Hilbert-Kunz multiplicity

Fix a Noetherian ring S and a finite colength ideal $J \subseteq S$, $\lambda(S/J) < \infty$.

<u>Monsky:</u>

$$\lambda\left(\frac{S}{J^{[p^n]}}\right) = c \cdot (p^n)^{\dim(S)} + O\left((p^n)^{\dim(S)-1}\right).$$

$J^{[p^n]}$ = Ideal generated by $\{f^{p^n} \mid f \in J\}$.

c = Hilbert-Kunz multiplicity of the pair (S, J)

$$c = \lim_{n \to \infty} \left(\frac{1}{p^n}\right)^{\dim(S)} \lambda\left(\frac{S}{J^{[p^n]}}\right)$$

$(S, J) \longrightarrow c = e_{HK}(S, J)$

$e_{HK}(S, J)$ detects sing of the pair (S, J)

- For eg when

 S is reg local,

 $J =$ max ideal

 $e_{HK}(S, J) = 1$

- In general $e_{HK}(S, J) > 1$.

<u>Thm</u> (Blickle - Enescu).

Let R be a Noeth local ring of +ve char $p > 0$.

Assume R is a complete domain.

If $e_{HK}(R, m)$

$$\leq 1 + \max\left\{\frac{1}{e(R)}, \frac{1}{(\dim(R))!}\right\}$$

Then ① (R, m) is Cohen-Macaulay

② R is F-rational.

Frobenius - Poincare Functions

Given a Noetherian graded k-alg R ( ) and a finite colength homogeneous ideal I, the Frob-Poincare function is a holomorphic function F,

$F(0) = e_{HK}(R, I)$.

R is a Noetherian $\mathbb{N}$-graded <u>domain</u>.

I : finite colength homogeneous ideal

R^{1/p^n} has a natural $1/p^n$ - $\mathbb{N}$ grading.

$\frac{R^{1/p^n}}{IR^{1/p^n}}$ - Take Hilb series.

$$\sum_{j} \lambda\left(\frac{R^{1/p^n}}{IR^{1/p^n}}\right)_{j/p^n} t^{j/p^n}$$

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$$\sum_{j} \lambda\left(\frac{R^{1/p^n}}{IR^{1/p^n}}\right)_{j/p^n} t^{j/p^n}$$

Replace $t = e^{-iy}$

- where y is the variable

$$t^{1/p^n} = e^{-iy/p^n}$$

$$G_n(y) = \sum \lambda\left(\frac{R^{1/p^n}}{IR^{1/p^n}}\right)_{j/p^n} \left(e^{-iy/p^n}\right)^{j}$$

<u>Thm</u> 1) $\left(\frac{1}{p^n}\right)^{\dim(R)} \sum \lambda\left(\frac{R^{1/p^n}}{IR^{1/p^n}}\right)_{j/p^n} e^{-iyj/p^n}$ converges for every $y \in \mathbb{C}$.

2) The limiting function is holomorphic everywhere on $\mathbb{C}$.

Note: If the underlying field is perfect,

$$\frac{R^{1/p^n}}{I R^{1/p^n}} \xrightarrow{\sim} \frac{R}{I^{[p^n]} R}.$$

Thm. (____) Let R be a Noetherian $\mathbb{N}$-graded ring, s.t $R_0 = k = \bar{k}$, I homo ideal $\lambda(R/I) < \infty$.

Let M be a f.g $\mathbb{Z}$-graded R module.

Then

1) $$\left(\frac{1}{p^n}\right)^{\dim(M)} \sum_j \lambda\left(\frac{M}{I^{[p^n]}M}\right)_j e^{-iyj/p^n}$$

converges for every $y \in \mathbb{C}$.

2) $$F(M,R,I) = \lim_{n\to\infty} \left(\frac{1}{p^n}\right)^{\dim(M)} \sum \lambda\left(\frac{M}{I^{[p^n]}M}\right)_j e^{-iyj/p^n}$$

is holomorphic on $\mathbb{C}$.

Note: $F(M, R, I)(0) = e_{HK}(M, R, I)$

Q. Even when R is a polynomial ring, there are ideals I, J s.t.

$$e_{HK}(R, I) = e_{HK}(R, J)$$

but $F(R, R, I) \neq F(R, R, J)$.

Question. Do Frobenius–Poincaré functions have special structures?

Examples 1) $R = k[x]$, $I = (x)$

$$\lim_{n\to\infty} \frac{1}{p^n} \sum_j \lambda\left(\left(\frac{k[x]}{x^{p^n}} \right)_j \right) e^{-iyj/p^n}$$

$$= \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} 1 \cdot e^{-iyj/p^n} = \lim_{n\to\infty} \frac{1}{p^n} \frac{1-\left(e^{-iy/p^n}\right)^{p^n}}{1-e^{-iy/p^n}}$$

$$= \frac{1-e^{-iy}}{iy}$$

2) $R = k[x_1, \ldots x_d]$, $I = (x_1 \ldots x_d)$

$$F(R, R, I) = \left(\frac{1-e^{-iy}}{iy} \right)^d.$$

Properties of Frob- Poincare functions

1) Take a triple (M, R, I) (Localization formula)

Let $p_1, p_2, \ldots, p_r$ be the minimal primes of $\dim = \dim(M)$ in $\mathrm{Supp}(M)$

Then $F(M, R, I) = \sum_{j=1}^{r} \lambda_{R_{p_j}}(M_{p_j})\, F\left(\frac{R}{p_j}, \frac{R}{p_j}, \frac{IR}{p_j}\right)$

2) Additivity: $0 \to M' \to M \to M'' \to 0$ short exact of graded modules.

a) if $\dim(M') < \dim(M)$, $F(M) = F(M'')$

b) if $\dim(M'') < \dim(M)$ $F(M) = F(M')$

c) if $\dim(M') = \dim(M'') = \dim(M)$

$$F(M) = F(M') + F(M'')$$

Prop. $I = (f_1, f_2, \ldots, f_d)$, $d = \dim(R)$

$\deg f_i = \delta_i$, $\lambda(R/I) < \infty$.

Then $$F(R, R, I) = e_R \cdot \frac{(1-e^{-iy\delta_1})(1-e^{-iy\delta_2})\cdots(1-e^{-iy\delta_d})}{(iy)^d}$$

$$e_R = \lim_{n\to\infty} \frac{d!}{n^d} \cdot \sum_{j=0}^{n} \lambda(R_j)$$

$=$ HS mult when R is std graded

Prop: Suppose R is one dimensional graded domain, I homo, $\lambda(R/I) < \infty$.

Then $$F(R, R, I) = e_R \cdot \left(\frac{1 - e^{-iyh}}{iy} \right)$$

$$h = \min \{ j \mid I_j \neq 0 \}.$$

Question. Are Frob. Poincare function always of the form

$$\frac{Q(e^{-iyr})}{(iy)^{\dim(R)}}, \text{ for some } r \in \mathbb{R}\,?$$

for f.g graded R-modules M, N

$$\chi^R(M,N)(t) = \sum_{j \in \mathbb{N}} (-1)^j H_{\mathrm{Tor}_j(M,N)}(t)$$

<u>Thm</u>: (R, I) as before. Let $S \hookrightarrow R$ be a graded Noeth normalization.

<u>Then</u> $F(R,R,I)(y) = \dfrac{\lim_{n\to\infty} \chi^S\left(\frac{R}{I^{[p^n]}}, k\right)\left(e^{-iy/p^n}\right)}{(iy)^{\dim(R)}}$

Thm: (R, I) as before. Let $S \hookrightarrow R$ be a graded Noeth normalization.

Then

$$F(R, R, I)(y) = \frac{\lim_{n \to \infty} \chi^S\left(\frac{R}{I^{[p^n]}}, k\right)\left(e^{-iy/p^n}\right)}{(iy)^{\dim(R)}}$$

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\oplus b_{d,j}} \to \cdots \to \bigoplus S(-j)^{\oplus b_{0,j}} \to R/I^{[p^n]} \to 0$$

$$\mathbb{B}^S(j, n) = \sum_{\alpha=0}^{d} (-1)^{\alpha} b_{\alpha,j}$$

Then $\chi^S\left(R/I^{[p^n]}, k\right)\left(e^{-iy/p^n}\right)$

$$= \sum_{j \in \mathbb{N}} \mathbb{B}^S(j, n)\left(e^{-iy/p^n}\right)^j$$

Consider $R/I^{[p^n]}$ as S-modules.

<u>Thm</u> (Miller, Rahmati, Rebecca R.G.).

fix an odd prime p, d = even integer.

$$\underset{\substack{\cup\\ \text{collection of deg } d \text{ pol}}}{S_d} \subseteq k[x,y,z] = A$$

The A module resolution of $\dfrac{A}{(f, x^{p^n}, y^{p^n}, z^{p^n})}$ for very general $f \in S_d$ and $n >> 0$ is as follows

$$0 \to A^{\oplus 2d}\left(-\tfrac{3}{2}p^n - \tfrac{1}{2}d - \tfrac{1}{2}\right) \to \begin{matrix} A^{\oplus 2d}\left(-\tfrac{3}{2}p^n - \tfrac{1}{2}d + \tfrac{1}{2}\right) \\ \oplus \\ A^{\oplus 3}(-p^n - d) \end{matrix} \to \begin{matrix} A^{\oplus 3}(-p^n) \\ \oplus \\ A(-d) \end{matrix} \to A \longrightarrow 0$$

Question: (R, I) as before

$S \hookrightarrow R$ graded Noether normalisation.

$$\overset{?}{\leadsto} \quad \lim_{n \to \infty} \chi^S\left(R/I^{[p^n]}, k \right)\left(e^{-iy/p^n}\right) \Big|_{\mathbb{R}}$$

bounded ?