Midterm

Instructions: For full credit, turn in 5 problems in a pdf file. You cannot use any resources besides me or our course notes. In particular, you cannot discuss the problems with your classmates until after the due date, and you are not allowed to use the internet or any other textbooks as a resource.

Problem 1. For each of the following, either give an example with justification, or prove that no such example exists.

a) A ring R such that $\mathbb{C}[y^2] \subseteq R \subseteq \mathbb{C}[x,y]$ that is not noetherian.

b) A ring R such that $\mathbb{C}[x^2,y^2] \subseteq R \subseteq \mathbb{C}[x,y]$ that is not noetherian.

Problem 2. Let R be a noetherian ring and I and J be ideals in R. Show that $I \subseteq J$ if and only if $I_P \subseteq J_P$ for all $P \in \operatorname{Ass}(J)$.

Problem 3. Let $R \subseteq S$ be a module-finite extension of rings. Show that every $s \in S$ has a nonzero multiple in R.

Problem 4. Let R be a noetherian ring. Show that if P is a minimal prime of R, then

$$P \nsubseteq \bigcup_{\substack{Q \in \operatorname{Min}(R) \\ Q \neq P}} Q.$$

Problem 5. Let W be a multiplicatively closed subset of R that contains a nonunit. Show that R is not a direct summand of $W^{-1}R$.

Problem 6. In this problem we are going to show that localization is exact. Let R be a ring and W a multiplicatively closed set.

a) Show that if $M \xrightarrow{f} N$ is a homomorphism of R-modules, then $W^{-1}(\ker(f)) = \ker(W^{-1}(f))$.

b) Show that if $M \xrightarrow{f} N$ is a homomorphism of R-modules, then $W^{-1}(\operatorname{im}(f)) = \operatorname{im}(W^{-1}(f))$

c) Show that if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of R-modules, then

$$0 \longrightarrow W^{-1}A \xrightarrow{W^{-1}(f)} W^{-1}B \xrightarrow{W^{-1}(g)} W^{-1}C \longrightarrow 0$$

is also a short exact sequence.

Problem 7. Let k be a field, $R = k[x,y]$, and let $M = R/(x^2, xy)$. Find a prime filtration for M and determine $\operatorname{Ass}(M)$.