CHAMP 1/12/21

Perturbing Ideals in Arbitrary Noetherian Local Rings and the $\mathfrak{m}$-adic Continuity of HK-Multiplicity

$\boxed{\text{Introduction}}$

Hilbert-Kunz Multiplicity

$(R, \mathfrak{m}_R) \longleftarrow$ Noetherian local ring
Char $p > 0$

(in char $p>0$ $(x+y)^p = x^p + y^p$)

Frobenius: $F: R \to R$, $F(x) = x^p$

(for $e \geq 0$) $F^e: R \to R$, $F^e(x) = x^{p^e}$

$(F^0(x) = x)$

Given $I \subseteq R$,

$$I^{[p^e]} := F^e(I)R = (x^{p^e} \mid x \in I)$$
$$= (f_1^{p^e}, \dots, f_c^{p^e})$$
where $(f_1, \dots, f_c) = I$

R is F-finite if $R \xrightarrow{F} R$ is module finite.

Frobenius $\rightsquigarrow$ Inseparability... $(t^p - 1 = (t-1)^p)$

$\boxed{\text{Ernst Kunz '69}}$

(R, m_R) Noetherian, local, dim d. Char $p>0$.

(i) ("Kunz's Theorem")

R is regular $\Longleftrightarrow$ $F : R \longrightarrow R$ is flat

(ii) For every $e \geq 0$,

λ_R = length $\longrightarrow$ $\lambda_R\left(R/m_R^{[p^e]}\right) \geq p^{ed}$

And, this is

equality for some e $\Longleftrightarrow$ equality $\forall e$ $\Longleftrightarrow$ R Regular

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$\rightsquigarrow$ the values of $\left\{ \frac{1}{p^{ed}} \lambda_R\left(R/m_R^{[p^e]}\right) \mid e \geq 0 \right\}$ close to 1

$\rightsquigarrow$ R is close to Regular...

(Monsky '83) (R, m_R) Noetherian, local, char $p>0$, dim d.

The limit

$$e_{HK}(R) := \lim_{e\to\infty} \frac{1}{p^{ed}} \lambda\left(R/m_R^{[p^e]}\right)$$

exists. Called the <u>Hilbert-Kunz multiplicity</u> of R.

Shows more...

$$e_{HK}(J, M) := \lim_{e \to \infty} \frac{1}{p^{e \dim(M)}} \lambda_R\left(\frac{M}{J^{[p^e]}M}\right)$$

exists...

.... lots of interesting properties of e_{HK}...

- Additive on s.e.s.
- Associativity formula
- same as Hilbert-Samuel mult. if $\dim(R) = 1$, or if $J = (x_1, \dots, x_d)$ is generated by a s.o.p.
- If R is unmixed, ← No embedded primes ... equidim.

  $e_{HK}(R) = 1 \iff R$ is regular
- $e_{HK}(R)$ can be irrational or transcendental.

A not so nice property of HK-mult. is that it is extremely hard to compute...

$\boxed{\text{Notation}}$ (R, m_R) Noeth. and local.

If $I \subseteq R$ is generated by $f_1, \dots, f_c$, will write $I = (f_1, \dots, f_c) = (\underline{f})$.

<u>m_R-adic Perturbations</u>: Ideals of the form

$$(f_1 + \varepsilon_1, \dots, f_c + \varepsilon_c) = (\underline{f + \varepsilon})$$

where $I = (f_1, \dots, f_c)$ are minimal gens, and $\varepsilon_1, \dots, \varepsilon_c \in m_R^T$ $(T \geq 1)$, are <u>m_R-adic perturbations of I</u>.

(perturbation is "small" if T is "large")

For M an R-module, and we are perturbing some fixed I, write

$$\overline{M} := \frac{M}{IM}, \quad \overline{M}_{(\underline{f+\varepsilon})} := \frac{M}{(\underline{f+\varepsilon})M}$$

where $(\underline{f+\varepsilon})$ is a perturbation of I.

(may refer to $\overline{M}_{(\underline{f+\varepsilon})}$ as "perturbations" of $\overline{M}$)

Theorem (Polstra & Smirnov '18)

(R, m_R) d-dimensional, F-finite, CM local ring of char $p > 0$.

Let $I = (f_1, \dots, f_c)$ be an ideal in R generated by $c > 0$ parameters.

Assume $\hat{R}/I\hat{R}$ is reduced.

Then, for any $\delta > 0$, there is a $T \in \mathbb{N}$ such that for all $\varepsilon_1, \dots, \varepsilon_c \in m_R^T$,

$$\left| e_{HK}(\bar{R}) - e_{HK}(\bar{R}_{(\underline{f}+\underline{\varepsilon})}) \right| < \delta$$

- a kind of m_R-adic continuity for e_{HK}
- T does not depend on the choice of min gens of I.
- This works for $J \supseteq I$, m_R-primary ideals

Goal - drop the CM condition

In the P+S result, R is CM and I is generated by part of an s.o.p.

$\Longrightarrow$ I gen'd by reg. seq. + $R/I = \overline{R}$ is CM.

... It is not hard to show some condition on I is necessary... being a parameter ideal isn't enough outside the CM case (Example 5.2 of [-])

Notation:

M a f.g. R-module, $I \subseteq R$,

$H_1(I;M)$ = first Koszul Homology

$:= H_1((\underline{f});M)$

where $(f_1,\ldots,f_c)$ are min. gens. for I (in R)

Fact: (R,m_R) local + Noeth

$\Rightarrow$ different choice of min. gens. give isomorphic $H_1(I;M)$'s.

(**Note:** this $H_1(I;M)$ does not nec. commute with localization — min gens for I in R may no longer be min gens in R_p.)

PdS's condition is $\underline{H_1(I;R)=0}$. (Also have $\overline{R}$ CM)

Theorem [–]:

(R, m_R) Noetherian, local, char $p>0$, and F-finite. Let $I \subseteq R$, be such that $\dim R/I \geq 1$. Assume,

(i) $\hat{R}/I\hat{R}$ is equidimensional

(ii) $\dim H_1(I;R) < \dim \overline{R}$

(iii) $\dim \operatorname{nilrad}(\hat{R}/I\hat{R}) < \dim \overline{R}$

for Cohen-Gabber

Then, for any $\delta>0$, there is a $T \in \mathbb{N}$ such that for all minimal generators, $I=(f_1,\dots,f_c)$ and all $\varepsilon_1,\dots,\varepsilon_c \in m_R^T$,

$$\left| e_{HK}(\overline{R}) - e_{HK}(\overline{R}_{(\underline{f}+\underline{\varepsilon})}) \right| < \delta$$

(works for arbitrary m_R-primary $J \supseteq I$)

Corollary [-]: Suppose the conditions of the theorem are satisfied, and that R is reduced and complete. Let $(f_1, \dots, f_c) = I$ be min. gens. Then,

$$\lim_{n_1\to\infty,\dots,n_c\to\infty} e_{HK}(R[f_1^{1/n_1}, \dots, f_c^{1/n_c}]) = e_{HK}(R/I)$$

[.... .... Proof...]

The proof of P&S's theorem begins by reducing to $R = \hat{R}$, and envoking Cohen-Gabber to get a commutative diagram of local rings:

D:

$$\begin{tikzcd} & R \arrow[d, two heads] \\ A \arrow[ur, hook, bend left] \arrow[r, hook, "\text{finite}"'] & \overline{R} \end{tikzcd}$$

A an RLR, with $A \hookrightarrow \overline{R}$ finite + generically étale (Reduced CM)

in their setup,

(*) $\overline{R} = R/\text{reg. seq.}$ is CM, so it is free as an A-module.... They show, for $T \gg 0$, $(\underline{f+\varepsilon})$ are also parameters, so the $\overline{R}(\underline{f+\varepsilon})$ are also CM and thus free over A.

$\longrightarrow$ this allows them to show the discriminants of $\bar{R}$ and $\bar{R}_{(t+\epsilon)}$ are m_A-adically close in A (for $T \gg 0$).

$\rightsquigarrow$ Gives uniform control over convergence rates of the $e_{HK}(\bar{R}_{(t+\epsilon)})$.

<u>Discriminant</u>: A, S local, Noeth. with A a domain

$A \hookrightarrow S$ module finite,

and let $s_1, \dots, s_n = \underline{s} \in S$,

be elements that map to a $QF(A)$-basis for $S \otimes_A QF(A) \cong QF(A)^{\oplus n}$

The <u>discriminant of S over A</u> is,

$$D_A(S) := \underline{\det\left(\operatorname{trace}(s_i s_j)\right)_{i,j}}$$

- if A is normal $D_A(S) \in A$
- if $s_1', \ldots, s_n' = \underline{s}' \in S$ gives another basis for $S \otimes_A QF(A)$, then the discriminants differ by a unit in A.

$D_A(S) \neq 0 \iff A \hookrightarrow S$ is generically étale

and if A & S are F-finite, and A is an RLR, $D_A(S)$ can be used to explicitly control the convergence rate of $\lim_{e \to \infty} \frac{1}{p^{ed}} \lambda_S(S/m_S^{[p^e]})$

Plan: relate the $D_A(\bar{R})$ and $D_A(\bar{R}_{(t+\underline{\varepsilon})})$ for ε's $\in m_R^T$, $T \gg 0$

$$\begin{tikzcd} & R \arrow[d, two heads] \\ A \arrow[ur] \arrow[r, hook] & \bar{R} \end{tikzcd}$$

$\rightsquigarrow$ Show that, when $\dim H_1(I;R) < \dim \bar{R}$, we can find $\underline{r_1, \ldots, r_n} \in R$ that map to a $QF(A)$-basis for all $\bar{R}_{(t+\underline{\varepsilon})} \otimes_A QF(A)$ simultaneously (for $T \gg 0$)

This gives $D_A(\bar{R})$ & $D_A(\bar{R}_{(\underline{f}+\underline{\varepsilon})})$ m_A-adically close.

Lemma [–] Suppose (R, m_R) and (A, m_A) are complete Noetherian local rings, that $I \subseteq R$ is an ideal, and there is a commuting diagram of local rings

$$D: \quad \begin{tikzcd} & R \arrow[d, two heads] \\ A \arrow[ur] \arrow[r, hook, "\text{finite}"'] & \bar{R} = R/I \end{tikzcd}$$

(note: $I + m_A R$ is m_R-primary)

(A) Let $T \in \mathbb{N}$ be such that $m_R^T \subseteq m_R(I + m_A R)$.

Let $r_1, \dots, r_m \in R$ be elements which map to a minimal generating set for $\bar{R}$ as an A-module. Then, for any $(f_1, \dots, f_c) = I$, and any $\varepsilon_1, \dots, \varepsilon_c \in m_R^T$, the images of $r_1, \dots, r_m$ are minimal generators for $\bar{R}_{(\underline{f}+\underline{\varepsilon})}$ as an A-module

$$\left(\frac{\bar{R}}{m_A \bar{R}} = \frac{\bar{R}_{(\underline{f}+\underline{\varepsilon})}}{m_A \bar{R}_{(\underline{f}+\underline{\varepsilon})}} \right)$$

(B) Assume $\operatorname{depth}(A) \geq 1$, and that $d\, H_1(I; R) = 0$ for some n.z.d. $d \in A$.

Let $r_1, \dots, r_n \in R$ be any elements whose image in $\bar{R}$ span a free A module of rank n.

There is a $T = T(D, d, \underline{r})$, such that for any min. gens. $I = (f_1, \dots, f_c)$, and all $\varepsilon_1, \dots, \varepsilon_c \in m_R^T$, the images of $r_1, \dots, r_n$ span a free A-module of rank n in $\bar{R}_{(\underline{f}+\underline{\varepsilon})}$.

$\longrightarrow$ Complicated A-R argument

$\longrightarrow$ Notice: if $\dim H_1(I; R) < \dim \bar{R}$ then $A \cap \operatorname{Ann}_{\bar{R}}(H_1(I; R)) \neq 0$

Along the way... get a result about Hilbert-Samuel multiplicity:

<u>Theorem [-]</u>: (R, m_R) equicharacteristic Noetherian local ring. $I \subseteq R$ an ideal such that $\dim \bar{R} \geq 1$. Let M be a finite R-module.

Assume $\dim H_1(I; M) < \dim \bar{R}$.

Then, there is a $T \in \mathbb{N}$, such that for all minimal gens $(f_1, \dots, f_c) = I$, and all $\varepsilon_1, \dots, \varepsilon_c \in m_R{}^T$,

$$\boxed{e(\bar{M}, \bar{R}) = e(\bar{M}_{(\underline{f}+\underline{\varepsilon})}, \bar{R}_{(\underline{f}+\underline{\varepsilon})})}$$

(Also works for any m_R-primary $J \supseteq I$)

Ex: $f = xy \in \underbrace{k[[x,y,z]]}_{}\!\overset{R}{=}$, k a field, char $p>0$

$\rightsquigarrow$ $e_{HK}\left(R/(f)\right) = 2$,

(A$_n$-singularity) $e_{HK}\left(R/(f + z^n)\right) = 2 - \frac{1}{n}$

Thank You!

Archiv: m-adic Perturbations in Noetherian Local Rings

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