Problem Set 8

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, our course notes, and the textbook.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Do not refer to theorems by their number in the course notes or textbook.

Problem 1. Let F be a field. Recall that

$$a1_F = \underbrace{1 + \cdots + 1}_{a \text{ times}}.$$

The **prime field** of F is the subfield of F generated by 1_F, that is

$$K = \mathrm{Frac}\left(\{\mathrm{k1_F} \mid \mathrm{k} \in \mathbb{Z}\}\right).$$

Show that the prime field of F is isomorphic to exactly one of the fields $\mathbb{Q}$ or $\mathbb{Z}/p$ for some prime integer p.

Problem 2. In this problem, we will show that adjoining a finite set of elements to a field F is the same as adjoining its elements one at a time. More precisely, let L/F be a field extension, and let $a_1, \ldots, a_m \in L$. Set $L_0 = F$ and for each $1 \leqslant i \leqslant m$ define $L_i = L_{i-1}(a_i)$. Show that $F(a_1, \ldots, a_m) = L_m$.

Problem 3. Show that $x^3 + 3x + 2 \in \mathbb{Q}[x]$ is irreducible.

Problem 4. In each part, determine, with justification, the degree of the given field extension.

a) $[\mathbb{Q}(2 + \sqrt{3}) : \mathbb{Q}]$.

b) $[\mathbb{Q}(1 + \sqrt[3]{2} + \sqrt[3]{4}) : \mathbb{Q}]$.

Problem 5. Consider the two field extensions $\mathbb{Q} \subseteq \mathbb{Q}(i, \sqrt{3})$ and $\mathbb{Q} \subseteq \mathbb{Q}(i, \sqrt[3]{2})$.

a) Show that $\mathbb{Q} \subseteq \mathbb{Q}(i, \sqrt{3})$ has degree 4.

b) Show that $\mathbb{Q} \subseteq \mathbb{Q}(i, \sqrt[3]{2})$ has degree 6.

c) Find a primitive element γ for the extension $\mathbb{Q} \subseteq \mathbb{Q}(i, \sqrt{3})$.

d) Find $m_{\gamma,\mathbb{Q}}(x)$.

Problem 6. Let R be a domain and let F be its fraction field. Show that F has the following universal property: if K is any field and $f : R \to K$ is any injective ring homomorphism, then f extends to an injective ring homomorphism $\overline{f} \colon F \to K$, so that $\overline{f}|_R = f$.