Random Monomial Ideals

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Random Graphs

Question

What does an average graph look like?

Random Graphs

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What does an random graph look like?

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What does an random graph look like?

Models:

- Erdős-Rényi random graph: $G(n, p)$ with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$
- Random Regular Graphs
- Random scale-free graphs: Barabási–Albert random graphs and others

Theorem (Erdős, Rényi 1959)

Given $\epsilon > 0$, $n \in \mathbb{N}$ and p a function of n,

$$\mathbb{P}[G(n, p) \textit{ is connected}] \to \begin{cases} 1 & \textit{if } p > \frac{(1+\epsilon)\log n}{n} \\ 0 & \textit{if } p < \frac{(1-\epsilon)\log n}{n} \end{cases}$$

Random Monomial Ideals

Goal: Construct and study useful models of random monomial ideals to better understand the behavior of ideals.

Recent Results:

- Erdős–Rényi-type random monomial ideals (De Loera, Petrović, Silverstein, Stasi, Wilburne 2019).
- Random monomial ideals from random flag complexes (Erman, Y. 2018).
- Free resolutions of ER-type random monomial ideals (De Loera, Hoșten, Krone, Silverstein 2018).
- Regularity of random edge ideals (Banerjee, Yogeshwaran 2020).
- Torsion in syzygies of random monomial ideals (Booms, Erman, Y. 2020).
- Degrees of random monomial ideals (Silverstein, Wilburn, Y. 2020).

Basic Probability

- $\mathbb{P}[A]$: The probability of an event A.
- Asymptotically almost surely: Given a family of events $\{A_n\}_{n\in\mathbb{N}}$, then they occur asymptotically almost surely if $\mathbb{P}[A_n] \to 1$ as $n \to \infty$.
- Convergence in probability: Given a family of random variables $\{X_n\}_{n\in\mathbb{N}}$, and a random variable Y, $X_n \to Y$ in probability if $\mathbb{P}[|X_n - Y| \leq \epsilon] \to 0$ as $n \to \infty$ for all $\epsilon > 0$.

ER-type Random Monomial Ideals

Definition (De Loera, Petrović, Silverstein, Stasi, Wilburne 2019)

Let $\mathcal{I}(n, D, p)$ be the model of a random monomial ideal in $k[x_1, \ldots, x_n]$ given by choosing (possibly redundant) generators from the monomials of degree at most D each with probability p.

Question

What is the behavior of a random ideal in the model $\mathcal{I}(n, D, p)$ as $D \to \infty$?

A Hyperbola Law and Degrees of Random Monomial Ideals

Theorem (Silverstein-Wilburne-Y. 2020)

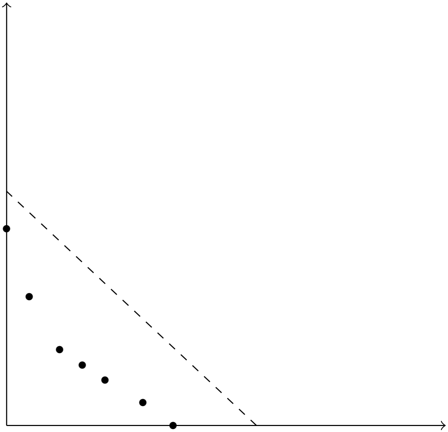
Let $\mathcal{I} \sim \mathcal{I}(n, D, p)$ with $D \to \infty$ and $p = D^{-k}$ for fixed $k \in (0, n)$. Then,

$$\mathbb{P}\left[D^{k-\epsilon} < \prod_{i=1}^{n}(\alpha_i + 1) < D^{k+\epsilon} \text{ for all } x^\alpha \in G(\mathcal{I})\right] \to 1,$$

and

$$\mathbb{P}\left[x^\alpha \in \mathcal{I} \text{ for all } x^\alpha \text{ s.t. } D^{k+\epsilon} < \prod_{i=1}^{n}(\alpha_i + 1) \text{ and } |\alpha| \leq D\right] \to 1.$$

A Hyperbola Law and Degrees of Random Monomial Ideals



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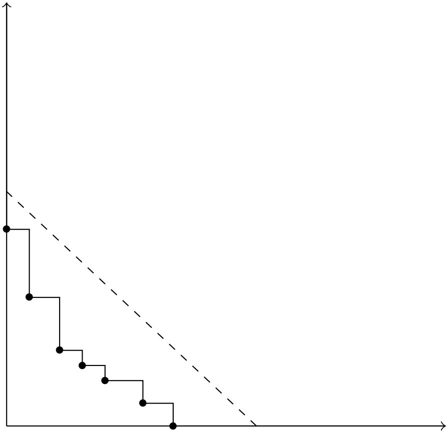

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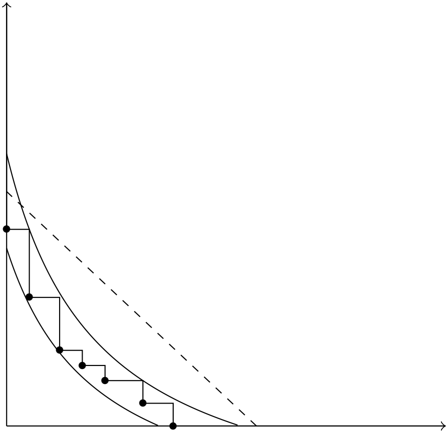

Degrees of Random Monomial Ideals

Theorem (Silverstein-Wilburne-Y. 2020)

Let $\mathcal{I} \sim \mathcal{I}(n, D, p)$, and suppose $p = D^{-k}$ for $k \in (0, n)$, not an integer. Let $s = \lfloor k \rfloor$. Then there exist constants $C_1, C_2 > 0$ such that asymptotically almost surely as $D \to \infty$,

$$C_1 D^{k-s-\epsilon}(\log D)^{n-1} < \deg(\mathcal{I}) < C_2 D^{k-s+\epsilon}(\log D)^{n-1}.$$

Random Flag Complexes

A flag complexes is a simplicial complex maximal among simplicial complexes with a fixed 1-skeleton.

Definition

Let $\Delta(n, p)$ is the model of a random flag complex given by taking the flag complex associated to a Erdős–Rényi random graph $G(n, p)$.

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Example

6

4 3 5

1 2

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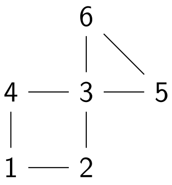

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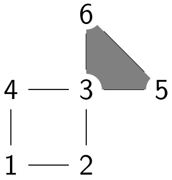


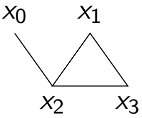
Stanley–Reisner Ideals

Definition

Given a polynomial ring $S = k[x_1, \ldots, x_n]$, and a simplicial complex Δ on the variables. The <u>Stanley–Reisner ideal</u> associated to Δ, denoted I_Δ is the squarefree monomial ideal generated by monomials corresponding to the minimal non-faces of Δ

Example

$\Delta =$



$$I_\Delta = \langle x_0x_1, x_0x_3, x_1x_2x_3 \rangle$$

Random Stanley–Reisner Ideals

Definition

A random Stanley–Reisner ideal is the distribution given by taking the Stanley–Reisner ideal associated with a random flag complex.

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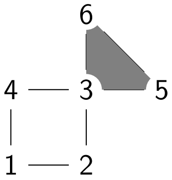

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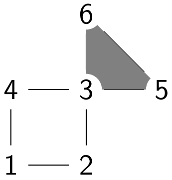


$$I_\Delta = (x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6)$$

Random Stanley–Reisner Ideals

Definition

A random Stanley–Reisner ideal is the distribution given by taking the Stanley–Reisner ideal associated with a random flag complex.

Question

What is the behavior of the Stanley–Reisner ideal associated to $\Delta(n, p)$ as $n \to \infty$.

Syzygies

Definition

Let k be a field and M a module over a ring $S = k[x_1, \ldots, x_n]$, then the Betti numbers are

$$\beta_{i,j}(M) = \dim Tor_i(M, k)_j$$

The Betti table is the following arrangement of the Betti numbers:

$$\begin{array}{|cccc}
\hline
\beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\
\beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

Asymptotic Syzygies

Question

Given a family of modules $\{M_n\}_{n\in\mathbb{N}}$ with increasing projective dimension, what can be said about the behavior of the Betti numbers $\beta_{i,j}(M_n)$ as $n \to \infty$?

Theorem (Ein, Lazarsfeld 2012)

Let X be a smooth, d-dimensional projective variety and let A be a very ample divisor on X. For any $n \geq 1$, let S_n be the homogeneous coordinate ring of X embedded by nA. For each $1 \leq k \leq d$, $\rho_k(S_n) \to 1$ as $n \to \infty$.

Where $\rho_k(M)$ is the "density" for the k-th row of the Betti table as given by the formula

$$\rho_k(M) := \frac{\#\{i \in [0, \mathrm{pdim}(M)] \text{ where } \beta_{i,i+k}(M) \neq 0\}}{\mathrm{pdim}(M) + 1}.$$

Asymptotic Syzygies of Random Monomial Ideals

Question

Let $S = k[x_1, \ldots, x_n]$, and $\Delta \sim \Delta(n, p)$ a random flag complex, what can be said about $\beta_{i,j}(S/I_\Delta)$ as $n \to \infty$?

Theorem (Erman, Y. 2018)

Fix some $r \geq 1$. *Let* $\Delta \sim \Delta(n, p)$ *with* $\frac{1}{n^{1/r}} \ll p \ll 1$. *For each* $1 \leq k \leq r+1$, *we have*

$$\rho_k(S/I_\Delta) \to 1$$

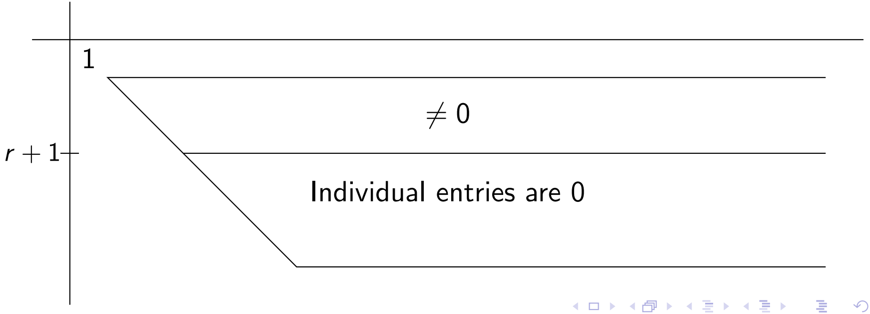
in probability.

Asymptotic Syzygies of Random Monomial Ideals

Theorem (Erman-Y., 2018)

Fix i, j with $1 \leq i$ and $i+1 \leq j \leq 2i$ and let $r := j - i - 1$. Fix some constant $0 < \epsilon \leq \frac{1}{2}$ and let $\Delta \sim \Delta(n, p)$.

1. *If $\frac{1}{n^{1/r}} \ll p \leq \epsilon$ then $\mathbb{P}[\beta_{i,v}(S/I_\Delta) \neq 0] \to 1$.*
2. *If $p \ll \frac{1}{n^{1/r}}$ then $\mathbb{P}[\beta_{i,v}(S/I_\Delta) = 0] \to 1$.*



A Heuristic and Normal Distribution of the Quadratic Strand

Heuristic

The asymptotic behavior of the Betti table for embeddings of smooth projective varieties resemble that of a random Stanley–Reisner ideal.

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The asymptotic behavior of the Betti table for embeddings of smooth projective varieties resemble that of a random Stanley–Reisner ideal.

Theorem

Fix a constant $0 < c < 1$ and let $\Delta \sim \Delta(n, \frac{c}{n})$ be a random flag complex. If $\{i_n\}$ is an integer sequence satisfying $i_n = n/2 + o(n)$, and if $C := \frac{1-c}{2}$,

$$\frac{\beta_{i_n, i_n+1}(S/I_\Delta)}{nC\binom{n}{i_n}} \longrightarrow 1$$

in probability.

This is a random analog to a normal distribution result by Ein, Erman, and Lazarsfeld in 2015.

Torsion in Syzygies

Question

Given a module M over $T = \mathbb{Z}[x_1, \ldots, x_n]$, and let $S = k[x_1, \ldots, x_n]$, does $\beta_{i,j}(M \otimes S)$ depend on the characteristic of k? In that case we say $\beta(M)$ has torsion.

For the Veronese module of $\mathbb{P}^r$ embedded by degree d, this is occurs for $r \geq 6$ and $d = 2$ (Andersen 1992, Reiner–Roberts 2000, Jonnson 2010) but not for $\mathbb{P}^1$ and any d.

Torsion in Random Syzygies

Question

Let Δ be a random flag complex chosen from $\Delta(n, p)$. For a field k, when does $\beta(k[x_1, \ldots, x_n]/I_\Delta)$ depend on the characteristic of k?

Theorem (Booms, Erman, Y. 2020)

Fix $\ell \geq 2$, and let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-1/6} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$. Asymptotically almost surely as $n \to \infty$, the Betti table of the Stanley–Reisner ideal of Δ has ℓ-torsion.

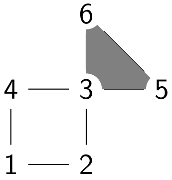
Hochster's Formula

Theorem (Hochster's Formula)

For Δ a simplicial complex and I_Δ the associated Stanley–Reisner Ideal

$$\beta_{i,j}(S/I_\Delta) = \sum_{|\alpha|=j} \dim \widetilde{H}_{i-j-1}(\Delta|_\alpha)$$

Example of Hochster's Formula



$$I_\Delta = (x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6)$$

	0	1	2	3	4
0	1	–	–	–	–
1	–	8	13	7	1
2	–	–	1	2	1

Outline of Proof

Induced subgraphs control the Betti table thus we wish to:

1. Find a “small” graph with the desired homology.
2. Show that that subgraph occurs in a random graph.

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1. Find a "small" graph with the desired homology.
2. Show that that subgraph occurs in a random graph.

Definition

The **essential density** of a graph G is

$$m(G) := \max\left\{\frac{|E(H)|}{|V(H)|} \;:\; H \subset G, |V(H)| > 0\right\},$$

and G is **strictly balanced** if $m(H) < m(G)$ for all proper subgraphs $H \subset G$.

Thresholds for Subgraphs

Proposition

Let G' be a fixed graph, and let $G(n,p)$ be a random graph. Suppose $p = p(n) \leq 1 - \epsilon$ for some $\epsilon > 0$. Then as $n \to \infty$, we have

$$\mathbb{P}\left[G' \overset{ind}{\subset} G(n,p)\right] \to \begin{cases} 0 & \text{if } p \ll n^{-1/m(G')} \\ 1 & \text{if } p \gg n^{-1/m(G')} \end{cases}.$$

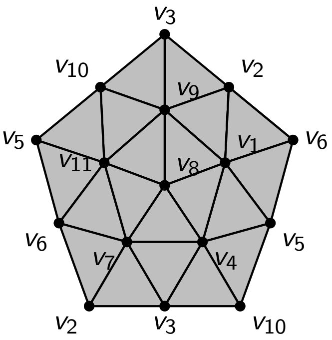
Remark

This proposition and its proof are based on a very similar theorem proven by Bollobás in 1981 for subgraphs.

Finding 2-torsion

Recall $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

$\mathbb{RP}^2 \cong$



The underlying graph G, is strictly balanced and $m(G) = 30/11$

Finding 2-torsion

Proposition

Let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-11/30} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$. Asymptotically almost surely as $n \to \infty$, the Betti table of the Stanley–Reisner ideal of Δ has 2-torsion.

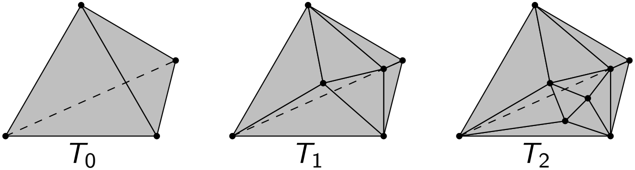
Finding ℓ-torsion

Goal: Construct a simplicial complex X_m such that $\mathbb{Z}/m\mathbb{Z} \subset H_1(X_m)$.
This techniques is based on a technique in a paper by Andrew Newman in 2018 for constructing (non-flag) simplicial complexes with prescribed homology.
Let $m = 2^{n_1} + \cdots + 2^{n_k}$ then

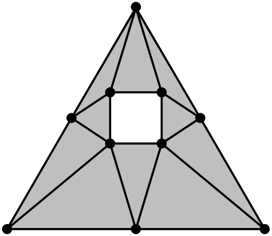
$$\mathbb{Z}/m\mathbb{Z} = \langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} \mid 2\gamma_i = \gamma_{i+1}, \gamma_{n_1} + \cdots + \gamma_{n_k} = 0 \rangle .$$

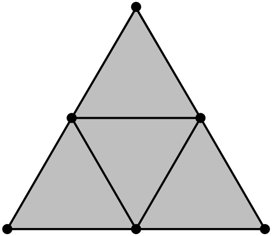
1. Construct a sphere with n_k punctures.
2. Glue the punctures in order.
3. Give a flag triangulation.

Creating a Sphere (with Holes)

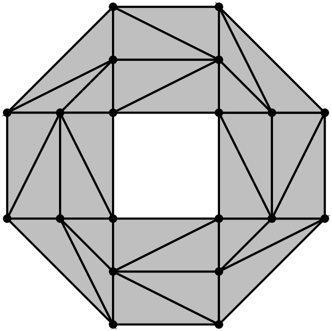


Making Holes





Telescopes



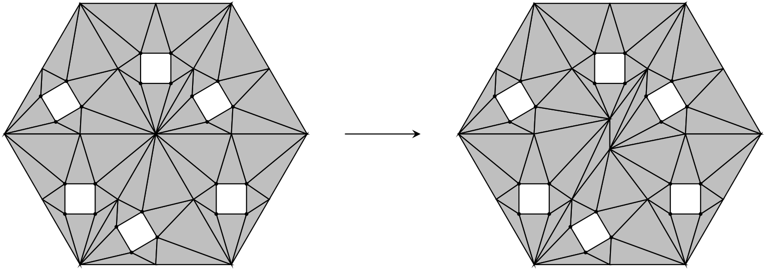
Degrees of vertices

In the worst case, the largest degree vertices have degrees 12,14. If used as is, it would give the following theorem

Theorem (Booms, Erman, Y. 2020)

Fix $\ell \geq 2$, and let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-1/7} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$. Asymptotically almost surely as $n \to \infty$, the Betti table of the Stanley–Reisner ideal of Δ has ℓ-torsion.

Removing Degree 14 Vertices



Main Result

Theorem (Booms, Erman, Y. 2020)

Fix $\ell \geq 2$, and let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-1/6} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$. Asymptotically almost surely as $n \to \infty$, the Betti table of the Stanley–Reisner ideal of Δ has ℓ-torsion.

Main Result

Theorem (Booms, Erman, Y. 2020)

Fix $\ell \geq 2$, and let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-1/6} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$. Asymptotically almost surely as $n \to \infty$, the Betti table of the Stanley–Reisner ideal of Δ has ℓ-torsion.

Heuristic (Booms, Erman, Y. 2020)

For a reasonable class of families of modules $\{M_n\}_{n \in \mathbb{N}}$, As $n \to \infty$, the number of primes ℓ such that $\beta(M_n)$ has ℓ-torsion is unbounded.

Thank You

Thank You

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