Problem Set 5

Problem 1. Let R be a domain and Q be its fraction field. Let $T(-)$ denote the torsion functor we introduced in Problem Set 3.

a) Show that $T(M) = \operatorname{Tor}_1^R(M, Q/R)$.[1]

b) Show that for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of R-modules gives rise to an exact sequence[2]

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow (Q/R) \otimes_R A \longrightarrow (Q/R) \otimes_R B \longrightarrow (Q/R) \otimes_R C \longrightarrow 0.$$

c) Show that the right derived functors of T are $R^1 T = (Q/R) \otimes_R -$ and $R^i T = 0$ for all $i \leqslant 2$.

Problem 2. Let I be an ideal in R. Show that

$$\operatorname{Ext}_R^n(I, M) \cong \operatorname{Ext}_R^{n+1}(R/I, M)$$

for all $n \geqslant 1$ and all R-modules M.

Problem 3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $r \in R$ and M and N be finitely generated R-modules.

a) Show that the map $\operatorname{Ext}_R^i(M, N) \to \operatorname{Ext}_R^i(M, N)$ induced by $M \xrightarrow{r} M$ is the map given by multiplication by r.

b) Show that if r is regular on M and $\operatorname{Ext}_R^i(M/rM, N) = 0$ for $i \gg 0$, then $\operatorname{Ext}_R^i(M, N) = 0$ for $i \gg 0$.

Problem 4. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

a) Show that for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules,

$$\operatorname{depth}(A) \geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(C) + 1\}.$$

b) Given any finitely generated R-module M, show that there exists $n \geqslant 1$ such that either $\operatorname{pdim}(M) < n$ or $\operatorname{depth}(\Omega_n M) = \operatorname{depth} R$.

[1] Hint: you want to look at some long exact sequence for Tor.
[2] Hint: apply the Snake Lemma to some nice diagram.

Problem 7. Consider the ring $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$ and the 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

Let P be the ideal in $S = \mathbb{Q}[x, y, z]$ defining the curve $\{(t^{13}, t^{42}, t^{73}) \mid t \in \mathbb{Q}\}$.

To solve this problem, you are not allowed to use any additional Macaulay2 packages besides the `Complexes` package and the ones that are automatically loaded with Macaulay2.

a) Find $\operatorname{pdim}_S(S/P)$ and $\operatorname{depth}(S/P)$.

b) Is P generated by a regular sequence?

c) Find $\operatorname{pdim}_R(M)$ and $\operatorname{depth}(M)$.

d) Is R a regular ring? Is it Cohen-Macaulay?