Problem Set 3

Turn in **4** of the following problems. You **must** pick at least **2 problems** involving tensor products. You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

Problem 1. Consider an exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{d} E.$$

a) Show a is surjective if and only if c is injective.

b) Show that if a and d are isos, then $C = 0$.

c) Show that every short exact sequence as above breaks into short exact sequences

$$0 \longrightarrow \operatorname{coker} a \xrightarrow{\alpha} C \xrightarrow{\beta} \ker d \longrightarrow 0$$

with

$$\alpha(x + \operatorname{im} a) = b(x) \qquad \text{and} \qquad \beta(x) = c(x).$$

Problem 2. Let $T\colon R\text{-}\mathbf{mod} \longrightarrow S\text{-}\mathbf{mod}$ be an additive functor. Show that if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a split short exact sequence of R-modules, then

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0$$

is a short exact sequence of S-modules.

Let R be a commutative ring and I be an ideal in R. The ***I*-torsion functor** is the functor $\Gamma_I\colon R\text{-}\mathbf{mod} \longrightarrow R\text{-}\mathbf{mod}$ that sends each R-module M to the R-module

$$\Gamma_I(M) := \bigcup_{n \geqslant 1} (0 :_M I^n) = \{m \in M \mid I^n m = 0 \text{ for some } n \geqslant 1\}$$

and that sends each R-module homomorphism $f\colon M \to N$ to its restriction to $\Gamma_I(M) \to \Gamma_I(N)$.

Problem 3. Let R be a commutative ring and I be an ideal in R.

a) Show that any R-module homomorphism $f\colon M \to N$ satisfies $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$.

b) Show that Γ_I is an indeed additive covariant functor.

c) Show that Γ_I is left exact.

d) Show that Γ_I is not right exact.

Problem 4. Let R be a ring, I and J ideals in R, and M be an R-module.

a) Show that $R/I \otimes_R R/J \cong R/(I+J)$.

b) Show that $R/I \otimes_R M \cong M/IM$.

c) There is an R-module map $I \otimes_R M \longrightarrow IM$ induced by the R-bilinear map $(a,m) \mapsto am$. This map is always clearly surjective; must it be injective?

Problem 5. Let $R = \mathbb{Z}[x]$, $I = (2,x)$, and consider the R-module $M = I \otimes_R I$.

a) Show that $2 \otimes 2 + x \otimes x$ is not a simple tensor in M.

b) Show that $m = 2 \otimes x - x \otimes 2$ is a nonzero torsion element in M.

c) Show that the submodule of $I \otimes_R I$ generated by m is isomorphic to R/I.

Let R be a domain and M be an R-module. The **torsion** of M is the submodule

$$T(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

The elements of $T(M)$ are called **torsion elements**, and we say that M **is torsion** if $T(M) = M$. Finally, M is **torsion free** if $T(M) = 0$.

Problem 6. By torsion abelian group we mean torsion $\mathbb{Z}$-module.

a) Show that if A is a divisible abelian group and T is a torsion abelian group, then $A \otimes_{\mathbb{Z}} T = 0$.

b) Prove that there is no exist nonzero (unital) ring R such that the underlying abelian group $(R,+)$ is both torsion and divisible.

For example, this shows that there is not is no ring whose underlying abelian group is $\mathbb{Q}/\mathbb{Z}$.

Problem 7. Let R be a domain with fraction field Q and M be an R-module.

a) Show that the R-module $M/T(M)$ is torsion free.

b) If $f \colon M \longrightarrow N$ is an R-module homomorphism, $f(T(M)) \subseteq T(N)$.

c) Show that the kernel of the map $M \longrightarrow Q \otimes_R M$ given by $m \mapsto 1 \otimes m$ is $T(M)$.

d) Show that torsion is a left exact covariant functor R-**Mod** $\to$ R-**Mod**.

Problem 8. Let R be a domain with fraction field Q.

a) Show that for every Q-vector space V and every R-module M, $V \otimes_R M \cong V \otimes_R (M/T(M))$.

b) Show that for every Q-vector space V and R-module M, $V \otimes_R M = 0$ if and only if M is torsion.

c) Show that $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \neq 0$.