Problem Set 3

To solve these problems, you are not allowed to use any additional Macaulay2 packages besides the `Complexes` package and the ones that are automatically loaded with Macaulay2.

Problem 1. Show that if R is a regular local ring then R_P is regular for every prime P.

Problem 2. Of the following rings, which ones are regular? Which ones are Cohen-Macaulay?

- $R = \mathbb{F}_{101}[X]/I_2(X)$, where X is a generic 2×3 matrix.
- $S = \mathbb{Q}[x^2, xy, y^2]$.
- $T = k[\![x, y, z, w]\!]/(x^2, xy, yz, zw, w^2)$, where k is any field.
- $U = k[\![x, y, z]\!]/(x + y + z)$, where k is any field.
- $V = \mathbb{Q}\begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{\mathbb{Q}[u,v,x,y,z]}{(x^3+y^3+z^3)}$.

Problem 3. Consider the ring $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$ and the 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

Let P be the ideal in $S = \mathbb{Q}[x, y, z]$ defining the curve parametrized by (t^{13}, t^{42}, t^{73}).

a) Find $\operatorname{pdim}(S/P)$ and $\operatorname{depth}(S/P)$.

b) Is there a regular sequence that generates P?

c) Find $\operatorname{pdim}_R(M)$ and $\operatorname{depth}(M)$.

d) Is R a regular ring? Is it Cohen-Macaulay?

Problem 4. Let k be any field and consider the prime ideal in $k[x, y, z]$

$$P = (x^3 - yz, y^2 - xz, z^3 - x^2y)$$

defining the curve parametrized by (t^3, t^4, t^5). Give (with proof!) two different ideals J such that $P^{(n)} = (P^n : J^\infty)$ for all $n \geqslant 1$, and test your proposed ideals J in Macaulay2 with your own choice of k and n.

Problem 5. Let R be a finitely generated k-algebra, and P a prime ideal in R. Show that $P^{\langle n \rangle}$ is P-primary for all $n \geqslant 1$.