Problem Set 4

Turn in **4** of the following problems. Slightly more challenging problems are indicated by $(\star)$.

Problem 1. Let $R = k[x, y]$, where k is a field, let $Q = \mathrm{frac}(R)$ be the fraction field of R. We are going to show that the R-module $M = Q/R$ is divisible but not injective.

a) Show[1] that if $ax + by = 0$ for some $a, b \in R$, we must have $b \in (x)$.

b) Show that $x \mapsto \frac{1}{y} + R$ and $y \mapsto 0$ induces a well-defined R-module homomorphism $(x, y) \xrightarrow{f} Q/R$.

c) Show that M is a divisible R-module, but not injective.

Problem 2. $(\star)$ Let R be a domain. Show that if R has a nonzero module M that is both injective and projective, then R must be a field.[2]

An R-module F is *faithfully flat* if F is flat and $F \otimes_R M \neq 0$ for every nonzero R-module M.

Problem 3. $(\star)$ Let R be a commutative ring. Show that the following are equivalent:

a) F is faithfully flat.

b) F is flat and for every proper ideal I, $IF \neq F$.

c) F is flat and for every maximal ideal $\mathfrak{m}$, $\mathfrak{m}F \neq F$.

d) The complex

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if

$$F \otimes_R A \xrightarrow{1 \otimes f} F \otimes_R B \xrightarrow{1 \otimes g} F \otimes_R C$$

is exact.

[1] If you know about regular sequences, this is easy to justify. But we aren't assuming anyone has seen regular sequences, so the challenge here is to give a clear, easy justification without invoking anything about regular sequences; though it's certainly ok to say the word regular.

[2] Hint: show that any nonzero R-module homomorphism $M \longrightarrow R$ must be surjective, and then show that such a homomorphism must exist.