Problem Set 3

Turn in **4** of the following problems. You **must** pick at least **2 problems** involving tensor products.

Problem 1. Consider an exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{d} E.$$

a) Show a is surjective if and only if c is injective.

b) Show that if a and d are isos, then $C = 0$.

c) Show that every short exact sequence breaks into short exact sequences

$$0 \longrightarrow \operatorname{coker} a \xrightarrow{\alpha} C \xrightarrow{\beta} \ker d \longrightarrow 0$$

with

$$\alpha(x + \operatorname{im} a) = b(x) \qquad \text{and} \qquad \beta(x) = c(x).$$

Problem 2. Let $T\colon R\text{-}\mathbf{mod} \longrightarrow S\text{-}\mathbf{mod}$ be an additive functor. Show that if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a split short exact sequence of R-modules, then

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0$$

is a short exact sequence of S-modules.

Let R be a commutative ring and I be an ideal in R. The ***I*-torsion functor** is the functor $\Gamma_I\colon R\text{-}\mathbf{mod} \longrightarrow R\text{-}\mathbf{mod}$ that sends each R-module M to the R-module

$$\Gamma_I(M) := \bigcup_{n \geqslant 1} (0 :_M I^n) = \{m \in M \mid I^n m = 0 \text{ for some } n \geqslant 1\}$$

and that sends each R-module homomorphism $f\colon M \to N$ to its restriction to $\Gamma_I(M) \to \Gamma_I(N)$.

Problem 3. Let R be a commutative ring and I be an ideal in R.

a) Show that any R-module homomorphism $f\colon M \to N$ satisfies $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$.

b) Show that Γ_I is an indeed additive covariant functor.

c) Show that Γ_I is left exact.

d) Show that Γ_I is not right exact.

Problem 4. Let R be a commutative ring, I and J ideals in R, and M be an R-module.

a) Show that $R/I \otimes_R R/J \cong R/(I+J)$.

b) Show that $R/I \otimes_R M \cong M/IM$.

c) There is an R-module map $I \otimes_R M \longrightarrow IM$ induced by the R-bilinear map $(a, m) \mapsto am$. This map is always clearly surjective; must it be injective?

Problem 5. Let $R = \mathbb{Z}[x]$, $I = (2, x)$, and consider the R-module $M = I \otimes_R I$.

a) Show that $2 \otimes 2 + x \otimes x$ is not a simple tensor in M.

b) Show that $m = 2 \otimes x - x \otimes 2$ is a nonzero torsion element in M.

c) Show that the submodule of $I \otimes_R I$ generated by m is isomorphic to R/I.

Let R be a domain and M be an R-module. The **torsion** of M is the submodule

$$T(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

The elements of $T(M)$ are called **torsion elements**, and we say that M **is torsion** if $T(M) = M$. Finally, M is **torsion free** if $T(M) = 0$.

Problem 6. By torsion abelian group we mean torsion $\mathbb{Z}$-module.

a) Show that if A is a divisible abelian group and T is a torsion abelian group, then $A \otimes_{\mathbb{Z}} T = 0$.

b) Prove that there is no nonzero (unital) ring R such that the underlying abelian group $(R, +)$ is both torsion and divisible.

For example, this shows that there is no ring whose underlying abelian group is $\mathbb{Q}/\mathbb{Z}$.

Problem 7. Let R be a domain with fraction field Q and M be an R-module.

a) Show that the R-module $M/T(M)$ is torsion free.

b) If $f : M \longrightarrow N$ is an R-module homomorphism, $f(T(M)) \subseteq T(N)$.

c) Show that the kernel of the map $M \longrightarrow Q \otimes_R M$ given by $m \mapsto 1 \otimes m$ is $T(M)$.

d) Show that torsion is a left exact covariant functor R-**Mod** $\to$ R-**Mod**.

Problem 8. Let R be a domain with fraction field Q.

a) Show that for every Q-vector space V and every R-module M, $V \otimes_R M \cong V \otimes_R (M/T(M))$.

b) PShow that for every Q-vector space $V \neq 0$ and every R-module M, $V \otimes_R M = 0$ if and only if M is torsion.

c) Show that $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \neq 0$.