Problem Set 1

Problem 0. This problem is mandatory.

a) Install Macaulay2.[1] Hardcore version: install emacs and run Macaulay2 through emacs.

b) Make an .m2 file setting up a polynomial ring R over a field k, a nontrivial ideal I in R, the R-module $M = R/I$ and the ring $S = R/I$.

Problem 1. Use Macaulay2 to find:

a) A presentation for the $\mathbb{Q}$-algebra $\mathbb{Q}[xy, xu, yv, uv] \subseteq \mathbb{Q}[x, y, u, v]$.

b) A presentation for the k-algebra U, where $k = \mathbb{Z}/101$ and

$$k \begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{k[u, v, x, y, z]}{(x^3 + y^3 + z^3)}.$$

Problem 2. Let K be a field, and $R := K[x^2, x^3] \subseteq S := K[x]$. Let $I = x^2R$ be the ideal generated by x^2 in R. Show that $IS \cap R \supsetneq I$, and conclude that R is not a direct summand of S.

Problem 3. Let k be a field. Is $k[x^{42}, y^{17} + y^7, z^{73} + z] \subseteq k[x, y, z]$ module-finite?

Problem 4. Find $\overline{\mathbb{Z}}$, the integral closure of $\mathbb{Z}$ in its field of fractions $\mathbb{Q}$.

Problem 5. If $R[x]$ is Noetherian, must R be Noetherian?

Problem 6. Suppose that every ascending chain of prime ideals in R stabilizes. Must R be a noetherian ring? Spoiler alert: there is a hint in the footnotes.[2]

Problem 7. Show that R is a Noetherian ring if and only if every prime ideal of R is finitely generated. Spoiler alert: there's a hint in the footnotes.[3]

[1] If you don't have access to a computer, or if your computer runs only Windows, come talk to me about it.

[2] Hint: it may be helpful to think of how we could construct rings with very few prime ideals.

[3] Use Zorn's Lemma to show that if R is a ring that is not noetherian, the set of ideals that are not finitely generated has a maximal element; show that element is a prime ideal.