Problem Set 5

Due Thursday, October 3

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let $f\colon G \to H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of G.

Problem 2. Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$. Prove that $xy = yx$ for all $x \in H, y \in K$.

Problem 3. Let $f\colon G \to H$ be a group homomorphism.

(3.1) Prove that if $K \trianglelefteq H$ then the preimage $f^{-1}(K)$ of K is a normal subgroup of G.

(3.2) Give an example showing that if $L \trianglelefteq G$ then $f(L)$ might not be a normal subgroup of H.

Problem 4. Let G be a group, S a subset of G, and $H = \langle S \rangle$.

(4.1) Prove that $H \triangleleft G$ if and only if $gsg^{-1} \in H$ for every $s \in S$ and $g \in G$.

(4.2) Let

$$[G,G] := \{aba^{-1}b^{-1} \mid a, b \in G\}$$

be the set of commutators in G. Prove that $H \trianglelefteq G$.

Problem 5. Show that any subgroup of index two is normal. More precisely: show that if G is a group, H is a subgroup of G, and $[G : H] = 2$, then H is normal.

Problem 6. Let G be any group. Show that if $G/\operatorname{Z}(G)$ is cyclic, then G is abelian.