$  is already maximal, so  $\mathfrak{m}$  is an associated prime of  $I$ .  $\square$

**Lemma 3.106.** *Let  $(R, \mathfrak{m})$  be a local ring or an  $\mathbb{N}$ -graded algebra over a field  $k$  with  $R_0 \cong k$ . Given an ideal  $I$  in  $R$ , which is homogeneous in the graded setting, if all the associated primes of  $I$  have height  $\dim(R) - 1$  then*

$$I^{(n)} = I^n \text{ if and only if } \text{depth}(R/I^n) > 0$$

for all  $n \geq 1$ .

For example, this applies if  $I$  is a prime of height  $\dim(R) - 1$ .

*Proof.* In the graded case, recall that all the associated primes of  $I$  must be homogeneous, by Theorem 1.44. Since all the associated primes of  $I$  have height  $\dim(R) - 1$ ,  $\mathfrak{m}$  is the only possible embedded prime of  $I^n$  for any  $n$ . By Theorem 3.105,  $\mathfrak{m}$  is associated to  $I$  if and only if  $\text{depth}(R/I^n) = 0$ .  $\square$

And finally, here is a large class of ideals  $I$  satisfying  $I^{(n)} = I^n$  for all  $n$ .

**Theorem 3.107.** *Let  $R$  be a noetherian ring and consider an ideal  $I$ . If  $I$  is generated by a regular sequence, then  $\text{Ass}(I^n) = \text{Ass}(I)$  for all  $n \geq 1$ , and thus  $I^{(n)} = I^n$ .*

*Proof.* We will construct a descending chain of ideals

$$I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

containing all the powers of  $I$  and such that

$$I_{n+1}/I_n \cong R/I$$

for all  $n$ . Given this sequence, the quotient maps

$$R/I_{n+1} \twoheadrightarrow R/I_n$$

all have kernel  $R/I$ , giving us short exact sequences

$$0 \longrightarrow R/I \longrightarrow R/I_{n+1} \longrightarrow R/I_n \longrightarrow 0.$$

By Lemma 1.40,

$$\text{Ass}(I) \subseteq \text{Ass}(I_{n+1}) \subseteq \text{Ass}(I) \cup \text{Ass}(I_n).$$

When  $n = 0$ ,  $I_n = I$ , so

$$\text{Ass}(I) \subseteq \text{Ass}(I_1) \subseteq \text{Ass}(I) \cup \text{Ass}(I) \implies \text{Ass}(I_1) = \text{Ass}(I).$$

Proceeding inductively, this shows that  $\text{Ass}(I_n) = \text{Ass}(I)$  for all  $n$ . In particular, all the powers  $I^n$  satisfy  $\text{Ass}(I^n) = \text{Ass}(I)$ .

So all that remains is to show the key technical point of the proof: the construction of the promised sequence. Let  $I = (f_1, \dots, f_t)$ , where  $f_1, \dots, f_t$  is a regular sequence. First, for each  $n$  we fix an order for the generators  $f_1^{a_1} \cdots f_t^{a_t}$  of  $I^n$ ; for example, consider the lexicographical order

$$f_{n,0} = f_1^n, f_{n,1} = f_1^{n-1}f_2, \dots, f_{n,t-1} = f_1^{n-1}f_t, f_{n,t} = f_1^{n-2}f_2^2, \dots, f_{n,\binom{n+t-1}{t-1}} = f_t^n.$$

For each  $n \geq 0$  and  $0 \leq m \leq \binom{n+t-1}{t-1}$ , set

$$I_{n,m} := I^{n+1} + (f_{n,k} \mid 0 \leq k \leq m).$$

By construction,  $I_{0,0} = I$ ,  $I_{n,m} \subseteq I_{n,m+1}$ , and

$$I_{n,\binom{n+t-1}{t-1}} = I^{n+1} + I^n = I^n = I_{n-1,0}.$$

So these give us a descending chain of ideals

$$I = I_{0,0} = I_{1,t} \supseteq I_{1,t-1} \supseteq \cdots \supseteq I_{1,0} = I^2 = I_{2,\binom{n+1}{t-1}} \supseteq \cdots.$$

If we can show that all the successive quotients are isomorphic to  $R/I$ , this chain of ideals will have all the promised features. To do that, it's sufficient to show that for all  $n \geq 1$  and  $0 \leq m < \binom{n+t-1}{t-1}$ ,

$$I_{n,m+1}/I_{n,m} \cong R/I.$$

By construction, the quotient  $I_{n,m+1}/I_{n,m}$  is a cyclic module, so

$$I_{n,m+1}/I_{n,m} \cong R/(I_{n,m} : I_{n,m+1}).$$

So all that remains to be shown is that  $(I_{n,m} : I_{n,m+1}) = I$ . One containment is immediate from the construction: since  $I_{n,m+1} \subseteq I^{n+1} + I^n$ , then

$$I \cdot I_{n,m+1} \subseteq I(I^{n+1} + I^n) \subseteq I^{n+2} + I^{n+1} \subseteq I^{n+1} \subseteq I_{n,m}.$$

Now suppose that  $g \in R$  satisfies  $gI_{n,m+1} \subseteq I_{n,m}$ . This is equivalent to the statement that

$$gf_{n,m+1} \subseteq I_{n,m} = I^{n+1} + (f_{n,k} \mid 0 \leq k \leq m).$$

Therefore, there exist  $r_0, \dots, r_m \in R$  such that

$$gf_{n,m+1} + r_0f_{n,0} + \dots + r_mf_{n,m} \in I^{n+1}.$$

The elements  $f_{n,0}, \dots, f_{n,m+1}$  are distinct monomials in  $f_1, \dots, f_t$  of degree  $n$ , so

$$gf_{n,m+1} + r_0f_{n,0} + \dots + r_mf_{n,m} = F(f_1, \dots, f_t)$$

for some homogeneous polynomial of degree  $n$ . By Theorem 3.50, the coefficients of  $F$  must be in  $I$ , so in particular  $g \in I$ . Since we took  $g$  to be any element in  $(I_{n,m} : I_{n,m+1})$ , we conclude that  $(I_{n,m} : I_{n,m+1}) = I$ .  $\square$

**Remark 3.108.** What we showed in Theorem 3.107 is that  $\text{Ass}(I^n) = \text{Ass}(I)$  for all  $n \geq 1$  whenever  $I$  is generated by a regular sequence. With our definition of symbolic powers, this gives the equality  $I^n = I^{(n)}$ . Notice, however, that if we choose the other definition of symbolic powers — meaning, by taking  $I^n R_P \cap R$  with  $P$  ranging over  $\text{Min}(I)$  instead of  $\text{Ass}(I)$  — we only obtain  $I^n = I^{(n)}$  in Cohen-Macaulay rings, since all complete intersections are unmixed in a Cohen-Macaulay ring. If  $R$  is not Cohen-Macaulay, we can still say that  $I^n = I^{(n)}$  whenever  $I$  is a complete intersection with no embedded primes.

# Chapter 4

## A geometric perspective

The symbolic powers of a radical ideal in  $k[x_1, \dots, x_d]$  have a geometric meaning. To explain that meaning, we will start with a brief recap of affine and projective varieties. For a nice and thorough computationally minded introduction to affine and projective varieties, see [CLO92].

### 4.1 Affine varieties

**Definition 4.1.** Given a field  $k$ , the **affine  $d$ -space over  $k$** , denoted  $\mathbb{A}_k^d$ , is given by

$$\mathbb{A}_k^d := \{(a_1, \dots, a_d) \mid a_i \in k\}.$$

For a subset  $T$  of  $k[x_1, \dots, x_d]$ , we define  $\mathbb{V}(T) \subseteq \mathbb{A}_k^d$  to be the set of common zeros or the *zero set* of the polynomials (equations) in  $T$ :

$$\mathbb{V}(T) = \{(a_1, \dots, a_d) \in \mathbb{A}_k^d \mid f(a_1, \dots, a_d) = 0 \text{ for all } f \in T\}.$$

Whenever we want to emphasize the role of  $k$ , we will write this as  $\mathbb{V}_k(T)$ .

A subset of  $\mathbb{A}_k^d$  of the form  $\mathbb{V}(T)$  for some subset  $T$  is called an **algebraic subset** of  $\mathbb{A}_k^d$ , or an **affine algebraic variety**. So a variety in  $\mathbb{A}_k^d$  is the set of common solutions of some (possibly infinite) collection of polynomial equations. A variety is **irreducible** if it cannot be written as the union of two proper subvarieties.

Note that some authors use the word *variety* to refer only to irreducible algebraic sets; it is always wise to check what definition of variety is being considered. Note also that the definitions given here are only completely standard when  $k$  is algebraically closed.

**Example 4.2.** Here are some simple examples of algebraic varieties:

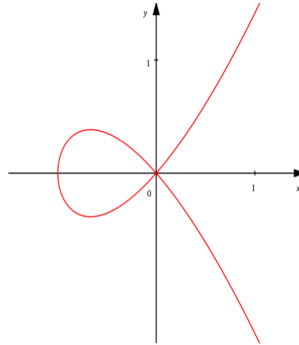
- a) For any field  $k$  and elements  $a_1, \dots, a_d \in k$ , we have

$$\mathbb{V}(x_1 - a_1, \dots, x_d - a_d) = \{(a_1, \dots, a_d)\}.$$

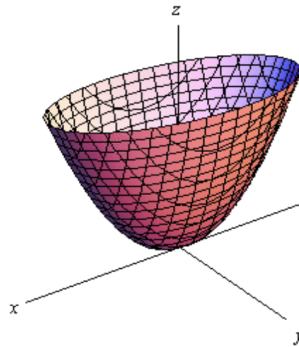
So, all one element subsets of  $\mathbb{A}_k^d$  are varieties.



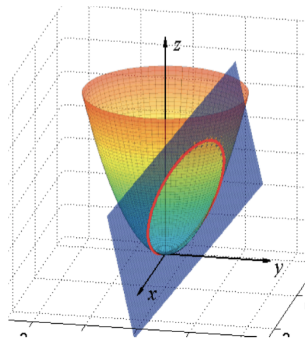
- b) For  $k = \mathbb{R}$  and  $n = 2$ ,  $\mathbb{V}(y^2 + x^2(x - 1))$  is a “nodal curve” in  $\mathbb{A}_{\mathbb{R}}^2$ , the real plane. Note that we have written  $x$  for  $x_1$  and  $y$  for  $x_2$  here.



- c) For  $k = \mathbb{R}$  and  $n = 3$ ,  $\mathbb{V}(z - x^2 - y^2)$  is a paraboloid in  $\mathbb{A}_{\mathbb{R}}^3$ , real three space.



- d) For  $k = \mathbb{R}$  and  $n = 3$ ,  $\mathbb{V}(z - x^2 - y^2, 3x - 2y + 7z - 7)$  is a circle in  $\mathbb{A}_{\mathbb{R}}^3$ .



- e) For  $k = \mathbb{R}$ ,  $\mathbb{V}_{\mathbb{R}}(x^2 + y^2 + 1) = \emptyset$ . Note that  $\mathbb{V}_{\mathbb{C}}(x^2 + y^2 + 1) \neq \emptyset$ .

We can consider the equations that a subset of affine space satisfies.

**Definition 4.3.** Given any subset  $X$  of  $\mathbb{A}_k^d$  for a field  $k$ , define

$$\mathbf{I}(X) = \{g(x_1, \dots, x_d) \in k[x_1, \dots, x_d] \mid g(a_1, \dots, a_d) = 0 \text{ for all } (a_1, \dots, a_d) \in X\}.$$

**Exercise 8.**  $\mathbf{I}(X)$  is an ideal in  $k[x_1, \dots, x_d]$  for any  $X \subseteq \mathbb{A}_k^d$ .

**Remark 4.4.** If  $I$  is an ideal in  $k[x_1, \dots, x_d]$ ,  $\mathbf{I}(\mathbb{V}(I)) \supseteq I$ , but we do not necessarily have equality. For example, when  $I = (x^2)$  in  $k[x]$ ,  $\mathbb{V}(I) = \{0\}$ , and thus  $\mathbf{I}(\mathbb{V}(I)) = (x)$ .

**Example 4.5.** The **twisted cubic (affine) curve** is the curve  $C$  parametrized by  $(t, t^2, t^3)$ , meaning it is the image of the map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^3. \\ t &\longmapsto (t, t^2, t^3) \end{aligned}$$

Consider the ideal  $I = (x^2 - y, x^3 - z)$ . It is clear that  $C \subseteq \mathbb{V}(I)$ . On the other hand, given a point  $(a, b, c) \in \mathbb{A}_{\mathbb{R}}^3$  in  $\mathbb{V}(I)$ , it must satisfy  $b = a^2$  and  $c = a^3$ , so  $(a, b, c) = (a, a^2, a^3) \in C$ . Therefore,  $C$  is a variety. To find  $\mathbf{I}(C)$ , we can get help from Macaulay2:

```
i1 : k = RR;

i2 : R = k[x,y,z];

i3 : f = map(k[t], R, {t, t^2, t^3});

i4 : ker f
      2          2
o4 = ideal (y  - x*z, x*y - z, x  - y)

o4 : Ideal of R
```

In our computation above,  $f$  sets  $x = t$ ,  $y = t^2$ , and  $z = t^3$ , and its kernel consists precisely of the polynomials that vanish at every point of this form. We conclude that  $\mathbf{I}(C) = (y - x^2, xz - y^2, z - xy)$ . Note that computations over the reals in Macaulay2 are experimental, and yet we obtain the correct answer; we can also run the same computation over  $k = \mathbb{Q}$ .

**Exercise 9.** Here are some properties of the functions  $\mathbb{V}$  and  $\mathbf{I}$ :

- a) For any field, we have  $\mathbb{V}(0) = \mathbb{A}_k^n$  and  $\mathbb{V}(1) = \emptyset$ .
- b)  $\mathbf{I}(\emptyset) = (1) = k[x_1, \dots, x_d]$  (the improper ideal).
- c)  $\mathbf{I}(\mathbb{A}_k^d) = (0)$  if and only if  $k$  is infinite.
- d) If  $I \subseteq J \subseteq k[x_1, \dots, x_d]$  then  $\mathbb{V}(I) \supseteq \mathbb{V}(J)$ .
- e) If  $S \subseteq T$  are subsets of  $\mathbb{A}_k^n$  then  $\mathbf{I}(S) \supseteq \mathbf{I}(T)$ .

- f) If  $I = (T)$  is the ideal generated by the elements of  $T \subseteq k[x_1, \dots, x_d]$ , then  $\mathbb{V}(T) = \mathbb{V}(I)$ .

So we will talk about the solution set of an ideal, rather than of an arbitrary set. Hilbert's Basis Theorem implies that every ideal in  $k[x_1, \dots, x_d]$  is finitely generated, so any system of equations in  $k[x_1, \dots, x_d]$  can be replaced with a system of *finitely many* equations.

**Example 4.6.** Let

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

be a  $2 \times 3$  matrix of variables — we usually call these *generic* matrices — and let

$$R = k[X] = k \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Let  $\Delta_1, \Delta_2, \Delta_3$  the  $2 \times 2$ -minors of  $X$ . Consider the ideal  $I = (\Delta_1, \Delta_2, \Delta_3)$ . Thinking of these generators as equations, a solution to the system corresponds to a choice of  $2 \times 3$  matrix whose  $2 \times 2$  minors all vanish — that is, a matrix of rank at most one. So  $\mathbb{V}(I)$  is the set of rank at most one matrices. Note that  $I \subseteq (x_1, x_2, x_3) =: J$ , and  $\mathbb{V}(J)$  is the set of  $2 \times 3$  matrices with top row zero. The containment  $\mathbb{V}(J) \subseteq \mathbb{V}(I)$  we obtain from  $I \subseteq J$  translates to the fact that a  $2 \times 3$  matrix with a zero row has rank at most 1.

Finally, the union and intersection of varieties is also a variety.

**Exercise 10.** Suppose that  $I$  and  $J$  are ideals in  $k[x_1, \dots, x_d]$ .

- a)  $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I + J)$ .
- b)  $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J) = \mathbb{V}(IJ)$ .

However, note that in general  $IJ \neq I \cap J$ .

**Remark 4.7.** We noted in Example 4.2 that a point is a variety. By Exercise 10, the finite union of varieties is a variety, so we conclude that any finite set of points is a variety.

**Theorem 4.8** (Weak Nullstellensatz). *Let  $k$  be an algebraically closed field. If  $I$  is a proper ideal in  $R = k[x_1, \dots, x_d]$ , then  $\mathbb{V}_k(I) \neq \emptyset$ .*

**Theorem 4.9** (Strong Nullstellensatz). *Let  $k$  be an algebraically closed field, and consider a polynomial ring  $R = k[x_1, \dots, x_d]$ . Let  $I \subseteq R$  be an ideal. The polynomial  $f$  vanishes on  $\mathbb{V}_k(I)$  if and only if  $f^n \in I$  for some  $n$ . Therefore,  $\mathbf{I}(\mathbb{V}(I)) = \sqrt{I}$ .*

This gives us a bijection between radical ideals and varieties. Given a variety  $V$ ,  $\mathbf{I}(V)$  is the unique radical ideal that determines  $V$ .

**Definition 4.10.** Given a variety  $X \subseteq \mathbb{A}_k^d$ , the **coordinate ring** of  $X$  is the ring

$$k[X] := k[x_1, \dots, x_d] / \mathbf{I}(X).$$

**Theorem 4.11.** *Given an algebraically closed field  $k$ ,  $\mathbf{I}$  and  $\mathbb{V}$  induce order-reversing bijections*

$$\begin{array}{ccc} \text{in } K[x_1, \dots, x_n] & & \text{in } \mathbb{A}_K^n \\ \{ \text{radical ideals} \} & \xleftrightarrow[\mathbf{I}]{\mathbb{V}} & \{ \text{varieties} \} \\ \{ \text{prime ideals} \} & \xleftrightarrow[\mathbf{I}]{\mathbb{V}} & \{ \text{irreducible varieties} \} \\ \{ \text{maximal ideals} \} & \xleftrightarrow[\mathbf{I}]{\mathbb{V}} & \{ \text{points} \}. \end{array}$$

*In particular, given ideals  $I$  and  $J$ , we have  $\mathbb{V}(I) = \mathbb{V}(J)$  if and only if  $\sqrt{I} = \sqrt{J}$ . Likewise, for any variety  $X$  over an algebraically closed field, we have order-reversing bijections*

$$\begin{array}{ccc} \text{in } k[X] & & \text{in } X \\ \{ \text{radical ideals} \} & \longleftrightarrow & \{ \text{subvarieties} \} \\ \{ \text{prime ideals} \} & \longleftrightarrow & \{ \text{irreducible subvarieties} \} \\ \{ \text{maximal ideals} \} & \longleftrightarrow & \{ \text{points} \}. \end{array}$$

**Example 4.12.** In Example 2.4, we showed that the kernel of the map

$$\begin{aligned} \mathbb{C}[x, y, z] &\longrightarrow \mathbb{C}[t^3, t^4, t^5] \\ (x, y, z) &\longmapsto (t^3, t^4, t^5) \end{aligned}$$

is the ideal

$$P = (x^3 - yz, y^2 - xz, z^2 - x^2y).$$

As a consequence, the set of points

$$X = \{(t^3, t^4, t^5) \in \mathbb{A}_{\mathbb{C}}^3 \mid t \in \mathbb{C}\}$$

satisfies  $\mathbf{I}(X) = P$ . In fact, we claim that for  $k = \mathbb{R}$  or  $\mathbb{C}$ , the set

$$X = \{(t^3, t^4, t^5) \mid t \in k\}$$

is an algebraic variety, though again it needs justification. Consider  $Y = \mathbb{V}(y^3 - x^4, z^3 - x^5)$ ; clearly,  $X \subseteq Y$ . Over  $\mathbb{R}$ , for  $(a, b, c) \in Y$ , take  $t = \sqrt[3]{a}$ ; then  $a = t^3$ ,  $b^3 = a^4$  means  $b = \sqrt[3]{a^4}$ , so  $b = t^4$ , and similarly  $c = t^5$ , so  $X = Y$ . We were using uniqueness of cube roots in this argument though, so we need to reconsider over  $\mathbb{C}$ . Indeed, if  $\omega$  is a cube root of unity, then  $(\omega, 1, 1) \in Y \setminus X$ , so we need to do better. Let's try  $Z = \mathbb{V}(y^3 - x^4, z^3 - x^5, z^4 - y^5)$ . Again,  $X \subseteq Z$ . Say that  $(a, b, c) \in \mathbb{A}_{\mathbb{C}}^3$  are in  $Z$ , and let  $s$  be a cube root of  $a$ . Then  $b^3 = a^4 = (s^4)^3$  implies that  $b = \omega s^4$  for some cube root of unity  $\omega$  (maybe 1, maybe not). Similarly  $c^3 = a^4 = (s^5)^3$  implies that  $c = \omega'' s^5$  for some cube root of unity  $\omega''$  (maybe 1, maybe  $\omega'$ , maybe not). So at least  $(a, b, c) = (s^3, \omega' s^4, \omega'' s^5)$ . Let  $t = \omega' s$ . Then  $(s^3, \omega' s^4, \omega'' s^5) = (t^3, t^4, \omega s^5)$ , where  $\omega = (\omega')^2 \omega''$  is again some cube root of unity. The

equation  $b^5 = c^4$  shows that  $t^{20} = \omega^4 t^{20}$ . If  $t \neq 0$ , this shows  $\omega = 1$ , so  $(a, b, c) = (t^3, t^4, t^5)$ ; if  $t = 0$ , then  $(a, b, c) = (0, 0, 0) = (0^3, 0^4, 0^5)$ . Thus,  $X = Z$ .

We have shown in particular that

$$X = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\}$$

is a variety, and that  $\mathbf{I}(X) = P$ . As we already saw in Example 2.4,  $P$  is a prime ideal, since  $\mathbb{C}[x, y, z]/P \cong \mathbb{C}[t^3, t^4, t^5] \subseteq \mathbb{C}[t]$  is a domain. Since  $P$  is prime, by ?? the variety  $X$  is irreducible.

Every radical ideal can be written as the intersection of its finitely many minimal primes, so in particular we can write  $I$  uniquely as

$$I = P_1 \cap \cdots \cap P_n$$

where each  $P_i$  is a prime ideal and  $P_i \not\subseteq P_j$  for each  $i \neq j$ . In fact, as we discussed in Remark 1.19, this is the unique way to write  $I$  as an intersection of finitely many incomparable primes. Translating this into the world of varieties, we obtain the following:

**Theorem 4.13.** *Every affine variety  $V \subseteq \mathbb{A}_k^d$  can be written uniquely as a finite union of irreducible affine varieties  $V = V_1 \cup \cdots \cup V_n$  with  $V_i \not\subseteq V_j$  for each  $i \neq j, s$ .*

In summary, over an algebraically closed field, we have the following dictionary between algebra and geometry:

| <u>Algebra</u>                                   | $\longleftrightarrow$ | <u>Geometry</u>                                     |
|--|-----------------------|---|
| radical ideals                                   | $\longleftrightarrow$ | varieties   |
| prime ideals                                     | $\longleftrightarrow$ | irreducible varieties                               |
| maximal ideals                                   | $\longleftrightarrow$ | points  |
| $(x_1 - a_1, \dots, x_d - a_d)$                  | $\longleftrightarrow$ | point $\{(a_1, \dots, a_d)\}$                       |
| $(0)$  | $\longleftrightarrow$ | variety $\mathbb{A}^d$                              |
| $k[x_1, \dots, x_d]$                             | $\longleftrightarrow$ | variety $\emptyset$                                 |
| smaller ideals                                   | $\longleftrightarrow$ | larger varieties                                    |
| larger ideals                                    | $\longleftrightarrow$ | smaller varieties                                   |
| sum of ideals                                    | $\longleftrightarrow$ | intersection of varieties                           |
| intersection of ideals                           | $\longleftrightarrow$ | union of varieties                                  |
| $I = P_1 \cap \cdots \cap P_k$                   | $\longleftrightarrow$ | $V = V_1 \cup \cdots \cup V_k$                      |
| unique decomposition into<br>incomparable primes | $\longleftrightarrow$ | unique decomposition into<br>irreducible components |

## 4.2 Projective varieties

**Definition 4.14.** Given a field  $k$  and  $d \geq 0$ , consider the equivalence relation  $\sim$  on  $k^{d+1}$  given by

$$(a_0, \dots, a_d) \sim (b_0, \dots, b_d) \text{ if there exists some } 0 \neq \lambda \in k \text{ such that } b_i = \lambda a_i \text{ for every } i.$$

The **projective  $d$ -space over  $k$** , denoted  $\mathbb{P}_k^d$ , is given by

$$\mathbb{P}_k^d := k^{d+1} / \sim.$$

The class of  $(a_0, \dots, a_d)$  in  $\mathbb{P}_k^d$  is denoted  $(a_0 : \dots : a_d)$ , and we call it a **point** in  $\mathbb{P}^d$ . Notice each point in  $\mathbb{P}_k^d$  can be represented by many different tuples in  $k^{d+1}$ ; given a point  $P$  in  $\mathbb{P}_k^d$ , any  $(a_0, \dots, a_d) \in k^{d+1}$  (which we can also think of as a point in  $\mathbb{A}_k^{d+1}$ ) such that  $P = (a_0 : \dots : a_d)$  is a set of **homogeneous coordinates** for  $P$ .

**Remark 4.15.** There is a one to one correspondence between  $\mathbb{P}_k^d$  and the set of lines through the origin in  $k^{n+1}$ .

**Remark 4.16.** Here is a typical trick one uses often. Given a point  $P = (a_0 : \dots : a_d)$ , we know that at least one of the coordinates  $a_i$  must be nonzero, so

$$P = (a_0 : \dots : a_d) = \left( \frac{a_0}{a_i} : \dots : \frac{a_i}{a_i} : \dots : \frac{a_d}{a_i} \right) = \left( \frac{a_0}{a_i} : \dots : 1 : \dots : \frac{a_d}{a_i} \right).$$

Therefore, we can always assume that  $P = (b_0 : \dots : b_{i-1} : 1 : b_{i+1} : \dots : \frac{a_d}{a_i})$ .

**Lemma 4.17.** Let  $k$  be a field and  $f \in k[x_0, \dots, x_d]$  be a homogeneous polynomial. Given any nonzero  $(a_0, \dots, a_d) \in k^{d+1}$ , if  $f(a_0, \dots, a_d) = 0$  then  $f(\lambda a_0, \dots, \lambda a_d) = 0$  for all nonzero  $\lambda \in k$ . In particular, the set

$$V(f) = \{p \in \mathbb{P}_k^d \mid f(p) = 0\}$$

is a well-defined subset of  $\mathbb{P}_k^d$ .

*Proof.* If  $f$  is homogeneous of degree  $d = n$ , we can write  $f$  as

$$f = \sum_{b_0 + \dots + b_d = n} c_b x_0^{b_0} \dots x_d^{b_d}$$

for some  $c_b \in k$ . Then

$$f(\lambda a_0, \dots, \lambda a_d) = \sum_{b_0 + \dots + b_d = n} c_b (\lambda a_0)^{b_0} \dots (\lambda a_d)^{b_d} = \sum_{b_0 + \dots + b_d = n} c_b \lambda^n a_0^{b_0} \dots a_d^{b_d} = \lambda^n f(a_0, \dots, a_d)$$

so  $f(a_0, \dots, a_d) = 0$  if and only if  $f(\lambda a_0, \dots, \lambda a_d) = 0$  for all  $\lambda$ .  $\square$

In contrast, if  $f$  is not homogeneous, then the equation  $f(p) = 0$  does not make sense in  $\mathbb{P}_k^d$ .

**Definition 4.18.** Let  $f_1, \dots, f_n$  be homogeneous polynomials in  $R = k[x_0, \dots, x_d]$ , where  $k$  is a field. The **projective variety** defined by  $f_1, \dots, f_n$  is

$$\mathbb{V}(f_1, \dots, f_n) := \{a \in \mathbb{P}_k^n \mid f_i(a) = 0 \text{ for all } 1 \leq i \leq n\}.$$

A **projective variety** is any subset  $V \subseteq \mathbb{P}_k^d$  which can be realized as  $V = \mathbb{V}(f_1, \dots, f_n)$  for some  $f_1, \dots, f_n$ .

**Notation 4.19.** If  $I$  is a homogeneous ideal in  $R$ , we set

$$\mathbb{V}(I) := \{p \in \mathbb{P}_k^d \mid f(p) = 0 \text{ for all } f \in I\}.$$

We will use the same notation for affine and projective varieties, hoping the distinction is clear from context.

**Remark 4.20.** Whenever  $I$  is a homogeneous ideal in  $R$ ,  $I$  is generated by homogeneous elements, say  $f_1, \dots, f_n$ . We claim that  $V(I) = V(f_1, \dots, f_n)$ . On the one hand, if  $p \in V(I)$  then in particular  $f_i(p) = 0$  for all  $i$ , so  $p \in V(f_1, \dots, f_n)$ . On the other hand, any element  $f \in I$  is of the form  $f = g_1 f_1 + \dots + g_n f_n$ , so if  $p \in V(f_1, \dots, f_n)$  then

$$f(p) = g_1(p) \underbrace{f_1(p)}_0 + \dots + g_n(p) \underbrace{f_n(p)}_0 = 0.$$

Therefore,  $V(I)$  is a projective variety.

**Definition 4.21.** Given a projective variety  $V \subseteq \mathbb{P}_k^d$ ,

$$\mathbf{I}(V) := \{f \in k[x_0, \dots, x_d] \mid f(p) = 0 \text{ for all } p \in V\}.$$

**Exercise 11.** Let  $V$  be a projective variety in  $\mathbb{P}_k^d$ . If  $k$  is infinite, then  $\mathbf{I}(V)$  is a homogeneous ideal.

**Remark 4.22.** When  $k$  is finite, there are certain polynomials that are in  $\mathbf{I}(V)$  for every variety  $V$ . Over  $\mathbb{F}_p$ , Fermat's Little Theorem implies that  $x_0^p - x_0, \dots, x_d^p - x_d$  vanish at every point in  $\mathbb{P}_{\mathbb{F}_p}^d$ , but these are not homogeneous polynomials.

**Exercise 12.**

- a) If  $I \subseteq J \subseteq k[x_0, \dots, x_d]$  are homogeneous ideals then  $\mathbb{V}(I) \supseteq \mathbb{V}(J)$ .
- b) If  $S \subseteq T$  are subsets of  $\mathbb{P}_k^d$  then  $\mathbf{I}(S) \supseteq \mathbf{I}(T)$ .

**Exercise 13.** If  $I$  is a homogeneous ideal, then  $\sqrt{I}$  is also homogeneous.

**Remark 4.23.** A homogeneous ideal  $I$  in  $R = k[x_0, \dots, x_d]$  gives us the projective variety  $V = \mathbb{V}(I) \subseteq \mathbb{P}_k^d$ , but it also determines the affine variety

$$A = \{a \in \mathbb{A}_k^{d+1} \mid f(a) = 0 \text{ for all } f \in I\} \subseteq \mathbb{A}_k^{d+1}.$$

If we think of each point in  $V$  as corresponding to a line through the origin in affine space, then  $A$  is the union of those lines; notice that  $A$  includes the origin. In particular, for every point  $a = (a_0 : \dots : a_d) \in V$ , whatever the choice of homogeneous coordinates  $a_0, \dots, a_d$ , we always have  $(a_0, \dots, a_d) \in A$ . This affine variety  $A$  is the **affine cone** of  $V$ .

Since a lot of the theory of projective varieties appears to be parallel to the theory of affine varieties, we might be expecting a projective version of Nullstellensatz. However, the details are a bit more complicated.

**Remark 4.24.** Not all homogeneous proper ideals in  $k[x_0, \dots, x_d]$  give rise to nonempty projective varieties. For example, the homogeneous maximal ideal  $(x_0, \dots, x_d)$  in  $k[x_0, \dots, x_d]$  determines an empty variety in  $\mathbb{P}_k^d$ . Why? One simple way to explain this is to note that the affine variety  $\mathbb{V}(x_0, \dots, x_d)$  is the origin.

Since  $(x_0, \dots, x_d)$  is essentially irrelevant from the perspective of projective varieties, we call it the **irrelevant (maximal) ideal** of  $k[x_0, \dots, x_d]$ .

**Theorem 4.25** (Projective Weak Nullstellensatz). *Let  $k$  be an algebraically closed field and let  $I$  be a homogeneous ideal in  $R = k[x_0, \dots, x_d]$ . Let  $\mathfrak{m}$  denote the irrelevant maximal ideal of  $R$ . The variety  $\mathbb{V}(I)$  is empty if and only if  $(I : \mathfrak{m}^\infty) = R$ .*

*Proof.* As described in Remark 4.23, we will consider two varieties: the projective variety  $V = \mathbb{V}(I) \subseteq \mathbb{P}_k^d$  and its affine cone, the affine variety

$$A = \{a \in \mathbb{A}_k^{d+1} \mid f(a) = 0 \text{ for all } f \in I\} \subseteq \mathbb{A}_k^{d+1}.$$

Since 0 does not determine a point in projective space,  $V$  is empty if and only if  $A \subseteq \{0\}$ . In affine space, we know that  $A = \{0\}$  if and only if  $\sqrt{I} = \mathfrak{m}$ , and in that case  $(I : \mathfrak{m}^\infty) = R$  by Exercise 1. Similarly,  $A$  is empty if and only if  $I = R$ , and in that case  $(I : \mathfrak{m}^\infty) = R$  is immediate. Conversely, by Exercise 1 the saturation  $(I : \mathfrak{m}^\infty)$  returns the intersection of primary components of  $I$  whose radicals do not contain  $\mathfrak{m}$ . If  $(I : \mathfrak{m}^\infty) = R$ , then all the associated primes of  $I$  contain  $\mathfrak{m}$ , so either  $I = R$  or  $\sqrt{I} = \mathfrak{m}$ . In both these cases,  $A \subseteq \{0\}$  and  $V$  is empty.  $\square$

As in the affine setting, a projective variety is irreducible if it cannot be decomposed as a finite union of proper subvarieties.

**Exercise 14.** A projective variety  $V \subseteq \mathbb{P}_k^d$  is irreducible if and only if  $\mathbf{I}(V)$  is a homogeneous prime ideal.

**Remark 4.26.** Let  $\mathfrak{m}$  be the irrelevant ideal in  $R = k[x_0, \dots, x_d]$ , where  $k$  is an algebraically closed field. If  $P \neq \mathfrak{m}$  is a homogeneous prime ideal, then by Exercise 1 we know that  $(P : \mathfrak{m}^\infty) = P$ , since  $P$  has no  $\mathfrak{m}$ -primary components, so by Theorem 4.25 the variety  $\mathbb{V}(P)$  is nonempty. Therefore, we have a bijective correspondence between homogeneous prime ideals  $P \neq \mathfrak{m}$  and nonempty irreducible projective varieties.

**Theorem 4.27** (Projective Strong Nullstellensatz). *Let  $k$  be an algebraically closed field and let  $I$  be a homogeneous ideal in  $R = k[x_0, \dots, x_d]$ . If  $V = \mathbb{V}(I)$  is a nonempty projective variety, then*

$$\mathbf{I}(\mathbb{V}(I)) = \sqrt{I}.$$

*Proof.* We will again consider projective variety  $V = \mathbb{V}(I) \subseteq \mathbb{P}_k^d$  and its affine cone

$$A = \{a \in \mathbb{A}_k^{d+1} \mid f(a) = 0 \text{ for all } f \in I\} \subseteq \mathbb{A}_k^{d+1}.$$



First, we claim that  $\mathbf{I}(A) = \mathbf{I}(V)$ . For each point  $P \in V$ , any choice of homogeneous coordinates  $(a_0 : \dots : a_d)$  for  $P$  must satisfy  $(a_0, \dots, a_d) \in A$ . Therefore, if  $f \in \mathbf{I}(A)$  then  $f(a_0, \dots, a_d) = 0$  for all homogeneous coordinates  $(a_0, \dots, a_d)$  for  $P$ , and thus  $f(P) = 0$ . This shows that  $\mathbf{I}(A) \subseteq \mathbf{I}(V)$ . Conversely, let  $f \in \mathbf{I}(V)$ . Any nonzero point in  $A$  gives homogeneous coordinates for some  $P \in V$ , and since  $f$  vanishes at  $V$ , we conclude that  $f$  vanishes at every nonzero point in  $A$ . To show that  $f(0) = 0$ , notice that  $f(0) = f_0$  is the homogeneous piece of  $f$  of degree 0. Since  $\mathbf{I}(V)$  is a homogeneous ideal and  $f \in \mathbf{I}(V)$ ,  $f_0 \in \mathbf{I}(V)$ . Now  $f_0$  is a constant and  $V$  is nonempty, so we must have  $f_0 = 0$ . Therefore,  $f(0) = f_0 = 0$ .

We have shown that  $\mathbf{I}(A) = \mathbf{I}(V)$ . By Theorem 4.9,  $\mathbf{I}(A) = \sqrt{I}$ . Therefore,

$$\mathbf{I}(\mathbb{V}(I)) = \mathbf{I}(V) = \mathbf{I}(A) = \sqrt{I}. \quad \square$$

In summary, over an algebraically closed field, we have the following dictionary:

| Algebra<br>of $k[x_0, \dots, x_d]$                                      | $\longleftrightarrow$ | Geometry<br>of $\mathbb{P}_k^d$              |
|---|-----------------------|--|
| radical homogeneous ideals<br>properly contained in $(x_0, \dots, x_d)$ | $\longleftrightarrow$ | nonempty projective<br>varieties             |
| homogeneous prime ideals<br>$P \neq (x_0, \dots, x_d)$                  | $\longleftrightarrow$ | irreducible nonempty<br>projective varieties |
| $(a_i x_j - a_j x_i \mid i, j)$   | $\longleftrightarrow$ | point $\{(a_0 : \dots : a_d)\}$              |
| $(0)$   | $\longleftrightarrow$ | variety $\mathbb{P}^d$                       |
| sum of ideals   | $\longleftrightarrow$ | intersection of varieties                    |
| intersection of ideals  | $\longleftrightarrow$ | union of varieties                           |

We close this section with a comment about finite sets of points in projective space.

**Remark 4.28.** In the affine setting, the ideal corresponding to the point  $(a_1, \dots, a_d)$  is  $(x_1 - a_1, \dots, x_d - a_d)$ . In projective space, the ideal corresponding  $P = (a_0 : \dots : a_d)$  is

$$I(P) = (a_i x_j - a_j x_i \mid 0 \leq i < j \leq d).$$

A point is of course irreducible, so  $I(P)$  is a homogeneous prime ideal by Exercise 14.

As described in Remark 4.16, we can assume that  $a_i = 1$  for some  $i$ . We claim that the set of  $d$  elements

$$\{x_j - a_j x_i \mid j \neq i\}$$

generate  $I(P)$ . And indeed, for each  $j, k \neq i$ ,

$$a_k x_j - a_j x_k = a_k (x_j - a_j x_i) - a_j (x_k - a_k x_i).$$

Notice that these  $d$  elements  $x_j - a_j x_i$  with  $j \neq i$  each involve a different variable that does not appear in the remaining ones, so they are linearly independent, and thus they form a regular sequence. This shows that the ideal  $I(P)$  is generated by a regular sequence of  $d$  elements, and thus by Theorem 3.48,  $\text{ht}(I(P)) = d$ .

The expression *ideal of points* is often used to refer to the radical ideal  $I(V)$  corresponding to a finite set of points  $V \subseteq \mathbb{P}^n$ . Given  $\{P_1, \dots, P_n\} \subseteq \mathbb{P}^d$ , the corresponding ideal of points is the ideal

$$I = \bigcap_{i=1}^n I(P_i)$$

of all polynomials that vanish at all the points  $P_1, \dots, P_n$ .

**Remark 4.29.** Let  $X = \{P_1, \dots, P_s\}$  be a finite set of points in  $\mathbb{P}_k^d$ , where  $k$  is any field. The ideal  $I = \mathbf{I}(X)$  can be written as

$$I = \mathbf{I}(X) = \bigcap_{i=1}^s \mathbf{I}(P_i).$$

By Remark 4.28, each  $\mathbf{I}(P_i)$  is a prime ideal of height  $d$ . Therefore, the  $\mathbf{I}(P_i)$  are the minimal primes of  $I$ , so

$$I^{(n)} = \bigcap_{i=1}^s \mathbf{I}(P_i)^{(n)}.$$

Also by Remark 4.28, each  $\mathbf{I}(P_i)$  is generated by a regular sequence, so by Theorem 3.107 we know that  $\mathbf{I}(P_i)^{(n)} = \mathbf{I}(P_i)^n$ . We conclude that

$$I^{(n)} = \bigcap_{i=1}^s \mathbf{I}(P_i)^n.$$

We can interpret this as saying that the symbolic powers of an ideal of points is the set of polynomials that vanish to order  $n$  at each point. Since a polynomial in  $R = k[x_0, \dots, x_d]$  determines a hypersurface in  $\mathbb{P}^d$ , we can think of the  $n$ th symbolic power of an ideal of points as corresponding to all the hypersurfaces that pass through our given points with multiplicity  $n$ .

**Example 4.30.** Consider the points  $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} \subseteq \mathbb{P}^2$ , which correspond to the homogeneous radical ideal

$$I = (x, y) \cap (y, z) \cap (x, z) = (xy, xz, yz)$$

in  $R = k[x, y, z]$ . The polynomials that vanish to order 2 at these points are

$$I^{(2)} = (x, y)^2 \cap (y, z)^2 \cap (x, z)^2,$$

which in particular contain  $xyz \notin I^2$ .

In the next section, we will show a much more general version of this: that in general, if  $k$  is an algebraically closed field and  $I$  is a radical ideal in  $k[x_1, \dots, x_d]$ , then  $I^{(n)}$  is the set of polynomials that vanish to order  $n$  along the variety determined by  $I$ .

### 4.3 Zariski–Nagata

The affine Nullstellensatz tells us that over an algebraically closed field  $k$ , a radical ideal  $I$  in  $R = k[x_1, \dots, x_d]$  is the set of polynomials that vanish at every point in  $\mathbb{V}(I)$ , and that the polynomials that vanish at the point  $(a_1, \dots, a_d)$  are those in the ideal  $(x_1 - a_1, \dots, x_d - a_d)$ . Therefore, every radical ideal  $I$  satisfies

$$I = \bigcap_{a \in \mathbb{V}(I)} (x_1 - a_1, \dots, x_d - a_d).$$

Moreover, Nullstellensatz tells us that all the maximal ideals  $\mathfrak{m}$  in  $R$  correspond to a point in  $\mathbb{A}^d$ , and so they are all of this form, and that  $I \subseteq \mathfrak{m}$  if and only if the point  $\mathbb{V}(\mathfrak{m})$  is in  $\mathbb{V}(I)$ . Therefore, every radical ideal  $I$  satisfies

$$I = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}.$$

The fact that this holds for all radical ideals  $I$  says that  $R = k[x_1, \dots, x_d]$  is a **Jacobson ring**. It turns out that all polynomial rings over a field are Jacobson rings, even if the field is not algebraically closed.

**Theorem 4.31.** *Let  $R$  be a finitely generated  $k$ -algebra over any field  $k$ . If  $I$  is a radical ideal in  $R$ , then*

$$I = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}.$$

*Proof.* The containment

$$I \subseteq \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}$$

is obvious. For the other containment, we need to show that for any  $f \notin I$  we can find a maximal ideal  $\mathfrak{m}$  containing  $I$  such that  $f \notin \mathfrak{m}$ . In order to do that, fix  $f \in I$ . In the ring  $(R/I)_f$ , we claim that  $1 \neq 0$ . Indeed,  $1 = 0$  would say that  $f^n + I = 0$  for some  $n$ , or equivalently that  $f^n \in I$ , but  $I$  is radical and  $f \notin I$ , so no such  $n$  exists. Therefore,  $(R/I)_f$  is a nonzero ring, and thus it has a maximal ideal  $J$ . Let  $\mathfrak{m}$  be an ideal in  $R$  whose image in  $(R/I)_f$  is our chosen maximal ideal; we can obtain  $\mathfrak{m}$  by considering the ideal  $J \cap (R/I)$  in  $R/I$ , which must necessarily be of the form  $\mathfrak{m}/I$  for some ideal  $\mathfrak{m}$  in  $I$ . Since the contraction of a prime is a prime, and the primes in  $R/I$  are of the form  $P/I$  with  $P$  prime in  $R$ , our chosen ideal  $\mathfrak{m}$  must be prime. By construction,  $\mathfrak{m}$  contains  $I$  and  $\mathfrak{m}/I$  becomes a maximal ideal when localizing at  $f$ , which means that  $f \notin \mathfrak{m}$ . We will show that  $\mathfrak{m}$  is a maximal ideal, which will finish the proof.

First, note that  $\mathfrak{m} \neq R$ , so  $\mathfrak{m}$  does not contain any element of  $k$ . Therefore, the quotient map  $R \rightarrow R/\mathfrak{m}$  induces an inclusion  $k \subseteq R/\mathfrak{m}$  by restriction. Let's write  $S = (R/I)_f$ , and recall that  $J = (\mathfrak{m}/I)_f$ . Now consider the localization map

$$R/\mathfrak{m} \cong \frac{R/I}{\mathfrak{m}/I} \longrightarrow \left( \frac{R/I}{\mathfrak{m}/I} \right)_f \cong \frac{(R/I)_f}{(\mathfrak{m}/I)_f} = S/J.$$

Since  $\mathfrak{m}$  is prime,  $R/\mathfrak{m}$  is a domain. Therefore, the localization at  $f$  is injective. We now have inclusions

$$k \subseteq R/\mathfrak{m} \subseteq S/J.$$

Notice that  $S/J \cong (R/\mathfrak{m})_f \cong R/\mathfrak{m}[s]/(sf-1)$  is algebra-finite over  $R/\mathfrak{m}$ . Since  $R$  is a finitely generated  $k$ -algebra, then the field extension  $k \subseteq S/J$  is algebra-finite. By Zariski's Lemma, which is the key ingredient in proving Nullstellensatz, the field extension  $k \subseteq R/\mathfrak{m} \subseteq S/J$  must be module-finite. Therefore, the extension of domains  $k \subseteq R/\mathfrak{m}$  must also be module-finite, and thus integral, by Theorem B.19. By Lemma B.20,  $R/\mathfrak{m}$  must be a field, so  $\mathfrak{m}$  is a maximal ideal.  $\square$

The Zariski–Nagata theorem is a higher order version of this result, which roughly speaking says that the symbolic powers of a radical ideal are the sets of polynomials that vanish up to order  $n$  on the corresponding variety. There are actually a few different results known as Zariski–Nagata; the first one is a theorem of Nagata's [Nag62]. To prove this theorem, we will need a few fundamental facts about Hilbert–Samuel multiplicity.

**Definition 4.32.** Let  $(R, \mathfrak{m})$  be a noetherian local ring with dimension  $d$ . Let  $\lambda(M)$  denote the length of the module  $M$ . The **Hilbert–Samuel multiplicity** of  $R$  is

$$e(R) := \lim_{n \rightarrow \infty} \frac{d! \lambda(R/\mathfrak{m}^n)}{n^d}.$$

The Hilbert–Samuel multiplicity is an important invariant which detects and measures singularities. The defining limit exists, and can also be described in terms of the Hilbert function of  $\mathfrak{m}$ : the Hilbert function is eventually a polynomial, and  $e(R)$  is essentially the coefficient of the highest order term in that polynomial (after some appropriate rescaling). We will need a few facts about  $e(R)$ , which we will not prove for now:

- If  $(R, \mathfrak{m})$  is a regular local ring and  $f \in \mathfrak{m}$ , then  $e(R) = \text{ord}(f) := \max\{t \mid f \in \mathfrak{m}^t\}$ .
- Under mild assumptions,  $e(R) \geq e(R_P)$ .

Using these two facts, we can now prove Nagata's version [Nag62] of the Zariski–Nagata theorem:

**Theorem 4.33** (Local Zariski–Nagata). *Let  $(R, \mathfrak{m})$  be a regular local ring. For every prime ideal  $P$  and every  $n \geq 1$ ,*

$$P^{(n)} \subseteq \mathfrak{m}^n.$$

*Proof.* Fix a prime ideal  $P$  and an element  $f \in \mathfrak{m}$ . First, note that  $R_P$  is also regular, and that  $f \in P^{(t)}$  if and only if  $\frac{f}{1} \in P^t R_P$ , so by the properties above,

$$\max\{t \mid f \in P^{(t)}\} = \max\{t \mid \frac{f}{1} \in P^t R_P\} = e((R/f)_P) \leq e(R/f) = \max\{t \mid f \in \mathfrak{m}^t\}.$$

So if  $f \in P^{(n)}$ , then we must have  $f \in \mathfrak{m}^n$ , showing that  $P^{(n)} \subseteq \mathfrak{m}^n$ .  $\square$

The assumption that  $R$  is regular is necessary: we cannot extend this result to any noetherian local ring.

**Example 4.34.** As in Example 1.54, when  $R = k[[x, y, z]]/(xy - z^c)$  and  $c \geq 2$ , the prime  $P = (x, z)$  satisfies  $x \in P^{(c)}$ , so in particular  $P^{(c)} \not\subseteq \mathfrak{m}^c$ . One can show that  $P^{(cn)} = (x^n)$ , so in fact  $P^{(cn)} \subseteq \mathfrak{m}^n$  for all  $n \geq 1$ .

This is a more general phenomenon.

**Theorem 4.35** (Huneke–Katz–Validashti, 2009 [HKV09]). *Let  $(R, \mathfrak{m})$  be a complete local domain. There exists a constant  $c$  such that for all primes  $P$  and all  $n \geq 1$ ,*

$$P^{(cn)} \subseteq \mathfrak{m}^n.$$

Finding effective bounds for this constant  $c$  is a difficult problem, and in general it is wide open; such bounds are known in the graded setting when  $k$  is a field [DDSG<sup>+</sup>18, Theorem 3.27] or a DVR with uniformizer  $p \in \mathbb{Z}$  [SGJ21] and  $R = k[f_1, \dots, f_n] \subseteq k[x_1, \dots, x_d]$  generated by homogeneous elements  $f_i$  and such that the inclusion of  $R$  into  $k[x_1, \dots, x_d]$  splits. In that case, one can take  $c = \max\{\deg(f_i)\}$ .

The theorem most commonly known as Zariski–Nagata is a result about polynomial rings, which we will prove via yet another version of the theorem, in terms of differential operators.

**Definition 4.36** (Differential operators). Given a finitely generated  $k$ -algebra  $R$ , the  $k$ -linear differential operators on  $R$  of order  $n$ ,  $D_R^n \subseteq \text{Hom}_k(R, R)$ , are defined as follows:

- The differential operators of order zero are the  $k$ -linear maps which are also  $R$ -linear:

$$D_{R|k}^0 := \text{Hom}_R(R, R) \cong R.$$

- We say that  $\delta \in \text{Hom}_k(R, R)$  is an operator of order up to  $n$ , meaning  $\delta \in D_R^n$ , if

$$[\delta, r] = \delta r - r\delta$$

is an operator of order up to  $n - 1$  for all  $r \in D_R^0$ .

The **ring of  $k$ -linear differential operators** is the subring of  $\text{Hom}_k(R, R)$  defined by

$$D_{R|k} := \bigcup_{n \in \mathbb{N}} D_{R|k}^n.$$

In particular, the multiplication on  $D_{R|k}$  is just composition.

If  $R$  or  $k$  are clear from the context, we may drop one of the subscripts, or both. Notice that  $D_{R|k}$  is almost always a noncommutative ring!

**Example 4.37.** Let  $k$  be a field and  $R = k[x_1, \dots, x_d]$  or  $R = k[[x_1, \dots, x_d]]$ . When  $k$  is a field of characteristic 0,

$$D_R^n = \bigoplus_{\alpha_1 + \dots + \alpha_d \leq n} R \cdot \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \quad \text{and} \quad D_{R|k} = R \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right\rangle.$$

When  $k$  has prime characteristic  $p$ , things are a little more complicated; notice that over  $R = k[x]$ ,  $\frac{\partial^p}{\partial x^p}(x^n) = 0$  for any  $n$ , but there are indeed nonzero differential operators of order  $p$ . To give a correct description of our differential operators on  $R = k[x_1, \dots, x_d]$  over any field  $k$  of any characteristic, we consider

$$D_\alpha := \frac{1}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!} \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \quad \text{where} \quad D_\alpha(x^\beta) = \begin{cases} \binom{\alpha}{\beta} x^{\alpha-\beta} & \text{if } \alpha_i \geq \beta_i \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

and now we have

$$D_R^n = \bigoplus_{\alpha_1 + \cdots + \alpha_d \leq n} D_\alpha.$$

For example, when  $R = \mathbb{F}_3[x]$ ,  $D_3(x^5) = \binom{5}{3} x^{5-3}$ ; since  $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = 10$ , and  $10 = 1$  in  $\mathbb{F}_3$ , this means that  $D_3(x^5) = x^2$ . On the other hand, notice that  $\frac{\partial^3}{\partial x^3}(x^5) = 5 \cdot 4 \cdot 3 \cdot x^2 = 0$ .

**Definition 4.38.** Let  $R$  be a finitely generated  $k$ -algebra,  $I$  an ideal of  $R$ , and  $n$  be a positive integer. The  $n$ th  $k$ -linear **differential power** of  $I$  is given by

$$I^{(n)} = \{f \in R \mid \delta(f) \in I \text{ for all } \delta \in D_R^{n-1}\}.$$

**Exercise 15.** Let  $\{I_\alpha\}_{\alpha \in A}$  be an indexed family of ideals. For every  $n \geq 0$  we have

$$\bigcap_{\alpha \in A} I_\alpha^{(n)} = \left( \bigcap_{\alpha \in A} I_\alpha \right)^{(n)}.$$

We will also use the fact that containments are local.

**Exercise 16.** (Containments are local statements) Given ideals  $I$  and  $J$  in a noetherian ring  $R$ , the following are equivalent:

- (a)  $I \subseteq J$ ;
- (b)  $I_P \subseteq J_P$  for all primes  $P \in \text{Supp}(R/J)$ ;
- (c)  $I_P \subseteq J_P$  for all primes  $P \in \text{Ass}(R/J)$ .

We note that in what follows and up to Proposition 4.42,  $k$  can be any ring.

**Remark 4.39.** Since  $D_R^{n-1} \subseteq D_R^n$ , it follows that  $I^{(n+1)} \subseteq I^{(n)}$ . Moreover, given any ideals  $I \subseteq J$ , we have  $I^{(n)} \subseteq J^{(n)}$  for every  $n \geq 1$ .

**Lemma 4.40.** Let  $R$  be a finitely generated  $k$ -algebra,  $I$  be an ideal of  $R$ , and  $n$  be a positive integer. The set  $I^{(n)}$  is an ideal.

*Proof.* Every  $f \in \text{Hom}_k(R, R)$  must satisfy  $f(0) = 0$ , and since  $D_{R|k} \subseteq \text{Hom}_k(R, R)$ , we conclude that  $D_{R|k} \cdot 0 = 0$ . In particular,  $0 \in I^{(n)}$  for all ideals  $I$  and all  $n$ .

If  $f, g \in I^{(n)}$  then  $f + g \in I^{(n)}$ , since for any  $\delta \in D_R^{n-1}$ ,

$$\delta(f + g) = \underbrace{\delta(f)}_{\in I} + \underbrace{\delta(g)}_{\in I} \in I.$$

Now we need to show that  $rf \in I^{(n)}$  for any  $r \in R$  and  $f \in I^{(n)}$ . For any  $\delta \in D^{n-1}$ , note that  $f \in I^{(n)} \subseteq I^{(n-1)}$ ,  $[\delta, r] \in D^{n-2}$ , and  $\delta(f) \in I$ , so

$$\delta(rf) = \underbrace{[\delta, r]}_{\in D^{n-2}} \underbrace{\left( \underbrace{f}_{\in I^{(n-1)}} \right)}_{\in I} + r \underbrace{\delta(f)}_{\in I} \in I.$$

We conclude that  $\delta(rf) \in I$ . Hence,  $rf \in I^{(n)}$ .  $\square$

**Proposition 4.41.** *Let  $R$  be a finitely generated  $k$ -algebra. If  $P$  is a prime ideal, then  $P^{(n)}$  is  $P$ -primary for all  $n \geq 1$ .*

**Proposition 4.42.** *Let  $R$  be a finitely generated  $k$ -algebra,  $I$  be an ideal of  $R$ , and  $n \geq 1$ . Then  $I^n \subseteq I^{(n)}$ .*

*Proof.* Induction on  $n$ . The base case is straightforward:  $I = I^{(1)}$  because  $D^0 = R$ .

Suppose that  $I^n \subseteq I^{(n)}$ . Notice that  $I^n$  is generated by the elements of the form  $fg$  where  $f \in I$ ,  $g \in I^n$ . In order to show that  $I^{n+1} \subseteq I^{(n+1)}$ , it is enough to show that  $fg \in I^{(n+1)}$  for any such  $f$  and  $g$ .

To do that, we consider any  $\delta \in D^n$ , and we will show that  $\delta(fg) \in I$ . And in fact, since by induction hypothesis  $g \in I^n \subseteq I^{(n)}$ , then

$$\delta(fg) = \underbrace{[\delta, f]}_{\in D^{n-1}} \underbrace{\left( \underbrace{g}_{\in I^{(n)}} \right)}_{\in I} + f \underbrace{\delta(g)}_{\in I} \in I.$$

Notice here we used the fact that  $\delta f = [\delta, f] + f\delta$ . We conclude that  $I^{n+1} \subseteq I^{(n+1)}$ .  $\square$

**Lemma 4.43.** *For any radical ideal  $I$  and prime ideal  $P$  in a  $k$ -algebra  $R$ ,  $(I_P)^{(n)} = (I^{(n)})_P$ .*

A lot more is true: taking differential powers commutes with localization at any multiplicative set  $W$  [BJNB19, Lemma 3.9]. The main technical point is that any  $k$ -linear differential operator of order  $n$  on  $R$  can be extended to a  $k$ -linear differential operator of order  $n$  on  $R_P$ , and that every  $k$ -linear differential operator on  $R_P$  can be obtained from a differential operator on  $R$ . We skip the proof to avoid a lengthier discussion on these technical details, but it can be found in [BJNB19, Lemma 3.9].

**Remark 4.44.** Let  $k$  be a field,  $R = k[x_1, \dots, x_d]$  or  $R = k[[x_1, \dots, x_d]]$ , and  $\mathfrak{m} = (x_1, \dots, x_d)$ . In this case,

$$D_R^n = R \left\langle \frac{1}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!} \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \mid \alpha_1 + \cdots + \alpha_d \leq n \right\rangle.$$

If  $f \notin \mathfrak{m}^n$ , then  $f$  has a monomial of the form  $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$  with nonzero coefficient  $\lambda \in k$  for some  $\alpha_1 + \cdots + \alpha_d < n$ . Fix such a monomial which is minimal among all monomials appearing in  $f$  under the graded lexicographical order. Applying the differential operator  $\frac{1}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!} \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$  maps  $\lambda x_1^{\alpha_1} \cdots x_d^{\alpha_d}$  to the nonzero element  $\lambda \in k$ , and any other monomial appearing in  $f$  either to a nonconstant monomial or to zero. Consequently,  $f \notin \mathfrak{m}^{(n)}$ . Hence,  $\mathfrak{m}^{(n)} \subseteq \mathfrak{m}^n$ . Since  $\mathfrak{m}^n \subseteq \mathfrak{m}^{(n)}$  by Proposition 4.42, we conclude that  $\mathfrak{m}^{(n)} = \mathfrak{m}^n$ .

There is a more technical version of this idea that proves that if we assume that  $k$  is perfect, and  $\mathfrak{m}$  is any maximal ideal in  $R$  a regular algebra essentially of finite type over  $k$ , then we still have  $\mathfrak{m}^n = \mathfrak{m}^{(n)}$ . This technical point, however, is difficult – this is the subtle part of the proof. For a complete proof, see [DSGJ20, Theorem 3.6].

**Definition 4.45.** A field  $k$  of prime characteristic  $p$  is **perfect** if every element of  $k$  is a  $p$ th power. More generally, a field  $k$  is **perfect** if  $k$  has characteristic 0 or  $k$  is a perfect field of characteristic  $p$ .

**Example 4.46.**

- 1) Every finite field is perfect, thanks to Fermat's Little Theorem.
- 2) If  $k$  is an algebraically closed field of characteristic  $p$ , then  $x^p - a$  has a root for every  $a \in k$ , which means that  $k$  is perfect. Therefore, every algebraically closed field is perfect.
- 3) For an example of an imperfect field, take  $\mathbb{F}_p(t)$ , where  $t$  does not have a  $p$ th root.

**Theorem 4.47.** Let  $k$  be a perfect field and let  $S = k[x_1, \dots, x_d]$ . If  $(R, \mathfrak{m})$  is a localization of  $S$  at a prime ideal, then  $\mathfrak{m}^n = \mathfrak{m}^{(n)}$  for all  $n \geq 1$ .

A detailed proof can be found in [DSGJ20, Theorem 3.6]. More surprisingly, this condition characterizes regularity, as shown by Brenner, Jeffries, and Núñez Betancourt [BJNB19, Theorem 10.2].

**Theorem 4.48** (Differential version of Zariski–Nagata, see [DDSG<sup>+</sup>18]). Let  $R = k[x_1, \dots, x_d]$ , where  $k$  is a perfect field, and let  $I$  be a radical ideal. For all  $n \geq 1$ ,

$$I^{(n)} = I^{\langle n \rangle}.$$

*Proof.* First, let  $I = P$  be a prime ideal.

- 1)  $P^{\langle n \rangle}$  is a  $P$ -primary ideal. (Lemma 4.40 and Proposition 4.41)
- 2)  $P^n \subseteq P^{\langle n \rangle}$ . (Proposition 4.42)
- 3)  $(P^{\langle n \rangle})_P = (P_P)^{\langle n \rangle}$ . (Lemma 4.43)

Now 1) and 2) together imply  $P^{\langle n \rangle} \subseteq P^{(n)}$ , since  $P^{\langle n \rangle}$  is the smallest  $P$ -primary ideal containing  $P^n$ . To show  $P^{\langle n \rangle} \subseteq P^{(n)}$ , we only need to show the containment holds after localizing at  $P$ , which is the only associated prime of  $P^{\langle n \rangle}$ , by Exercise 16. But 3) says differential powers commute with localization, and after localization  $P$  becomes the maximal ideal; so Theorem 4.47 completes the proof that  $P^{\langle n \rangle} \subseteq P^{(n)}$  for a prime ideal  $P$ .

Now if we take any radical ideal  $I$ , we can write  $I$  as the intersection of finitely many primes, say

$$I = P_1 \cap \dots \cap P_r.$$

Then

$$I^{(n)} = P_1^{\langle n \rangle} \cap \dots \cap P_r^{\langle n \rangle} = P_1^{\langle n \rangle} \cap \dots \cap P_r^{\langle n \rangle} = (P_1 \cap \dots \cap P_r)^{\langle n \rangle} = I^{\langle n \rangle}. \quad \square$$



In Theorem 4.48, we cannot replace  $k[x_1, \dots, x_d]$  with a finitely generated  $k$ -algebra.

**Example 4.49.** Let  $k$  be any field,  $R = k[x, y, z]/(xy - z^2)$ ,  $\mathfrak{m} = (x, y, z)$ , and  $P = (x, z)$ . By Example 1.54,  $x \in P^{(2)}$ , so  $P^{(2)} \not\subseteq \mathfrak{m}^2$ . On the other hand, it follows immediately from the definition that  $P^{(2)} \subseteq \mathfrak{m}^{(2)}$ . Thus  $x \in P^{(2)} \subseteq P^{(2)} \subseteq \mathfrak{m}^{(2)}$ , but since  $\mathfrak{m}$  is maximal,  $\mathfrak{m}^2 = \mathfrak{m}^{(2)}$ , so we can conclude that  $\mathfrak{m}^{(2)} \neq \mathfrak{m}^2$ .

We also cannot substitute  $k$  by any field.

**Example 4.50.** Let  $k = \mathbb{F}_p(t)$ , with  $p$  prime, and consider the ring  $R = k[x]$  and the prime ideal  $P = (x^p - t)$ . As we described in Example 4.37,  $D_{R|k}^1 = R \oplus R \frac{\partial}{\partial x}$ , and  $\frac{\partial}{\partial x}(x^p - t) = 0 \in P$ . Therefore,  $P^{(2)} = P$ . On the other hand,  $P$  is a principal ideal in a domain, so its symbolic powers are the powers; in particular,  $P^{(2)} = P^2 \neq P^{(2)}$ .

**Theorem 4.51** (Zariski–Nagata Theorem for polynomial rings [Zar49]). *Let  $k$  be a perfect field and  $R = k[x_1, \dots, x_d]$ . For any radical ideal  $I$ , we have*

$$I^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \in \text{mSpec}(R)}} \mathfrak{m}^n.$$

*Proof.* On the one hand,

$$\begin{array}{ccccc} I^{(n)} & \subseteq & I^{(n)} & \subseteq & \mathfrak{m}^{(n)} = \mathfrak{m}^n. \\ 4.41 & & 4.39 & & 4.44 \end{array}$$

For the converse, take  $f \in \mathfrak{m}^n$  for all the maximal ideals  $\mathfrak{m} \supseteq I$ . For each maximal ideal  $\mathfrak{m}$  containing  $I$ , we have  $f \in \mathfrak{m}^{(n)}$  by Remark 4.44, so for every  $\delta \in D^{n-1}$ ,  $\delta(f) \in \mathfrak{m}$ . But by Theorem 4.31, every radical ideal in  $R$  is an intersection of maximal ideals:

$$I = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \in \text{maximal ideal}}} \mathfrak{m}.$$

Thus  $\partial(f) \in I$  for every  $\delta \in D^{n-1}$ , so  $f \in I^{(n)}$ . By Theorem 4.48,  $I^{(n)} = I^{(n)}$ .  $\square$

Over  $\mathbb{C}$  or any perfect field, the polynomials in  $\mathfrak{m}^n$  are those that vanish to order  $n$  at the point corresponding to  $\mathfrak{m}$ . So Zariski–Nagata says that  $I^{(n)}$  is the set of polynomials that vanish to order  $n$  along the variety  $\mathbb{V}(I)$ .

We can also give such an interpretation to the symbolic powers of ideals corresponding to finite sets of points in projective space. We have seen that the polynomials that vanish to order  $n$  on a finite set of points  $X$  in  $\mathbb{P}^d$  are precisely the polynomials in  $\mathbf{I}(X)^{(n)}$ . More generally, this also holds for any projective variety.

**Theorem 4.52.** *Let  $k$  be an algebraically closed field, and let  $X \subseteq \mathbb{P}_k^d$  be a nonempty projective variety with corresponding ideal  $I = \mathbb{I}(X) \subseteq k[x_1, \dots, x_d]$ . For all  $n \geq 1$ ,*

$$I^{(n)} = \bigcap_{P \in X} I(P)^n.$$

*Proof.* Let  $C$  be the cone of  $X$ , meaning that  $C$  is the union of the lines through the origin of  $\mathbb{A}_k^{d+1}$  that correspond to points in  $X$ . First, we claim that

$$\mathcal{N} := \{\mathfrak{m} \in \text{mSpec}(R) \mid \mathfrak{m} \supseteq I\} = \{\mathfrak{m} \in \text{mSpec}(R) \mid \mathfrak{m} \supseteq I(P) \text{ for some } P \in X\}.$$

Since  $k$  is algebraically closed, the affine Nullstellensatz tells us that the maximal ideals containing  $I$  are precisely the maximal ideals corresponding to points in  $C$ . For each  $P \in X$ , consider the cone of  $P$  in  $\mathbb{A}_k^{d+1}$ , meaning the line  $L$  through the origin of  $\mathbb{A}_k^{d+1}$  that corresponds to  $P$ . Since  $k$  is algebraically closed, a maximal ideal  $\mathfrak{m}$  contains  $I(P)$  if and only if  $\mathfrak{m}$  corresponds to a point on the line  $L$ . Since  $C$  is the union of all such lines, this proves our claim.

By Theorem 4.31,

$$I(P) = \bigcap_{\substack{\mathfrak{m} \in \text{mSpec}(R) \\ \mathfrak{m} \supseteq I(P)}} \mathfrak{m}, \quad \text{so} \quad \bigcap_{P \in X} I(P) = \bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}.$$

Also by Theorem 4.31,

$$I = \bigcap_{\substack{\mathfrak{m} \in \text{mSpec}(R) \\ \mathfrak{m} \supseteq I}} \mathfrak{m} = \bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}, \quad \text{so} \quad I = \bigcap_{P \in X} I(P).$$

Now fix  $n \geq 1$ . By Theorem 4.51,

$$I^{(n)} = \bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^n,$$

and for each  $P \in X$ ,

$$I(P)^{(n)} = \bigcap_{\substack{\mathfrak{m} \in \text{mSpec}(R) \\ \mathfrak{m} \supseteq I(P)}} \mathfrak{m}^n.$$

By Remark 4.29,  $I(P)^{(n)} = I(P)^n$ . By definition of  $\mathcal{N}$ ,

$$\bigcap_{P \in X} I(P)^n = \bigcap_{P \in X} I(P)^{(n)} = \bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^n.$$

We conclude that

$$I^{(n)} = \bigcap_{P \in X} I(P)^n. \quad \square$$

There are several extensions of Zariski–Nagata. In 1979, Eisenbud and Hochster [EH79] gave a more general version of Theorem 4.51; Zariski’s original result [Zar49] then follows as a corollary. As for the differential operators version of the theorem, Yairon Cid Ruiz [CR21] recently gave a more general version of the theorem for finitely generated  $k$ -algebras, which uses differential operators that are  $k$ -linear maps from  $R$  to  $R/P$ . In the next section, we will briefly discuss a version of the differential version of Zariski–Nagata when we replace  $k$  by  $\mathbb{Z}$  or a DVR.

## 4.4 Mixed characteristic

As for the differential operators description of symbolic powers, if we replace  $k$  by  $\mathbb{Z}$  or some other ring of mixed characteristic, this description no longer holds; roughly speaking, the differential operators cannot see what happens in the arithmetic direction.

**Example 4.53.** In  $R = \mathbb{Z}[x]$ , the symbolic powers of the maximal ideal  $\mathfrak{m} = (2, x)$  coincide with its powers, so  $2 \notin \mathfrak{m}^n$  for any  $n > 1$ . However, any differential operator  $\partial \in D_{R|\mathbb{Z}}^n$  of any order is  $\mathbb{Z}$ -linear, so  $\partial(2) = 2 \cdot \partial(1) \in \mathfrak{m}$ . Therefore,  $2 \in \mathfrak{m}^{(n)}$  for all  $n$ , and thus  $\mathfrak{m}^{(n)} \neq \mathfrak{m}^{(n)}$ .

To describe symbolic powers in mixed characteristic, we need to consider differential operators together with  $p$ -derivations, a tool from arithmetic geometry introduced independently in [Joy85] and [Bui95]; for a thorough development of the theory of  $p$ -derivations, see [Bui05].

**Definition 4.54** ( $p$ -derivation). Fix a prime  $p \in \mathbb{Z}$ , and let  $S$  be a ring on which  $p$  is a nonzerodivisor. A set-theoretic map  $\delta : R \rightarrow R$  is a  **$p$ -derivation** if  $\phi_p(x) := x^p + p\delta(x)$  is a ring homomorphism. Equivalently,  $\delta$  is a  $p$ -derivation if  $\delta(1) = 0$  and  $\delta$  satisfies the following identities for all  $x, y \in R$ :

- 1)  $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$ ,
- 2)  $\delta(x + y) = \delta(x) + \delta(y) + \mathcal{C}_p(x, y)$

where  $\mathcal{C}_p(X, Y) = \frac{X^p + Y^p - (X+Y)^p}{p} \in \mathbb{Z}[X, Y]$ . If  $\delta$  is a  $p$ -derivation, we set  $\delta^a$  to be the  $a$ -fold self-composition of  $\delta$ ; in particular,  $\delta^0$  is the identity.

These are very nonintuitive maps; for example, they are not even additive!

**Example 4.55.** Let  $R = \mathbb{Z}$  and fix a prime  $p$ . To give a  $p$ -derivation is the same as finding a lift  $\Phi$  of the Frobenius map on  $R/p$  to  $R$ : the  $p$ -derivation  $\delta$  satisfies  $\Phi(n) = n^p + p\delta(n)$ . But the Frobenius map on  $\mathbb{Z}/p$  is the identity, by Fermat's Little Theorem, so a  $p$ -derivation  $\delta_p$  must satisfy  $n^p + p\delta_p(n) = n$ . Again by Fermat's Little Theorem  $x^p - x$  is always divisible by  $p$  for any integer  $x$ , so  $\delta_p(n) = \frac{n - n^p}{p}$ . In particular, there is a unique  $p$ -derivation on  $\mathbb{Z}$ .

**Example 4.56.** If  $S$  is a ring with a  $p$ -derivation  $\delta$ , then  $R = S[x]$  also has  $p$ -derivations: we can extend  $\delta$  to  $R$  by setting  $\delta(x)$  to be any element in  $S$ , and this always determines a unique  $p$ -derivation on  $S$  for each choice of  $\delta(x)$ .

Roughly speaking, a  $p$ -derivation and its powers play the role of differential operators in the arithmetic direction. The following theorem applies, for example, to  $R = \mathbb{Z}[x_1, \dots, x_d]$ .

**Theorem 4.57** (De Stefani – Grifo – Jeffries, 2020 [DSGJ20]). *Let  $p$  be a prime. Let  $A = \mathbb{Z}$  or a DVR with uniformizer  $p$ . Let  $R$  be an essentially smooth  $A$ -algebra that has a  $p$ -derivation  $\delta$ . Let  $Q$  be a prime ideal of  $R$  that contains  $p$ , and assume that  $A/pA$  is perfect, or more generally that the field extension  $A/pA \subseteq R_Q/pR_Q$  is separable. Then*

$$Q^{(n)} = \{f \in S \mid (\delta^s \circ \partial)(f) \in I \text{ for all } \partial \in D_{R|A}^t \text{ with } s + t \leq n - 1\}.$$

For prime ideals that do not contain  $p$ , the usual description using only differential operators, as in Theorem 4.48, still holds [DSGJ20, Theorem 3.9].

**Example 4.58.** The maximal ideal  $\mathfrak{m} = (2, x)$  in  $R = \mathbb{Z}[x]$  contains the prime 2, so to describe its symbolic powers we need to consider a 2-derivation. The map  $\delta_2: R \rightarrow R$

$$\delta_2(f(x)) = \frac{f(x^2) - f(x)^2}{2}$$

is a 2-derivation on  $R$ . By Theorem 4.57, the symbolic powers of  $\mathfrak{m} = (2, x)$  are given by

$$\mathfrak{m}^{(n)} = \left\{ f \in \mathbb{Z}[x] \mid \delta_2^a \left( \frac{\partial^b f}{\partial x^b} \right) \in (2, x), \text{ for } a + b \leq n - 1 \right\}.$$

In particular, we can now see that  $2 \notin \mathfrak{m}^{(2)}$ , since

$$\delta_2(2) = \frac{2 - 2^2}{2} = -1 \notin \mathfrak{m},$$

while as we saw in Example 4.53 there are no  $\mathbb{Z}$ -linear differential operators  $\partial$  of order up to 1 (or even any order!) satisfying  $\partial(2) \notin \mathfrak{m}$ .

# Chapter 5

## Monomial ideals

Throughout this chapter, we fix a field  $k$  and focus on the ring  $R = k[x_1, \dots, x_d]$ . We will briefly discuss some of the nice properties of monomial ideals and how they stand at the heart of the connection between combinatorics and commutative algebra. Most importantly, we will discuss how to compute their symbolic powers, which will allow us to prove other nice results about monomial ideals in the later chapters.

This chapter is not intended as an exhaustive discussion of the symbolic powers of monomial ideals. The book [HH11] discusses monomial ideals in detail, the paper [FMS14] gives an overview of the Stanley-Reisner correspondence in commutative algebra, the paper [Tuy13] gives a gentle introduction to edge ideals, and the paper [CEHH16] discusses the symbolic powers of monomial ideals.

### 5.1 A short introduction to monomial ideals

**Definition 5.1.** Let  $k$  be a field and let  $R = k[x_1, \dots, x_n]$ . A **monomial** in  $R$  is any element of the form  $x_1^{a_1} \cdots x_n^{a_n}$ . We will sometimes write  $\underline{x}^{\underline{a}}$  to refer to the monomial  $x_1^{a_1} \cdots x_n^{a_n}$ . An ideal  $I$  in  $R$  is a **monomial ideal** if it is generated by monomials.

Proofs of the following elementary facts can be found in [HH11]:

- The set of monomials in a monomial ideal  $I$  forms a  $k$ -basis for  $I$ .
- To check if a given polynomial  $f$  is in  $I$ , we need only to check if each monomial appearing in  $f$  is in  $I$ .
- The residue classes in  $R/I$  of the monomials that are not in  $I$  form a  $k$ -basis for  $R/I$ .
- Each monomial ideal has a unique set of minimal monomial generators.
- If  $f$  is a monomial and  $I$  is a monomial ideal with minimal monomial generating set  $\{g_1, \dots, g_n\}$ ,  $f \in I$  if and only if  $g_i | f$  for some  $i$ .
- The sum, product, intersection, radical, colon, and saturation of monomial ideals is a monomial ideal.

**Notation 5.2.** Given a monomial ideal  $I$ , we write  $G(I)$  for the minimal generating set for  $I$  consisting of monomials, which is unique.

**Definition 5.3.** A monomial  $\underline{x}^a$  in  $R = k[x_1, \dots, x_d]$  is **squarefree** if  $a_i \leq 1$  for all  $i$ . A monomial ideal  $I$  is a **squarefree monomial ideal** if it is generated by squarefree monomials.

**Example 5.4.** The monomial ideal  $I = (x^2, xy)$  is not squarefree, since  $x^2$  is not a squarefree monomial while  $(xy, xz)$  is a squarefree monomial ideal.

**Notation 5.5.** Given a monomial  $f = \underline{x}^a$ , we will write  $\sqrt{f}$  for the monomial  $\sqrt{f} = \underline{x}^b$  given by

$$b_i = \begin{cases} 1 & \text{if } a_i \geq 1 \\ 0 & \text{if } a_i = 0 \end{cases}$$

So  $\sqrt{f}$  is essentially the squarefree part of  $f$ ;  $\sqrt{f}$  is what we obtain from  $f$  when we lower all the nonzero powers to 1.

**Example 5.6.** In  $k[x, y, z]$ ,  $\sqrt{x^7 y^2} = xy$ .

**Lemma 5.7.** Let  $I$  be a monomial ideal in  $R = k[x_1, \dots, x_k]$ . Then

$$\sqrt{I} = \left( \sqrt{f} \mid f \in I \text{ is a monomial} \right).$$

In particular,  $I$  is a radical ideal if and only if  $I$  is squarefree.

*Proof.* Since both ideals are monomial, it is sufficient to show that the monomials in both ideals coincide.

For each monomial  $f = \underline{x}^a \in I$ , set  $n = \max_i a_i$ ; then  $f | (\sqrt{f})^n$ , so  $\sqrt{f} \in \sqrt{I}$ . On the other hand, if  $f$  is a monomial with  $f^n \in I$  for some  $n$ , then there exists a monomial  $g \in I$  such that  $g | f^n$ , and thus every variable appearing in  $g$  appears in  $f$ , so  $\sqrt{g} | f$ .  $\square$

Notice that as an immediate consequence, if  $I$  has minimal monomial generating set  $G(I) = (g_1, \dots, g_n)$ , then

$$\sqrt{I} = (\sqrt{g_1}, \dots, \sqrt{g_n}).$$

**Notation 5.8.** Given two monomials  $\underline{x}^a$  and  $\underline{x}^b$ , we set

$$\text{lcm}(\underline{x}^a, \underline{x}^b) := \underline{x}^c \text{ with } c_i := \max\{a_i, b_i\} \text{ and } \text{gcd}(\underline{x}^a, \underline{x}^b) := \underline{x}^d \text{ with } d_i := \min\{a_i, b_i\}.$$

**Lemma 5.9.** Given monomial ideals  $I$  and  $J$  be monomial ideals in  $R = k[x_1, \dots, x_d]$ ,

$$I \cap J = (\text{lcm}(f, g) \mid f \in G(I) \text{ and } g \in G(J)).$$

*Proof.* On the one hand,  $\text{lcm}(f, g) \in I \cap J$  because it is both a multiple of  $f \in I$  and  $g \in J$ . On the other hand, if  $h \in I \cap J$  is a monomial, then  $h$  must be a multiple of some  $f \in G(I)$  and some  $g \in G(J)$ , and thus  $\text{lcm}(f, g) | h$ . This shows that

$$\{\text{lcm}(f, g) \mid f \in G(I) \text{ and } g \in G(J)\}$$

is a generating set for  $I \cap J$ .  $\square$

**Lemma 5.10.** *Given monomial ideals  $I$  and  $J$  in  $R = k[x_1, \dots, x_d]$ ,*

$$(I : J) = \bigcap_{f \in G(J)} (I : f).$$

*Moreover,*

$$(I : f) = \left( \frac{g}{\gcd(f, g)} \mid g \in G(I) \right).$$

*Proof.* The first claim holds for any two ideals  $I$  and  $J$ , where  $G(J)$  takes the role of some generating set for  $J$ . Let  $G(J) = \{f_1, \dots, f_n\}$ . By definition,  $(I : J) \subseteq (I : f_i)$  for each  $i$ . On the other hand, if  $gf_i \in I$  for every  $i$ , then  $gJ \in I$ . This shows the first claim.

Now fix  $f$ . For any  $g \in G(I)$ ,  $\frac{f}{\gcd(f, g)}$  is a monomial in  $R$ , so

$$f \frac{g}{\gcd(f, g)} = \frac{f}{\gcd(f, g)} g \in I.$$

Therefore,  $\frac{g}{\gcd(f, g)} \in (I : f)$ . Now for any monomial  $h \in (I : f)$ , the monomial  $fh \in I$  must be divisible by some  $g \in G(I)$ . But then  $\frac{g}{\gcd(g, f)}$  divides  $h$ , as desired.  $\square$

## 5.2 Primary decomposition of monomial ideals

**Lemma 5.11.** *A monomial ideal  $I$  is prime if and only if it is generated by variables.*

*Proof.* Any monomial ideal  $I$  generated by variables is prime since the quotient of  $R/I$  is the polynomial ring in the other variables. On the other hand, if  $x$  is a variable and  $xf \in G(I)$  for some monomial  $f \neq 1$ , then  $x$  is not in  $I$ , and thus  $I$  is not prime.  $\square$

**Lemma 5.12.** *The ideal  $I = (x_{i_1}^{a_{i_1}}, \dots, x_{i_n}^{a_{i_n}})$  with  $a_{i_j} \geq 1$  for all  $j$  is  $(x_{i_1}, \dots, x_{i_n})$ -primary.*

*Proof.* Every prime containing  $I$  must contain  $x_{i_j}$  for each  $j$ , and thus it must contain the prime ideal  $(x_{i_1}, \dots, x_{i_n})$ . On the other hand, the prime ideal  $(x_{i_1}, \dots, x_{i_n})$  contains  $I$ . It follows that  $(x_{i_1}, \dots, x_{i_n})$  is the unique minimal prime of  $I$ .

Now suppose  $P$  is an embedded prime of  $I$ . Then  $P = (I : g)$  for some  $g \notin I$ , and  $P$  contains some polynomial  $f \notin (x_{i_1}, \dots, x_{i_n})$ . After subtracting elements in  $P$ , we may assume that none of the monomials appearing in  $f$  contain the variables  $x_{i_j}$ . But then  $f$  is a regular element on  $R/I$ , and since  $fg \in I$  this means that  $g \in I$ , which is a contradiction. We conclude that  $P = (x_{i_1}, \dots, x_{i_n})$  is the only associated prime of  $I$ , and thus  $I$  is primary to  $(x_{i_1}, \dots, x_{i_n})$ .  $\square$

**Theorem 5.13.** *Every monomial ideal  $I$  in  $R = k[x_1, \dots, x_d]$  can be decomposed as an intersection of ideals of the form  $(x_{i_1}^{a_{i_1}}, \dots, x_{i_n}^{a_{i_n}})$ . Moreover, any decomposition in ideals of this form where no component can be deleted is unique.*

*Proof.* Let  $G(I) = \{f_1, \dots, f_n\}$ . If all of the  $f_i$  are of the form  $x_i^{a_i}$ , we are done, so suppose not. Then we can write  $f_1$  in the form  $f_1 = gh$ , where  $g$  and  $h$  are coprime monomials, meaning they contain disjoint sets of variables. We claim that  $I = (g, f_2, \dots, f_n) \cap (h, f_2, \dots, f_n)$ .

The containment  $I \subseteq (g, f_2, \dots, f_n) \cap (h, f_2, \dots, f_n)$  is obvious, since  $f_1 = gh \in (g) \cap (h)$  and  $f_2, \dots, f_n \in (g, f_2, \dots, f_n) \cap (h, f_2, \dots, f_n)$ . On the other hand, consider a monomial  $u \in (g, f_2, \dots, f_n) \cap (h, f_2, \dots, f_n)$ . If  $u$  is a multiple of some  $f_i$ , then  $u \in I$  and we are done. If not, then  $u$  must be a multiple of both  $g$  and  $h$ , and hence of  $f_1 = gh$ , since  $g$  and  $h$  were chosen to be coprime. In either case,  $u \in I$ .

Now we keep proceeding in this way. Whenever we have a component that is not generated by powers of primes, we decompose it as above, and keep repeating the process until each component is generated by powers of variables. This process must stop, since there are finitely many variables and in each step we are decomposing a product of at least two variables in two coprime components.

Given such a decomposition, we can turn it into an irredundant decomposition by first deleting components which contain others, and then intersecting components with the same radical. By Lemma 5.12, two components of this form have the same radical if they are of the form  $(x_{i_1}^{a_{i_1}}, \dots, x_{i_n}^{a_{i_n}})$  and  $(x_{i_1}^{b_{i_1}}, \dots, x_{i_n}^{b_{i_n}})$  for some  $a_{i_j}, b_{i_j} \geq 1$ .

It remains to show that this decomposition is unique. Consider two such decompositions  $I = Q_1 \cap \dots \cap Q_r = Q'_1 \cap \dots \cap Q'_s$ . Fix  $i$ , and suppose that  $Q_i = (x_1^{a_1}, \dots, x_k^{a_k})$ , perhaps after changing the names of the variables. If  $Q'_j \not\subseteq Q_i$  for all  $j$ , then for each  $j$  we can find  $x_{\ell_j}^{b_j} \in Q'_j$  with  $x_{\ell_j}^{b_j} \notin Q_i$ . Notice that this implies that either  $\ell_j > k$  or  $b_j < a_{\ell_j}$ . Now consider

$$f := \text{lcm}(x_{\ell_1}^{b_1}, \dots, x_{\ell_s}^{b_s}) \in Q'_1 \cap \dots \cap Q'_s = I \subseteq Q_i.$$

But this would mean that  $x_i^{a_i}$  divides  $f$  for some  $i$ , which is a contradiction.

Therefore, there exists  $j$  such that  $Q'_j \subseteq Q_i$ . Applying the argument in reverse, we conclude that there exists  $l$  such that  $Q_l \subseteq Q'_j$ , and thus  $Q_l \subseteq Q_i$ . But both decompositions were irredundant, so  $Q_l = Q'_j = Q_i$ .  $\square$

**Definition 5.14.** Given a monomial ideal  $I$ , its unique decomposition in ideals generated by pure powers of variables is called the **standard primary decomposition** of  $I$ .

**Remark 5.15.** Notice that we are *not* saying that monomial ideals have a unique primary decomposition; only that the decomposition in primary ideals of the form  $(x_{i_1}^{a_{i_1}}, \dots, x_{i_n}^{a_{i_n}})$  are unique. In fact, in Example 1.66 we showed that  $(x^2, xy)$  has infinitely many primary decompositions.

In fact, the standard primary decomposition of a monomial ideal might not be an irredundant primary decomposition. The primary monomial ideal  $(x^2, xy, y^2)$  has standard primary decomposition  $(x^2, xy, y^2) = (x^2, y) \cap (x, y^2)$ .

Theorem 5.13 says in particular that every monomial ideal has a primary decomposition into monomial ideals. Since symbolic powers are obtained by collecting certain primary components of  $I^n$ , we conclude that the symbolic powers of a monomial ideal are monomial.

**Corollary 5.16.** *The symbolic powers of a monomial ideal are monomial ideals.*

**Corollary 5.17.** *The associated primes of a monomial ideal are generated by variables.*

*Proof.* By Theorem 5.13, every monomial ideal can be decomposed as an intersection of ideals of the form  $(x_{i_1}^{a_{i_1}}, \dots, x_{i_n}^{a_{i_n}})$ , which are primary by Lemma 5.12. Also by Lemma 5.12, the radicals of these primary ideals are all generated by variables. By Theorem 1.67, these monomial primes coincide with the set of associated primes of our given monomial ideal.  $\square$



**Corollary 5.18.** *An ideal in  $R = k[x_1, \dots, x_d]$  is a squarefree monomial ideal if and only if it is a finite intersection of ideals generated by variables.*

*Proof.* A squarefree monomial ideal is radical, by Lemma 5.7, and thus it must coincide with the intersection of its minimal primes, which by Corollary 5.17 are generated by variables. On the other hand, finite intersections of monomial ideals are monomial, any ideal generated by variables is prime, and any finite intersection of primes is radical, by Lemma 1.18.  $\square$

**Remark 5.19.** The proof of Theorem 5.13 also shows that the variables appearing as generators of the associated primes of  $I$  must all be variables appearing in the minimal generators of  $I$ , and that every such variable appears as a generator of at least one of the associated primes.

Finally, we can now say something more concrete about the symbolic powers of monomial ideals.

**Corollary 5.20.** *Let  $I = P_1 \cap \dots \cap P_k$  be a squarefree monomial ideal with minimal primes  $P_1, \dots, P_k$ . Then for all  $n \geq 1$ ,*

$$I^{(n)} = \bigcap_{i=1}^k P_i^n.$$

*Proof.* Since  $I$  is radical and the  $P_i$  are its minimal primes,  $I^{(n)} = P_1^{(n)} \cap \dots \cap P_k^{(n)}$ . Since each  $P_i$  is generated by variables,  $P_i^{(n)} = P_i^n$  by Lemma 2.5.  $\square$

### 5.3 The Stanley-Reisner correspondance

**Definition 5.21.** A **simplicial complex**  $\Delta$  on  $[n] := \{1, \dots, n\}$  is a collection of subsets of  $[n]$  such that if  $F \in \Delta$  and  $F' \subseteq F$ , then  $F' \in \Delta$ .<sup>1</sup>

An element  $F \in \Delta$  is called a **face** of  $\Delta$ . A **facet** of  $\Delta$  is a face that is maximal with respect to inclusion. The set of facets of  $\Delta$  is denoted  $\mathcal{F}(\Delta)$ . A **nonface** of  $\Delta$  is a subset  $F$  of  $[n]$  such that  $F \notin \Delta$ ; the set of minimal nonfaces of  $\Delta$  is denoted  $\mathcal{N}(\Delta)$ .

Note that the facets  $\mathcal{F}(\Delta)$  of  $\Delta$  determine  $\Delta$  completely: by definition, if  $F \in \mathcal{F}(\Delta)$ , then any  $F' \subseteq F$  must be a face of  $\Delta$ , and every face of  $\Delta$  is necessarily contained in some facet. Similarly, the set  $\mathcal{N}(\Delta)$  of minimal nonfaces of  $\Delta$  also determines  $\Delta$  completely.

**Definition 5.22.** Given a simplicial complex  $\Delta$  on  $[n]$  and field  $k$ , set  $R = k[x_1, \dots, x_n]$ . Given a subset  $F \subseteq [n]$ ,  $x_F := \prod_{i \in F} x_i$  is the squarefree monomial corresponding to  $F$  in  $R$ . The **Stanley-Reisner ideal** of  $\Delta$  is the ideal of  $R$  given by

$$I_\Delta := \left( \prod_{i \in F} x_i \mid F \in \mathcal{N}(\Delta) \right).$$

The **Stanley-Reisner ring** of  $\Delta$  is the ring  $k[\Delta] = R/I_\Delta$ .

---

<sup>1</sup>Some authors also require that  $\{i\} \in \Delta$  for all  $i \in [n]$ .

**Lemma 5.23.** *The monomials of the form  $x_1^{a_1} \cdots x_n^{a_n}$  with  $\{i \in [n] \mid a_i \neq 0\} \in \Delta$  are a  $k$ -basis for  $k[\Delta]$ .*

*Proof.* First, note that by construction,  $\Delta$  is a squarefree monomial ideal, and thus radical, by Lemma 5.7. Given a monomial  $f = x_1^{a_1} \cdots x_n^{a_n}$ , if  $\{i \in [n] \mid a_i \neq 0\} \notin \Delta$  then the monomial  $\sqrt{f}$  is in  $I_\Delta$  by definition. Therefore, by Lemma 5.7,  $f \in \sqrt{I_\Delta} = I_\Delta$ , so  $f + I_\Delta$  is zero in  $k[\Delta]$ . On the other hand, if  $f = x_1^{a_1} \cdots x_n^{a_n} \in I_\Delta$ , then  $\sqrt{f} \in \sqrt{I_\Delta} = I_\Delta$  as well. Therefore,  $\sqrt{f}$  must be a multiple of one of the generators of  $I_\Delta$ , which is equivalently to saying that there is a subset of  $\{i \in [n] \mid a_i \neq 0\}$  in  $\mathcal{N}(\Delta)$ . Since  $\Delta$  is a simplicial complex,  $\{i \in [n] \mid a_i \neq 0\} \notin \Delta$ . This shows that a monomial  $f = x_1^{a_1} \cdots x_n^{a_n}$  is in  $I_\Delta$  if and only if  $\{i \in [n] \mid a_i \neq 0\} \notin \Delta$ . The monomials not in  $I_\Delta$  form a  $k$ -basis for  $k[\Delta]$ .  $\square$

**Lemma 5.24.** *Let  $\Delta$  be a simplicial complex in  $[n]$ . The set of minimal nonfaces of  $\Delta$  corresponds to a minimal generating set for  $I_\Delta$ .*

*Proof.* If  $F \subseteq F'$ , then  $x_F \mid x_{F'}$ , so the monomials corresponding to minimal nonfaces generate  $I_\Delta$ . On the other hand, two minimal nonfaces of  $\Delta$  are necessarily incomparable, and thus the corresponding squarefree monomials do not divide each other. A monomial is in a given monomial ideal if and only if it is divisible by one of the generators, so we conclude that the minimal nonfaces form a minimal generating set for  $I_\Delta$ .  $\square$

By construction, each simplicial complex gives rise to a squarefree monomial ideal  $I_\Delta$ . On the other hand, every squarefree monomial ideal is the Stanley-Reisner ideal of some simplicial complex, and this correspondence is bijective.

**Theorem 5.25** (Stanley-Reisner correspondence). *Let  $n \geq 1$  and  $k$  be a field. There is a bijective correspondence between simplicial complexes on  $[n]$  and squarefree monomial ideals in  $k[x_1, \dots, x_n]$  given by  $\Delta \mapsto I_\Delta$ .*

*Proof.* Given a squarefree monomial ideal  $I$  in  $k[x_1, \dots, x_n]$ , set

$$\Delta_I := \{\{i_1, \dots, i_j\} \mid x_{i_1} \cdots x_{i_j} \notin I\}.$$

Suppose that  $F = \{i_1, \dots, i_j\} \in \Delta$ , and consider any subset  $F' = \{i_1, \dots, i_t\}$  of  $F$ . By definition,  $x_F \notin I$ , but since  $x_{F'}$  divides  $x_F$ , we must have  $x_{F'} \notin I$ , and thus  $F' \in \Delta$ . This shows that  $\Delta$  is a simplicial complex.

It follows immediately from the definition that  $\Delta_{I_\Delta} = \Delta$  for any simplicial complex  $\Delta$  and that  $I_{\Delta_I} = I$  for any squarefree monomial ideal.  $\square$

This correspondence is very rich, and hides a dictionary between combinatorial properties of the simplicial complex and algebraic properties of its Stanley-Reisner ring. For example, we can find a primary decomposition for any squarefree monomial ideal via its corresponding simplicial complex. This is best described via the Alexander dual of  $\Delta$ .

**Definition 5.26.** Let  $\Delta$  be a simplicial complex on  $[n]$ . The **Alexander dual**  $\Delta^\vee$  of  $\Delta$  is the simplicial complex

$$\Delta^\vee := \{[n] \setminus F \mid F \notin \Delta\}.$$

The **Alexander dual**  $I^\vee$  of  $I$  is the Stanley-Reisner ideal of  $\Delta_I^\vee$ .

**Lemma 5.27.** *Given any simplicial complex  $\Delta$  on  $[n]$ ,  $\Delta^\vee$  is a simplicial complex, and  $(\Delta^\vee)^\vee = \Delta$ .*

*Proof.* If  $F \in \Delta^\vee$  and  $F' \subseteq F$ , then  $[n] \setminus F \notin \Delta$ . Since  $[n] \setminus F \subseteq [n] \setminus F'$  and  $\Delta$  is a simplicial complex,  $[n] \setminus F' \notin \Delta$ , so  $F' \in \Delta^\vee$ . Now the equality  $(\Delta^\vee)^\vee = \Delta$  is immediate from the definition.  $\square$

Each generator  $x_{i_1} \cdots x_{i_j}$  of  $I_{\Delta^\vee}$  corresponds to a minimal prime  $(x_{i_1}, \dots, x_{i_j})$  of  $I_\Delta$ .

**Lemma 5.28.** *Let  $\Delta$  be a simplicial complex on  $[n]$ . Then*

$$I_{\Delta^\vee} = (x_{[n] \setminus F} \mid F \in \mathcal{F}(\Delta))$$

*Proof.* The minimal monomial generators of  $\Delta^\vee$  are the squarefree monomials that correspond to minimal nonfaces of  $\Delta^\vee$ . A given  $F \subseteq [n]$  is a nonface of  $\Delta^\vee$  if and only if  $[n] \setminus F$  is a face of  $\Delta$ . Moreover, such an  $F$  is a minimal nonface of  $\Delta^\vee$  if and only if no  $F' \subseteq [n]$  which properly contains  $[n] \setminus F$  is a face of  $\Delta$ , so  $F$  must be a facet of  $\Delta$ .  $\square$

In particular, note that  $\mathcal{F}(\Delta^\vee) = \{[n] \setminus F \mid F \in \mathcal{N}(\Delta)\}$ .

**Theorem 5.29.** *Let  $\Delta$  be a simplicial complex on  $[n]$ . The standard decomposition of  $I_\Delta$ , which in this case is the unique irredundant primary decomposition of  $I_\Delta$ , is given by*

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \in [n] \setminus F).$$

Therefore, if the Alexander dual of the squarefree monomial ideal  $I$  has minimal generators

$$I^\vee = (x_{i,1} \cdots x_{i,s_i} \mid 1 \leq i \leq t),$$

then the unique irredundant decomposition of  $I$  is

$$I = \bigcap_{i=1}^t (x_{i,1}, \dots, x_{i,s_i}).$$

*Proof.* Fix any monomial  $f = x_1^{a_1} \cdots x_n^{a_n}$ . If  $f \in I_\Delta$ , then by Lemma 5.23 we know that  $\{i \in [n] \mid a_i \neq 0\} \notin \Delta$ , and thus no facet of  $\Delta$  contains  $\{i \in [n] \mid a_i \neq 0\}$ . Therefore, for every facet  $F \in \mathcal{F}(\Delta)$ ,

$$\{i \in [n] \mid a_i \neq 0\} \cap ([n] \setminus F) \neq \emptyset,$$

and thus

$$f \in (x_i \mid i \in [n] \setminus F).$$

We conclude that

$$I_\Delta \subseteq \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \in [n] \setminus F).$$

On the other hand, if  $f \notin I_\Delta$ , then again by Lemma 5.23 we have  $\{i \in [n] \mid a_i \neq 0\} \in \Delta$ . Thus there is a facet  $F \in \mathcal{F}(\Delta)$  that contains  $\{i \in [n] \mid a_i \neq 0\}$ , so that

$$f \notin (x_i \mid i \in [n] \setminus F)$$

for that facet  $F$ , so

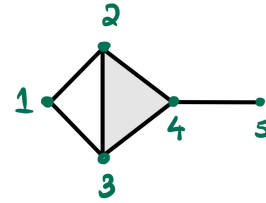
$$f \notin \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \in [n] \setminus F). \quad \square$$

**Example 5.30.** Consider the squarefree monomial ideal  $I = (xy, xz)$  in  $k[x, y, z]$ . By Theorem 5.29, the Alexander dual of  $I$  is  $I^\vee = (x, y) \cap (x, z) = (x, yz)$ . By Lemma 5.27,  $I$  is the Alexander dual of  $I^\vee$ , so applying Theorem 5.29 again gives us the primary decomposition  $I = (x) \cap (y, z)$ .

**Macaulay2.** In Macaulay2, we can compute the Alexander dual of a squarefree monomial ideal  $I$  via the command `dual`. This command receives inputs of type `monomialIdeal`, which means that it is not sufficient to set up an ideal that is generated by monomials, you must set it up to be something of the right type by using the command `monomialIdeal`.

**Example 5.31.** The picture below represents the simplicial complex  $\Delta$  on  $[5]$  with facets

$$\mathcal{F}(\Delta) = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{4, 5\}\}.$$



The shaded region indicates that  $\{2, 3, 4\} \in \Delta$ . The minimal nonfaces of  $\Delta$  are

$$\mathcal{N}(\Delta) = \{\{1, 4\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{1, 2, 3\}\}.$$

The corresponding Stanley-Reisner ideal is

$$I_\Delta = (x_1x_4, x_1x_5, x_2x_5, x_3x_5, x_1x_2x_3) = (x_1, x_5) \cap (x_1, x_2, x_3) \cap (x_2, x_4, x_5) \cap (x_3, x_4, x_5),$$

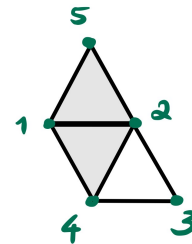
and its Alexander dual is

$$I_{\Delta^\vee} = (x_1x_5, x_1x_2x_3, x_2x_4x_5, x_3x_4x_5).$$

Notice in particular that  $I_\Delta$  and  $I_{\Delta'}$  can have minimal generators in common.

**Example 5.32.** The picture below represents the simplicial complex  $\Delta$  on  $[5]$  with facets

$$\mathcal{F}(\Delta) = \{\{1, 2, 4\}, \{1, 2, 5\}, \{2, 3\}, \{3, 4\}\}.$$



The minimal nonfaces of  $\Delta$  are

$$\mathcal{N}(\Delta) = \{\{1, 3\}, \{3, 5\}, \{4, 5\}, \{2, 3, 4\}\}.$$

The corresponding Stanley-Reisner ideal is

$$I_\Delta = (x_3, x_4) \cap (x_3, x_5) \cap (x_1, x_4, x_5) \cap (x_1, x_2, x_5).$$

and its Alexander dual is

$$I_{\Delta^\vee} = (x_3x_4, x_3x_5, x_1x_4x_5, x_1x_2x_5).$$

## 5.4 Edge ideals

**Definition 5.33.** Let  $G$  be a finite simple graph with  $n$  vertices  $V(G) = \{1, \dots, n\}$  and with edges  $E(G)$ . Let  $R = k[x_1, \dots, x_n]$ , where  $k$  is a field. The **edge ideal** of  $G$  is the ideal

$$I(G) := (x_i x_j \mid \{i, j\} \in E(G)).$$

The **cover ideal** of  $G$  is the Alexander dual of  $I(G)$ ,

$$J(G) := \bigcap_{\{i,j\} \in E(G)} (x_i, x_j).$$

The edge ideal of a finite simple graph is a squarefree monomial ideal, and thus the Stanley-Reisner ideal of some simplicial complex. To describe that complex, we need the independent sets of  $G$ .

**Definition 5.34.** Let  $G$  be a finite simple graph. A subset  $W \subseteq V(G)$  of the vertices of  $G$  is called **independent** if no edge of  $G$  has both of its endpoints in  $W$ .

The edge ideal of  $G$  coincides with the Stanley-Reisner ideal of the simplicial complex

$$\Delta(G) := \{W \subseteq [n] \mid W \text{ is an independent set}\}.$$

Once again, there is a bijection between finite simple graphs and all quadratic squarefree monomial ideals, and there are many theorems relating the algebra of  $I(G)$  and the combinatorics of  $G$ . For example, we can now describe the standard primary decomposition of  $I(G)$  from the combinatorics of  $G$ .

**Definition 5.35.** Let  $G$  be a finite simple graph. A set of vertices  $W \subseteq V(G)$  is a **vertex cover** for  $G$  if for every edge  $e \in E(G)$ ,  $W$  contains at least one of the vertices in  $e$ . We say  $W$  is a **minimal vertex cover** of  $G$  if  $W$  is a vertex cover that is minimal with respect to containment.

**Theorem 5.36.** *Let  $G$  be a finite simple graph. Then*

$$J(G) = (x_W \mid W \subseteq V(G) \text{ is a minimal vertex cover of } G).$$

*Proof.* Let  $W$  be a minimal vertex cover of  $G$ , and let

$$x_W := \prod_{i \in W} x_i.$$

Given any edge  $\{i, j\}$  of  $I(G)$ ,  $W$  must contain at least one of  $i$  or  $j$ , so  $x_W \in (x_i, x_j)$ . Therefore,

$$x_W \in \bigcap_{\{i,j\} \in E(G)} (x_i, x_j) = J(G).$$

Conversely, consider a minimal monomial generator  $m$  of  $J(G)$ , say  $m = x_{i_1} \cdots x_{i_r}$ , and set

$$W := \{i_1, \dots, i_r\}.$$

For each edge  $\{j, k\} \in E(G)$ , we must have  $x_j|m$  or  $x_k|m$ , since  $m \in J(G)$ , so either  $j \in W$  or  $k \in W$ . Therefore,  $W$  is a vertex cover of  $G$ . Now if  $W'$  is a minimal vertex cover contained in  $W$ , we still have  $x_{W'}|x_W = m$ , and thus

$$m \in (x_W \mid W \subseteq V(G) \text{ is a minimal vertex cover of } G). \quad \square$$

**Corollary 5.37.** *Let  $W_1, \dots, W_t$  be minimal vertex covers of  $G$ . Then*

$$I(G) = \bigcap_{i=1}^t (x_j \mid j \in W_i)$$

*is the unique irredundant primary decomposition of  $I(G)$ .*

*Proof.* This is an immediate consequence of Theorem 5.29 and Theorem 5.36, noting that

$$\{x_W \mid W \subseteq V(G) \text{ is a minimal vertex cover of } G\}$$

is necessarily a minimal generating set for  $J(G)$ , given that all the minimal vertex covers are incomparable.  $\square$

**Example 5.38.** The edge ideal  $I(G)$  of the 5-cycle graph



has irredundant primary decomposition

$$I(G) = (x_1, x_3, x_4) \cap (x_1, x_3, x_5) \cap (x_2, x_4, x_5) \cap (x_2, x_3, x_5) \cap (x_1, x_2, x_4).$$

## 5.5 Symbolic powers of monomial ideals

When  $I$  is a radical monomial ideal, we have discussed how to compute its symbolic powers: we first decompose  $I$  as a finite intersection of monomial primes  $I = P_1 \cap \dots \cap P_k$  — which are generated by variables — using the methods discussed in the previous sections, and then

$$I^{(n)} = P_1^n \cap \dots \cap P_k^n.$$

While our main interest is in radical ideals, let us briefly discuss how to compute symbolic powers of general monomial ideals. A lot more about symbolic powers of monomial ideals can be found in the survey [CEHH16].

Suppose that  $I$  is any monomial ideal, and let  $P_1, \dots, P_k$  be the maximal associated primes of  $I$ . For each of these primes  $P_i$ , we compute the intersection  $Q_i$  of the primary components of  $I$  associated to primes with contained in  $P_i$ . Each of these separate primary

components is not unique, but the intersection of all the components contained in  $P_i$  is indeed unique, and equal to  $IR_{P_i} \cap R$ . If we are given the standard primary decomposition of  $I$ , where all the components are of the form  $(x_{i_1}^{a_1}, \dots, x_{i_s}^{a_s})$ , to compute  $Q_i$  we need only to collect the components involving only the variables that generate  $P_i$ . Now  $I = Q_1 \cap \dots \cap Q_k$  is an intersection of monomial ideals with only one maximal associated prime.

As we discussed in Discussion 2.13,  $I = Q_1^{(n)} \cap \dots \cap Q_k^{(n)}$ . We claim that  $I = Q_1^n \cap \dots \cap Q_k^n$ , which follows from the following lemma.

**Lemma 5.39.** *If  $I$  is a monomial ideal with only one maximal associated prime, then  $I^{(n)} = I^n$ .*

*Proof.* Let  $R = k[x_1, \dots, x_d]$  be our polynomial ring. Without loss of generality, assume that  $P = (x_1, \dots, x_s)$  is the unique maximal associated prime of  $I$ . Note that every associated prime of  $I$  is generated by a subset of the variables  $x_1, \dots, x_s$ . In particular, this holds for the minimal primes of  $I$ , and thus the generators of  $\sqrt{I}$  involve only the variables  $x_1, \dots, x_s$ . Given the description of  $\sqrt{I}$  from Lemma 5.7, we conclude that the generators of  $I$  involve only the variables  $x_1, \dots, x_s$ .

We want to show that  $I^{(n)} = I^n$ . Since  $I^n \subseteq I^{(n)}$  are both monomial ideals, it is sufficient to show that every monomial  $f \in I^{(n)}$  is in  $I^n$ . As discussed in Discussion 2.13, there exists  $s \notin P$  such that  $sf \in I^n$ . Since  $I^n$  is a monomial ideal, every nonzero monomial term in  $sf$  must be in  $I^n$ ; since at least one of the monomials appearing in  $s$  must be outside of  $P$ , we can in fact find a monomial  $g \notin P$  such that  $gf \in I^n$ . Since  $g \notin P$ , none of the variables  $x_1, \dots, x_s$  can divide  $g$ , and thus  $g$  must be of the form  $g = x_{s+1}^{a_{s+1}} \dots x_n^{a_n}$ . On the other hand,  $I^n$  is generated by monomials that involve only the variables  $x_1, \dots, x_s$ , so we conclude that  $f \in I^n$ .  $\square$

Powers and intersections of monomial ideals are monomial, and thus this also shows that the symbolic powers of a monomial ideal are monomial.

**Example 5.40.** Consider the monomial ideal  $I = (xy, y^2w, x^2z, x^3)$ . Its standard primary decomposition is

$$I = (x^2, y) \cap (x, y^2) \cap (x^3, y, z) \cap (x, w).$$

Notice that this is not an irredundant primary decomposition, as the first two components are both  $(x, y)$ -primary. There are two maximal associated primes:  $(x, y, z)$  and  $(x, w)$ . The intersection of the components contained in  $(x, y, z)$  is  $(y^2, xy, x^2z, x^3)$ , while the intersection of the components contained in  $(x, w)$  is simply  $(x, w)$ . Thus

$$I = (y^2, xy, x^2z, x^3) \cap (x, w),$$

and for all  $n \geq 1$

$$I^{(n)} = (y^2, xy, x^2z, x^3)^n \cap (x, w)^n.$$

# Chapter 6

## Rees algebras

Rather than studying each (symbolic) power at a time, it is sometimes convenient to study them all at once. The Rees algebra of an ideal  $I$  is obtained by essentially packaging together all the powers of  $I$  and constructing a graded algebra. In this chapter we will introduce Rees algebras and other related concepts, with an eye towards applications to symbolic powers.

### 6.1 Rees algebras

**Definition 6.1.** Let  $I$  be an ideal in a ring  $R$ . The **Rees algebra** of  $I$  is the graded ring

$$\mathcal{R}(I) := \bigoplus_n I^n t^n = R[It] \subseteq R[t].$$

The grading is given by setting  $\deg t = 1$  and  $\deg r = 0$  for all  $r \in R$ .

Here the variable  $t$  serves the purpose of keeping track of degrees. As an  $R$  module, the Rees algebra of  $I$  is isomorphic to

$$\bigoplus_n I^n,$$

but  $t$  helps help us distinguish an element  $f \in I^{(n)}$  in the piece of degree  $n$  from the element  $f \in I^{(n)} \subseteq I$  in the piece of degree 0.

**Remark 6.2.** If  $R$  is a noetherian ring, then  $I$  is a finitely generated ideal, so consider a finite generating set  $I = (f_1, \dots, f_n)$ . Then  $\mathcal{R}(I) = R[f_1 t, \dots, f_n t]$  is a finitely generated  $R$ -algebra. Therefore, if  $R$  is noetherian then  $R[It]$  is noetherian.

The key technical point that allows us to construct a graded ring from the powers of  $I$  is the fact that they form a graded family of ideals.

**Definition 6.3.** Let  $\mathcal{I} = \{I_n\}_{n \geq 0}$  be a family of ideals in a ring  $R$ . We say that  $\mathcal{I}$  forms a **graded family of ideals** if  $I_0 = R$  and

$$I_n I_m \subseteq I_{n+m} \text{ for all } m, n.$$

If  $\mathcal{I}$  is a graded family of ideals and  $I_n \supseteq I_{n+1}$  for all  $n$ , we say that  $\mathcal{I}$  is a **filtration**.



**Example 6.4.** The powers of any ideal form a graded family, since indeed  $I^n I^m \subseteq I^{n+m}$ .

**Definition 6.5.** Let  $\mathcal{I} = \{I_n\}_{n \geq 0}$  be a graded family of ideals. The **Rees algebra** of  $\mathcal{I}$  is the  $\mathbb{N}$ -graded ring

$$\mathcal{R}(\mathcal{I}) := \bigoplus_{n \geq 0} I_n t^n \subseteq R[t].$$

The grading is given by setting  $\deg t = 1$  and  $\deg r = 0$  for all  $r \in R$ .

**Example 6.6.** We showed in Lemma 2.9 that the symbolic powers of an ideal form a graded family. The corresponding Rees algebra is called the **symbolic Rees algebra**, which we will discuss in detail later.

Sometimes it is convenient to extend the Rees algebra of an ideal, and form what is called the extended Rees algebra of  $I$ . In fact, some authors call the extended Rees algebra the Rees algebra of  $I$ .

**Definition 6.7.** The **extended Rees algebra** of an ideal  $I$  is the  $\mathbb{Z}$ -graded ring

$$R[It, t^{-1}] = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2 t^2 \oplus \cdots \subseteq R[t, t^{-1}].$$

Again, the grading is given by setting  $\deg t = 1$  and  $\deg r = 0$  for all  $r \in R$ . This may also be written as

$$\bigoplus_{n \in \mathbb{Z}} I^n t^n,$$

where the convention is that  $I^n = R$  for all  $n \leq 0$ .

Note that as long as  $I$  is a proper ideal,  $t \notin R[It, t^{-1}]$  since  $1 \notin I$ , so  $t^{-1}$  is not a unit in the extended Rees algebra, even though it looks like one.

**Remark 6.8.** If  $R$  is a noetherian ring, then  $I$  is a finitely generated ideal, so consider a finite generating set  $I = (f_1, \dots, f_n)$ . Now  $R[It, t^{-1}] = R[f_1 t, \dots, f_n t, t^{-1}]$  is a finitely generated  $R$ -algebra. Therefore, if  $R$  is noetherian then  $R[It, t^{-1}]$  is noetherian.

**Remark 6.9.** Let  $I$  and  $J$  be ideals in a ring  $R$ . Then

$$J \subseteq JR[It] \cap R \subseteq JR[It, t^{-1}] \cap R \subseteq JR[t, t^{-1}] \cap R,$$

and since

$$JR[t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} Jt^n,$$

the piece of degree 0 in  $JR[t, t^{-1}]$  is  $J$ , so  $JR[t, t^{-1}] \cap R = J$ . We conclude that all the containments above must actually be equalities:

$$J = JR[It] \cap R = JR[It, t^{-1}] \cap R.$$

Therefore, every ideal in  $R$  is contracted from an ideal in  $R[It]$  and from an ideal in  $R[t, t^{-1}]$ , and every prime ideal in  $R$  is contracted from a prime ideal in  $R[It]$ .

We can now compute the dimension of the Rees algebra of an ideal.

**Lemma 6.10.** *Let  $R$  be a Noetherian ring, and  $I$  an ideal in  $R$ .*

- 1) *The minimal primes of  $R[It, t^{-1}]$  are exactly the primes of the form  $\mathfrak{p}R[t, t^{-1}] \cap R[It, t^{-1}]$  for  $\mathfrak{p} \in \text{Min}(R)$ .*
- 2) *The minimal primes of  $R[It]$  are exactly the primes of the form  $\mathfrak{p}R[t, t^{-1}] \cap R[It]$  for  $\mathfrak{p} \in \text{Min}(R)$ .*

*Proof.* Let  $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$  be a minimal primary decomposition of  $(0)$  in  $R$ , and  $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ . One can check (exercise!) that  $\mathfrak{q}_i R[t]$  is primary with radical  $\mathfrak{p}_i R[t]$ . Then the same is true in  $R[t, t^{-1}]$ , by localizing at  $\{1, t, t^2, \dots\}$  and using Theorem 1.45. By Lemma 1.59, the contraction of a primary ideal is primary. Contracting to  $R[It, t^{-1}]$  or  $R[It]$ , we get primary ideals that intersect to  $(0)$ ; none is contained in the intersection of the others, since this is the case after contracting to  $R$ , and likewise the radicals are distinct since they contract to different primes in  $R$ .

Therefore, setting  $\mathfrak{q}'_i = \mathfrak{q}_i R[t, t^{-1}] \cap R[It, t^{-1}]$ ,  $\mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_t$  is a minimal primary decomposition of  $(0)$  in  $R[It, t^{-1}]$ , and thus the minimal primes in  $R[It, t^{-1}]$  are precisely  $\mathfrak{p}_i R[t, t^{-1}] \cap R[It, t^{-1}]$ . Repeating the argument with  $R[It]$  gives us that the minimal primes in  $R[It]$  are precisely  $\mathfrak{p}_i R[t, t^{-1}] \cap R[It]$  for each  $i$ .  $\square$

**Theorem 6.11.** *Let  $R$  be a finite dimensional noetherian ring, and  $I$  be an ideal of  $R$ . Then*

$$\dim(R[It]) = \begin{cases} \dim(R) + 1 & \text{if } I \not\subseteq P \text{ for some prime } P \text{ with } \dim(R/P) = \dim(R) \\ \dim(R) & \text{otherwise.} \end{cases}$$

*Proof.* By Lemma 6.10, the minimal primes in  $R[It]$  are of the form  $PR[t, t^{-1}] \cap R[It]$  with  $P$  a minimal prime of  $R$ . So

$$\dim(R[It]) = \max \left\{ \dim \left( \frac{R[It]}{PR[t, t^{-1}] \cap R[It]} \right) \mid P \in \text{Min}(R) \right\}.$$

Now notice that

$$\frac{R[It]}{PR[t, t^{-1}] \cap R[It]} \cong \frac{R}{P} \left[ \frac{I+P}{P} t \right],$$

the Rees algebra of the image of  $I$  in  $R/P$ . Therefore,

$$\dim(R[It]) = \max \left\{ \dim \left( \frac{R}{P} \left[ \frac{I+P}{P} t \right] \right) \mid P \in \text{Min}(R) \right\}.$$

By quotienting by each minimal prime of  $R$ , we can now reduce the problem to the case when  $R$  is a domain. For a domain  $R$ , we want to show that

$$\dim(R[It]) = \begin{cases} \dim(R) + 1 & \text{if } I \neq 0 \\ \dim(R) & \text{if } I = 0. \end{cases}$$

When  $I = 0$ ,  $R[It] \cong R$ , so of course that  $\dim(R[It]) = \dim(R)$ .

By the [Dimension Inequality](#), every prime ideal  $P$  in  $R[It]$  satisfies  $\text{ht}(P) \leq \text{ht}(P \cap R) + 1$ , so  $\dim(R[It]) \leq \dim(R) + 1$ .

Now consider the ideal  $P = ItR[It]$ . Since  $R[It]/P \cong R$  and  $R$  is a domain,  $P$  is prime. Moreover, every nonzero homogeneous element in  $P$  has positive degree, so  $P \cap R = 0$ . Finally, note that  $R[It] \subseteq R[t]$  is also a domain, and  $P \neq 0$  since  $P \supseteq It$ , so we conclude that  $\text{ht}(P) > 0$ . Therefore,

$$\dim(R[It]) \geq \dim(R[It]/P) + \text{ht}(P) = \dim(R) + \text{ht}(P) \geq \dim(R) + 1,$$

and we conclude that  $\dim(R[It]) = \dim(R) + 1$ .  $\square$

**Definition 6.12.** The **associated graded ring** of an ideal  $I$  in a ring  $R$  is the ring

$$\text{gr}_I(R) := \bigoplus_{n \geq 0} I^n / I^{n+1}$$

with  $n$ th graded piece  $I^n / I^{n+1}$  and multiplication

$$(a + I^{n+1})(b + I^{m+1}) = ab + I^{m+n+1} \text{ for } a \in I^n, b \in I^m.$$

If  $(R, \mathfrak{m})$  is local, then  $\text{gr}(R) := \text{gr}_{\mathfrak{m}}(R)$  will be called the associated graded ring of  $R$ .

The associated graded ring of a graded family of ideals is defined in the obvious way.

**Remark 6.13.** The associated graded ring of  $I$  is also isomorphic to

$$\frac{R[It]}{IR[It]} \cong \frac{R[It, t^{-1}]}{t^{-1}R[It, t^{-1}]}.$$

By [Remark 6.2](#), this is a quotient of a noetherian ring, and thus  $\text{gr}_I(R)$  is also a noetherian ring.

**Remark 6.14.** If  $a \in I^n$ ,  $b \in I^m$ ,  $u \in I^{n+1}$ ,  $v \in I^{m+1}$ , that is,  $a + u \in a + I^{n+1}$  and  $b + v \in b + I^{m+1}$  then

$$(a + u)(b + v) = ab = av + bu + uv \in ab + I^{m+n+1}.$$

This says that the multiplication on the associated graded ring is well-defined.

**Remark 6.15.**

- 1)  $[\text{gr}_I(R)]_0 = R/I$ , so if  $(R, \mathfrak{m}, k)$  is a local ring then  $[\text{gr}(R)]_0 = R/\mathfrak{m} = k$ .
- 2) Each graded piece  $[\text{gr}_I(R)]_n = I^n / I^{n+1}$  is an  $R$ -module annihilated by  $I$ , and thus it is an  $R/I$ -module. In particular, if  $(R, \mathfrak{m}, k)$  is local then  $[\text{gr}(R)]_n$  is a  $k$ -vector space.
- 3) Let  $R$  be Noetherian and  $I = (f_1, \dots, f_n)$ . Then  $f_1 + I^2, \dots, f_n + I^2 \in [\text{gr}_I(R)]_1$  and  $\text{gr}_I(R) = (R/I)[f_1, \dots, f_n]$  is finitely generated as a  $[\text{gr}(R)]_0$ -algebra by elements of degree one. If  $(R, \mathfrak{m}, k)$  is Noetherian and local, then by [NAK](#) a basis for  $[\text{gr}(R)]_n$  corresponds to a minimal set of generators for  $\mathfrak{m}^n$ , and in particular  $\dim_k [\text{gr}(R)]_n = \mu(\mathfrak{m}^n) < \infty$ .

**Example 6.16.** Take  $R = k[x, y]$  and  $I = (x, y)$ . Then

$$\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots = k \oplus (kx \oplus ky) \oplus (kx^2 \oplus kxy \oplus ky^2) \oplus \cdots$$

so we see that in fact  $\mathrm{gr}_{(x,y)}(k[x, y]) = k[x, y]$ . Similarly, for any graded  $k$ -algebra generated in degree 1,  $\mathrm{gr}_{R_+}(R) = R$ .

**Example 6.17.** Let  $R = k[x, y]_{(x,y)}$ .

a) Let  $I = (x, y)$ . The same computation as above applies to show

$$\mathrm{gr}_{(x,y)} k[x, y]_{(x,y)} = k[x, y].$$

b) Now take  $I = (x^2, y^2)$ . Then

$$\mathrm{gr}_I(R) = R/I \oplus (x^2R/I \oplus y^2R/I) \oplus (x^2R/I \oplus x^2y^2R/I \oplus y^2R/I) \oplus \cdots$$

so we get

$$\mathrm{gr}_I(R) = (R/I)[x^2, y^2]$$

with  $\deg(x^2) = \deg(y^2) = 1$  and  $\deg(r + I) = 0$  for all  $r \in R$ . The  $R/I$ -algebra generators for  $\mathrm{gr}_I(R)$  are algebraically independent.

c) Finally, take  $I = (x^2, xy)$ . Then

$$\mathrm{gr}_I(R) = (R/I)[x^2, xy],$$

with  $\deg(x^2) = \deg(xy) = 1$  and  $\deg(r + I) = 0$  for all  $r \in R$ . However, in this case the algebra generators  $x^2, xy$  are not algebraically independent over  $R/I$ . For example,  $\bar{y}x^2 - \bar{x}xy = 0$ .

**Example 6.18.** If  $R = k[x, y]$  and  $I = (x^2, y^2)$ , then  $R[It, t^{-1}] = R[x^2t, y^2t, t^{-1}]$ . Think about  $t$  as being a constant which is allowed to vary in  $k$ . Then the extended Rees algebra of  $I$  can be viewed as a family of  $R$ -algebras, one for each value of  $t^{-1}$ . Let's explore some of the algebras in this family by plugging in values for  $t^{-1}$ :

- if  $t^{-1} = 0$ , which is ok to do since  $t^{-1}$  is not actually a unit, then we get

$$R[It, t^{-1}]|_{t^{-1}=0} = R[x^2t, y^2t] \cong \mathrm{gr}_I[t].$$

- If  $t^{-1} = 1$  then we get  $x^2t = x^2t \cdot 1 = x^2tt^{-1} = x^2 \in R$  and similarly  $y^2t = y^2 \in R$  so

$$R[It, t^{-1}]|_{t^{-1}=1} = R[x_R^2, y_R^2] \cong R.$$

In fact the same is true for every value  $t^{-1} \in k^\times$ .

The following lemma makes these observations rigorous.

**Lemma 6.19.** *There are isomorphisms*

$$R[It, t^{-1}]/(t^{-1}) \cong \mathrm{gr}_I(R) \quad \text{and} \quad R[It, t^{-1}]/(t^{-1} - 1) \cong R.$$

*Proof.* For the first isomorphism, since  $t$  is homogeneous, we can use the graded structure. We have

$$t^{-1}R[It, t^{-1}] = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus I \oplus I^2t \oplus I^3t^2 \oplus \cdots,$$

so matching the graded pieces, we see that

$$\begin{aligned} R[It, t^{-1}]/(t^{-1}) &= \frac{\cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \cdots}{\cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus I \oplus I^2t \oplus I^3t^2 \oplus \cdots} \\ &\cong R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \\ &= \text{gr}_I(R). \end{aligned}$$

For the second isomorphism, we consider the map  $R[It, t^{-1}] \rightarrow R$  given by sending  $t \mapsto 1$ . This is surjective, and the kernel is the set of elements  $a_mt^m + \cdots + a_nt^n$  such that  $a_m + \cdots + a_n = 0$ . We claim that this ideal is generated by  $(t^{-1} - 1)$ . We proceed by induction on  $n - m$ . The case  $n - m = 0$  corresponds to there being at most one nonzero term, say  $at^m$ , in which case  $at^m$  is in the kernel if and only if  $a = 0$ . In  $n - m = 1$ , we have an element of the form  $at^{n-1} - at^n$  for some  $a$ , which is of the form  $(at^n)(t^{-1} - 1)$ . For the inductive step, if  $a_m + \cdots + a_n = 0$ , write

$$a_mt^m + \cdots + a_nt^n = (a_mt^m + \cdots + (a_{n-1} + a_n)t^{n-1}) + (-a_nt^{n-1} + a_nt^n).$$

Observe that  $-a_nt^{n-1} + a_nt^n \mapsto -a_n + a_n = 0$ , and thus  $a_mt^m + \cdots + (a_{n-1} + a_n)t^{n-1}$  must also be in the kernel. The induction hypothesis now applies to both  $-a_nt^{n-1} + a_nt^n$  and  $a_mt^m + \cdots + (a_{n-1} + a_n)t^{n-1}$ , which must then be in  $(t^{-1} - 1)$ . Therefore, so is  $a_mt^m + \cdots + a_nt^n$ , and we are done.  $\square$

**Theorem 6.20.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring, and  $I \subseteq \mathfrak{m}$  an ideal. Then*

$$\dim(R) = \dim(R[It, t^{-1}]) - 1 = \dim(\text{gr}_I(R)).$$

*Proof.* First, let's show  $\dim(R) = \dim(R[It, t^{-1}]) - 1$ . By Lemma 6.10, we can reduce to the case when  $R$  is a domain by localizing at each of the minimal primes of  $R$ . In particular,  $R[It, t^{-1}]$  is also a domain.

By Lemma 6.19,  $R[It, t^{-1}]/(t^{-1} - 1) \cong R$ , so  $\dim(R[It, t^{-1}]) \geq \dim(R)$ . Also, since  $(t^{-1} - 1)$  is principal,  $\text{ht}(t^{-1} - 1) \leq 1$ , by Krull's Height Theorem. But  $R[It, t^{-1}]$  is a domain, so  $\text{ht}(t^{-1} - 1) = 1$ . By Theorem 3.20,

$$\dim R = \dim(R[It, t^{-1}]) - \text{ht}(t^{-1} - 1) = \dim(R[It, t^{-1}]) - 1.$$

Now, we claim that

$$Q = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus \mathfrak{m} \oplus It \oplus I^2t^2 \oplus \cdots = (\mathfrak{m}, It, t^{-1})R[It, t^{-1}]$$

is a maximal ideal of height  $\dim(R) + 1$  in  $R[It, t^{-1}]$ . The quotient ring is  $R/\mathfrak{m}$ , so it is clearly maximal. Given a chain  $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathfrak{m}$  of length  $h = \dim(R)$ , let  $\mathfrak{q}_i =$

$\mathfrak{p}_i R[t, t^{-1}] \cap R[It, t^{-1}]$ . Since  $\mathfrak{q}_i \cap R = \mathfrak{p}_i[t, t^{-1}] \cap R = \mathfrak{p}_i$ , this is a proper chain of primes in  $R[It, t^{-1}]$ . We have

$$\mathfrak{q}_h = \cdots \oplus \mathfrak{m}t^{-2} \oplus \mathfrak{m}t^{-1} \oplus \mathfrak{m} \oplus It \oplus I^2t^2 \oplus \cdots = (\mathfrak{m}, It)R[It, t^{-1}] \subsetneq Q$$

so the height of  $Q$  is at least  $\dim(R) + 1$ , and hence equal to  $\dim(R) + 1$  using the previous upper bound on the dimension.

For the last equality, since  $t^{-1}$  is a nonzerodivisor on  $R[It, t^{-1}]$ , we have

$$\dim(\operatorname{gr}_I(R)) \leq \dim(R[It, t^{-1}]) - 1.$$

For the other inequality, let  $\overline{Q} = Q/(t^{-1})$ . Then

$$\begin{aligned} \dim(\operatorname{gr}_I(R)) &\geq \dim(\operatorname{gr}_I(R)_{\overline{Q}}) = \dim(R[It, t^{-1}]_Q/(t^{-1})) \\ &\geq \dim(R[It, t^{-1}]_Q) - 1 \\ &= \operatorname{height}(Q) - 1 \\ &= \dim(R). \end{aligned}$$

□

## 6.2 Defining equations of Rees algebras

When  $I = (f_1, \dots, f_n)$  is an ideal in a ring  $R$ . Then  $\mathcal{R}(I) = R[f_1t, \dots, f_nt]$ . Thus  $\mathcal{R}(I)$  can be written as a quotient of the polynomial ring  $R[T_1, \dots, T_n]$ , by the quotient map

$$\begin{array}{ccc} R[T_1, \dots, T_n] & \xrightarrow{\pi} & \mathcal{R}(I) \\ T_i & \longmapsto & f_it \end{array}$$

If we can describe the generators of the kernel  $L$  of  $\pi$ , we will have a complete understanding of  $\mathcal{R}(I)$  as a finitely generated  $R$ -algebra. This homogeneous ideal  $L$  is given by

$$L = (F(T_1, \dots, T_n) \in R[T_1, \dots, T_n] \mid F(f_1, \dots, f_n) = 0).$$

We call a set of generators of  $L$  the **defining equations** of  $\mathcal{R}(I)$ . Unfortunately, finding defining equations for the Rees algebra of a given ideal is incredibly difficult in practice.

We can also view the Rees algebra of  $I$  as a quotient of the symmetric algebra. Recall that the **symmetric algebra**  $\operatorname{Sym}(I)$  of  $I$  is the quotient of the tensor algebra  $\bigoplus_{n \geq 0} I^{\otimes n}$  by the ideal generated by all elements of the form  $u \otimes v - v \otimes u$ . The symmetric algebra is also a quotient of  $R[T_1, \dots, T_n]$ , the quotient  $\operatorname{Sym}(I) \cong R[T_1, \dots, T_n]/L_1$ , where  $L_1$  is the ideal

$$L_1 = (a_1T_1 + \dots + a_nT_n \in R[T_1, \dots, T_n] \mid a_1, \dots, a_n \in R \text{ satisfy } a_1f_1, \dots, a_nf_n = 0).$$

As the notation suggests, this is indeed the ideal generated by the degree 1 elements in ideal  $L$  defining  $\mathcal{R}(I)$ . As a consequence, we can write  $\mathcal{R}(I)$  as a graded quotient of  $\operatorname{Sym}(I)$ , by taking  $R[T_1, \dots, T_n]/L_1 \twoheadrightarrow R[T_1, \dots, T_n]/L$ . Alternatively, the multiplication map on  $I \otimes I$  induces a surjective graded map  $\operatorname{Sym} \twoheadrightarrow \mathcal{R}(I)$ .

**Definition 6.21.** An ideal  $I$  is said to be of **linear type** if the natural surjective graded map  $\text{Sym} \rightarrow \mathcal{R}(I)$  is an isomorphism. Equivalently, the kernel  $L$  of the canonical projection  $\pi : R[T_1, \dots, T_n] \rightarrow \mathcal{R}(I)$  satisfies  $L_1 = L$ .

So the Rees algebra of an ideals of linear type is determined by defining equations of degree 1. The best case scenario when we want to study the Rees algebra of  $I$  is that  $I$  is of linear type. For ideals that are not of linear type, describing  $\mathcal{R}(I)$  is typically very difficult.

### 6.3 Stability of associated primes of powers

Rees algebras can be quite useful when studying symbolic powers. The first application we will see is that the set of primes associated to some power of a given ideal is finite.

Before we prove this, we should note that this is not at all obvious. In fact, the behavior of the function  $n \mapsto \text{Ass}(I^n)$  can be quite complicated. For example, the number of associated primes of the powers of a prime ideal can be arbitrarily large.

**Example 6.22** (Kim–Swanson, [KS19]). Let  $k$  be any field, and fix positive integers  $m$  and  $v$ . There exists an  $m$ -generated prime ideal  $P$  of height  $m - 1$  in the polynomial ring  $k[x_1, \dots, x_{mv}]$  in  $mv$  variables such that  $P^e$  has  $v^m$  associated primes for all  $e \geq 2$ .

Moreover, it is *not* true that  $\text{Ass}(I^n) \subseteq \text{Ass}(I^{n+1})$  for all  $n$ ; though by [Bro79], this is *eventually* true.

**Definition 6.23.** We say an ideal  $I$  has the **persistence property** if  $\text{Ass}(I^n) \subseteq \text{Ass}(I^{n+1})$  for all  $n$ .

Here are some examples of ideals without the persistence property.

**Example 6.24** (Sathaye, see the example on page 2 in [McA83]). Let  $k$  be a field and fix  $s \geq 1$ . In the polynomial ring  $S = k[x, z_1, \dots, z_{2s}]$ , consider the ideal

$$I = (xz_{2i-1}^{2i-1} - z_{2i}^{2i} \mid 1 \leq i \leq s) + (z_j^j z_i \mid 1 \leq i, j \leq 2s),$$

and let  $R = S/I$ . Let  $P = (z_1, \dots, z_{2s}) \subseteq \mathfrak{m} = (x, z_1, \dots, z_{2s})$  in  $R$ . For all  $1 \leq n \leq 2s$ , one can show that  $\mathfrak{m} \in \text{Ass}(I^n)$  if  $n$  is odd, but  $\mathfrak{m} \notin \text{Ass}(I^n)$  whenever  $n$  is even. In particular,  $\text{Ass}(I^n) \not\subseteq \text{Ass}(I^{n+1})$  for all odd  $3 \leq n \leq 2s$ .

One can even construct monomial ideals without the persistence property.

**Example 6.25** (Bandari–Herzog–Hibi [BHH14]). Fix a field  $k$  and an integer  $s \geq 0$ , and let  $\mathfrak{m}$  be the homogeneous maximal ideal in  $R = k[a, b, c, d, x_1, y_1, \dots, x_s, y_s]$ . The monomial ideal

$$I = (a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d, a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4, x_sy_s^2, b^4x_s^2y_s)$$

has  $\mathfrak{m} \in \text{Ass}(I^n)$  for all odd  $n \leq 2s + 1$ , but  $\mathfrak{m} \notin \text{Ass}(I^n)$  for  $n \leq 2s$  even.

In [BHH14], the authors showed even more: that function  $n \mapsto \text{depth}(R/I^n)$  can have arbitrarily many local maxima. Recall that  $\mathfrak{m} \in \text{Ass}(J)$  if and only if  $\text{depth}(R/J) = 0$ , by Theorem 3.105. The function  $n \mapsto \text{depth}(R/I^n)$  then encodes information about the associated primes of  $I^n$ . This depth function must necessarily be constant for  $n \gg 0$ . Hà, Nguyen, Trung, and Trung [HNTT21] recently settled a conjecture of Herzog and Hibi by showing that  $n \mapsto \text{depth}(R/I^n)$  can be any function  $\mathbb{N} \rightarrow \mathbb{N}$  that is eventually constant.

**Remark 6.26.** Note that the  $R$ -modules  $I^n/I^{n+1}$  are all killed by  $I$ , so by Lemma 1.50

$$\text{Ass}_{R/I}(I^n/I^{n+1}) = \{P/I \mid P \in \text{Ass}_R(I^n/I^{n+1})\}.$$

With this in mind, we will abuse notation and identify the associated primes of  $I^n/I^{n+1}$  as an  $R$ -module and as an  $R/I$ -module.

Any ideal of positive height with a Cohen-Macaulay associated graded ring has the persistence property.

**Theorem 6.27** (Morey, Lemma 2.3 in [Mor99]). *Let  $R$  be a Cohen-Macaulay ring and let  $I$  be an ideal with  $\text{ht}(I) \geq 1$  and such that the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay. Then the sets of associated primes  $\text{Ass}(I^n)$  form an ascending chain. Moreover, the sets of associated primes  $\text{Ass}(I^n/I^{n+1})$  also form an ascending chain.*

*Proof.* Since  $\text{ht}(I) \geq 1$ , we must have  $\dim(\text{gr}_I(R)) \geq 1$ , by Theorem 6.20. Since  $\text{gr}_I(R)$  is Cohen-Macaulay, its homogeneous maximal ideal must have depth at least 1. In particular, we can choose a regular element that is homogeneous of degree 1, say  $a^* := a + I^2 \in I/I^2$  for some  $a \in I$ . Given  $P \in \text{Ass}(I^n/I^{n+1})$ , fix  $c \in I^n/I^{n+1}$  such that  $P = (0 :_{R/I} c)$ . Since  $a^*$  is regular,  $a^*c \neq 0$ , and  $P = (0 :_{R/I} a^*c)$ . But now  $a^*c + I^{n+2} \in I^{n+1}/I^{n+2}$ , which shows that  $P \in \text{Ass}(I^{n+1}/I^{n+2})$ .

Since  $I^0 = R$  by convention, for  $n = 1$  we have  $I^{n-1}/I^n = R/I$  and  $\text{Ass}(I) = \text{Ass}(R/I) = \text{Ass}(I^{n-1}/I^n)$ . We claim that  $\text{Ass}(R/I^n) = \text{Ass}(I^n/I^{n+1})$  for all  $n \geq 1$ , and since we have shown the case  $n = 1$ , we now proceed by induction.

Suppose that  $\text{Ass}(R/I^t) = \text{Ass}(I^{t-1}/I^t)$  for some  $t$ . Applying Lemma 1.40 to the short exact sequence

$$0 \longrightarrow I^t/I^{t+1} \longrightarrow R/I^t \longrightarrow R/I^t \longrightarrow 0,$$

we obtain

$$\text{Ass}(R/I^{t+1}) \subseteq \text{Ass}(I^t/I^{t+1}) \cup \text{Ass}(R/I^t).$$

Then given any  $P \in \text{Ass}(R/I^{t+1})$ , either  $P \in \text{Ass}(I^t/I^{t+1})$  or  $P \in \text{Ass}(R/I^t) = \text{Ass}(I^{t-1}/I^t) \subseteq \text{Ass}(I^t/I^{t+1})$ . By induction, we conclude that  $\text{Ass}(R/I^n) = \text{Ass}(I^n/I^{n+1})$  for all  $n \geq 1$ . Since the sets  $\text{Ass}(I^n/I^{n+1})$  form an ascending chain, so do the sets  $\text{Ass}(R/I^n)$ .  $\square$

So when is the associated graded ring Cohen-Macaulay? In [Hun81], Huneke showed that if  $R$  is a Cohen-Macaulay ring and  $I$  is generated by  $n + 1$  elements and  $\text{ht}(I) = n$ , if  $R/I$  is Cohen-Macaulay then  $\text{gr}(I)$  is Cohen-Macaulay. Moreover, the converse holds if  $R$  is Gorenstein.

We are now ready to show that despite the fact that  $\text{Ass}(I^n)$  can have a very complicated behavior, there are only finitely many primes that are associated to  $I^n$  for some  $n$ .



**Theorem 6.28** (Ratliff, [Rat76]). *Let  $I$  be an ideal in a noetherian ring  $R$ . Then the set*

$$\mathcal{A} = \bigcup_{n \geq 1} \text{Ass}(I^n)$$

*is finite.*

*Proof.* Fix  $n \geq 1$ . In the extended Rees algebra  $S = R[It, t^{-1}]$  of  $I$ , the ideal

$$t^{-n}R[It, t^{-1}] = \bigoplus_{k \geq n+1} t^{-k} \oplus \bigoplus_{k \geq 0} I^k t^{k-n}$$

has degree 0 piece  $I^n$ , so  $t^{-n}R[It, t^{-1}] \cap R = I^n$ . On the other hand,  $t^{-1}$  is a regular element in  $S = R[It, t^{-1}]$ , so by Theorem 3.107

$$\text{Ass}_S(t^{-n}S) = \text{Ass}_S(t^{-1}S).$$

Now let  $Q \in \text{Ass}_R(I^n)$ . Since  $t^{-n}S \cap R = I^n$ , we have inclusions

$$R/Q \hookrightarrow R/I^n \hookrightarrow S/t^{-n}S,$$

so  $Q \in \text{Ass}_R(S/t^{-n}S)$ , when we view  $S/t^{-n}S$  as an  $R$ -module. By Theorem 1.49,

$$\text{Ass}_R(S/t^{-n}S) = \{P \cap R \mid P \in \text{Ass}_S(S/t^{-n}S)\}.$$

We conclude that  $Q = P \cap R$  for some prime  $P$  in  $S$  that is associated to the ideal generated by  $t^{-n}$ . But as we saw above,

$$\text{Ass}_S(t^{-n}S) = \text{Ass}_S(t^{-1}S),$$

and since  $S$  is a noetherian ring, by Remark 6.8,  $\text{Ass}_S(t^{-1}S)$  is a finite set of primes, by Theorem 1.44. Then

$$\mathcal{A} = \bigcup_{n \geq 1} \text{Ass}(I^n) \subseteq \{P \cap R \mid P \in \text{Ass}_S(t^{-1}S)\}$$

is a finite set. □

As we mentioned in Chapter 1, more is true: Brodmann [Bro79] showed that in fact,  $\text{Ass}(I^n)$  eventually stabilizes, meaning that there exists  $n$  such that  $\text{Ass}(I^n) = \text{Ass}(I^m)$  for all  $m \geq n$ . Brodmann's proof, however, is rather technical, as it uses a technical lemma by Ratliff [Rat76] involving superficial elements, which we have not discussed. Instead, we will follow McAdam's proof [McA83].

**Corollary 6.29.** *Let  $I$  be an ideal in a noetherian ring  $R$ . Then*

$$\bigcup_{n \geq 0} \text{Ass}(I^n/I^{n+1})$$

*is a finite set.*

As we discussed in Remark 6.26, the associated primes of  $I^n/I^{n+1}$  as a module over  $R$  can be identified with its associated primes as a module over  $R$ , so we are really showing that both

$$\bigcup_{n \geq 0} \text{Ass}_R(I^n/I^{n+1}) \quad \text{and} \quad \bigcup_{n \geq 0} \text{Ass}_{R/I}(I^n/I^{n+1})$$

are finite sets.

*Proof.* Since  $I^n/I^{n+1}$  is a submodule of  $R/I^{n+1}$ , by Lemma 1.40 we have

$$\text{Ass}_R(I^n/I^{n+1}) \subseteq \text{Ass}_R(R/I^{n+1}).$$

Recall that we write  $\text{Ass}(I^n)$  for  $\text{Ass}(R/I^{n+1})$ , so we have

$$\bigcup_{n \geq 0} \text{Ass}_R(I^n/I^{n+1}) \subseteq \bigcup_{n \geq 0} \text{Ass}(I^n),$$

which is finite by Theorem 6.28. The  $R$ -modules  $I^n/I^{n+1}$  are all killed by  $I$ , so by Lemma 1.50

$$\text{Ass}_{R/I}(I^n/I^{n+1}) = \{P/I \mid P \in \text{Ass}_R(I^n/I^{n+1})\}.$$

Therefore,

$$\bigcup_{n \geq 0} \text{Ass}_{R/I}(I^n/I^{n+1})$$

is a finite set. □

**Lemma 6.30.** *Let  $R = \bigoplus_{n \geq 0} R_n$  be a graded ring with  $R = R_0[R_1]$ , meaning that  $R$  is generated as an  $R_0$ -algebra by the elements of degree 1. There exists an integer  $n$  such that for all  $s \geq n$ ,*

$$(0 : R_1) \cap R_n = (0).$$

*In other words, all elements in  $R$  that annihilate the degree 1 piece have degree at most  $n$ .*

*Proof.* The colon of two homogeneous ideals in a graded ring is a homogeneous ideal (exercise!), so  $J = (0 : R_1)$  is a homogeneous ideal. Since  $R$  is generated by elements of degree 1,  $J$  is also the annihilator of the ideal  $R_+$  generated by all positive degree elements.

Pick homogeneous generators  $F_1, \dots, F_s$  for  $J$ , and set  $n := 1 + \max\{\deg(F_i)\}$ . Any homogeneous element  $F \in J$  can be written as  $F = r_1 F_1 + \dots + r_s F_s$  for some homogeneous  $r_i \in R$ . In fact, if  $\deg(F) \geq n$ , then necessarily we must have  $\deg(r_1), \dots, \deg(r_s) \geq 1$ . But each  $F_i \in J$  must annihilate any homogeneous element of positive degree, so  $r_i F_i = 0$  for all  $i$ . We conclude that  $F = 0$ , and thus the nonzero homogeneous elements in  $J$  must all have degree less than  $n$ . □

We will be applying this lemma to the associated graded ring of an ideal  $I$  in a noetherian ring  $R$ , which is a noetherian ring by Remark 6.13. More precisely, we will use the following corollary of Lemma 6.30: given an ideal  $I$  in a noetherian ring  $R$ , there exists an integer  $n$  such that for all  $s \geq n$ ,

$$(0 :_{\text{gr}_I(R)} [\text{gr}_I(R)]_1) \cap [\text{gr}_I(R)]_s = (0).$$

Lemma 6.30 applies here because  $\text{gr}_I(R)$  is indeed generated in degree 1: every element in  $I^n$  can be written as a linear combination of products of  $n$  elements in  $I$ , so any element in  $[\text{gr}_I(R)]_d = I^d/I^{d+1}$  is a linear combination of products of elements in  $[\text{gr}_I(R)]_1 = I/I^2$ .

**Lemma 6.31.** *Let  $I$  be an ideal in a noetherian ring  $R$ . There exists an integer  $N$  such that*

$$\text{Ass}_{R/I}(I^n/I^{n+1}) \subseteq \text{Ass}_{R/I}(I^{n+1}/I^{n+2})$$

for all  $n \geq N$ .

*Proof.* Let  $N$  be as in Lemma 6.30: all homogeneous elements in  $\text{gr}_I(R)$  that kill  $[\text{gr}_I(R)]_1$  must have degree at most  $N$ . Fix  $n > N$ , and a prime ideal  $P \in \text{Ass}_{R/I}(I^n/I^{n+1})$ . By definition, there exists an element  $c \in I^n$ ,  $c \notin I^{n+1}$ , which we can think of as giving a homogeneous element  $c + I^{n+1} \in I^n/I^{n+1}$  such that

$$P = (0 + I^{n+1} :_{R/I} c + I^{n+1}).$$

Now consider the submodule of  $I^{n+1}/I^{n+2}$  given by  $(c + I^{n+1})I/I^2$ . We claim that

$$P = (0 + I^{n+1} :_{R/I} c + I^{n+1}) = (0 + I^{n+1} :_{R/I} (c + I^{n+1})I/I^2).$$

The containment  $\subseteq$  is immediate, so we just need to show that

$$(r + I) \cdot (c + I^{n+1})I/I^2 = 0 \implies (r + I) \cdot (c + I^{n+1}) = 0.$$

First, note that

$$(r + I) \cdot (c + I^{n+1}) = rc + I^{n+1} \in I^n/I^{n+1},$$

so our assumption says that

$$rc + I^{n+1} \in (0 + I^{n+2} :_{R/I} I/I^2).$$

We can think of  $R/I$  as the degree 0 piece of  $\text{gr}_I(R)$ , and of  $I/I^2$  as the degree 1 piece of  $\text{gr}_I(R)$ , so we must have

$$rc + I^{n+1} \in (0 :_{\text{gr}_I(R)} [\text{gr}_I(R)]_1).$$

Moreover, we can now view  $rc + I^{n+1}$  as a homogeneous element in  $\text{gr}_I(R)$  of degree  $n$ , but by construction any element in  $\text{gr}_I(R)$  that kills  $[\text{gr}_I(R)]_1$  must have degree less than  $n$ . We conclude that  $rc + I^{n+1} = 0$ , so indeed we proved our claim that

$$r + I \in (0 :_{R/I} c + I^{n+1}).$$

So we have shown that

$$P = (0 + I^{n+1} :_{R/I} c + I^{n+1}) = (0 + I^{n+1} :_{R/I} (c + I^{n+1})I/I^2).$$

Our goal was to show that  $P$  is associated to  $I^{n+1}/I^{n+2}$ . By Lemma 1.36, it is sufficient to show this holds after we localize at  $P$ , so we may as well assume that  $P$  is the unique maximal ideal of  $R/I$ .

Now to find an explicit element that is annihilated by  $P$ , we claim that there exists an element  $r + I^2 \in I/I^2$  such that

$$cr + I^{n+2} = (r + I^2)(c + I^{n+1}) \neq 0.$$

If no such element exists, then  $(c + I^{n+1})I/I^2 = 0$ , so

$$P = (0 + I^{n+1} :_{R/I} (c + I^{n+1})I/I^2) = R/I,$$

which is absurd. Now given  $r + I^2 \in I/I^2$  with  $cr + I^{n+2} \neq 0$ ,

$$P = (0 + I^{n+1} :_{R/I} (c + I^{n+1})I/I^2) \subseteq (0 + I^{n+1} :_{R/I} cr + I^{n+2}) \neq R/I,$$

but since  $P$  is maximal, we must have

$$P = (0 + I^{n+1} :_{R/I} (c + I^{n+1})I/I^2) = (0 + I^{n+1} :_{R/I} cr + I^{n+2}).$$

Therefore,  $P$  is the annihilator of  $cr + I^{n+2} \in I^{n+1}/I^{n+2}$ , and thus  $P$  is associated to  $I^{n+1}/I^{n+2}$ .  $\square$

**Theorem 6.32.** *Let  $I$  be an ideal in the noetherian ring  $R$ . There exists  $N$  such that for all  $n \geq N$ ,*

$$\text{Ass}(I^n/I^{n+1}) = \text{Ass}(I^{n+1}/I^{n+2}).$$

*Proof.* By Lemma 1.50,  $P \in \text{Ass}_R(I^n/I^{n+1})$  if and only if  $P/I \in \text{Ass}_{R/I}(I^n/I^{n+1})$ , so it is sufficient to show that

$$\text{Ass}_{R/I}(I^n/I^{n+1}) = \text{Ass}_{R/I}(I^{n+1}/I^{n+2}).$$

By Lemma 6.31, there exists  $N$  such that

$$\text{Ass}_{R/I}(I^n/I^{n+1}) \subseteq \text{Ass}_{R/I}(I^{n+1}/I^{n+2}) \subseteq \dots$$

for all  $n \geq N$ . These are all contained in

$$\bigcup_{t \geq 1} \text{Ass}_{R/I}(I^t/I^{t+1}),$$

which is finite by Corollary 6.29. Whenever we have an increasing sequence of subsets of a finite set, it must eventually stabilize, and thus

$$\text{Ass}_{R/I}(I^n/I^{n+1}) = \text{Ass}_{R/I}(I^{n+1}/I^{n+2})$$

for all  $n \geq N$ .  $\square$

We are finally ready to prove that the associated primes of  $I^n$  stabilize for large  $n$  [Rat76].

**Theorem 6.33** (Stability of associated primes of powers). *Let  $I$  be an ideal in a noetherian ring  $R$ . There exists  $N$  such that  $\text{Ass}(I^n) = \text{Ass}(I^N)$  for all  $n \geq N$ .*

*Proof.* For all  $n$ , we have a short exact sequence

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \longrightarrow 0,$$

so by Lemma 1.40 we have

$$\text{Ass}(I^n/I^{n+1}) \subseteq \text{Ass}(R/I^{n+1}) \subseteq \text{Ass}(R/I^n) \cup \text{Ass}(I^n/I^{n+1}).$$

Notice here we chose to use the module notation (as opposed to the ideal notation) for associated primes to avoid confusion.

By Theorem 6.32, there exists  $N$  such that for all  $n \geq N$ ,

$$\text{Ass}(I^n/I^{n+1}) = \text{Ass}(I^N/I^{N+1}).$$

For any  $n \geq N + 1$ , we have

$$\text{Ass}(I^n/I^{n+1}) \subseteq \text{Ass}(R/I^{n+1}) \subseteq \text{Ass}(R/I^n) \cup \text{Ass}(I^n/I^{n+1}),$$

but since

$$\text{Ass}(I^n/I^{n+1}) = \text{Ass}(I^{n-1}/I^n) \subseteq \text{Ass}(R/I^n),$$

we can now say that for all  $n \geq N + 1$

$$\text{Ass}(R/I^{n+1}) \subseteq \text{Ass}(R/I^n).$$

This we have a chain

$$\cdots \supseteq \text{Ass}(R/I^{N+t}) \supseteq \text{Ass}(R/I^{N+t-1}) \supseteq \text{Ass}(R/I^{N+t-2}) \supseteq \cdots \supseteq \text{Ass}(R/I^N),$$

but since  $R$  is noetherian,  $\text{Ass}(R/I^N)$  is finite, by Theorem 1.44. Therefore, only finitely many of these inclusions can be strict, and thus for all  $t$  sufficiently large, we must have  $\text{Ass}(R/I^{N+t}) = \text{Ass}(R/I^{N+t+1})$ .  $\square$

Finding the value  $N$  at which  $\text{Ass}(I^n)$  stabilizes is not easy. For some results on this stability index, see Susan Morey's work in [Mor99].

# Chapter 7

## When is $I^{(n)} = I^n$ ?

Given a noetherian ring  $R$ , which ideals  $I$  satisfy  $I^n = I^{(n)}$  for all  $n$ ? This is one of the most fundamental questions we may ask about symbolic powers, and yet there is much that we do not know.

When  $R$  is a ring of dimension 0, every prime is maximal, and thus there are no ideals with embedded primes; therefore, every ideal  $I$  satisfies  $I^n = I^{(n)}$  for all  $n$ . When  $R$  is a domain of dimension 1, every nonzero ideal has height 1, so again there are no ideals with no embedded primes, and every ideal  $I$  satisfies  $I^n = I^{(n)}$  for all  $n$ . If we drop the domain assumption, then a ring of dimension 1 that is not a domain might have minimal primes whose symbolic powers are not the powers. For example, consider the ring  $R = k[x, y]/(xy, y^3)$ , which has dimension 1, but is not a domain, and the height 0 prime  $P = (y)$ . Then  $P^2 = (y) \cap (x, y^2)$  has an embedded component, and thus  $P^2 \neq P^{(2)}$ .

When  $R$  has dimension 2, even if  $R$  is a domain, things can get even more interesting: for example, we saw in Example 2.3 that the symbolic powers of the height 1 prime  $P = (x, z)$  in the hypersurface  $k[x, y, z]/(xy - z^c)$  are interesting; in fact, one can show in that example that  $P^{(n)} \neq P^n$  for all  $n$ . If we consider a UFD, all the height 1 primes must be principal, and thus generated by a regular element, so their symbolic powers are the ordinary powers, by Theorem 3.107. In a UFD of dimension 1, we must then have  $P^{(n)} = P^n$  for all  $n \geq 1$  and all primes  $P$ . In a UFD of dimension 2, all primes are either maximal or have height 1, so  $P^{(n)} = P^n$  for all  $n \geq 1$  and all primes  $P$ . But in a UFD of dimension 3 – in fact, even over a regular ring of dimension 3 – we no longer have equality between symbolic and ordinary powers of all prime ideals. For example, in  $k[x, y, z]$ , we saw in Example 2.4 that the height 2 prime  $P$  defining the curve  $(t^3, t^4, t^5)$  has  $P^{(2)} \neq P^2$ .

In this chapter, we will discuss some of the many interesting variations on this question:

- Given a ring  $R$ , what are necessary or sufficient conditions on  $I$  so that  $I^{(n)} = I^n$  for all  $n \geq 1$ ?
- Given an ideal  $I$ , for which  $n$  do we have  $I^{(n)} = I^n$ ?
- Is there a constant  $d$ , perhaps depending on the ideal  $I$ , such that  $I^{(n)} = I^n$  for all  $n \leq d$  implies  $I^{(n)} = I^n$  for all  $n$ ?

We will also give a new, more sophisticated proof of the equality of symbolic and ordinary powers of ideals generated by a regular sequence in a Cohen-Macaulay ring, and collect some interesting results and open questions.

## 7.1 Complete intersections

We now have the tools to give a shorter, cleaner proof of the equality of symbolic and ordinary powers of ideals generated by regular sequences. To do that, we will consider the associated graded ring of  $I$ .

**Theorem 7.1.** *Let  $I = (f_1, \dots, f_t)$  be an ideal in a noetherian ring  $R$ . If  $f_1, \dots, f_t$  is a regular sequence, then  $\text{gr}_I(R)$  is a polynomial ring. More precisely,  $\text{gr}_I(R) \cong R/I[T_1, \dots, T_n]$ .*

*Proof.* Consider the degree 0 ring homomorphism induced by

$$\begin{aligned} R/I[T_1, \dots, T_n] &\xrightarrow{\pi} \text{gr}_I(R) \\ r + I \in R/I &\longmapsto r + I \in R/I \\ T_i &\longmapsto f_i + I \in I/I^2 \end{aligned}$$

which is also a homomorphism of  $R$ -modules. By construction, and since  $I = (f_1, \dots, f_n)$ ,  $\pi$  is surjective. Notice that  $\pi$  is an isomorphism in degree 0. Now consider a homogeneous element  $F(T_1, \dots, T_n) \in \ker \pi$ , which we may as well assume has positive degree. After choosing lifts of the coefficients of  $F$  to  $R$ , we may think of  $F$  as a polynomial in  $R[T_1, \dots, T_n]$ ; any two such choices of lifts of  $F$  will lead to the same image in  $\text{gr}_I(R)$ . If  $F$  is in the kernel of  $\pi$ , that means that  $F$  has degree  $d$  but  $F(f_1, \dots, f_n)$  is zero in  $I^d/I^{d+1}$ , meaning  $F(f_1, \dots, f_n) \in I^{d+1}$ . By Theorem 3.50, this implies that the coefficients of  $F$  are all in  $I$ . When we view  $F$  as a polynomial with coefficients in  $R/I$ , this says that  $F = 0$ . Therefore,  $\pi$  is an isomorphism.  $\square$

Here is our second proof of Theorem 3.107.

**Theorem.** *Let  $R$  be a noetherian ring and consider an ideal  $I = (f_1, \dots, f_t)$ . If  $f_1, \dots, f_t$  is a regular sequence, then  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

*Proof.* By Theorem 7.1, we have an isomorphism  $\text{gr}_I(R) \cong R/I[T_1, \dots, T_n]$ . Therefore, as an  $R$ -module  $I^d/I^{d+1}$  is isomorphic to a sum of copies of  $R/I$ . In fact,  $I^d/I^{d+1} \cong (R/I)^{\binom{n+d-1}{d}}$ . Therefore, by Corollary 1.43

$$\text{Ass}(I^d/I^{d+1}) = \text{Ass}(R/I),$$

which we have been denoting by  $\text{Ass}(I)$ . On the other hand, we have short exact sequences

$$0 \longrightarrow I^d/I^{d+1} \longrightarrow R/I^{d+1} \longrightarrow R/I^d \longrightarrow 0,$$

so by Lemma 1.40

$$\text{Ass}(I) = \text{Ass}(I^d/I^{d+1}) \subseteq \text{Ass}(I^{d+1}) \subseteq \text{Ass}(I^d) \cup \text{Ass}(I^d/I^{d+1}) = \text{Ass}(I^d) \cup \text{Ass}(I).$$

Simplifying, we have

$$\text{Ass}(I) \subseteq \text{Ass}(I^{d+1}) \subseteq \text{Ass}(I^d) \cup \text{Ass}(I).$$

When  $d = 1$ , we get  $\text{Ass}(I^2) = \text{Ass}(I)$ . Proceeding by induction,  $\text{Ass}(I^n) = \text{Ass}(I)$  for all  $n$ , and thus  $I^{(n)} = I^n$  for all  $n$ .  $\square$

The equality  $I^n = I^{(n)}$  for all  $n$  does not imply  $I$  is generated by a regular sequence. Later we will give other classes of ideals whose symbolic powers are all equal to the powers. We will see some examples soon.

## 7.2 The associated graded ring and $I^n = I^{(n)}$

Our last proof of the equality of symbolic and ordinary powers of complete intersection ideals relied on the associated graded ring of  $I$ . The associated graded ring is closely connected to the equality of symbolic and ordinary powers in a more general way.

**Theorem 7.2.** *Let  $I$  be an ideal in a noetherian ring  $R$ . If  $\text{gr}_I(R)$  is a domain, then  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

*Proof.* Suppose that  $I^{(n)} \neq I^n$  for some  $n \geq 2$ . Consider an element  $a \in I^{(n)}$  but not in  $I^n$ . By definition, as described in Remark 2.8, there exists some  $s$  not in any associated prime of  $I$  such that  $sa \in I^n$ . In particular,  $s \notin I$ .

If we consider the associated graded ring  $\text{gr}_I(R)$ , the fact that  $s \notin I$  means that the homogeneous element  $s + I \in R/I$  of degree 0 in  $\text{gr}_I(R)$  is nonzero. Let  $m$  be such that  $a \in I^m$  but  $a \notin I^{m+1}$ , which necessarily satisfies  $m < n$  since  $a \notin I^n$ . In the associated graded ring  $\text{gr}_I(R)$ , the homogeneous element  $a + I^m \in I^m/I^{m+1}$  is also nonzero. But the product of  $s + I$  in degree 0 and  $a + I^m$  in degree  $m$  is the homogeneous element

$$(s + I)(a + I^m) = sa + I^{m+1} \in I^m/I^{m+1}$$

also in degree  $m$ , which is 0 in  $I^m/I^{m+1}$  since  $sa \in I^n \subseteq I^{m+1}$ . Therefore,  $\text{gr}_I(R)$  is not a domain.  $\square$

**Theorem 7.3.** *Let  $P$  be a prime ideal in a noetherian ring  $R$ . If  $R_P$  is regular, then  $\text{gr}_P(R)$  is a domain if and only if  $P^{(n)} = P^n$  for all  $n \geq 1$ .*

*Proof.* One implication is Theorem 7.2: if  $\text{gr}_P(R)$  is a domain, then  $P^{(n)} = P^n$  for all  $n \geq 1$ . For the other implication, suppose that  $R_P$  is regular and that  $P^{(n)} = P^n$  for all  $n \geq 1$ . For  $s \notin P$ , if  $sa \in P^n$  for some  $a \in R$  and some  $n \geq 1$  then  $a \in P^{(n)} = P^n$ . Translating this into the associated graded ring  $\text{gr}_P(R)$ , the nonzero homogeneous element  $s + P \in R/P$  of degree 0 has the property that for any homogeneous element  $a + P^n \in P^{n-1}/P^n$ ,

$$(s + P)(a + P^n) = 0 \implies sa + P^n = 0 \implies sa \in P^n \implies a \in P^n \implies a + P^n = 0.$$

Therefore, all the nonzero homogeneous elements of degree 0 in  $\text{gr}_P(R)$  are regular in  $\text{gr}_P(R)$ .

On the other hand, the nonzero homogeneous elements in  $\text{gr}_P(R)$  of degree 0 form a multiplicatively closed set, since they correspond to the image of  $R \setminus P$  in  $R/P$  and  $P$  is prime. Let  $W$  be this multiplicatively closed set. The localization of  $\text{gr}_P(R)$  at  $W$  is isomorphic to the localization  $\text{gr}_P(R)R_P$  of the  $R$ -module  $\text{gr}_P(R)$  at the prime  $P$ . Localization commutes with direct sums and quotients, so in fact these localizations are isomorphic to the associated graded ring  $\text{gr}_{P_P}(R_P)$  of the maximal ideal  $P_P$  in  $R_P$ . Since  $R_P$  is a regular ring, by Theorem 3.70 the ideal  $P_P$  is generated by a regular sequence. By Theorem 7.1,  $\text{gr}_{R_P}(R_P)$  is a polynomial ring, and thus a domain.

Now this domain is the localization of  $S = \text{gr}_P(R)$  at the multiplicatively closed set  $W$ . If  $S$  has zerodivisors, say  $st = 0$  for some nonzero  $s, t \in S$ , then  $\frac{s}{1}$  and  $\frac{t}{1}$  are still zerodivisors in  $W^{-1}S$  unless  $\frac{t}{1} = 0$  or  $\frac{s}{1} = 0$ . But all the elements in  $W$  are regular in  $S$ , so by Remark 1.25 the localization of  $S$  at  $W$  is injective. Therefore, any zerodivisor in  $S$  gives rise to a zerodivisor in  $W^{-1}S$ . Since  $W^{-1}S$  is a domain, we conclude that  $\text{gr}_P(R) = S$  is also a domain.  $\square$



In the proof of Theorem 7.3, we made a simple but useful observation. Before we can explore it further, we need to recall some elementary definitions.

**Definition 7.4.** Let  $R$  be a ring and  $M$  be an  $R$ -module. An element  $m \in M$  is **torsion** if  $rm = 0$  for some element  $r \in R$  that is regular on  $R$ . An  $R$ -module  $M$  is **torsion-free** if there are no nonzero elements in  $R$  that torsion.

Since the union of the associated primes of a module is its set of zerodivisors, a module is torsion free if and only if its associated primes are contained in the associated primes of  $R$ .

**Definition 7.5.** A ring  $R$  is **reduced** if it has no nonzero nilpotents; equivalently,  $\sqrt{(0)} = (0)$ , which can be rephrased as saying that  $(0)$  is a radical ideal.

**Remark 7.6.** Let  $I$  be a radical ideal in a noetherian ring  $R$ . The condition  $I^{(n)} = I^n$  for all  $n$  is equivalent to saying that for every  $s \in R$  that is not in any minimal prime of  $I$ , and every  $r \in R$ , if  $sr \in I^n$ , then  $r \in I^n$ . Since  $I$  is radical, an element not in any minimal prime of  $I$  corresponds to an element in  $R/I$  that is regular. We can rewrite this in terms of the associated graded ring  $\text{gr}_I(R)$ : for every  $n \geq 1$ , every regular element  $s + I \in R/I$ , and every  $r + I^n \in I^{n-1}/I^n$ ,

$$(s + I)(r + I^n) = 0 \implies r + I^n = 0.$$

Equivalently, this says that  $\text{gr}_I(R)$  is torsion-free over  $R/I$ . So  $I^{(n)} = I^n$  for all  $n \geq 1$  if and only if  $\text{gr}_I(R)$  is torsion-free over  $R/I$ .

**Theorem 7.7** (Huneke–Simis–Vasconcelos, 1984, Corollary 1.10 in [HSV89]). *Let  $R$  be a Cohen-Macaulay ring and  $I$  an ideal in  $R$  of finite projective dimension. Then  $\text{gr}_I(R)$  is reduced if and only if it is torsion-free over  $R/I$ .*

*Proof.* Suppose that  $\text{gr}_I(R)$  is reduced. The key technical point of the proof is [HSV89, Theorem 1.2], which says that if  $Q$  is a minimal prime of  $\text{gr}_I(R)$ , then  $Q \cap R/I = q/I$  for some minimal prime of  $I$ . The proof is rather technical, and we will skip it.

Since  $\text{gr}_I(R)$  is reduced, every associated prime of  $\text{gr}_I(R)$  is minimal. If  $r$  is a regular element in  $R/I$ , it is not in any minimal prime of  $R/I$ , and thus it cannot be in any associated prime of  $\text{gr}_I(R)$ . Therefore,  $\text{gr}_I(R)$  is torsion-free over  $R/I$ .

Suppose that  $\text{gr}_I(R)$  is torsion-free over  $R/I$ . Let  $Q \in \text{Ass}(\text{gr}_I(R))$  and  $Q \cap R/I = q/I$ . Since  $\text{gr}_I(R)$  is torsion-free over  $R/I$ , this prime  $q$  in  $R$  must be in  $\text{Ass}(R/I)$ . But  $I$  is radical, so all of its associated primes are minimal, and thus  $q$  is minimal over  $I$ , and thus  $I_q = q_q$ . Since  $I$  has finite projective dimension, so does  $q_q$ , but then by Theorem 3.70  $R_q$  must be regular. Therefore,  $q_q$  is generated by a regular sequence, and by Theorem 7.1,  $\text{gr}_{R_P}(R_P)$  is a polynomial ring, and thus a domain.

Localization commutes with direct sums, so  $\text{gr}_{q_q}(R_q) \cong \text{gr}_{I_q}(R_q)$ , and  $\text{gr}_I(R)_Q$  must also be a domain. We started off with  $Q$  being any prime  $\text{Ass}(\text{gr}_I(R))$ , and showed that  $\text{gr}_I(R)_Q$  is a domain; we claim that this implies that  $\text{gr}_I(R)$  is reduced. Indeed, if  $\text{gr}_I(R)$  were not reduced, then it would have a nonzero associated prime  $Q$ , which would contain only zerodivisors, and  $\text{gr}_I(R)_Q$  would not be a domain.  $\square$

Remark 7.6 now gives us the following corollary directly:

**Corollary 7.8.** *Let  $R$  be a Cohen-Macaulay ring and  $I$  an ideal in  $R$  of finite projective dimension. Then  $\text{gr}_I(R)$  is reduced if and only if  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

### 7.3 Hochster's criteria

In the 1970s, Hochster gave criteria for the equality  $P^{(n)} = P^n$  to hold for all  $n$  when  $P$  is a prime ideal. He used this to prove the equality holds for prime ideals generated by regular sequences, and for ideals of maximal minors of generic  $(n-1) \times n$  matrices. He also used his criterion to give an example of a prime  $P$  such that  $R/P$  is not Cohen-Macaulay, and still  $P^{(n)} = P^n$  for all  $n$ . To write the criteria, we need the following definition:

**Definition 7.9.** Let  $I = (f_1, \dots, f_n)$  be a radical ideal in a reduced noetherian ring  $R$ . Set  $S := R[x_1, \dots, x_n]$ . The ideal  $J_m(I)$  are the following ascending chain of ideals in  $S$ :

- $J_0(I) := (0)$ .
- For each  $m \geq 0$ ,  $J_{n+1}(I) := (\sum_{i=1}^n s_i x_i \mid s_i \in S \text{ and } \sum_{i=1}^n s_i f_i \in J_m(I))$ .

**Theorem 7.10** (Hochster, [Hoc73]). *Let  $P$  be a prime ideal in a noetherian domain  $R$ . Fix a set of generators  $f_1, \dots, f_n$  for  $P$ , set  $S := R[x_1, \dots, x_n]$ , and consider the ascending chain  $J_m(I)$  from Definition 7.9. Let  $J := \bigcup_{m \geq 0} J_m$ . The following are equivalent:*

- 1)  $P^{(m)} = P^m$  for all  $m$ , and the associated graded ring of  $R_P$  is a domain.
- 2)  $PS + J$  is a prime ideal.
- 3) For some  $m \geq 1$ ,  $PS + J_m$  is a prime ideal of height  $n$ . In this case,  $PS + J_m = PS + J$ .
- 4) There is a prime  $Q$  in  $S$  of height  $n$  such that  $Q \subseteq PS + J$ . In this case,  $Q = PS + J$ .
- 5) The element  $t^{-1}$  in the extended Rees algebra  $R[Pt, t^{-1}]$  is a prime element.

This was generalized by Li and Swanson to radical ideals [LS06].

**Theorem 7.11** (Hochster, [Hoc73], Li-Swanson [LS06]). *Let  $I = (f_1, \dots, f_n)$  be a radical ideal in a reduced noetherian ring  $R$ . Set  $S := R[x_1, \dots, x_n]$ , and consider the ascending chain  $J_m(I)$  from Definition 7.9. Let  $J := \bigcup_{n \geq 0} J_n(I)$  and  $U := R \setminus \bigcup_{P \in \text{Min}(I)} P$ . The following are equivalent:*

- 1)  $I^{(m)} = I^m$  for all  $m$ , and the associated graded ring  $\text{gr}_{U^{-1}I}(U^{-1}R)$  is a reduced.
- 2)  $IS + J$  is a radical ideal and  $U \cap \mathcal{Z}(S/(IS + J)) = \emptyset$ .
- 3) For some  $m \geq 1$ ,  $IS + J_m$  is a radical ideal of height  $n$  which has the same number of minimal primes as  $IS + J$ , and  $U \cap \mathcal{Z}(S/(IS + J)) = \emptyset$ . In this case,  $IS + J_m = IS + J$ .
- 4) There is a radical ideal  $Q \subseteq IS + J$  in  $S$  of height  $n$  which has the same number of minimal primes as  $IS + J$ , and such that  $U \cap \mathcal{Z}(S/Q) = \emptyset$ . In this case,  $Q = IS + J$ .
- 5) In  $R' = R[Pt, t^{-1}]$ , the ideal  $t^{-1}R[Pt, t^{-1}]$  is radical and  $U \cap \mathcal{Z}(S'/(t^{-1})) = \emptyset$ .

This sequence  $J_m(I)$  can be constant.

**Theorem 7.12.** *Let  $R$  be a noetherian ring and let  $I$  be an ideal generated by the regular sequence  $a_1, \dots, a_n$ . Then*

$$J_1(I) = (x_j a_k - x_k a_j \mid 1 \leq j < k \leq m) = J_2 = \dots = J.$$

*Proof.* By definition, the generators of  $J_1$  are the elements of the form  $\sum_i s_i x_i$  where  $\sum_i s_i a_i = 0$  in  $R$ . These are relations among our generators; but since  $R/I$  is resolved by the Koszul complex on the  $a_i$ , this means that each relation  $(s_1, \dots, s_n)$  must be in the  $R$ -submodule of  $R^n$  generated by the Koszul relations  $(0, \dots, a_i, \dots, -a_k, \dots, 0)$  with  $k < j$ . Thus  $J_1$  is also generated by  $a_j x_k - a_k x_j$ .

Now we claim that  $J_2 = J_1$ , which will imply that  $J_s = J_1$  for all  $s \geq 2$ . So consider a generator  $\sum_i s_i x_i$  for  $J_2$ , which must necessarily come from a relation  $\sum_i s_i a_i \in J_1$ . We can rewrite this last element of  $J_1$  using the generators we found for  $J_1$ , so in the form  $\sum_i s_i a_i = \sum_{j < k} l_{jk} (x_j a_k - x_k a_j)$  for some  $l_{jk} \in S$ . Then

$$\sum_{i=1}^m \left( s_i - \sum_{j=1}^{i-1} l_{ji} x_j + \sum_{j=i+1}^m l_{ji} x_j \right) a_i = 0$$

and thus

$$\sum_{i=1}^m s_i x_i = \sum_{i=1}^m \left( s_i - \sum_{j=1}^{i-1} l_{ji} x_j + \sum_{j=i+1}^m l_{ji} x_j \right) x_i \in J_1.$$

□

**Theorem 7.13.** *Let  $I$  be a radical ideal in a noetherian ring  $R$ . If  $I$  is generated by a regular sequence, then  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

*Proof.* Let  $a_1, \dots, a_m$  be a regular sequence generating  $I$ , and set  $S = R[x_1, \dots, x_m]$  as before. If  $\text{Min}(I) = \{P_1, \dots, P_t\}$ , then  $IS$  is a radical ideal with minimal primes  $P_1 S, \dots, P_t S$ . Each  $P_i S$  is a minimal prime over an ideal generated by  $m$  elements, which must then have height at most  $m$ , by [Krull's Height Theorem](#). Moreover, when viewed as elements of  $S$ ,  $a_1, \dots, a_m$  still form a regular sequence, and thus each  $P_i S$  has height exactly  $m$ .

By Theorem 7.12,  $J \subseteq IS$ , so  $J + IS = IS$ . Therefore, every minimal prime over the radical ideal  $J + IS$  has height exactly  $m = \text{ht}(I)$ . In order to apply Theorem 7.11, we need to show that  $U = R \setminus P_1 \cup \dots \cup P_t$  satisfies  $U \cap \mathcal{Z}(S/PS) = \emptyset$ . This follows from the isomorphism  $S/IS \cong R/I[x_1, \dots, x_m]$ . □

The sequence  $J_i(I)$  might not stabilize at  $J_1$ . To see that, here is a nice example from Li and Swanson's paper [\[LS06\]](#):

**Example 7.14.** Let  $k$  be a field,  $R = k[y_1, y_2]$ , and let  $I = (y_1^2, y_1 y_2, y_2^2)$ . The module of relations on  $a_1 = y_1^2, a_2 = y_1 y_2, a_3 = y_2^2$  in  $S = R[x_1, x_2, x_3]$  is generated by  $(y_2, -y_1, 0)$  and  $(0, y_2, -y_1)$ , so  $J_1 = (y_2 x_1 - y_1 x_2, y_2 x_2 - y_1 x_3) \subseteq (y_1, y_2)S$ . Notice in particular that  $x_1 x_3 - x_2^2$  is not in  $J_1$ , but it is in  $J_2$ , since

$$x_1 a_3 - x_2 a_2 = x_1 y_2^2 - x_2 y_1 y_2 = y_2 (x_1 y_2 - x_2 y_1) \in J_1.$$

## 7.4 Regular rings of dimension 3

For primes of height 2 in a regular local ring, Huneke showed that  $P^{(n)} = P^n$  for all  $n$  is in fact equivalent to  $P$  being a complete intersection, and that if  $P$  is not a complete intersection, then  $P^{(n)} \neq P^n$  for all  $n \geq 2$ . To prove that, we will use the following fact:

**Theorem 7.15** (Hilbert, Burch, [Bur68, Hil90]). *Let  $R$  be a noetherian local ring and  $I$  be an ideal with  $\text{pdim}(R/I) = 2$ . Then the minimal free resolution of  $R/I$  is of the form*

$$0 \longrightarrow R^{n-1} \xrightarrow{A} R^n \xrightarrow{B} R \longrightarrow R/I \longrightarrow 0$$

*and there exists a regular element  $a$  such that  $I = aI_{n-1}(A)$ , where  $I_{n-1}(A)$  is the ideal of maximal minors of the  $(n-1) \times n$  matrix  $A$ . In fact, the map  $B$  in degree 0 is the row with  $(-1)^i a$  times the determinant of the matrix obtained from  $A$  by deleting the  $i$ th column of  $A$ .*

**Remark 7.16.** Notice that if  $P$  is a prime ideal and  $\text{pdim}(R/P) = 2$ , then in fact Theorem 7.15 says that  $P$  is the ideal of maximal minors of some  $(n-1) \times n$  matrix, since for a nonunit,  $P = aI_{n-1}(A)$  would be a product of two ideals.

We will also use the following useful observation:

**Lemma 7.17.** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$ , and let  $P$  be a prime ideal of height  $d-1$ . For each  $n$ ,  $P^{(n)} = P^n$  if and only if  $R/P^n$  is Cohen-Macaulay.*

*Proof.* Since  $R$  is local and  $\text{ht}(P) = \dim(R) - 1$ ,  $\mathfrak{m}$  is the only potential embedded prime of  $P^n$ , so we have  $P^{(n)} = P^n$  if and only if  $\mathfrak{m} \in \text{Ass}(P^n)$ . On the other hand,  $R/P^n$  is a local ring with  $\dim(R/P^n) = 1$ , so  $R/P^n$  is Cohen-Macaulay if and only if  $\text{depth}(R/P^n) \neq 0$ , or equivalently,  $\mathfrak{m} \notin \text{Ass}(P^n)$ .  $\square$

Notice that the proof also applies to any radical ideal  $I$  of pure height  $\dim(R) - 1$ . And ideal  $I$  has **pure** height if all the minimal primes of  $I$  have height  $\dim(R) - 1$ .

The key technical point in Huneke's proof is the following theorem; the proof involves intersection multiplicities, which we have not discussed.

**Theorem 7.18** (Huneke, 1986, Theorem 2.1 in [Hun86]). *Let  $S$  be a Cohen-Macaulay local ring and  $Q$  be a prime ideal in  $S$  such that  $S/Q$  is Cohen-Macaulay and  $S_Q$  is regular. Suppose that  $\underline{x} = x_1, \dots, x_t$  is a sequence of elements in  $S$  that form a regular sequence on  $S$  and  $S/Q$ . Let  $R = S/(\underline{x})$ , and suppose that  $P = Q/(\bar{x})$  is a prime ideal such that  $R_P$  is regular. Then for each  $n$ ,  $R/P^n$  is Cohen-Macaulay if and only if  $R/P^n$  is Cohen-Macaulay.*

Taking the previous theorem as fact, we are finally ready to prove Huneke's result about regular rings of dimension 3:

**Theorem 7.19** (Huneke, 1986, Corollary 2.2 in [Hun86]). *Let  $(R, \mathfrak{m})$  be a regular local ring of dimension 3 and let  $P$  be a prime ideal of height 2 in  $R$ . If  $P$  is not a complete intersection, meaning it is not generated by a regular sequence, then  $P^{(n)} \neq P^n$  for all  $n \geq 2$ .*

*Proof.* Since  $R$  is regular,  $R/P$  has finite projective dimension, by Theorem 3.70. Since  $R/P$  is a 1-dimensional domain, the maximal ideal must necessarily contain a regular element (any nonzero nonunit will do), and so in particular  $\text{depth}(R/P) = 1$ . By the Auslander–Buchsbaum formula,  $\text{pdim}(R/P) = \text{depth}(R) - \text{depth}(R/P) = 3 - 1 = 2$ . By the Hilbert–Burch Theorem,  $P$  is the ideal of maximal minors of some  $(n-1) \times n$  matrix, where  $n$  is the minimal number of generators of  $P$ . When  $n = 2$ ,  $P$  is a complete intersection and  $P^{(t)} = P^t$  for all  $t$ , so we are interested in the case when  $n \geq 3$ .

Let  $X = (x_{ij})$  be a generic  $(n-1) \times n$  matrix, and let  $R := R[X]_{(x_{ij})+\mathfrak{m}}$ . Set  $Q = I_{n-1}(X)$  to be the ideal of  $(n-1)$ -minors of  $X$ . By construction,  $S$  is a regular local ring, and thus  $S_Q$  is regular, and  $Q$  is a prime ideal. Moreover, it is well-known that  $S/Q$  is Cohen-Macaulay. Notice also that  $\{x_{ij} - a_{ij}\}_{i,j}$ , where  $a_{ij}$  are the entries in the matrix  $A$  defining our original ideal  $P$ , is a regular sequence – in particular, because each element involves a different  $x_{ij}$ , and  $\{x_{ij}\}_{i,j}$  is also a regular sequence in  $S$ . Moreover,  $S/Q \otimes_S R = R/P$ .

By Lemma 7.17, for each fixed  $t$  we have  $P^{(t)} = P^t$  if and only if  $R/P^t$  is Cohen-Macaulay. By Theorem 7.18,  $R/P^t$  is Cohen-Macaulay if and only if  $S/Q^t$  is Cohen-Macaulay. We are trying to show that  $R/P^t$  is not Cohen-Macaulay when  $n \geq 3$  and  $t \geq 2$ , which we will prove by showing that  $S/Q^t$  is not Cohen-Macaulay for all  $n \geq 3$  and  $t \geq 2$ .

Since  $\dim(S/Q^t) = \dim(S/Q)$ , we only need to prove that  $\text{depth}(S/Q^t) < \text{depth}(S/Q)$ . Since  $S$  is regular,  $S/Q^t$  must have finite projective dimension. By the Auslander–Buchsbaum Formula,  $\text{depth}(S/Q^t) < \text{depth}(S/Q)$  is equivalent to  $\text{pdim}(S/Q^t) > \text{pdim}(S/Q)$ . In [Wey79], Weyman gave explicit minimal resolutions for  $S/Q^t$ , and those resolutions show that  $S/Q^t$  are never Cohen-Macaulay when  $t > 1$  and  $n \geq 3$ .  $\square$

In the proof, we reduced the problem to the Cohen-Macaulayness of powers of certain ideals of minors, and then we needed to compute the resolutions of the powers of those ideals. In general, finding resolutions for the powers of an ideal is not easy; the beauty of Huneke’s idea is that we can reduce to a much simpler case of generic minors, where we can indeed compute the minimal resolutions of the powers. This can be shown through several different techniques; for example, it is one of the resolutions obtained by Weyman in [Wey79]. One can also find these resolutions through the Rees algebra.

**Theorem 7.20.** *Let  $k$  be a field,  $X$  be a generic  $(n-1) \times n$  generic matrix, and consider the ideal  $I = I_{n-1}(X)$  of maximal minors of  $X$  in  $R = k[X]$ . If  $n > 2$ ,  $\text{pdim}(I^t) > \text{pdim}(I)$  for all  $t \geq 2$ .*

*Proof.* Let  $m_i$  be the minor of  $X$  corresponding to  $(-1)^{i+1}$  times the determinant of the matrix obtained by deleting the  $i$ th column of  $X$ , so that  $I = (m_1, \dots, m_n)$ . As we described before, the Rees algebra  $R[It]$  of  $I$  is a quotient of  $U = R[T_1, \dots, T_n]$ , given by the surjection

$$\begin{aligned} R[T_1, \dots, T_n] &\longrightarrow R[It] \\ T_i &\longmapsto m_i. \end{aligned}$$

with kernel  $L$ , so that  $R[It] \cong R[T_1, \dots, T_n]/L$ . By [Hun82, Example 1.1],  $I$  is an ideal of linear type (since it is generated by a  $d$ -sequence); this means that  $L$  is generated by the elements of degree 1. These correspond to the relations among generators  $m_1, \dots, m_n$  of the original ideal  $I$ . But since  $X$  is generic, the relations among the generators are precisely

$$x_{i,1}m_1 + \dots + x_{i,n}m_n = 0$$

for each  $i = 1, \dots, n-1$ , and thus

$$L = L_1 = (x_{i,1}T_1 + \dots + x_{i,n}T_n \mid 1 \leq i \leq n-1).$$

Notice moreover that each of the generators for  $L$  above involve completely different variables  $x_{i,j}$ , and thus in fact they form a regular sequence. Therefore, we can resolve  $R[It] \cong U/L$  by the Koszul complex on the regular sequence

$$F_1 := x_{1,1}T_1 + \dots + x_{1,n}T_n, \dots, F_{n-1} := x_{n-1,1}T_1 + \dots + x_{n-1,n}T_n$$

with  $n-1$  elements, which has the form

$$C = 0 \longrightarrow U(-(n-1)) \longrightarrow U(-(n-2))^{n-1} \longrightarrow \dots \longrightarrow U(-1)^{n-1} \longrightarrow U \longrightarrow 0.$$

$\qquad\qquad\qquad n-1 \qquad\qquad\qquad n-2 \qquad\qquad\qquad \dots \qquad\qquad\qquad 1 \qquad\qquad\qquad 0$

In fact, this is a graded resolution of  $R[It]$ , and each differential is a map of degree 1. For each fixed degree  $d$ ,  $U_d$  is the free  $R$ -module generated by the monomials in  $T_1, \dots, T_n$  of degree  $d$ , so if we break the resolution into strands in each fixed degree, we will obtain free resolutions for the degree  $d$  piece of  $R[It]$ , which is precisely  $I^d$ . The linear strand in degree  $d$  is thus a free resolution for  $I^d$  over  $R$ .

When we read off the degree  $d$  strand of  $C$ , let's call it  $C_d$ , in homological degree  $i$  we have

$$U(-i)_d = U_{d-i} = R\{T_1^{a_1} \dots T_n^{a_n} \mid a_1 + \dots + a_n = d-i\} = \begin{cases} R^{\binom{d-i+n-1}{n-1}} & \text{if } i \leq d \\ 0 & \text{if } i > d. \end{cases}$$

The nonzero entries in the matrices in the resolution  $C$  are all of the form  $\pm F_i$ . In  $C_d$ , the nonzero entries in each matrix are induced by multiplication by  $\pm F_i$ , and each basis element in degree  $i \leq d$  corresponds to a monomial of degree  $d-i$  on  $T_1 \dots T_n$ . Since

$$F_i T_1^{a_1} \dots T_n^{a_n} = \sum_{j=1}^n x_{i,j} T_1^{a_1} + \dots + T_j^{a_j+1} \dots + x_{i,n} T_n^{a_n},$$

the nonzero entries in the matrices in  $C_d$  are of the form  $\pm x_{i,j}$ . We conclude that  $C_d$  is a minimal free resolution for  $I^d$ .

In particular, we can now compute the projective dimension of  $I^d$  for each  $d$ . By the description above, we see that the module in  $C_d$  in homological degree  $i$  is zero unless  $i \leq \min\{d, n-1\}$ . We conclude that  $\text{pdim}(I) = \min\{1, n-1\} = 1$ , and for all  $d \leq 2$ ,  $\text{pdim}(I^d) > 1 = \text{pdim}(I)$ , as desired.  $\square$

Notice that for any ideal  $J$  in a ring  $R$ ,  $\text{pdim}(R/J) = \text{pdim}(J) + 1$ , so the statement of Theorem 7.19 is equivalent to  $\text{pdim}(R/I^t) > \text{pdim}(R/I)$  for all  $t \geq 2$ .

In the proof of Theorem 7.19, we actually used a version of Theorem 7.20 where  $k$  is replaced by a ring  $R$ : if  $X$  be a generic  $(n-1) \times n$  generic matrix, and  $I = I_{n-1}(X)$  is the ideal of maximal minors of  $X$  in  $S = R[X]$ , then  $\text{pdim}(I^t) > \text{pdim}(I)$  for all  $t \geq 2$ . When  $R$  contains a field  $k$ , this is an easy corollary of Theorem 7.20: given a minimal free resolution for the ideal of minors of the generic matrix over  $k$ , the functor  $- \otimes_k R$  is exact, since  $R$  is a flat  $k$ -module, and thus we obtain a free resolution over  $R$ . Since in this case the entries in the differentials of the resolutions over  $k$  involve only the variables  $x_{i,j}$ , the resulting resolutions over  $R$  are still minimal. In fact, the same reason implies that we still get minimal resolutions when we localize and  $\mathfrak{m} + (x_{i,j})$ , as did in the proof of Theorem 7.19.



## 7.5 Higher dimension

When  $R$  is a regular local ring of dimension 3, we now know everything there is to know about the equality of symbolic and ordinary powers of prime ideals, thanks to Huneke's Theorem 7.19:

- If  $\text{ht}(P) = 1$  or  $\text{ht}(P) = 3$ , then  $P^{(n)} = P^n$  for all  $n$ .
- If  $\text{ht } P = 2$  and  $P$  is a complete intersection, then  $P^{(n)} = P^n$  for all  $n$ .
- If  $\text{ht}(P) = 2$  and  $P$  is not a complete intersection, then  $P^{(n)} \neq P^n$  for all  $n \geq 2$ .

We can view this as a result about minimal number of generators: if  $P$  has height 2 and the minimal number of generators of  $P$  is 3 or more, then  $P^{(n)} \neq P^n$  for all  $n \geq 2$ . Or we can view this as saying that is sufficient to test the equality  $P^{(2)} = P^2$ : if it holds, then  $P^{(n)} = P^n$  for all  $n \geq 2$ , otherwise,  $P^{(n)} \neq P^n$  for all  $n \geq 2$ .

We saw in Example 2.12 that the equality  $I^{(n)} = I^n$  can hold for low values of  $n$  without holding for all  $n$ ; this can happen even over a regular ring. Nevertheless, under some reasonable conditions it is sufficient to reduce the test of equality up to  $n = \dim(R) - 1$ . To state such a result, we need a few definitions.

**Definition 7.21.** Let  $I$  be an ideal in a noetherian local ring  $R$ . We say  $I$  is **perfect** if  $\text{pdim}(R/I) = \text{depth}_I(R)$ .

So an ideal is perfect if the largest length of a regular sequence inside  $I$  coincides with the projective dimension of  $R/I$ .

**Remark 7.22.** When  $R$  is Cohen-Macaulay, Theorem 3.103 says that the maximal length of a regular sequence inside  $I$  is precisely  $\text{ht}(I)$ , and thus an ideal  $I$  is perfect ideal if and only if it satisfies  $\text{pdim}(R/I) = \text{ht}(I)$ . The Cohen-Macaulay assumption also gives  $\text{ht}(I) = \dim(R) - \dim(R/I)$ . On the other hand,  $\text{pdim}(R/I) = \text{depth}(R) - \text{depth}(R/I)$ , by the Auslander–Buchsbaum formula, so over a Cohen-Macaulay ring, an ideal  $I$  is perfect if and only if

$$\text{depth}(R) - \text{depth}(R/I) = \dim(R) - \dim(R/I) \Leftrightarrow \dim(R/I) = \text{depth}(R/I),$$

meaning that  $R/I$  is Cohen-Macaulay.

**Definition 7.23.** We say a ring  $I$  is **generically a complete intersection** if  $I_P$  is generated by a regular sequence for all  $P \in \text{Min}(I)$ .

If  $I$  is a radical ideal in a regular ring, then for all  $P \in \text{Min}(I)$ ,  $R_P$  is a regular local ring with maximal ideal  $I_P = P_P$ , and thus  $I_P$  is generated by a regular sequence, by Theorem 3.70. Therefore, any radical ideal in a regular ring is generically a complete intersection.

The remainder of this section will contain several references to Gorenstein rings. While we did not discuss these in detail, they are a very important class of Cohen-Macaulay rings that includes all complete intersections, and in particular all regular rings. For more on Gorenstein rings, see [BH93].

**Theorem 7.24** (Morey, 1999 [Mor99]). *Let  $I$  be a perfect ideal in a noetherian local ring  $R$ , and suppose that  $I$  is generically a complete intersection. Assume that either*

- $\text{ht}(I) = 2$  and  $R/I$  is Cohen-Macaulay, or
- $\text{ht}(I) = 3$  and  $R/I$  is Gorenstein.

*If  $I^{(n)} = I^n$  for all  $n \leq \dim(R) - 1$ , then  $I^{(n)} = I^n$  for all  $n \geq 1$ . Moreover, if  $I^{(n)} = I^n$  for all  $n$ , then  $\mu(I) \leq \dim(R) - 1$ .*

**Remark 7.25.** We claim that Theorem 7.24 recovers Huneke's Theorem 7.19. Huneke's theorem is about height 2 primes in a 3-dimensional regular local ring, which as we discussed above are all automatically generically a complete intersection. Moreover, if  $P$  is such a height 2 prime, then as we discussed in the previous section,  $R/P$  is a 1-dimensional Cohen-Macaulay ring, and thus  $P$  is perfect. Finally, the condition  $I^{(n)} = I^n$  for all  $n \leq \dim(R) - 1$  is now equivalent to  $I^{(2)} = I^2$ . The discussion in the beginning of this section should now make it clear that indeed, Huneke's theorem coincides with Morey's in these circumstances.

One of the main points is that these conditions imply that  $I$  is strongly Cohen-Macaulay, which is a technical but very nice condition that in short allows one a certain amount of control over the powers of  $I$ .

**Definition 7.26.** An ideal  $I$  is **strongly Cohen-Macaulay** if the Koszul homology modules  $H_i(\underline{f})$ , where  $\underline{f} = f_1, \dots, f_n$  is a generating set for  $I$ , are Cohen-Macaulay.

In fact, the conditions  $\text{ht}(I) = 3$  and  $R/I$  is Gorenstein imply that  $I$  is licci. Linkage is a vast, important topic, which we will not discuss in detail; however, here is a definition.

**Definition 7.27.** Two ideals  $I$  and  $J$  in a Cohen-Macaulay ring  $R$  are said to be **linked** if there is a regular sequence  $\underline{a} = a_1, \dots, a_g$  such that  $((\underline{a}) : I) = J$  and  $((\underline{a}) : J) = I$ , and we write  $I \sim J$  to denote that  $I$  and  $J$  are linked. If there exists a sequence of ideals  $I \sim I_1 \sim I_2 \sim \dots \sim I_n \sim J$ , we say that  $I$  is **in the linkage class** of  $J$ . Finally,  $I$  is **licci** if it is in the linkage class of a complete intersection.

**Theorem 7.28** (Huneke–Ulrich, 1987, Corollary 2.9 in [HU89]). *Let  $R$  be a regular local ring and  $P$  a prime ideal in  $R$  which is not a complete intersection. If  $P$  is licci and  $\dim(R/P) = 1$ , then the following holds:*

- 1) *If  $R/P$  is not Gorenstein, then  $P^{(n)} \neq P^n$  for all  $n \geq 2$ .*
- 2) *If  $R/P$  is Gorenstein, then  $P^{(2)} = P^2$ , but  $P^{(n)} \neq P^n$  for all  $n \geq 3$ .*

Ultimately, the equality  $I^{(n)} = I^n$  is about what primes are associated to  $I^n$  for each  $n$ , which is a statement about depth:  $I^{(n)} = I^n$  if and only if

$$\text{depth}((R/I^n)_P) > 0 \quad \text{for every prime } P \supseteq I, P \notin \text{Min}(I).$$

As we discussed in Section 6.3, computing the depth of the powers of a given ideal is very challenging. However, there are lower bounds on  $\text{depth}(R/I^n)$  which can be exploited to give sufficient conditions for equality. For more on this, see Ulrich's work in [Ulr94, Corollary 2.13].



## 7.6 The Packing Problem

When  $I$  is a squarefree monomial ideal, there is a beautiful conjecture, first made by Conforti and Cornuéjols in the context of max-flow min-cut properties, which was reworked [GVV07, GRV05a] into a commutative algebra statement that proposes an equivalent condition to the equality  $I^{(n)} = I^n$  for all  $n$ . To state the conjecture, we first need a few definitions.

**Definition 7.29.** A squarefree monomial ideal  $I$  of height  $c$  is **König** if there exists a regular sequence of monomials in  $I$  of length  $c$ .

**Remark 7.30.** Since the polynomial ring  $R = k[x_1, \dots, x_d]$  over a field  $k$  is Cohen-Macaulay, every squarefree monomial ideal  $I$  of height  $c$  does indeed contain a regular sequence of length  $c$ . However, there might be no such regular sequence consisting of only monomials.

**Lemma 7.31.** Let  $k$  be a field and  $R = k[x_1, \dots, x_d]$ . A sequence of monomials  $f_1, \dots, f_n$  is regular if and only if there is no variable  $x_j$  dividing two or more monomials  $f_i$ .

*Proof.* Since monomials are homogenous elements, by Corollary 3.45 the order of the  $f_i$  does not matter, so we can reduce to proving that  $f_n$  is regular on  $R/(f_1, \dots, f_{n-1})$  if and only if there is no variable  $x_j$  and no  $i < n$  such that  $x_j$  divides  $f_i$  and  $f_n$ .

In general, for any monomials  $f_1, \dots, f_n$  the minimal monomial generating set  $G(f_1, \dots, f_n)$  is a subset of  $\{f_1, \dots, f_n\}$ . If one of the  $f_i$  is in the ideal generated by the remaining ones, then  $f_1, \dots, f_{n-1}$  is not a regular sequence and there are variables in common among  $f_1, \dots, f_n$ , so we can assume without loss of generality that  $f_i \notin (f_j \mid i \neq j)$  for all  $i$ .

By the proof of Theorem 5.13, the associated primes of a monomial ideal  $I$  with  $G(I) = (f_1, \dots, f_n)$  are generated by subsets of the variables appearing in  $f_1, \dots, f_n$ , and every variable appearing in some  $f_i$  must be a generator of at least one associated prime of  $(f_1, \dots, f_n)$ . Therefore, the monomials in the union of the associated primes of  $I$  are the monomials in  $R$  that are multiples of at least one of the variables appearing in  $f_1, \dots, f_n$ .

By Theorem 1.39,  $f_n$  is regular on  $R/(f_1, \dots, f_{n-1})$  if and only if  $f_n$  is not in the union of the associated primes of  $(f_1, \dots, f_{n-1})$ . We conclude that  $f_n$  is regular on  $R/(f_1, \dots, f_{n-1})$  if and only if none of the variables appearing in  $f_1, \dots, f_{n-1}$  appears in  $f_n$ .  $\square$

We can now rephrase our definition of König: a squarefree monomial ideal  $I$  of height  $c$  is König if and only if  $c$  monomials in  $G(I)$  with no variables in common.

**Example 7.32.** The ideal  $I = (xy, xz, yz)$  has height 2, and it contains regular sequences of length 2, such as  $xy, xz + yz$ . However, it is not König, since every two generators have a common variable.

In contrast, the ideal  $J = (xy, xz)$  is König, since it has height 1, and  $xy$  is a regular sequence of length 1.

**Definition 7.33.** Let  $I$  be a squarefree monomial ideal in  $k[x_1, \dots, x_d]$ . We say  $I$  is **packed** if every ideal obtained from  $I$  by setting any number of variables equal to 0 or 1 is König.

Notice we are also allowed to do nothing to some or all of the variables. In particular, any ideal that is not König cannot be packed. Notice moreover that setting a variable  $x_i$  equal to 0 is the same as looking at the image of  $I$  in  $R/(x_i)$ ; setting a variable equal to 1

corresponds to looking at the image of  $I$  in the localization  $R_{x_i}$ . The ideals obtained from  $I$  by localizing at a collection of the variables, quotienting by an ideal generated by variables, or a combination of the two, are called **minors** of  $I$ .

**Example 7.34.** We saw in Example 7.32 that  $I = (xy, xz, yz)$  is not König, so it is also not packed.

We also saw that  $I = (xy, xz)$  is König. Here are the ideals we can obtain from  $I$ :

- Any option that involves  $x = 0$  leads to  $(0)$ , which is König for trivial reasons.
- If we set  $x = 1$ , we end up with  $(y, z)$ , which is König. If we further set  $y$  or  $z$  equal to 1 or 0 we end up with  $R$ ,  $(y)$ ,  $(z)$ , or  $(0)$ , all of which are König.
- If we do not do anything to  $x$ ,  $y$  and  $z$  have a symmetric role, so up to change of variables, we can end up with  $(x)$ ,  $(xy)$ , or  $(0)$ , all of which are König.

Therefore,  $(xy, xz)$  is a packed ideal.

Here is the commutative algebra version of Conforti and Cornuéjols' conjecture about max-cut min-flow properties:

**Conjecture 7.35** (Packing Problem). *A squarefree monomial ideal  $I$  satisfies  $I^{(n)} = I^n$  for all  $n$  if and only if  $I$  is packed.*

One implication is relatively easy. To do that, we first need a lemma.

**Lemma 7.36.** *Let  $I$  be a squarefree monomial ideal. The property  $I^{(n)} = I^n$  for all  $n$  is preserved by setting any variables equal to 1 or 0.*

*Proof.* We will write  $I(x_i = 0)$  or  $I(x_i = 0, x_j = 1)$  to indicate the ideal obtained from  $I$  by setting the corresponding variables equal to the given variables.

To prove our claim, let  $I = (m_1, \dots, m_n)$ , and let  $P_1, \dots, P_s$  be the minimal primes of  $I$ , so that  $I = P_1 \cap \dots \cap P_s$ . We will first show that  $I(x_i = a) = P_1(a) \cap \dots \cap P_s(a)$ . In the proof of Theorem 5.13, we showed that the minimal components can be obtained via the following process: given  $(f_1, \dots, f_n)$ , if  $f_1 = gh$  with  $\gcd(g, h) = 1$ , then  $(f_1, \dots, f_n) = (g, f_2, \dots, f_n) \cap (h, f_2, \dots, f_n)$ . Since  $P_1, \dots, P_s$  are obtained from  $I$  by successive applications of this process, it is sufficient to show that

$$(f_1, \dots, f_n)(x_i = a) = (g, f_2, \dots, f_n)(x_i = a) \cap (h, f_2, \dots, f_n)(x_i = a).$$

If  $x_i$  does not divide  $f_1$ , the statement is clear, so we might as well assume that  $g = x_i^d g'$  and  $x_i$  does not divide  $g'$  nor  $h$ . When  $a = 0$ ,

$$(f_1, \dots, f_n)(x_i = 0) = (f_2, \dots, f_n)(x_i = 0)$$

and

$$(g, f_2, \dots, f_n)(x_i = 0) = (f_2, \dots, f_n)(x_i = 0), \quad (h, f_2, \dots, f_n)(x_i = 0) = (h) + (f_2, \dots, f_n)(x_i = 0),$$

so

$$\begin{aligned} & ((g, f_2, \dots, f_n)(x_i = 0)) \cap ((h, f_2, \dots, f_n)(x_i = 0)) \\ &= (f_2, \dots, f_n)(x_i = 0) \cap ((h) + (f_2, \dots, f_n)(x_i = 0)) \\ &= (f_2, \dots, f_n)(x_i = 0) \end{aligned}$$

When  $a = 1$ ,

$$(f_1, \dots, f_n)(x_i = 1) = (g'h) + (f_2, \dots, f_n)(x_i = 1)$$

and

$$\begin{aligned} & ((g, f_2, \dots, f_n)(x_i = 1)) \cap ((h, f_2, \dots, f_n)(x_i = 1)) \\ &= ((g') + (f_2, \dots, f_n)(x_i = 1)) \cap ((h) + (f_2, \dots, f_n)(x_i = 1)) \\ &= (g'h) + (f_2, \dots, f_n)(x_i = 1). \end{aligned}$$

This shows that if  $\text{Min}(I) = \{P_1, \dots, P_s\}$ , then  $I(x_i = a) = P_1(x_i = a) \cap \dots \cap P_s(x_i = a)$ . Now note that setting a variable equal to 1 or 0 commutes with taking powers, and since  $I^{(n)} = P_1^n \cap \dots \cap P_s^n$ , we conclude that

$$I(x_i = a)^{(n)} = I^{(n)}(x_i = a).$$

Therefore, the equality  $I^{(n)} = I^n$  is preserved by setting variables equal to 0 or 1.  $\square$

**Theorem 7.37.** *Let  $I$  be a squarefree monomial ideal. If  $I^{(n)} = I^n$  for all  $n$ , then  $I$  is packed.*

*Proof.* By Lemma 7.36, the property  $I^{(n)} = I^n$  for all  $n$  is preserved by setting any variables equal to 1 or 0. This reduces the problem to showing that if  $I^{(n)} = I^n$  for all  $n$  then  $I$  must be König. After setting all the variables that do not divide any minimal monomial generator of  $I$  equal to 0, we may as well assume that every variable does indeed appear in at least one of the monomial generators of  $I$ ; say that the ring is now  $k[x_1, \dots, x_d]$ . Therefore, by Remark 5.19 we know that every variable appears as a generator of some minimal prime of  $I$ . Since  $I$  has height  $c$ , every minimal prime  $P$  of  $I$  has height at least  $c$ , so  $x_1 \cdots x_d \in P^c$ . Therefore, since

$$I^{(c)} = \bigcap_{P \in \text{Min}(I)} P^c,$$

by Corollary 5.20, we conclude that  $x_1 \cdots x_d \in I^{(c)}$ . By assumption,  $I^{(c)} = I^c$ , which means that there exist monomials  $m_1, \dots, m_c \in I$  such that  $m_1 \cdots m_c = x_1 \cdots x_d$ . By construction, these monomials involve disjoint sets of variables, and thus  $I$  is König.  $\square$

Therefore, to solve the Packing Problem one needs to prove that any packed ideal must satisfy  $I^{(n)} = I^n$  for all  $n$ . This is, however, an open question.

For the case of edge ideals, however, the Packing Problem is solved. First, we note that the conditions that the edge ideal of the graph  $G$  is König or packed can be rephrased in terms of the graph.

**Definition 7.38.** Let  $G$  be a finite simple graph. The **covering number** of  $G$  is the smallest number of vertices in any minimal vertex cover of  $G$ . A set of edges  $F$  of  $G$  is called **independent** if no two edges in  $F$  share a common vertex. The **independence number** of  $G$  is the maximum number of independent edges.

It turns out (see [GRV09], Section 2 and Proposition 2.2) that if  $I$  is the edge ideal of a graph  $G$ , the covering number of  $G$  is equal to the height of  $G$ , while the independence number of  $G$  is equal to the **monomial grade** of  $I$ , which is the largest length of a regular sequence of monomials in  $I$ . One then defines a **König** graph as a graph  $G$  whose covering number and independence number coincide, and it is immediate from these facts that  $G$  is König if and only if  $I(G)$  is König.

**Definition 7.39.** A finite simple graph  $G$  is **bipartite** if there is a partition its vertex set  $V$  in two disjoint subsets  $V_1 \cup V_2$  such that every edge in  $G$  connects a vertex in  $V_1$  to a vertex in  $V_2$ .

**Theorem 7.40** (Simis–Vasconcelos–Villarreal, 1994, Theorem 5.9 in [SVV94]). *Let  $I = I(G)$  be the edge ideal of a graph  $G$ . Then  $I^{(n)} = I^n$  for all  $n$  if and only if  $G$  is bipartite.*

Note that in their paper, instead of  $I^{(n)} = I^n$  for all  $n$ , Simis, Vasconcelos, and Villarreal write the condition that  $I$  is *normally torsion free*, which means that  $\text{Ass}(I^t) = \text{Ass}(I)$  for all  $t$ . As we know, this is equivalent to the equality of symbolic and ordinary powers.

In [GRV05b, Proposition 4.27], Gitler, Reyes, and Villarreal show that if  $I$  is the edge ideal of a graph,  $G$  is bipartite if and only if  $I$  is packed, which completely solves the Packing Problem in this case.

For monomial ideals in general, the question of equality and the packing problem in particular remain open, though as we will see in the next section, Montaña and Núñez Betancourt have recently given a sufficient condition for monomial ideal to satisfy  $I^{(n)} = I^n$  for all  $n$ .

The Packing Problem originated in the context of optimization. To describe the corresponding combinatorial problem, we need to discuss clutters.

**Definition 7.41.** A **clutter**  $C = (V, E)$  is a collection of subsets  $E$  of  $V = \{v_1, \dots, v_n\}$  such that every two elements of  $E$  are incomparable with respect to inclusion. The **incidence matrix** of a clutter  $C$  is an  $m \times n$  matrix  $M(C)$  where each row corresponds to a vertex  $v_i \in V$  and each column corresponds to an edge  $e \in E(V)$ , and in the entry corresponding to the vertex  $v$  and the edge  $e$  we have a 1 or 0 according to whether or not  $v \in e$ .

One can define the covering number and independence numbers of a clutter just as above, and extend the definition of König to a hypergraph.

**Definition 7.42.** Let  $C$  be a clutter with vertices  $V = \{v_1, \dots, v_n\}$  and edges  $E(C)$ . Given a field  $k$ , the **edge ideal** of  $C$  is the monomial ideal in  $k[x_1, \dots, x_n]$  given by

$$I(C) := \left( \prod_{v_i \in e} x_i \mid e \in E \right).$$

The restriction that the edges should be incomparable automatically implies that the monomials corresponding to each edge of  $C$  form a minimal generating set for  $I(C)$ , and by construction,  $I(C)$  is a squarefree monomial ideal. On the other hand, given a squarefree monomial ideal  $I$  in  $n$  variables, one can define a clutter  $C$  on  $\{v_1, \dots, v_n\}$  with edges

$$E(C) = \{\{v_{i_1}, \dots, v_{i_s}\} \mid x_{i_1} \cdots x_{i_s} \in G(I)\}.$$

Since the minimal generators of  $I$  must be incomparable, this gives us a clutter, and thus we have a correspondence between clutters and squarefree monomial ideals.

We can also think of a clutter as the collection of facets of a simplicial complex. From that perspective, the edge ideal of a clutter is the facet ideal of the corresponding simplicial complex.

**Definition 7.43.** Let  $C = (V, E)$  be a clutter with  $n$  vertices  $V = \{v_1, \dots, v_n\}$  and incidence matrix  $A$ . Given any  $c \in \mathbb{Z}_{\geq 0}^n$ , the Strong Duality Theorem, which is a well-known result in Optimization, says that dual linear programs have the same optimal solution:

$$\min\{c \cdot x \mid x \in \mathbb{R}_{\geq 0}^n, M^t x \geq \mathbb{1}_m\} = \max\{\mathbb{1}_m \cdot y \mid y \in \mathbb{R}_{\geq 0}^m, My \leq c\}.$$

Here  $\mathbb{1}_n$  denotes the vector  $(1, \dots, 1) \in \mathbb{Z}^n$ . We say that  $C$  has the **max-flow min-cut property (MCMF)** if for every  $c \in \mathbb{Z}_{\geq 0}^n$  there exist integer solutions  $x \in \mathbb{Z}_{\geq 0}^n$  and  $y \in \mathbb{Z}_{\geq 0}^m$  to the linear programs above.

It turns out a clutter has the MFMC property if and only if its edge ideal  $I$  satisfies  $I^{(n)} = I^n$  for all  $n \geq 1$  (see [GRV05b, Corollary 3.14] and [HHTZ08, Corollary 1.6]).

**Definition 7.44.** A clutter  $C$  has the **packing property** if every minor of  $I(C)$  is König.

The Conjecture of Conforti and Cornuéjols is that  $C$  has the MFMC property if and only if it has the packing property.

## 7.7 A sufficient condition for monomial ideals

Morey's Theorem 7.24 gives an answer to the following question of Huneke's in a nice setting, though the general question is open:

**Question 7.45.** Let  $I$  be an ideal in a noetherian ring  $R$ . Is there a constant  $d$  such that if  $I^{(n)} = I^n$  for all  $n \leq d$ , then  $I^{(n)} = I^n$  for all  $n \geq 1$ ?

Since Brodmann showed that  $\text{Ass}(I^n)$  stabilizes, the naive answer is yes — one can take  $d$  to be the stability index such that  $\text{Ass}(I^n) = \text{Ass}(I^d)$  for all  $n \geq d$ . But in practical terms, this stability value is unknown, and indeed should be  $n = 1$  when the equality  $I^{(n)} = I^n$  holds for all  $n \geq 1$ . So we should interpret the question as asking for a simple, computable invariant of  $I$ , which does not require knowledge of the associated primes of the powers of  $I$ .

When  $I$  is a monomial ideal, such an invariant does exist, thanks to a recent theorem of Jonathan Montaño and Luis Núñez Betancourt [MnNb21]. The tools they use to do this are inspired by standard prime characteristic tools which can be adapted to monomial ideals in any characteristic.

**Definition 7.46.** Let  $R = k[x_1, \dots, x_d]$  be a standard graded polynomial ring over the field  $k$ . For each  $m \geq 1$ , let  $R^{1/m} := k[x_1^{1/m}, \dots, x_d^{1/m}]$ , and let  $\Phi_m : R^{1/m} \rightarrow R$  be the map induced by

$$\Phi_m(\underline{x}^{a/m}) = \begin{cases} \underline{x}^{a/m} & \text{if } a_i \equiv 0 \pmod{m} \\ 0 & \text{otherwise.} \end{cases}$$

where  $\underline{a} \in \mathbb{Z}_{\geq 0}^d$ .

**Remark 7.47.** When we restrict  $\Phi_m$  to  $R$ , we get the identity map on  $R$ . Therefore, this map  $\Phi_m$  is a splitting of the inclusion map  $R \subseteq R^{1/m}$ , and  $R$  is isomorphic to a direct summand of  $R^{1/m}$ .

**Remark 7.48.** Let  $I$  be a squarefree monomial ideal with minimal primes  $P_1, \dots, P_s$ . Since  $\Phi_m$  commutes with intersections of ideals, for all  $n$  and  $m$  we have

$$\begin{aligned} I^{(n)} &= P_1^n \cap \dots \cap P_s^n \\ &= (P_1^{nm})^{1/m} \cap \dots \cap (P_s^{nm})^{1/m} \\ &= (P_1^{nm} \cap \dots \cap P_s^{nm})^{1/m} \\ &= (I^{(nm)})^{1/m}. \end{aligned}$$

**Lemma 7.49** (Lemma 3.3 in [MnNb21]). *Let  $I$  be a squarefree monomial ideal. For all  $n \geq 0$ ,  $m \geq 1$ , and  $1 \leq j \leq m$ , we have*

$$\Phi_m((I^{(nm+j)})^{1/m}) = I^{(n+1)}.$$

*Proof.* By Remark 7.48,

$$I^{(n+1)} \subseteq (I^{((n+1)m)})^{1/m} \subseteq (I^{(mn+j)})^{1/m}.$$

On the other hand,  $\Phi_m$  is an  $R$ -linear map, and thus it preserves inclusions, and since it is the identity on  $R$ , we must have

$$I^{(n+1)} = \Phi_m(I^{(n+1)}) \subseteq \Phi_m((I^{(mn+j)})^{1/m}).$$

It remains to show that  $\Phi_m((I^{(mn+j)})^{1/m}) \subseteq I^{(n+1)}$ . First, we show the claim when  $I$  is prime, which implies that  $I$  is generated by variables. Without loss of generality, we can assume that  $I = (x_1, \dots, x_s)$  for some  $s$ . As a  $k$ -vector space,  $(I^{(mn+j)})^{1/m}$  is generated by

$$\{(\underline{x}^a)^{1/m} \mid a_1 + \dots + a_s \geq nm + j\}.$$

Given such an element  $(\underline{x}^a)^{1/m} \in (I^{(mn+j)})^{1/m}$ , if  $\Phi_m((\underline{x}^a)^{1/m}) \neq 0$ , then we must have both  $a_1 + \dots + a_s \geq nm + j$  and that each  $a_i$  is a multiple of  $m$ . Thus  $b_i = \frac{a_i}{m}$  is an integer for each  $i$ , and

$$b_1 + \dots + b_s = \frac{1}{m}(a_1 + \dots + a_s) \geq n + \frac{j}{m},$$

but since all the  $b_i$  are integers and  $j \geq 1$ , we must have

$$b_1 + \dots + b_s \geq n + 1.$$

Therefore,  $(\underline{x}^a)^{1/m} \in (\underline{x}^b) \subseteq I^{(n+1)}$ . Since this holds for all  $\underline{x}^a$  that is not sent to 0 by  $\Phi_m$ , we conclude that  $\Phi_m((I^{(mn+j)})^{1/m}) \subseteq I^{(n+1)}$ .

Now consider any radical squarefree monomial ideal  $I$ , and let  $P_1, \dots, P_s$  be its minimal primes. As in Remark 7.48,

$$(I^{(mn+j)})^{1/m} = (P_1^{(mn+j)})^{1/m} \cap \dots \cap (P_s^{(mn+j)})^{1/m},$$

so

$$\begin{aligned} \Phi_m \left( (I^{(mn+j)})^{1/m} \right) &= \Phi_m \left( (P_1^{(mn+j)})^{1/m} \cap \dots \cap (P_s^{(mn+j)})^{1/m} \right) \\ &\subseteq P_1^{(n+1)} \cap \dots \cap P_s^{(n+1)} \\ &= I^{(n+1)}. \end{aligned} \quad \square$$

**Lemma 7.50** (Theorem 4.5 in [MnNb21]). *Let  $I$  be a squarefree monomial ideal. The following are equivalent:*

- 1)  $\Phi_m((I^{nm+1})^{1/m}) \subseteq I^{n+1}$  for all  $m \geq 2$  and all  $n \geq 1$ .
- 2)  $\Phi_m((I^{nm+1})^{1/m}) \subseteq I^{n+1}$  for some  $m \geq 2$  and all  $n \geq 1$ .
- 3)  $I^{(n)} = I^n$  for all  $n \geq 1$ .

*Proof.* The fact that 1) implies 2) is obvious. By Lemma 7.49,

$$\Phi_m((I^{nm+1})^{1/m}) \subseteq I^{(n+1)}$$

holds for all  $m$  and all  $n$ , so if we assume that all symbolic powers are the powers, this simplifies to 1). Thus we only need to show that 2) implies 3).

Consider the ideal  $J$  in the Rees algebra  $R[It]$  given by

$$J := IR[It] = \bigoplus_{n \geq 0} I^{n+1}t^n \subseteq R[It].$$

Since  $\text{gr}_I(R) \cong R[It]/J$ , the ideal  $J$  is radical if and only if  $\text{gr}_I(R)$  is reduced. We will show that  $J$  is radical, which by Corollary 7.8 implies  $I^{(n)} = I^n$  for all  $n \geq 1$ .

To show that  $J$  is radical, consider a monomial  $f \in I^n$  such that  $ft^n \in \sqrt{J}$ . Fix  $e$  such that  $(ft^n)^{m^e} \in J$ , which must necessarily exist since  $(ft^n)^s \in J$  for all  $s \gg 0$ .

We can extend  $\Phi_m$  naturally to  $R[t]$ , which induces a map on  $R[It]^{1/m} := \bigoplus I^{n/m}t^{n/m}$  with

$$\Phi(r^{1/m}t^{n/m}) = \begin{cases} \Phi(r^{1/m})t^{n/m} & \text{if } m \text{ divides } n \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} f^{m^{e-1}}t^{nm^{e-1}} &= \Phi_m \left( (f^{m^e}t^{nm^e})^{1/m} \right) \\ &\in \Phi_m \left( (I^{nm^e+1}t^{nm^e})^{1/m} \right) \quad \text{since } f^{m^e}t^{nm^e} \in J_{nm^e} = I^{nm^e+1} \\ &= \Phi_m \left( (I^{nm^e+1})^{1/m} \right) t^{nm^{e-1}} \\ &\subseteq I^{nm^{e-1}+1}t^{nm^{e-1}} \quad \text{by assumption} \\ &\subseteq J. \end{aligned}$$

If we keep repeating the argument, in a sort of reverse induction, we eventually get to  $ft^n \in J$ . This shows that  $J$  is radical, and as described above this completes the proof.  $\square$

**Theorem 7.51** (Montaño – Núñez Betancourt, Theorem 4.8 in [MnNb21]). *Let  $I$  be a squarefree monomial ideal. If  $I^{(n)} = I^n$  for all  $n \leq \left\lceil \frac{\mu(I)}{2} \right\rceil$ , then  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

*Proof.* Set  $\mu := \mu(I)$ . First, we are going to show that

$$\Phi_2 \left( (I^{2n+1})^{1/2} \right) \subseteq I^{n+1}$$

for all  $n$ . By Lemma 7.49,

$$\Phi_2 \left( (I^{2n+1})^{1/2} \right) \subseteq \Phi_2 \left( (I^{(2n+1)})^{1/2} \right) \subseteq I^{(n+1)}.$$

When  $n < \left\lceil \frac{\mu(I)}{2} \right\rceil$ ,  $I^{(n+1)} = I^{n+1}$ , so

$$\Phi_2 \left( (I^{2n+1})^{1/2} \right) \subseteq I^{n+1}.$$

Now fix  $n \geq \left\lceil \frac{\mu(I)}{2} \right\rceil$ , and note that  $2n + 1 \geq \mu(I) + 1$ . Since  $I^{2n+1}$  is generated by elements of the form

$$f_1^{a_1} \cdots f_\mu^{a_\mu}$$

with  $a_1 + \cdots + a_\mu \geq 2n + 1 > \mu(I)$ , necessarily there exists an  $i$  such that  $a_i \geq 2$ . Therefore, the power  $I^{2n+1}$  is generated by elements of the form  $fg^2$ , where  $g \in I$  and  $f \in I^{2n-1}$  are monomials. Note that  $(fg^2)^{1/2} = f^{1/2}g$  and

$$\Phi_2 \left( (fg^2)^{1/2} \right) = \Phi_2(f^{1/2})g,$$

so

$$\Phi_2 \left( (I^{2n+1})^{1/2} \right) = \Phi_2 \left( (I^{2n-1})^{1/2} \right) I.$$

By induction, we may assume that  $\Phi_2 \left( (I^{2n-1})^{1/2} \right) \subseteq I^n$ , so

$$\Phi_2 \left( (I^{2n+1})^{1/2} \right) = \Phi_2 \left( (I^{2n-1})^{1/2} \right) I \subseteq I^n I = I^{n+1}.$$

By Lemma 7.50, this implies  $I^{(n)} = I^n$  for all  $n \geq 1$ . □



# Chapter 8

## A crash course on prime characteristic tools

The subject of prime characteristic commutative algebra is vast and interesting in its own right. Here we cover only a tiny corner of the subject, with an eye towards proving theorems about symbolic powers. For more on characteristic  $p$ , there are a few sets of lecture notes I strongly recommend:

- Karen Smith’s lecture notes;
- Karl Schwede’s lecture notes;
- Mel Hochster’s lecture notes (of which there are multiple versions, all accessible on his website);
- Linquan Ma and Thomas Polstra’s notes.

These short notes draw from all those sources.

### 8.1 The Frobenius map

The advantages of being in prime characteristic  $p$  are many, and they all start with the Frobenius map.

**Definition 8.1.** Let  $R$  be a ring of prime characteristic  $p$ . The **Frobenius map** on  $R$  is the ring homomorphism  $F: R \rightarrow R$  given by  $F(x) = x^p$ .

The fact that this is a ring homomorphism is a simple exercise; the crucial point is that in characteristic  $p$ ,  $(a + b)^p = a^p + b^p$ , a fact often known as the *Freshman’s Dream*.

Many properties of  $R$  are encoded in the Frobenius map. Depending on the circumstances, we may think of  $R$  and its relationship with the Frobenius map in different ways, which help us identify those nice properties.

We can view  $R$  as an  $R$ -module over itself with a new structure, the one induced by the Frobenius map  $F$ , via restriction of scalars. This can be confusing, since the underlying abelian group is  $R$  but the  $R$ -module structure is not the usual one, so we denote this module by  $F_*(R)$ .

**Notation 8.2.** The  $R$ -module on the abelian group  $R$  with  $R$ -module structure

$$s \cdot r := F(s)r = s^p r$$

is denoted by  $F_*$ . More generally, the  $R$ -module on the abelian group  $R$  with  $R$ -module structure given by

$$s \cdot r := F^e(s)r = s^{p^e} r$$

is denoted by  $F_*^e(R)$ .

In this notation, the iterated Frobenius map  $F^e: R \rightarrow F_*^e(R)$  is an  $R$ -module homomorphism.

We may also think of the Frobenius map as a functor from  $R$ -modules to  $R$ -modules, where an  $R$ -module  $M$  is taken to the  $R$ -module  $F_*(M)$  with the same underlying abelian group  $M$  but with  $R$ -action given by

$$rF_*(m) = F(r)m = r^p m.$$

This is the restriction of scalars functor induced by  $F$ . Note that this is an exact functor, since it is the identity at the level of abelian groups.

When  $R$  is a reduced ring, we may also consider the subring of  $R$  of  $p^e$ th powers, denoted

$$R^{p^e} := \{r^{p^e} \mid r \in R\},$$

which is the image of the ring homomorphism given by the iterated Frobenius map  $F^e: R \rightarrow R$ . This map then factors through the inclusion  $R^{p^e} \subseteq R$ , so one may sometimes identify  $F$  with this inclusion.

Finally, when  $R$  is a domain, one may fix an algebraic closure of the fraction field of  $R$ , and consider the subring of  $p^e$ th roots of elements of  $R$ :

$$R^{1/p^e} := \{r^{1/p^e} \mid r \in R\}.$$

This ring  $R^{1/p^e}$  is the inspiration for the rings used by Monataño and Núñez Betancourt that we discussed in Section 7.7. The Frobenius map on  $R^{1/p^e}$  has image  $R \subseteq R^{1/p^e}$ , so we can identify the Frobenius map with the inclusion  $R \subseteq R^{1/p^e}$ .

Ultimately, these three different points of view are all equivalent:

- The structure of  $F_*^e(R)$  as an  $R$ -module.
- The structure of  $R^{1/p^e}$  as an  $R$ -module.
- The structure of  $R$  as an  $R^{p^e}$ -module.

Each perspective might be more or less convenient depending to the context.

**Definition 8.3.** Let  $I$  be an ideal in a ring  $R$  of prime characteristic  $p$ . The  $e$ th **Frobenius power of  $I$**  is the ideal

$$I^{[p^e]} := (f^{p^e} \mid f \in I).$$

This is the image of  $I$  via the Frobenius map. Note that  $F_*^e(I^{[p^e]}) = IF_*^e(R)$ .

The first thing one may ask about the module structure induced by the Frobenius map is whether it is finitely generated.

**Definition 8.4.** Let  $R$  be a ring of prime characteristic  $p$ . We say  $R$  is  *$F$ -finite* if  $F_*(R)$  is a finitely generated  $R$ -module.

**Example 8.5.** If  $k$  is a perfect field, then  $k$  is an  $F$ -finite ring. Therefore, any finitely generated algebra over a perfect field is an  $F$ -finite ring.

There are many conditions one might impose on the Frobenius map — or equivalently, on the  $R$ -module structure of  $F_*^e(R)$  — which are collectively known as  *$F$ -singularities*. Ultimately,  $F$ -singularities detect how *singular* (read: not nice) the ring is. Some of these conditions are not new, but old conditions rephrased in terms of Frobenius.

**Lemma 8.6.**  *$R$  is reduced if and only if the Frobenius map is injective.*

*Proof.* If  $R$  is reduced, then there are no nilpotent elements. In particular, for any  $r \in R$ ,  $F(r) = r^p \neq 0$  unless  $r = 0$ , so  $F$  is injective.

Conversely, supposed  $F$  is injective, and let  $r \in R$ . If  $r$  is nilpotent, then  $r^n = 0$  for some  $n$ . Let  $e$  be such that  $p^e > n$ . Then  $F^e(r) = r^{p^e} = r^n r^{p^e-n} \in (r^n) = (0)$ . But  $F$  is injective, and thus so is  $F^e$ , so we must have  $r = 0$ .  $\square$

We note that there is a type of  $F$ -singularity called  $F$ -injective, and it does not mean reduced — more precisely, it does not mean that  $F$  is injective.

One of our main goals is to prove Theorem 9.13, and for that we will be focusing on regular rings of prime characteristic. One of the main facts we will need is that over a regular ring, the Frobenius map is flat. This is also one of the points where the assumption that we are working over a regular ring is crucial: the flatness of Frobenius *characterizes* regular rings.

**Theorem 8.7** (Kunz, 1969 [Kun69]). *If  $R$  is a reduced local ring of prime characteristic  $p$ . Then  $R$  is regular if and only if  $F_*^e(R)$  is a flat  $R$ -module for some  $e$ .*

When  $R$  is a noetherian local ring, any finitely generated flat  $R$ -module must be free. Therefore, if  $R$  is  $F$ -finite, Kunz's theorem says that  $R$  is regular if and only if  $F_*^e(R)$  is free.

**Example 8.8.** Let  $k$  be a perfect field and let  $R = k[x_1, \dots, x_d]$ . Then  $R$  is the free module over  $R^p$  given by

$$\bigoplus_{0 \leq a_i \leq p-1} R^p \cdot x_1^{a_1} \cdots x_d^{a_d}.$$

Notice that this is equivalent to saying that  $F_*(R)$  is a free  $R$ -module. More generally, for any  $e \geq 1$ ,

$$R = \bigoplus_{0 \leq a_i \leq p^e-1} R^{p^e} \cdot x_1^{a_1} \cdots x_d^{a_d}.$$

## 8.2 Splittings

In prime characteristic, we can measure how far away the ring is from being regular by asking how far  $F_*^e(R)$  is from being free.

**Definition 8.9.** Let  $R$  be a ring of prime characteristic  $p$ . We say  $R$  is **F-split** if the Frobenius map **splits** as a map of  $R$ -modules, meaning that there exists an  $R$ -module map  $\beta$ , called a **splitting** such that

$$R \xrightleftharpoons[F]{\beta} F_*(R)$$

$$\beta \circ F = \text{id}_R.$$

The concept of a split map is very important in characteristic  $p$  commutative algebra. The fact that the inclusion map  $f: A \rightarrow B$  splits is equivalent to asking that if  $C = \text{coker } f$  and  $g$  denotes the quotient map  $B \rightarrow C$ , then the short exact sequence with

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

splits. There is a very well-known result in homological algebra (the Splitting Lemma) which says that the sequence splits if and only if  $B \cong A \oplus \text{coker } f$ , which happens if and only if there is a map  $\beta: C \rightarrow B$  such that  $g \circ \beta = \text{id}_C$ .

**Remark 8.10.** Any homomorphism of  $R$ -modules  $\beta \in \text{Hom}_R(F_*(R), R)$  such that  $\beta(1) = 1$  is a splitting of the Frobenius map, since by  $R$ -linearity

$$\beta(F(r)) = \beta(rF(1)) = r\beta(F(1)) = r\beta(1) = r$$

for all  $r \in R$ .

We will make use of the fact that splitting is a local property, which we explain in detail below.

**Remark 8.11.** Let  $M \xrightarrow{f} N$  be a map of finitely generated  $R$ -modules, where  $R$  is a noetherian ring. We claim that this map splits if and only if the induced map  $f_P: M_P \rightarrow N_P$  splits for all primes  $P$ ; in fact, it is sufficient to consider whether  $f_{\mathfrak{m}}$  splits for maximal ideals  $\mathfrak{m}$ . One implication is immediate: if the map splits, then the localization of a splitting induces a splitting on the localization.

To prove the converse, we first note a general fact about split exact sequences: a short exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

splits if and only if there exists  $\beta \in \text{Hom}_R(C, B)$  such that  $g \circ \beta = \text{id}_C$ , which is equivalent to saying that  $\text{id}_C$  is in the image of the induced map

$$\text{Hom}_R(C, B) \xrightarrow{g_*} \text{Hom}_R(C, C).$$

Now we ready to show that splitting is a local property. Suppose that  $f$  does not splits, and let  $Q$  be the cokernel of  $f$ , so that

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0$$

is a short exact sequence. We claim that there exists a maximal ideal  $\mathfrak{m}$  such that

$$0 \longrightarrow M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}} \longrightarrow Q_{\mathfrak{m}} \longrightarrow 0$$

does not split. Let

$$C := \text{coker}(\text{Hom}_R(Q, M) \rightarrow \text{Hom}_R(Q, Q)).$$

Note that  $C$  is a quotient of  $\text{Hom}_R(Q, Q)$ , so let  $g$  denote the image of  $\text{id}_Q$  in  $C$ . Since the original sequence does not split,  $g \neq 0$ . Therefore,  $\text{ann}_R(g) \neq R$ , and thus we can find a maximal ideal  $\mathfrak{m} \supseteq \text{ann}_R(g)$ . Since all our modules are finitely generated and  $R$  is noetherian, localization commutes with  $\text{Hom}$ , and since localization is exact and  $\text{Hom}_R(Q, -)$  is left exact, we obtain the following exact sequence:

$$0 \longrightarrow \text{Hom}(Q_{\mathfrak{m}}, M_{\mathfrak{m}}) \longrightarrow \text{Hom}(Q_{\mathfrak{m}}, N_{\mathfrak{m}}) \longrightarrow \text{Hom}(Q_{\mathfrak{m}}, Q_{\mathfrak{m}}) \longrightarrow C_{\mathfrak{m}} \longrightarrow 0$$

By construction,  $\frac{g}{1} \neq 0$ , and thus the image of the identity map on  $Q_{\mathfrak{m}}$  in  $C_{\mathfrak{m}}$  is nonzero. Therefore, our comment above shows that

$$0 \longrightarrow M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}} \longrightarrow Q_{\mathfrak{m}} \longrightarrow 0$$

is not split.

**Remark 8.12.** The localization of the Frobenius map is the Frobenius map of the localization, and thus the previous remark shows that the F-split property is a local property.

When  $R$  is  $F$ -finite, the F-split condition is equivalent to  $F$ -purity, which was first defined by Hochster and Roberts in the same paper where they show rings of invariants of linearly reductive groups are Cohen-Macaulay [HR74]. Funny enough, that result follows directly if one proves these are strongly F-regular, a notion that Hochster and Huneke would invent more than a decade later [HH89b].

**Definition 8.13** (Hochster—Roberts, 1974 [HR74]). Let  $M \xrightarrow{f} N$  be a homomorphism of  $R$ -modules. We say  $f$  is **pure** if for every  $R$ -module  $W$ , the induced map

$$\begin{aligned} M &\longrightarrow S \otimes_R M \\ m &\longmapsto f(1) \otimes m \end{aligned}$$

is injective. A ring extension  $R \rightarrow S$  is **pure** if it is pure as a map of  $R$ -modules. Finally, let  $R$  be a ring of prime characteristic  $p$ . We say  $R$  is **F-pure** if the Frobenius map is pure, meaning that for every  $R$ -module  $M$ ,  $M \rightarrow M \otimes_R F_*(R)$  is injective.

**Remark 8.14.** Suppose that the ring homomorphism  $f: R \rightarrow S$  splits, meaning that there exists a splitting  $\beta$  such that  $\beta \circ f = \text{id}_R$ . In particular, this implies that  $f$  is injective. Moreover, this also implies that  $S = R \otimes N$  for some  $R$ -module  $N = \ker \beta$ , and thus for any  $R$ -module  $M$ ,  $S \otimes_R M \cong M \oplus N \otimes_R M$ . In particular, the map  $M \rightarrow S \otimes_R M$  induced by  $f$  is injective.

As a consequence, we conclude that any F-split ring is F-pure.

When  $R$  is  $F$ -finite, the two notions coincide.

**Lemma 8.15.** *Let  $R$  be an  $F$ -finite ring of characteristic  $p$ . Then  $R$  is  $F$ -pure if and only if  $R$  is  $F$ -split.*

We will be focusing on  $F$ -finite rings, so we will not need to worry about the distinction. Nevertheless, we note that the two notions do indeed differ when  $R$  is not  $F$ -finite. As it often happens in the literature, we will prefer the term F-pure to F-split, though again they are completely synonymous in our setting.

**Lemma 8.16.** *Let  $(R, \mathfrak{m})$  be an  $F$ -finite local ring. If  $R$  is regular then  $R$  is  $F$ -split.*

*Proof.* Let  $(R, \mathfrak{m})$  be an  $F$ -finite regular local ring. By [Kunz's Theorem](#),  $F_*(R)$  is flat, and thus free. We claim that  $F_*(1)$  is part of a free basis. On the one hand,  $1 \notin \mathfrak{m}^{[p]}$ , while  $F_*(\mathfrak{m}^{[p]}) = \mathfrak{m}F_*(R)$ , so  $F_*(1) \notin \mathfrak{m}F_*(R)$  is indeed a minimal generator of  $F_*(R)$ . The projection map from  $F_*(R)$  to the free submodule generated by  $F_*(1)$  is a homomorphism of  $R$ -modules  $F_*(R) \rightarrow R$  sending  $F_*(1)$  to 1, and thus it is a splitting for the Frobenius map.  $\square$

Recall that we say that a ring  $R$  is a **direct summand** of a ring  $S$  if the inclusion  $R \subseteq S$  splits as a map of  $R$ -modules, or equivalently, that there exists an  $R$ -module  $M$  such that  $S = R \oplus M$ .

**Theorem 8.17.** *Let  $R \subseteq S$  be an inclusion of rings of prime characteristic  $p$  such that  $R$  is a direct summand of  $S$ . If  $S$  is  $F$ -split, then so is  $R$ .*

*Proof.* Identifying the Frobenius map on  $R$  with the inclusion  $R^p \subseteq R$  and the Frobenius map on  $S$  with the inclusion  $S^p \subseteq S$ , we want to show that the inclusion  $R^p \subseteq R$  splits.

Consider the following diagram, where  $\beta$  is a splitting of the inclusion  $R \rightarrow S$  and  $\phi$  is a splitting of the Frobenius map  $S^p \rightarrow S$ , and the  $\oplus$  signs indicate a map splits:

$$\begin{array}{ccc}
 R & \xleftarrow{\beta} & S \\
 \downarrow \beta \circ \phi \circ i & \searrow i & \uparrow \oplus 1 \phi \\
 R^p & \xrightarrow{\beta} & S^p
 \end{array}$$

For each  $r \in R$ ,  $r^p \in R^p$  is an element of  $S^p$ , and thus  $\phi(r^p) = r^p$ , since  $\phi$  is a splitting of the Frobenius map on  $S$ . Similarly,  $\beta(r^p) = r^p$ , since  $\beta$  is the identity on  $R$ . Identifying  $\beta$  with its restriction to  $S^p$ , we see that

$$(\beta \circ \phi \circ i)(r^p) = \beta(\phi(r^p)) = \beta(r^p) = r^p,$$

so  $\beta \circ \phi \circ i$  is a splitting of  $R^p \subseteq R$ .  $\square$

**Definition 8.18.** Let  $k$  be a field and  $R = k[x_1, \dots, x_d]$ . The  $n$ th **Veronese subring** of  $R$  is the  $k$ -algebra generated by all the degree  $n$  monomials in  $R$ , meaning

$$R^{(n)} := k[x_1^{a_1} \cdots x_d^{a_d} \mid a_1 + \cdots + a_d = n].$$

More generally, if  $R$  is a graded  $k$ -algebra, the  $n$ th Veronese of  $R$  is the  $k$ -subalgebra of  $R$  spanned by all the homogeneous elements in  $R$  whose degree is a multiple of  $n$ , meaning that

$$R^{(n)} := \bigoplus_i R_{in}.$$

**Corollary 8.19.** Let  $R = k[x_1, \dots, x_d]$ , where  $k$  is a perfect field of characteristic  $p > 0$ , and let  $R^{(n)}$  be the  $n$ th Veronese of  $R$ . Then  $R$  is  $F$ -split.

*Proof.* The main point is that Veronese rings are direct summands of the corresponding polynomial ring. The graded map  $R \rightarrow R^{(n)}$  defined on homogeneous elements as

$$x_1^{a_1} \cdots x_d^{a_d} \mapsto \begin{cases} x_1^{a_1} \cdots x_d^{a_d} & \text{if } a_1 + \cdots + a_d \equiv 0 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

is a splitting. By Lemma 8.16,  $R$  is  $F$ -split, and thus by Theorem 8.17 so is  $R^{(n)}$ .  $\square$

$F$ -purity can be detected in a very precise and computable way.

**Theorem 8.20** (Fedder's Criterion, 1983 [Fed83]). Let  $(R, \mathfrak{m})$  be a regular local ring of characteristic  $p$ , and let  $I$  be a radical ideal  $I$  in  $R$ . The ring  $R/I$  is  $F$ -pure if and only if  $(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$  for all (or equivalently, some)  $q = p^e$ .

This is especially simple to use for a hypersurface: if  $R$  is a regular local ring and  $f \in R$ , the hypersurface  $R/f$  is  $F$ -pure if and only if  $f^{p-1} \notin \mathfrak{m}^{[p]}$ . Notice also that Fedder's criterion can easily be used to test for  $F$ -purity in Macaulay2. The only downside is that if  $p$  is large, the computations can get considerably slow.

**Exercise 17.** Let  $I$  be a squarefree monomial ideal in  $R = k[x_1, \dots, x_d]$ , where  $k$  is a perfect field. Show that  $R/I$  is  $F$ -pure.

**Metatheorem 8.21** (Luis Núñez Betancourt). Any fact about squarefree monomial ideals probably extends to any ideal defining an  $F$ -pure ring.

We will later see an example of such a theorem in the context of symbolic powers. This philosophy extends even to applying characteristic  $p$  like techniques to proving results about monomial ideals — a great example of this is Montaña and Núñez Betancourt's Theorem 7.51] we proved in the previous chapter.

The  $F$ -purity condition is a measure of *nice* singularities, and yet the class of  $F$ -pure rings is quite large, containing all Stanley-Reisner rings and all direct summands of  $F$ -finite regular rings. In an  $R$ -pure ring, the Frobenius map has *one* splitting; if it has *many* splittings, our rings has very nice singularities.

**Definition 8.22** (Hochster–Huneke [HH89b]). Let  $R$  be a reduced  $F$ -finite ring of prime characteristic  $p$ . We say  $R$  is **strongly  $F$ -regular** if for every  $c \in R$  that is not in any minimal prime of  $R$ , there exists  $e \gg 0$  such that the map  $R \rightarrow R^{1/p}$  sending  $1 \mapsto c^{1/p}$  splits.

**Remark 8.23.** Strongly F-regular rings are F-split: if  $R$  is strongly F-regular, then in particular the map  $R \rightarrow R^{1/p^e}$  sending  $1 \rightarrow 1$  splits, and this is just  $F^e$ , so  $R$  is also F-pure.

**Lemma 8.24.** *Every F-finite regular ring is strongly F-regular.*

*Proof.* Regularity and strong F-regularity properties both localize, so we can reduce to the local case. Let  $(R, \mathfrak{m})$  be a regular local ring of prime characteristic  $p$ . Recall that regular rings are domains. Given any nonzero  $c \in R$ , if  $c^{1/p^e} \in \mathfrak{m}R^{1/p^e}$  for all  $e$  then  $c \in \mathfrak{m}^{[p^e]}$  for all  $e$ . Now note that

$$\bigcap_{e \geq 0} \mathfrak{m}^{[p^e]} \subseteq \bigcap_{e \geq 0} \mathfrak{m}^{p^e} = 0,$$

where the intersection is 0 by [Krull's Intersection Theorem](#). Therefore,  $c^{1/p^e} \notin \mathfrak{m}R^{1/p^e}$  for some  $e$ , and thus it is a minimal generator for  $R^{1/p^e}$ . Since  $R$  is regular and local, by [Theorem 8.7](#)  $F_*^e(R)$  is free, and thus  $c^{1/p^e}$  is a basis element for this free module. The map  $R \rightarrow R^{1/p}$  sending  $1 \mapsto c^{1/p}$  splits, since the projection  $R^{1/p} \rightarrow R$  onto the summand corresponding to  $c^{1/p}$  send  $c^{1/p}$  to 1.  $\square$

Strongly F-regular rings are very nice; in particular, they are normal and Cohen-Macaulay. There is a criterion similar [Fedder's Criterion](#) for strongly F-regular rings:

**Theorem 8.25** (Glassbrenner's Criterion for strong F-regularity, 1996 [[Gla96](#)]). *Let  $(R, \mathfrak{m})$  be an F-finite regular local ring of prime characteristic  $p$ . Given a proper radical ideal  $I$  of  $R$ ,  $R/I$  is strongly F-regular if and only if for each element  $c \in R$  not in any minimal prime of  $I$ , we have  $c(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$  for all  $q = p^e \gg 0$ .*

Fun fact: Richard Fedder and Donna Glassbrenner were both Mel Hochster's students, and [Fedder's Criterion](#) was part of Richard Fedder's PhD thesis, while [Glassbrenner's Criterion](#) was part of Donna Glassbrenner's Phd thesis.

**Lemma 8.26** (Theorem 5.5 c) in [[HH94a](#)]). *Strongly F-regular rings are products of domains. In particular, any local strongly F-regular ring must be a domain.*

**Exercise 18.** Let  $I$  be a squarefree monomial ideal in  $R = k[x_1, \dots, x_d]$ , where  $k$  is a perfect field. Show that  $R/I$  is strongly F-regular if and only if  $I$  is generated by variables.

One can adapt the proof of [Theorem 8.17](#) to show that direct summands of strongly F-regular rings are strongly F-regular:

**Theorem 8.27.** *Let  $R \subseteq S$  be an inclusion of rings of prime characteristic  $p$  such that  $R$  is a direct summand of  $S$ . If  $S$  is strongly F-regular, then so is  $R$ .*

Strongly F-regular rings can be roughly described as *very nice* singularities, but again there are many interesting classes of rings that are strongly F-regular.

**Example 8.28.** Let  $k$  be a field of characteristic  $p$  and  $R = k[x_1, \dots, x_n]$ . Combining [Theorem 8.27](#) and the proof of [Corollary 8.19](#), we see that the  $n$ th Veronese subring of  $R$  is strongly F-regular. More generally, given a homogeneous ideal  $J$  in  $A$ , if  $A/J$  is strongly F-regular, then any Veronese subring of  $A/J$  is also strongly F-regular.



**Example 8.29.** Let  $k$  be a field of characteristic  $p$ , and consider a generic  $m \times n$  matrix  $M$ . Given any  $t \leq m, n$ , the  $k$ -algebra generated by all the  $t$ -minors of  $M$  is strongly F-regular. Moreover, if  $R$  is the polynomial ring over  $k$  generated by the entries of  $M$ , and  $I = I_t(M)$ , then  $R/I$  is strongly F-regular. More generally, rings of invariants of classical groups, after reduction modulo  $p$ , are strongly F-regular [SVdB97, Theorem 5.2.3].

Locally acyclic cluster algebras [BMRS15] and certain ladder determinantal varieties [GS95] are also strongly F-regular. See also [BT06].

## 8.3 Tight closure

Tight closure is a very useful tool introduced by Hochster and Huneke in 1989 [HH89a] which revolutionized characteristic  $p$  commutative algebra.

**Definition 8.30** (Tight Closure). Let  $R$  be a domain of prime characteristic  $p$ . Given an ideal  $I$  in  $R$ , the *tight closure* of  $I$  is the ideal

$$I^* = \{z \in R \mid \text{there exists a nonzero } c \in R \text{ such that } cz^q \in I^{[q]} \text{ for all } q = p^e \gg 0\}.$$

When  $R$  is reduced but not a domain, one takes  $c$  not in any minimal prime of  $R$ .

**Remark 8.31.** Notice that  $I \subseteq I^*$ .

It is sometimes easier to prove something is contained in the tight closure of an ideal than in the ideal itself. This idea is especially useful if we are working over a regular ring, since all ideals are tightly closed. The main point is that taking Frobenius powers commutes with the formation of colon ideals.

**Exercise 19.** Let  $R$  be a regular ring and let  $I$  and  $J$  be ideals in  $R$ . Then for all  $q = p^e$ ,  $(I : J)^{[q]} = (I^{[q]} : J^{[q]})$ .

**Theorem 8.32** (Theorem (4.4) in [HH90]). *Let  $R$  be a regular ring containing a field of prime characteristic. Then  $I = I^*$  for every ideal  $I$  in  $R$ .*

*Proof.* Let  $I$  be a proper ideal in  $R$ , and suppose that  $x \in I^*$  but  $x \notin I$ . By assumption,  $(I : x)$  is a proper ideal, so let  $\mathfrak{m}$  be a maximal ideal containing  $(I : x)$  and localize at  $\mathfrak{m}$ . Replacing  $R$  with  $R_{\mathfrak{m}}$ , we may as well assume that  $R$  is local with maximal ideal  $\mathfrak{m}$ . By assumption, there exists a nonzero  $c$  and a positive integer  $e'$  such that  $cx^q \in I^{[q]}$  for all  $q = p^e$  with  $e \geq e'$ . Therefore,

$$c \in \bigcap_{q \geq q'} (I^{[q]} : x^{[q]}) = \bigcap_{q \geq q'} (I : x)^{[q]} \subseteq \bigcap_{q \geq q'} \mathfrak{m}^{[q]} = 0,$$

where the last equality is just [Krull's Intersection Theorem](#). We conclude that  $c = 0$ , which contradicts our assumption that  $c \neq 0$ . We conclude that no such  $x$  exists, and thus  $I^* = I$ .  $\square$

A ring in which every ideal is tightly closed is called **weakly F-regular**. In that language, the theorem above says that regular rings are weakly F-regular. Every strongly F-regular ring is weakly F-regular. A famous open question in tight closure theory, known as the *Weak implies strong conjecture*, asks whether every weakly F-regular ring is strongly F-regular. Hochster and Huneke showed this holds when  $R$  is Gorenstein and  $F$ -finite [HH89a], and Lyubeznik and Smith showed it holds when  $R$  is graded over a field of prime characteristic [LS01], though the general case remains open.

Another famous conjecture was whether tight closure commutes with localization; turns out this is false! Brenner and Monsky found a counterexample, in a paper aptly called *Tight closure does not commute with localization*<sup>1</sup> [BM10]. Since strong F-regularity is preserved by localization, the Weak implies strong conjecture would imply that weak F-regularity also commutes with localization.

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<sup>1</sup>Metatheorem: the best paper titles are those that contain the main result of the paper.

# Chapter 9

## The Containment Problem

This chapter is dedicated to the Containment Problem:

**Question 9.1** (Containment Problem). Let  $I$  be an ideal in a noetherian ring  $R$ . For what values of  $a$  and  $b$  is  $I^{(a)} \subseteq I^b$ ?

This question first appeared in work of Schenzel in the 1980s [Sch86], but gained more serious attention in the new millennium, in large part due to Irena Swanson's answers [Swa00] to some of Schenzel's questions, which then inspired work of Ein–Lazarsfeld–Smith and Hochster–Huneke [ELS01, HH02] in the early 2000s.

Giving an answer to the containment problem is a way to answer various other questions. If  $I^{(n)} \subseteq I^n$ , then we must have  $I^n = I^{(n)}$ , so from that perspective the containment problem also contains the equality question as a subproblem. Moreover, if  $I$  is homogeneous and  $a$  and  $b$  are such that  $I^{(a)} \subseteq I^b$ , then we also obtain a lower bound for the smallest degree  $\alpha(I^{(a)})$  nonzero homogeneous elements in  $I^{(a)}$ , since we must necessarily have  $\alpha(I^{(a)}) \subseteq I^b$ . We can also think of the Containment Problem as an attempt to compare the symbolic and ordinary powers of  $I$  when they are not equal.

Ultimately, the Containment Problem is interesting in its own right, as we hope to show in this chapter. In a way, this is an open ended question: there is no hope of ever completely determining the best possible answer for all ideals in all rings. In fact, even if we restrict to regular rings, there is much that we do not understand even about the smallest symbolic power contained in  $I^2$ . Nevertheless, one may hope to give best possible answers that work for all radical ideals in a fixed ring, and we will indeed see that there are many things we can say. In this chapter, we will discuss some beautiful theorems and many interesting open questions; this is a subject of very active current research, with many fun open problems to work on.

### 9.1 Linear equivalence of topologies

Before we even try to answer the Containment Problem, the first question we should is whether an answer even *exists*: given an ideal  $I$  and a positive integer  $b$ , must there exist  $a$  such that  $I^{(a)} \subseteq I^b$ ? This question is equivalent to asking if the topologies determined by the filtrations  $\{I^n\}$  and  $\{I^{(n)}\}$  are equivalent. To make that clear, we first, we need to recall a few elementary things about filtrations and completion.

**Definition 9.2.** Let  $I$  be an ideal on a ring  $R$ . The  $I$ -**adic topology** on  $R$  is the topology that has the following basis of open sets:

$$\{r + I^n \mid r \in R, n \geq 1\}.$$

The  $I$ -**symbolic topology** on  $R$  is the topology with the following basis of open sets:

$$\{r + I^{(n)} \mid r \in R, n \geq 1\}.$$

More generally, any filtration  $\{I_n\}$  determines a topology on  $R$  with basis elements of the form  $r + I_n$  for  $r \in R$  and  $n \geq 1$ .

**Definition 9.3.** A **Cauchy sequence** on  $R$  with respect to the  $I$ -adic topology is a sequence of elements  $\{r_i\}_i$  such that for each  $n$  there exists  $N$  such that  $r_i - r_j \in I^n$  for all  $i, j > N$ . Two Cauchy sequences  $\{r_i\}$  and  $\{s_i\}$  are **equivalent** if for all  $n$  there exists  $N$  such that  $r_i - s_i \in I^n$  for all  $i > N$ . The set of equivalence classes of Cauchy sequences on  $R$  forms a ring, the  $I$ -**adic completion** of  $R$ , denoted  $\widehat{R}^I$ . If  $(R, \mathfrak{m})$  is a local ring, we write  $\widehat{R}$  for  $\widehat{R}^{\mathfrak{m}}$ , and call it the **completion** of  $R$ .

**Remark 9.4.** Let  $\mathcal{I} = \{I_n\}$  and  $\mathcal{J} = \{J_n\}$  be two filtrations on a ring  $R$ . If for every  $n$  there exists  $m$  such that  $J_m \subseteq I_n$ , then the topology determined by  $\mathcal{J}$  is **finer** than the topology defined by  $\mathcal{I}$ , any open set in the topology determined by  $\mathcal{I}$  must contain an open set in the finer topology.

**Definition 9.5.** We say the two filtrations  $\mathcal{I} = \{I_n\}$  and  $\mathcal{J} = \{J_n\}$  are **cofinal** if:

- for every  $n$  there exists  $m$  such that  $J_m \subseteq I_n$ , and
- for every  $n$  there exists  $m$  such that  $I_m \subseteq J_n$ .

**Remark 9.6.** If two filtrations  $\mathcal{I} = \{I_n\}$  and  $\mathcal{J} = \{J_n\}$  on  $R$  are cofinal, they determine equivalent topologies: any open set in one of these topologies contains an open set in the other topology.

The Containment Problem first appeared in work on Schenzel [Sch86], who was the first one to ask about the function  $f: \mathbb{N} \rightarrow \mathbb{N}$  that for each  $n$  returns the smallest possible  $f(n)$  such that

$$I^{(f(n))} \subseteq I^n.$$

It is not even clear that this function is in fact defined; given an  $n$ , there may be no symbolic power contained in  $I^n$ . Thus the first part of the Containment Problem is to determine when this function  $f$  exists, which is the same as asking if the  $I$ -symbolic topology is finer than the  $I$ -adic topology. On the other hand, we always have  $I^n \subseteq I^{(n)}$  for all  $n \geq 1$  by definition, so the  $I$ -adic topology is always finer than the  $I$ -symbolic topology. Therefore, the function  $f$  above exists if and only if the  $I$ -adic and  $I$ -symbolic topologies are equivalent. Schenzel completely characterized when the  $I$ -adic and  $I$ -symbolic topologies are equivalent in the 1980s.

**Theorem 9.7** (Schenzel, 1985 [Sch85]). *For a prime ideal  $P$  in a noetherian ring  $R$ , the following conditions are equivalent:*

- (1) The  $P$ -adic topology is equivalent to the  $P$ -symbolic topology.
- (2) For all  $Q \in \mathcal{A}(P) \setminus \{P\}$  and all  $p \in \text{Ass}(\widehat{R_Q})$ ,  $\text{ht}(\widehat{PR_Q} + p/p) < \dim(\widehat{R_Q}/p)$ .

Here  $\widehat{R_Q}$  denotes completion of  $R_Q$  with respect to its maximal ideal  $QR_Q$ .

Schenzel also asked if the equivalence of the  $I$ -adic and  $I$ -symbolic topologies must imply that the function  $f$  above is actually linear. Fifteen years later, Swanson said yes.

**Theorem 9.8** (Swanson, 2000 [Swa00]). *Let  $R$  be a noetherian local ring and  $I$  an ideal in  $R$ . The following are equivalent:*

- (1) The  $I$ -adic and  $I$ -symbolic topologies are equivalent.
- (2) There exists a constant  $s$  such that  $I^{(sn)} \subseteq I^n$  for all  $n$ .

She even proved the more general statement when we replace  $I^{(n)}$  by  $(I^n : J^\infty)$  for some ideal  $J$ . This is both a spectacular and surprising result, though the constant  $s$  is not constructed explicitly, and Swanson dedicates the last section of her paper to explicit examples where one can find such  $s$ .

**Definition 9.9.** Let  $I$  be an ideal in a noetherian ring  $R$  such that the  $I$ -adic and  $I$ -symbolic topologies are equivalent. The smallest integer  $s$  such that  $I^{(sn)} \subseteq I^n$  is the **Swanson constant** of  $I$ .

It is natural to refer to the condition  $I^{(kn)} \subseteq I^n$  as a *linear equivalence* between the  $I$ -adic and the  $I$ -symbolic topologies. However, a word of caution: in papers from the 1980s and 1990s, one finds references<sup>1</sup> to *linearly equivalent* topologies in this context to mean that there exists a constant  $k$  such that  $I^{(n+k)} \subseteq I^n$  for all  $n$ , which is a much stronger requirement.

For every prime ideal in a complete local domain, the symbolic and -adic topologies are indeed equivalent.

**Theorem 9.10** (Huneke–Katz–Validashti, Proposition 2.4 in [HKV09]). *Let  $R$  be a complete local domain. For all prime ideals  $P$ , there exists a constant  $h$  such that  $P^{(hn)} \subseteq P^n$ .*

In the next section, we will discuss a situation where we can find explicit upper bounds for Swanson’s constant – and where it can be taken uniformly on  $R$ .

## 9.2 A uniform constant

The remainder of the story will often rely on the following important invariant:

**Definition 9.11.** Let  $I$  be an ideal in a noetherian ring  $R$ . The **big height** of  $I$  is the maximum height of an associated prime of  $I$ .

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<sup>1</sup>For example, see [Sch85].

**Example 9.12.** If  $P$  is prime, then its big height is just its height. In  $R = k[x, y, z]$ , where  $k$  is a field, the ideal  $I = (xy, xz) = (x) \cap (y, z)$  has height 1 but big height 2.

Soon after Swanson's theorem [Swa00] was announced, Lawrence Ein, Robert Lazarsfeld, and Karen Smith [ELS01] found that Swanson's constant can be computed very explicitly when  $R = \mathbb{C}[x_1, \dots, x_d]$ , or more generally when  $R$  is any smooth  $\mathbb{C}$ -algebra. Their proof relies on multiplier ideals and their wonderful properties. Proving beautiful theorems is contagious, it appears, since soon after that Mel Hochster and Craig Huneke [HH02] extended the result to any regular ring containing a field, but using completely different techniques. This time, Frobenius (and, as we will see later, our old friend the Pigeonhole Principle!) came to the rescue: Hochster and Huneke proved the result in prime characteristic, and then extended it to equicharacteristic zero via reduction to characteristic  $p$ . It wasn't until recently, when Yves André proved the Direct Summand Conjecture using Scholze's perfectoid spaces techniques [And18], that Linquan Ma and Karl Schwede [MS18] complete the story by proving the result holds in mixed characteristic. The key idea is to establish a mixed characteristic version of multiplier ideals, which then allows one to follow Ein, Lazarsfeld, and Smith's original strategy. One technical detail remained to be settled: Ma and Schwede's proof needed  $R$  to be excellent. But just a few months ago, Takumi Murayama [Mur21] settled the final technical details that allows us to state the following beautiful theorem:

**Theorem 9.13** (Ein–Lazarsfeld–Smith, Hochster–Huneke, Ma–Schwede, Murayama). *Let  $R$  be a regular ring, and  $I$  be a radical ideal. If  $I$  has big height  $h$ , then*

$$I^{(hn)} \subseteq I^n \quad \text{for all } n \geq 1.$$

Let us outline Ein, Lazarsfeld, and Smith's proof, which as we mentioned above uses multiplier ideals. Given an ideal  $I$  and  $t \geq 0$ , the multiplier ideal  $\mathcal{J}(R, I^t)$  measures the singularities of  $V(I) \subseteq \text{Spec}(R)$ , scaled by  $t$  in a certain sense. We refer to [ELS01, MS18] for the definition. The proof of Theorem 9.13 in the equicharacteristic 0 case relies on a few key properties of multiplier ideals:

- $I \subseteq \mathcal{J}(R, I)$ ;
- For all  $n \geq 1$ ,  $\mathcal{J}\left(R, (P^{(nh)})^{\frac{1}{n}}\right) \subseteq P$  whenever  $P$  is a prime of height  $h$ ;
- For all integers  $n \geq 1$ ,  $\mathcal{J}(R, I^{tn}) \subseteq \mathcal{J}(R, I^t)^n$ .

Then, given a prime ideal  $P$  of height  $h$ ,

$$P^{(hn)} \subseteq \mathcal{J}(R, (P^{(nh)})) \subseteq \mathcal{J}\left(R, (P^{(nh)})^{\frac{1}{n}}\right)^n \subseteq P^n.$$

In characteristic  $p$ , a similar proof works, replacing multiplier ideals by test ideals. We will later see Hochster and Huneke's proof [HH02], which does not involve multiplier/test ideals.

**Example 9.14.** Let  $P$  be the prime ideal defining the curve  $(t^3, t^4, t^5)$  in  $k[x, y, z]$ , which we discussed in Example 2.4. Since this is a prime of height 2, Theorem 9.13 says that  $P^{(2n)} \subseteq P^n$  for all  $n \geq 1$ .

**Example 9.15.** Let  $I = (xy, xz, yz)$  in  $k[x, y, z]$ , where  $k$  is a field. The minimal primes of this radical ideal all have height 2, so the big height of  $I$  is 2, and thus by Theorem 9.13  $I^{(2n)} \subseteq I^n$  for all  $n \geq 1$ .

**Remark 9.16.** As a corollary of Theorem 9.13, we obtain a uniform constant  $c$  as in Swanson's Theorem 9.8. Indeed, the big height of any ideal is at most the dimension  $d$  of the ring, so that  $I^{(dn)} \subseteq I^n$  for all  $n$ . This constant can be improved to  $d - 1$ , since the ordinary and symbolic powers of any maximal ideal coincide.

In the non-regular setting, we do know that the two topologies are equivalent for all prime ideals in a complete local domain. However, the question of whether this constant  $h$  can be taken independently of  $P$  is still open in the non-regular setting. Since this asks for a uniform comparison between the symbolic and adic topologies, rings with this property are said to satisfy the Uniform Symbolic Topologies Property.

**Question 9.17** (Uniform Symbolic Topologies). Let  $R$  be a complete local domain. Is there a uniform constant  $h$  depending only on  $R$  such that

$$P^{(hn)} \subseteq P^n$$

for all primes  $P$  and all  $n \geq 1$ ?

This is an open question, though the answer is known to be yes in some special settings.

**Definition 9.18.** A local ring  $(R, \mathfrak{m})$  is an **isolated singularity** if  $R_P$  is regular for all  $P \neq \mathfrak{m}$ .

**Theorem 9.19** (Huneke–Katz–Validashti, 2009, [HKV09]). *Let  $R$  be an equicharacteristic reduced local ring such that  $R$  is an isolated singularity. Assume either that  $R$  is equidimensional and essentially of finite type over a field of prime characteristic zero, or that  $R$  has positive characteristic and is  $F$ -finite. Then there exists  $h \geq 1$  with the following property: for all ideals  $I$  with positive grade for which the  $I$ -symbolic and  $I$ -adic topologies are equivalent,  $I^{(hn)} \subseteq I^n$  holds for all  $n \geq 1$ .*

Their result does not provide effective bounds for what  $h$  might be. In general, finding explicit best possible bounds for this constant is a very difficult question. For the case of monomial prime ideals over normal toric rings, see the work of Robert M. Walker [Wal16, Wal18].

**Example 9.20** (Carvajal-Rojas — Smolkin, 2018). Let  $k$  be a field of characteristic  $p$  and consider  $R = k[a, b, c, d]/(ad - bc)$ . Then for all primes  $P$  in  $R$ ,  $P^{(2n)} \subseteq P^n$  for all  $n \geq 1$ .

### 9.3 A beautiful proof by Hochster and Huneke

In this section, we will discuss Hochster and Huneke's proof of Theorem 9.13 in prime characteristic. One of the main advantages of characteristic  $p$  is that while we cannot control the associated primes of the powers of an ideal, when the ring is regular we can control the associated primes of the Frobenius powers. The following is a corollary of [PS73, 1.7].

**Theorem 9.21.** *Let  $R$  be a regular ring of prime characteristic  $p$  and  $I$  be an ideal in  $R$ . Then  $\text{Ass}(I^{[q]}) = \text{Ass}(I)$  for all  $q = p^e$ .*

*Proof.* Fix a prime  $Q$  and  $q = p^e$ . By Theorem 3.105, given an ideal  $J$ ,  $Q \in \text{Ass}(J)$  if and only if  $\text{depth}((R/J)_Q) = 0$ . By Theorem 3.70, since  $R$  is regular, so is  $R_Q$ , and thus all ideals over  $R_Q$  have finite projective dimension. By the Auslander Buchsbaum formula, we have  $Q \in \text{Ass}(J)$  if and only if  $\text{pdim}_{R_Q}(R_Q/J_Q) = \dim(R_Q)$ .

Frobenius commutes with localization. To show that  $\text{Ass}(I^{[q]}) = \text{Ass}(I)$ , we need only to show that whenever  $(R, \mathfrak{m})$  is a local ring,  $\text{pdim}(I) = \text{pdim}(I^{[q]})$ .

Let  $C$  be a minimal free resolution of  $I^{[q]}$ . Frobenius is an exact functor, and thus  $F_*(C)$  is exact. Since  $R$  is regular,  $F_*^e(R)$  is a free  $R$ -module, by Theorem 8.7, and thus  $F_*(C)$  is a free resolution of  $F_*^e(I^{[q]}) = IF_*^e(R)$ . Moreover, this is a minimal free resolution, since the entries in each map simply got raised to  $p^e$ . Since  $F_*^e(R)$  is a free  $R$ -module,  $IF_*^e(R)$  is just a finite direct sum of copies of  $I$ , and thus  $F_*^e(R)$  is a sum of copies of a minimal free resolution of  $I$ . In particular, their lengths are the same, meaning that  $\text{pdim}(I) = \text{pdim}(I^{[q]})$ . We conclude that  $Q \in \text{Ass}(I^{[q]})$  if and only if  $Q \in \text{Ass}(I)$ .  $\square$

The following theorem is the key piece of the puzzle, even though it is beautifully elementary: it is just a fancy version of the Pigeonhole Principle.

**Theorem 9.22.** *Let  $I$  be a radical ideal of big height  $h$  in a regular ring  $R$  of prime characteristic  $p$ . For all  $q = p^e$ ,  $I^{(hq)} \subseteq I^{[q]}$ .*

*Proof.* Fix  $q = p^e$ . By Exercise 16, it is sufficient to show the containment holds after localizing at all the associated primes of  $I^{[q]}$ . By Theorem 9.21, and  $\text{Ass}(I^{[q]}) = \text{Ass}(I)$ .

Let  $Q \in \text{Ass}(I)$ . We know that  $I_Q = Q_Q$ ,  $(I^{[q]})_Q = Q_Q^{[q]}$ , and  $(I^{(hq)})_Q = Q_Q^{hq}$ . By assumption,  $\dim(R_Q) = \text{ht}(Q) \leq h$ . So we have a regular local ring of dimension  $h$  with maximal ideal  $\mathfrak{m}$  and we want to show that  $\mathfrak{m}^{hq} \subseteq \mathfrak{m}^{[q]}$ . Since our ring is regular,  $\mathfrak{m}$  is generated by dimension many elements, so at most  $h$  elements. Let  $\mathfrak{m} = (x_1, \dots, x_h)$ . The power  $\mathfrak{m}^{hq}$  is generated by all monomials of the form  $x_1^{a_1} \cdots x_h^{a_h}$  with

$$a_1 + \cdots + a_h \geq hq,$$

so by the Pigeonhole Principle, there exists  $i$  such that  $a_i \geq q$ . Therefore,

$$x_1^{a_1} \cdots x_h^{a_h} \in (x_i^q) \subseteq \mathfrak{m}^{[q]}. \quad \square$$

**Remark 9.23.** While the statement of Theorem 9.22 is all we need to prove Theorem 9.13, we note that if we apply the full power of the Pigeonhole Principle we can do even better: indeed, as long as

$$a_1 + \cdots + a_h \geq h(q-1) + 1,$$

at least one  $a_i \geq q$ . We conclude that any radical ideal of big height  $h$  satisfies  $I^{(hq-h+1)} \subseteq I^{[q]}$  for all  $q = p^e$ .

We now have all the tools we need to give Hochster and Huneke's proof of Theorem 9.13 in prime characteristic  $p$ . To show the theorem in equicharacteristic zero, one uses standard reduction to characteristic  $p$  techniques — which while standard are quite technical, and thus we will not discuss them here. For more on reduction to prime characteristic, see [HH99].



**Theorem 9.24.** *Let  $R$  be a regular ring of prime characteristic  $p$ , and  $I$  a radical ideal in  $R$ . If  $I$  has big height  $h$ , then*

$$I^{(hn)} \subseteq I^n \text{ for all } n \geq 1.$$

*Proof.* Fix  $n$ . We will show that if  $u \in I^{(hn)}$ , then  $u \in (I^n)^*$ , and since  $R$  is regular, by Theorem 8.32 we conclude that  $u \in I^n$ . To do that, we will find a nonzero element  $c$  such that  $cr^q \in (I^n)^q$  for all  $q = p^e$ .

Given  $q = p^e$ , we can write  $q = an + r$  for some integers  $a, r \geq 0$  with  $r < n$ . Then  $u^a \in (I^{(hn)})^a \subseteq I^{(han)}$ , and

$$I^{hn}u^a \subseteq I^{hr}u^a \subseteq I^{hr}I^{(han)} \subseteq I^{(han+hr)} = I^{(hq)}.$$

By Theorem 9.22,  $I^{(hq)} \subseteq I^{[q]}$ , and thus we have  $I^{hn}u^a \subseteq I^{[q]}$ . Now take powers of  $n$  on both sides:

$$I^{hn^2}u^{an} \subseteq (I^{[q]})^n = (I^n)^{[q]}.$$

By choice of  $a$ , we know  $q \geq an$ , so that

$$I^{hn^2}u^q \subseteq I^{hn^2}u^{an} \subseteq (I^n)^{[q]}.$$

Since  $R$  is a domain, there exists a nonzero element  $c \in I^{hn^2}$ , which does not depend on the choice of  $q$ . Such  $c$  satisfies  $cu^q \in (I^n)^{[q]}$ , and thus as we described above we must have  $u \in I^n$ .  $\square$

The following generalization is [ELS01, Theorem 2.2] in the case of smooth complex varieties, [HH02, Theorem 2.6] in equicharacteristic, and [Mur21] in mixed characteristic:

**Theorem 9.25** (Ein–Lazarsfeld–Smith, Hochster–Huneke, Murayama). *Let  $I$  be a radical ideal of a regular ring, and let  $h$  be the big height of  $I$ . Then for all  $n \geq 1$  and all  $k \geq 0$ ,  $I^{(hn+kn)} \subseteq (I^{(k+1)})^n$ .*

One can even do better! If we reinterpret  $kn$  as a sum of  $n$  terms all equal to  $k$ , we can extend this further and allow the sum of any  $n$  terms:

**Theorem 9.26** (Johnson, Murayama [Joh14, Mur21]). *Let  $I$  be a radical ideal of a regular ring, and let  $h$  be the big height of  $I$ . Then for all  $n \geq 1$  and all  $a_1, \dots, a_n \geq 0$ ,*

$$I^{(hn+a_1+\dots+a_n)} \subseteq I^{(a_1)} \dots I^{(a_n)}.$$

**Exercise 20.** Show Theorem 9.26 in prime characteristic, essentially by repeating the argument we just gave.

When  $k = 0$ , Theorem 9.25 says that  $I^{(hn)} \subseteq I^n$ , so it is indeed a generalization of Theorem 9.13. Moreover, in characteristic  $p$ , we can obtain a generalized version of Harbourne’s Conjecture, which we will discuss in the next section, for powers of  $p$ :

**Lemma 9.27** (Lemma 2.4 (a) in [HH02]). *Let  $I$  be a radical ideal in a regular ring  $R$  of prime characteristic  $p$  and  $h$  the big height of  $I$ . For all  $q = p^e$ ,*

$$I^{(hq+kq-h+1)} \subseteq (I^{(k+1)})^{[q]}.$$

*Proof.* Since containments are local, it is enough to show that the statement holds once we localize at each associated prime of  $(I^{(k+1)})^{[q]}$ . Since

$$\text{Ass} \left( (I^{(k+1)})^{[q]} \right) = \text{Ass} (I^{(k+1)}) = \text{Ass}(I),$$

it is enough to show that the statement holds at the associated primes of  $I$ . If  $P$  is an associated prime of  $I$ , the statement we need to show over  $R_P$  is the following:

$$P_P^{hq+kq-h+1} \subseteq (P_P^{k+1})^{[q]}.$$

To simplify notation, note that we just need to prove that if  $(R, \mathfrak{m})$  is a regular local ring, then

$$\mathfrak{m}^{hq+kq-h+1} \subseteq (\mathfrak{m}^{k+1})^{[q]}.$$

Now we claim that Frobenius powers commute with localization. To check that, suppose that we are given any two ideals  $I$  and  $J$ . On the one hand, the ideal  $IJ$  is generated by products of the form  $fg$  with  $f \in I$  and  $g \in J$ , so  $(IJ)^{[q]}$  is generated by  $(fg)^q = f^q g^q$  with  $f \in I$  and  $g \in J$ . On the other hand,  $I^{[q]}$  is generated by elements of the form  $f^q$  with  $f \in I$ , and thus  $I^{[q]}J^{[q]}$  is generated by products of the form  $f^q g^q$  with  $f \in I$  and  $g \in J$ . This shows that  $(IJ)^{[q]} = I^{[q]}J^{[q]}$ , and as a consequence, Frobenius powers commute with products. Therefore, to achieve our goal we only need to show that

$$\mathfrak{m}^{hq+kq-h+1} \subseteq (\mathfrak{m}^{[q]})^{k+1}.$$

The claim now follows by a similar argument as Theorem 9.22. If  $\mathfrak{m} = (x_1, \dots, x_h)$ , then  $\mathfrak{m}^{hq+kq-h+1}$  is generated by monomials  $x_1^{a_1} \cdots x_h^{a_h}$  with

$$a_1 + \cdots + a_h \geq h(q-1) + kq + 1,$$

then at least one of the  $a_i$  must be at least  $q$ . Subtracting  $q$  from  $a_i$ , we now have

$$a_1 + \cdots + a_h \geq h(q-1) + (k-1)q + 1,$$

which once more implies that some  $a_j$  must be at least  $q$ . We can repeat this  $k$  times, until we have decomposed our monomial as product of an element in  $(\mathfrak{m}^{[q]})^k$  and a monomial  $x_1^{b_1} \cdots x_h^{b_h}$  with

$$b_1 + \cdots + b_h \geq h(q-1) + 1,$$

which once more implies some  $b_s \geq q$ . We conclude that our original monomial was in  $(\mathfrak{m}^{[q]})^{k+1}$ , as desired.  $\square$

Finally, we note that Theorem 9.13 also holds more generally. When  $I$  is any ideal in a regular ring  $R$ , we can take the constant  $h$  such that  $I^{(hn)} \subseteq I^n$  to be any of the following:

- the largest number of generators of  $I_P$ , where  $P \in \text{Ass}(I)$ ;
- the big height of  $I$ , taken to be the largest height of an associated prime of  $I$ ;
- the largest analytic spread of  $I_P$ , where  $P \in \text{Ass}(I)$ .

When  $I$  is radical, these invariants all coincide. In general, the last invariant is the smallest, and the point is that one can always replace  $I_P$  in the containment by a *minimal reduction*, which is generated by  $h$  many elements.

## 9.4 Harbourne's Conjecture

Theorem 9.13 says that if  $I$  is any radical ideal of big height  $h$ , then  $I^{(hn)} \subseteq I^n$  for all  $n \geq 1$ . But does this completely answer the Containment Problem? Our goal was to find the smallest  $f(n)$  such that  $I^{(f(n))} \subseteq I^n$ , and the theorem tells us only that  $f(n) \leq hn$ . But can we do better?

**Example 9.28.** Consider the monomial ideal  $I = (xy, xz, yz)$  in  $R = k[x, y, z]$ , where  $k$  is a field. Since all the minimal primes of  $I$  have height 2, the big height of  $I$  is 2. We saw in Example 3.56 that  $I^{(2)} \neq I^2$ , and Theorem 9.13 says that  $I^{(2n)} \subseteq I^n$  for all  $n$ . To solve the containment problem for  $I^2$  we need only to determine whether  $I^{(3)} \subseteq I^2$ . And indeed, one can easily show that this does hold.

**Example 9.29.** Consider the prime ideal  $P$  defining the curve  $(t^3, t^4, t^5)$  over a field  $k$ . As we already pointed out in Example 9.14,  $P^{(2n)} \subseteq P^n$  for all  $n$ . We also saw in Example 9.14 that  $P^{(2)} \neq P^2$ . But once more, we can do better than Theorem 9.13: it turns out that  $P^{(3)} \subseteq P^2$ .

**Question 9.30** (Huneke, 2000). Let  $P$  be a prime ideal of height 2 in a regular local ring  $R$ . Is  $P^{(3)} \subseteq P^2$ ?

**Remark 9.31.** If  $R$  is a regular ring of prime characteristic 2, then by Remark 9.23 we must have  $I^{(3)} \subseteq I^2$  for any radical ideal of big height 2. Note that  $3 = hn - h + 1$  for  $h = 2$  and  $n = 2$ .

This question remains open. As we will discuss in the next chapter, the symbolic powers of the primes determining curves of the form  $(t^a, t^b, t^c)$  exhibit lots of interesting behavior, and would be a natural place for a counterexample. Nevertheless, they do satisfy the result.

**Theorem 9.32** (Grifo, 2020 [Gri20]). *Let  $k$  be a field of characteristic not 3. If  $P$  is the prime ideal in  $k[x, y, z]$  defining the curve  $(t^a, t^b, t^c)$ , then  $P^{(3)} \subseteq P^2$ .*

Brian Harbourne extended Huneke's question to a much more general setting. We note that his original question, which first appeared in print in [BRH<sup>+</sup>09, 8.4.3], was about homogeneous ideals in  $R = k[x_1, \dots, x_d]$ , though we present here a slightly modified version of his question:

**Conjecture 9.33** (Harbourne, 2008). *Let  $I$  be a radical ideal in a regular ring  $R$ . If  $I$  has big height  $h$ , then*

$$I^{(hn-h+1)} \subseteq I^n \quad \text{for all } n \geq 1.$$

The value suggested by this conjecture is very natural. In fact, by Remark 9.23, if  $R$  has characteristic  $p$  and we take  $n = q = p^e$  for some  $e$ , then the containment in Harbourne's Conjecture holds, and it is simply the value suggested by the Pigeonhole Principle.

But in a strange turn of events, the conjecture does not hold for all radical ideals — not even for homogeneous ideals in a polynomial ring. The first counterexample was found by Dumnicki, Szemberg, and Tutaj-Gasińska [DSTG13], and it is the radical ideal defining a certain nice configuration of twelve points in  $\mathbb{P}^2$  over  $\mathbb{C}$ . Harbourne and Secleanu extended their example to a family of examples in any characteristic other than 2 [HS15].

**Example 9.34** (Dumnicki—Szemberg—Tutaj-Gasińska, 2013, Harbourne—Secreanu, 2015 [DSTG13, HS15]). Fix  $n \geq 3$ . Let  $k$  be a field of characteristic not 2 and containing  $n$  distinct roots of unity, and let  $R = k[x, y, z]$ . The ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$$

is a radical ideal of pure height 2, and yet  $I^{(3)} \not\subseteq I^2$ . In fact, the element

$$f = (y^n - z^n)(z^n - x^n)(x^n - y^n)$$

satisfies  $f \in I^{(3)}$ , but  $f \notin I^2$ .

This ideal is the homogeneous radical ideal corresponding to a particularly nice configuration of points in  $\mathbb{P}^2$ , known as the **Fermat configuration**. When  $n = 3$ , the corresponding picture is the following:

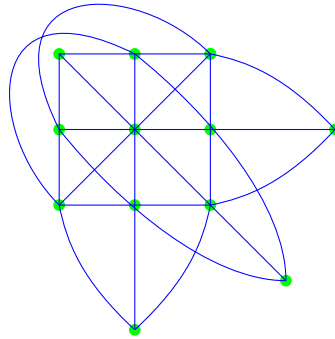


Figure 9.1: Fermat configuration of points when  $n = 3$ .

Other counterexamples to Harbourne's Conjecture have since been found. There is much we do not understand about the conjecture, though many of the known counterexamples arise as the singular loci of hyperplane arrangements, and Ben Drabkin and Alexandra Seceleanu [DS21] have completely classified which finite complex reflection groups lead to counterexamples to  $I^{(3)} \subseteq I^2$ . Notice, however, that these do not provide any counterexamples to Huneke's Question 9.30.

However, all the known examples are, in a way, very special; nice classes of ideals do satisfy Harbourne's Conjecture.

**Definition 9.35.** A set  $S$  of points in  $\mathbb{P}_k^n$  is **generic** if the coordinates of the points in  $S$  are algebraically independent over the prime field of  $k$ ; this is the smallest subring of  $k$  containing 1, which is either isomorphic to  $\mathbb{F}_p$  or  $\mathbb{Q}$  depending on the characteristic of  $k$ .

**Theorem 9.36** (Bocci—Harbourne, 2010, Theorems 4.1 and 4.2 in [BH10]). *Let  $S$  be a set of generic points in  $\mathbb{P}^2$ , and let  $I = I(S)$ . Then  $I^{(3)} \subseteq I^2$ . Moreover,  $I^{(a)} \subseteq I^b$  whenever  $\frac{a}{b} > \frac{3}{2}$ .*

Notice that  $2n - 1 > \frac{3}{2}n$  for all  $n \geq 2$ , so in particular generic points in  $\mathbb{P}^2$  satisfy Harbourne's Conjecture.

**Theorem 9.37** (Dumnicki, 2015 [Dum15]). *Let  $S$  be a set of generic points in  $\mathbb{P}^3$ , and let  $I = I(S)$ . Then  $I$  satisfies Harbourne's Conjecture:  $I^{(3n-2)} \subseteq I^n$  for all  $n \geq 1$ .*

Certain special configurations of points in  $\mathbb{P}^n$  also satisfy Harbourne's Conjecture.

**Definition 9.38.** Let  $H = \{H_1, \dots, H_s\}$  be a collection of  $s \geq 1$  distinct hyperplanes in  $\mathbb{P}^n$  corresponding to linear forms  $L_1, \dots, L_s$ . Suppose that these hyperplanes meet properly, meaning that the intersection of any  $i$  of the hyperplanes is either empty or has codimension  $i$ . The ideal

$$I = \bigcap_{1 \leq i_1 < \dots < i_c \leq s} (L_{i_1}, \dots, L_{i_c}),$$

where  $1 \leq i_1 < \dots < i_c \leq s$  range over all the indices such that  $H_{i_1} \cap \dots \cap H_{i_s}$  has codimension  $c$ , is called a **codimension  $c$  star configuration**.

By construction, each  $(L_{i_1}, \dots, L_{i_c})$  is an ideal of height  $c$ , and thus a complete intersection. In fact, this is the radical ideal corresponding to the intersection of hyperplanes  $H_{i_1} \cap \dots \cap H_{i_c}$ . Therefore, the symbolic powers of this star configuration are given by

$$I^{(n)} = \bigcap_{1 \leq i_1 < \dots < i_c \leq s} (L_{i_1}, \dots, L_{i_c})^n.$$

Star configurations satisfy Harbourne's Conjecture [BRH<sup>+</sup>09, Example 8.4.8].

Next, we will show that squarefree monomial ideals satisfy Harbourne's Conjecture. In order to do that, we need a definition inspired by prime characteristic ideals — when  $I$  is a monomial ideal, we can consider a sort of fake Frobenius power for any  $m$ , independently of characteristic.

**Definition 9.39.** Let  $f_1, \dots, f_s$  be monomials in  $R = k[x_1, \dots, x_d]$  and let  $I = (f_1, \dots, f_s)$ . For each  $m \geq 1$ ,  $I^{[m]}$  denotes the ideal

$$I^{[m]} := (f_1^m, \dots, f_s^m).$$

**Lemma 9.40.** *If  $I$  and  $J$  are monomial ideals, then  $I^{[t]} \cap J^{[t]} = (I \cap J)^{[t]}$ .*

*Proof.* Let  $G(I) = \{f_1, \dots, f_n\}$  and  $G(J) = \{g_1, \dots, g_m\}$ . By Lemma 5.9,

$$I \cap J = (\text{lcm}(f_i, g_j) \mid 1 \leq i \leq n, 1 \leq j \leq m)$$

so

$$(I \cap J)^{[t]} = (\text{lcm}(f_i, g_j)^t \mid 1 \leq i \leq n, 1 \leq j \leq m),$$

while

$$I^{[t]} \cap J^{[t]} = (\text{lcm}(f_i^t, g_j^t) \mid 1 \leq i \leq n, 1 \leq j \leq m).$$

Since  $\text{lcm}(f_i^t, g_j^t) = \text{lcm}(f_i, g_j)^t$ , we conclude that  $I^{[t]} \cap J^{[t]} = (I \cap J)^{[t]}$ .  $\square$

**Lemma 9.41** (see Example 8.4.5 in [BRH<sup>+</sup>09]). *Let  $I$  be a squarefree monomial ideal in  $R = k[x_1, \dots, x_d]$ , where  $k$  is any field. If  $I$  has big height  $h$ , then  $I^{(hn-h+1)} \subseteq I^n$  for all  $n \geq 1$ .*

*Proof.* Let  $\text{Min}(I) = P_1, \dots, P_s$ , so that

$$I^{(m)} = P_1^m \cap \dots \cap P_s^m$$

for all  $m \geq 1$ , by Corollary 5.20. Each  $P_i$  is generated by at most  $h$  variables. Given the prime ideal  $P = (x_1, \dots, x_h)$ , the same argument in Remark 9.23 shows that  $P^{hn-h+1} \subseteq (x_1^n, \dots, x_h^n)$ : the power  $P^{hn-h+1}$  is generated by monomials  $x_1^{a_1} \cdots x_h^{a_h}$  with

$$a_1 + \dots + a_h \geq h(n-1) + 1,$$

and thus the Pigeonhole Principle guarantees that there exists an  $i$  such that  $a_i \geq n$ . For each  $m$  and each monomial ideal  $J$ , if  $J = (f_1, \dots, f_s)$ , then let  $I^{[m]} := (f_1^m, \dots, f_s^m)$ . Therefore,

$$\begin{aligned} I^{(hn-h+1)} &= \bigcap_{i=1}^s P_i^{hn-h+1} \\ &\subseteq \bigcap_{i=1}^s P_i^{[n]} && \text{what we have just shown} \\ &= \left( \bigcap_{i=1}^s P_i \right)^{[n]} && \text{by Lemma 9.40} \\ &= I^{[n]} \\ &\subseteq I^n. \quad \square \end{aligned}$$

As we learned from Luis Núñez Betancourt's philosophy in Metatheorem 8.21, any fact about squarefree monomial ideals probably extends to any ideal defining an F-pure ring. With that in mind, do ideals defining F-pure rings satisfy Harbourne's Conjecture? Indeed, they do!

This is what we will show next. Naively, the idea of the proof is to study the colon ideal  $(I^n : I^{(hn-h+1)})$ . The colon ideal  $(J : I)$  measures the failure of  $I \subseteq J$ , and  $(J : I) = R$  precisely when  $I \subseteq J$ . In order to show that  $(I^n : I^{(hn-h+1)}) = R$ , we need to show that this ideal contains some *large* ideal; Fedder's Criterion 8.20 provides the perfect candidate. The proof in [GH19] does just that — we show that

$$(I^{[q]} : I) \subseteq (II^{(n)} : I^{(n+h)})^{[q]},$$

for all ideals  $I$  and all  $q = p^e \gg 0$ , and when  $R/I$  is F-pure that implies Harbourne's Conjecture. The proof we will follow here uses the same techniques, but instead we will show a slightly more powerful lemma.

**Lemma 9.42.** *Let  $R$  be a regular ring of prime characteristic  $p$ . Let  $I$  be a radical ideal in  $R$  and  $h$  the big height of  $I$ . For all  $n \geq 1$ ,*

$$(I^{[q]} : I) \subseteq (II^{(n)} : I^{(n+h)})^{[q]}$$

for all  $q = p^e \gg 0$ .

*Proof.* By Exercise 19,

$$(II^{(n)} : I^{(n+h)})^{[q]} = \left( (II^{(n)})^{[q]} : (I^{(n+h)})^{[q]} \right).$$

Take  $s \in (I^{[q]} : I)$ . Then  $sI^{(n+h)} \subseteq sI \subseteq I^{[q]}$ , so

$$s(I^{(n+h)})^{[q]} \subseteq (sI^{(n+h)})(I^{(n+h)})^{q-1} \subseteq I^{[q]}(I^{(n+h)})^{q-1}.$$

We will show that

$$(I^{(n+h)})^{q-1} \subseteq (I^{(n)})^{[q]},$$

which implies that

$$s(I^{(n+h)})^{[q]} \subseteq (II^{(n)})^{[q]},$$

completing the proof.

By Lemma 2.9,

$$(I^{(n+h)})^{q-1} \subseteq I^{((n+h)(q-1))}.$$

By Lemma 9.27 with  $k = n - 1$ , we obtain the following containment:

$$I^{(hq+(n-1)q-h+1)} \subseteq (I^{(n)})^{[q]}.$$

We claim that for all  $q \gg 0$ ,  $(I^{(n+h)})^{q-1} \subseteq I^{(hq+(n-1)q-h+1)}$ , which would conclude the proof that  $(I^{(n+h)})^{q-1} \subseteq (I^{(n)})^{[q]}$ . To show that the claim, it is enough to prove that

$$(n+h)(q-1) \geq hq + (n-1)q - h + 1$$

for large values of  $q$ . This can be seen by comparing the coefficients in  $q$ , and noticing that  $n+h \geq n+h-1$ , or by explicitly solving the inequality. In particular, it holds as long as  $q \geq n+1$ .  $\square$

**Corollary 9.43.** *Let  $R$  be a regular ring of prime characteristic  $p$ . Let  $I$  be an ideal in  $R$  of big height  $h$ . If  $R/I$  is F-pure, then for all  $n \geq 1$  we have*

$$I^{(n+h)} \subseteq II^{(n)}.$$

*Proof.* First, note that we can reduce to the local case:

- containments are local statements, by Exercise 16;
- the big height of an ideal does not increase under localization;
- all localizations of an F-pure ring are F-pure [HR74, 6.2].

So suppose that  $(R, \mathfrak{m})$  is a regular local ring, and that  $R/I$  is F-pure. Fix  $n \geq 1$ , and consider  $q \gg 0$  as in Lemma 9.42. Then

$$(I^{[q]} : I) \subseteq (II^{(n)} : I^{(n+h)})^{[q]}.$$

If  $I^{(n+h)} \not\subseteq II^{(n)}$ , then  $(II^{(n)} : I^{(n+h)})^{[q]} \subseteq \mathfrak{m}^{[q]}$ , and thus  $(I^{[q]} : I) \subseteq \mathfrak{m}^{[q]}$ , contradicting Fedder's Criterion.  $\square$

We can now show that Harbourne's conjecture holds for ideals defining F-pure rings.

**Theorem 9.44** (Grifo–Huneke, 2019 [GH19]). *Let  $R$  be a regular ring of prime characteristic  $p$ . Let  $I$  be an ideal in  $R$  of big height  $h$ . If  $R/I$  is F-pure, then for all  $n \geq 1$ ,  $I^{(hn-h+1)} \subseteq I^n$ .*

*Proof.* By Corollary 9.43,

$$I^{(n+h)} \subseteq II^{(n)}$$

for all  $n \geq 1$ . Inductively, we conclude that

$$I^{(h(n-1)+1)} \subseteq II^{(h(n-2)+1)} \subseteq I^2 I^{(h(n-3)+1)} \subseteq \dots \subseteq I^{n-1} I = I^n.$$

□

In fact, Corollary 9.43 implies more than just Harbourne's Conjecture.

**Exercise 21.** Let  $R$  be a regular ring of prime characteristic  $p$ , and consider an ideal  $I$  in  $R$  such that  $R/I$  is F-pure. Show that given any integer  $c \geq 1$ , if  $I^{(hk-c)} \subseteq I^k$  for some  $n$ , then  $I^{(hn-c)} \subseteq I^n$  for all  $n \gg 0$ .

There are examples — even of monomial ideals — that show that we cannot do better than Harbourne's Conjecture.

**Exercise 22.** Let  $R = k[x_1, \dots, x_d]$  and consider the squarefree monomial ideal

$$I = \bigcap_{i < j} (x_i, x_j).$$

Show that while  $I^{(2n-1)} \subseteq I^n$  holds for all  $n \geq 1$ ,  $I^{(2n-2)} \not\subseteq I^n$  for  $n < d$ , so we cannot do better than Harbourne's Conjecture in this case. What happens when  $n = d$ ? How does this example generalize to higher height?

**Exercise 23.** Let  $I$  be a radical ideal in a ring  $R$ , and fix  $n \geq 1$ . Show that  $(I^n : I^{(n)})$  always contains an element not in any minimal prime of  $I$ .

If we ask for more than F-pure — if we ask for  $R/I$  to be strongly F-regular — then we can in fact do better: not only does Harbourne's Conjecture hold, but we can replace the big height  $h$  of  $I$  by  $h - 1$ . In a way, this says that when  $R/I$  is strongly F-regular, then  $I$  behaves like an ideal of smaller height. This is particularly interesting when  $h = 2$ . To prove this, we need the following lemma:

**Lemma 9.45** (Lemma 4.2 in [GH19]). *Let  $R$  be a regular ring of characteristic  $p > 0$ ,  $I$  an ideal in  $R$ , and  $h \geq 2$  the big height of  $I$ . Then for all  $d \geq h - 1$  and for all  $q = p^e$ ,*

$$(I^d : I^{(d)}) (I^{[q]} : I) \subseteq (II^{(d+1-h)} : I^{(d)})^{[q]}.$$

*Proof.* Let  $t \in (I^d : I^{(d)})$  and  $s \in (I^{[q]} : I)$ . Again by Exercise 19, it suffices to prove that

$$st (I^{(d)})^{[q]} \subseteq (II^{(d+1-h)})^{[q]}.$$



First, note that

$$st(I^{(d)})^{[q]} \subseteq s(tI^{(d)})(I^{(d)})^{q-1} \subseteq sI^d(I^{(d)})^{q-1} \subseteq I^{[q]}I^{d-1}(I^{(d)})^{q-1}.$$

Now since  $d - 1 + d(q - 1) = dq - 1$ , we have  $I^{d-1}(I^{(d)})^{q-1} \subseteq I^{(dq-1)}$ . By Lemma 9.27, we get

$$I^{(qd-h+1)} \subseteq (I^{(d+1-h)})^{[q]}.$$

As long as  $h \geq 2$ , we have  $qd - 1 \geq qd - h + 1$ , which implies that  $I^{(qd-1)} \subseteq (I^{(d+1-h)})^{[q]}$ . Then

$$st(I^{(d)})^{[q]} \subseteq I^{[q]}I^{d-1}(I^{(d)})^{q-1} \subseteq (II^{(d+1-h)})^{[q]},$$

as desired.  $\square$

We can now show the main result on ideals defining strongly F-regular rings:

**Theorem 9.46** (Grifo–Huneke, Theorem 4.1 in [GH19]). *Let  $R$  be an F-finite regular ring of characteristic  $p > 0$ , and  $I$  an ideal of big height  $h \geq 2$  such that  $R/I$  is strongly F-regular. Then  $I^{(d)} \subseteq II^{(d+1-h)}$  for all  $d \geq h - 1$ . In particular,*

$$I^{((h-1)n-(h-1)+1)} \subseteq I^n$$

for all  $n \geq 1$ .

*Proof.* We first note that the second statement follows from the first, by induction. To see this, assume we have shown that  $I^{(d)} \subseteq II^{(d+1-h)}$  for all  $d \geq h - 1$ . When  $d = h$ , this means that  $I^{(h)} \subseteq II^{(1)} \subseteq I^2$ , which is the statement we are trying to show for the case  $n = 1$ . The induction step follows from choosing  $d = (h - 1)(n + 1) + 1$ .

We prove the first statement by contradiction. As before, we can reduce to the case where  $(R, \mathfrak{m})$  is a regular local ring. Note that strong F-regularity is a local property, that is,  $R$  is strongly F-regular if and only if all of its localizations are strongly F-regular [HH89b, 3.1 (a)].

Suppose that  $(II^{(d+1-h)} : I^{(d)}) \subseteq \mathfrak{m}$ . By Exercise 23, we can always find an element  $t \in (I^d : I^{(d)})$  not in any minimal prime of  $I$ . By Lemma 9.45,

$$t(I^{[q]} : I) \subseteq (II^{(d+1-h)} : I^{(d)})^{[q]} \subseteq \mathfrak{m}^{[q]}.$$

By Glassbrenner's Criterion, this contradicts the fact that  $R/I$  is strongly F-regular.  $\square$

**Remark 9.47.** Note that if  $R$  is a local ring and  $R/I$  is strongly F-regular, then  $R/I$  is a domain and  $I$  is a prime ideal, so that the big height of  $I$  is in fact the height of  $I$ .

For primes of height 2, Theorem 9.46 actually gives equality:

**Corollary 9.48.** *Let  $R$  be a regular ring of characteristic  $p > 0$ , and  $I$  a height 2 prime such that  $R/I$  is strongly F-regular. Then  $I^{(n)} = I^n$  for all  $n \geq 1$ .*

This gives non-trivial classes of ideals with  $I^{(n)} = I^n$  for all  $n \geq 1$ .

**Example 9.49.** Let  $S = k[s^3, s^2t, st^2, t^3] \subseteq k[s, t]$ , where  $k$  is a field of characteristic  $p > 3$ . This is a Veronese subring of  $k[s, t]$ , and thus strongly F-regular. We can write  $S$  as a quotient of  $k[a, b, c, d]$  by a 3-generated height 2 prime ideal,

$$P = (b^2 - ac, c^2 - bd, bc - ad).$$

By Corollary 9.48,  $P^{(n)} = P^n$  for all  $n \geq 1$ .

**Example 9.50.** Let  $R = k[a, b, c, d]$ , where  $k$  is a field of prime characteristic  $p > 2$ . Let  $n$  be an integer, and let  $I$  be the ideal of  $2 \times 2$  minors of the  $2 \times 3$  matrix

$$\begin{pmatrix} a^2 & b & d \\ c & a^2 & b^n - d \end{pmatrix}.$$

By [Sin99, Proposition 4.3],  $R/I$  is strongly F-regular. Since  $I$  has height 2, Corollary 9.48 says that  $I^{(k)} = I^k$  for all  $k$ .

We do not know if Theorem 9.46 is sharp when  $h > 2$ . There are, however, subclasses of ideals defining strongly F-regular rings for which we can obtain better containments. The following is Example 3 in [GH19].

**Example 9.51** (Determinantal ideals). Consider a generic  $n \times n$  matrix  $X$ , a field  $k$  of characteristic 0 or  $p > \min\{t, n - t\}$ , and let  $I = I_t(X)$  denote the ideal of  $t$ -minors of  $X$  in  $R = k[X]$ . These ideals of minors define strongly F-regular rings in characteristic  $p$ , by [HH94b, 7.14]. For which values of  $k$  and  $m$  do we have  $I_t^{(k)} \subseteq I_t^m$ ? We claim that this holds when  $\frac{n}{t(n-t+1)}k \geq m$ .

The key tool that allows us to make such a precise statement is that we have an explicit description of the symbolic and ordinary powers of generic determinantal ideals [DEP80]. In fact, such a description also exists for ideals of minors of generic symmetric matrices and for pfaffians of skew-symmetric matrices [JMnV15, DN96]. Any such ideal and all of its symbolic powers are generated by minors of  $X$  of various sizes (not just  $t$ ). Moreover, there is a basis for  $k[X]$  given by certain products of minors of  $X$ , called *standard monomials*, and thus one can explicitly describe  $I^{(n)}$  for all  $n$  by determining which standard monomials live in  $I^{(n)}$  [DEP80, JMnV15, DN96].

Fix  $k$  and consider  $m \leq \frac{nk}{t(n-t+1)}$ . By [BV88, Theorem 10.4], given  $s_i$ -minors  $\delta_i$  of  $X$ , we have

$$\delta_1 \cdots \delta_u \in I_t^{(k)} \text{ if and only if } \sum_{i=1}^u \max\{0, s_i - t + 1\} \geq k,$$

and moreover,  $I_t^{(k)}$  is generated by such products. Notice that according to this formula, any factors corresponding to minors of size less than  $t$  are irrelevant to determine whether or not the given product is in  $I_t^{(k)}$ , since their contribution to the sum is 0, and thus we might as well ignore minors of small size.

So consider  $\delta_1, \dots, \delta_u$ , with  $\delta_i$  an  $s_i$ -minor for each  $i$ , such that  $\delta_1 \cdots \delta_u \in I_t^{(k)}$ , and write  $s = s_1 + \cdots + s_u$ . Using the remark above, we may assume that  $s_i \geq t$  for each  $i$ . We want to show that for all such possible choices of  $\delta_i$ ,  $\delta_1 \cdots \delta_u \in I_t^m$ . Since  $\delta_1 \cdots \delta_u \in I_t^{(k)}$ , then

$$\sum_{i=1}^u \max\{0, s_i - t + 1\} \geq k$$

which, since we are assuming  $s_i \geq t$  for all  $i$ , can be rewritten as

$$(*) \quad s \geq k + u(t - 1).$$

Moreover,  $s_i$  is the size of a minor of an  $n \times n$  matrix, and thus each  $s_i \leq n$ . In particular, this implies that  $un \geq s$ . Therefore, we must have

$$un \geq k + u(t - 1),$$

so that

$$u \geq \frac{k}{n - t + 1}.$$

Combining this with  $(*)$ , we obtain

$$s \geq k + \frac{k}{n - t + 1}(t - 1) = \frac{nk}{n - t + 1}.$$

By [DEP80, 7.3],  $I_t^m = I_t^{(m)} \cap I_{t-1}^{(2m)} \cap \dots \cap I_1^{(tm)}$ . Thus it suffices to show that  $\delta_1 \cdots \delta_u \in I_t^m$ . In particular, for each  $1 \leq j \leq t$ , we need to show that  $\delta_1 \cdots \delta_u \in I_j^{((t-j+1)m)}$ , which is equivalent to the following inequalities:

$$\begin{aligned} s &\geq tm \\ s &\geq (t-1)m + u = tm + (u-m) \\ s &\geq (t-2)m + 2u = tm + 2(u-m) \\ &\vdots \\ s &\geq m + (t-1)u = tm + (t-1)(u-m). \end{aligned}$$

Note that all these inequalities are convex combinations of the first and the last one, so that they are verified as long as the first and last one are, that is, if and only if

$$\begin{aligned} s &\geq tm \\ s &\geq m + (t-1)u. \end{aligned}$$

Since we are assuming that  $s \geq k + (t-1)u$ , then  $s \geq m + (t-1)u$  as long as  $k \geq m$ , which must be satisfied by any  $m$  with  $I_t^{(k)} \subseteq I_t^m$ . It remains to check that  $s \geq tm$ .

We have seen above that  $s \geq \frac{nk}{n-t+1}$ . Given this,  $s \geq tm$  is satisfied as long as  $\frac{nk}{n-t+1} \geq tm$ , which can be rewritten as  $k \geq \frac{(n-t+1)t}{n} m$ , which was our claim.

We conclude that  $I^{(k)} \subseteq I^m$  as long as  $k \geq \frac{t(n-t+1)}{n} m$ . This bound is much better than that of Theorem 9.46 when  $n$  is large and  $t$  is close to  $\frac{n}{2}$ . We also note that the computations above can be used to show that, given  $k$ , this is actually the best possible value of  $m$ .

Note that we can obtain the same containment results for the ideal of  $t \times t$  minors of a symmetric  $n \times n$  matrix or the ideal of  $2t$ -Pfaffians of a generic  $n \times n$  matrix, using Proposition 4.3 and Theorem 4.4 in [JMnV15] for the symmetric case and Theorem 2.1 and Theorem 2.4 in [DN96] for the Pfaffians.

The key ingredient that allowed us to completely answer the Containment Problem for such ideals is the explicit description of the symbolic and ordinary powers given by [DEP80, JMnV15, DN96]. In general, there is no such nice characterization of symbolic powers.

Theorems 9.44 and 9.46 actually extend beyond regular rings:

**Theorem 9.52** (Grifo–Ma–Schwede, 2022 [GMS22]). *Let  $R$  be an  $F$ -finite Gorenstein ring of prime characteristic  $p$  and let  $I$  be an ideal of finite projective dimension with big height  $h$ . Then*

- *If  $R/I$  is  $F$ -pure, then  $I^{(hn-h+1)} \subseteq I^n$  for all  $n \geq 1$ .*
- *If  $R/I$  is strongly  $F$ -regular, then  $I^{((h-1)n-(h-1)+1)} \subseteq I^n$  for all  $n \geq 1$ .*

To prove the theorem above, one needs to develop a Fedder-like criterion for ideals of finite projective dimension. The colon of interest in that case is no longer  $(I^{[q]} : I)$ , but an analogous colon where Frobenius powers are replaced by a slightly more technical ideal whose symbolic powers coincide with the associated primes of  $I$ .

We may also ask about Harbourne’s Conjecture in singular rings.

**Question 9.53.** Let  $R$  be a noetherian ring, and  $I$  a radical ideal. If  $I$  has Swanson constant  $s$ , must  $I^{(sn-s+1)} \subseteq I^n$  for all  $n \geq 1$ ?

# Appendix A

## Macaulay2

There are several computer algebra systems dedicated to algebraic geometry and commutative algebra computations, such as [Singular](#) (more popular among algebraic geometers), [CoCoA](#) (which is more popular with european commutative algebraists, having originated in Genova, Italy), and [Macaulay2](#). There are many computations you could run on any of these systems (and others), but we will focus on Macaulay2 since it's the most popular computer algebra system among US based commutative algebraists.

Macaulay2, as the name suggests, is a successor of a previous computer algebra system named Macaulay. Macaulay was first developed in 1983 by Dave Bayer and Mike Stillman, and while some still use it today, the system has not been updated since its final release in 2000. In 1993, Daniel Grayson and Mike Stillman released the first version of Macaulay2, and the current stable version is Macaulay2 1.16.

Macaulay2, or M2 for short, is an open-source project, with many contributors writing packages that are then released with the newest Macaulay2 version. Journals like the *Journal of Software for Algebra and Geometry* publish peer-refereed short articles that describe and explain the functionality of new packages, with the package source code being peer reviewed as well.

The National Science Foundation has funded Macaulay2 since 1992. Besides funding the project through direct grants, the NSF has also funded several Macaulay2 workshops — conferences where Macaulay2 package developers gather to work on new packages, and to share updates to the Macaulay2 core code and recent packages.

### A.1 Getting started

A Macaulay2 session often starts with defining some ambient ring we will be doing computations over. Common rings such as the rationals and the integers can be defined using the commands `QQ` and `ZZ`; one can easily take quotients or build polynomial rings (in finitely many variables) over these. For example,

```
i1 : R = ZZ/101[x,y]
```

```
o1 = R
```

```
o1 : PolynomialRing
```

```
and
```

```
i1 : k = ZZ/101;
```

```
i2 : R = k[x,y];
```

both store the ring  $\mathbb{Z}/101$  as  $R$ , with the small difference that in the second example Macaulay2 has named the coefficient field  $k$ . One quirk that might make a difference later is that if we use the first option and later set  $k$  to be the field  $\mathbb{Z}/101$ , our ring  $R$  is *not* a polynomial ring over  $k$ . Also, in the second example we ended each line with a `;`, which tells Macaulay2 to run the command but not display the result of the computation — which is in this case was simply an assignment, so the result is not relevant.

We can now do all sorts of computations over our ring  $R$ . For example, we can define an ideal in  $R$ , as follows:

```
i3 : I = ideal(x^2,y^2,x*y)
```

```
o3 = ideal (x2, y2, x*y)
```

```
o3 : Ideal of R
```

Above we have set  $I$  to be the ideal in  $R$  that is generated by  $x^2, y^2, xy$ . The notation `ideal( )` requires the usage of `^` for powers and `*` for products; alternatively, we can define the exact same ideal with the notation `ideal" "`, as follows:

```
i3 : I = ideal"x2,y2,xy"
```

```
o3 = ideal (x2, y2, x*y)
```

```
o3 : Ideal of R
```

Now we can use this ideal  $I$  to either define a quotient ring  $S = R/I$  or the  $R$ -module  $M = R/I$ , as follows:

```
i4 : M = R^1/I
```

```
o4 = cokernel | x2 y2 xy |  
1
```

```
o4 : R-module, quotient of R
```

```
i5 : S = R/I
```

```
o5 = S
```

```
o5 : QuotientRing
```

It's important to note that while  $R$  is a ring,  $R^1$  is the  $R$ -module  $R$  — this is a very important difference for Macaulay2, since these two objects have different types. So  $S$  defined above is a ring, while  $M$  is a module. Notice that Macaulay2 stored the module  $M$  as the cokernel of the map

$$R^3 \xrightarrow{\begin{bmatrix} x^2 & y^2 & xy \end{bmatrix}} R.$$

When you make a new definition in Macaulay2, you might want to pay attention to what ring your new object is defined over. For example, now that we defined this ring  $S$ , Macaulay2 has automatically taken  $S$  to be our current ambient ring, and any calculation or definition we run next will be considered over  $S$  and not  $R$ . If you want to return to the original ring  $R$ , you must first run the command `use R`.

If you want to work over a finitely generated algebra over one of the basic rings you can define in Macaulay2, and your ring is not a quotient of a polynomial ring, you want to rewrite this algebra as a quotient of a polynomial ring. For example, suppose you want to work over the second Veronese in 2 variables over our field  $k$  from before, meaning the algebra  $k[x^2, xy, y^2]$ . We need 3 algebra generators, which we will call  $a, b, c$ , corresponding to  $x^2$ ,  $xy$ , and  $y^2$ :

```
i6 : U = k[a,b,c]

o6 = U

o6 : PolynomialRing

i7 : f = map(R,U,{x^2,x*y,y^2})
          2      2
o7 = map(R,U,{x , x*y, y })

o7 : RingMap R <--- U

i8 : J = ker f
          2
o8 = ideal(b  - a*c)

o8 : Ideal of U

i9 : T = U/J

o9 = T

o9 : QuotientRing
```

Our ring  $T$  at the end is isomorphic to the 2nd Veronese of  $R$ , which is the ring we wanted. Note the syntax order in `map`: first target, then source, then a list with the images of each algebra generator.

## A.2 Asking Macaulay2 for help

As you're learning how to use Macaulay2, you will often find yourself needing some help. Luckily, Macaulay2 can help you directly! For example, suppose you know the name of a command, but do not remember the syntax to use it. You can ask `?command`, and Macaulay2 will show you the different usages of the command you want to know about.

```
i10 : ?primaryDecomposition
```

```
primaryDecomposition -- irredundant primary decomposition of an ideal
```

```
* Usage:
    primaryDecomposition I
* Inputs:
    * I, an ideal, in a (quotient of a) polynomial ring R
* Optional inputs:
    * MinimalGenerators => a Boolean value, default value true, if false, the
      components will not be minimalized
    * Strategy => ..., default value null,
* Outputs:
    * a list, containing a minimal list of primary ideals whose intersection
      is I
```

```
Ways to use primaryDecomposition :
```

```
=====
```

```
* "primaryDecomposition(Ideal)" -- see "primaryDecomposition" -- irredundant
  primary decomposition of an ideal
* "primaryDecomposition(Module)" -- irredundant primary decomposition of a
  module
* "primaryDecomposition(Ring)" -- see "primaryDecomposition(Module)" --
  irredundant primary decomposition of a module
```

```
For the programmer
```

```
=====
```

The object `"primaryDecomposition"` is a method function with options.

If instead you'd rather read the complete Macaulay2 documentation on the command you are interested in, you can use the `viewHelp` command, which will open an html page with the documentation you asked for. So running

```
i11 : viewHelp "primaryDecomposition"
```

will open an html page dedicate to the method `primaryDecomposition`, which includes examples and links to related methods.



## A.3 Basic commands

Many Macaulay2 commands are easy to guess, and named exactly what you would expect them to be named. Often, googling “Macaulay2” followed by a few descriptive words will easily land you on the documentation for whatever you are trying to do.

Here are some basic commands you will likely use:

- `ideal( $f_1, \dots, f_n$ )` will return the ideal generated by  $f_1, \dots, f_n$ . Here products should be indicated by `*`, and powers with `^`. If you’d rather not use `^` (this might be nice if you have lots of powers), you can write `ideal( $f_1, \dots, f_n$ )` instead.
- `map( $S, R, f_1, \dots, f_n$ )` gives a ring map  $R \rightarrow S$  if  $R$  and  $S$  are rings, and  $R$  is a quotient of  $k[x_1, \dots, x_n]$ . The resulting ring map will send  $x_i \mapsto f_i$ . There are many variations of `map` — for example, you can use it to define  $R$ -module homomorphisms — but you should carefully input the information in the required format. Try `viewHelp map` in Macaulay2 for more details
- `ker( $f$ )` returns the kernel of the map  $f$ .
- `I + J` and `I * J` return the sum and product of the ideals  $I$  and  $J$ , respectively.
- `A = matrix{{ $a_{1,1}, \dots, a_{1,n}$ }, ..., { $a_{m,1}, \dots, a_{m,n}$ }}` returns the matrix

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}$$

If you are familiar with any other programming language, many of the basics are still the same. For example, some of the commands we will use return lists, and we might often need to do operations on lists. As with many other programming languages, a list is indicated by `{ }` with the elements separated by commas.

```
i6 : w = {ZZ, 3, ideal"xy3"}
          3
o6 = {ZZ, 3, ideal(x*y )}

o6 : List
```

As in most programming languages, Macaulay2 follows the convention that the first position in a list is the 0th position.

The method `primaryDecomposition` returns a list of primary ideals whose intersection is the input ideal, and `associatedPrimes` returns the list of associated primes of the given ideal or module. Operations on lists are often intuitive. For example, let’s say we want to find the primary component of an ideal with a particular radical.

```

i1 : R = QQ[x,y];

i2 : I = ideal"x2,xy";

o2 : Ideal of R

i3 : prim = primaryDecomposition I
          2
o3 = {ideal x, ideal (y, x )}

o3 : List

i4 : L = select(prim, Q -> radical(Q) == ideal"x,y")
          2
o4 = {ideal (y, x )}

o4 : List

```

The method `select` returns a list of all the elements in our list with the required properties. In this case, if we actually want the primary ideal we just selected, as opposed to a list containing it, we need to extract the first component of our list  $L$ .

```

i5 : L_0
          2
o5 = ideal (y, x )

o5 : Ideal of R

```

## A.4 Graded rings

Polynomial rings in Macaulay2 are graded with the standard grading by default, meaning that all the variables have degree 1. To define a different grading, we give Macaulay2 a list with the grading of each of the variables:

```

i1 : R = ZZ/101[a,b,c,Degrees=>{{1,2},{2,1},{1,0}}];

```

We can check whether an element of  $R$  is `isHomogeneous`, and the function `degree` applied to an element of  $R$  returns the least upper bound of the degrees of its monomials:

```

i2 : degree (a+b)
o2 = {2, 2}
o2 : List

i3 : isHomogeneous(a+b)
o3 = false

```

## A.5 Complexes and homology in Macaulay2

There are two different ways to do computations involving complexes in Macaulay2: using `ChainComplexes`, or the new (and still under construction) `Complexes` package. To use `Complexes`, you must first load the `Complexes` package, while the `ChainComplexes` methods are automatically loaded with Macaulay2.

### A.5.1 Chain Complexes

To create a new chain complex by hand, we start by setting up  $R$ -module maps.

```
i1 : R = QQ[a,b];

i2 : d1 = map(R^1, R^2, {{a,b}})

o2 = | a b |
      1      2
o2 : Matrix R  <--- R

i3 : d2 = map(R^2, R^1, {{-b},{a}})

o3 = | -b |
      | a  |
      2      1
o3 : Matrix R  <--- R
```

Keep in mind that the syntax of `map` is a bit funny: we write `map(target,source,matrix)`. To make sure we set up the next map in a way that is composable with  $d_1$ , we can use the methods `source` and `target`:

```
i3 : d1 = map(source d0, R^1, {{-b},{a}})

o3 = | -b |
      | a  |
      2      1
o3 : Matrix R  <--- R
```

We can also double check our maps do indeed map a complex, by checking the composition  $d_1 \circ d_2$ :

```
i4 : d1 * d2 == 0

o4 = true
```

So now we are ready to set up our new chain complex.

```
i5 : C = new ChainComplex
```

```
o5 = 0
```

```
o5 : ChainComplex
```

```
i6 : C#0 = target d1
```

```
      1
o6 = R
```

```
o6 : R-module, free
```

```
i7 : C#1 = target d2
```

```
      2
o7 = R
```

```
o7 : R-module, free
```

```
i8 : C#2 = source d2
```

```
      1
o8 = R
```

```
o8 : R-module, free
```

Given a chain complex  $C$ , we can ask Macaulay2 what our complex is by simply running the name of the complex:

```
i9 : C
```

```
      1      2      1
o9 = R  <-- R  <-- R
```

```
      0      1      2
```

```
o9 : ChainComplex
```

Or we can ask for a better visual description of the maps, using  $C.dd$ :

```
i10 : C.dd
```

```
      1      2
o10 = 0 : R  <----- R  : 1
              0
```

$$\begin{array}{ccc} & 2 & 1 \\ 1 : R & \xleftarrow{\quad} & R : 2 \\ & 0 & \end{array}$$

o10 : ChainComplexMap

We can also set up the same complex in a more compact way, by simply feeding the maps we want in order. Macaulay2 will automatically place the first map with the target in homological degree 0 and the source in degree 1.

11 : D = chainComplex(d1,d2)

$$\begin{array}{ccccc} & 1 & & 2 & & 1 \\ \text{o11} = R & \xleftarrow{\quad} & R & \xleftarrow{\quad} & R \\ & 0 & & 1 & & 2 \end{array}$$

o11 : ChainComplex

Notice this is indeed the same complex.

i12 : D.dd

$$\begin{array}{ccc} & 1 & 2 \\ \text{o12} = 0 : R & \xleftarrow{\quad} & R : 1 \\ & | \ a \ b \ | & \end{array}$$

$$\begin{array}{ccc} & 2 & 1 \\ 1 : R & \xleftarrow{\quad} & R : 2 \\ & | \ -b \ | & \\ & | \ a \ | & \end{array}$$

o12 : ChainComplexMap

We can also ask Macaulay2 to compute the homology of our complex:

i13 : HH D

$$\text{o13} = 0 : \text{cokernel } | \ a \ b \ |$$

$$\begin{array}{l} 1 : \text{subquotient } (| \ b \ |, | \ -b \ |) \\ \qquad \qquad \qquad | \ -a \ | \ | \ a \ | \\ 2 : \text{image } 0 \end{array}$$

o13 : GradedModule

Or we could simply ask for the homology in a specific degree:

```
i14 : HH_0 D

o14 = cokernel | a b |
              1
o14 : R-module, quotient of R
```

### A.5.2 The Complexes package

To use this functionality, you must first load the `Complexes` package.

```
i15 : needsPackage "Complexes";

o15 = Complexes

o15 : Package
```

We can use our maps from above to set up a complex with the same maps. We feed a list of the maps we want to use to the method `complex`.

```
i16 : F = complex({d1,d2})

      1      2      1
o16 = R  <-- R  <-- R

      0      1      2

o16 : Complex
```

We can read off the maps and the homology in our complex using the same commands as we use with `chainComplexes`, although the information returned gets presented in a slightly different fashion.

```
i17 : HH F

o17 = cokernel | a b | <-- subquotient (| b |, | -b |) <-- image 0
              | -a | | a |
              0              2

              1

o17 : Complex

i18 : F.dd

      1      2
```

```
o18 = 0 : R  <----- R   : 1
          | a b |
```

```
      2           1
1 : R  <----- R   : 2
      | -b |
      | a  |
```

```
o18 : ComplexMap
```

If we want to set up our complex starting in a different homological degree, we can do the following:

```
i19 : G = complex({d1,d2}, Base => 7)
```

```
      1      2      1
o19 = R  <-- R  <-- R
      7      8      9
```

```
o19 : Complex
```

```
i20 : H = complex({d1,d2}, Base => -13)
```

```
      1      2      1
o20 = R  <-- R  <-- R
      -13    -12    -11
```

```
o20 : Complex
```

### A.5.3 Maps of complexes

Suppose we are given two complexes  $C$  and  $D$  and a map of complexes  $f : C \rightarrow D$ . The routine `map` can be used to define  $f$  using `chainComplexes`: it receives the target  $D$ , the source  $C$ , and a function `f` that returns  $f_i$  when we compute `f(i)`.

```
i1 : R = QQ[a,b];
```

```
i2 : c1 = map(R^0,R^1,0);
```

```
      1
o2 : Matrix 0 <--- R
```

```
i3 : c2 = map(R^1, R^2, {{a,b}});
```

```

      1      2
o3 : Matrix R <--- R

i4 : c3 = map(R^2, R^1, {{-b},{a}});

      2      1
o4 : Matrix R <--- R

i5 : c4 = map(R^1, R^0, 0);

      1
o5 : Matrix R <--- 0

i6 : C = chainComplex(c1,c2,c3,c4);

i7 :
    d1 = map(R^0,R^1,0);

      1
o7 : Matrix 0 <--- R

i8 : d2 = id_(R^1);

      1      1
o8 : Matrix R <--- R

i9 : d3 = map(R^1, R^0, 0);

      1
o9 : Matrix R <--- 0

i10 : d4 = map(R^0, R^0, 0);

o10 : Matrix 0 <--- 0

i11 : D = chainComplex(d1,d2,d3,d4)

      1      1
o11 = 0 <-- R <-- R <-- 0 <-- 0

      0      1      2      3      4

o11 : ChainComplex

i12 :
```



```

      f0 = map(R^0, R^0, 0);

o12 : Matrix 0 <--- 0

i13 : f1 = map(R^1, R^1, matrix{{0_R}});

      1      1
o13 : Matrix R  <--- R

i14 : f2 = map(R^2, R^1, {{b},{-a}});

      2      1
o14 : Matrix R  <--- R

i15 : f3 = map(R^1, R^0, 0);

      1
o15 : Matrix R  <--- 0

i16 : f4 = map(R^0, R^0, 0);

o16 : Matrix 0 <--- 0

i17 : f = map(C,D,i -> if i==0 then f0 else(
      if i==1 then f1 else (
      if i==2 then f2 else (
      if i == 3 then f3 else (
      if i==4 then f4))))))

o17 = 0 : 0 <----- 0 : 0
      0

      1      1
1 : R  <----- R  : 1
      0

      2      1
2 : R  <----- R  : 2
      | b |
      | -a |

      1
3 : R  <----- 0 : 3
      0

4 : 0 <----- 0 : 4

```

$$0$$

o17 : ChainComplexMap

Here's what we can do if we prefer to write a list with the maps in **f**:

i18 : f = map(C,D,i -> {f0,f1,f2,f3,f4}\_i)

o18 = 0 : 0 <----- 0 : 0  

$$0$$

$$\begin{array}{ccc} 1 & & 1 \\ 1 : R & \xleftarrow{\quad} & R : 1 \\ & 0 & \end{array}$$

$$\begin{array}{ccc} 2 & & 1 \\ 2 : R & \xleftarrow{\quad} & R : 2 \\ & \begin{array}{cc} | & b & | \\ | & -a & | \end{array} & \end{array}$$

$$\begin{array}{ccc} 1 & & \\ 3 : R & \xleftarrow{\quad} & 0 : 3 \\ & 0 & \end{array}$$

$$\begin{array}{ccc} 4 : 0 & \xleftarrow{\quad} & 0 : 4 \\ & 0 & \end{array}$$

o18 : ChainComplexMap

If we prefer to do the same with the **Complexes** package, one advantage is that **map** *does* receive (target, source, list of maps).

i42 : C = complex({c1,c2,c3,c4});

i43 : D = complex({d1,d2,d3,d4});

i44 : f = map(C,D,{f0,f1,f2,f3,f4})

$$\begin{array}{ccc} 2 & & 1 \\ \text{o44} = 2 : R & \xleftarrow{\quad} & R : 2 \\ & \begin{array}{cc} | & b & | \\ | & -a & | \end{array} & \end{array}$$

o44 : ComplexMap

# Appendix B

## Commutative algebra background

### B.1 Prime Avoidance

**Theorem B.1** (Prime avoidance). *Let  $R$  be a ring,  $I_1, \dots, I_n, J$  be ideals, and suppose that  $I_i$  is prime for  $i > 2$ .<sup>1</sup> If  $J \not\subseteq I_i$  for all  $i$ , then  $J \not\subseteq \bigcup_i I_i$ . Equivalently, if  $J \subseteq \bigcup_i I_i$ , then  $J \subseteq I_i$  for some  $i$ . So if  $J \not\subseteq I_i$  for all  $i$ , then we can find an element  $x \in J$  such that  $x \notin I_i$  for all  $i$ .*

*Moreover, if  $R$  is  $\mathbb{N}$ -graded, and all of the ideals are homogeneous, all  $I_i$  are prime, and  $J \not\subseteq I_i$  for all  $i$ , then there is a homogeneous element in  $J$  that is not in  $\bigcup_i I_i$ .*

*Proof.* We proceed by induction on  $n$ . If  $n = 1$ , there is nothing to show.

By induction hypothesis, we can find elements  $a_i$  such that

$$a_i \notin \bigcup_{j \neq i} I_j \text{ and } a_i \in J$$

for each  $i$ . If some  $a_i \notin I_i$ , we are done, so let's assume that  $a_i \in I_i$  for each  $i$ . Consider  $a = a_n + a_1 \cdots a_{n-1} \in J$ . Notice that  $a_1 \cdots a_{n-1} = a_i(a_1 \cdots \widehat{a_i} \cdots a_{n-1}) \in I_i$ . If  $a \in I_i$  for  $i < n$ , then we also have  $a_n \in I_i$ , a contradiction. If  $a \in I_n$ , then we also have  $a_1 \cdots a_{n-1} = a - a_n \in I_n$ , since  $a_n \in I_n$ . If  $n = 2$ , this says  $a_1 \in I_2$ , a contradiction. If  $n > 2$ , our assumption is that  $I_n$  is prime, so one of  $a_1, \dots, a_{n-1} \in I_n$ , which is a contradiction. So  $a$  is the element we were searching for, meaning  $a \notin I_i$  for all  $i$ .

If all  $I_i$  are homogeneous and prime, then we proceed as above but replacing  $a_n$  and  $a_1, \dots, a_{n-1}$  with suitable powers so that  $a_n + a_1 \cdots a_{n-1}$  is homogeneous. For example, we could take

$$a := a_n^{\deg(a_1) + \cdots + \deg(a_{n-1})} + (a_1 \cdots a_{n-1})^{\deg(a_n)}.$$

The primeness assumption guarantees that noncontainments in ideals is preserved.  $\square$

**Corollary B.2.** *Let  $I$  be an ideal and  $M$  a finitely generated module over a Noetherian ring  $R$ . If  $I$  consists of zerodivisors on  $M$ , then  $Im = 0$  for some nonzero  $m \in M$ .*

---

<sup>1</sup>So all the ideals are prime, except we may allow two of them to not be prime.

*Proof.* The assumption says that

$$I \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}(M)} (\mathfrak{p}).$$

By the assumptions, Theorem 1.44 applies, and it guarantees that this is a finite set of primes. By Prime Avoidance,  $I \subseteq \mathfrak{p}$  for some  $\mathfrak{p} \in \text{Ass}(M)$ . Equivalently,  $I \subseteq \text{ann}_R(m)$  for some nonzero  $m \in M$ .  $\square$

Later on, we will also need a slightly stronger version of Prime Avoidance, so we record it now.

**Theorem B.3.** *Let  $R$  be a ring,  $P_1, \dots, P_n$  prime ideals,  $x \in R$  and  $I$  be an ideal in  $R$ . If  $(x) + I \not\subseteq P_i$  for each  $i$ , then there exists  $y \in I$  such that*

$$x + y \notin \bigcup_{i=1}^n P_i.$$

*Proof.* We proceed by induction on  $n$ . When  $n = 1$ , if every element of the form  $x + y$  with  $y \in R$  is in  $P = P_1$ , then multiplying by  $r \in R$  we conclude that every  $rx + y \in P$ , meaning  $(x) + I \subseteq P$ .

Now suppose  $n > 1$  and that we have shown the statement for  $n - 1$  primes. If  $P_i \subseteq P_j$  for some  $i \neq j$ , then we might as well exclude  $P_i$  from our list of primes, and the statement follows by induction. So assume that all our primes  $P_i$  are incomparable.

If  $x \notin P_i$  for all  $i$ , we are done, since we can take  $x + 0$  for the element we are searching for. So suppose  $x$  is in some  $P_i$ , which we assume without loss of generality to be  $P_n$ . Our induction hypothesis says that we can find  $y \in I$  such that  $x + y \notin P_1 \cup \dots \cup P_{n-1}$ . If  $x + y \notin P_n$ , we are done, so suppose  $x + y \in P_n$ . Since we assumed  $x \in P_n$ , we must have  $I \not\subseteq P_n$ , or else we would have had  $(x) + I \subseteq P_n$ . Now  $P_n$  is a prime ideal that does not contain  $P_1, \dots, P_{n-1}$ , nor  $I$ , so

$$P \not\supseteq IP_1 \cdots P_{n-1}.$$

Choose  $z \in IP_1 \cdots P_{n-1}$  not in  $P_n$ . Then  $x + y + z \notin P_n$ , since  $z \notin P_n$  but  $x + y \in P_n$ . Moreover, for all  $i < n$  we have  $x + y + z \notin P_i$ , since  $z \in P_i$  and  $x + y \notin P_i$ .  $\square$

## B.2 NAK

We will now show a very simple but extremely useful result known as Nakayama's Lemma. As noted in [Mat89, page 8], Nakayama himself claimed that this should be attributed to Krull and Azumaya, but it's not clear which of the three actually had the commutative ring statement first. So some authors (eg, Matsumura) prefer to refer to it as NAK. There are actually a range of statements, rather than just one, that go under the banner of Nakayama's Lemma a.k.a. NAK.

**Proposition B.4.** *Let  $R$  be a ring,  $I$  an ideal, and  $M$  a finitely generated  $R$ -module. If  $IM = M$ , then*

a) *there is an element  $r \in 1 + I$  such that  $rM = 0$ , and*

b) there is an element  $a \in I$  such that  $am = m$  for all  $m \in M$ .

*Proof.* Let  $M = Rm_1 + \cdots + Rm_s$ . By assumption, we have equations

$$m_1 = a_{11}m_1 + \cdots + a_{1s}m_s, \quad \dots, \quad m_s = a_{s1}m_1 + \cdots + a_{ss}m_s,$$

with  $a_{ij} \in I$ . Setting  $A = [a_{ij}]$  and  $v = [x_i]$  we have a matrix equations  $Av = v$ . By the determinantal trick, Lemma B.18, the element  $\det(I_{s \times s} - A) \in R$  kills each  $m_i$ , and hence  $M$ . Since  $\det(I_{s \times s} - A) \equiv \det(I_{s \times s}) \equiv 1 \pmod{I}$ , this determinant is the element  $r$  we seek for the first statement.

For the latter statement, set  $a = 1 - r$ ; this is in  $I$  and satisfies  $am = m - rm = m$  for all  $m \in M$ .  $\square$

**Proposition B.5.** *Let  $(R, \mathfrak{m}, k)$  be a local ring, and  $M$  be a finitely generated module. If  $M = \mathfrak{m}M$ , then  $M = 0$ .*

*Proof.* By the Proposition B.4, there exists an element  $r \in 1 + \mathfrak{m}$  that annihilates  $M$ . Notice that  $1 \notin \mathfrak{m}$ , so any such  $r$  must be outside of  $\mathfrak{m}$ , and thus a unit. Multiplying by its inverse, we conclude that 1 annihilates  $M$ , or equivalently, that  $M = 0$ .  $\square$

**Proposition B.6.** *Let  $(R, \mathfrak{m}, k)$  be a local ring, and  $M$  be a finitely generated module, and  $N$  a submodule of  $M$ . If  $M = N + \mathfrak{m}M$ , then  $M = N$ .*

*Proof.* By taking the quotient by  $N$ , we see that

$$M/N = (N + \mathfrak{m}M)/N = \mathfrak{m}(M/N).$$

By Proposition B.5,  $M = N$ .  $\square$

**Proposition B.7.** *Let  $(R, \mathfrak{m}, k)$  be a local ring, and  $M$  be a finitely generated module. For  $m_1, \dots, m_s \in M$ ,*

$$m_1, \dots, m_s \text{ generate } M \iff \overline{m_1}, \dots, \overline{m_s} \text{ generate } M/\mathfrak{m}M.$$

*Thus, any generating set for  $M$  consists of at least  $\dim_k(M/\mathfrak{m}M)$  elements.*

*Proof.* The implication  $(\Rightarrow)$  is clear. If  $m_1, \dots, m_s \in M$  are such that  $\overline{m_1}, \dots, \overline{m_s}$  generate  $M/\mathfrak{m}M$ , let  $N = Rm_1 + \cdots + Rm_s \subseteq M$ . By Proposition B.5,  $M/N = 0$  if and only if  $M/N = \mathfrak{m}(M/N)$ . The latter statement is equivalent to  $M = \mathfrak{m}M + N$ , which is equivalent to saying that  $M/\mathfrak{m}M$  is generated by the image of  $N$ .  $\square$

**Remark B.8.** Since  $R/\mathfrak{m}$  is a field,  $M/\mathfrak{m}M$  is a vector space over the field  $R/\mathfrak{m}$ .

**Definition B.9.** Let  $(R, \mathfrak{m})$  be a local ring, and  $M$  a finitely generated module. A set of elements  $\{m_1, \dots, m_t\}$  is a **minimal generating set** of  $M$  if the images of  $m_1, \dots, m_t$  form a basis for the  $R/\mathfrak{m}$  vector space  $M/\mathfrak{m}M$ .

As a consequence of basic facts about basis for vector spaces, we conclude that any generating set for  $M$  contains a minimal generating set, and that every minimal generating set has the same cardinality.

**Definition B.10.** Let  $(R, \mathfrak{m})$  be a local ring, and  $N$  an  $R$ -module. The **minimal number of generators** of  $M$  is

$$\mu(M) := \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M).$$

Equivalently, this is the number of elements in a minimal generating set for  $M$ .

We commented before that graded rings behave a lot like local rings, so now we want to give graded analogues for the results above.

**Proposition B.11.** *Let  $R$  be an  $\mathbb{N}$ -graded ring, and  $M$  a  $\mathbb{Z}$ -graded module such that  $M_{<a} = 0$  for some  $a$ . If  $M = (R_+)M$ , then  $M = 0$ .*

*Proof.* On the one hand, the homogeneous elements in  $M$  live in degrees at least  $a$ , but  $(R_+)M$  lives in degrees strictly bigger than  $a$ . If  $M$  has a nonzero element, it has a nonzero homogeneous element, and we obtain a contradiction.  $\square$

This condition includes all finitely generated  $\mathbb{Z}$ -graded  $R$ -modules.

**Remark B.12.** If  $M$  is finitely generated, then it can be generated by finitely many homogeneous elements, the homogeneous components of some finite generating set. If  $a$  is the smallest degree of a homogeneous element in a homogeneous generating set, since  $R$  lives only in positive degrees we must have  $M \subseteq RM_{\geq a} \subseteq M_{\geq a}$ , so  $M_{<a} = 0$ .

Just as above, we obtain the following:

**Proposition B.13.** *Let  $R$  be an  $\mathbb{N}$ -graded ring, with  $R_0$  a field, and  $M$  a  $\mathbb{Z}$ -graded module such that  $M_{<a} = 0$  for some degree  $a$ . A set of elements of  $M$  generates  $M$  if and only if their images in  $M/(R_+)M$  spans as a vector space. Since  $M$  and  $(R_+)M$  are graded,  $M/(R_+)M$  admits a basis of homogeneous elements.*

In particular, if  $k$  is a field,  $R$  is a positively graded  $k$ -algebra, and  $I$  is a homogeneous ideal, then  $I$  has a minimal generating set by homogeneous elements, and this set is unique up to  $k$ -linear combinations.

**Definition B.14.** Let  $R$  be an  $\mathbb{N}$ -graded ring with  $R_0$  a field, and  $M$  a finitely generated  $\mathbb{Z}$ -graded  $R$ -module. The **minimal number of generators** of  $M$  is

$$\mu(M) := \dim_{R/R_+}(M/R_+M).$$

We can use Macaulay2 to compute (the) minimal (number of) generators of graded modules over graded  $k$ -algebras, using the commands `mingens` and `numgens`.

Note that we can use NAK to prove that certain modules are finitely generated in the graded case; in the local case, we cannot.

## B.3 Krull's Intersection Theorem

**Theorem B.15** (Krull intersection theorem). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring. Then*

$$\bigcap_{n \geq 1} \mathfrak{m}^n = 0.$$

*Proof.* Let  $J = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n$ . First, we claim that  $J \subseteq \mathfrak{m}J$ .

Let  $\mathfrak{m}J = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$  be a primary decomposition. To show that  $J \subseteq \mathfrak{m}J$ , it is sufficient to prove that  $J \subseteq \mathfrak{q}_i$  for each  $i$ . If  $\sqrt{\mathfrak{q}_i} \neq \mathfrak{m}$ , pick  $x \in \mathfrak{m}$  such that  $x \notin \sqrt{\mathfrak{q}_i}$ . Then  $xJ \subseteq \mathfrak{m}J \subseteq \mathfrak{q}_i$ , but  $x \notin \sqrt{\mathfrak{q}_i}$ , so  $J \subseteq \mathfrak{q}_i$  by definition of primary. If instead  $\sqrt{\mathfrak{q}_i} = \mathfrak{m}$ , there is some  $N$  with  $\mathfrak{m}^N \subseteq \mathfrak{q}_i$  by ???. By definition of  $J$ , we have  $J \subseteq \mathfrak{m}^N \subseteq \mathfrak{q}_i$ , and we are done.

We showed that  $J \subseteq \mathfrak{m}J$ , hence  $J = \mathfrak{m}J$ , and thus  $J = 0$  by [NAK](#).  $\square$

This also holds over any domain without the local assumption.

**Theorem B.16** (Krull Intersection Theorem for domains). *If  $R$  is a domain, then*

$$\bigcap_{n \geq 1} I^n = 0.$$

*for any proper ideal  $I$  in  $R$ .*

*Proof.* Let  $\mathfrak{m}$  be a maximal ideal in  $R$ . The only minimal prime over  $\mathfrak{m}^n$  is  $\mathfrak{m}$ , so the powers of  $\mathfrak{m}$  have no embedded primes and must then all be  $\mathfrak{m}$ -primary. In particular,  $\mathfrak{m}^n = \mathfrak{m}^n R_{\mathfrak{m}} \cap R$ . Notice that taking pre-images commutes with taking intersections, so

$$\bigcap_{n \geq 1} \mathfrak{m}^n = \bigcap_{n \geq 1} (\mathfrak{m}^n R_{\mathfrak{m}} \cap R) = \left( \bigcap_{n \geq 1} \mathfrak{m}^n R_{\mathfrak{m}} \right) \cap R = \left( \bigcap_{n \geq 1} \mathfrak{m}_{\mathfrak{m}}^n \right) \cap R.$$

By Theorem [B.15](#),  $\bigcap_{n \geq 1} \mathfrak{m}_{\mathfrak{m}}^n = 0$ . Since  $R$  is a domain, the localization map is injective, and we conclude that

$$\bigcap_{n \geq 1} \mathfrak{m}^n = 0.$$

Now if  $I$  is any proper ideal in  $R$ ,  $I \subseteq \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$ , and

$$\bigcap_{n \geq 1} I^n = \bigcap_{n \geq 1} \mathfrak{m}^n = 0. \quad \square$$

## B.4 Ring extensions

In field theory, there is a close relationship between (vector space-)finite field extensions and algebraic equations. The situation for rings is similar.

**Definition B.17** (Integral element/extension). Let  $R$  be an  $A$ -algebra. The element  $r \in R$  is **integral** over  $A$  if there are elements  $a_0, \dots, a_{n-1} \in A$  such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0;$$

i.e.,  $r$  satisfies an **equation of integral dependence** over  $A$ . We say that  $R$  is **integral over**  $A$  if every  $r \in R$  is integral over  $A$ .

Integral automatically implies algebraic, but the condition that there exists an equation of algebraic dependence that is *monic* is stronger in the setting of rings.

**Lemma B.18** (Determinantal trick). *Let  $R$  be a ring,  $B \in M_{n \times n}(R)$ ,  $v \in R^{\oplus n}$ , and  $r \in R$ .*

- 1)  $\text{adj}(B)B = \det(B)I_{n \times n}$ .
- 2) *If  $Bv = rv$ , then  $\det(rI_{n \times n} - B)v = 0$ .*

*Proof.*

- 1) When  $R$  is a field, this is a basic linear algebra fact. We deduce the case of a general ring from the field case.

The ring  $R$  is a  $\mathbb{Z}$ -algebra, so we can write  $R$  as a quotient of some polynomial ring  $\mathbb{Z}[X]$ . Let  $\psi : \mathbb{Z}[X] \twoheadrightarrow R$  be a surjection,  $a_{ij} \in \mathbb{Z}[X]$  be such that  $\psi(a_{ij}) = b_{ij}$ , and let  $A = [a_{ij}]$ . Note that

$$\psi(\text{adj}(A)_{ij}) = \text{adj}(B)_{ij} \quad \text{and} \quad \psi((\text{adj}(A)A)_{ij}) = (\text{adj}(B)B)_{ij},$$

since  $\psi$  is a homomorphism, and the entries are the same polynomial functions of the entries of the matrices  $A$  and  $B$ , respectively. Thus, it suffices to establish

$$\text{adj}(B)B = \det(B)I_{n \times n}$$

in the case when  $R = \mathbb{Z}[X]$ , and we can do this entry by entry. Now,  $R = \mathbb{Z}[X]$  is an integral domain, hence a subring of a field (its fraction field). Since both sides of the equation

$$(\text{adj}(B)B)_{ij} = (\det(B)I_{n \times n})_{ij}$$

live in  $R$  and are equal in the fraction field (by linear algebra) they are equal in  $R$ . This holds for all  $i, j$ , and thus 1) holds.

- 2) We have  $(rI_{n \times n} - B)v = 0$ , so by part 1)

$$\det(rI_{n \times n} - B)v = \text{adj}(rI_{n \times n} - B)(rI_{n \times n} - B)v = 0. \quad \square$$

**Theorem B.19.** *Let  $A \subseteq R$  be module-finite. Then  $R$  is integral over  $A$ .*



*Proof.* Given  $r \in R$ , we want to show that  $r$  is integral over  $A$ . The idea is to show that multiplication by  $r$ , realized as a linear transformation over  $A$ , satisfies the characteristic polynomial of that linear transformation.

Write  $R = Ar_1 + \cdots Ar_t$ . We may assume that  $r_1 = 1$ , perhaps by adding module generators. By assumption, we can find  $a_{ij} \in A$  such that

$$rr_i = \sum_{j=1}^t a_{ij}r_j$$

for each  $i$ . Let  $C = [a_{ij}]$ , and  $v$  be the column vector  $(r_1, \dots, r_t)$ . We have  $rv = Cv$ , so by the determinant trick,  $\det(rI_{n \times n} - C)v = 0$ . Since we chose one of the entries of  $v$  to be 1, we have in particular that  $\det(rI_{n \times n} - C) = 0$ . Expanding this determinant as a polynomial in  $r$ , this is a monic equation with coefficients in  $A$ .  $\square$

**Lemma B.20.** *If  $R \subseteq S$  is an integral extension of domains,  $R$  is a field if and only if  $S$  is a field.*

*Proof.* Take any nonzero  $s \in S$ , and consider an equation of integral dependence of  $s$  over  $R$ , say

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0.$$

Since  $a_0$  is a unit in  $R \subseteq S$ , we can divide by  $a_0$ , so that

$$-s(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1) = 1.$$

The  $s$  is a unit, and  $S$  is a field.

If  $S$  is a field, and  $r \in R$  is nonzero, then there exists an inverse  $s$  for  $r$  in  $S$ , which is integral over  $R$ . Then

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0$$

for some  $a_i \in R$ , and multiplying through by  $r^{n-1}$  gives

$$s = -(a_{n-1} + a_{n-2}r \cdots + a_1r^{n-2} + a_0r^{n-1}) \in R.$$

Then  $R$  is a field.  $\square$

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