

Final Exam practice

Here is a selection of some old qualifying exam problems to practice for the final exam.

Problem 1 (January 2014). Let E be a subfield of \mathbb{C} and assume that every element of E is a root of a polynomial of degree 10 in $\mathbb{Q}[x]$. Prove that $[E : \mathbb{Q}] \leq 10$.

Problem 2 (January 2016). Let L be a finite Galois field extension of \mathbb{Q} . Let E and F be subfields of L such that $EF = L$, E/\mathbb{Q} is Galois, and $E \cap F = \mathbb{Q}$. Prove that $[L : \mathbb{Q}] = [E : \mathbb{Q}][F : \mathbb{Q}]$.

Problem 3 (May 2022). Let L be the splitting field of $x^4 - 2022$ over \mathbb{Q} . Prove there exists a unique intermediate field $\mathbb{Q} \subseteq K \subseteq L$ such that $[K : \mathbb{Q}] = 4$ and K/\mathbb{Q} is a Galois extension.

Problem 4. Let

$$A = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & 0 \\ 2 & 4 & 0 \end{pmatrix} \in M_3(\mathbb{R}) \text{ and } B = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix} \in M_3(\mathbb{R}).$$

For each of the matrices A and B , determine the following:

- a) Find the rational canonical form for A and B .
- b) Find the Jordan canonical form for A and B , if they exist.
- c) Is A diagonalizable? Is B diagonalizable?

Problem 5 (May 2017). Make \mathbb{R}^3 into an $\mathbb{R}[x]$ -module as follows: given any $f(x) \in \mathbb{R}[x]$ and any $v \in \mathbb{R}^3$, let $f(x)v = Av$, where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

This makes \mathbb{R}^3 into an $\mathbb{R}[x]$ -module isomorphic to $\mathbb{R}[x]^3 / \text{im}(t_A)$, where $t_A : \mathbb{R}[x]^3 \rightarrow \mathbb{R}[x]^3$ is given by $\varphi(v) = (Ix - A)v$. It turns out that this module is cyclic; find an explicit polynomial $p(x)$ such that $\mathbb{R}^3 \cong \mathbb{R}[x]/(p(x))$ as $\mathbb{R}[x]$ -modules.

Problem 6. List all possible rational canonical forms over \mathbb{Q} and Jordan canonical forms over \mathbb{C} for 8×8 matrices with determinant 81 and minimal polynomial $(x - 3)^2(x^2 + 1)$. Carefully justify.