Symbolic powers of prime ideals

Eloísa Grifo

University of Virginia

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Symbolic Power

The n-th symbolic power of a prime ideal P is the ideal given by

$$P^{(n)} = P^n R_P \cap R = \{f \in R : sf \in P^n \text{ for some } s \notin P\}.$$

For any ideal I, the n-th symbolic power of I is given by

$$I^{(n)} = \bigcap_{Q \in \operatorname{Ass}(R/I)} I^n R_Q \cap R.$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) If I is generated by a regular sequence in a Cohen-Macaulay ring, then $I^n = I^{(n)}$.

(3) In general, $I^n \neq I^{(n)}$.

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Example: $I = \left(y^2 - xz, x^2y - z^2, x^3 - yz\right)$ in $R = \mathbb{C}[x,y,z]$, which is the kernel of the map

$$\begin{aligned} \mathbb{C}[x,y,z] &\longrightarrow \mathbb{C}[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^4 \\ z &\longmapsto t^5 \end{aligned}$$

For this prime ideal, $I^{(2)} \neq I^2$. However, $I^{(3)} \subseteq I^2$.

Main question

When is $I^{(b)} \subseteq I^a$?

We could also study

$$\rho(I) = \sup \left\{ \frac{b}{a} : I^{(b)} \not\subseteq I^a \right\} < \infty.$$

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Theorem (Ein-Lazersfeld-Smith, 2001)

Let P be a prime ideal in a smooth algebraic variety over $\mathbb{C}$ of dimension d, and $c = \max\{1, \dim(R) - 1\}$. Then for all $n \geqslant 1$,

$$P^{(cn)} \subseteq P^n.$$

Theorem (Hochster-Huneke, 2002)

Let P be a prime in a RLR containing a field, R, and $c = \max\{1, \dim(R) - 1\}$. Then for all $n \geqslant 1$,

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What do these theorems say for a polynomial ring in 3 variables?

Let $R = k[x, y, z]$, k a field, and P be a prime ideal in R. Then for all $n \geqslant 1$

$$P^{(2n)} \subseteq P^n.$$

In particular, $P^{(4)} \subseteq P^2$.

Question (Huneke, 2000)

Let P be a codimension 2 prime in a RLR. Is

$$P^{(3)} \subseteq P^2?$$

Question (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in $K[x_0, \ldots, x_N]$. Is

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The question has a positive answer for

- Arbitrary ideals in characteristic 2. (Huneke)
- Monomial ideals in arbitrary characteristic.
- General points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).

If $I \subseteq R = k[x_1, \ldots, x_n]$ is such that R/I is F-pure, then $I^{(3)} \subseteq I^2$.

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Counterexample (Dumnicki, Szemberg, Tutaj-Gasińska, 2015)

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \not\subseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

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Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in $K[x_0, \ldots, x_N]$. Then for all $k \geqslant 1$,

$$I^{(Nk-(N-1))} \subseteq I^k.$$

Is all lost?

- The conjecture could still hold for $k \geqslant 3$.
- There are no known counterexamples for prime ideals.

Theorem (Seceleanu, 2015)

Let I be a quasihomogeneous ideal of height 2 in $R = k[x, y, z]$, and $b \geqslant a$. Consider the natural inclusion $\iota : I^b \hookrightarrow I^a$. The following are equivalent:

(1) $I^{(b)} \subseteq I^a$.

(2) *The induced map* $\operatorname{Ext}^2_R(\iota)\colon \operatorname{Ext}^2_R(I^a, R) \to \operatorname{Ext}^2_R\left(I^b, R\right)$ *is* 0.

How can we study $\operatorname{Ext}^2_R(\iota)\colon \operatorname{Ext}^2_R\left(I^a, R\right) \to \operatorname{Ext}^2_R\left(I^b, R\right)$?

We have two problems to solve:

- We need to find free resolutions for the powers of I.
- Given resolutions for I^a and I^b, we need to find extensions of the inclusion map:

$$\begin{tikzcd}
\cdots \arrow[r] & R^{\beta_2} \arrow[r] & R^{\beta_1} \arrow[r] & R^{\beta_0} \arrow[r] & I^a \arrow[r] & 0 \\
\cdots \arrow[r] & R^{\gamma_2} \arrow[r] \arrow[u, dashed] & R^{\gamma_1} \arrow[r] \arrow[u, dashed] & R^{\gamma_0} \arrow[r] \arrow[u, dashed] & I^b \arrow[r] \arrow[u, hook, "\iota"'] & 0
\end{tikzcd}$$

A nice class of primes

We will focus on the primes defining monomial curves as follows: we are looking at the kernels of maps of the form

$$\begin{aligned} K[x,y,z] &\longrightarrow K[t] \\ x &\longmapsto t^a \\ y &\longmapsto t^b \\ z &\longmapsto t^c \end{aligned}$$

A nice class of primes

We are looking at height 2 primes of the form $P = \ker f$, where $f : K[x, y, z] \longrightarrow K[t]$ is given by

$$f(x) = t^a, f(y) = t^b, f(z) = t^c.$$

If P is not generated by a regular sequence, then P is the ideal of 2×2 minors of

$$\begin{bmatrix} x^{\alpha} & y^{\beta} & z^{\gamma} \\ z^{\gamma'} & x^{\alpha'} & y^{\beta'} \end{bmatrix}.$$

Rees Algebra

The Rees algebra of P is given by $\mathcal{R}(P) = \bigoplus_{n \geqslant 0} P^n t^n$. This is a quotient of $S = R[T_1, T_2, T_3]$.

When P is the ideal of 2×2 minors of a 2×3 matrix, we can resolve $\mathcal{R}(P)$ over S, and by restricting to degree n, we get a free resolution for P^n.

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Theorem (–)

Let $R = K[x, y, z]$, where K is a field of characteristic not 3, and P be a prime defining a monomial curve as before. Then $P^{(3)} \subseteq P^2$.

What else can we say about these primes?

Questions:

(1) Harbourne's question: $P^{(2k-1)} \subseteq P^k$?

(2) Is $P^{(5)} \subseteq P^3$? In general, $P^{(4)} \nsubseteq P^3$.

(3) What is

$$\rho(P) = \sup \left\{ \frac{b}{a} : P^{(b)} \nsubseteq P^a \right\}$$

for these primes?

Thank you!