Problems on Symbolic Powers

All rings are commutative and have 1. Problems marked with a $(*)$ are a bit more difficult, or might be easier to do after taking a first course in commutative algebra.

Commutative algebra preliminaries

The problems in this section are here to be used as tools in the other sections. You can skip this section and use these as theorems if you like, but don't forget to read them — you might find them helpful.

Problem 1. Given two ideals I and J, the *product* of I and J is the ideal

$$IJ := (fg \mid f \in I \text{ and } g \in J),$$

while their sum is the ideal

$$I + J := \{f + g \mid f \in I \text{ and } g \in J\}.$$

a) Show that $IJ \subseteq I \cap J$.

b) If I and J satisfy $I + J = R$, then $IJ = I \cap J$.

c) In general, $IJ \neq I \cap J$. Find an example of a ring R and ideals I and J with $IJ \neq I \cap J$.

Problem 2. Let R be a ring. The following are equivalent:

1) Every ascending chain of ideals
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
eventually stabilizes: there is some N for which $I_n = I_{n+1}$ for all $n \geqslant N$.

2) Every nonempty family of ideals in R has a maximal element.

3) Every ideal of R is finitely generated.

Problem 3. Let P and I be ideals in R, with P prime. Show that

$$IR_P \cap R := \{f \in R \mid \frac{f}{1} \in IR_P\} = \{f \in R \mid sf \in I \text{ for some } P\}.$$

Problem 4. Let M be a module over a Noetherian ring R. For any nonzero $m \in M$, $\operatorname{ann}(m)$ is contained in some associated prime of M.

Problem 5. Given any module M over a Noetherian ring R,

$$\{r \in R \mid rm = 0 \text{ for some nonzero } m \in M\} = \bigcup_{P \in \operatorname{Ass}(M)} P.$$

Problem 6. Show that $\sqrt{I^n} = \sqrt{I}$ for any ideal I. In particular, if P is prime, then $\sqrt{P^n} = P$.

Problem 7. Let I be an ideal in R and M an R-module. Show that there exists a non-zero element $m \in M$ such that $I = \operatorname{ann}_R(m)$ if and only if R/I includes into M.

Primary decompositions exist

This is a guide to Noether's proof that every ideal in a Noetherian ring has a primary decomposition.

An ideal I is **irreducible** if if it cannot be written as a proper intersection of larger ideals.

Problem 8. Show that every ideal in a Noetherian ring can be written as a finite intersection of irreducible ideals.[1]

Problem 9. Let Q be an ideal in a Noetherian ring R, and let $xy \in Q$ with $x \notin Q$ and $y \notin \sqrt{Q}$.

a) Show that there exists n with $y^{n+1} f \in Q \implies y^n f \in Q$.[2]

b) Show that $Q = (Q + (y^n)) \cap (Q + (x))$.

Problem 10. Show that every ideal in a Noetherian ring has a primary decomposition.

Problems about primary decomposition

Problem 11. ($*$) Let Q be an ideal in a Noetherian ring R. Show that the following are equivalent:

1) Q is P-primary.

2) Every zerodivisor in the ring R/Q must be nilpotent.

3) $\operatorname{Ass}(R/Q) = \{P\}$ is a singleton.

4) $P := \sqrt{Q}$ is prime, and $QR_P \cap R = Q$.

Problem 12. Show that if the radical of an ideal I is maximal, then I is primary.

Problem 13. Show that if $\mathfrak{m}$ is any maximal ideal in R, then $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for any $n \geqslant 1$. Conclude that $\mathfrak{m}^{(n)} = \mathfrak{m}^n$ for all n.

Problem 14. Show that a finite intersection of P-primary ideals is a P-primary ideal.

Problem 15. Let $R = \mathbb{Z}[\sqrt{-5}]$. While $6 \in R$ is not a unique product of irreducibles, we are going to show that the ideal $I = (6)$ does have a unique primary decomposition.

a) Prove that (2) is a primary ideal.

b) Prove that (3) is *not* a primary ideal.

c) Prove that $(3, 1 + \sqrt{-5})$ and $(3, 1 - \sqrt{-5})$ are both primary.

d) Show that $(6) = (2) \cap (3, 1 + \sqrt{-5}) \cap (3, 1 - \sqrt{-5})$.

e) ($*$) Why is this primary decomposition unique? In fact, every primary decomposition in this ring is unique!

[1] Hint: consider the set of all ideals that are not a finite intersection of irreducible ideals.
[2] Hint: construct an appropriate increasing chain of ideals in R.

Problems about symbolic powers

Problem 16. Show that if P is prime, $P^{(n)}$ is the smallest P-primary ideal containing P^n.

Problem 17. Let I be a radical ideal in a noetherian ring R. Prove the following basic properties:

(a) $I^{(1)} = I$.

(b) For all $n \geqslant 1$, $I^n \subseteq I^{(n)}$.

(c) $I^a \subseteq I^{(b)}$ if and only if $a \geqslant b$.

(d) If $a \geqslant b$, then $I^{(a)} \subseteq I^{(b)}$.

(e) For all $a, b \geqslant 1$, $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$.

(f) $I^n = I^{(n)}$ if and only if I^n has no embedded primes.

Problem 18. Consider the ideal $I = I_2(X)$ of 2×2 minors of a generic 3×3 matrix

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}$$

in the polynomial ring $R = k[X] = k\left[x_{i,j} \,|\, 1 \leqslant i, j \leqslant 3\right]$ generated by the variables in X over a field k. Show that $g := \det X \in P^{(2)}$, while $g \notin P^2$.

The **colon** of two ideals I and J in R is the ideal

$$(I : J) := \{r \in R \mid rJ \subseteq I\}.$$

Problem 19. Let I be an ideal in a noetherian ring. Show that $\left(I^d : I^{(d)}\right)$ always contains an element that is not in any minimal prime of I.

Problem 20. Let R be a Noetherian ring. Let I be an ideal in R, and $x \in R$. The *saturation* of I with respect to x is the ideal

$$(I : x^\infty) := \bigcup_{n=1}^{\infty} (I : x^n).$$

a) Let Q be a P-primary ideal. Show that

$$(Q : x^\infty) = \begin{cases} Q & \text{if } x \notin P \\ R & \text{if } x \in P \end{cases}.$$

b) Show that $(I : x^\infty) = (I : x^n)$ for some n.

c) Show that $(I \cap J : x^\infty) = (I : x^\infty) \cap (J : x^\infty)$ for any ideals I and J.

d) Let $I = Q_1 \cap \cdots \cap Q_k$ be a primary decomposition, and $x \in R$. Show that

$$(I : x^\infty) = \bigcap_{x \notin \sqrt{Q_i}} Q_i.$$

e) (∗) Let I be a radical ideal in a Noetherian ring R. A beautiful theorem of Brodmann says that the set

$$\bigcup_{n \geqslant 1} \operatorname{Ass}\,(R/I^n)$$

is finite. Show that there exists an element x such that:

- x is contained in every embedded prime of I^n for every n, and
- $x \notin P$ for all $P \in \operatorname{Min}(I)$.

Conclude that there exists $x \in R$ such that $I^{(n)} = (I^n : x^\infty)$ for all $n \geqslant 1$.

Problem 21. Consider s points $P_1 = (a_{11}, \ldots, a_{1d}), \ldots, P_s = (a_{s1}, \ldots, a_{sd})$ in $\mathbb{C}^d$, and let I be the corresponding radical ideal in $\mathbb{C}[x_1, \ldots, x_d]$. Show that for all $n \geqslant 1$,

$$I^{(n)} = \bigcap_{i=1}^{s} (x_1 - a_{i1}, \ldots, x_d - a_{id})^n.$$

Computational problems

These problems require using a computer. You can try Macaulay2 online here without installing. Macaulay2 is a computer algebra software for computations in commutative algebra and algebraic geometry, primarily used by researchers

Problem 22. Consider the ideal $I = (x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3))$ in $R = \mathbb{Q}[x, y, z]$.

a) Find a primary decomposition for I.

b) Show that I is radical.

c) Find $I^{(2)}$, and show that $I^{(2)} \neq I^2$.

d) Show that $I^{(4)} \subseteq I^2$, but $I^{(3)} \nsubseteq I^2$.

Problem 23. The ring $S = \mathbb{Q}[s^3, s^2t, st^2, t^3]$ is a domain, so we can write it as a quotient of $R = \mathbb{Q}[a, b, c, d]$ by a prime ideal P. Find $P^{(2)}$, and compare it to P^2. Is P a complete intersection?

Problem 24. Find the ideal P defining the curve parametrized by (t^{13}, t^{42}, t^{73}) over $\mathbb{Q}$. Check that this is a prime ideal. What does that say about this curve? What is the smallest degree of a polynomial in $P^{(2)}$ (where you might want to think about what we should mean by degree here)?