Symbolic Powers
joint work with Craig Huneke
Clemson Algebra Seminar, March 9th, 2017

Setup $R = \mathbb{C}[x_1, \ldots, x_d]$ Algebra $\longleftrightarrow$ Geometry

I ideal in R ideal I $\rightsquigarrow$ points in $\mathbb{C}^d$ that are zeros of all $f \in I$ $= X$

$R = \mathbb{C}[x_0, \ldots, x_d]$ Algebra $\longleftrightarrow$ Geometry

I homogeneous ideal   homogeneous ideal I $\rightsquigarrow$ points in $\mathbb{P}^d$ that are zeros of all $f \in I$ $= X$

n-th symbolic power of I $\rightsquigarrow$ $I^{(n)}$ = polynomials that vanish up to order n along X

Given $f \in I$, what does it mean to say that f vanishes up to order n along X?

1) Say we have only one variable, and we want to know how how many times f vanishes at 0. We can:

$$f(x) = x^a (x-\alpha_1)^{a_1} \cdots (x-\alpha_k)^{a_k} \longrightarrow f \text{ vanishes with multiplicity } a \text{ at } 0$$

$$\Leftrightarrow \quad f \in (x-\alpha)^b, \; f \notin (x-\alpha)^{b+1} \qquad b = \text{multiplicity} = a$$

Can we ask this locally? (Locally in CA means localizing!)

think of f as a function in a small neighbourhood of 0.

$\longrightarrow$ Look in $R_{(x)} = \{ \frac{g}{h} : h(0) \neq 0 \}$, where all ideals look like (x^n)

$f R_{(x)} = (x^c) \longrightarrow$ say that f vanishes up to order c

So what should we do for an ideal I?

$$I^{(n)} = \bigcap_{\alpha \in X} (x_1 - \alpha_1, \ldots, x_d - \alpha_d)^n \longleftarrow \text{if } X \text{ is infinite, this is not very practical}$$

2) Any number of variables: <u>Algebra</u> $\longleftrightarrow$ <u>Geometry</u>

prime ideal P $\leftrightsquigarrow$ irreducible variety X

f vanishes of degree up to n $\Leftrightarrow$ when we localize at P, $f \in P^n R_P$

$\Leftrightarrow$ $sf \in P^n$ for some $s \notin P$

Over any field:

$$\boxed{\begin{aligned} P^{(n)} &= P^n R_P \cap R = \{sf \in P^n, s \notin P\} \quad n\text{-th symbolic power of } P \\ &= \text{all polynomials that vanish of degree up to } n \text{ along } X \end{aligned}}$$

How about for more general ideals I? I radical ($f^n \in I \Rightarrow f \in I$)

$I = P_1 \cap \ldots \cap P_k$ finite intersection of prime ideals

precisely the condition we need for our algebra - geometry dictionary to be bijective

$$I^{(n)} = \bigcap_{i=1}^{n} \left(I^n R_{P_i} \cap R\right)$$

<u>Remark</u> By a theorem of Zariski and Nagata (Zariski 49, Nagata 62) this definition coincides with the one we gave before.

<u>Examples</u>:

1) If the ideal is generated by variables, or more generally by a regular sequence, then symbolic powers and ordinary powers coincide.

2) In general, symbolic powers and ordinary powers can be quite different, and the symbolic powers can be extremely difficult to compute.

But we can see from the definition that $\boxed{P^n \subseteq P^{(n)}}$

Example $I = (x,y) \cap (x,z) \cap (y,z) = (xy, xz, yz)$

$$I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 \ni xyz$$

But $xyz \notin I^2 \rightsquigarrow I^2 \neq I^{(2)}$

However, $I^{(3)} \subseteq I^2$.

Main Question When is $I^{(a)} \subseteq I^b$?

Theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002)
$I \subseteq R = k[x_0, \dots, x_d]$ homogeneous ideal. Then for all $n \geq 1$,

$$I^{(dn)} \subseteq I^n$$

More precisely, $I^{(hn)} \subseteq I^n$, where $h = \max\{ht\, P_i\}$, $I = P_1 \cap \dots \cap P_k$.
$ht\, P_i = \operatorname{codim} P_i$

Example $I = (x,y) \cap (y,z) \cap (x,z) \longrightarrow h = 2$
Note: $(x,y) \rightsquigarrow (0:0:1)$ in $\mathbb{P}^2$, so $\operatorname{codim}(x,y) = 2 - 0 = 2$

theorem $\Rightarrow I^{(2n)} \subseteq I^n \longrightarrow I^{(4)} \subseteq I^2$. Actually, $I^{(3)} \subseteq I^2$.

Question (Huneke, 2000) If P is a prime of codim 2, is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, ≤ 2008) $I^{(h(n-1)+1)} \subseteq I^n$.

Example In our example, $h=2 \rightsquigarrow$ so the conjecture says $I^{(2n-1)} \subseteq I^n$.

the conjecture holds for:

- → I in char 2 (Huneke)
- → Monomial ideals in any characteristic
- → General points in $\mathbb{P}^2$ (HaHu) and $\mathbb{P}^3$ (Dumnicki)

Bad news: there exist ideals in $k[x,y,z]$ with $h=2$ for which $I^{(3)} \not\subseteq I^2$
(first example found by Dumnicki, Szemberg and Tutaj-Gasińska)

Good news: the known counterexamples are very special

there are no known counterexamples for primes.

So maybe the conjecture just needs the assumptions on I to be tuned.

From now on, $R = K[x_0, \dots, x_d]$, char $K = p > 0$, $m = (x_0, \dots, x_d)$
In char p, we have the Frobenius map $F(a) = a^p$, which is a homomorphism.

Freshman's dream $(a+b)^p = a^p + b^p$

theorem (HH 2007, Takagi-Yoshida 2007)

R/I F-pure $\Rightarrow$ $I^{(hn-1)} \subseteq I^n$.

What is an F-pure ring? this is some condition on the singularities of X.

$S = R/I$ is F-pure if $S \xrightarrow{F} S$ splits as a map of S-modules.

Example: If I is a squarefree monomial ideal, R/I is F-pure.

Theorem (–, Huneke) R/I F-pure $\Rightarrow I^{(h(n-1)+1)} \subseteq I^n \quad \forall n \geq 1$.

Is the result sharp? Yes!
For each h, $\exists I =$ intersection of primes of height h with $I^{(h(n-1))} \not\subseteq I^n$ for n as large as we like.

Can we get stronger containments by tightening the restrictions on I?

Yes! When R/I is strongly F-regular.
Strongly F-regular $\equiv$ lots of splittings, rather than just one

Example: Veronese subrings of $K[x_0, \dots, x_n]$ are strongly F-regular.
k-Algebra generated by all monomials of a fixed degree.

Example: Determinantal rings: take a generic matrix (where the entries are all distinct variables) and consider the ideal generated by all minors of a certain order) $\longrightarrow$ call R to the polynomial ring, $I =$ ideal of minors $\rightsquigarrow R/I$ is strongly F-regular.

Example Monomial ideals are not strongly F-regular.

Theorem (–, Huneke) R/I strongly F-regular $\Rightarrow I^{((h-1)(n-1)+1)} \subseteq I^n$

this is Harbourne's Conjecture with h replaced by $h-1$.

Corollary P prime of height 2, R/P strongly F-regular $\Rightarrow P^{(n)} = P^n \quad \forall n \geq 1$.

Example (Singh) $P = \underbrace{I_2\begin{pmatrix} a^2 & b & d \\ c & a^2 & b^n - d \end{pmatrix}}_{\text{2×2 minors of this matrix}} \subseteq K[a,b,c,d]$ any n

$P^{(k)} = P^k \quad \forall k \geq 1$

Example $P = (\underbrace{xz - y^2}_{f}, \underbrace{x^3 - yz}_{g}, \underbrace{x^2y - z^2}_{h}) \subseteq K[x,y,z]$ prime ideal

Check: $fg - h^2 = xq$ where this $q \notin P^2$, but $q \in P^{(2)} \Rightarrow$ So $P^{(2)} \neq P^2$.

Example (not in a polynomial ring)

$P = (x,y) \subseteq R = k[x,y,z]/(x^n - yz)$ for some fixed n.

Check: $yz = x^n \in P^n$, $z \notin P \Rightarrow y \in P^{(n)}$ but $y \notin P^2$

$\therefore P^{(n)} \nsubseteq P^2$

Fedder's Criterion $R = K[x_0, \ldots, x_n]$, $m = (x_0, \ldots, x_n)$, I homogeneous.

R/I F-pure $\Leftrightarrow (I^{[p]} : I) \nsubseteq m^{[p]}$

Here $I^{[p]} = (a^p : a \in I) = (f_1^p, \ldots, f_n^p)$ if $I = (f_1, \ldots, f_n)$

Glassbrenner's Criterion $R = K[x_0, \ldots, x_n]$, $m = (x_0, \ldots, x_n)$, I homogeneous.

R/I strongly F-regular $\Leftrightarrow c\,(I^{[p]} : I) \nsubseteq m^{[p]}$ c no zerodivisor on R/I