A stable version of Harbourne's Conjecture

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Background

Symbolic Power

The n-th **symbolic power** of a radical ideal I in a regular ring R is

$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(R/I)} (I^n R_P \cap R).$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.

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(3) If I is generated by a regular sequence, then $I^n = I^{(n)}$ for all n.

(4) In general, $I^n \neq I^{(n)}$.

Containment Problem (Schenzel)

When is $I^{(b)} \subseteq I^a$?

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a[1] *regular ring R and h be the big height of I. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.*

[1] Excellent in the mixed characteristic case.

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EXAMPLE

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

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$$\text{In fact, } P^{(3)} \subseteq P^2.$$

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Question (Huneke, 2000)

Let P be a height 2 prime in a regular local ring. Is $P^{(3)} \subseteq P^2$?

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Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in a regular ring, and let h be the big height of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Theorem (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]} \subseteq I^q.$$

Notation: $I^{[q]} = (f^q \mid f \in I)$.

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Harbourne's Conjecture

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Dumnicki, Szemberg, Tutaj-Gasińska, 2015

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \not\subseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

Harbourne's Conjecture

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When does Harbourne's Conjecture hold?

- For general points in $\mathbb{P}^2$ (Harbourne–Huneke), $\mathbb{P}^3$ (Dumnicki).
- If R/I is an F-pure ring (G–Huneke).
  Eg, when I is a squarefree monomial ideal, or when R/I is direct summand of a polynomial ring over a perfect field.

Theorem (G–Ma–Schwede)

Let $(R, \mathfrak{m})$ be an F-finite Gorenstein local ring of characteristic $p > 0$ and $Q \subseteq R$ be a radical ideal of finite projective dimension with big height h.

1) *If R/Q is F-pure, then $Q^{(hn-h+1)} \subseteq Q^n$ for all $n \geqslant 1$.*
2) *If R/Q is strongly F-regular, then $Q^{((h-1)(n-1)+1)} \subseteq Q^n$ for all $n \geqslant 1$.*

Harbourne's Conjecture (stable version)

Main Question

Does Harbourne's Conjecture always hold *eventually*?

Evidence for the Stable Harbourne Conjecture

Let $a \geqslant 3$, k be a field, and the Fermat ideal

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)).$$

This is a well-known counterexample to $I^{(3)} \subseteq I^2$. However,

$$I^{(2n-1)} \subseteq I^n$$

for all $n \geqslant 3$, which follows from work of Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska.

Main Question

Does Harbourne's Conjecture always hold *eventually*?

Harbourne's Conjecture (stable version)

Given a radical ideal I of big height h in a regular ring, does

$$I^{(hn-h+1)} \subseteq I^n$$

for all $n \gg 0$?

Question

If there exists a value of m such that

$$I^{(hm-h+1)} \subseteq I^m,$$

does that imply that

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Good enough in prime characteristic

In characteristic p, this would prove the stable version of Harbourne's Conjecture, since $I^{(hp-h+1)} \subseteq I^p$.

Theorem

Let I be a radical ideal of big height h in a regular ring ~~containing a field~~. If there exists a value of n such that

$$I^{(hn-h)} \subseteq I^n,$$

then

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Theorem

Let I be a radical ideal of big height h in a regular ring ~~containing a field~~. If there exists a value of n such that

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Example

The defining ideal of $k[t^3, t^4, t^5]$ in $k[x, y, z]$ verifies $P^{(2\times 3-2=4)} \subseteq P^3$, and thus $P^{(2m-2)} \subseteq P^m$ for all $m \geqslant 6$.

Theorem

Let k be a field of characteristic not 2 nor 3, let $a = 3$ or $a = 4$, and let $a < b < c$ be integers. If P is the defining ideal of $k[\![t^a, t^b, t^c]\!]$ or $k[t^a, t^b, t^c]$ in $R = k[\![x, y, z]\!]$ or $R = k[x, y, z]$, repectively. Then

$$P^{(4)} \subseteq P^3.$$

As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The defining ideal P of $k[t^9, t^{11}, t^{14}]$ fails $P^{(4)} \subseteq P^3$, but Macaulay2 computations show that

$$P^{(2\times 4-2=6)} \subseteq P^4,$$

so $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The squarefree monomial ideal

$$I = \bigcap_{i \neq j} (x_i, x_j) \subseteq k[x_1, \ldots, x_v].$$

has $I^{(2n-2)} \not\subseteq I^n$ for $n < v$, but $I^{(2v-2)} \subseteq I^v$. Therefore,

$$I^{(2n-2)} \subseteq I^n \text{ for all } n \gg 0.$$

Definition (Bocci-Harbourne)

The resurgence of an ideal I is given by

$$\rho(I) = \sup \left\{ \frac{a}{b} : I^{(a)} \not\subseteq I^{b} \right\}.$$

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$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^{b}\right\}.$$

$$1 \leqslant \rho(I) \leqslant h.$$

If $\frac{a}{b} > \rho(I)$, then $I^{(a)} \subseteq I^{b}$.

Observation

Let I is a radical ideal, and h be the big height of I. If $\rho(I) < h$, then for every constant $C > 0$,

$$I^{(hn-C)} \subseteq I^n$$

for all $n \gg 0$.

Observation

Let I is a radical ideal, and h be the big height of I. If $\rho(I) < h$, then for every constant $C > 0$,

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for all $n \gg 0$.

Question

Is there an ideal I with $\rho(I) = h$?

Example

Let $a \geqslant 3$, k be a field, and Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska showed that

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a))\,.$$

has resurgence $\frac{3}{2}$, so $I^{(2n-1)} \subseteq I^n$ for all $n \geqslant 3$.

Question

Let I be a radical ideal of big height h in a regular ring R. Fix an integer $C > 0$. Does

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hold for all $n \gg 0$?

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Let I be a radical ideal of big height h in a regular ring R. Fix an integer $C > 0$. Does

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hold for all $n \gg 0$?

Yes, if

- if $\rho(I) < h$, and
- if $I^{(hm-C)} \subseteq I^m$ for some m and $I^{(n+h)} \subseteq I\, I^{(n)}$ for all $n \geqslant 1$.

Example (Seceleanu)

The ideal $I = \big(x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3)\big) \subseteq \mathbb{C}[x, y, z]$ fails $I^{(n+2)} \subseteq I I^{(n)}$ for n arbitrarily large.

Yet in this example it is still true that given any C, $I^{(2n-C)} \subseteq I^n$ for all $n \gg 0$.

Theorem

Let R be a regular ring of characteristic $p > 0$. Let I be an ideal in R such that R/I is an F-pure ring, and let h be the big height of I. Then for all $n \geqslant 1$,

$$I^{(n+h)} \subseteq I I^{(n)}.$$

In particular, if $I^{(hk-C)} \subseteq I^k$ for some k and C, then $I^{(hn-C)} \subseteq I^n$ for all $n \geqslant k$.

Obrigada!