Homological Algebra

Math 915 Fall 2023

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Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, I will be most grateful to you for letting me know. If you are looking for a place where to learn homological algebra or category theory, I strongly recommend the following excellent resources:

- Rotman's *An introduction to homological algebra*, second edition. [Rot09]
- Weibel's *Homological Algebra* [Wei94].
- Mac Lane's *Categories for the working mathematician* [ML98].
- Emily Riehl's *Category Theory in context*

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Chapter 0

Where are we going?

Homological algebra first appeared in the study of topological spaces. Roughly speaking, homology is a way of associating a sequence of abelian groups (or modules, or other more sophisticated algebraic objects) to another object, for example a topological space. The homology of a topological space encodes topological information about the space in algebraic language — this is what algebraic topology is all about.

More formally, we will study *complexes* and their homology from a more abstract perspective. While algebraic topologists are often concerned with complexes of abelian groups, we will work a bit more generally with complexes of R-modules. The basic assumptions and notation about rings and modules we will use in this class can be found in Appendix A. As an appetizer, we begin with some basic homological algebra definitions.

Definition 0.1. A **chain complex** of R-modules $(C_\bullet, \delta_\bullet)$, also referred to simply as a **complex**, is a sequence of R-modules C_i and R-module homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots$$

such that $\delta_n \delta_{n+1} = 0$ for all n. We refer to C_n as the module in **homological degree** n. The maps δ_n are the **differentials** of our complex. We may sometimes omit the differentials δ_n and simply refer to the complex $C_\bullet$ or even C; we may also sometimes refer to $\delta_\bullet$ as *the* differential of $C_\bullet$.

In some contexts, it is important to make a distinction between chain complexes and co-chain complexes, where the arrows go the opposite way: a co-chain complex would look like

$$\cdots \longrightarrow C_{n-1} \xrightarrow{\delta_n} C_n \xrightarrow{\delta_{n+1}} C_{n+1} \longrightarrow \cdots .$$

We will not need to make such a distinction, so we will call both of these complexes and most often follow the convention in the definition above. We will say a complex is **bounded above** if $F_n = 0$ for all $n \gg 0$, and **bounded below** if $F_n = 0$ for all $n \ll 0$. A **bounded complex** is one that is both bounded above and below. If a complex is bounded, we may sometimes simply write it as a finite complex, say

$$C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_m.$$

Remark 0.2. The condition that $\delta_n\delta_{n+1} = 0$ for all n implies that $\operatorname{im}\delta_{n+1} \subseteq \ker\delta_n$.

Definition 0.3. The complex $(C_\bullet, \delta_\bullet)$ is **exact** at n if $\operatorname{im}\delta_{n+1} = \ker\delta_n$. An **exact sequence** is a complex that is exact everywhere. More precisely, an **exact sequence** of R-modules is a sequence

$$\cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \cdots$$

of R-modules and R-module homomorphisms such that $\operatorname{im} f_n = \ker f_{n+1}$ for all n. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a **short exact sequence**, sometimes written **ses**.

Remark 0.4. The sequence

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact if and only if f is injective. Similarly,

$$M \xrightarrow{f} N \longrightarrow 0$$

is exact if and only if f is surjective. So

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence if and only if

- f is injective
- g is surjective
- $\operatorname{im} f = \ker g$.

Requiring that f be injective is the same as asking that $\ker f = 0$, while g is surjective if and only if $\operatorname{coker} g = 0$.[1] When this is indeed a short exact sequence, we can identify A with its image $f(A)$, and $A = \ker g$. Moreover, since g is surjective, by the First Isomorphism Theorem we conclude that $C \cong B/A$, so we might abuse notation and identify C with B/A.

Example 0.5. Let π be the canonical projection $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. The following is a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0\ .$$

We will most often be interested in **complexes of R-modules**, where the abelian groups that show up are all modules over the same ring R.

Example 0.6. Let $R = k[x]$ be a polynomial ring over the field k. Then following is a short exact sequence:

$$0 \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\pi} R/(x) \longrightarrow 0\ .$$

The first map is multiplication by x, and the second map is the canonical projection.

[1]The **cokernel** of a homomorphism $f\colon M \longrightarrow N$ is the R-module $N/\operatorname{im} f$.

Example 0.7. Given an ideal I in a ring R, the inclusion map $\iota : I \to R$ and the canonical projection $\pi : R \to R/I$ give us the following short exact sequence:

$$0 \longrightarrow I \xrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0.$$

Example 0.8. Let $R = k[x]/(x^2)$. The following complex is exact:

$$\cdots \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \longrightarrow \cdots.$$

Indeed, the image and the kernel of multiplication by x are both (x).

Sometimes we can show that certain modules vanish or compute them explicitly when they do not vanish by seeing that they fit in some naturally constructed exact sequence involving other modules we understand better. We will discuss this in more detail when we talk about long exact sequences.

Remark 0.9. The complex $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an isomorphism.

Remark 0.10. The complex $0 \longrightarrow M \longrightarrow 0$ is exact if and only if $M = 0$.

Historically, chain complexes first appeared in topology. To study a topological space, one constructs a particular chain complex that arises naturally from information from the space, and then calculates its homology, which ends up encoding important topological information in the form of a sequence of abelian groups.

Definition 0.11 (Homology)**.** The **homology** of the complex $(C_\bullet, \delta_\bullet)$ is the sequence of R-modules

$$\mathrm{H}_n(C_\bullet) = \mathrm{H}_n(C) := \frac{\ker \delta_n}{\operatorname{im} \delta_{n+1}}.$$

The n**th homology** of $(C_\bullet, \delta_\bullet)$ is $\mathrm{H}_n(C)$. The submodules $Z_n(C_\bullet) = Z_n(C) := \ker \delta_n \subseteq C_n$ are called **cycles**, while the submodules $B_n(C_\bullet) = B_n(C) := \operatorname{im} \delta_{n+1} \subseteq C_n$ are called **boundaries**. One sometimes uses the word boundary to refer an element of $B_n(C)$ (an n-boundary), and the word cycle to refer to an element of $Z_n(C)$ (an n-cycle).

The homology of a complex measures how far our complex is from being exact at each point. Again, we can talk about the **cohomology** of a cochain complex instead, which we write as $\mathrm{H}^n(C)$; we will for now not worry about the distinction.

Remark 0.12. Note that $(C_\bullet, \delta_\bullet)$ is exact at n if and only if $\mathrm{H}_n(C_\bullet) = 0$.

Example 0.13. Let $R = k[x]/(x^3)$. Consider the following complex:

$$F_\bullet = \ \cdots \longrightarrow R \xrightarrow{\cdot x^2} R \xrightarrow{\cdot x^2} R \longrightarrow \cdots.$$

The image of multiplication by x^2 is (x^2), while the the kernel of multiplication by x^2 is $(x) \supseteq (x^2)$. For all n,

$$\mathrm{H}_n(F_\bullet) = (x)/(x^2) \cong R/(x).$$

Example 0.14. Let $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$ be the canonical projection map. Then

$$C = \underset{2}{\mathbb{Z}} \xrightarrow{4} \underset{1}{\mathbb{Z}} \xrightarrow{\pi} \underset{0}{\mathbb{Z}/2\mathbb{Z}}$$

is a complex, since the image of multiplication by 4 is $4\mathbb{Z}$, and that is certainly contained in $\ker \pi = 2\mathbb{Z}$. The homology of C is

$$\begin{aligned}
\mathrm{H}_n(C) &= 0 && \text{for } n \geqslant 3 \\
\mathrm{H}_2(C) &= \frac{\ker(\mathbb{Z} \xrightarrow{4} \mathbb{Z})}{\mathrm{im}(0 \longrightarrow \mathbb{Z})} = \frac{0}{0} = 0 \\
\mathrm{H}_1(C) &= \frac{\ker(\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z})}{\mathrm{im}(\mathbb{Z} \xrightarrow{4} \mathbb{Z})} = \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \\
\mathrm{H}_0(C) &= \frac{\ker(\mathbb{Z}/2\mathbb{Z} \longrightarrow 0)}{\mathrm{im}(\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z})} = \frac{\mathbb{Z}/2\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} = 0 \\
\mathrm{H}_n(C) &= 0 && \text{for } n < 0
\end{aligned}$$

Notice that our complex is exact at 2 and 0. The exactness at 2 says that the map $\mathbb{Z} \xrightarrow{4} \mathbb{Z}$ is injective, while exactness at 0 says that π is surjective.

Before we can continue any further into the world of homological algebra, we will need some categorical language. We will take a short break to introduce category theory, and then armed with that knowledge we will be ready to study homological algebra.

Chapter 1

Categories for the working homological algebraist

Most fields in modern mathematics follow the same basic recipe: there is a main type of object one wants to study – groups, rings, modules, topological spaces, etc – and a natural notion of arrows between these – group homomorphisms, ring homomorphisms, module homomorphisms, continuous maps, etc. The objects are often sets with some extra structure, and the arrows are often maps between the objects that preserve whatever that extra structure is. Category theory is born of this realization, by abstracting the basic notions that make math and studying them all at the same time. How many times have we felt a sense of déjà vu when learning about a new field of math? Category theory unifies all those ideas we have over and over in different contexts.

Category theory is an entire field of mathematics in its own right. As such, there is a lot to say about category theory, and unfortunately it doesn't all fit in the little time we have to cover it in this course. We include here some basic definitions and ideas from category theory we will need throughout the course, but you are strongly encouraged to learn more about category theory, for example from [ML98] or [Rie17].

First, we want to note that there is a long and fun story about why we used the word *c*ollection when describing the objects in a category. Not all collections are allowed to be sets, an issue that was first discovered by Russel with his famous Russel's Paradox. Russel exposed the fact that one has to be careful with how we formalize set theory. We follow the ZFC (Zermelo–Fraenkel with choice, short for the Zermelo–Fraenkel axioms plus the Axiom of Choice) axiomatization of set theory, and while we will not discuss the details of this formalization here, you are encouraged to read more on the subject.

1.1 Categories

A category consists of a collection of objects and arrows or morphisms between those objects. While these are often sets and some kind of functions between them, beware that this will not always be the case. We will use the words morphism and arrows interchangeably, though *arrow* has the advantage of reminding us we are not necessarily talking about functions.

Definition 1.1. A **category** $\mathscr{C}$ consists of three different pieces of data:

- a collection of **objects**, $\mathbf{ob}(\mathscr{C})$,
- for each two objects, say A and B, a collection $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of **arrows** or **morphisms** between A and B, and
- for each three objects A, B, and C, a composition

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)\ . \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

We will often drop the $\circ$ and write simply fg for $f \circ g$.

These ingredients satisfy the following axioms:

1) The $\mathrm{Hom}_{\mathscr{C}}(A, B)$ are all disjoint. In particular, if f is an arrow in $\mathscr{C}$, we can talk about its **source** A and its **target** B as the objects such that $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

2) For each object A, there is an **identity arrow** $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and all $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

3) Composition is associative, meaning $f \circ (g \circ h) = (f \circ g) \circ h$ for all appropriately chosen arrows.

Exercise 1. Prove that every element in a category has a unique identity morphism.

Here are some categories you have likely encountered before:

Example 1.2.

1) The category **Set** with objects all sets and arrows all functions between sets.

2) The category **Grp** whose objects are the collection of all groups, and whose arrows are all the homomorphisms of groups. The identity arrows are the identity homomorphisms.

3) The category **Ab** whose objects are the collection of all abelian groups, and whose arrows are the homomorphisms of abelian groups. The identity arrows are the identity homomorphisms.

4) The category **Ring** of rings and ring homomorphisms. Contrary to what you may expect, this is not nearly as important as the next one.

5) The category R-**mod** of modules over a fixed ring R and with R-module homomorphisms. Sometimes one writes R-**Mod** for this category, and reserve R-**mod** for the category of finitely generated R-modules with R-module homomorphisms. When $R = k$ is a field, the objects in the category k-**mod** are k-vector spaces, and the arrows are linear transformations; we may instead refer to this category as **Vect**-k.

6) The category **Top** of topological spaces and continuous functions.

One may consider many variations of the categories above. Here are some variations on vector spaces:

Example 1.3. Let k be a field.

1) The collection of finite dimensional k-vector spaces with all linear transformations is a category.

2) The collection of all n-dimensional K-vector spaces with all linear transformations is a category.

3) The collection of all K-vector spaces (or n-dimensional vector spaces) with linear isomorphisms is a category.

4) The collection of all k-vector spaces (or n-dimensional vector spaces) with nonzero linear transformations is not a category, since it is not closed under composition.

5) The collection of all n-dimensional vector spaces with linear transformations of determinant 0 is not a category, since it does not have identity maps.

Here is an important variation of **Set**:

Example 1.4. There is a category $\mathbf{Set}^*$ of pointed sets, where the objects are pairs (X, x) of sets X and points $x \in X$, and where for two pointed sets (X, x) and (Y, y), the morphisms from (X, x) to (Y, y) are functions $f \colon X \to Y$ such that $f(x) = y$, with the usual composition of functions.

While the collections of objects and arrows might not actually be sets, sometimes they are.

Definition 1.5. A category $\mathscr{C}$ is **locally small** if for all objects A and B in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set. A category $\mathscr{C}$ is **small** if it is locally small and the collection of all objects in $\mathscr{C}$ is a set.

In fact, one can define locally small category as one where the collection of all arrows is a set. It follows immediately that the collection of all objects is also a set, since it must be a subset of the set of arrows – for each object, there is an identity arrow.

Many important categories are at least locally small. For example, **Set** is locally small but not small. In a locally small category, we can now refer to its Hom-sets.

Categories where the objects are sets with some extra structure and the arrows are some kind of functions between the objects are called **concrete**. Not all categories are concrete.

Example 1.6. Given a partially ordered set $(X, \leqslant)$, we can regard X itself as a category: the objects are the elements of X, and for each x and y in X, $\mathrm{Hom}_X(x, y)$ is either a singleton if $x \leqslant y$ or empty if $x \nleqslant y$. There is only one possible way to define composition, and the transitive property of $\leqslant$ guarantees that the composition of arrows is indeed well-defined: if there is an arrow $i \to j$ and an arrow $j \to k$, then $i \leqslant j$ and $j \leqslant k$, so $i \leqslant k$ and thus there is a unique arrow $i \to k$. This category is clearly locally small, since all nonempty Hom-sets are in fact singletons.

Example 1.7. For each positive integer n, the category $\mathbf{n}$ has n objects $0, 1, \ldots, n-1$ and $\mathrm{Hom}(i, j)$ is either empty if $i > j$ or a singleton if $i \leqslant j$. As Example 1.6, composition is defined in the only way possible, and things work out.

Example 1.8. Fix a field k. We define a category **Mat-**k with objects all positive integers, and given two positive integers a and b, the Hom-set $\mathrm{Hom}(a, b)$ consists of all $b \times a$ matrices with entries in k. The composition rule is given by product of matrices: given $A \in \mathrm{Hom}(a, b)$ and $B \in \mathrm{Hom}(b, c)$, the composition $B \circ A$ is the matrix $BA \in \mathrm{Hom}(a, c)$. For each object a, its identity arrow is given by the $a \times a$ identity matrix.

Example 1.9. Let G be a directed graph. We can construct a category from G as follows: the objects are the vertices of G, and the arrows are directed paths in the graph G. In this category, composition of arrows corresponds to concatenation of paths. For each object A, the identity arrow corresponds to the empty path from A to A.

Remark 1.10. A locally small category with just one element is completely determined by its unique Hom-set; it thus consists of a set S with an associative operation that has an identity element, which in this class is what we call a **semigroup**.[1]

A key insights that category theory brings is that many important concepts can be understood through diagrams.

Definition 1.11. A **diagram** in a category $\mathscr{C}$ is a directed multigraph whose vertices are objects in C and whose arrows/edges are morphisms in $\mathscr{C}$. A commutative diagram in $\mathscr{C}$ is a diagram in which for each pair of vertices A and B, any two paths from A to B compose to the same morphism.

Another way to formalize what a diagram is involves talking about functors, which we will discuss in the next section. Homological algebra is in many ways the study of commutative diagrams.

Example 1.12. The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

commutes if and only if $gf = vu$.

There are some special types of arrows we will want to consider.

Definition 1.13. Let $\mathscr{C}$ be any category.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is **left invertible** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $gf = 1_A$. In this case, we say that g is the **left inverse** of f. So g is a left inverse of f if the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd}$$

  commutes.

[1] Some authors prefer the term monoid.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is **right invertible** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $fg = 1_B$. In this case, we say that g is the **right inverse** of f. So g is a right inverse of f if the diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

  commutes.

- An arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is an **isomorphism** if there exists $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$. Unsurprisingly, such an arrow g is called the **inverse** of f.

- An arrow $f \in \mathrm{Hom}(B, C)$ is **monic**, a **monomorphism**, or a **mono** if for all arrows

$$\begin{tikzcd} A \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

  if $fg_1 = fg_2$ then $g_1 = g_2$.

- Similarly, an arrow $f \in \mathrm{Hom}(A, B)$ is an **epi** or an **epimorphism** if for all arrows

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

  if $g_1 f = g_2 f$ then $g_1 = g_2$.

Here are some examples:

Exercise 2. Show that in **Set**, the monos coincide with the injective functions and the epis coincide with the surjective functions.

Example 1.14.

1. In **Grp**, **Ring**, and R-**Mod** the isomorphisms are the morphisms that are bijective functions.

2. In contrast, in **Top** the isomorphisms are the homeomorphisms, which are the bijective continuous functions with continuous inverses. These are *not* the same thing as just the bijective continuous functions.

Exercise 3. Show that in any category, every isomorphism is both epi and mono.

Exercise 4. Show that the usual inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epi in the category **Ring**.

This *should* feel weird: it says being epi and being surjective are *not* the same thing. Similarly, being monic and being injective are *not* the same thing.

Exercise 5. Show that the canonical projection $\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a mono in the category of divisible abelian groups.[2]

[2]An abelian group A is divisible if for every $a \in A$ and every positive integer n there exists $b \in A$ such that $nb = a$.

Exercise 6. Show that given any poset P, in the poset category of P every morphism is both monic and epic, but no nonidentity morphism has a left or right inverse.

There are some special types of objects we will want to consider.

Definition 1.15. Let $\mathscr{C}$ be a category. An **initial object** in $\mathscr{C}$ is an object i such that for every object x in $\mathscr{C}$, $\operatorname{Hom}_{\mathscr{C}}(i, x)$ is a singleton, meaning there exists a unique arrow $i \longrightarrow x$. A **terminal object** in $\mathscr{C}$ is an object t such that for every object x in $\mathscr{C}$, $\operatorname{Hom}_{\mathscr{C}}(x, t)$ is a singleton, meaning there exists a unique arrow $x \longrightarrow t$. A **zero object** in $\mathscr{C}$ is an object that is both initial and terminal.

Exercise 7. Initial objects are unique up to unique isomorphism. Terminal objects are unique up to unique isomorphism.

So we can talk about *the* initial object, *the* terminal object, and *the* zero object, if they exist.

Example 1.16.

a) The empty set if initial in **Set**. Any singleton is terminal. Since the empty set and a singleton are not isomorphic in **Set**, there is no zero object in **Set**.

b) The 0 module is the zero object in R-**mod**.

c) The trivial group $\{e\}$ is the zero object in **Grp**.

d) In the category of rings, $\mathbb{Z}$ is the initial object, but there is no terminal object unless we allow the 0 ring.

e) There are no initial nor terminal objects in the category of fields.

We will now continue to follow a familiar pattern and define the related concepts one can guess should be defined.

Definition 1.17. A **subcategory** $\mathscr{C}$ of a category $\mathscr{D}$ consists of a subcollection of the objects of $\mathscr{D}$ and a subcollection of the morphisms of $\mathscr{D}$ such that the following hold:

- For every object C in $\mathscr{C}$, the arrow $1_C \in \operatorname{Hom}_{\mathscr{D}}(C, C)$ is an arrow in $\mathscr{C}$.
- For every arrow in $\mathscr{C}$, its source and target in $\mathscr{D}$ are objects in $\mathscr{C}$.
- For every pair of arrows f and g in $\mathscr{C}$ such that fg is an arrow that makes sense in $\mathscr{D}$, fg is an arrow in $\mathscr{C}$.

In particular, $\mathscr{C}$ is a category in its own right.

Example 1.18. The category of finitely generated R-modules with R-module homomorphisms is a subcategory of R-**mod**.

Definition 1.19. A subcategory $\mathscr{C}$ of $\mathscr{D}$ is a **full subcategory** if $\mathscr{C}$ includes *all* of the arrows in $\mathscr{D}$ between any two objects in $\mathscr{C}$.

Example 1.20.

a) The category **Ab** of abelian groups is a full subcategory of **Grp**.

b) Since every group is a set, and every homomorphism is a function, **Grp** is a subcategory of **Set**. However, not every function between two groups is a group homomorphism, so **Grp** is not a full subcategory of **Set**.

c) The category whose objects are all sets and with arrows all bijections is a subcategory of **Set** that is not full.

Here is another way of constructing a new category out of an old one.

Definition 1.21. Let $\mathscr{C}$ be a category. The **opposite category** of $\mathscr{C}$, denoted $\mathscr{C}^{\mathrm{op}}$, is a category whose objects are the objects of $\mathscr{C}$, and such that each arrow $f \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B)$ is the same as some arrow in $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B)$. The composition of two morphisms fg in $\mathscr{C}^{\mathrm{op}}$ is defined as the composition gf in $\mathscr{C}$.

Many objects and concepts one might want to describe are obtained from existing ones by flipping the arrows. Opposite categories give us the formal framework to talk about such things. We will often want to refer to **dual** notions, which will essentially mean considering the same notion in a category $\mathscr{C}$ and in the opposite category $\mathscr{C}^{\mathrm{op}}$; in practice, this means we should flip all the arrows involved. We will see examples of this later on.

The dual category construction gives us a formal framework to talk about **dual notions**. We will often make a statement in a category $\mathscr{C}$ and make comments about the **dual statement**; in practice, this corresponds to simply switching the way all arrows go. Here are some examples of dual notions and statements:

source	target
epi	mono
g is a right inverse for f	g is a left inverse for f
f is invertible	f is invertible
initial objects	terminal objects
homology	cohomology

Note that the dual of the dual is the original statement; we can make this more formal by saying that $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$. Sometimes we can easily prove a statement by dualizing; however, this is not always straightforward, and one needs to carefully dualize all portions of the statement in question. Nevertheless, Sanders MacLane, one of the fathers of category theory, wrote that "If any statement about a category is deducible from the axioms for a category, the dual statement is likely deducible" [Mac50]. One of the upshots of duality is that any theorem in category theory must simultaneously prove two theorems: the original statement and its dual. But for this to hold, we need proofs that use the abstraction of a purely categorical proof.

Opposite categories are more interesting than they might appear at first; there is more than just flipping all the arrows. For example, consider the opposite category of **Set**. For any nonempty set X, there is a unique morphism in **Set** (a function) $i : \emptyset \to X$, but there are no functions $X \to \emptyset$, so $i^{\mathrm{op}} : X \to \emptyset$ is not a function. Thus thinking about $\mathbf{Set}^{\mathbf{op}}$ is a bit difficult. One can show that this is the category of complete atomic Boolean algebras – but we won't concern ourselves with what that means.

1.2 Functors

Many mathematical constructions are *functorial*, in the sense that they behave well with respect to morphisms. In the formalism of category theory, this means that we can think of a functorial construction as a functor.

Definition 1.22. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a mapping that assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$, such that

- F preserves the composition of maps, meaning $F(fg) = F(f)F(g)$ for all arrows f and g in $\mathscr{C}$, and
- F preserves the identity arrows, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

A **contravariant functor** $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a mapping that assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(B), F(A))$, such that

- F preserves the composition of maps, meaning $F(fg) = F(g)F(f)$ for all composable arrows f and g in $\mathscr{C}$, and
- F preserves the identity arrows, meaning $F(1_A) = 1_{F(A)}$ for all objects A in $\mathscr{C}$.

So a contravariant functor is a functor that flips all the arrows. We can also describe a contravariant functor as a covariant functor from $\mathscr{C}$ to the opposite category of $\mathscr{D}$, $\mathscr{D}^{\mathrm{op}}$.

Remark 1.23. A contravariant functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ can be thought of as a covariant functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$, or also as a covariant functor $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$. If using one of these conventions, one needs to be careful, however, when composing functors, so that the respective sources and targets match up correctly. While we haven't specially discussed how one composes functors, it should be clear that applying a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$ is the same as applying a functor $\mathscr{C} \longrightarrow \mathscr{D}$, which we can write as GF.

For example, if $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$ are both contravariant functors, the composition $GF\colon \mathscr{C} \longrightarrow \mathscr{E}$ is a covariant functor, since

$$\begin{tikzcd} A \arrow[d, "f"'] & \rightsquigarrow & F(A) & \rightsquigarrow & GF(A) \arrow[d, "GF(f)"'] \\ B & & F(B) \arrow[u, "F(f)"] & & GF(B) \end{tikzcd}$$

So we could think of F as a covariant functor $\mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$ and G as a covariant functor $\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{E}$. Similarly, if $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a covariant functor and $G : \mathscr{D} \longrightarrow \mathscr{E}$ is a contravariant functor, $GF\colon \mathscr{C} \longrightarrow \mathscr{E}$ is a contravariant functor. In this case, we can think of G as a covariant functor $\mathscr{D} \longrightarrow \mathscr{E}^{\mathrm{op}}$, so that GF is now a covariant functor $\mathscr{C} \longrightarrow \mathscr{E}^{\mathrm{op}}$.

Exercise 8. Show that functors preserve isomorphisms.

Example 1.24. Here are some examples of functors you may have encountered before.

a) Many categories one may think about are concrete categories, where the objects are sets with some extra structure, and the arrows are functions between those sets that preserved that extra structure. The **forgetful functor** from such a category to **Set** is the functor that, just as the name says, *forgets* that extra structure, and sees only the underlying sets and functions of sets. For example, the forgetful functor $\mathbf{Gr} \longrightarrow \mathbf{Set}$ sends each group to its underlying set, and each group homomorphism to the corresponding function of sets.

b) The identity functor on any category $\mathscr{C}$ does what the name suggests: it sends each object to itself and each arrow to itself.

c) Given a group G, the subgroup $[G, G]$ of G generated by the set of commutators

$$\{ghg^{-1}h^{-1} \mid g, h \in G\}$$

is a normal subgroup, and the quotient $G^{\mathrm{ab}} := G/[G, G]$ is called the **abelianization** of G. The group G^{ab} is abelian. Given a group homomorphism $f \colon G \to H$, f automatically takes commutators to commutators, so it induces a homomorphism $\tilde{f} \colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. More precisely, abelianization gives a covariant functor from **Grp** to **Ab**.

d) The unit group functor $-^* : \mathbf{Ring} \ \to \mathbf{Grp}$ sends a ring R to its group of units R^*. To see this is indeed a functor, we should check it behaves well on morphisms; and indeed if $f \colon R \to S$ is a ring homomorphism, and $u \in R^*$ is a unit in R, then

$$f(u)f(u^{-1}) = f(uu^{-1}) = f(1_R) = 1_S,$$

so $f(u)$ is a unit in S. Thus f induces a function $R^* \to S^*$ given by restriction of f to R^*, which must therefore be a group homomorphism since f preserves products.

e) Fix a field k. Given a vector space V , the collection V^* of linear transformations from V to k is again a k-vector space, the **dual vector space** of V. If $\varphi \colon W \to V$ is a linear transformation and $\ell \colon V \to K$ is an element of V^*, then $\ell \circ \varphi \colon W \to k$ is in W^*. Doing this for all elements $\ell \in V^*$ gives a function $\varphi^* : V^* \to W^*$, and one can show that φ^* is a linear transformation. Finally, the assignment that sends each vector space V to its dual vector space V^* and each linear transformation φ to φ^* is a contravariant functor on **Vect-**k.

f) Localization is a functor. Let R be a ring and W be a multiplicatively closed set in R. There is localization at W induces a a functor R-mod $\longrightarrow W^{-1}R$-mod that sends each R-module M to $W^{-1}M$, and each R-module homomorphism $\alpha : M \longrightarrow N$ to the R-module homomorphism $W^{-1}\alpha : W^{-1}M \longrightarrow W^{-1}N$.

Remark 1.25. Any functor sends isomorphisms to isomorphisms, since it preserves compositions and identities.

Remark 1.26. If we apply a covariant functor to a diagram, then we get a diagram of the same shape:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\
C \arrow[r, "v"'] & D
\end{tikzcd}
\quad \overset{F}{\rightsquigarrow} \quad
\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "F(u)"'] & F(B) \arrow[d, "F(g)"] \\
F(C) \arrow[r, "F(v)"'] & F(D)
\end{tikzcd}$$

However, if we apply a contravariant functor to the same diagram, we get a similar diagram but with the arrows reversed:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C & D \arrow[l, "v"] \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F(A) & F(B) \arrow[l, "F(f)"'] \\ F(C) \arrow[u, "F(u)"] & F(D) \arrow[l, "F(v)"] \arrow[u, "F(g)"'] \end{tikzcd}$$

If we think about functors as functions between categories, it's natural to consider what would be the appropriate versions of the notions of injective or surjective.

Definition 1.27. A covariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ between locally small categories is

- **faithful** if all the functions of sets
$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(A, B) &\longrightarrow \operatorname{Hom}_{\mathscr{D}}(F(A), F(B)) \\ f &\longmapsto F(f) \end{aligned}$$
are injective.

- **full** if all the functions of sets
$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(A, B) &\longrightarrow \operatorname{Hom}_{\mathscr{D}}(F(A), F(B)) \\ f &\longmapsto F(f) \end{aligned}$$
are surjective.

- **fully faithful** if it is full and faithful.

- an **embedding** if it is fully faithful and injective on objects.

Example 1.28. The forgetful functor R-**mod** $\longrightarrow$ **Set** is faithful since any two maps of R-modules with the same source and target coincide if and only if they are the same function of sets. This functor is not full, since there not every functions between the underlying sets of two R-modules is an R-module homomorphism.

Remark 1.29. A fully faithful functor is not necessarily injective on objects, but it is injective on objects up to isomorphism.

Remark 1.30. A subcategory $\mathscr{C}$ of $\mathscr{D}$ is full if the inclusion functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is full.

Example 1.31.

a) The category **Ab** of abelian groups is a full subcategory of **Grp**.

b) The category whose objects are all sets and with arrows all bijections is a subcategory of **Set** that is not full.

We close this section by introducing two of the most important functors we will discuss this semester:

Definition 1.32. Let $\mathscr{C}$ be a locally small category. An object A in $\mathscr{C}$ induces two Hom functors:

- The covariant functor $\mathrm{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} \longrightarrow \mathbf{Set}$ is defined as follows:

$$\mathscr{C} \longrightarrow \mathbf{Set}$$

on objects: $$X \longmapsto \mathrm{Hom}_{\mathscr{C}}(A,X)$$

on arrows:
$$\begin{tikzcd} B \arrow[d, "f"'] & & \mathrm{Hom}_{\mathscr{C}}(A,B) \arrow[d] & \ni g \arrow[d, mapsto] \\ C \arrow[r, phantom, "\rightsquigarrow"] & {} & \mathrm{Hom}_{\mathscr{C}}(A,C) & \ni f \circ g \end{tikzcd}$$

  We may refer to this functor as the covariant functor **represented by** A. Given an arrow f in $\mathscr{C}$, we write $f_* := \mathrm{Hom}_{\mathscr{C}}(A,f)$. It is easier to see what f_* does through the following commutative diagram:

$$f_* = \mathrm{Hom}_{\mathscr{C}}(A,f) : \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed, "f_*(g)=fg"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

- The contravariant functor $\mathrm{Hom}_{\mathscr{C}}(-,B) : \mathscr{C} \longrightarrow \mathbf{Set}$ is defined as follows:

$$\mathscr{C} \longrightarrow \mathbf{Set}$$

on objects: $$X \longmapsto \mathrm{Hom}_{\mathscr{C}}(X,B)$$

on arrows:
$$\begin{tikzcd} A \arrow[d, "f"'] & & \mathrm{Hom}_{\mathscr{C}}(A,B) & \ni g \circ f \\ C \arrow[r, phantom, "\rightsquigarrow"] & {} & \mathrm{Hom}_{\mathscr{C}}(C,B) \arrow[u] & \ni g \arrow[u, mapsto] \end{tikzcd}$$

  We may refer to this functor as the contravariant functor **represented by** B. Given an arrow f in $\mathscr{C}$, we write $f^* := \mathrm{Hom}_{\mathscr{C}}(A,-)$. It is easier to see what f^* does through the following commutative diagram:

$$f^* = \mathrm{Hom}_{\mathscr{C}}(f,B) : \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "f^*(g)=gf"'] & C \arrow[d, "g"] \\ & B \end{tikzcd}$$

Exercise 9. Check that $\mathrm{Hom}(A,-)$ and $\mathrm{Hom}(-,B)$ are indeed functors.

We will be particularly interested in the Hom-functors in the category R-**mod**, which we will study in detail in a later chapter.

1.3 Natural transformations

Definition 1.33. Let F and G be covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A **natural transformation** between F and G is a mapping that to each object A in $\mathscr{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes. A **natural isomorphism** is a natural transformation η where each η_A is an isomorphism. We sometimes write

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathscr{D} \end{tikzcd}$$

or simply $\eta \colon F \implies G$.

Definition 1.34. Let F and G be contravariant functors $\mathscr{C} \longrightarrow \mathscr{D}$. A **natural transformation** between F and G is a mapping that to each object A in $\mathscr{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] & G(A) \\ F(B) \arrow[r, "\eta_B"'] \arrow[u, "F(f)"] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

commutes.

Often, when studying a particular topic, we sometimes say a certain map is *natural* to mean that there is actually a natural transformation behind it.

Example 1.35. Recall the abelianization functor we discussed in Example 1.24. The abelianization comes equipped with a natural projection map $\pi_G : G \longrightarrow G^{\mathrm{ab}}$, the usual quotient map from G to a normal subgroup. Here we mean natural in two different ways: both that this is common sense map to consider, and that this is in fact coming from a natural transformation. What's happening behind the scenes is that abelianization is a functor $\mathrm{ab} \colon \mathbf{Grp} \longrightarrow \mathbf{Grp}$. On objects, the abelianizations functor is defined as $G \mapsto G^{\mathrm{ab}}$. Given an arrow, meaning a group homomorphism $G \xrightarrow{f} H$, one can check that $[G, G]$ is contained in the kernel of $\pi_H f$, so $\pi_H f$ factors through G^{ab}, and there exists a group homomorphism f^{ab} making the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d, "f"'] & G^{\mathrm{ab}} \arrow[d, dashed, "f^{\mathrm{ab}}"] \\ H \arrow[r, "\pi_H"'] & H^{\mathrm{ab}} \end{tikzcd}.$$

So the abelianization functor takes the arrow f to f^{ab}. The commutativity of the diagram above says that $\pi_{\_}$ is a natural transformation between the identity functor on **Grp** and the abelianization functor, which we can write more compactly as

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{ab}"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathbf{Grp} \end{tikzcd}.$$

Definition 1.36. Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be two functors between the categories $\mathscr{C}$ and $\mathscr{D}$. We write

$$\mathrm{Nat}(F, G) = \{\text{natural transformations } F \longrightarrow G\}.$$

Given two categories $\mathscr{C}$ and $\mathscr{D}$, one can build a **functor category**[3] with objects all covariant functors $\mathscr{C} \longrightarrow \mathscr{D}$, and arrows the corresponding natural transformations. This category is denoted $\mathscr{D}^{\mathscr{C}}$. Sometimes one writes $\mathrm{Hom}(F, G)$ for $\mathrm{Nat}(F, G)$, but we will avoid that, as it might make things even more confusing.

For the functor category to truly be a category, though, we need to know how to compose natural transformations.

Remark 1.37. Consider natural transformations

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathscr{D} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{D} \end{tikzcd}.$$

We can compose them for form a new natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta\varphi"] & \mathscr{D} \end{tikzcd}$$

We should think of this composition as happening *vertically*. For each object C in $\mathscr{C}$, $\eta\varphi$ sends C to the arrow $F(A) \xrightarrow{\varphi_A} G(A) \xrightarrow{\eta_A} H(A)$. This makes the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varphi_A"'] & F(B) \arrow[d, "\varphi_B"] \\ G(A) \arrow[r, "G(f)"] \arrow[d, "\eta_A"'] & G(B) \arrow[d, "\eta_B"] \\ H(A) \arrow[r, "H(f)"] & H(B) \end{tikzcd}$$

commute.

Exercise 10. Show that a natural transformation $\eta\colon \mathscr{C} \implies \mathscr{D}$ is a natural isomorphism if and only if there exists a natural transformation $\mu\colon \mathscr{D} \implies \mathscr{C}$ such that $\eta \circ \mu$ is the identity natural isomorphism on G and $\mu \circ \eta$ is the identity natural isomorphism on F.

[3] Yes, the madness is neverending.

Even though this is only a short introduction to category theory, we would be remiss not to mention the Yoneda Lemma, arguably the most important statement in category theory.

Theorem 1.38 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a locally small category, and fix an object A in $\mathscr{C}$. Let $F\colon \mathscr{C} \longrightarrow$ **Set** be a covariant functor. Then there is a bijection*

$$Nat(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \xrightarrow{\gamma} F(A)\ .$$

Moreover, this correspondence is natural in both A and F.

Proof. Let φ be a natural transformation in $\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F)$. The proof of Yoneda's Lemma is essentially the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(A,A) \arrow[rrr, "\mathrm{Hom}_{\mathscr{C}}(A,f)"] \arrow[ddd, "\varphi_A"'] & & & \mathrm{Hom}_{\mathscr{C}}(A,X) \arrow[ddd, "\varphi_X"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & (F(f))u = \varphi_X(f) & \\
F(A) \arrow[rrr, "F(f)"'] & & & F(X)
\end{tikzcd}$$

Our bijection will be defined by $\gamma(\varphi) := \varphi_A(1_A)$. We should first check that this makes sense: arrows in **Set** are just functions between sets, and so φ_A is a function of sets $\mathrm{Hom}_{\mathscr{C}}(A,A) \longrightarrow F(A)$. Also, $\mathrm{Hom}_{\mathscr{C}}(A,A)$ is a set that contains at least the element 1_A, and $\varphi_A(1_A)$ is some element in the set $F(A)$.

Given any fixed $f \in \mathrm{Hom}_{\mathscr{C}}(A,X)$, the fact that φ is a natural transformation translates into the outer commutative diagram. In particular, the maps of sets $F(f)\varphi_A$ and $\varphi_X \mathrm{Hom}_{\mathscr{C}}(A,f)$ coincide, and must in particular take 1_A to the same element in $F(X)$. This is the commutativity of the inner diagram, with $u := \varphi_A(1_A)$.

The commutativity of the diagram above says that φ is completely determined by $\varphi_A(1_A)$, since for any other object X in $\mathscr{C}$ and any arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A,X)$, we necessarily have $\varphi_X(f) = F(f)\varphi_A(1_A)$. In particular, our map $\gamma(\varphi) = \varphi_A(1_A)$ is injective. Moreover, note that each choice of $u \in F(A)$ gives rise to a different natural transformation φ by setting $\varphi_X(f) = F(f)u$. So our map γ is indeed a bijection.

We now have two naturality statements to prove. Naturality in the functor means that given a natural isomorphism $\eta\colon F \longrightarrow G$, the diagram

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\gamma_F"] \arrow[d, "\eta_*"'] & F(A) \arrow[d, "\eta_A"] \\
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),G) \arrow[r, "\gamma_G"'] & G(A)
\end{tikzcd}$$

commutes. Given a natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F,

$$\begin{aligned}
\eta_A \circ \gamma_F(\varphi) &= \eta_A(\varphi_A(1_A)) && \text{by definition of } \gamma \\
&= (\eta \circ \varphi)_A(1_A) && \text{by definition of composition of natural transformations} \\
&= \gamma_G(\eta \circ \varphi) && \text{by definition of } \gamma \\
&= \gamma_G \circ \eta_*(\varphi) && \text{by definition of } \eta_*
\end{aligned}$$

so commutativity does hold. Naturality on the object means that given an arrow $f\colon A \longrightarrow B$, the diagram

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\gamma"] \arrow[d, "(f^*)^*"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(B,-),F) \arrow[r, "\gamma"'] & F(B)
\end{tikzcd}$$

commutes. Given a natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F,

$$F(f) \circ \gamma_A(\varphi) = F(f)(\varphi_A(1_A)),$$

while

$$\gamma_B \circ (f^*)^*(\varphi) = \gamma_B(\varphi \circ f^*) = (\varphi \circ f^*)_B(1_B).$$

Now notice that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(B,B) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathscr{C}}(A,B) \arrow[r, "\varphi_B"] & F(B) \ . \\
1_B \arrow[r, mapsto] & f \arrow[r, mapsto] & \varphi_B(f)
\end{tikzcd}$$

Let's look back at the big commutative diagram we started our proof with. It says, in particular, that $\varphi_B(f) = F(f)(\varphi_A(1_A))$. So again commutativity does hold, and we are done. $\square$

One can naturally (pun intended) define the notion of functor category of contravariant functors, and then prove the corresponding Yoneda Lemma, which will instead use the contravariant Hom functor.

Exercise 11 (Contravariant version of the Yoneda Lemma)**.** Let $\mathscr{C}$ be a locally small category, and fix an object B in $\mathscr{C}$. Let $F\colon \mathscr{C} \longrightarrow \mathbf{Set}$ be a contravariant functor. Then there is a bijection

$$\mathrm{Nat}\ (\mathrm{Hom}_{\mathscr{C}}(-,B),F) \xrightarrow{\ \gamma\ } F(B)\ .$$

In a way, the Yoneda Lemma says that to give a natural transformation between the functors $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and F is choosing an element in $F(A)$.

Remark 1.39. Notice that the Yoneda Lemma says in particular that the collection of all natural transformations from $\mathrm{Hom}_{\mathscr{C}}(A,-)$ to F is a set. This wasn't clear a priori, since the collection of objects in $\mathscr{C}$ is not necessarily a set.

Remark 1.40. If we apply the Yoneda Lemma to the case when F itself is also a Hom functor, say $F = \mathrm{Hom}_{\mathscr{C}}(B,-)$, the Yoneda Lemma says that there is a bijection between $\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-),\mathrm{Hom}_{\mathscr{C}}(B,-))$ and $\mathrm{Hom}_{\mathscr{C}}(B,A)$. In particular, each arrow in $\mathscr{C}$ determines a natural transformation between Hom functors.

One of the consequences of the Yoneda Lemma is the Yoneda Embedding, which roughly says that every locally small category can be embedded into the category of contravariant functors from $\mathscr{C}$ to **Set**. In particular, the Yoneda embedding says that natural transformations between representable functors correspond to arrows between the representing objects.

Theorem 1.41 (Yoneda Embedding)**.** *Let $\mathscr{C}$ be a locally small category. The covariant functor*

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \textbf{\textit{Set}}^{\mathscr{C}^{op}} \\ & & \\ A & & \mathrm{Hom}_{\mathscr{C}}(-,A) \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle f_*} \\ B & & \mathrm{Hom}_{\mathscr{C}}(-,B) \end{array}$$

*from $\mathscr{C}$ to the category of contravariant functors $\mathscr{C} \longrightarrow$ **Set** is an embedding. Moreover, the contravariant functor*

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \textbf{\textit{Set}}^{\mathscr{C}} \\ & & \\ A & & \mathrm{Hom}_{\mathscr{C}}(A,-) \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\uparrow{\scriptstyle f^*} \\ B & & \mathrm{Hom}_{\mathscr{C}}(B,-) \end{array}$$

*from the category $\mathscr{C}$ to the category of covariant functors $\mathscr{C} \longrightarrow$ **Set** is also an embedding.*

Proof. First, note that our functors are injective on objects because the Hom-sets in our category are all disjoint. We need to check that given objects A and B in $\mathscr{C}$, we have bijections

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(-,A), \mathrm{Hom}_{\mathscr{C}}(-,B))$$

and

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(A,-), \mathrm{Hom}_{\mathscr{C}}(B,-)).$$

We will do the details for the first one, and leave the second as an exercise.

First, let us take a sanity check and confirm that indeed our proposed functors take arrows $f\colon A \longrightarrow B$ in $\mathscr{C}$ to natural transformations between $\mathrm{Hom}_{\mathscr{C}}(-,A)$ and $\mathrm{Hom}_{\mathscr{C}}(-,B)$. This is essentially the content of Remark 1.40, but let's carefully check the details. The Yoneda Lemma 1.38 applied here tells us that each natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(-,A)$ and $F = \mathrm{Hom}_{\mathscr{C}}(-,B)$ corresponds to an element $u \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, which we obtain by taking $u := \varphi_A(1_A)$. As we discussed in the proof of the Yoneda Lemma 1.38, we can recover φ from u by taking the natural transformation φ that for each object X in $\mathscr{C}$ has $\varphi_X\colon \mathrm{Hom}_{\mathscr{C}}(X,A) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,B)$ given by $\varphi_X(f) = \mathrm{Hom}_{\mathscr{C}}(f,B)(u) = f_*(u)$.

We can see that different arrows f give rise to different natural transformations by applying the resulting natural transformation $f_;*$ to the identity arrow 1_A, which takes it to f. Moreover, the Yoneda Lemma 1.38 tells us that every natural transformation φ between $\mathrm{Hom}_{\mathscr{C}}(-,A)$ and $\mathrm{Hom}_{\mathscr{C}}(-,B)$ is the image of some u, as described above. $\square$

The functors that are naturally isomorphic to some Hom functor are important.

Definition 1.42. A covariant functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists an object A in $\mathscr{C}$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(A, -)$. A contravariant functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists an object B in $\mathscr{C}$ such that F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, B)$.

Example 1.43. We claim that the identity functor $\mathbf{Set} \longrightarrow \mathbf{Set}$ is representable. Let $\mathbf{1}$ be a singleton set. Given any set X, there is a bijection between elements $x \in X$ and functions $\mathbf{1} \longrightarrow X$ sending the one element in $\mathbf{1}$ to each x. Moreover, given any other set Y, and a function $f : X \longrightarrow Y$, our bijections make the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\
\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, Y) \arrow[r, "\cong"] & Y.
\end{tikzcd}$$

This data gives a natural isomorphism between the identity functor and $\mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, -)$.

A representable functor encodes a *universal property* of the object that represents it. For example, in Example 1.43, mapping out of the singleton set is the same as choosing an element x in a set X. We have all seen constructions that are at first a bit messy but that end up satisfying some nice universal property that makes everything work out. At the end of the day, a universal property allows us to ignore the messy details and focus on the universal property, which usually says everything we need to know about the construction. And as you may have already realized, universal properties are *everywhere*. Before we give a formal definition, we want to discuss some important examples.

1.4 Products and coproducts

Definition 1.44. Let $\mathscr{C}$ be a locally small category, and consider a family of objects $\{A_i\}_{i \in I}$ in $\mathscr{C}$. The **product** of the A_i is an object in $\mathscr{C}$, denoted by $\prod_i A_i$ or $A_1 \times \cdots \times A_n$ if I is finite, together with arrows $\pi_j \in \mathrm{Hom}_{\mathscr{C}}(\prod_i A_i, A_j)$ for each j, satisfying the following universal property: given any object B in $\mathscr{C}$ and arrows $f_i : B \longrightarrow A_i$ for each i,

$$\begin{tikzcd}
 & B \arrow[dl, "f_i"'] \arrow[d, dashed, "f"] \arrow[dr, "f_j"] & \\
A_i & \prod_i A_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & A_j
\end{tikzcd}$$

The **coproduct** of the A_i is an object in $\mathscr{C}$, denoted by $\coprod_i A_i$ or in some contexts $\bigoplus_i A_i$, together with arrows $\iota_j \in \mathrm{Hom}_{\mathscr{C}}(A_j, \coprod_i A_i,)$ for each j, satisfying the following universal property: given any object B in $\mathscr{C}$ and arrows $f_i : A_i \longrightarrow B$ for each i, the following diagram commutes:

$$\begin{tikzcd}
 & B & \\
A_i \arrow[ur, "f_i"] \arrow[r, "\iota_i"'] & \coprod_i A_i \arrow[u, dashed, "f"] & A_j \arrow[ul, "f_j"'] \arrow[l, "\iota_j"]
\end{tikzcd}$$

Here is a larger diagram for the (first few) maps involved in a product when the indexing set $I = \mathbb{N}$ is countable:

$$\begin{tikzcd}
& \prod_i A_i \arrow[dr, "\pi_1"] \arrow[ddr, "\pi_2"] \arrow[dddr, "\pi_3"] \arrow[ddddr] & \\
& & A_1 \\
B \arrow[uur, dashed, "f"] \arrow[urr, "f_1"] \arrow[rr, "f_2"] \arrow[drr, "f_3"] \arrow[ddrr] & & A_2 \\
& & A_3 \\
& & \vdots
\end{tikzcd}$$

We can also take a “big picture” view of this universal property of the product:

$$\begin{tikzcd}
& \prod_i A_i \arrow[dr, squiggly, "\{\pi_i\}"] & \\
B \arrow[ur, dashed, "f"] \arrow[rr, squiggly, "\{f_i\}"'] & & \{A_i\},
\end{tikzcd}$$

where the squiggly arrows are again collections of maps instead of maps.

Here is a diagram for the (first few) maps involved in a coproduct when $\Lambda = \mathbb{N}$ is countable:

$$\begin{tikzcd}
& \coprod_i A_i \arrow[ddr, dashed, "f"] & \\
\vdots \arrow[ur] \arrow[drr] & & \\
A_3 \arrow[uur, "i_3"] \arrow[rr, "f_3"] & & B \\
A_2 \arrow[uuur, "i_2"] \arrow[urr, "f_2"] & & \\
A_1 \arrow[uuuur, "i_1"] \arrow[uurr, "f_1"] & &
\end{tikzcd}$$

We can also take a “big picture” view of the universal property of the coproduct:

$$\begin{tikzcd}
& \coprod_i A_i \arrow[dr, dashed, "f"] & \\
\{A_i\} \arrow[ur, squiggly, "\{i_\lambda\}"] \arrow[rr, squiggly, "\{f_i\}"'] & & B,
\end{tikzcd}$$

where the squiggly arrows are now collections of maps instead of maps.

In summary, these universal properties are as follows:

- Mapping *in* to a product is completely determined by mapping in to each of the factors.
- Mapping *out* of a coproduct is completely determined by mapping out of each factor.

Theorem 1.45. *If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda\in\Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a category $\mathscr{C}$, then there is a unique isomorphism $\alpha\colon P \xrightarrow{\sim} P'$ such that $p'_\lambda \circ \alpha = p_\lambda$ for all λ. The analogous statement holds for coproducts.*

Proof. We will just deal with products. The following picture is a rough guide:

$$\begin{tikzcd}
P \arrow[r, dashed, "\alpha"] \arrow[drrr, squiggly, "\{p_\lambda\}"'] & P' \arrow[r, dashed, "\beta"] \arrow[drr, squiggly, "\{p'_\lambda\}"] & P \arrow[dr, squiggly, "\{p_\lambda\}"] & \\
& & & \{X_\lambda\}
\end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta : P' \to P$ such that $p_\lambda \circ \beta = p'_\lambda$. Switching roles, we obtain a unique map $\alpha : P \to P'$ such that $p'_\lambda \circ \alpha = p_\lambda$.

Consider the composition $\beta \circ \alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ for all λ. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ for all λ, so by the uniqueness property of products, $\beta \circ \alpha = 1_P$. We can again switch roles to see that $\alpha \circ \beta = 1_{P'}$. Thus α is an isomorphism. The uniqueness of α in the statement is part of the universal property. $\square$

This explains why the notation $\prod_i A_i$ and $\coprod_i A_i$ makes sense.

Exercise 12. Prove the analogous statement for coproducts.

Example 1.46. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The product of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the cartesian product of sets along with the canonical projection maps.

The familiar notion of Cartesian product or direct product serves as a product in many of our favorite categories. Let's note first that given a family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in any of the categories **Sgrp**, **Grp**, **Ring**, R-**mod**, **Top**, the usual direct product $\prod_{\lambda\in\Lambda} X_\lambda$ is an object of the same category:

- for semigroups, groups, and rings, take the operation coordinate by coordinate:
$$(x_\lambda)_{\lambda\in\Lambda} \cdot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda};$$
- for modules, addition is coordinate by coordinate, and the action is the same on each coordinate: $r \cdot (x_\lambda)_{\lambda\in\Lambda} = (r \cdot x_\lambda)_{\lambda\in\Lambda}$;
- for topological spaces, use the product topology.

Note that this is not true for fields! The usual product of fields is not a field. In fact, there is no product in this category.

Theorem 1.47. *In each of the categories* **Set**, **Sgrp**, **Grp**, **Ring**, R-***mod***, **Top***, given a family* $\{X_\lambda\}_{\lambda\in\Lambda}$*, the direct product* $\prod_{\lambda\in\Lambda} X_\lambda$ *along with the projection maps* $\pi_\lambda : \prod_{\gamma\in\Lambda} X_\gamma \to X_\lambda$ *forms a product in the category.*

Proof. We observe that in each category, the direct product is an object, and the projection maps π_λ are morphisms in the category.

Let $\mathscr{C}$ be one of these categories, and suppose that we have morphisms $g_\lambda : Y \to X_\lambda$ for all λ in $\mathscr{C}$. We need to show there is a unique morphism $\phi\colon Y \to \prod_{\lambda\in\Lambda} X_\lambda$ such that $\pi_\lambda \circ \phi = g_\lambda$ for all λ. The last condition is equivalent to $(\phi(y))_\lambda = (\pi_\lambda \circ \phi)(y) = g_\lambda(y)$ for all λ, which is equivalent to $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$, so if this is a valid morphism, it is unique. Thus, it suffices to show that the map $\phi(y) = (g_\lambda(y))_{\lambda\in\Lambda}$ is a morphism in $\mathscr{C}$, which is easy to see in each case. □

Example 1.48. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ is given by the disjoint union with the various inclusion maps. By disjoint union, we simply mean union if the sets are disjoint; in general do something like replace X_λ with $X_\lambda \times \{\lambda\}$ to make them disjoint.

Theorem 1.49. *Let R be a ring, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left R-modules. A coproduct for the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is $\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, \{\iota_\lambda\}_{\lambda\in\Lambda}\right)$, where*

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \textit{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda$$

is the direct sum of the modules M_λ, and ι_λ is the inclusion map to the λ coordinate.

Proof. Given R-module homomorphisms $g_\lambda : M_\lambda \to N$ for each λ, we need to show that there is a unique R-module homomorphism $\alpha : \bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ such that $\alpha \circ \iota_\lambda = g_\lambda$. We define

$$\alpha((m_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} g_\lambda(m_\lambda).$$

Note that since $(m_\lambda)_{\lambda\in\Lambda}$ is in the direct sum, at most finitely many m_λ are nonzero, so the sum on the right hand side is finite, and hence makes sense in N. We need to check that α is R-linear; indeed,

$$\alpha((m_\lambda)+(n_\lambda)) = \alpha((m_\lambda+n_\lambda)) = \sum g_\lambda(m_\lambda+n_\lambda) = \sum g_\lambda(m_\lambda)+\sum g_\lambda(n_\lambda) = \alpha((m_\lambda))+\alpha((n_\lambda)),$$

and the check for scalar multiplication is similar. For uniqueness of α, note that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is generated by the elements $\iota_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha' \circ \iota_\lambda = g_\lambda$ for all λ, then $\alpha(\iota_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(\iota_\lambda(m_\lambda))$ so the maps must be equal. □

Remark 1.50. If the index set Λ is finite, then the objects $\prod_{\lambda\in\Lambda} M_\lambda$ and $\bigoplus_{\lambda\in\Lambda} M_\lambda$ are identical, but the product and coproduct are not the same since one involves projection maps and the other involves inclusion maps. When Λ is infinite, the two objects are truly distinct, and in fact the direct sum is a submodule of the product.

Remark 1.51. For any indexing set Λ, $\coprod_{\lambda\in\Lambda} R$ is a free R-module. If $R = k$ happens to be a field, then $\prod_{\lambda\in\Lambda} k$ is free, since all vector spaces are free modules, but in general, $\prod_{\lambda\in\Lambda} R$ is not free for an infinite set Λ.

Example 1.52.

1) In **Top**, disjoint unions serve as coproducts.

2) In **Sgrp** and **Grp**, coproducts exist, and are given as free products. You may see or have seen them in topology in the context of Van Kampen's theorem.

3) In **Ring**, the story is more complicated. Let's note first that disjoint unions won't work, since they are not rings. Direct sums of infinitely many rings do not have 1, so they are not rings in this class, but even finite direct sums or products will not work, since the inclusion maps does not send 1 to 1. We will later on construct coproducts in the full subcategory of **Ring** consisting of commutative rings.

1.5 Limits and colimits

1.6 Universal properties

Appendix A

Rings and modules

We will study complexes of R-modules; to make sure we are all speaking the same language, we record here our basic assumptions on rings and modules. You can learn more about the basic theory of rings and modules in any introductory algebra book, such as [DF04].

A.1 Rings and why they have 1

In this class, all rings have a multiplicative identity, written as 1 or 1_R is we want to emphasize that we are referring to the ring R. This is what some authors call *unital rings*; since for us all rings are unital, we will omit the adjective. Moreover, we will think of 1 as part of the structure of the ring, and thus require it be preserved by all natural constructions. As such, a subring S of R must share the same multiplicative identity with R, meaning $1_R = 1_S$. Moreover, any ring homomorphism must preserve the multiplicative identity. To clear any possible confusion, we include below the relevant definitions.

Definition A.1. A **ring** is a set R equipped with two binary operations, $+$ and $\cdot$, satisfying:

1) $(R, +)$ is an abelian group with identity element denoted 0 or 0_R.

2) The operation $\cdot$ is associative, so that $(R, \cdot)$ is a semigroup.

3) For all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

4) there is a multiplicative identity, written as 1 or 1_R, such that $1 \cdot a = a = a \cdot 1$ for all $a \in R$.

To simplify notation, we will often drop the $\cdot$ when writing the multiplication of two elements, so that ab will mean $a \cdot b$.

Definition A.2. A ring R is a **commutative ring** if for all $a, b \in R$ we have $a \cdot b = b \cdot a$.

Definition A.3. A ring R is a **division ring** if $1 \neq 0$ and $R \setminus \{0\}$ is a group under $\cdot$, so every nonzero $r \in R$ has a multiplicative inverse. A **field** is a commutative division ring.

Definition A.4. A commutative ring R is a **domain**, sometimes called an **integral domain** if it has no zerodivisors: $ab = 0 \Rightarrow a = 0$ or $b = 0$.

For some familiar examples, $\mathrm{M}_n(R)$ (the set of $n \times n$ matrices) is a ring with the usual addition and multiplication of matrices, $\mathbb{Z}$ and $\mathbb{Z}/n$ are commutative rings, $\mathbb{C}$ and $\mathbb{Q}$ are fields, and the real Hamiltonian quaternion ring $\mathbb{H}$ is a division ring.

Definition A.5. A **ring homomorphism** is a function $f \colon R \to S$ satisfying the following:

- $f(a+b) = f(a) + f(b)$ for all $a, b \in R$.
- $f(ab) = f(a)f(b)$ for all $a, b \in R$.
- $f(1_R) = 1_S$.

Under this definition, the map $f \colon \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$ sending $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ preserves addition and multiplication but not the multiplicative identities, and thus it is not a ring homomorphism.

Exercise 13. For any ring R, there exists a unique homomorphism $\mathbb{Z} \to R$.

Definition A.6. A subset S of a ring R is a **subring** of R if it is a ring under the same addition and multiplication operations and $1_R = 1_S$.

So under this definition, $2\mathbb{Z}$, the set of even integers, is not a subring of $\mathbb{Z}$; in fact, it is not even a ring, since it does not have a multiplicative identity!

Definition A.7. Let R be a ring. A subset I of R is an **ideal** if:

- I is nonempty.
- $(I, +)$ is a subgroup of $(R, +)$.
- For every $a \in I$ and every $r \in R$, we have $ra \in I$ and $ar \in I$.

The final property is often called **absorption**. A **left ideal** satisfies only absorption on the left, meaning that we require only that $ra \in I$ for all $r \in R$ and $a \in I$. Similarly, a **right ideal** satisfies only absorption on the right, meaning that $ar \in I$ for all $r \in R$ and $a \in I$.

When R is a commutative ring, the left ideals, right ideals, and ideals over R are all the same. However, if R is not commutative, then these can be very different classes.

One key distinction between unital rings and nonunital rings is that if one requires every ring to have a 1, as we do, then the ideals and subrings of a ring R are very different creatures. In fact, the *only* subring of R that is also an ideal is R itself. The change lies in what constitutes a subring; notice that nothing has changed in the definition of ideal.

Remark A.8. Every ring R has two **trivial ideals**: R itself and the zero ideal $(0) = \{0\}$.

A **nontrivial ideal** I of R is an ideal that $I \neq R$ and $I \neq (0)$. An ideal I of R is a **proper ideal** if $I \neq R$.

A.2 Basic definitions: modules

You can learn more about the basic theory of (commutative) rings and R-modules in any introductory algebra book, such as [DF04].

Definition A.9. Let R be a ring with $1 \neq 0$. A **left R-module** is an abelian group $(M, +)$ together with an action $R \times M \to M$ of R on M, written as $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$ we have the following:

- $(r + s)m = rm + sm$,
- $(rs)m = r(sm)$,
- $r(m + n) = rm + rn$, and
- $1m = m$.

A **right R-module** is an abelian group $(M, +)$ together with an action of R on M, written as $M \times R \to M, (m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$ we have

- $m(r + s) = mr + ms$,
- $m(rs) = (mr)s$,
- $(m + n)r = mr + nr$, and
- $m1 = m$.

By default, we will be studying left R-modules. To make the writing less heavy, we will sometimes say **R-module** rather than left R-module whenever there is no ambiguity.

Remark A.10. If R is a commutative ring, then any left R-module M may be regarded as a right R-module by setting $mr := rm$. Likewise, any right R-module may be regarded as a left R-module. Thus for commutative rings, we just refer to modules, and not left or right modules.

The definitions of submodule, quotient of modules, and homomorphism of modules are very natural and easy to guess, but here they are.

Definition A.11. If $N \subseteq M$ are R-modules with compatible structures, we say that N is a **submodule** of M.

A map $M \xrightarrow{f} N$ between R-modules is a **homomorphism of R-modules** if it is a homomorphism of abelian groups that preserves the R-action, meaning $f(ra) = rf(a)$ for all $r \in R$ and all $a \in M$. We sometimes refer to R-module homomorphisms as **R-module maps**, or **maps of R-modules**. An isomorphism of R-modules is a bijective homomorphism, which we really should think about as a relabeling of the elements in our module. If two modules M and N are isomorphic, we write $M \cong N$.

Given an R-module M and a submodule $N \subseteq M$, the **quotient** M/N is an R-module whose elements are the equivalence classes determined by the relation on M given by $a \sim b \Leftrightarrow a - b \in N$. One can check that this set naturally inherits an R-module structure from the R-module structure on M, and it comes equipped with a natural **canonical map** $M \longrightarrow M/N$ induced by sending 1 to its equivalence class.

Example A.12. The modules over a field k are precisely all the k-vector spaces. Linear transformations are precisely all the k-module maps.

While vector spaces make for a great first example, be warned that many of the basic facts we are used to from linear algebra are often a little more subtle in commutative algebra. These differences are features, not bugs.

Example A.13. The $\mathbb{Z}$-modules are precisely all the abelian groups.

Example A.14. When we think of the ring R as a module over itself, the submodules of R are precisely the ideals of R.

Theorem A.15 (First Isomorphism Theorem)**.** *Any R-module homomorphism $M \xrightarrow{f} N$ satisfies $M/\ker f \cong \operatorname{im} f$.*

The first big noticeable difference between vector spaces and more general R-modules is that while every vector space has a basis, most R-modules do not.

Definition A.16. A subset $\Gamma \subseteq M$ of an R-module M is a **generating set**, or a **set of generators**, if every element in M can be written as a finite linear combination of elements in M with coefficients in R. A **basis** for an R-module M is a generating set Γ for M such that $\sum_i a_i \gamma_i = 0$ implies $a_i = 0$ for all i. An R-module is **free** if it has a basis.

Remark A.17. Every vector space is a free module.

Remark A.18. Every free R-module is isomorphic to a direct sum of copies of R. Indeed, let's construct such an isomorphism for a given free R-module M. Given a basis $\Gamma = \{\gamma_i\}_{i \in I}$ for M, let

$$\begin{array}{ccc} \bigoplus_{i \in I} R & \xrightarrow{\pi} & M \\ (r_i)_{i \in I} & \longrightarrow & \sum_i r_i \gamma_i \end{array} .$$

The condition that Γ is a basis for M can be restated into the statement that π is an isomorphism of R-modules.

One of the key things that makes commutative algebra so rich and beautiful is that most modules are in fact *not* free. In general, every R-module has a generating set — for example, M itself. Given some generating set Γ for M, we can always repeat the idea above and write a **presentation** $\oplus_{i \in I} R \xrightarrow{\pi} M$ for M, but in general the resulting map π will have a nontrivial kernel. A nonzero kernel element $(r_i)_{i \in I} \in \ker \pi$ corresponds to a **relation** between the generators of M.

Remark A.19. Given a set of generators for an R-module M, any homomorphism of R-modules $M \longrightarrow N$ is determined by the images of the generators.

We say that a module is **finitely generated** if we can find a finite generating set for M. The simplest finitely generated modules are the cyclic modules.

Example A.20. An R-module is **cyclic** if it can be generated by one element. Equivalently, we can write M as a quotient of R by some ideal I. Indeed, given a generator m for M, the kernel of the map $R \xrightarrow{\pi} M$ induced by $1 \mapsto m$ is some ideal I. Since we assumed that m generates M, π is automatically surjective, and thus induces an isomorphism $R/I \cong M$.

Similarly, if an R-module has n generators, we can naturally think about it as a quotient of R^n by the submodule of relations among those n generators.

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