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PhD Thesis Defense

Symbolic Powers and the Containment Problem

Thesis outline

Chapter 1 Symbolic powers

Chapter 2 Symbolic powers in characteristic p

Chapter 3 Height 2 ideals in 3 variables

Chapter 4 A stable version of Harbourne's Conjecture

Chapter 5 Symbolic Rees algebras

Chapter 6 Reduction to positive characteristic

Chapter 7 Algorithms for computing symbolic powers

What are symbolic powers?

All rings are noetherian (with identity).

Ideal

An ideal I in a ring R is a subset of R that is closed under addition and closed for products by elements in the ring.

Prime ideal

A proper ideal P is prime if R/P is an integral domain.

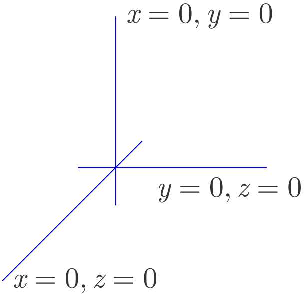
Hilbert's Nullstellensatz

$$\begin{array}{ccc} \mathbb{C}[x_1,\ldots,x_d] & & \mathbb{C}^d \end{array}$$

$$\text{ideal } I \xmapsto{\text{zeroes of all the } f\in I} \text{variety } V$$

$$\text{ideal } I \xleftarrow{\text{polynomials that vanish along } V} \text{variety } V$$

$$\text{prime ideal } P \longleftrightarrow \text{irreducible variety } V$$



Variety corresponding to the ideal (xy, xz, yz) in $k[x, y, z]$.

Powers of ideals

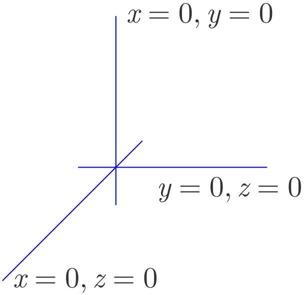
Given an ideal I, I^n is the ideal generated by all elements of the form $f_1 \cdots f_n$, where each $f_i \in I$. General elements in I^n are

$$\sum_i c_i f_{i,1} \cdots f_{i,n}, \text{ where each } c_i \in R \text{ and } f_{i,j} \in I.$$

This is a natural algebraic notion of power.

Example

The square of $I = (xy, xz, yz)$ is generated by x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2.



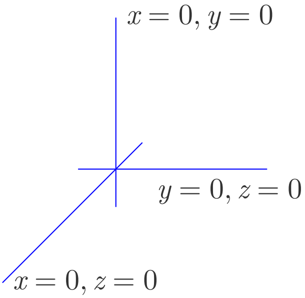
Variety given by $(xy, xz, yz)^2 = (x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2)$.

Radical ideal

The radical $\sqrt{I}$ of an ideal I is the ideal generated by all elements f such that $f^n \in I$ for some n. An ideal I is radical if $\sqrt{I} = I$.

Hilbert's Nullstellensatz

$$\begin{array}{lcl} \mathbb{C}[x_1, \ldots, x_d] & & \mathbb{C}^d \\ \text{radical ideal } I & \longleftrightarrow & \text{variety } V \\ \text{prime ideal } P & \longleftrightarrow & \text{irreducible variety} \end{array}$$



Variety given by $(xy, xz, yz)^2 = (x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2)$.

Bad news

This notion of power is not meaningful geometrically.

Question

What is a meaningful geometric notion of power?

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What is a meaningful geometric notion of power?

We want to consider the functions that vanish up to order n on a given variety.

The n-th symbolic power of an ideal I is denoted by $I^{(n)}$.

Theorem (Zariski–Nagata: order of vanishing)

Let Q be a prime ideal in $\mathbb{C}[x_1, \ldots, x_d]$. Then $Q^{(n)}$ is the ideal of polynomials that vanish up to order n in the variety defined by Q.

$$Q^{(n)} = \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \mathfrak{m} \supseteq Q}} \mathfrak{m}^n.$$

The n-th symbolic power of an ideal I is denoted by $I^{(n)}$.

Theorem (Zariski–Nagata: differential operators)

Let I be a radical ideal in $\mathbb{C}[x_1, \ldots, x_d]$. A polynomial f is in $I^{(n)}$ if and only if $\frac{\partial}{\partial x_i}(f) \in I^{(n-1)}$ for all $i = 1, \ldots, d$.

Example

In the variety defined by $I = (xy, xz, yz)$, the polynomial xyz vanishes up to order 2, since $\frac{\partial}{\partial x}(xyz)$, $\frac{\partial}{\partial y}(xyz)$, $\frac{\partial}{\partial z}(xyz) \in I$.

In the language of symbolic powers, this means that $xyz \in I^{(2)}$.

Note that xyz has degree 3 and all elements in I^2 have degree at least 4, so $xyz \notin I^2$.

Properties:

- $I^n \subseteq I^{(n)}$.
- $I^{(n+1)} \subseteq I^{(n)}$.
- $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$.

Very difficult open question

Given a noetherian ring R, characterize the ideals I for which $I^n = I^{(n)}$ for all $n \geqslant 1$.

Example

If I is an ideal in $k[x_1, \ldots, x_d]$ generated by some of the variables, then we do have $I^n = I^{(n)}$ for all n.

Symbolic powers arise naturally from the theory of primary decomposition.

Let P be a prime ideal in a noetherian ring R.

Definition (Symbolic Powers)

The n-th symbolic power of P is the ideal

$$\begin{aligned} P^{(n)} &= P^n R_P \cap R. \\ &= \{f \in R \mid f/g \in P^n R_P \text{ for some } g \notin P\} \\ &= \{f \in R \mid gf \in P^n \text{ for some } g \notin P\} \\ &= \ P\text{-primary component in a decomposition of } P^n \end{aligned}$$

Example

$P \subseteq R = k[x,y,z]$ the defining ideal of $k[t^3,t^4,t^5]$, meaning the kernel of the map $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$, with $R/P \cong k[t^3,t^4,t^5]$. Set $\deg x = 3$, $\deg y = 4$, $\deg z = 5$.

$$P = (\underbrace{x^2y - z^2}_{\substack{f \\ \deg 10}}, \underbrace{xz - y^2}_{\substack{g \\ \deg 8}}, \underbrace{yz - x^3}_{\substack{h \\ \deg 9}})$$

Since $\underbrace{fg - h^2}_{\text{in } P^2} = xq$ for some q and $x \notin P$, $q \in P^{(2)}$.

Since $\deg(fg - h^2) = 18$, $\deg q = 18 - 3 = 15$.

Elements in P^2 have degree at least 16, so $q \notin P^2$.

Question

How do we compare symbolic and ordinary powers?

The Containment Problem

Containment Problem

When is $I^{(b)} \subseteq I^a$?

Definition (Height)

The height of a prime ideal P in a regular ring R is

- $\dim(R) - \dim(R/P)$
- the codimension of the variety corresponding to P
- the largest size n of a prime chain $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n = P$.

Definition (Big height)

The **big height** of the radical ideal $I = P_1 \cap \cdots \cap P_k$ is

$$\begin{aligned} h = &\ \text{maximal height among the } P_i \\ = &\ \text{maximal codimension of an irreducible} \\ &\ \quad \text{component of the variety corresponding to } I \end{aligned}$$

Theorem (Ein-Lazarsfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal of big height h in a regular ring R. Then for all $n \geqslant 1$,

$$I^{(hn)} \subseteq I^n.$$

In particular, $I^{(dn)} \subseteq I^n$ for $d = \dim R$.

Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede

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Example

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$$h = 2 \rightsquigarrow P^{(2n)} \subseteq P^n \rightsquigarrow P^{(4)} \subseteq P^2.$$

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$$h = 2 \rightsquigarrow P^{(2n)} \subseteq P^n \rightsquigarrow P^{(4)} \subseteq P^2.$$

In fact, $P^{(3)} \subseteq P^2$.

Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

If f vanishes up to order 3 on some variety V, can we write it algebraically in a simpler way, using combinations of products of 2 functions that vanish along V?

Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal of big height h in a regular ring.

For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Key Lemma (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]}.$$

Notation: $I^{[q]} = (f^q \mid f \in I)$.

Theorem (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

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Notation: $I^{[q]} = (f^q \mid f \in I)$.

Harbourne's Conjecture

Let I be a radical ideal in a regular ring of big height h.

For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Dumnicki, Szemberg, Tutaj-Gasińska, 2013

There exists a radical ideal in $\mathbb{C}[x, y, z]$ with $h = 2$ and $I^{(3)} \nsubseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

This corresponds to the Fermat configuration of 12 points in $\mathbb{P}^2$.

Harbourne's Conjecture

Let I be a radical ideal of big height h in a regular ring.

For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

When does Harbourne's Conjecture hold?

- For monomial ideals.
- For general points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).
- For star configurations (Harbourne–Huneke).

Goals:

- Study the smallest open case of this open question:

Open Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

- Find versions of Harbourne's Conjecture that hold:

Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal of big height h in a regular ring. For all $n \geqslant 1$,

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Symbolic powers in char p

Harbourne's Conjecture

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- For star configurations (Harbourne–Huneke).

Theorem (G–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic p. If R/I is an F-pure ring, then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Theorem (G–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic p. If R/I is an F-pure ring, then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

This includes the case when I is a monomial ideal.

Some F-pure rings include determinantal rings, Veronese rings.

Theorem (Fedder's Criterion)

Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p, and I a radical ideal in R. The ring R/I is F-pure if and only if

$$\left(I^{[p]} : I\right) \not\subseteq \mathfrak{m}^{[p]}.$$

$I^{[p]} = (f^p \,|\, f \in I)$.

$(J : I) = \{r \in R \,|\, rI \subseteq J\}$.

Theorem (Fedder's Criterion)

Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p, and I a radical ideal in R. The ring R/I is F-pure if and only if for all $q = p^e$

$$\left(I^{[q]} : I\right) \not\subseteq \mathfrak{m}^{[q]}.$$

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- Show that

$$\left(I^{[q]} : I\right) \subseteq \left(I^n : I^{(hn-h+1)}\right)^{[q]}$$

for all $q = p^e \gg 0$.

Proof sketch (part II)

For all $q = p^e \gg 0$,

$$\left(I^{[q]} : I\right) \subseteq \left(I^n : I^{(hn-h+1)}\right)^{[q]}.$$

If $I^{(hn-h+1)} \not\subseteq I^n$ for some n, then there exists $q = p^e$ such that

$$\left(I^{[q]} : I\right) \subseteq \left(I^n : I^{(hn-h+1)}\right)^{[q]} \subseteq \mathfrak{m}^{[q]},$$

and R/I is not F-pure.

Theorem (G–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic p. If R/I is an F-pure ring, then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

This is in fact best possible over the class of squarefree monomial ideals, and thus more generally for the class of ideals defining F-pure rings.

Example

The squarefree monomial ideal

$$I = \bigcap_{i \neq j} (x_i, x_j) \subseteq k[x_1, \ldots, x_v].$$

verifies $I^{(2n-2)} \not\subseteq I^n$ for $n < v$.

But we can do better if we restrict our class of ideals.

$$\{F\text{-pure rings}\} \supseteq \{ \text{ strongly F-regular rings } \}$$

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$$\{F\text{-pure rings}\} \supseteq \{\text{ strongly F-regular rings }\}$$

Good news! There is a Fedder-like criterion for strongly F-regular rings by Donna Glassbrenner.

Theorem (G–Huneke)

Let I be a radical ideal of big height h in an F-finite regular ring of characteristic p. If R/I is a strongly F-regular ring, then for all $n \geqslant 1$,

$$I^{((h-1)n+1)} \subseteq I^{n+1}.$$

Determinantal rings, Veronese rings.

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Open question

Is this best possible?

Corollary

Let I be a radical ideal of big height 2 in an F-finite regular ring of characteristic p.

If R/I is a strongly F-regular ring, then for all $n \geqslant 1$, $I^n = I^{(n)}$.

Height 2 ideals in 3 variables

Open Question (Huneke, 2000)

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Theorem (–)

Let k be a field of characteristic not 3, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2.$$

Monomial space curves

Let k be a field. The kernel of the map

$$k[x, y, z] \longrightarrow k[t^a, t^b, t^c] \subseteq k[t]$$

is a prime ideal of height 2, generated by the maximal minors of

$$\begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}.$$

More generally

We study the ideals I generated by the 2×2 minors of

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

So $I = (f_1 = a_2 b_3 - a_3 b_2, f_2 = a_3 b_1 - a_1 b_3, f_3 = a_1 b_2 - a_2 b_1)$.

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The key ideas in this chapter follow work by Alexandra Seceleanu.

She studied examples with $I^{(3)} \not\subseteq I^2$, but her ideas can also be applied to any $I^{(a)} \subseteq I^b$ and to obtain positive results.

Seceleanu's Ingredients

- $\mathrm{H}^0_{\mathfrak{m}}(R/I^n) = I^{(n)}/I^n$.

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- $I^{(a)} \subseteq I^b$ if and only if the map induced by $I^a \subseteq I^b$ on Ext

$$\mathrm{Ext}^3(R/I^b, R) \longrightarrow \mathrm{Ext}^3\,(R/I^a, R)$$

  is the 0 map.

Seceleanu's ingredients

- $I^{(a)} \subseteq I^b$ if and only if the map induced by $I^a \subseteq I^b$ on Ext

$$\mathrm{Ext}^2(I^b, R) \longrightarrow \mathrm{Ext}^2\left(I^a, R\right)$$

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Seceleanu's ingredients

- $I^{(a)} \subseteq I^b$ if and only if the map induced by $I^a \subseteq I^b$ on Ext

$$\mathrm{Ext}^2(I^b, R) \longrightarrow \mathrm{Ext}^2\left(I^a, R\right)$$

  is the 0 map.
- Get resolutions for all I^n and a lifting of the map $I^{n+1} \subseteq I^n$ from the Rees algebra of I, $\bigoplus_n I^n t^n \subseteq R[t]$.

$$
\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & I^2 \arrow[r] & 0 \\
0 \arrow[r] & G_2 \arrow[r] \arrow[u, dashed] & G_1 \arrow[r] \arrow[u, dashed] & G_0 \arrow[r] \arrow[u, dashed] & I^3 \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & I^2 \arrow[r] & 0 \\
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\end{tikzcd}$$

Apply $\mathrm{Hom}_R(-, R)$.

$$
\begin{tikzcd}
0 & F_2 \arrow[l] \arrow[d, "B"] & F_1 \arrow[l] \arrow[d] & F_0 \arrow[l] \arrow[d] & I^2 \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & G_2 \arrow[l] & G_1 \arrow[l, "A"'] & G_0 \arrow[l] & I^3 \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

$I^{(3)} \subseteq I^2$ if and only if all the columns of B are in the image of A.

We need to solve an explicit linear algebra question.

Theorem (Seceleanu)

The containment $I^{(3)} \subseteq I^2$ is equivalent to

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathrm{im} \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}.$$

$$I = I_2(M) \text{ for } M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Theorem (–)

Let k be a field of characteristic not 3, and I be the ideal of 2×2 minors of

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

in $k[x, y, z]$. If $a_1 | b_2$, then $I^{(3)} \subseteq I^2$.

Harbourne's Conjecture

Is $I^{(2n-1)} \subseteq I^n$ for all n?

Harbourne's Conjecture

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Theorem (–)

Let k be a field of characteristic not 2, 3 or 5, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2 \textit{ and } P^{(2\times 3-1=5)} \subseteq P^3.$$

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$$P^{(3)} \subseteq P^2 \text{ and } P^{(2\times 3-1=5)} \subseteq P^3.$$

We can also give sufficient conditions for $I^{(4)} \subseteq I^3$.

A stable version of Harbourne's Conjecture

Main Question

Does Harbourne's Conjecture always hold *eventually*?

Evidence for the Stable Harbourne Conjecture

Let $a \geqslant 3$, k be a field, and the Fermat ideal

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) \, .$$

This is a well-known counterexample to $I^{(3)} \subseteq I^2$. However,

$$I^{(2n-1)} \subseteq I^n$$

for all $n \geqslant 3$, which follows from work of Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska.

Main Question

Does Harbourne's Conjecture always hold *eventually*?

Stable Harbourne Conjecture

Let I be a radical ideal of big height h in a regular ring. For all $n \gg 0$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Question

If there exists a value of m such that

$$I^{(hm-h+1)} \subseteq I^m,$$

does that imply that

$$I^{(hn-h+1)} \subseteq I^n,$$

for all $n \gg 0$?

Theorem (–)

Let I be a radical ideal of big height h in a regular ring containing a field. If there exists a value of m such that

$$I^{(hm-h)} \subseteq I^m,$$

then

$$I^{(hn-h)} \subseteq I^n,$$

for all $n \gg 0$.

Theorem (–)

Let I be a radical ideal of big height h in a regular ring containing a field. If there exists a value of m such that

$$I^{(hm)} \subseteq I^{m+1},$$

then

$$I^{(hn)} \subseteq I^{n+1},$$

for all $n \gg 0$.

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then

$$I^{(hm-h)} \subseteq I^m,$$

for all $m \gg 0$.

Example

The defining ideal of $k[t^3, t^4, t^5]$ in $k[x, y, z]$ verifies $P^{(2\times 3-2=4)} \subseteq P^3$, and thus $P^{(2m-2)} \subseteq P^m$ for all $m \gg 0$.

Theorem (–)

Let k be a field of characteristic not 2 nor 3, let $a = 3$ or $a = 4$, and let b and c be integers with $a < b < c$. If P is the defining ideal of $k[t^a, t^b, t^c]$, then

$$P^{(4)} \subseteq P^3.$$

As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

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As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

In Chapter 3, we give sufficient conditions on a 2×3 matrix M in $k[x, y, z]$ that imply $I^{(4)} \subseteq I^3$ for $I = I_2(M)$.

Example

The defining ideal P of $k[t^9, t^{11}, t^{14}]$ fails $P^{(4)} \subseteq P^3$, but Macaulay2 computations show that

$$P^{(2\times 4-2=6)} \subseteq P^4,$$

so $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The squarefree monomial ideal

$$I = \bigcap_{i \neq j} (x_i, x_j) \subseteq k[x_1, \ldots, x_v].$$

has $I^{(2n-2)} \not\subseteq I^n$ for $n < v$, but $I^{(2v-2)} \subseteq I^v$. Therefore,

$$I^{(2n-2)} \subseteq I^n \text{ for all } n \gg 0.$$

Symbolic Rees algebras

Definition (Symbolic Rees algebra)

The symbolic Rees algebra of I is the graded algebra

$$\mathcal{R}_s(I) = \bigoplus_{n \geqslant 0} I^{(n)} t^n \subseteq R[t].$$

Very difficult open question (Cowsik)

When is the symbolic Rees algebra of I finitely generated over R?

Example

The symbolic Rees algebra of the defining ideal P of $\mathbb{C}[t^3, t^4, t^5]$ is noetherian, but the symbolic Rees algebra of $\mathbb{C}[t^{25}, t^{29}, t^{72}]$ is not noetherian (by a result of Goto, Nishida and Watanabe). In both cases, $P^{(2n-2)} \subseteq P^n$ for all $n \geqslant 6$.

Questions we will try to tackle

Suppose symbolic Rees algebra of I is finitely generated.

- When is $I^{(a)} \subseteq I^b$?
- Must I verify the stable version of Harbourne's Conjecture?

Theorem (–)

If $\mathcal{R}_s(I) = R\left[It, I^{(2)}t^2, \ldots, I^{(d)}t^d\right]$*, and* $I^{(hn-h+1)} \subseteq I^n$ *for all* $hn - h + 1 \leqslant d$*, then*

$$I^{(hn-h+1)} \subseteq I^n$$

for all $n \geqslant 1$*.*

Theorem (–)

If $\mathcal{R}_s(I) = R\left[It, I^{(2)}t^2, \ldots, I^{(d)}t^d\right]$, *then*

$$I^{(dn-d+1)} \subseteq I^n$$

for all $n \geqslant 1$.

Theorem (–)

Let R be a regular ring of characteristic $p \equiv 2 \pmod{3}$. If I is a radical ideal of big height 2 such that the symbolic Rees algebra of I is generated in degree up to 3, then $I^{(2n-1)} \subseteq I^n$ for all $n \geqslant p^2 + p - 2$.

Theorem (–)

Suppose that h and k are integers such that h and $h-1$ are both coprime with k. There is an infinite set of prime ideals p with the following property:

Given a regular ring R of characteristic p, if I a radical ideal of big height h and such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$, then

$$I^{(hn-h+1)} \subseteq I^n$$

for all $n \geqslant N$, where N only depends on h, k and p.

Theorem (–)

Let I be a radical ideal of big height h in a regular ring R containing a field. If I is such that $I^{(hm)} = \left(I^{(h)}\right)^m$ for all $m \geqslant 1$, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

Open Questions

Huneke's Question

If P is a prime ideal of height 2 in a RLR, must $P^{(3)} \subseteq P^2$?

A stable version of Harbourne's Conjecture

If I is a radical ideal of big height h in a regular ring, is $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$?

Can we ask for more?

If I is a radical ideal of big height h in a regular ring, and given a constant C, must $I^{(hn-C)} \subseteq I^n$ hold for all $n \gg 0$?

Obrigada!