Problem Set 3

Instructions: You are welcome to work together with your classmates on all the problems, and I will be happy to give you hints or discuss the problems with you, but you should write up your solutions by yourself. You cannot use any resources besides me, your classmates, our course notes, and the Macaulay2 documentation.

Problem 1. Let k be a field, x an indeterminate, and $R := k[x^2, x^3] \subseteq S := k[x]$. Find an ideal I in R such that $IS \cap R \supsetneq I$.

Problem 2. Let k be an infinite field, $m \geqslant 1$, and let $R = k[x_1, \dots, x_n]$ be a polynomial ring. Let $G = (k^\times)^m$ act on R as follows:

$$\begin{aligned} (\lambda_1, \dots, \lambda_m) \cdot c &= c && \text{for } c \in k, \\ (\lambda_1, \dots, \lambda_m) \cdot x_i &= \lambda_1^{a_{1i}} \cdots \lambda_m^{a_{mi}} x_i && i = 1, \dots, n \end{aligned}$$

for some $m \times n$ matrix of integers $A = [a_{ij}]$.

a) Show that R^G has a k-vector space basis given by the set of monomials $x_1^{b_1} \cdots x_n^{b_n}$ such that, for $b = (b_1, \dots, b_n)$, $Ab = 0$.

b) Consider the polynomial ring R with a (nonstandard) $\mathbb{Z}^m$-grading given by setting

$$\deg(x_i) = (a_{1i}, \dots, a_{mi})$$

for each i. Show that R^G is the degree zero piece of R under this grading.

c) Show that R^G is a direct summand of R, and conclude that R^G is a finitely generated k-algebra.

d) Let $R = k[x, y, z, w]$ and consider the k-linear action of $G = k^\times$ on R given by

$$\lambda \cdot x = \lambda x \qquad \lambda \cdot y = \lambda y \qquad \lambda \cdot z = \lambda^{-1} z \qquad \lambda \cdot w = \lambda^{-1} w.$$

Find a finite set of generators for R^G as a k-algebra.