A stable version of Harbourne's Conjecture

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Symbolic Power

The n-th symbolic power of an ideal I in R is given by

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}(R/I)} I^n R_P \cap R.$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) If I is generated by a regular sequence, then $I^n = I^{(n)}$.

(3) In general, $I^n \neq I^{(n)}$.

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Containment Problem

When is $I^{(b)} \subseteq I^a$?

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a regular ring, R, and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$,

$$I^{(hn)} \subseteq I^n.$$

Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

In fact, $P^{(3)} \subseteq P^2$.

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Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

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Counterexample (Dumnicki, Szemberg, Tutaj-Gasińska, 2015)

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \nsubseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

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When does Harbourne's Conjecture hold?

- For monomial ideals.
- For general points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).
- For star configurations (Harbourne–Huneke).

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When does Harbourne's Conjecture hold?

- In characteristic p, if R/I defines an F-pure ring. (G–Huneke) Determinantal rings, Veronese rings, nice rings of invariants.
- For general points in $\mathbb{P}^2$ and $\mathbb{P}^3$.
- For star configurations (Harbourne–Huneke).

Evidence for the Stable Harbourne Conjecture

Let $a \geqslant 3$, k be a field, and

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) .$$

This is a well-known counter-example to $I^{(3)} \subseteq I^2$. However, work of Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska shows that

$$I^{(2n-1)} \subseteq I^n$$

for all $n \geqslant 3$.

Stable Harbourne Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \gg 0$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Question

If there exists a value of n such that

$$I^{(hn-h+1)} \subseteq I^n,$$

does that imply that

$$I^{(hm-h+1)} \subseteq I^m,$$

for all $m \gg 0$?

Theorem (–)

Let I be a radical ideal in a regular ring containing a field, and let h be the maximal height of a minimal prime of I. If there exists a value of n such that

$$I^{(hn-h)} \subseteq I^n,$$

then

$$I^{(hm-h)} \subseteq I^m,$$

for all $m \gg 0$.

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Let I be a radical ideal in a regular ring containing a field, and let h be the maximal height of a minimal prime of I. If there exists a value of n such that

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for all $m \gg 0$.

Example

The defining ideal of $k[t^3, t^4, t^5]$ in $k[x, y, z]$ verifies $P^{(2\times 3-2=4)} \subseteq P^3$, and thus $P^{(2m-2)} \subseteq P^m$ for all $m \gg 0$.

Definition (Bocci-Harbourne)

The resurgence of an ideal I is given by

$$\rho(I) = \sup \left\{ \frac{a}{b} : I^{(a)} \not\subseteq I^b \right\}.$$

$$1 \leqslant \rho(I) \leqslant h.$$

If $\frac{a}{b} > \rho(I)$, then $I^{(a)} \subseteq I^b$.

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$$1 \leqslant \rho(I) \leqslant h.$$

$$\text{If } \tfrac{a}{b} > \rho(I), \text{ then } I^{(a)} \subseteq I^b.$$

Observation

Let I is a radical ideal, and h be the maximal height of a minimal prime of I. If $\rho(I) < h$, then for every constant $C > 0$,

$$I^{(hn-C)} \subseteq I^n$$

for all $n \gg 0$.

Question

Is there an ideal I with $\rho(I) = h$?

Observation

Let I is a radical ideal, and h be the maximal height of a minimal prime of I. If $\rho(I) < h$, then for every constant $C > 0$,

$$I^{(hn-C)} \subseteq I^n$$

for all $n \gg 0$.

Question

Is there an ideal I with $\rho(I) = h$?

Example

Let $a \geqslant 3$, k be a field, and Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska showed that

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)).$$

has resurgence $\frac{3}{2}$, so $I^{(2n-1)} \subseteq I^n$ for all $n \geqslant 3$.

Harbourne's Conjecture for primes

Let P be a prime ideal in a regular ring of height h. For all $n \geqslant 1$,

$$P^{(hn-h+1)} \subseteq P^n.$$

Harbourne's Conjecture for primes of height 2

Let P be a prime ideal of height 2 in a regular ring. For all $n \geqslant 1$,

$$P^{(2n-1)} \subseteq P^n.$$

First step

If P is a prime of height 2, is $P^{(3)} \subseteq P^2$?

Monomial space curves

Let k be a field. The kernel of the map

$$k[x, y, z] \longrightarrow k[t^a, t^b, t^c] \subseteq k[t]$$

is a prime ideal of height 2, generated by the maximal minors of

$$\begin{bmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{bmatrix}.$$

Theorem (–)

Let k be a field of characteristic not 3, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2.$$

Theorem (–)

Let k be a field of characteristic not 2, 3 or 5, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2.$$

Moreover,

$$P^{(2\times 3-1=5)} \subseteq P^3.$$

Theorem (–)

Let k be a field of characteristic not 2, 3 or 5, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2 \text{ and } P^{(5)} \subseteq P^3.$$

Ingredients in the proof

- $I^{(a)} \subseteq I^b$ if and only if the map induced by $I^a \subseteq I^b$ on *Ext* $\mathrm{Ext}^2\left(R/I^b, R\right) \longrightarrow \mathrm{Ext}^2\left(R/I^a, R\right)$, is the 0 map.
- Use Rees Algebra techniques to find the resolutions of I^n.

Theorem (–)

Let k be a field of characteristic not 2 nor 3, let a, b and c be integers, and let P be the defining ideal of $k[t^4, t^b, t^c]$. Then

$$P^{(4)} \subseteq P^3.$$

As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The defining ideal P of $k[t^9, t^{11}, t^{14}]$ fails the containment $P^{(4)} \subseteq P^3$, but Macaulay2 computations show that

$$P^{(2\times 4-2=6)} \subseteq P^4,$$

so $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Thank you!