Some old qualifying exam questions

Here are some old qualifying exam problems you are already ready to solve.

Problem 1 (January 2024)**.** Let H be a subgroup of a group G. Show that $[G, G] \leq H$ if and only if H is a normal subgroup of G and G/H is abelian.

Problem 2 (June 2023)**.** Let G be a group and let H be a subgroup of G. The following sets are subgroups of G

$$N_G(H) = \{g \in G : gHg^{-1} \subseteq H\} \quad \text{and} \quad C_G(H) = \{g \in G : gh = hg, \forall h \in H\},$$

a fact which you may use without proof. Prove that $C_G(H)$ is a normal subgroup of $N_G(H)$.

Note: Above I wrote only part (a); while you know what you need for part (b), that is a problem best left for later in the semester.

Problem 3 (June 2023)**.** Let G be a group with center $\mathrm{Z}(G)$. Prove that if the quotient group $G/\mathrm{Z}(G)$ is cyclic, then G is abelian.

Problem 4 (May 2021)**.** Let G be a group (not necessarily finite) and H a nonempty subset of G that is closed under multiplication. Suppose that for all $g \in G$ we have $g^2 \in H$.

(a) Show that H must be a subgroup of G.

(b) Show that H must be a normal subgroup of G.

(c) Show that G/H must be abelian.