Symbolic Powers

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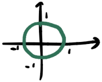
Mulheres Matemáticas: uma tarde de encontro

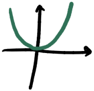
Algebra $\longleftrightarrow$ Geometry

$$xy = 0$$

$$x^2 + y^2 - 1 = 0$$

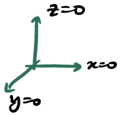


$$y - x^2 = 0$$



Algebra $\longleftrightarrow$ Geometry

$$\begin{cases} xy = 0 \\ xz = 0 \\ yz = 0 \end{cases}$$



$$\begin{cases} x_1 - a_1 = 0 \\ \quad\vdots \\ x_d - a_d = 0 \end{cases}$$

point $(a_1, \ldots, a_d)$
in d-dimensional space

systems of polynomial equations in d variables

nice subsets of d-dimensional space

Algebra	$\longleftrightarrow$	Geometry
systems of polynomial equations in d variables	$\longleftrightarrow$	nice subsets of d-dimensional space

Definition (Variety)

A *variety* is a subset $V \subseteq \mathbb{C}^d$ that consists of precisely all the common zeroes of a system of polynomial equations.

Theorem (Hilbert's Basis Theorem)

Every system of polynomial equations in d variables with coefficients in $\mathbb{R}$, $\mathbb{C}$, or more generally any field can be described by a finite number of equations.

Algebra $\longleftrightarrow$ Geometry

$f_1, f_2, \ldots$ polynomials
in d variables

$$\begin{cases} f_1 = 0 \\ f_2 = 0 \\ \quad\vdots \end{cases} \longmapsto$$

variety V
in d-space
all common
zeros of all f_i

all polynomials f with
$f(v) = 0$ for all $v \in V$

$$\text{Hilbert gives} \ \Big\downarrow \ \text{finitely many } f$$

$$I = (f_1, \ldots, f_n) := \{g_1 f_1 + \cdots + g_n f_n : g_i \text{ polynomial}\}$$ ideal

Ideals

An **ideal** I of the **ring** of polynomials in d variables, $R = \mathbb{C}[x_1, \ldots, x_d]$, is a nice set of polynomials with good algebraic properties.

- $0 \in I$
- $a, b \in I \Rightarrow a + b \in I$
- $a \in I$ and $r \in R \Rightarrow ra \in I$

Algebra	$\longleftrightarrow$	Geometry
algebra of ideals	$\longleftrightarrow$	geometry of varieties
algebra of $R = \mathbb{C}[x_1, \ldots, x_d]$	$\longleftrightarrow$	geometry of $\mathbb{C}^d$
<u>radical</u> ideals ($f^n \in I \Rightarrow f \in I$)	$\xleftrightarrow{1:1}$	varieties

Hilbert's Nullstellensatz

Algebra	$\longleftrightarrow$	Geometry
$(0) = \{0\}$	$\longleftrightarrow$	variety $\mathbb{C}^d$
$\mathbb{C}[x_1, \ldots, x_d]$	$\longleftrightarrow$	variety $\varnothing$
$(x_1 - a_1, \ldots, x_d - a_d)$	$\longleftrightarrow$	point $\{(a_1, \ldots, a_d)\}$
smaller ideals	$\longleftrightarrow$	larger varieties
prime ideals $(fg \in I \Rightarrow f \in I \text{ or } g \in I)$	$\longleftrightarrow$	irreducible varieties (not the union of smaller varieties)

variety V $\longleftrightarrow$ ideal of all polynomials f that vanish at every point $v \in V$

Question

How do we measure vanishing?

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Example

The polynomial $f = x^3$ vanishes *more* at the point 0 than $g = x$.

Vanishing to order n

A polynomial f vanishes to order n at P if all terms of order $< n$ in its power series expansion around P vanish.

Definition (Algebraic Powers)

For an ideal I, its nth power I^n is the ideal

$$I^n = (f_1 \cdots f_n : f_i \in I).$$

Example

In $\mathbb{C}[x, y]$, $(x, y)^2 = (x^2, xy, y^2)$.

Symbolic Powers

For an ideal I, its *nth symbolic power* $I^{(n)}$ can be defined via *primary decomposition*. Roughly speaking, primary decomposition is an ideal version of the fundamental theorem of algebra, which says (for $\mathbb{Z}$) that we can write things as products of primes.

Definition

The nth symbolic power of a prime P (so the corresponding variety is irreducible) in $R = \mathbb{C}[x_1, \ldots, x_d]$ is

$$P^{(n)} = \{r \in R \mid sr \in P^n \text{ for some } s \notin P\}.$$

In general, if $I = P_1 \cap \cdots \cap P_t$,

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_t^{(n)}.$$

Algebraic Powers

The algebraic powers I^n are very easy to describe algebraically, but have no clear geometric meaning.

Symbolic Powers

The symbolic powers $I^{(n)}$ are very hard to describe algebraically, even with a computer, but have a very important geometric meaning.

Theorem (Zariski–Nagata)

I ideal $\longleftrightarrow$ variety V

$$I^{(n)} = \{f \in I : f \text{ vanishes to order } n \text{ at every } v \in V\}$$

Some simple facts

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4. In general, $I^n \neq I^{(n)}$.

Example

The ideal I in $\mathbb{C}[x, y, z]$ corresponding to the curve parametrized by (t^3, t^4, t^5) in $\mathbb{C}^3$ has $I^n \neq I^{(n)}$ for all $n \geqslant 2$.

Question

Given an ideal I (or the corresponding variety V), describe $I^{(n)}$.

This can be very difficult to do, even with a powerful computer. This is why many innocent sounding questions about symbolic powers are still open.

Question

When is $I^n = I^{(n)}$?

For which I is $I^n = I^{(n)}$ for all n? Given I, is $I^n = I^{(n)}$ for some n?

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Theorem (G–Huneke, 2019)

If I has codimension 2 and R/I has nice* singularities, then $I^{(n)} = I^n$ for all n.

For example, this applies to

- I generated by the minors of a generic matrix
- R/I Veronese ($k[\text{all monomials of degree } d \text{ in } v \text{ variables}]$)
- R/I ring of invariants of a linearly reductive group

* = strongly F-regular in char p / of dense strong F-regular type in char 0

Question

What is the smallest degree $\alpha(I^{(n)})$ of an element in $I^{(n)}$? That is, what is the lowest degree of a polynomial that vanishes to order n on a given variety?

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Conjecture (Chudnovsky, 1981)

If V is a finite set of points in $\mathbb{P}^n$, then

$$\frac{\alpha(I^{(m)})}{m} \geqslant \frac{\alpha(I)+n-1}{n}.$$

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Theorem (Bisui–G–Hà–Nguyễn, 2020)

Chudnovsky's Conjecture holds for $\geqslant 4^n$ general points in $\mathbb{P}^n$.

Containment Problem

When is $I^{(a)} \subseteq I^b$?

Conjecture (Harbourne, 2008)

If the largest codimension of a component of V is h, then

$$I^{(hn-h+1)} \subseteq I^n$$

for all $n \geqslant 1$.

(Dumnicki–Szemberg–Tutaj-Gasińska): this fails for very special configurations of points in $\mathbb{P}^2$.

Theorem (G–Huneke, 2019)

If R/I has ok singularities, then I satisfies Harbourne's Conjecture.*

* = F-pure in char p / of dense F-pure type in char 0

Obrigada!

Symbolic Power

For a prime ideal P in $R = \mathbb{C}[x_1, \ldots, x_d]$, the n-th **symbolic power** of P is

$$P^{(n)} \quad = \{f \in R : sf \in P^n \text{ for some } s \notin P\}$$