Problem Set 1

Turn in any **4** of the following problems. Slightly more challenging problems are indicated by $(\star)$. You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

Problem 1. Consider the category R-**mod**.

a) Show that a homomorphism of R-modules is injective if and only if it is a mono in R-**mod**, and surjective if and only if it is an epi in R-**mod**.

b) Show that the homomorphism of $\mathbb{Z}$-modules $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is monic but has no left inverse in $\mathbb{Z}$-**mod**.

c) Show that the canonical homomorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is epic but has no right inverse in $\mathbb{Z}$-**mod**.

Problem 2.

a) Show that in any category, every isomorphism is both an epi and a mono.

b) Show that the usual inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi in the category **Ring**.

This *should* feel weird: it says being epi and being surjective are *not* the same thing.

c) Show that the canonical projection $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is a mono in the category of divisible abelian groups.[1]

Again, this is very strange: it says being monic and being injective are *not* the same thing.

Problem 3. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are concrete categories and $F\colon \mathscr{C} \to \mathscr{D}$ is a covariant functor.

a) Show that if α is an arrow in $\mathscr{C}$ that has a left inverse, then $F(\alpha)$ is an injective function.

b) Show that if α is an arrow in $\mathscr{C}$ that has a right inverse, then $F(\alpha)$ is a surjective function.

c) Use part (a) to show[2] that there is no covariant functor $F\colon \mathbf{Grp} \to \mathbf{Grp}$ that, on objects, maps a group to its center.

Problem 4. We will show that every functor creates isos, and fully faithful functors reflect isos. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor.

a) Show that if f is an iso in $\mathscr{C}$, then $F(f)$ is an iso in $\mathscr{D}$.

b) Show that if X and Y are isomorphic objects in $\mathscr{C}$, then $F(X)$ and $F(Y)$ are isomorphic in $\mathscr{D}$.

c) Suppose F is fully faithful. Show that if $F(f)$ is an iso, then f is an iso.

d) Let F be fully faithful. Show that if $F(X)$ and $F(Y)$ are isomorphic in $\mathscr{D}$, then X and Y are isomorphic in $\mathscr{C}$.

e) Find an example to show that a faithful functor need not reflect isomorphisms.

[1] An abelian group A is divisible if for every $a \in A$ and every positive integer n there exists $b \in A$ such that $nb = a$.

[2] Hint: You might consider some appropriate inclusion of the group $\mathbb{Z}/2$ into the symmetric group $\mathbb{S}_3$.