Homological Algebra

Math 915 Fall 2023

Eloísa Grifo

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Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, I will be most grateful to you for letting me know. If you are looking for a place where to learn homological algebra or category theory, I strongly recommend the following excellent resources:

- Rotman's *An introduction to homological algebra*, second edition. [Rot09]
- Weibel's *Homological Algebra* [Wei94].
- Mac Lane's *Categories for the working mathematician* [ML98].
- Emily Riehl's *Category Theory in context*

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Chapter 0

Where are we going?

Homological algebra first appeared in the study of topological spaces. Roughly speaking, homology is a way of associating a sequence of abelian groups (or modules, or other more sophisticated algebraic objects) to another object, for example a topological space. The homology of a topological space encodes topological information about the space in algebraic language — this is what algebraic topology is all about.

More formally, we will study *complexes* and their homology from a more abstract perspective. While algebraic topologists are often concerned with complexes of abelian groups, we will work a bit more generally with complexes of R-modules. The basic assumptions and notation about rings and modules we will use in this class can be found in Appendix A. As an appetizer, we begin with some basic homological algebra definitions.

Definition 0.1. A **chain complex** of R-modules $(C_\bullet, \partial_\bullet)$, also referred to simply as a **complex**, is a sequence of R-modules C_i and R-module homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

such that $\partial_n \partial_{n+1} = 0$ for all n. We refer to C_n as the module in **homological degree** n. The maps ∂_n are the **differentials** of our complex. We may sometimes omit the differentials ∂_n and simply refer to the complex $C_\bullet$ or even C; we may also sometimes refer to $\partial_\bullet$ as *the* differential of $C_\bullet$.

In some contexts, it is important to make a distinction between chain complexes and co-chain complexes, where the arrows go the opposite way: a co-chain complex would look like

$$\cdots \longrightarrow C_{n-1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n+1}} C_{n+1} \longrightarrow \cdots.$$

We will not need to make such a distinction, so we will call both of these complexes and most often follow the convention in the definition above. We will say a complex C is **bounded above** if $C_n = 0$ for all $n \gg 0$, and **bounded below** if $C_n = 0$ for all $n \ll 0$. A **bounded complex** is one that is both bounded above and below. If a complex is bounded, we may sometimes simply write it as a finite complex, say

$$C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_m.$$

Remark 0.2. The condition that $\partial_n \partial_{n+1} = 0$ for all n implies that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$.

Definition 0.3. The complex $(C_\bullet, \partial_\bullet)$ is **exact** at n if $\operatorname{im} \partial_{n+1} = \ker \partial_n$. An **exact sequence** is a complex that is exact everywhere. More precisely, an **exact sequence** of R-modules is a sequence

$$\cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \cdots$$

of R-modules and R-module homomorphisms such that $\operatorname{im} f_n = \ker f_{n+1}$ for all n. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a **short exact sequence**, sometimes written **ses**.

Remark 0.4. The sequence

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact if and only if f is injective. Similarly,

$$M \xrightarrow{f} N \longrightarrow 0$$

is exact if and only if f is surjective. So

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence if and only if

- f is injective
- g is surjective
- $\operatorname{im} f = \ker g$.

When this is indeed a short exact sequence, we can identify A with its image $f(A)$, and $A = \ker g$. Moreover, since g is surjective, by the First Isomorphism Theorem we conclude that $C \cong B/f(A)$, so we might abuse notation and identify C with B/A.

Notation 0.5. We write $A \twoheadrightarrow B$ to denote a surjective map, and $A \hookrightarrow B$ to denote an injective map.

Definition 0.6. The **cokernel** of a map of R-modules $A \xrightarrow{f} B$ is the module

$$\operatorname{coker} f := B/\operatorname{im}(f).$$

Remark 0.7. We can rephrase Remark 0.4 in a fancier language: if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence, then $A = \ker g$ and $C = \operatorname{coker} f$.

Example 0.8. Let π be the canonical projection $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. The following is a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0\ .$$

We will most often be interested in **complexes of R-modules**, where the abelian groups that show up are all modules over the same ring R.

Example 0.9. Let $R = k[x]$ be a polynomial ring over the field k. The following is a short exact sequence:

$$0 \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\pi} R/(x) \longrightarrow 0\,.$$

The first map is multiplication by x, and the second map is the canonical projection.

Example 0.10. Given an ideal I in a ring R, the inclusion map $\iota : I \to R$ and the canonical projection $\pi : R \to R/I$ give us the following short exact sequence:

$$0 \longrightarrow I \xrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0.$$

Example 0.11. Let $R = k[x]/(x^2)$. The following complex is exact:

$$\cdots \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \longrightarrow \cdots\,.$$

Indeed, the image and the kernel of multiplication by x are both (x).

Sometimes we can show that certain modules vanish or compute them explicitly when they do not vanish by seeing that they fit in some naturally constructed exact sequence involving other modules we understand better. We will discuss this in more detail when we talk about long exact sequences.

Remark 0.12. The complex $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an isomorphism.

Remark 0.13. The complex $0 \longrightarrow M \longrightarrow 0$ is exact if and only if $M = 0$.

Historically, chain complexes first appeared in topology. To study a topological space, one constructs a particular chain complex that arises naturally from information from the space, and then calculates its homology, which ends up encoding important topological information in the form of a sequence of abelian groups.

Definition 0.14 (Homology)**.** The **homology** of the complex $(C_\bullet, \partial_\bullet)$ is the sequence of R-modules

$$\mathrm{H}_n(C_\bullet) = \mathrm{H}_n(C) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

The n**th homology** of $(C_\bullet, \partial_\bullet)$ is $\mathrm{H}_n(C)$. The submodules $Z_n(C_\bullet) = Z_n(C) := \ker \partial_n \subseteq C_n$ are called **cycles**, while the submodules $B_n(C_\bullet) = B_n(C) := \operatorname{im} \partial_{n+1} \subseteq C_n$ are called **boundaries**. One sometimes uses the word boundary to refer an element of $B_n(C)$ (an n-boundary), and the word cycle to refer to an element of $Z_n(C)$ (an n-cycle).

The homology of a complex measures how far our complex is from being exact at each point. Again, we can talk about the **cohomology** of a cochain complex instead, which we write as $\mathrm{H}^n(C)$; we will for now not worry about the distinction.

Remark 0.15. Note that $(C_\bullet, \partial_\bullet)$ is exact at n if and only if $\mathrm{H}_n(C_\bullet) = 0$.

Example 0.16. Let $R = k[x]/(x^3)$. Consider the following complex:

$$F_\bullet = \cdots \longrightarrow R \xrightarrow{\cdot x^2} R \xrightarrow{\cdot x^2} R \longrightarrow \cdots .$$

The image of multiplication by x^2 is (x^2), while the the kernel of multiplication by x^2 is $(x) \supseteq (x^2)$. For all n,

$$\mathrm{H}_n(F_\bullet) = (x)/(x^2) \cong R/(x).$$

Example 0.17. Let $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$ be the canonical projection map. Then

$$C = \underset{2}{\mathbb{Z}} \xrightarrow{4} \underset{1}{\mathbb{Z}} \xrightarrow{\pi} \underset{0}{\mathbb{Z}/2\mathbb{Z}}$$

is a complex of abelian groups, since the image of multiplication by 4 is $4\mathbb{Z}$, and that is certainly contained in $\ker \pi = 2\mathbb{Z}$. The homology of C is

$$\begin{aligned}
&\mathrm{H}_n(C) = 0 && \text{for } n \geqslant 3 \\
&\mathrm{H}_2(C) = \frac{\ker(\mathbb{Z} \xrightarrow{4} \mathbb{Z})}{\mathrm{im}(0 \longrightarrow \mathbb{Z})} = \frac{0}{0} = 0 && \\
&\mathrm{H}_1(C) = \frac{\ker(\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z})}{\mathrm{im}(\mathbb{Z} \xrightarrow{4} \mathbb{Z})} = \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} && \\
&\mathrm{H}_0(C) = \frac{\ker(\mathbb{Z}/2\mathbb{Z} \longrightarrow 0)}{\mathrm{im}(\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z})} = \frac{\mathbb{Z}/2\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} = 0 && \\
&\mathrm{H}_n(C) = 0 && \text{for } n < 0
\end{aligned}$$

Notice that our complex is exact at 2 and 0. The exactness at 2 says that the map $\mathbb{Z} \xrightarrow{4} \mathbb{Z}$ is injective, while exactness at 0 says that π is surjective.

Before we can continue any further into the world of homological algebra, we will need some categorical language. We will take a short break to introduce category theory, and then armed with that knowledge we will be ready to study homological algebra.

Chapter 1

Categories for the working homological algebraist

Most fields in modern mathematics follow the same basic recipe: there is a main type of object one wants to study – groups, rings, modules, topological spaces, etc – and a natural notion of arrows between these – group homomorphisms, ring homomorphisms, module homomorphisms, continuous maps, etc. The objects are often sets with some extra structure, and the arrows are often maps between the objects that preserve whatever that extra structure is. Category theory is born of this realization, by abstracting the basic notions that make math and studying them all at the same time. How many times have we felt a sense of déjà vu when learning about a new field of math? Category theory unifies all those ideas we have seen over and over in different contexts.

Category theory is an entire field of mathematics in its own right. As such, there is a lot to say about category theory, and unfortunately it doesn't all fit in the little time we have to cover it in this course. You are strongly encouraged to learn more about category theory, for example from [ML98] or [Rie17].

Before we go any further, note that there is a long and fun story about why we use the word *collection* when describing the objects in a category. Not all collections are allowed to be sets, an issue that was first discovered by Russel with his famous Russel's Paradox.[1] Russel exposed the fact that one has to be careful with how we formalize set theory. We follow the ZFC (Zermelo–Fraenkel with choice, short for the Zermelo–Fraenkel axioms plus the Axiom of Choice) axiomatization of set theory, and while we will not discuss the details of this formalization here, you are encouraged to read more on the subject.

1.1 Categories

A category consists of a collection of objects and arrows or morphisms between those objects. While these are often sets and some kind of functions between them, beware that this will not always be the case. We will use the words morphism and arrows interchangeably, though *arrow* has the advantage of reminding us we are not necessarily talking about functions.

[1] The collection of all sets that don't contain themselves cannot be a set. Do you see why?

Definition 1.1. A **category** $\mathscr{C}$ consists of three different pieces of data:

- a collection of **objects**, $\mathbf{ob}(\mathscr{C})$,
- for each two objects, say A and B, a collection $\mathrm{Hom}_{\mathscr{C}}(A,B)$ of **arrows** or **morphisms** from A to B, and
- for each three objects A, B, and C, a composition

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,C) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A,C) \ . \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

  We will often drop the $\circ$ and write simply gf for $g \circ f$.

These ingredients satisfy the following axioms:

1) The $\mathrm{Hom}_{\mathscr{C}}(A,B)$ are all disjoint. In particular, if f is an arrow in $\mathscr{C}$, we can talk about its **source** A and its **target** B as the objects such that $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$.

2) For each object A, there is an **identity arrow** $1_A \in \mathrm{Hom}_{\mathscr{C}}(A,A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B,A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A,B)$.

3) Composition is **associative**: $f \circ (g \circ h) = (f \circ g) \circ h$ for all appropriately chosen arrows.

Notation 1.2. We sometimes write $f \colon A \to B$ or $A \xrightarrow{f} B$ for an arrow $f \in \mathrm{Hom}(A,B)$.

Exercise 1. Prove that every element in a category has a unique identity morphism.

Here are some categories you have likely encountered before:

Example 1.3.

1) The category **Set** with objects all sets and arrows all functions between sets.

2) The category **Grp** whose objects are the collection of all groups, and whose arrows are all the homomorphisms of groups. The identity arrows are the identity homomorphisms.

3) The category **Ab** with objects all abelian groups, and arrows the homomorphisms of abelian groups. The identity arrows are the identity homomorphisms.

4) The category **Ring** of rings and ring homomorphisms. Contrary to what you may expect, this is not nearly as important as the next one.

5) The category R-**mod** of left modules over a fixed ring R and with R-module homomorphisms. Sometimes one writes R-**Mod** for this category, and reserve R-**mod** for the category of finitely generated R-modules with R-module homomorphisms. When $R = k$ is a field, the objects in the category k-**Mod** are k-vector spaces, and the arrows are linear transformations; we may instead refer to this category as **Vect**-k.

6) The category **Top** of topological spaces and continuous functions.

One may consider many variations of the categories above. Here are some variations on vector spaces:

Example 1.4. Let k be a field.

1) The collection of finite dimensional k-vector spaces with all linear transformations is a category.

2) The collection of all n-dimensional k-vector spaces with all linear transformations is a category.

3) The collection of all k-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.

4) The collection of all k-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it is not closed under composition.

5) The collection of all n-dimensional vector spaces with linear transformations of determinant 0 is not a category, since it does not have identity maps.

Here is an important variation of **Set**:

Example 1.5. The category $\mathbf{Set}^*$ of pointed sets has objects all pairs (X, x) of sets X and points $x \in X$, and for two pointed sets (X, x) and (Y, y), the morphisms from (X, x) to (Y, y) are functions $f \colon X \to Y$ such that $f(x) = y$, with the usual composition of functions.

Example 1.6. The empty category has no objects and no arrows.

While the collections of objects and arrows might not actually be sets, sometimes they are.

Definition 1.7. A category $\mathscr{C}$ is **locally small** if for all objects A and B in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set. A category $\mathscr{C}$ is **small** if it is locally small and the collection of all objects in $\mathscr{C}$ is a set.

In fact, one can define a small category as one where the collection of all arrows is a set. It follows immediately that the collection of all objects is also a set, since it must be a subset of the set of arrows – for each object, there is an identity arrow.

Many important categories are at least locally small. For example, **Set** is locally small but not small. In a locally small category, we can now refer to its Hom-sets.

Categories where the objects are sets with some extra structure and the arrows are some kind of functions between the objects are called **concrete**. Not all categories are concrete.

Example 1.8. Given a partially ordered set $(X, \leqslant)$, we can regard X itself as a category: the objects are the elements of X, and for each x and y in X, $\mathrm{Hom}_X(x, y)$ is either a singleton if $x \leqslant y$ or empty if $x \nleqslant y$. There is only one possible way to define composition, and the transitive property of $\leqslant$ guarantees that the composition of arrows is indeed well-defined: if there is an arrow $i \to j$ and an arrow $j \to k$, then $i \leqslant j$ and $j \leqslant k$, so $i \leqslant k$ and thus there is a unique arrow $i \to k$. This category is clearly locally small, since all nonempty Hom-sets are in fact singletons. It is in fact small, since the objects are by construction the set X. We will denote this poset category by $\mathbf{PO}(X)$.

Example 1.9. For each positive integer n, the category $\mathbf{n}$ has n objects $0, 1, \dots, n-1$ and $\mathrm{Hom}(i, j)$ is either empty if $i > j$ or a singleton if $i \leqslant j$. As Example 1.8, composition is defined in the only way possible, and things work out. This is the poset category for the poset $(\{0, 1, \dots, n-1\}, \leqslant)$ with the usual $\leqslant$.

Example 1.10. Fix a field k. We define a category **Mat-k** with objects all positive integers, and given two positive integers a and b, the Hom-set $\mathrm{Hom}(a, b)$ consists of all $b \times a$ matrices with entries in k. The composition rule is given by product of matrices: given $A \in \mathrm{Hom}(a, b)$ and $B \in \mathrm{Hom}(b, c)$, the composition $B \circ A$ is the matrix $BA \in \mathrm{Hom}(a, c)$. For each object a, its identity arrow is given by the $a \times a$ identity matrix.

Example 1.11. Let G be a directed graph. We can construct a category from G as follows: the objects are the vertices of G, and the arrows are directed paths in the graph G. In this category, composition of arrows corresponds to concatenation of paths. For each object A, the identity arrow corresponds to the empty path from A to A.

Remark 1.12. A locally small category with just one element is completely determined by its unique Hom-set; it thus consists of a set S with an associative operation that has an identity element, which in this class is what we call a **semigroup**.[2]

A key insight we get from category theory is that many important concepts can be understood through diagrams. Homological algebra is in many ways the study of commutative diagrams. One way to formalize what a diagram is involves talking about functors, which we will discuss in Section 1.2; here is a more down to earth definition.

Definition 1.13. A **diagram** in a category $\mathscr{C}$ is a directed multigraph whose vertices are objects in C and whose arrows/edges are morphisms in $\mathscr{C}$. A commutative diagram in $\mathscr{C}$ is a diagram in which for each pair of vertices A and B, any two paths from A to B compose to the same morphism.

Example 1.14. The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

commutes if and only if $gf = vu$.

There are some special types of arrows we will want to consider.

Definition 1.15. Let $\mathscr{C}$ be any category.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is **left invertible** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $gf = 1_A$. In this case, we say that g is the **left inverse** of f. So g is a left inverse of f if the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

[2]Some authors prefer the term monoid.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is **right invertible** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $fg = 1_B$. In this case, we say that g is the **right inverse** of f. So g is a right inverse of f if the diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

  commutes.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is an **isomorphism** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$. Unsurprisingly, such an arrow g is called the **inverse** of f.

- An arrow $f \in \mathrm{Hom}(B, C)$ is **monic**, a **monomorphism**, or a **mono** if for all arrows

$$\begin{tikzcd} A \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

  if $fg_1 = fg_2$ then $g_1 = g_2$.

- Similarly, an arrow $f \in \mathrm{Hom}(A, B)$ is an **epi** or an **epimorphism** if for all arrows

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

  if $g_1 f = g_2 f$ then $g_1 = g_2$.

Here are some examples:

Exercise 2. Show that in **Set**, the monos coincide with the injective functions and the epis coincide with the surjective functions.

Example 1.16.

a) In **Grp**, **Ring**, and R-**Mod** the isomorphisms are the morphisms that are bijective functions.

b) In contrast, in **Top** the isomorphisms are the homeomorphisms, which are the bijective continuous functions with continuous inverses. These are *not* the same thing as just the bijective continuous functions.

Exercise 3. Show that in any category, every isomorphism is both epi and mono.

Exercise 4. Show that the usual inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epi in the category **Ring**.

This *should* feel weird: it says being epi and being surjective are *not* the same thing. Similarly, being monic and being injective are *not* the same thing.

Exercise 5. Show that the canonical projection $\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a mono in the category of divisible abelian groups.[3]

[3] An abelian group A is divisible if for every $a \in A$ and every positive integer n there exists $b \in A$ such that $nb = a$.

Exercise 6. Show that given any poset P, in the poset category of P every morphism is both monic and epic, but no nonidentity morphism has a left or right inverse.

There are some special types of objects we will want to consider.

Definition 1.17. Let $\mathscr{C}$ be a category. An **initial object** in $\mathscr{C}$ is an object i such that for every object x in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(i, x)$ is a singleton, meaning there exists a unique arrow $i \longrightarrow x$. A **terminal object** in $\mathscr{C}$ is an object t such that for every object x in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(x, t)$ is a singleton, meaning there exists a unique arrow $x \longrightarrow t$. A **zero object** in $\mathscr{C}$ is an object that is both initial and terminal.

Exercise 7. Initial objects are unique up to unique isomorphism. Terminal objects are unique up to unique isomorphism.

So we can talk about *the* initial object, *the* terminal object, and *the* zero object, if they exist.

Example 1.18.

a) The empty set is initial in **Set**. Any singleton is terminal. Since the empty set and a singleton are not isomorphic in **Set**, there is no zero object in **Set**.

b) The 0 module is the zero object in R-**Mod**.

c) The trivial group $\{e\}$ is the zero object in **Grp**.

d) In the category of rings, $\mathbb{Z}$ is the initial object, but there is no terminal object unless we allow the 0 ring.

e) There are no initial nor terminal objects in the category of fields.

We will now continue to follow a familiar pattern and define the related concepts one can guess should be defined.

Definition 1.19. A **subcategory** $\mathscr{C}$ of a category $\mathscr{D}$ consists of a subcollection of the objects of $\mathscr{D}$ and a subcollection of the morphisms of $\mathscr{D}$ such that the following hold:

- For every object C in $\mathscr{C}$, the arrow $1_C \in \mathrm{Hom}_{\mathscr{D}}(C, C)$ is an arrow in $\mathscr{C}$.
- For every arrow in $\mathscr{C}$, its source and target in $\mathscr{D}$ are objects in $\mathscr{C}$.
- For every pair of arrows f and g in $\mathscr{C}$ such that fg is an arrow that makes sense in $\mathscr{D}$, fg is an arrow in $\mathscr{C}$.

In particular, $\mathscr{C}$ is a category in its own right.

Example 1.20. The category of finitely generated R-modules with R-module homomorphisms is a subcategory of R-**Mod**.

Definition 1.21. A subcategory $\mathscr{C}$ of $\mathscr{D}$ is a **full subcategory** if $\mathscr{C}$ includes *all* of the arrows in $\mathscr{D}$ between any two objects in $\mathscr{C}$.

Example 1.22.

a) The category **Ab** of abelian groups is a full subcategory of **Grp**.

b) Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, not every function between two groups is a group homomorphism, so **Grp** is not a full subcategory of **Set**.

c) The category whose objects are all sets and with arrows all bijections is a subcategory of **Set** that is not full.

Here is another way of constructing a new category out of an old one.

Definition 1.23. Let $\mathscr{C}$ be a category. The **opposite category** of $\mathscr{C}$, denoted $\mathscr{C}^{\mathrm{op}}$, is a category whose objects are the objects of $\mathscr{C}$, and such that each arrow $f \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B)$ is the same as some arrow in $\mathrm{Hom}_{\mathscr{C}}(B, A)$. The composition fg of two morphisms f and g in $\mathscr{C}^{\mathrm{op}}$ is defined as the composition gf in $\mathscr{C}$.

Many objects and concepts one might want to describe are obtained from existing ones by flipping the arrows. Opposite categories give us the formal framework to talk about such things. We will often want to refer to **dual** notions, which will essentially mean considering the same notion in a category $\mathscr{C}$ and in the opposite category $\mathscr{C}^{\mathrm{op}}$; in practice, this means we should flip all the arrows involved. We will see examples of this later on.

The dual category construction gives us a formal framework to talk about **dual notions**. We will often make a statement in a category $\mathscr{C}$ and make comments about the **dual statement**; in practice, this corresponds to simply switching the way all arrows go. Here are some examples of dual notions and statements:

source	target
epi	mono
g is a right inverse for f	g is a left inverse for f
f is invertible	f is invertible
initial objects	terminal objects
homology	cohomology

The prefix co- is often used to denote the dual of something, such as in *co*homology. Note that the dual of the dual is the original statement; formally, $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$. Sometimes we can easily prove a statement by dualizing; however, this is not always straightforward, and one needs to carefully dualize all portions of the statement in question. Nevertheless, Sanders MacLane, one of the fathers of category theory, wrote that "If any statement about a category is deducible from the axioms for a category, the dual statement is likely deducible" [Mac50]. One of the upshots of duality is that any theorem in category theory must simultaneously prove two theorems: the original statement and its dual. But for this to hold, we need proofs that use the abstraction of a purely categorical proof.

Opposite categories are more interesting than they might appear at first; there is more than just flipping all the arrows. For example, consider the opposite category of **Set**. For any nonempty set X, there is a unique morphism in **Set** (a function) $i : \emptyset \to X$, but there are no functions $X \to \emptyset$, so $i^{\mathrm{op}}\colon \emptyset \to X$ is not a function. Thus thinking about $\mathbf{Set^{op}}$ is a bit difficult. One can show that this is the category of complete atomic Boolean algebras – but we won't concern ourselves with what that means.

1.2 Functors

Many mathematical constructions are *functorial*, in the sense that they behave well with respect to morphisms. In the formalism of category theory, this means that we can think of a functorial construction as a functor.

Definition 1.24. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a mapping that assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$ an arrow $F(f) \in \operatorname{Hom}_{\mathscr{D}}(F(A), F(B))$, such that

- F preserves the composition of maps, meaning $F(fg) = F(f)F(g)$ for all arrows f and g in $\mathscr{C}$, and
- F preserves the identity arrows, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

A **contravariant functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a mapping that assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$ an arrow $F(f) \in \operatorname{Hom}_{\mathscr{D}}(F(B), F(A))$, such that

- F preserves the composition of maps, meaning $F(fg) = F(g)F(f)$ for all composable arrows f and g in $\mathscr{C}$, and
- F preserves the identity arrows, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

So a contravariant functor is a functor that flips all the arrows. We can also describe a contravariant functor as a covariant functor from $\mathscr{C}$ to the opposite category of $\mathscr{D}$, $\mathscr{D}^{\mathrm{op}}$.

Remark 1.25. A contravariant functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ can be thought of as a covariant functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$, or also as a covariant functor $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$. If using one of these conventions, one needs to be careful, however, when composing functors, so that the respective sources and targets match up correctly. While we haven't specially discussed how one composes functors, it should be clear that applying a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$ is the same as applying a functor $\mathscr{C} \longrightarrow \mathscr{D}$, which we can write as GF.

For example, if $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$ are both contravariant functors, the composition $GF\colon \mathscr{C} \longrightarrow \mathscr{E}$ is a covariant functor, since

$$\begin{tikzcd}
A \arrow[d, "f"'] & \rightsquigarrow & F(A) & \rightsquigarrow & GF(A) \arrow[d, "GF(f)"'] \\
B & & F(B) \arrow[u, "F(f)"] & & GF(B)
\end{tikzcd}$$

So we could think of F as a covariant functor $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$ and G as a covariant functor $\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{E}$. Similarly, if $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a covariant functor and $G : \mathscr{D} \longrightarrow \mathscr{E}$ is a contravariant functor, $GF\colon \mathscr{C} \longrightarrow \mathscr{E}$ is a contravariant functor. In this case, we can think of G as a covariant functor $\mathscr{D} \longrightarrow \mathscr{E}^{\mathrm{op}}$, so that GF is now a covariant functor $\mathscr{C} \longrightarrow \mathscr{E}^{\mathrm{op}}$.

Exercise 8. Show that functors preserve isomorphisms.

Remark 1.26. Any functor sends isos to isos, since it preserves compositions and identities.

Example 1.27. Here are some examples of functors you may have encountered before.

a) Many categories one may think about are concrete categories, where the objects are sets with some extra structure, and the arrows are functions between those sets that preserved that extra structure. The **forgetful functor** from such a category to **Set** is the functor that, just as the name says, *forgets* that extra structure, and sees only the underlying sets and functions of sets. For example, the forgetful functor **Gr** $\longrightarrow$ **Set** sends each group to its underlying set, and each group homomorphism to the corresponding function of sets.

b) The identity functor $1_{\mathscr{C}}$ on any category $\mathscr{C}$ does what the name suggests: it sends each object to itself and each arrow to itself.

c) Given an object C in a category $\mathscr{C}$, the **constant functor** at C is the functor $\Delta C : \mathscr{C} \to \mathscr{C}$ that sends every object to C every arrow to 1_C.

d) Given a group G, the subgroup $[G, G]$ of G generated by the set of commutators
$$\{ghg^{-1}h^{-1} \mid g, h \in G\}$$
is a normal subgroup, and the quotient $G^{\text{ab}} := G/[G, G]$ is called the **abelianization** of G. The group G^{ab} is abelian. Given a group homomorphism $f : G \to H$, f automatically takes commutators to commutators, so it induces a homomorphism $\tilde{f} : G^{\text{ab}} \to H^{\text{ab}}$. More precisely, abelianization gives a covariant functor from **Grp** to **Ab**.

e) The unit group functor $-^* :$ **Ring** $\to$ **Grp** sends a ring R to its group of units R^*. To see this is indeed a functor, we should check it behaves well on morphisms; and indeed if $f : R \to S$ is a ring homomorphism, and $u \in R^*$ is a unit in R, then
$$f(u)f(u^{-1}) = f(uu^{-1}) = f(1_R) = 1_S,$$
so $f(u)$ is a unit in S. Thus f induces a function $R^* \to S^*$ given by restriction of f to R^*, which must therefore be a group homomorphism since f preserves products.

f) Fix a field k. Given a vector space V, the collection V^* of linear transformations from V to k is again a k-vector space, the **dual vector space** of V. If $\varphi : W \to V$ is a linear transformation and $\ell : V \to K$ is an element of V^*, then $\ell \circ \varphi : W \to k$ is in W^*. Doing this for all elements $\ell \in V^*$ gives a function $\varphi^* : V^* \to W^*$, and one can show that φ^* is a linear transformation. The assignment that sends each vector space V to its dual vector space V^* and each linear transformation φ to φ^* is a contravariant functor on **Vect-k**.

g) Localization is a functor. Let R be a ring and W be a multiplicatively closed set in R. The localization at W induces a a functor R-mod $\longrightarrow W^{-1}R$-mod: this functor sends each R-module M to $W^{-1}M$, and each R-module homomorphism $\alpha : M \to N$ to the R-module homomorphism $W^{-1}\alpha : W^{-1}M \to W^{-1}N$.

Remark 1.28. If we apply a covariant functor to a diagram, then we get a diagram of the same shape:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] & & F(A) \arrow[r, "F(f)"] \arrow[d, "F(u)"'] & F(B) \arrow[d, "F(g)"] \\
C \arrow[r, "v"'] & D \arrow[r, phantom, "\overset{F}{\rightsquigarrow}"] & & F(C) \arrow[r, "F(v)"'] & F(D)
\end{tikzcd}$$

However, if we apply a contravariant functor to the same diagram, we get a similar diagram but with the arrows reversed:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C & D \arrow[l, "v"] \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F(A) & F(B) \arrow[l, "F(f)"'] \\ F(C) \arrow[u, "F(u)"] & F(D) \arrow[l, "F(v)"] \arrow[u, "F(g)"'] \end{tikzcd}$$

Definition 1.29. The category **Cat** has objects all small categories and arrows all functors between them.

If we think about functors as functions between categories, it's natural to consider what would be the appropriate versions of the notions of injective or surjective.

Definition 1.30. A covariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ between locally small categories is

- **faithful** if all the functions of sets

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A, B) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)) \\ f &\longmapsto F(f) \end{aligned}$$

  are injective.

- **full** if all the functions of sets

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A, B) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)) \\ f &\longmapsto F(f) \end{aligned}$$

  are surjective.

- **fully faithful** if it is full and faithful.

- **essentially surjective** if every object d in $\mathscr{D}$ is isomorphic to Fc for some c in $\mathscr{C}$.

- an **embedding** if it is fully faithful and injective on objects.

Example 1.31. The forgetful functor R**-Mod** $\longrightarrow$ **Set** is faithful since any two maps of R-modules with the same source and target coincide if and only if they are the same function of sets. This functor is not full, since there not every functions between the underlying sets of two R-modules is an R-module homomorphism.

Remark 1.32. A fully faithful functor is not necessarily injective on objects, but it is injective on objects up to isomorphism.

Remark 1.33. A subcategory $\mathscr{C}$ of $\mathscr{D}$ is full if the inclusion functor $\mathscr{C} \longrightarrow \mathscr{D}$ is full.

Exercise 9. Show that every fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ reflects and creates isos:

a) If f is an arrow in $\mathscr{C}$ such that $F(f)$ is an iso, then f is an iso.

b) If $F(X)$ and $F(Y)$ are isomorphic, then the objects X and Y are isomorphic in $\mathscr{C}$.

Note that the converses of these statements hold for any functor.

To close this section, here are the two of the most important functors we will discuss this semester:

Definition 1.34. Let $\mathscr{C}$ be a locally small category. An object A in $\mathscr{C}$ induces two Hom functors:

- The covariant functor $\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C} \longrightarrow \mathbf{Set}$ is defined as follows:

$$\begin{array}{lccc} & \mathscr{C} & \longrightarrow & \mathbf{Set} \\ \text{on objects:} & X & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(A, X) \\ \text{on arrows:} & \begin{array}{c} B \\ {\scriptstyle f}\big\downarrow \\ C \end{array} & \rightsquigarrow & \begin{array}{c} \mathrm{Hom}_{\mathscr{C}}(A, B) \\ \big\downarrow \\ \mathrm{Hom}_{\mathscr{C}}(A, C) \end{array} \quad \begin{array}{c} \ni g \\ \big\downarrow \\ \ni f \circ g \end{array} \end{array}$$

  We may refer to this functor as the covariant functor **represented by** A. Given an arrow f in $\mathscr{C}$, we write $f_* := \mathrm{Hom}_{\mathscr{C}}(A, f)$. It is easier to see what f_* does through the following commutative diagram:

$$f_* = \mathrm{Hom}_{\mathscr{C}}(A, f) : \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed, "f_*(g)=fg"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

- The contravariant functor $\mathrm{Hom}_{\mathscr{C}}(-, B) : \mathscr{C} \longrightarrow \mathbf{Set}$ is defined as follows:

$$\begin{array}{lccc} & \mathscr{C} & \longrightarrow & \mathbf{Set} \\ \text{on objects:} & X & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(X, B) \\ \text{on arrows:} & \begin{array}{c} A \\ {\scriptstyle f}\big\downarrow \\ C \end{array} & \rightsquigarrow & \begin{array}{c} \mathrm{Hom}_{\mathscr{C}}(A, B) \\ \big\uparrow \\ \mathrm{Hom}_{\mathscr{C}}(C, B) \end{array} \quad \begin{array}{c} \ni g \circ f \\ \big\uparrow \\ \ni g \end{array} \end{array}$$

  We may refer to this functor as the contravariant functor **represented by** B. Given an arrow f in $\mathscr{C}$, we write $f^* := \mathrm{Hom}_{\mathscr{C}}(A, -)$. It is easier to see what f^* does through the following commutative diagram:

$$f^* = \mathrm{Hom}_{\mathscr{C}}(f, B) : \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "f^*(g)=gf"'] & C \arrow[d, "g"] \\ & B \end{tikzcd}$$

Exercise 10. Check that $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$ are indeed functors.

We will be particularly interested in the Hom-functors in the category R-**mod**, which we will study in detail in a later chapter.

1.3 Natural transformations

Definition 1.35. Let F and G be covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A **natural transformation** between F and G is a mapping that to each object A in $\mathscr{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes. A **natural isomorphism** is a natural transformation η where each η_A is an isomorphism. We sometimes write

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathscr{D} \end{tikzcd}$$

or simply $\eta \colon F \implies G$.

Definition 1.36. Let F and G be contravariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A **natural transformation** between F and G is a mapping that to each object A in $\mathscr{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] & G(A) \\ F(B) \arrow[r, "\eta_B"'] \arrow[u, "F(f)"] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

commutes.

Often, when studying a particular topic, we sometimes say a certain map is *natural* to mean that there is actually a natural transformation behind it.

Example 1.37. Recall the abelianization functor we discussed in Example 1.27. The abelianization comes equipped with a natural projection map $\pi_G : G \longrightarrow G^{\mathrm{ab}}$, the usual quotient map from G to a normal subgroup. Here we mean natural in two different ways: both that this is common sense map to consider, and that this is in fact coming from a natural transformation. What's happening behind the scenes is that abelianization is a functor $\mathrm{ab} \colon \mathbf{Grp} \longrightarrow \mathbf{Grp}$. On objects, the abelianizations functor is defined as $G \mapsto G^{\mathrm{ab}}$. Given an arrow, meaning a group homomorphism $G \xrightarrow{f} H$, one can check that $[G, G]$ is contained in the kernel of $\pi_H f$, so $\pi_H f$ factors through G^{ab}, and there exists a group homomorphism f^{ab} making the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d, "f"'] & G^{\mathrm{ab}} \arrow[d, dashed, "f^{\mathrm{ab}}"] \\ H \arrow[r, "\pi_H"'] & H^{\mathrm{ab}} \end{tikzcd} .$$

So the abelianization functor takes the arrow f to f^{ab}. The commutativity of the diagram above says that $\pi_{\_}$ is a natural transformation between the identity functor on **Grp** and the abelianization functor, which we can write more compactly as

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{ab}"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathbf{Grp} \end{tikzcd} .$$

Definition 1.38. Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be two functors between the categories $\mathscr{C}$ and $\mathscr{D}$. We write

$$\mathrm{Nat}(F, G) = \{\text{natural transformations } F \longrightarrow G\}.$$

Given two categories $\mathscr{C}$ and $\mathscr{D}$, one can build a **functor category**[4] with objects all covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$, and arrows the corresponding natural transformations. This category is denoted $\mathscr{D}^{\mathscr{C}}$. Sometimes one writes $\mathrm{Hom}(F, G)$ for $\mathrm{Nat}(F, G)$, but we will avoid that, as it might make things even more confusing.

For the functor category to truly be a category, though, we need to know how to compose natural transformations.

Remark 1.39. Consider natural transformations

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathscr{D} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{D} \end{tikzcd}.$$

We can compose them for form a new natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta\varphi"] & \mathscr{D} \end{tikzcd}$$

We should think of this composition as happening *vertically*. For each object C in $\mathscr{C}$, $\eta\varphi$ sends C to the arrow $F(A) \xrightarrow{\varphi_A} G(A) \xrightarrow{\eta_A} H(A)$. This makes the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varphi_A"'] & F(B) \arrow[d, "\varphi_B"] \\ G(A) \arrow[r, "G(f)"] \arrow[d, "\eta_A"'] & G(B) \arrow[d, "\eta_B"] \\ H(A) \arrow[r, "H(f)"] & H(B) \end{tikzcd}$$

commute.

Exercise 11. Show that a natural transformation $\eta\colon \mathscr{C} \implies \mathscr{D}$ is a natural isomorphism if and only if there exists a natural transformation $\mu\colon \mathscr{D} \implies \mathscr{C}$ such that $\eta \circ \mu$ is the identity natural isomorphism on G and $\mu \circ \eta$ is the identity natural isomorphism on F.

[4] Yes, the madness is neverending.

Definition 1.40. Two categories $\mathscr{C}$ and $\mathscr{D}$ are **equivalent** if there exist functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ and two natural isomorphisms $\alpha\colon GF \implies 1_{\mathscr{C}}$ and $\beta\colon FG \implies 1_{\mathscr{D}}$. We say that a functor $F\colon \mathscr{C} \to \mathscr{D}$ is an **equivalence of categories** if there exists a functor G and natural isomorphisms α and β as above.

If one assumes the Axiom of Choice, this is the right notion of isomorphism of two categories (though not in the categorical sense!); better said, two categories that are equivalent are essentially the same. Note that this does not mean that there is a bijection between the objects of $\mathscr{C}$ and the objects of $\mathscr{D}$. In fact, one can show that a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective – though this fact requires the Axiom of Choice!

Exercise 12. Let $\mathscr{C}$ be the category with one object C and a unique arrow 1_C. Let $\mathscr{D}$ be the category with two objects D_1 and D_2 and four arrows: the identities 1_{D_i} and two isomorphisms $\alpha\colon D_1 \to D_2$ and $\beta\colon D_2 \to D_1$. Let $\mathscr{E}$ be the category with two objects E_1 and E_2 and only two arrows, 1_{E_1} and 1_{E_2}.

a) Show that $\mathscr{C}$ and $\mathscr{D}$ are equivalent categories.

b) Show that $\mathscr{C}$ and $\mathscr{E}$ are not equivalent categories.

The functors that are naturally isomorphic to some Hom functor are important.

Definition 1.41. A covariant functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists an object A in $\mathscr{C}$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(A, -)$. A contravariant functor $F\colon \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists an object B in $\mathscr{C}$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, B)$.

Example 1.42. We claim that the identity functor $\mathbf{Set} \longrightarrow \mathbf{Set}$ is representable. Let $\mathbf{1}$ be a singleton set. Given any set X, there is a bijection between elements $x \in X$ and functions $\mathbf{1} \longrightarrow X$ sending the one element in $\mathbf{1}$ to each x. Moreover, given any other set Y, and a function $f\colon X \longrightarrow Y$, our bijections make the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"'] \\
\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, Y) \arrow[r, "\cong"] & Y.
\end{tikzcd}$$

This data gives a natural isomorphism between the identity functor and $\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, -)$.

Exercise 13. Show that the forgetful functor $\mathbf{Grp} \longrightarrow \mathbf{Set}$ is representable.

Exercise 14. Given a ring R, show that the forgetful functor R-**mod** $\longrightarrow \mathbf{Set}$ is representable.

The Yoneda Lemma tells us that in order to study a locally small category $\mathscr{C}$, it is in many ways sufficient to study the category of functors from $\mathscr{C}$ to **Set**, and that representable functors are the most important functors of all.

1.4 The Yoneda Lemma

Even though this is only a short introduction to category theory, we would be remiss not to mention the Yoneda Lemma, arguably the most important statement in category theory.

The Yoneda Lemma has many consequences. First, it answers an important question about representable functors: given a representable functor $F : \mathscr{C} \to \mathbf{Set}$, which data do we need to find a natural isomorphism between F and $\mathrm{Hom}(A, -)$? And more generally, given any functor $F : \mathscr{C} \to \mathbf{Set}$, what does it mean to give a natural transformation from $\mathrm{Hom}(A, -)$ to F?

Theorem 1.43 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a locally small category, and fix an object A in $\mathscr{C}$. Let $F : \mathscr{C} \longrightarrow \mathbf{Set}$ be a covariant functor. Then there is a bijection*

$$Nat(\mathrm{Hom}_{\mathscr{C}}(A, -), F) \xrightarrow{\gamma} F(A) \, .$$

Moreover, this correspondence is natural in both A and F.

Proof. Let φ be a natural transformation in $\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A, -), F)$. The proof of Yoneda's Lemma is essentially the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(A, A) \arrow[rrr, "\mathrm{Hom}_{\mathscr{C}}(A,f)"] \arrow[ddd, "\varphi_A"'] & & & \mathrm{Hom}_{\mathscr{C}}(A, X) \arrow[ddd, "\varphi_X"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & (F(f))u = \varphi_X(f) & \\
F(A) \arrow[rrr, "F(f)"'] & & & F(X)
\end{tikzcd}$$

Our bijection will be defined by $\gamma(\varphi) := \varphi_A(1_A)$. We should first check that this makes sense: arrows in **Set** are just functions between sets, and so φ_A is a function of sets $\mathrm{Hom}_{\mathscr{C}}(A, A) \longrightarrow F(A)$. Also, $\mathrm{Hom}_{\mathscr{C}}(A, A)$ is a set that contains at least the element 1_A, and $\varphi_A(1_A)$ is some element in the set $F(A)$.

Given any fixed $f \in \mathrm{Hom}_{\mathscr{C}}(A, X)$, the fact that φ is a natural transformation translates into the outer commutative diagram. In particular, the maps of sets $F(f)\varphi_A$ and $\varphi_X \,\mathrm{Hom}_{\mathscr{C}}(A, f)$ coincide, and must in particular take 1_A to the same element in $F(X)$. This is the commutativity of the inner diagram, with $u := \varphi_A(1_A)$.

The commutativity of the diagram above says that φ is completely determined by $\varphi_A(1_A)$, since for any other object X in $\mathscr{C}$ and any arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, X)$, we necessarily have $\varphi_X(f) = F(f)\varphi_A(1_A)$. In particular, our map $\gamma(\varphi) = \varphi_A(1_A)$ is injective. Moreover, note that each choice of $u \in F(A)$ gives rise to a different natural transformation φ by setting $\varphi_X(f) = F(f)u$. So our map γ is indeed a bijection.

We now have two naturality statements to prove. Naturality in the functor means that given a natural isomorphism $\eta\colon F \longrightarrow G$, the following diagram must commute:

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\gamma_F"] \arrow[d, "\eta_*"'] & F(A) \arrow[d, "\eta_A"] \\
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),G) \arrow[r, "\gamma_G"'] & G(A)
\end{tikzcd}$$

Given a natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F,

$$\begin{aligned}
\eta_A \circ \gamma_F(\varphi) &= \eta_A(\varphi_A(1_A)) && \text{by definition of } \gamma \\
&= (\eta \circ \varphi)_A(1_A) && \text{by definition of composition of natural transformations} \\
&= \gamma_G(\eta \circ \varphi) && \text{by definition of } \gamma \\
&= \gamma_G \circ \eta_*(\varphi) && \text{by definition of } \eta_*
\end{aligned}$$

so commutativity does hold. Naturality on the object means that given an arrow $f\colon A \longrightarrow B$, the diagram

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\gamma"] \arrow[d, "(f^*)^*"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(B,-),F) \arrow[r, "\gamma"'] & F(B)
\end{tikzcd}$$

commutes. Given a natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F,

$$F(f) \circ \gamma_A(\varphi) = F(f)(\varphi_A(1_A)),$$

while

$$\gamma_B \circ (f^*)^*(\varphi) = \gamma_B(\varphi \circ f^*) = (\varphi \circ f^*)_B(1_B).$$

Now notice that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(B,B) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathscr{C}}(A,B) \arrow[r, "\varphi_B"] & F(B) \\
1_B \arrow[r, mapsto] & f \arrow[r, mapsto] & \varphi_B(f)
\end{tikzcd}.$$

Let's look back at the big commutative diagram we started our proof with: it says in particular that $\varphi_B(f) = F(f)(\varphi_A(1_A))$. So commutativity does hold, and we are done. $\square$

One can naturally (pun intended) define the notion of functor category of contravariant functors, and then prove the corresponding Yoneda Lemma, which will instead use the contravariant Hom functor.

Exercise 15 (Contravariant version of the Yoneda Lemma)**.** Let $\mathscr{C}$ be a locally small category, and fix an object B in $\mathscr{C}$. Let $F\colon \mathscr{C} \longrightarrow \mathbf{Set}$ be a contravariant functor. Then there is a bijection

$$\mathrm{Nat}\ (\mathrm{Hom}_{\mathscr{C}}(-,B),F) \xrightarrow{\gamma} F(B)$$

which is natural on both B and F.

The Yoneda Lemma says that to give a natural transformation between the functors $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F is choosing an element in the set $F(A)$.

Remark 1.44. Notice that the Yoneda Lemma says in particular that the collection of all natural transformations from $\mathrm{Hom}_{\mathscr{C}}(A,-)$ to F is a set. This wasn't clear a priori, since the collection of objects in $\mathscr{C}$ is not necessarily a set.

The Yoneda Lemma says that natural transformations between representable functors correspond to arrows between the representing objects.

Remark 1.45. If we apply the Yoneda Lemma to the case when F itself is also a Hom functor, say $F = \mathrm{Hom}_{\mathscr{C}}(B,-)$, the Yoneda Lemma says that there is a bijection between $\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-), \mathrm{Hom}_{\mathscr{C}}(B,-))$ and $\mathrm{Hom}_{\mathscr{C}}(B,A)$. In particular, each arrow in $\mathscr{C}$ determines a natural transformation between Hom functors.

The Yoneda Embedding formalizes the remark above. It roughly says that every locally small category can be embedded into the category of contravariant functors from $\mathscr{C}$ to **Set**. It is common to refer to both Theorem 1.43 and Theorem 1.46 as the Yoneda Lemma.

Theorem 1.46 (Yoneda Embedding)**.** *Let $\mathscr{C}$ be a locally small category. The covariant functor*

$$\begin{tikzcd}
\mathscr{C} \arrow[rr] & & \boldsymbol{Set}^{\mathscr{C}^{op}} \\
A \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(-,A) \arrow[d, "f_*"] \\
B & & \mathrm{Hom}_{\mathscr{C}}(-,B)
\end{tikzcd}$$

*from $\mathscr{C}$ to the category of contravariant functors $\mathscr{C} \longrightarrow$ **Set** is an embedding. Moreover, the contravariant functor*

$$\begin{tikzcd}
\mathscr{C} \arrow[rr] & & \boldsymbol{Set}^{\mathscr{C}} \\
A \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(A,-) \\
B & & \mathrm{Hom}_{\mathscr{C}}(B,-) \arrow[u, "f^*"']
\end{tikzcd}$$

*from the category $\mathscr{C}$ to the category of covariant functors $\mathscr{C} \longrightarrow$ **Set** is also an embedding.*

Remark 1.47. Before the proof, there is a high potential for confusion with the notation, which we clarify right away. We are used to thinking of $f_* = \mathrm{Hom}(-,f)$ as the manifestation of the contravariant Hom-functor on arrows. However, here $f \mapsto f_*$ is a covariant assignment, with f_* indicating the natural transformation $\mathrm{Hom}_{\mathscr{C}}(-,A) \implies \mathrm{Hom}_{\mathscr{C}}(-,B)$ that sends each object X to the arrow (function)

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(X,A) &\xrightarrow{f_*} \mathrm{Hom}_{\mathscr{C}}(X,B) \\
g &\longmapsto fg.
\end{aligned}$$

We might as well take this opportunity for a sanity check and confirm that indeed our proposed functors take arrows $f\colon A \longrightarrow B$ in $\mathscr{C}$ to natural transformations between $\mathrm{Hom}_{\mathscr{C}}(-,A)$

and $\mathrm{Hom}_{\mathscr{C}}(-,B)$. The fact that the image is a natural transformation is encoded in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X,A) \arrow[r, "f_*"] & \mathrm{Hom}_{\mathscr{C}}(X,B) \\
\mathrm{Hom}_{\mathscr{C}}(Y,A) \arrow[u, "\mathrm{Hom}_{\mathscr{C}}(g,A)=g^*"] \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathscr{C}}(Y,B) \arrow[u, "\mathrm{Hom}_{\mathscr{C}}(g,B)=g^*"']
\end{tikzcd}$$

This diagram commutes since

$$g^* f_*(h) = g^*(fh) = (fh)g = f(hg) = f_*(hg) = f_* g^*(h).$$

Proof. First, note that our functors are injective on objects because the Hom-sets in our category are all disjoint. So all we need to check is that given objects A and B in $\mathscr{C}$, we have bijections

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(-,A), \mathrm{Hom}_{\mathscr{C}}(-,B))$$

and

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-), \mathrm{Hom}_{\mathscr{C}}(B,-)).$$

We will do the details for the first one, and leave the second as an exercise.

This follows from Remark 1.45, but let's carefully check the details. First, in Remark 1.47 we have already checked that each arrow is indeed taken to a natural transformation, so we just need to check injectivity and surjectivy at the level of arrows.

The Yoneda Lemma applied here tells us that each natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(-,A)$ and $F = \mathrm{Hom}_{\mathscr{C}}(-,B)$ corresponds to an element $u \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, which we obtain by taking $u := \varphi_A(1_A)$. As we discussed in the proof of the Yoneda Lemma 1.43, we can recover φ from u by taking the natural transformation φ that for each object X in $\mathscr{C}$ has $\varphi_X : \mathrm{Hom}_{\mathscr{C}}(X,A) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,B)$ given by $\varphi_X(f) = \mathrm{Hom}_{\mathscr{C}}(f,B)(u) = f_*(u)$. This shows surjectivity on arrows.

Finally, different arrows f give rise to different natural transformations by applying the resulting natural transformation f_* to the identity arrow 1_A, which takes it to f. This shows injectivity on arrows. $\square$

Finally, the Yoneda Embedding says that you can essentially recover an object in a category by knowing the maps from it or into it.

Theorem 1.48. *Let X and Y be objects in a locally small category $\mathscr{C}$. If $\mathrm{Hom}_{\mathscr{C}}(-,X)$ and $\mathrm{Hom}_{\mathscr{C}}(-,Y)$ are naturally isomorphic, or if $\mathrm{Hom}_{\mathscr{C}}(X,-)$ and $\mathrm{Hom}_{\mathscr{C}}(Y,-)$ are naturally isomorphic, then X and Y are isomorphic objects.*

Proof. The Yoneda Embeddings from Theorem 1.46 are fully faithful, and thus by Exercise 9 they must reflect isomorphisms. A natural isomorphism between the functors $\mathrm{Hom}_{\mathscr{C}}(X,-)$ and $\mathrm{Hom}_{\mathscr{C}}(Y,-)$ (or the functors $\mathrm{Hom}_{\mathscr{C}}(-,X)$ and $\mathrm{Hom}_{\mathscr{C}}(-,Y)$) is an isomorphism in the target functor category, and it corresponds to f_* for some arrow f from Y to X. By Exercise 9, f must be an isomorphism. In particular, X and Y are isomorphic. $\square$

1.5 Products and coproducts

Definition 1.49. Let $\mathscr{C}$ be a locally small category, and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathscr{C}$. The **product** of the A_i is an object in $\mathscr{C}$, denoted by $\prod_i A_i$ or $A_1 \times \cdots \times A_n$ if I is finite, together with arrows $\pi_j \in \mathrm{Hom}_{\mathscr{C}}(\prod_i A_i, A_j)$ for each j, called **projections**, satisfying the following universal property: given any object B in $\mathscr{C}$ and arrows $f_i \colon B \longrightarrow A_i$ for each i,

$$\begin{tikzcd}
 & B \arrow[dl, "f_i"'] \arrow[d, dashed, "f"] \arrow[dr, "f_j"] & \\
A_i & \prod_i A_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & A_j
\end{tikzcd}$$

Here is a larger diagram for the (first few) maps involved in a product when the indexing set $I = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & \prod_i A_i \arrow[dr, "\pi_1"] \arrow[ddr, "\pi_2"] \arrow[dddr, "\pi_3"] \arrow[ddddr] & \\
 & & A_1 \\
B \arrow[uur, dashed, "f"] \arrow[urr, "f_1"] \arrow[rr, "f_2"] \arrow[drr, "f_3"] \arrow[ddrr] & & A_2 \\
 & & A_3 \\
 & & \vdots
\end{tikzcd}$$

We can also take a "big picture" view of this universal property of the product:

$$\begin{tikzcd}
 & \prod_i A_i \arrow[dr, rightsquigarrow, "\{\pi_i\}"] & \\
B \arrow[ur, dashed, "f"] \arrow[rr, rightsquigarrow, "\{f_i\}"'] & & \{A_i\},
\end{tikzcd}$$

where the squiggly arrows are again collections of maps instead of maps.

The dual notion is the coproduct.

Definition 1.50. Let $\mathscr{C}$ be a locally small category, and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathscr{C}$. The **coproduct** of the A_i is an object in $\mathscr{C}$, denoted by $\coprod_i A_i$ or in some contexts $\bigoplus_i A_i$, together with arrows $\iota_j \in \mathrm{Hom}_{\mathscr{C}}(A_j, \coprod_i A_i,)$ for each j, satisfying the following universal property: given any object B in $\mathscr{C}$ and arrows $f_i : A_i \longrightarrow B$ for each i, the following diagram commutes:

$$\begin{tikzcd}
 & B & \\
A_i \arrow[ur, "f_i"] \arrow[r, "\iota_i"'] & \coprod_i A_i \arrow[u, dashed, "f"] & A_j \arrow[ul, "f_j"'] \arrow[l, "\iota_j"]
\end{tikzcd}$$

Here is a diagram for the (first few) maps involved in a coproduct when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
 & \coprod_i A_i \arrow[dd, dashed, "f"] \\
\vdots \arrow[ur] \arrow[dr] & \\
A_3 \arrow[uur, "i_3"] \arrow[r, "f_3"] & B \\
A_2 \arrow[uuur, "i_2"] \arrow[ur, "f_2"] & \\
A_1 \arrow[uuuur, "i_1"] \arrow[uur, "f_1"] &
\end{tikzcd}$$

We can also take a "big picture" view of the universal property of the coproduct:

$$\begin{tikzcd}
 & \coprod_i A_i \arrow[dr, dashed, "f"] & \\
\{A_i\} \arrow[ur, squiggly, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_i\}"'] & & B,
\end{tikzcd}$$

where the squiggly arrows are now collections of maps instead of maps.

Theorem 1.51. *If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda\in\Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a category $\mathscr{C}$, then there is a unique isomorphism $\alpha\colon P \xrightarrow{\sim} P'$ such that $p'_\lambda \circ \alpha = p_\lambda$ for all λ. The analogous statement holds for coproducts.*

Proof. We will just deal with products. The following picture is a rough guide:

$$\begin{tikzcd}
P \arrow[r, dashed, "\alpha"] \arrow[drr, squiggly, "\{p_\lambda\}"'] & P' \arrow[r, dashed, "\beta"] \arrow[dr, squiggly, "\{p'_\lambda\}"] & P \arrow[d, squiggly, "\{p_\lambda\}"] \\
 & & \{X_\lambda\}
\end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta : P' \to P$ such that $p_\lambda \circ \beta = p'_\lambda$. Switching roles, we obtain a unique map $\alpha : P \to P'$ such that $p'_\lambda \circ \alpha = p_\lambda$.

Consider the composition $\beta \circ \alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ for all λ. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ for all λ, so by the uniqueness property of products, $\beta \circ \alpha = 1_P$. We can again switch roles to see that $\alpha \circ \beta = 1_{P'}$. Thus α is an isomorphism. The uniqueness of α in the statement is part of the universal property. $\square$

Exercise 16. Prove the analogous statement to Theorem 1.51 for coproducts.

This explains why the notations $\prod_i A_i$ and $\coprod_i A_i$ make sense: we can talk about *the* product and *the* coproduct of the A_i, if they exist.

In summary, the key point of these universal properties is the following:

- Mapping *into* a product is completely determined by mapping into each of the factors.
- Mapping *out* of a coproduct is completely determined by mapping out of each factor.

Example 1.52. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The product of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the cartesian product of sets along with the canonical projection maps.

The familiar notion of Cartesian product or direct product serves as a product in many of our favorite categories. Let's note first that given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in any of the categories $\mathbf{Sgrp}, \mathbf{Grp}, \mathbf{Ring}, R\text{-}\mathbf{Mod}, \mathbf{Top}$, the usual direct product $\prod_{\lambda\in\Lambda} X_\lambda$ is an object of the same category:

- for semigroups, groups, and rings, take the operation coordinate by coordinate:

$$(x_\lambda)_{\lambda\in\Lambda} \cdot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda};$$

- for modules, addition is coordinate by coordinate, and the action is the same on each coordinate: $r \cdot (x_\lambda)_{\lambda\in\Lambda} = (r \cdot x_\lambda)_{\lambda\in\Lambda}$;
- for topological spaces, use the product topology.

Note that this is not true for fields! The usual product of fields is not a field. In fact, there is no product in this category.

Theorem 1.53. *In each of the categories* **Set**, **Grp**, **Ring**, R-***Mod***, *and* **Top**, *given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$, the object $\prod_{\lambda\in\Lambda} X_\lambda$ given by the usual direct product along with the usual projection maps $\pi_\lambda \colon \prod_{\gamma\in\Lambda} X_\gamma \to X_\lambda$ forms a product in the category.*

Proof. We observe that in each category, the direct product is an object, and the projection maps π_λ are morphisms in the category.

Let $\mathscr{C}$ be one of these categories, and suppose that we have morphisms $g_\lambda \colon Y \to X_\lambda$ for all λ in $\mathscr{C}$. We need to show there is a unique morphism $\phi \colon Y \to \prod_{\lambda\in\Lambda} X_\lambda$ such that $\pi_\lambda \circ \phi = g_\lambda$ for all λ. The last condition is equivalent to

$$(\phi(y))_\lambda = (\pi_\lambda \circ \phi)(y) = g_\lambda(y)$$

for all λ, which is equivalent to $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$, so if this is a valid morphism, it is unique. Thus, it suffices to show that the map $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$ is a morphism in $\mathscr{C}$; we leave the details as an exercise. $\square$

Example 1.54. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the disjoint union with the various inclusion maps. By disjoint union, we simply mean union if the sets are disjoint; in general do something like replace X_λ with $X_\lambda \times \{\lambda\}$ to make them disjoint.

Theorem 1.55. *Let R be a ring, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left R-modules. A coproduct for the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is given by the direct sum of modules*

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \textit{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda$$

together with the inclusion maps

$$M_\lambda \xrightarrow{\iota_\lambda} \bigoplus_{\lambda \in \Lambda} M_\lambda$$

that send each $m \in M_\lambda$ to the tuple that has m in coordinate λ and zeroes elsewhere.

Proof. Given R-module homomorphisms $g_\lambda : M_\lambda \to N$ for each λ, we need to show that there is a unique R-module homomorphism $\alpha : \bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ such that $\alpha \circ \iota_\lambda = g_\lambda$. We define

$$\alpha((m_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} g_\lambda(m_\lambda).$$

Note that since $(m_\lambda)_{\lambda\in\Lambda}$ is in the direct sum, at most finitely many m_λ are nonzero, so the sum on the right hand side is finite, and hence makes sense in N. We need to check that α is R-linear; indeed,

$$\begin{aligned}
\alpha((m_\lambda) + (n_\lambda)) &= \alpha((m_\lambda + n_\lambda)) \\
&= \sum g_\lambda(m_\lambda + n_\lambda) \\
&= \sum g_\lambda(m_\lambda) + \sum g_\lambda(n_\lambda) \\
&= \alpha((m_\lambda)) + \alpha((n_\lambda)),
\end{aligned}$$

and the check for scalar multiplication is similar. For uniqueness of α, note that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is generated by the elements $\iota_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha' \circ \iota_\lambda = g_\lambda$ for all λ, then $\alpha(\iota_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(\iota_\lambda(m_\lambda))$ so the maps must be equal. □

Remark 1.56. If the index set Λ is finite, then the objects $\prod_{\lambda\in\Lambda} M_\lambda$ and $\bigoplus_{\lambda\in\Lambda} M_\lambda$ are identical, but the product and coproduct are not the same since one involves projection maps and the other involves inclusion maps. When Λ is infinite, the two objects are truly distinct, and in fact the direct sum is a submodule of the product.

Remark 1.57. For any indexing set Λ, $\coprod_{\lambda\in\Lambda} R$ is a free R-module. If $R = k$ happens to be a field, then $\prod_{\lambda\in\Lambda} k$ is free, since all vector spaces are free modules, but in general, $\prod_{\lambda\in\Lambda} R$ is not free for an infinite set Λ.

Example 1.58.

1) In **Top**, disjoint unions serve as coproducts.

2) In **Sgrp** and **Grp**, coproducts exist, and are given as free products. You may see or have seen them in topology in the context of Van Kampen's theorem.

3) In **Ring**, the story is more complicated. Let's note first that disjoint unions won't work, since they are not rings. Direct sums of infinitely many rings do not have 1, so they are not rings in this class, but even finite direct sums or products will not work, since the inclusion maps does not send 1 to 1. We will later on construct coproducts in the full subcategory of **Ring** consisting of commutative rings.

1.6 Limits and colimits

Definition 1.59. Let $(I, \geqslant)$ be a partially ordered set and let $\mathscr{C}$ be a category. An **inverse system** in $\mathscr{C}$ indexed by I is a contravariant functor $\mathbf{PO}(I) \to \mathscr{C}$.

Remark 1.60. Let's unwrap the definition of inverse system a bit. For each $i \in I$, we get an object M_i in $\mathscr{C}$. Moreover, in the category $\mathbf{PO}(I)$, there is exactly one arrow $i \to j$ for each $i \leqslant j$, and the image of this arrow under any contravariant functor $\mathbf{PO}(I) \to \mathscr{C}$ is an arrow $M_j \to M_i$. Finally, our functor must preserve compositions of arrows, so whenever $k \geqslant j \geqslant i$, the arrow $M_k \to M_i$ should match the composition of arrows through j. Thus an inverse system in $\mathscr{C}$ indexed by I consists of the following data:

- for each $i \in I$, an object M_i in $\mathscr{C}$, and
- for each $j \geqslant i$, an arrow $\varphi_i^j \colon M_j \to M_i$ in $\mathscr{C}$

such that whenever $k \geqslant j \geqslant i$, the following diagram must commute:

$$\begin{tikzcd} M_k \arrow[rr, "\varphi_i^k"] \arrow[dr, "\varphi_j^k"'] & & M_i \\ & M_j \arrow[ur, "\varphi_i^j"'] & \end{tikzcd}$$

Note moreover that $\varphi_i^i = \mathrm{id}_{M_i}$, since functors preserve identities. We sometimes denote this data by saying that $\{M_i, \varphi_i^j\}$ is an inverse system.

Example 1.61.

a) An inverse system in a category $\mathscr{C}$ indexed by $\mathbb{N}$ is determined by a diagram of the form

$$X_0 \xleftarrow{a_0} X_1 \xleftarrow{a_1} X_2 \xleftarrow{a_2} X_3 \xleftarrow{a_3} X_4 \xleftarrow{a_4} X_5 \leftarrow \cdots .$$

All the other arrows $X_i \to X_j$ for $i < j$ are given by composition.

b) Let I be a family of submodules of an R-module M. Then we can think of I as a partially ordered set with the reverse inclusion $\supseteq$, so that $N \leqslant L$ if and only if $N \supseteq L$. Whenever $N \subseteq L$, we have an inclusion map $N \to L$, and the family of submodules I together with the inclusion maps forms an inverse system of R-modules.

A special case of this is when we have a descending chain of submodules of M

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

which is also a special case of an inverse system indexed by $\mathbb{N}$.

c) If I is a poset with the **discrete partial order**, meaning $i \geqslant j$ if and only if $i = j$, then an inverse system indexed by I is just a family of objects indexed by I.

d) If $I = \{1, 2, 3\}$ is a poset with $1 \leqslant 2$ and $1 \leqslant 3$, then an inverse system indexed by I is just a diagram of the form

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A. \end{tikzcd}$$

Exercise 17. Let J be an ideal in a commutative ring R, and consider its nth power, which is the ideal

$$J^n := (f_1 \cdots f_n \mid f_i \in J)$$

generated by all n-fold products of elements in R. For each $m \geqslant n$, consider the maps

$$\begin{aligned} R/J^m &\xrightarrow{\varphi_n^m} R/J^n \\ r + J^m &\longmapsto r + J^n. \end{aligned}$$

Show that these form an inverse system in R-**Mod** indexed by $\mathbb{N}_{>0}$. Note that this can be represented as

$$R/J \xleftarrow{\varphi_1^2} R/J^2 \xleftarrow{\varphi_2^3} R/J^3 \xleftarrow{\varphi_3^4} R/J^4 \xleftarrow{\varphi_4^5} R/J^5 \longleftarrow \cdots.$$

Definition 1.62. Let $\mathscr{C}$ be a category and let $\{M_i, \varphi_i^j\}_i$ be an inverse system on $\mathscr{C}$ indexed by I. The **limit** or **inverse limit** of $\{M_i, \varphi_i^j\}$ consists of an object

$$\varprojlim M_i$$

and arrows

$$\pi_i \colon \varprojlim M_i \to M_i$$

called **projections** such that

$$\varphi_i^j \pi_j = \pi_i \quad \text{for all } i, j \in I$$

satisfying the following universal property: for all arrows $f_i \colon X \to M_i$ such that $\varphi_i^j f_j = f_i$ for all i, j, meaning that the diagram

$$\begin{tikzcd} M_i & X \arrow[l, "f_i"'] \arrow[dl, "f_j"] \\ M_j \arrow[u, "\varphi_i^j"] & \end{tikzcd}$$

commutes, there exists a unique arrow $f \colon X \to \varprojlim M_i$ such that

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"] \arrow[ddr, bend right, "\pi_j"'] & & X \arrow[ll, dashed, "\exists! f"'] \arrow[dl, "f_i"'] \arrow[ddl, bend left, "f_j"] \\ & M_i & \\ & M_j \arrow[u, "\varphi_i^j"] & \end{tikzcd}$$

commutes.

One can show that if it exists, the object $\varprojlim M_i$ is unique up to isomorphism; in fact, this is the terminal object in some appropriate (and technical) category. So we can refer to *the* limit of an inverse system.

Remark 1.63. Given an inverse system $\{M_i, \varphi_i^j\}$ indexed by I in a category $\mathscr{C}$, say corresponding to the contravariant functor $\varphi\colon I \to \mathscr{C}$, suppose that its limit exists, and let $L = \varprojlim M_i$. The projections π_i give us commutative diagrams

$$\begin{tikzcd} L \arrow[r, "1_L"] \arrow[d, "\pi_i"'] & L \arrow[d, "\pi_j"] \\ M_i \arrow[r, "\varphi_i^j"'] & M_j \end{tikzcd}$$

This is the same data as a natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "\Delta L"] \arrow[r, bend right, "\varphi"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C} \end{tikzcd}.$$

In other words, a limit for α consists of an object and a natural transformation from the constant functor on that object to the functor α.

Example 1.64. A terminal object can be viewed as a limit of the empty diagram: since there are no objects in an inverse limit from the empty category, the limit is an object L that must satisfy the condition that for every object X, there is a unique arrow $X \to L$.

Exercise 18. Show that if I is a partially ordered set with the discrete order, then the limit of any inverse system just the product.

Theorem 1.65. *Let R be any ring. The limit of any inverse system of left R-modules over any partially ordered set exists.*

Proof. Let I be a partially ordered set and consider an inverse system of R-modules indexed by I, say with modules M_i and homomorphisms $\varphi_i^j\colon M_j \to M_i$. Let

$$L := \{(m_i) \in \prod_i M_i \mid \varphi_i^j(m_j) = m_i \text{ for all } i \leqslant j\}.$$

One can show (exercise!) that this is a submodule of the product of the M_i. For each i, let $\pi_i\colon L \to M_i$ be the restriction of the projection maps $\prod M_i \to M_i$ to L. We claim that L is a limit for the inverse system, together with the projection maps π_i.

First, note that

$$\varphi_i^j \pi_j((m_k)_k) = \varphi_i^j(m_j) = m_i = \pi_i((m_k)_k),$$

by construction, so $\varphi_i^j \pi_j = \pi_i$.

Moreover, suppose that we are given an R-module X and R-module homomorphisms $f_i\colon X \to M_i$ such that $\varphi_i^j f_j = f_i$ for all $i \leqslant j$. Define

$$\begin{aligned} X &\xrightarrow{\ g\ } \textstyle\prod_i M_i \\ x &\longmapsto (f_i(x))_i. \end{aligned}$$

First, note that $\pi(f(x)) = f_i(x)$ for all i by construction. Moreover, this is an R-module homomorphism; it is induced by the universal property of the product. We claim that the image of g is contained in L, and thus that we can restrict g to an R-module homomorphism $f\colon X \to L$. Indeed, given any $x \in X$,

$$\varphi_i^j(\pi_j(g(x))) = \varphi_i^j(f_j(x)) = f_i(x) = \pi_i(g(x)).$$

This says that $g(x) \in L$, so we get an R-module homomorphism $f\colon X \to L$ given by

$$f(x) = (f_i(x)).$$

Finally, we claim that L and f satisfy the desired universal property, and for that, we need first to check that

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"'] & & X \arrow[ll, dashed, "f"'] \arrow[dl, "f_i"] \\ & M_i & \end{tikzcd}$$

commutes, and we need to check that such f is unique. The commutativity is immediate, since as noted above $\pi(f(x)) = f_i(x)$ for all $x \in X$ by construction. For uniqueness, suppose that h is any other R-module homomorphism $X \to L$ such that

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"'] & & X \arrow[ll, dashed, "h"'] \arrow[dl, "f_i"] \\ & M_i & \end{tikzcd}$$

also commutes. Given any $x \in X$, let $h(x) = (m_i)$. Then

$$m_i = \pi_i(h(x)) = f_i(x)$$

for all i, so

$$h(x) = (m_i) = (f_i(x)) = f(x),$$

and thus $h = f$. This completes the proof that L is a limit for the given inverse system. $\square$

Remark 1.66. One can adapt the proof of Theorem 1.65 to show that all limits in **Set** exist, and can be constructed explicitly as a subset of the product of the sets forming the inverse system.

Example 1.67.

a) If I is a partially ordered set with the discrete order, then the limit of any inverse system just the product.

b) Given a ring R and an ideal J, the limit of the inverse system

$$R/J \longleftarrow R/J^2 \longleftarrow R/J^3 \longleftarrow R/J^4 \longleftarrow R/J^5 \longleftarrow \cdots$$

is the J-adic completion of R.

The dual construction to limits is the notion of a colimit.

Definition 1.68. Let $(I, \geqslant)$ be a partially ordered set and let $\mathscr{C}$ be a category. An **direct system** in $\mathscr{C}$ indexed by I is a covariant functor $\mathbf{PO}(I) \to \mathscr{C}$.

Remark 1.69. An inverse system in $\mathscr{C}$ indexed by I consists of the following data:

- for each $i \in I$, an object M_i in $\mathscr{C}$, and
- for each $i \geqslant j$, an arrow $\varphi^i_j \colon M_i \to M_j$ in $\mathscr{C}$

such that whenever $k \geqslant j \geqslant i$, the following diagram must commute:

$$\begin{tikzcd} M_i \arrow[rr, "\varphi^i_k"] \arrow[dr, "\varphi^j_k"'] & & M_k \\ & M_j \arrow[ur, "\varphi^k_j"'] & \end{tikzcd}$$

Note moreover that $\varphi^i_i = \mathrm{id}_{M_i}$, since functors preserve identities. We sometimes denote this data by saying that $\{M_i, \varphi^j_i\}$ is an inverse system.

Example 1.70.

a) A direct system in a category $\mathscr{C}$ indexed by $\mathbb{N}$ is determined by a diagram of the form

$$X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3 \xrightarrow{a_3} X_4 \xrightarrow{a_4} X_5 \to \cdots .$$

All the other arrows $X_i \to X_j$ for $i < j$ are given by composition: $a_{j-1} \circ \cdots \circ a_i$.

b) Let I be a family of submodules of an R-module M. Then we can think of I as a partially ordered set with $\subseteq$. Whenever $N \subseteq L$, we have an inclusion map $N \to L$, and the family of submodules I together with the inclusion maps forms a direct system of R-modules.

A special case of this is when we have an ascending chain of submodules of M

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

which is also a special case of a direct system indexed by $\mathbb{N}$.

c) If I is a poset with the discrete partial order, then an inverse system indexed by I is just a family of objects indexed by I.

d) If $I = \{1, 2, 3\}$ is a poset with $1 \leqslant 2$ and $1 \leqslant 3$, then a direct system indexed by I is just a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C. & \end{tikzcd}$$

Definition 1.71. Let $\mathscr{C}$ be a category and let $\{M_i, \varphi_j^i\}_i$ be a direct system on $\mathscr{C}$ indexed by I. The **colimit** or **direct limit** of $\{M_i, \varphi_j^i\}$ consists of an object

$$\varinjlim M_i$$

and arrows

$$\alpha_i \colon M_i \to \varinjlim M_i$$

called **insertion arrows** such that

$$\alpha_j \varphi_j^i = \alpha_i \quad \text{for all } i, j \in I$$

satisfying the following universal property: for all arrows $f_i \colon M_i \to X$ such that $f_j \varphi_j^i = f_i$ for all i, j, meaning that the diagram

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "\varphi_j^i"'] & X \\ M_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

commutes, there exists a unique arrow $f \colon X \to \varinjlim M_i$ such that

$$\begin{tikzcd} \varinjlim M_i \arrow[rr, dashed, "\exists! f"] & & X \\ & M_i \arrow[ul, "\alpha_i"] \arrow[ur, "f_i"'] \arrow[d, "\varphi_i^j"] & \\ & M_j \arrow[uul, bend left, "\pi_j"] \arrow[uur, bend right, "f_j"'] & \end{tikzcd}$$

commutes.

One can show that if it exists, the object $\varinjlim M_i$ is unique up to isomorphism; in fact, this is the initial object in some appropriate (and technical) category. So we can refer to *the* colimit of a direct system.

Remark 1.72. Given a direct system $\{M_i, \varphi_j^i\}$ indexed by I in a category $\mathscr{C}$, say corresponding to the covariant functor $\varphi \colon I \to \mathscr{C}$, suppose that its colimit exists, and let $L = \varinjlim M_i$. The α_i give us commutative diagrams

$$\begin{tikzcd} L \arrow[r, "1_L"] & L \\ M_i \arrow[u, "\alpha_i"] \arrow[r, "\varphi_j^i"'] & M_j \arrow[u, "\alpha_j"'] \end{tikzcd}$$

This is the same data as a natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\Delta L"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C} \end{tikzcd}.$$

In other words, a limit for α consists of an object and a natural transformation from α to the constant functor on that object.

Example 1.73. An initial object can be viewed as a colimit of the empty diagram: since there are no objects in a direct limit from the empty category, the colimit is an object C that must satisfy the condition that for every object X, there is a unique arrow $C \to X$.

Exercise 19. Show that if I is a partially ordered set with the discrete order, then the colimit of any inverse system just the coproduct.

Theorem 1.74. *Let R be any ring. The colimit of any direct system of left R-modules over any partially ordered set exists.*

Proof. Let I be a partially ordered set and consider a direct system of R-modules indexed by I, say with modules M_i and homomorphisms $\varphi^i_j \colon M_j \to M_i$. Let $\iota_i \colon M_i \to \oplus_j M_j$ be the inclusions into the direct sum, let S be the submodule of $\oplus M_i$ generated by all elements of the form

$$\iota_i(\varphi^i_j(m_i)) - \iota_i(m_i),$$

and define

$$C := \oplus_i M_i / S.$$

For each i, let

$$\begin{aligned} M_i &\xrightarrow{\alpha_i} C \\ m &\longmapsto \iota_i(m) + S. \end{aligned}$$

We claim that C together with the maps α_i is a colimit for the direct system; we leave the details as an exercise. □

Remark 1.75. One can adapt the proof of Theorem 1.65 to show that all limits in **Set** exist, and can be constructed explicitly as a subset of the product of the sets forming the inverse system.

Definition 1.76. A covariant functor $F \colon \mathscr{C} \to \mathscr{D}$

- **preserves colimits** if

  $$F(\varinjlim M_i) = \varinjlim F(M_i),$$

  meaning that if $\varinjlim M_i$ is the colimit of a direct system $\{M_i, \varphi^i_j\}$ with insertion morphisms $\alpha_i \colon M_i \to \varinjlim M_i$, then $F(\varinjlim M_i)$ is a colimit of the direct system $\{F(M_i), F(\varphi^i_j)\}$ with insertion morphisms $F(\alpha_i) \colon F(M_i) \to F(\varinjlim M_i)$.

- **preserves limits** if

  $$F(\varprojlim M_i) = \varprojlim F(M_i),$$

  meaning that if $\varprojlim M_i$ is the limit of an inverse system $\{M_i, \varphi^j_i\}$ with projections $\pi_i \colon \varprojlim M_i \to M_i$, then $F(\varprojlim M_i)$ is a limit of the inverse system $\{F(M_i), F(\varphi^j_i)\}$ with projections $F(\pi_i) \colon F(\varprojlim M_i) \to F(M_i)$.

Definition 1.77. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ **converts limits to colimits** or **sends limits to colimits** if

$$F(\varinjlim M_i) = \varprojlim F(M_i),$$

meaning that if $\varprojlim M_i$ is the limit of an inverse system $\{M_i, \varphi_i^j\}$ with projections $\pi_i : \varprojlim M_i \to M_i$, then $F(\varprojlim M_i)$ is a colimit of the direct system $\{F(M_i), F(\varphi_i^j)\}$ with insertion morphisms $F(\pi_i) \colon F(M_i) \to F(\varprojlim M_i)$.

Similarly, a contravariant functor $F : \mathscr{C} \to \mathscr{D}$ **converts colimits to limits** or **sends colimits to limits** if

$$F(\varprojlim M_i) = \varinjlim F(M_i).$$

Later we will see some important examples of functors with some of these properties.

There are many other important constructions that arise as special cases of limits and colimits, some of which we will study later in the class. We close this section by giving one more example:

Definition 1.78. Let $\mathscr{C}$ be a category. A **pullback** of the arrows f and g consists of an object P and arrows p_1 and p_2 such that

$$\begin{tikzcd} P \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes, and satisfying the following universal property: for all objects Q and arrows q_1 and q_2 such that

$$\begin{tikzcd} A \arrow[r, "q_1"] \arrow[d, "q_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes, there exists a unique u such that

$$\begin{tikzcd} Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes. One sometimes refers to the diagram

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

as a **pullback diagram**.

The dual construction is the pushout.

Definition 1.79. Let $\mathscr{C}$ be a category. A **pushout** of the arrows f and g consists of an object P and arrows p_1 and p_2 such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dashed, "p_1"] \\ C \arrow[r, dashed, "p_2"'] & P \end{tikzcd}$$

commutes, and satisfying the following universal property: for all objects Q and arrows q_1 and q_2 such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q_1"] \\ C \arrow[r, "q_2"'] & Q \end{tikzcd}$$

commutes, there exists a unique u such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "p_1"] \arrow[ddr, bend left, "q_2"] & \\ B \arrow[r, "p_2"'] \arrow[drr, bend right, "q_1"'] & P \arrow[dr, dashed, "u"] & \\ & & Q \end{tikzcd}$$

commutes. One sometimes refers to the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_1"] \\ C \arrow[r, "p_2"'] & P \end{tikzcd}$$

as a **pushout diagram**.

Pullbacks and pushouts are special cases of limits and colimits.

Exercise 20. Interpret the notion of pullback as a limit and a pushout as a colimit. More precisely, describe a partially ordered set and corresponding inverse system or direct system whose limit or colimit is the same as a pushout or pullback.

We showed in Theorem 1.65 and Theorem 1.74 that R-**Mod** has all limits and colimits. In the case of pullbacks and pushouts, one can describe the corresponding module in a more manageable way.

Exercise 21. Explicitly describe pullbacks and pushouts in R-**Mod**.

1.7 Universal properties

We have all seen constructions that are at first a bit messy but that end up satisfying some nice universal property that makes everything work out. At the end of the day, a universal property allows us to ignore the messy details and focus on the universal property, which usually says everything we need to know about the construction.

Universal properties are *everywhere*. Limits and colimits are a big example; products and coproducts are a special case of limits and colimits. A representable functor encodes a *universal property* of the object that represents it: for example, in Example 1.42, mapping out of the singleton set is the same as choosing an element x in a set X.

Definition 1.80. Let $\mathscr{C}$ be a locally small category. A **universal property** of an object C in $\mathscr{C}$ consists of a representable functor $F\colon \mathscr{C} \longrightarrow \mathbf{Set}$ together with a **universal element** $X \in F(C)$ such that F is naturally isomorphic to either $\mathrm{Hom}_{\mathscr{C}}(C,-)$ (if F is covariant) or $\mathrm{Hom}_{\mathscr{C}}(-,C)$ (if F is contravariant), via the natural isomorphism that corresponds to X via the bijection in the Yoneda Lemma 1.43.

We can rephrase this in terms of universal arrows.

Definition 1.81. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be covariant functor. Given an object $D \in \mathscr{D}$, a **universal arrow from** D **to** F is a unique pair (U,u) where U is an object in $\mathscr{C}$ and a unique arrow $u \in \mathrm{Hom}_{\mathscr{D}}(D,F(U))$ with the following **universal property**: for any arrow $f \in \mathrm{Hom}_{\mathscr{D}}(D,F(Y))$, there exists a unique arrow $h \in \mathrm{Hom}_{\mathscr{C}}(U,Y)$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, dashed, "h"'] & & D \arrow[r, "u"] \arrow[dr, "f"'] & F(U) \arrow[d, dashed, "F(h)"] \\ Y & & & F(Y) \end{tikzcd}$$

There is a dual to this definition. A **universal arrow from** F **to** D is a unique pair (U,u), where C is an object in $\mathscr{C}$ and $u \in \mathrm{Hom}_{\mathscr{D}}(F(U),D)$ that satisfy the following **universal property**: for any arrow $f \in \mathrm{Hom}_{\mathscr{D}}(F(Y),D)$, there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(Y,U)$ such that the following diagram commutes:

$$\begin{tikzcd} U & & D & F(U) \arrow[l, "u"'] \\ Y \arrow[u, dashed, "h"] & & & F(Y) \arrow[ul, "f"] \arrow[u, dashed, "F(h)"'] \end{tikzcd}$$

Remark 1.82. Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a covariant functor, and fix an object U in $\mathscr{C}$, an object DD in $\mathscr{D}$, and an arrow $u \in \mathrm{Hom}_{\mathscr{D}}(D,F(U))$. Notice that $\mathrm{Hom}_{\mathscr{D}}(D,F(-))$ determines a covariant functor $\mathscr{C} \longrightarrow \mathbf{Set}$. By the Yoneda Lemma 1.43, the following is a recipe for a natural transformation between $\mathrm{Hom}_{\mathscr{C}}(U,-)$ and $\mathrm{Hom}_{\mathscr{D}}(D,F(-))$: for each object Y in $\mathscr{C}$ and each arrow $h \in \mathrm{Hom}_{\mathscr{C}}(U,Y)$, set $\varphi_X(h) := \mathrm{Hom}_{\mathscr{D}}(D,F(h))(u)$. Notice that

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(D,F(U)) \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(D,F(h))"] & \mathrm{Hom}_{\mathscr{D}}(D,F(Y)) \, , \\ f \arrow[r, mapsto] & F(h) \circ u \end{tikzcd}$$

so $\varphi_X(h)(f) = F(h) \circ u$.

We get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(U,U) \arrow[rrr, "\mathrm{Hom}_{\mathscr{C}}(U,h)"] \arrow[ddd, "\varphi_U"'] & & & \mathrm{Hom}_{\mathscr{C}}(U,Y) \arrow[ddd, "\varphi_Y"] \\
& 1_U \arrow[r, mapsto] \arrow[d, mapsto] & h \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & F(h) \circ u =: \varphi_Y(h) & \\
\mathrm{Hom}_{\mathscr{D}}(D,F(U)) \arrow[rrr, "\mathrm{Hom}_{\mathscr{D}}(D,F(h))"'] & & & \mathrm{Hom}_{\mathscr{D}}(D,F(Y))
\end{tikzcd}$$

Given an arrow $f \in \mathrm{Hom}_{\mathscr{D}}(D, F(Y))$, $\varphi_Y(h) = f$ for some $h \in \mathrm{Hom}_{\mathscr{C}}(U, Y)$ if and only if $F(h) \circ u = f$.

On the one hand, φ is a natural isomorphism if and only if for every object Y in $\mathscr{C}$ and every $f \in \mathrm{Hom}_{\mathscr{D}}(D, F(Y))$ there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(U, Y)$ such that $F(h) \circ u = f$. On the other hand, that is exactly the condition required for (U, u) to be a universal arrow from D to F. So we have shown that the following are equivalent:

- (U, u) is a universal arrow from D to F.
- U represents the functor $\mathrm{Hom}_{\mathscr{D}}(D, F(-)) \colon \mathscr{C} \longrightarrow \mathbf{Set}$, via $u \in \mathrm{Hom}_{\mathscr{D}}(D, F(U))$.

Similarly, one can prove the dual statement:

- (U, u) is a universal arrow from F to D.
- U represents the functor $\mathrm{Hom}_{\mathscr{D}}(F(-), D) \colon \mathscr{C} \longrightarrow \mathbf{Set}$, via $u \in \mathrm{Hom}_{\mathscr{D}}(F(U), D)$.

Let's phrase the universal property of products as a universal property in this formal sense, at least in the case of the product of two object C_1 and C_2 in $\mathscr{C}$. To do that, we need to consider the **product category** $\mathscr{C} \times \mathscr{C}$ with objects given by pairs (C_1, C_2) of objects in $\mathscr{C}$ and arrows in $(C_1, C_2) \longrightarrow (C_3, C_4)$ given by pairs of arrows (f_1, f_2) with $f_1 \in \mathrm{Hom}_{\mathscr{C}}(C_1, C_3)$ and $f_2 \in \mathrm{Hom}_{\mathscr{C}}(C_2, C_4)$. The diagonal functor $\Delta \colon \mathscr{C} \longrightarrow \mathscr{C} \times \mathscr{C}$ is exactly what it sounds like: $\Delta(C) = (C, C)$ for every object C in $\mathscr{C}$ and $\Delta(f) = (f, f)$ for every arrow f in $\mathscr{C}$.

Given objects X and Y in $\mathscr{C}$, consider the projection arrows $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 \colon X \times Y \longrightarrow Y$. We claim that the object $X \times Y$ together with the arrow (π_1, π_2) in $\mathscr{C} \times \mathscr{C}$ form a universal arrow from Δ to (X, Y) in $\mathscr{C} \times \mathscr{C}$. Why? This means that given any arrow $(f_1, f_2) \in \mathrm{Hom}_{\mathscr{C} \times \mathscr{C}}((X_1, X_2), \Delta(Y))$, there exists a unique $h \in \mathscr{C}(X_1 \times X_2, Y)$ such that

$$\begin{tikzcd}
X_1 \times X_2 \\
Y \arrow[u, dashed, "h"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(X_1, X_2) & \Delta(X_1 \times X_2) \arrow[l, "(\pi_1,\pi_2)"'] \\
& \Delta(Y) \arrow[u, dashed, "\Delta(h)"'] \arrow[ul, "(f_1,f_2)"]
\end{tikzcd}$$

commutes. This is indeed the universal property of products we just described less formally above: given $f_1 \colon Y \longrightarrow X_1$ and $f_2 \colon Y \longrightarrow X_2$, there is a unique $h \colon Y \longrightarrow X_1 \times X_2$ such that

$$\begin{tikzcd} X_1 \times X_2 \\ Y \arrow[u, dashed, "h"] \end{tikzcd} \qquad \begin{tikzcd} (X_1, X_2) & (X_1 \times X_2, X_1 \times X_2) \arrow[l, "(\pi_1,\pi_2)"'] \\ & (Y,Y) \arrow[ul, "(f_1,f_2)"] \arrow[u, dashed, "\Delta(h)"'] \end{tikzcd}$$

Equivalently, following the recipe we described in Remark 1.82, the universal property of the product is encoded in the representable functor $\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(-),(X_1,X_2))$, which is represented by $X_1 \times X_2$ via (π_1,π_2). So more precisely, that says that there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(-, X_1 \times X_2) \cong \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(-),(X_1,X_2)),$$

and that is precisely the natural transformation that the Yoneda bijection we constructed in Theorem 1.43 takes to $(\pi_1,\pi_2) \in \mathrm{Hom}_{\mathscr{C}}(\Delta(X_1 \times X_2),(X_1,X_2))$. If we follow that bijection, our natural isomorphism φ sends an object Y in $\mathscr{C}$ to the arrow

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(Y, X_1 \times X_2) \arrow[r, "\varphi_Y"] & \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(Y),(X_1,X_2)) \\ f \arrow[r, mapsto] & \left(\Delta(Y) \xrightarrow{(f,f)} \Delta(X_1 \times X_2) \xrightarrow{(\pi_1,\pi_2)} (X_1,X_2)\right). \end{tikzcd}$$

Since φ_Y is a bijection, every arrow $(f_1,f_2) \in \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}(\Delta(Y),(X_1,X_2))$ is $\varphi_Y(f)$ for some $f \in \mathrm{Hom}_{\mathscr{C}}(Y, X_1 \times X_2)$. Ultimately, this means that there exists f such that $f_1 = \pi_1 f$ and $f_2 = \pi_2 f$. And suprise surprise: we just rediscovered the universal property of the product!

1.8 Adjointness

Universal properties are closely related to adjoint functors.

Definition 1.83. Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Two covariant functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

form an **adjoint** pair (F,G) if given any objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$, there is a bijection between the Hom-sets

$$\mathrm{Hom}_{\mathscr{D}}(F(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(C, G(D))$$

which is natural on both objects, meaning that for all $f \in \mathrm{Hom}_{\mathscr{C}}(C_1, C_2)$ and $g \in \mathrm{Hom}_{\mathscr{D}}(D_1, D_2)$, the diagrams

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(F(C_2), D) \arrow[r, "\cong"] \arrow[d, "F(f)^*"'] & \mathrm{Hom}_{\mathscr{C}}(C_2, G(D)) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathscr{D}}(F(C_1), D) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(C_1, G(D)) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(F(C), D_1) \arrow[r, "\cong"] \arrow[d, "g_*"'] & \mathrm{Hom}_{\mathscr{C}}(C, G(D_1)) \arrow[d, "G(g)_*"] \\ \mathrm{Hom}_{\mathscr{D}}(F(C), D_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(C, G(D_2)) \end{tikzcd}$$

commute for all $C \in \mathscr{C}$ and all $D \in \mathscr{D}$. We say that F is the **left adjoint** of G, or that F has a **right adjoint**, and that G is the **right adjoint** of F, or that G has a **left adjoint**.

We can think of adjoint functors as solutions to optimization problems. A particular adjoint functor gives the most efficient functorial solution to some problem.

Example 1.84. Given a set I, what is the most efficient way to assign an R-module to I in a functorial way? The solution to this problem is the construction of free modules, the functor $\mathbf{Free}\colon \mathbf{Set} \longrightarrow R\text{-}\mathbf{Mod}$ that sends each set I to the free R-module R^I on I. The free functor is precisely a left adjoint to the forgetful functor $R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$.

As Mac Lane said [ML98], "the slogan is *adjoint functors arise everywhere*".

Remark 1.85. We can rephrase the condition that $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ as follows: for every object C in $\mathscr{C}$, there is a universal arrow from C to G, and for every object D in $\mathscr{D}$ there exists a universal arrow from F to D. To see that, let $\eta_D \in \operatorname{Hom}_{\mathscr{D}}(F(G(D)), D)$ be the image of the identity on $\operatorname{Hom}_{\mathscr{D}}(G(D), G(D))$ via the bijection

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(G(D), G(D)) \arrow[r, "\cong"] & \operatorname{Hom}_{\mathscr{D}}(F(G(D)), D) \\
\mathrm{id}_{G(D)} \arrow[r, mapsto] & \eta_D
\end{tikzcd}$$

given by the definition of adjoint functors, and let $\varepsilon_C \in \operatorname{Hom}_{\mathscr{C}}(C, GF(C))$ be be the image of the identity on $\operatorname{Hom}_{\mathscr{C}}(F(C), F(C))$ via the bijection

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{D}}(F(C), F(C)) \arrow[r, "\cong"] & \operatorname{Hom}_{\mathscr{D}}(C, GF(C))\ . \\
\mathrm{id}_{F(C)} \arrow[r, mapsto] & \varepsilon_C
\end{tikzcd}$$

We claim that $(F(C), \varepsilon_C)$ is a universal arrow from C to G. That would mean that given arrow $f \in \operatorname{Hom}_{\mathscr{C}}(C, G(Y))$, there must exist a unique arrow $h \in \operatorname{Hom}_{\mathscr{D}}(F(C), Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
F(C) \arrow[d, dashed, "h"'] & & D \arrow[r, "\varepsilon_C"] \arrow[dr, "f"'] & G(F(C)) \arrow[d, dashed, "G(h)"] \\
Y & & & G(Y).
\end{tikzcd}$$

This says that $G(h)_*(\varepsilon_C) = G(h) \circ \varepsilon_C = f$, which means that

$$\begin{tikzcd}
1_{F(C)} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \varepsilon_C \arrow[ddd, mapsto] \\
& \operatorname{Hom}_{\mathscr{D}}(F(C), F(C)) \arrow[r, "\cong"] \arrow[d, "h_*"'] & \operatorname{Hom}_{\mathscr{C}}(C, GF(C)) \arrow[d, "G(h)_*"] & \\
& \operatorname{Hom}_{\mathscr{D}}(F(C), Y) \arrow[r, "\cong"'] & \operatorname{Hom}_{\mathscr{C}}(C, G(Y)) & \\
h \arrow[rrr, mapsto] & & & f
\end{tikzcd}$$

On the one hand, such an h does exist: just take $h \in \operatorname{Hom}_{\mathscr{C}}(F(C), Y)$ that is sent to f via the bijection between $\operatorname{Hom}_{\mathscr{D}}(F(C), Y)$ and $\operatorname{Hom}_{\mathscr{C}}(C, G(Y))$. Since this map is a bijection, such an h is unique.

Similarly, we claim that $(G(D), \eta_D)$ is a universal arrow from F to D. That would mean that for any arrow $f \in \mathrm{Hom}_{\mathscr{D}}(F(Y), D)$, there exists a unique $h \in \mathrm{Hom}_{\mathscr{C}}(Y, G(D))$ such that the following diagram commutes:

$$\begin{tikzcd} G(D) & & D & F(G(D)) \arrow[l, "\eta_D"'] \\ Y \arrow[u, dashed, "h"] & & & F(Y) \arrow[ul, "f"] \arrow[u, dashed, "F(h)"'] \end{tikzcd}$$

This means that $(F(h))^*(\eta_D) = \eta_D \circ F(h) = f$, which means that

$$\begin{tikzcd} 1_{G(D)} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \eta_D \arrow[ddd, mapsto] \\ & \mathrm{Hom}_{\mathscr{C}}(G(D), G(D)) \arrow[r, "\cong"] \arrow[d, "h^*"'] & \mathrm{Hom}_{\mathscr{D}}(D, FG(D)) \arrow[d, "F(h)^*"] & \\ & \mathrm{Hom}_{\mathscr{C}}(G(D), Y) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathscr{D}}(D, F(Y)) & \\ h \arrow[rrr, mapsto] & & & f \end{tikzcd}$$

Again, such an h exists and it is unique because it must correspond to f via the bijection between $\mathrm{Hom}_{\mathscr{D}}(D, F(Y))$ and $\mathrm{Hom}_{\mathscr{C}}(G(D), Y)$.

We can talk about *the* left or right adjoint to a given functor.

Exercise 22. Left and right adjoints are unique up to natural isomorphism. More precisely, given an adjoint pair of functors (F, G), show that if G' is also a right adjoint to F, then G' and G are naturally isomorphic. Similarly, one can show that if F' is also a left adjoint to G, then F and F' are naturally isomorphic.

Example 1.86. Fix a ring R. The forgetful functor $F\colon R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Set}$ has a left adjoint. That left adjoint is the functor $G\colon \mathbf{Set} \longrightarrow R\text{-}\mathbf{Mod}$ that takes a set S to the free R-module $\oplus_S R$ on S, meaning the R-module whose elements are finite formal linear combinations $r_1 s_1 + \cdots + r_n s_n$ of elements $s_i \in S$ with coefficients $r_i \in R$. This is the same free module we described much earlier. Each function of sets defines an R-module map by sending each basis element to its image via the given function.

Even without any category theory, one often describes the free R-module on a set S by the following universal property: given a function f from a set S to an R-module M, there exists a unique R-module homomorphism from the free module $\oplus_S R$ to M that agrees with f on the basis elements. And indeed, one can check that this is the universal property we formally obtain from the fact that the free R-module functor is left adjoint to the forgetful functor from R-**Mod**.

This type of *free* construction is quite common, and often gives rise to adjunctions. We can think about the free functor from **Set** to R-**Mod** as the most efficient way of defining an R-module from a given set. It's efficient because it comes with a nice universal property.

We close this short detour into the wonderful world of category theory to point out that if we wanted to sound really obscure, we could have defined chain complexes in this categorical language.

Remark 1.87. First, we view $\mathbb{Z}$ as a partially ordered set under $\geqslant$. As in Example 1.3 1.8, $\mathbb{Z}$ now gives us a category whose objects are the integers, and where we have an arrow in $\mathrm{Hom}_{\mathbb{Z}}(n,m)$ if $n \geqslant m$. If we ignore the identity maps $\mathrm{Hom}_{\mathbb{Z}}(n,n)$ and composite maps, we can represent this category in the following diagram:

$$\cdots \longrightarrow n+1 \longrightarrow n \longrightarrow n-1 \longrightarrow \cdots.$$

From this perspective, a chain complex is a functor $F\colon \mathbb{Z} \longrightarrow \mathbf{Ab}$: for each $n \in \mathbb{Z}$, we get an R-module F_n, and we also get an R-module homomorphisms $F_{n+1} \longrightarrow F_n$ for each n. Indeed, this can all be represented as a sequence

$$\cdots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots.$$

For our functor to truly be a complex, though, we must require that all compositions $F_{n+1} \longrightarrow F_n \longrightarrow F_{n-1}$ be 0. A map of complexes, also known as a chain map, is a natural transformation between two such functors.

Chapter 2

The category of chain complexes

We are finally ready to introduce the category of chain complexes, and to talk more about exact sequences and homology.

2.1 Maps of complexes

Unsurprisingly, we can form a category of complexes, but to do that we need the right definition of maps between complexes. We also take this section as a chance to set up some definitions we will need later.

Definition 2.1. Let $(F_\bullet, \partial_\bullet^F)$ and $(G_\bullet, \partial_\bullet^G)$ be complexes. A **map of complexes** or a **chain map**, which we write as $h\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ or simply $h\colon F_\bullet \longrightarrow G_\bullet$, is a sequence of homomorphisms of R-modules $h_n\colon F_n \longrightarrow G_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, "h_{n+1}"'] & F_n \arrow[r] \arrow[d, "h_n"'] & F_{n-1} \arrow[r] \arrow[d, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r] & G_n \arrow[r] & G_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

This means that $h_n \partial_{n+1}^F = \partial_{n+1}^G h_{n+1}$ for all n.

Example 2.2. The zero and the identity maps of complexes $(F_\bullet, \partial_\bullet) \longrightarrow (F_\bullet, \partial_\bullet)$ are exactly what they sound like: the zero map $0_{F_\bullet}$ is 0 in every homological degree, and the identity map $1_{F_\bullet}$ is the identity in every homological degree.

This is the notion of morphism we would want to form a category of chain complexes.

Definition 2.3. Let R be a ring. The **category of chain complexes** of R-modules, denoted $\mathrm{Ch}(R\textbf{-mod})$ or simply $\mathrm{Ch}(R)$, is the category with objects all chain complexes of R-modules and arrows all maps of complexes of R-modules. When $R = \mathbb{Z}$, we write $\mathrm{Ch}(\mathbf{Ab})$ for $\mathrm{Ch}(\mathbb{Z})$, the category of chain complexes of abelian groups.

Exercise 23. Show that the isomorphisms in the category $\mathrm{Ch}(R)$ are precisely the maps of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, "h_{n+1}"'] & F_n \arrow[r] \arrow[d, "h_n"'] & F_{n-1} \arrow[r] \arrow[d, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r] & G_n \arrow[r] & G_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that h_n is an isomorphism for all n.

This is a good notion of map of complexes: it induces homomorphisms in homology.

Lemma 2.4. *Let $h\colon (F_\bullet, \partial^F_\bullet) \longrightarrow (G_\bullet, \partial^G_\bullet)$ be a map of complexes. For all n, h_n restricts to homomorphisms $B_n(h) : B_n(F_\bullet) \longrightarrow B_n(G_\bullet)$ and $Z_n(h) : Z_n(F_\bullet) \longrightarrow Z_n(G_\bullet)$. As a consequence, h induces homomorphisms on homology $\mathrm{H}_n(h) : \mathrm{H}_n(F_\bullet) \longrightarrow \mathrm{H}_n(G_\bullet)$.*

Proof. Since $h_n \partial^F_{n+1} = \partial^G_{n+1} h_{n+1}$, any element $a \in B_n(F_\bullet)$, say $a = \partial^F_{n+1}(b)$, is taken to

$$h_n(a) = h_n \partial^F_{n+1}(b) = \partial^G_{n+1} h_{n+1}(b) \in \operatorname{im} \partial^G_{n+1} = B_n(G_\bullet).$$

Similarly, if $a \in Z_n(F_\bullet) = \ker \partial^F_n$, then

$$\partial_n h_n(a) = h_{n-1} \partial^F_n(a) = 0,$$

so $h_n(a) \in \ker \partial^G_n = Z_n(G_\bullet)$. Finally, the restriction of h_n to $Z_n(F_\bullet) \longrightarrow Z_n(G_\bullet)$ sends $B_n(F_\bullet)$ into $B_n(G_\bullet)$, and thus it induces a well-defined homomorphism on the quotients $H_n(F_\bullet) \longrightarrow \mathrm{H}_n(G_\bullet)$. $\square$

In particular, this says that taking nth homology is a functor $\mathrm{H}_n \colon \mathrm{Ch}(R) \longrightarrow R$-**Mod**, which takes each map of complexes $h\colon F_\bullet, \longrightarrow G_\bullet$ to the R-module homomorphism $\mathrm{H}_n(h)$: $\mathrm{H}_n(F_\bullet) \longrightarrow \mathrm{H}_n(G_\bullet)$.

Definition 2.5. A map of chain complexes h is a **quasi-isomorphism** if it induces an isomorphism in homology, meaning $\mathrm{H}_n(h)$ is an isomorphism of R-modules for all n.

Remark 2.6. Note that saying that if f is a quasi-isomorphism between F and G says more than just that $\mathrm{H}_n(F) \cong \mathrm{H}_n(G)$ for al n: it says that there are such isomorphisms that are all induced by f.

Exercise 24. Let π denote the projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. The chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism.

Definition 2.7. Let $f, g\colon F \longrightarrow G$ be maps complexes. A **homotopy**, sometimes referred to as a **chain homotopy**, between f and g is a sequence of maps $h_n\colon F_n \longrightarrow G_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & F_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "g_{n+1}"'] \arrow[d, shift left, "f_{n+1}"] & F_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"] \arrow[d, shift right, "g_n"'] \arrow[d, shift left, "f_n"] & F_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "h_{n-1}"] \arrow[d, shift right, "g_{n-1}"'] \arrow[d, shift left, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial_{n+2}"'] & G_{n+1} \arrow[r, "\partial_{n+1}"'] & G_n \arrow[r, "\partial_n"'] & G_{n-1} \arrow[r, "\partial_{n-1}"'] & \cdots
\end{tikzcd}$$

such that

$$\partial_{n+1} h_n + h_{n-1} \partial_n = f_n - g_n$$

for all n. If there exists a homotopy between f and g, we say that f and g are **homotopic**. If f is homotopic to the zero map, we say it is **null-homotopic**. If $f\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ and $g\colon (G_\bullet, \partial_\bullet^G) \longrightarrow (F_\bullet, \partial_\bullet^F)$ are maps of complexes such that fg is homotopic to the identity map on $(G_\bullet, \partial_\bullet^G)$ and gf is homotopic to the identity chain map on $(F_\bullet, \partial_\bullet^F)$, we say that f and g are **homotopy equivalences** and $(F_\bullet, \partial_\bullet^F)$ and $(G_\bullet, \partial_\bullet^G)$ are **homotopy equivalent**.

Exercise 25. Homotopy is an equivalence relation.

This is an interesting relation because homotopic maps induce the same map on homology.

Lemma 2.8. *Homotopic maps of complexes induce the same map on homology.*

Proof. Let $f, g : (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ be homotopic maps of complexes, and let h be a homotopy between f and g. We claim that the map of complexes $f - g$ (defined in the obvious way) sends cycles to boundaries. If $a \in Z_n(F_\bullet)$, then

$$(f - g)_n(a) = \partial_{n+1} h_n + h_{n-1} \underbrace{\partial_n(a)}_{0} = \partial_{n+1}(h_n(a)) \in B_n(G_\bullet).$$

The map on homology induced by $f - g$ must then be the 0 map, so f and g induce the same map on homology. □

Corollary 2.9. *Homotopy equivalences are quasi-isomorphisms.*

Proof. If $f\colon (F_\bullet, \partial_\bullet^F) \longrightarrow (G_\bullet, \partial_\bullet^G)$ and $g\colon (G_\bullet, \partial_\bullet^G) \longrightarrow (F_\bullet, \partial_\bullet^F)$ are such that fg is homotopic to $1_{G_\bullet}$ and gf is homotopic to $1_{F_\bullet}$, then by Lemma 2.8 fg induces the identity map on homology. Then for each n, $\mathrm{H}_n(f)\,\mathrm{H}_n(g) = \mathrm{H}_n(fg)$ is an isomorphism, and thus $\mathrm{H}_n(f)$ and $\mathrm{H}_n(g)$ must both be isomorphisms. □

The converse is false.

Exercise 26. Let π denote the projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. The chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism but not a homotopy equivalence.

Remark 2.10. In fact, the relation "there is a quasi-isomorphism from F to G" is not symmetric: in Exercise 26, there is no quasi-isomorphism going in the opposite direction of the one given.

Now that we know about maps between complexes, it's time to point out that we can also talk about complexes and exact sequences of complexes. While we will later formalize this a little better when we discover that $\mathrm{Ch}(R)$ is an abelian category, let's for now give quick definitions that we can use.

Definition 2.11. Given complexes B and C, B is a **subcomplex** of C if B_n is a submodule of C_n for all n, and the inclusion maps $\iota_n \colon B_n \subseteq C_n$ define a map of complexes $\iota : B \longrightarrow C$. Given a subcomplex B of C, the **quotient** of C by B is the complex C/B that has C_n/B_n in homological degree n, with differential induced by the differential on C_n.

Exercise 27. If B is a subcomplex of C, then the differential d on C satisfies $d_n(B_n) \subseteq B_{n-1}$. Therefore, d_n induces a map of R-modules $C_n/B_n \longrightarrow C_{n-1}/B_{n-1}$ for all n, so that our definition of the differential on C/B actually makes sense.

We can also talk about kernels and cokernels of maps of complexes.

Definition 2.12. Given any map of complexes $f \colon B_\bullet \longrightarrow C_\bullet$, the **kernel** of f is the subcomplex $\ker f$ of $B_\bullet$ that we can assemble from the the kernels $\ker f_n$. More precisely, $\ker f$ is the complex

$$\cdots \longrightarrow \ker f_{n+1} \longrightarrow \ker f_n \longrightarrow \ker f_{n-1} \longrightarrow \cdots$$

where the differentials are simply the corresponding restrictions of the differentials on $B_\bullet$. Similarly, the **image** of f is the subcomplex of $C_\bullet$

$$\cdots \longrightarrow \operatorname{im} f_{n+1} \longrightarrow \operatorname{im} f_n \longrightarrow \operatorname{im} f_{n-1} \longrightarrow \cdots$$

where the differentials are given by restriction of the corresponding differentials in $C_\bullet$. The **cokernel** of f is the quotient complex $C_\bullet/\operatorname{im} f$.

Again, there are some details to check.

Exercise 28. Show that the kernel, image, and cokernel of a complex map are indeed complexes.

Definition 2.13. A **complex** in $\mathrm{Ch}(R)$ is a sequence of complexes of R-modules C^n and chain maps $d_n \colon C^n \longrightarrow C^{n-1}$ between them

$$\cdots \longrightarrow C^{n+1} \xrightarrow{d_{n+1}} C^n \xrightarrow{d_n} C^{n-1} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all n.

Given a complex C in $\mathrm{Ch}(R)$, we can talk about cycles and boundaries, which are a sequence of subcomplexes of the complexes in C, and thus its homology. Such a complex is exact if $\operatorname{im} d_{n+1} = \ker d_n$ for all n.

Definition 2.14. A **short exact sequence** of complexes is an exact complex in $\mathrm{Ch}(R)$ of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Equivalently, a short exact sequence of complexes is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\partial_{i+1}"'] & B_{i+1} \arrow[r, "g_{i+1}"] \arrow[d, "\partial_{i+1}"'] & C_{i+1} \arrow[r] \arrow[d, "\partial_{i+1}"'] & 0 \\
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "\partial_i"'] & B_i \arrow[r, "g_i"] \arrow[d, "\partial_i"'] & C_i \arrow[r] \arrow[d, "\partial_i"'] & \cdots \\
\cdots \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] & B_{i-1} \arrow[r, "g_{i-1}"] & C_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

where the rows are exact and the columns are complexes.

2.2 Long exact sequences

A long exact sequence is just what it sounds like: an exact sequence that is, well, long. Long exact sequences arise naturally in various ways, and are often induced by some short exact sequence. The first long exact sequence one encounters is the long exact sequence on homology. All other long exact sequences are, in some way, a special case of this one. The main tool we need to build it is the Snake Lemma.

Theorem 2.15 (Snake Lemma)**.** *Consider the commutative diagram of R-modules*

$$\begin{tikzcd}
 & A' \arrow[r, "i'"] \arrow[d, "f"'] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C &
\end{tikzcd}.$$

If the rows of the diagram are exact, then there exists an exact sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

Given $c' \in C'$, pick $b' \in B'$ such that $p'(b') = c'$, and $a \in A$ such that $i(a) = g(b')$. Then

$$\partial(c') = a + \operatorname{im} f \in \operatorname{coker} f.$$

The picture to keep in mind (and which explains the name of the lemma) is the following:

$$\begin{tikzcd}
\ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\partial"] \\
A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r] \arrow[d, "g"'] & C' \arrow[d, "h"'] \\
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
\operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h
\end{tikzcd}$$

Definition 2.16. The map ∂ in the Snake Lemma is the **connecting homomorphism**.

Proof. If $a' \in \ker f$, then $b' := i'(a')$ must satisfy $g(b') = if(a') = 0$, by commutativity, so $b' \in \ker g$. Similarly, the image of $b' \in \ker g$ by p' is in the kernel of h. So the maps

$$\ker f \longrightarrow \ker g \longrightarrow \ker h$$

are restrictions of the maps $A' \xrightarrow{i'} B' \xrightarrow{p'} C'$, so $i'(\ker f) \subseteq \ker(B' \xrightarrow{p'} C')$. If $b' \in \ker g$ is such that $p'(b') = 0$, then there exists $a' \in A'$ such that $i'(a') = b'$; we only need to check that $a' \in \ker f$. An indeed, by commutativity we have

$$if(a') = gi'(a') = g(b') = 0,$$

and since i is injective, we must have $f(a') = 0$.

Similarly, if $a \in \operatorname{im} f$, the commutativity of the diagram guarantees that $i(a) \in \operatorname{im} g$, and if $b \in \operatorname{im} g$, then $p(b) \in \operatorname{im} h$. So the maps $A \xrightarrow{i} B \xrightarrow{p} C$ restrict to maps

$$\operatorname{im} f \longrightarrow \operatorname{im} g \longrightarrow \operatorname{im} h \,,$$

which then induce maps

$$\operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \,.$$

Again, we automatically get $i(\operatorname{coker} f) \subseteq \ker(\operatorname{coker} g \longrightarrow \operatorname{coker} h)$, so we only need to check equality. If $b \in B$ is such that $p(b) = 0$ in $\operatorname{coker} h$, meaning $p(b) \in \operatorname{im} h$, let $c' \in C$ be such that $h(c') = p(b)$. Since p is surjective, there exists $b' \in B'$ such that $p'(b') = c'$, and by commutativity,

$$pg(b') = hp'(b') = h(c') = p(b).$$

Then $b - g(b') \in \ker p = \operatorname{im} i$. Since $b = b - g(b')$ in $\operatorname{coker} g$, this shows that the class of b in $\operatorname{coker} g$ is in $i(\operatorname{coker} f)$. So we have shown exactness at $\ker g$ and $\operatorname{coker} g$.

So everything we need to prove concerns the connecting homomoprhism ∂. Our definition of ∂ can be visualized as follows:

$$\begin{tikzcd}
 & & c' \in \ker h \arrow[d] & \\
A' \arrow[r, "i'"] \arrow[d, "f"] & b' \in B' \arrow[r, mapsto, dashed, "p'"] \arrow[d, mapsto, "g"] & c' \in C' \arrow[d, "h"] & \\
a \in A \arrow[r, mapsto, "i"'] \arrow[d, mapsto] & g(b') \in B \arrow[r, "p"'] & 0 \in C \arrow[r, Rightarrow] & g(b') \in \ker p = \operatorname{im} i \\
a + \operatorname{im} f \in \operatorname{coker} f & & &
\end{tikzcd}$$

We need to show the following:

1) ∂ is well-defined.

2) $p'(\ker g) = \ker \partial$.

3) $\operatorname{im} \partial = \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$.

The last two points together say that the sequence

$$\ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g$$

is exact.

To show that ∂ is well-defined, let's fix some $c' \in \ker h \subseteq C'$. Since p' is surjective, $c' \in \operatorname{im} p'$. Consider $b'_1, b'_2 \in B'$ such that $p'(b'_1) = p'(b'_2) = c'$. Then $p'(b'_1 - b'_2) = 0$. We will show that our definition of $\delta(0)$ is independent of the choice of $b' \in \ker g$, which implies that our definition of $\delta(c')$ is independent of our choice of b'_1 or b'_2. Given $b' \in \ker p' = \operatorname{im} i'$, there exists $a' \in A'$ such that $i'(a') = b'$. Notice that $a := f(a') \in A$ is such that

$$i(a) = if(a') = gi'(b')$$

so $\delta(0)$ is defined as $a + \operatorname{im} f \in \operatorname{coker} f$. Since $a = f(a') \in \operatorname{im} f$, we conclude that $\delta(0) = 0$ for any choice of b'. This shows that δ is well-defined, and 1) holds.

If $b' \in \ker g$, then $g(b') = 0$ and the only $a \in A$ such that $i(a) = g(b') = 0$ is $a = 0$. Therefore, $\delta(p'(b')) = 0$, so $p'(\ker g) \subseteq \ker \delta$. On the other hand, let $c' \in \ker h$ be such that $\partial(c') = 0$. That means that for any $b' \in B'$ such that $p'(b') = c'$ we must have $g(b') = i(a)$ for some $a \in \operatorname{im} f$. Let $a' \in A'$ be such that $f(a') = a$. Then

$$gi'(a') = if(a') = i(a) = g(b')$$

so $b' - i'(a') \in \ker g$. Since $p'i' = 0$, $p'(b' - i'(a')) = p'(b') = c'$, so $c' \in \operatorname{im} p'$. We conclude that $\ker \partial = p'(\ker g)$, and this shows 2).

Let $a \in A$. The statement $i(a + \operatorname{im} f) = 0$ lifts to B as $i(a) \in \operatorname{im} g$, so we can choose $b' \in B'$ such that $g(b') = i(a)$. Then $\partial(p'(b')) = a + \operatorname{im} f$, and moreover $p'(b')$ is in $\ker h$, since by commutativity we have $hp'(b') = pg(b') = pi(b') = 0$. This shows that $\ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g) \subseteq \operatorname{im} \partial$. Finally, if c', b', and a are as in the diagram above, $i(a + \operatorname{im} f) = g(b') + \operatorname{im} g = 0$, so $\operatorname{im} \partial \subseteq \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$. This shows 3). $\square$

This proof is what we call a *diagram chase*, for reasons that may be obvious by now: we followed the diagram in the natural way, and everything worked out in the end.

Now that we have the Snake Lemma, we can construct the long exact sequence in homology:

Theorem 2.17 (Long exact sequence in homology)**.** *Given a short exact sequence in* $\mathrm{Ch}(R)$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

there are connecting homomorphisms $\partial : \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A)$ *such that*

$$\cdots \longrightarrow \mathrm{H}_{n+1}(C) \xrightarrow{\partial} \mathrm{H}_n(A) \xrightarrow{f} \mathrm{H}_n(B) \xrightarrow{g} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \longrightarrow \cdots$$

is an exact sequence.

Proof. For each n, we have short exact sequences

$$0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0.$$

The condition that f and g are maps of complexes implies, by Lemma 2.4, that f and g both take boundaries to boundaries, so that we get exact sequences

$$A_n/\operatorname{im} d^A_{n+1} \longrightarrow B_n/\operatorname{im} d^B_{n+1} \longrightarrow C_n/\operatorname{im} d^C_{n+1} \longrightarrow 0\,.$$

Again by Lemma 2.4, the condition that f and g are maps of complexes also implies that f and g take cycles to cycles, so we get exact sequences

$$0 \longrightarrow Z_n(A) \longrightarrow Z_n(B) \longrightarrow Z_n(C)\,.$$

Let F be one of A, B, of C. The boundary maps on F induce maps $F_n \longrightarrow Z_{n-1}(F)$ that send $\operatorname{im} d_{n+1}$ to 0, so we get induced maps $F_n/\operatorname{im} d_{n+1} \longrightarrow Z_{n-1}(F)$. Putting all this together, we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & A_n/\operatorname{im} d^A_{n+1} \arrow[r] \arrow[d, "d^A_n"'] & B_n/\operatorname{im} d^B_{n+1} \arrow[r] \arrow[d, "d^B_n"'] & C_n/\operatorname{im} d^C_{n+1} \arrow[r] \arrow[d, "d^C_n"'] & 0\,. \\
0 \arrow[r] & Z_n(A) \arrow[r] & Z_n(B) \arrow[r] & Z_n(C) &
\end{tikzcd}$$

For each $F = A, B, C$, the kernel of $F_n/\operatorname{im} d^F_{n+1} \xrightarrow{d^F_n} Z_{n-1}(F)$ is $\mathrm{H}_n(F)$, and its cokernel is $Z_{n-1}(F)/\operatorname{im} d^F_n = \mathrm{H}_{n-1}(F)$. The Snake Lemma now gives us exact sequences

$$\mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \longrightarrow \mathrm{H}_{n-1}(B) \longrightarrow \mathrm{H}_{n-1}(C)$$

Finally, we glue all these together to obtain the long exact sequence in homology. $\square$

Remark 2.18. It's helpful to carefully consider how we compute the connecting homomorphisms in the long exact sequence, which we can easily put together from the proof of the Snake Lemma. Suppose that $c \in \ker d^C_{n+1}$. When we view c as an element in C_{n+1}, we can find $b \in B_{n+1}$ such that $g_{n+1}(b) = c$, since g_{n+1} is surjective by assumption. Since $d^B_{n+1}(b) \in \ker g_n$, we can find $a \in A_n$ with $f_n(a) = d^B_{n+1}(b)$. Finally, $\partial(c) = a + \operatorname{im} d^A_{n+1}$.

We will soon see that long exact sequences appear everywhere, and that they are very helpful. Before we see more examples, we want to highlight a connection between long and short exact sequences.

Remark 2.19. Suppose that

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots$$

is a long exact sequence. This long exact sequence breaks into the short exact sequences

$$0 \longrightarrow \ker f_n \xrightarrow{i} C_n \xrightarrow{\pi} \operatorname{coker} f_{n+1} \longrightarrow 0\,.$$

The first map i is simply the inclusion of the submodule $\ker f_n$ into C_n, while the second map π is the canonical projection onto the quotient. While it is clear that i is injective and π is surjective, exactness at the middle is less obvious. This follows from the exactness of the original complex, which gives $\operatorname{im} i = \ker f_n = \operatorname{im} f_{n+1} = \ker \pi$.

The long exact sequence in homology is natural.

Theorem 2.20 (Naturality of the long exact sequence in homology)**.** *Any commutative diagram in* $\mathrm{Ch}(R)$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "p'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows induces a commutative diagram where the rows are long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{n+1}(C) \arrow[r, "\partial"] \arrow[d, "h"'] & \mathrm{H}_n(A) \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{H}_n(B) \arrow[r, "p"] \arrow[d, "g"'] & \mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "h"'] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d, "f"'] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_{n+1}(C') \arrow[r, "\partial'"'] & \mathrm{H}_n(A') \arrow[r, "i'"'] & \mathrm{H}_n(B) \arrow[r, "p'"'] & \mathrm{H}_n(C) \arrow[r, "\partial'"'] & \mathrm{H}_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The rows of the resulting diagram are the long exact sequences in homology induced by each row of the original diagram, as in Theorem 2.17. So the content of the theorem is that the maps induced in homology by f, g, and h make the diagram commute. The commutativity of

$$\begin{tikzcd}
\mathrm{H}_n(A) \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{H}_n(B) \arrow[r, "p"] \arrow[d, "g"'] & \mathrm{H}_n(C) \arrow[d, "h"'] \\
\mathrm{H}_n(A') \arrow[r, "i'"'] & \mathrm{H}_n(B) \arrow[r, "p'"'] & \mathrm{H}_n(C)
\end{tikzcd}$$

follows from the fact that H_n is a functor, so we only need to check commutativity of the square

$$\begin{tikzcd}
\mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "h"'] & \mathrm{H}_{n-1}(A) \arrow[d, "f"] \\
\mathrm{H}_n(C) \arrow[r, "\partial'"'] & \mathrm{H}_{n-1}(A)
\end{tikzcd}$$

that involves the connecting homomorphisms ∂ and ∂'. Consider the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[rr] & & A_n \arrow[rr, "i"] \arrow[dl, "d"'] \arrow[dd, dashed] & & B_n \arrow[rr, "p"] \arrow[dl, "d"'] \arrow[dd, dashed] & & C_n \arrow[rr] \arrow[dl, "d"'] \arrow[dd, dashed] & & 0 \\
0 \arrow[rr] & & A_{n-1} \arrow[rr, crossing over] & & B_{n-1} \arrow[rr, crossing over] & & C_{n-1} \arrow[rr, crossing over] & & 0 & \\
 & 0 \arrow[rr] & & A'_n \arrow[rr] \arrow[dl, "d"'] & & B'_n \arrow[rr] \arrow[dl, "d"'] & & C'_n \arrow[rr] \arrow[dl, "d"'] & & 0 \\
0 \arrow[rr] & & A'_{n-1} \arrow[rr, "i'"'] \arrow[from=uu, crossing over] & & B'_{n-1} \arrow[rr, "p'"'] \arrow[from=uu, crossing over] & & C'_{n-1} \arrow[rr] \arrow[from=uu, crossing over] & & 0 &
\end{tikzcd}$$

Given $c \in \ker(d_n\colon C_n \longrightarrow C_{n-1})$, we need to check that $f_{n-1}(\partial(c)) = \partial' h_n(c)$ in $\mathrm{H}_{n-1}(A)$. To compute $\partial(c)$, we find a lift $b \in B_n$ such that $p_n(b) = c$, and $a \in A_{n-1}$ with $i_{n-1}(a) = d_n(b)$, and set $\partial(c) = a + \operatorname{im} d_n \in \mathrm{H}_{n-1}(A)$. So $f_{n-1}\partial(c) = f_{n-1}(a) + \operatorname{im} d_n$. On the other hand, to compute $\partial' h_n(c)$, we start by finding $b' \in B'_n$ such that $p'_n(b') = h_n(c)$. By commutativity of the top square

$$\begin{tikzcd} B_n \arrow[r, "p_n"] \arrow[d, "g_n"'] & C_n \arrow[d, "h_n"] \\ B'_n \arrow[r, "p'_n"'] & C'_n \end{tikzcd}$$

we can choose $b' = g_n(b)$, since

$$p'_n(b') = p'_n g_n(b) = h_n p_n(b) = h_n(c).$$

Next we take $a' \in A'_{n-1}$ such that $i'_{n-1}(a') = d_n(b')$, and set $\partial'(h(c)) = a' + \operatorname{im} d_n \in \mathrm{H}_{n-1}(A')$. By commutativity of the middle square

$$\begin{tikzcd} B_n \arrow[r, "d_n"] \arrow[d, "g_n"'] & B_{n-1} \arrow[d, "g_{n-1}"] \\ B'_n \arrow[r, "d_n"'] & B'_{n-1} \end{tikzcd}$$

we have

$$d_n(b') = d_n g_n(b) = g_{n-1} d_n(b).$$

By our choice of a, we have

$$d_n(b') = g_{n-1} d_n(b) = g_{n-1} i_{n-1}(a),$$

and by commutativity of the front left square

$$\begin{tikzcd} A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"'] & B_{n-1} \arrow[d, "g_{n-1}"] \\ A'_{n-1} \arrow[r, "i'_{n-1}"'] & B'_{n-1} \end{tikzcd}$$

we have

$$i'_{n-1} f_{n-1}(a) = g_{n-1} i_{n-1}(a) = d_n(b').$$

So we can take $a' = f_{n-1}(a)$. Finally, this means $\partial'(h_n(c)) = f_{n-1}(a) + \operatorname{im} d_{n-1}$, as we wanted to prove. $\square$

Remark 2.21. Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence in $\mathrm{Ch}(R)$. We can think of Theorem 2.20 as saying that the induced maps on homology $i_*\colon \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B)$ and $p_*\colon \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C)$ and the connecting homomorphism $\partial\colon \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A)$ are all natural transformations. More precisely, consider the category **SES** of short exact sequences of R-modules, which is a full subcategory of $\mathrm{Ch}(R)$. Homology gives us functors $\mathbf{SES} \longrightarrow R\text{-}\mathbf{Mod}$ that given a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

return the R-modules $\mathrm{H}_n(A)$, $\mathrm{H}_n(B)$, or $\mathrm{H}_n(C)$). A map between two short exact sequences then induces R-module homomorphisms between the corresponding homologies. With this framework, Theorem 2.20 says that $i_*\colon \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B)$, and $p_*\colon \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C)$ and the connecting homomorphism $\partial : \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A)$ are all natural transformations between the corresponding homology functors.

Chapter 3

R-Mod

Before we study abelian categories in general, we want to understand our best prototype for what an abelian category looks like: the category R-**Mod** of R-modules and R-module homomorphisms.

Appendix A

Rings and modules

We will study complexes of R-modules; to make sure we are all speaking the same language, we record here our basic assumptions on rings and modules. You can learn more about the basic theory of rings and modules in any introductory algebra book, such as [DF04].

A.1 Rings and why they have 1

In this class, all rings have a multiplicative identity, written as 1 or 1_R is we want to emphasize that we are referring to the ring R. This is what some authors call *unital rings*; since for us all rings are unital, we will omit the adjective. Moreover, we will think of 1 as part of the structure of the ring, and thus require it be preserved by all natural constructions. As such, a subring S of R must share the same multiplicative identity with R, meaning $1_R = 1_S$. Moreover, any ring homomorphism must preserve the multiplicative identity. To clear any possible confusion, we include below the relevant definitions.

Definition A.1. A **ring** is a set R equipped with two binary operations, $+$ and $\cdot$, satisfying:

1) $(R, +)$ is an abelian group with identity element denoted 0 or 0_R.

2) The operation $\cdot$ is associative, so that $(R, \cdot)$ is a semigroup.

3) For all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

4) there is a multiplicative identity, written as 1 or 1_R, such that $1 \neq 0$ and $1 \cdot a = a = a \cdot 1$ for all $a \in R$.

To simplify notation, we will often drop the $\cdot$ when writing the multiplication of two elements, so that ab will mean $a \cdot b$.

Note that the requirement that $1 \neq 0$ makes it so that the *zero ring* is not a ring.

Definition A.2. A ring R is a **commutative ring** if for all $a, b \in R$ we have $a \cdot b = b \cdot a$.

Definition A.3. A ring R is a **division ring** if $1 \neq 0$ and $R \setminus \{0\}$ is a group under $\cdot$, so every nonzero $r \in R$ has a multiplicative inverse. A **field** is a commutative division ring.

Definition A.4. A commutative ring R is a **domain**, sometimes called an **integral domain** if it has no zerodivisors: $ab = 0 \Rightarrow a = 0$ or $b = 0$.

For some familiar examples, $\mathrm{M}_n(R)$ (the set of $n \times n$ matrices) is a ring with the usual addition and multiplication of matrices, $\mathbb{Z}$ and $\mathbb{Z}/n$ are commutative rings, $\mathbb{C}$ and $\mathbb{Q}$ are fields, and the real Hamiltonian quaternion ring $\mathbb{H}$ is a division ring.

Definition A.5. A **ring homomorphism** is a function $f\colon R \to S$ satisfying the following:

- $f(a+b) = f(a) + f(b)$ for all $a, b \in R$.
- $f(ab) = f(a)f(b)$ for all $a, b \in R$.
- $f(1_R) = 1_S$.

Under this definition, the map $f\colon \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$ sending $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ preserves addition and multiplication but not the multiplicative identities, and thus it is not a ring homomorphism.

Exercise 29. For any ring R, there exists a unique homomorphism $\mathbb{Z} \to R$.

Definition A.6. A subset S of a ring R is a **subring** of R if it is a ring under the same addition and multiplication operations and $1_R = 1_S$.

So under this definition, $2\mathbb{Z}$, the set of even integers, is not a subring of $\mathbb{Z}$; in fact, it is not even a ring, since it does not have a multiplicative identity!

Definition A.7. Let R be a ring. A subset I of R is an **ideal** if:

- I is nonempty.
- $(I, +)$ is a subgroup of $(R, +)$.
- For every $a \in I$ and every $r \in R$, we have $ra \in I$ and $ar \in I$.

The final property is often called **absorption**. A **left ideal** satisfies only absorption on the left, meaning that we require only that $ra \in I$ for all $r \in R$ and $a \in I$. Similarly, a **right ideal** satisfies only absorption on the right, meaning that $ar \in I$ for all $r \in R$ and $a \in I$.

When R is a commutative ring, the left ideals, right ideals, and ideals over R are all the same. However, if R is not commutative, then these can be very different classes.

One key distinction between unital rings and nonunital rings is that if one requires every ring to have a 1, as we do, then the ideals and subrings of a ring R are very different creatures. In fact, the *only* subring of R that is also an ideal is R itself. The change lies in what constitutes a subring; notice that nothing has changed in the definition of ideal.

Remark A.8. Every ring R has two **trivial ideals**: R itself and the zero ideal $(0) = \{0\}$.

A **nontrivial ideal** I of R is an ideal that $I \neq R$ and $I \neq (0)$. An ideal I of R is a **proper ideal** if $I \neq R$.

A.2 Basic definitions: modules

You can learn more about the basic theory of (commutative) rings and R-modules in any introductory algebra book, such as [DF04].

Definition A.9. Let R be a ring with $1 \neq 0$. A **left R-module** is an abelian group $(M, +)$ together with an action $R \times M \to M$ of R on M, written as $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$ we have the following:

- $(r+s)m = rm + sm$,
- $(rs)m = r(sm)$,
- $r(m+n) = rm + rn$, and
- $1m = m$.

A **right R-module** is an abelian group $(M, +)$ together with an action of R on M, written as $M \times R \to M, (m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$ we have

- $m(r+s) = mr + ms$,
- $m(rs) = (mr)s$,
- $(m+n)r = mr + nr$, and
- $m1 = m$.

By default, we will be studying left R-modules. To make the writing less heavy, we will sometimes say **R-module** rather than left R-module whenever there is no ambiguity.

Remark A.10. If R is a commutative ring, then any left R-module M may be regarded as a right R-module by setting $mr := rm$. Likewise, any right R-module may be regarded as a left R-module. Thus for commutative rings, we just refer to modules, and not left or right modules.

The definitions of submodule, quotient of modules, and homomorphism of modules are very natural and easy to guess, but here they are.

Definition A.11. If $N \subseteq M$ are R-modules with compatible structures, we say that N is a **submodule** of M.

A map $M \xrightarrow{f} N$ between R-modules is a **homomorphism of R-modules** if it is a homomorphism of abelian groups that preserves the R-action, meaning $f(ra) = rf(a)$ for all $r \in R$ and all $a \in M$. We sometimes refer to R-module homomorphisms as **R-module maps**, or **maps of R-modules**. An isomorphism of R-modules is a bijective homomorphism, which we really should think about as a relabeling of the elements in our module. If two modules M and N are isomorphic, we write $M \cong N$.

Given an R-module M and a submodule $N \subseteq M$, the **quotient** M/N is an R-module whose elements are the equivalence classes determined by the relation on M given by $a \sim b \Leftrightarrow a - b \in N$. One can check that this set naturally inherits an R-module structure from the R-module structure on M, and it comes equipped with a natural **canonical map** $M \longrightarrow M/N$ induced by sending 1 to its equivalence class.

Example A.12. The modules over a field k are precisely all the k-vector spaces. Linear transformations are precisely all the k-module maps.

While vector spaces make for a great first example, be warned that many of the basic facts we are used to from linear algebra are often a little more subtle in commutative algebra. These differences are features, not bugs.

Example A.13. The $\mathbb{Z}$-modules are precisely all the abelian groups.

Example A.14. When we think of the ring R as a module over itself, the submodules of R are precisely the ideals of R.

Theorem A.15 (First Isomorphism Theorem)**.** *Any R-module homomorphism $M \xrightarrow{f} N$ satisfies $M/\ker f \cong \operatorname{im} f$.*

The first big noticeable difference between vector spaces and more general R-modules is that while every vector space has a basis, most R-modules do not.

Definition A.16. A subset $\Gamma \subseteq M$ of an R-module M is a **generating set**, or a **set of generators**, if every element in M can be written as a finite linear combination of elements in M with coefficients in R. A **basis** for an R-module M is a generating set Γ for M such that $\sum_i a_i\gamma_i = 0$ implies $a_i = 0$ for all i. An R-module is **free** if it has a basis.

Remark A.17. Every vector space is a free module.

Remark A.18. Every free R-module is isomorphic to a direct sum of copies of R. Indeed, let's construct such an isomorphism for a given free R-module M. Given a basis $\Gamma = \{\gamma_i\}_{i\in I}$ for M, let

$$\begin{aligned} \textstyle\bigoplus_{i\in I} R &\xrightarrow{\ \pi\ } M \\ (r_i)_{i\in I} &\longrightarrow \sum_i r_i\gamma_i \end{aligned} \quad .$$

The condition that Γ is a basis for M can be restated into the statement that π is an isomorphism of R-modules.

One of the key things that makes commutative algebra so rich and beautiful is that most modules are in fact *not* free. In general, every R-module has a generating set — for example, M itself. Given some generating set Γ for M, we can always repeat the idea above and write a **presentation** $\oplus_{i\in I} R \xrightarrow{\pi} M$ for M, but in general the resulting map π will have a nontrivial kernel. A nonzero kernel element $(r_i)_{i\in I} \in \ker \pi$ corresponds to a **relation** between the generators of M.

Remark A.19. Given a set of generators for an R-module M, any homomorphism of R-modules $M \longrightarrow N$ is determined by the images of the generators.

We say that a module is **finitely generated** if we can find a finite generating set for M. The simplest finitely generated modules are the cyclic modules.

Example A.20. An R-module is **cyclic** if it can be generated by one element. Equivalently, we can write M as a quotient of R by some ideal I. Indeed, given a generator m for M, the kernel of the map $R \xrightarrow{\pi} M$ induced by $1 \mapsto m$ is some ideal I. Since we assumed that m generates M, π is automatically surjective, and thus induces an isomorphism $R/I \cong M$.

Similarly, if an R-module has n generators, we can naturally think about it as a quotient of R^n by the submodule of relations among those n generators.

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