Research Statement (Eloísa Grifo)

My research interests lie in Commutative Algebra and Homological Algebra. I am interested in questions that relate to infinite free resolutions, DG algebras, local cohomology and symbolic powers, and in applying Homological Algebra and characteristic p techniques to these topics. Some of the projects I have been working on recently relate to

(1) Symbolic powers of ideals and the Containment Problem;
(2) DG algebras and deviations;
(3) Levels of complexes.

1. Symbolic Powers and the Containment Problem

1.1. **Symbolic Powers.** In a polynomial ring over a perfect field k, radical ideals correspond to varieties in k^n. The Zariski–Nagata Theorem [Zar49, Nag62] says that the n-th symbolic power of a given prime ideal consists of the elements that vanish up to order n on the corresponding variety.

Symbolic powers are defined algebraically, in terms of primary decomposition. To give a primary decomposition of I consists of finding primary ideals $Q_1, \ldots, Q_k$ such that I can be written as $I = Q_1 \cap \cdots \cap Q_k$. A classical result in Commutative Algebra states that every ideal in a Noetherian ring has a primary decomposition. Primary ideals can be thought of as a generalization of prime ideals; prime ideals are primary, but not all primary ideals are prime. The radical of a primary ideal, however, is always prime.

While primary decompositions are not unique, there are some unicity aspects to those that are irredundant, meaning that no component can be deleted and that the radicals of each Q_i are distinct. These radicals turn out to be the associated primes of the original ideal, and the components corresponding to minimal primes are indeed unique. If P is a minimal prime of the ideal I in the regular ring R, the P-primary component of I is given by all elements a such that $sa \in I$ for some $s \notin P$. In other words, the pre-image in R of the localization at P of the ideal I, which we write $IR_P \cap R$.

In particular, given a prime ideal P, its (ordinary) n-th power P^n, which is the ideal generated by all products of n elements in P, is not necessarily primary, but we can find a primary decomposition for each P^n. The primary component of P^n with radical P is the n-th symbolic power of P, which we denote by $P^{(n)}$, and it can be computed by taking $P^n R_P \cap R$. The remaining components correspond to *embedded* primes of P, which are associated primes that are not minimal.

More generally, given an ideal I with no embedded components, its powers I^n might have embedded primes. We define the n-th **symbolic power** of I by

$$I^{(n)} = \bigcap_{P \in \operatorname{Ass}(I)} I^n R_P \cap R.$$

The symbolic powers of I are a sequence of ideals containing I that form a graded family, meaning that $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$. Moreover, $I^{(n+1)} \subseteq I^{(n)}$ for all n. While symbolic powers have good geometric properties, they can be very difficult to compute; on the other hand, ordinary powers are easily computable, but do not enjoy good geometric properties.

Although powers and symbolic powers do not, in general, coincide, there are ways to compare these two notions. Geometrically, consider points $p_1, \ldots, p_k$ in $\mathbb{P}^n$ with corresponding ideal $I = I(p_1) \cap \cdots \cap I(p_k)$ in $\mathbb{C}[x_0, \ldots, x_n]$. After a change of coordinates, each $I(p_i)$ is generated by variables, so that $I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_k)^m$ — these are the polynomials that vanish up to order m at each point p_i. As an example, consider $I = (x, y) \cap (x, z) \cap (y, z)$ in $\mathbb{C}[x, y, z]$, with $I^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2$, and note that $xyz \in I^{(2)}$ but $xyz \notin I^2$. However, one can show that $I^{(3)} \subseteq I^2$. Containments of this type are the subject of the Containment Problem, which I am very interested in.

Even though the subject of primary decomposition has been around for a century, there are many open questions related to symbolic powers, even over a polynomial ring. When does equality of ordinary and symbolic powers hold? Given a homogeneous ideal in a polynomial ring, what degrees are the symbolic powers generated in? What is a minimal set of generators for the n-th symbolic power of a given ideal? These are some of the reasons why the Containment Problem is so difficult. For a survey on symbolic powers, see [DDSG$^+$17].

1.2. **The Containment Problem.** The Containment Problem asks for which values of a and b the containment $I^{(a)} \subseteq I^b$ holds. Over a regular ring R, there is a famous answer of Ein-Lazersfeld-Smith [ELS01] and Hochster-Huneke [HH02], recently generalized to the mixed characteristic setting by Ma-Schwede [MS17]: $I^{(hn)} \subseteq I^n$ if I is a prime ideal of height h, or more generally if I is a radical ideal and h is the largest height of an associated prime of I, known as the **big height** of I.

However, this containment can often be improved. In the example we discussed above, where $I = (x, y) \cap (x, z) \cap (y, z)$ in $\mathbb{C}[x, y, z]$, the theorem says that $I^{(2n)} \subseteq I^n$, and in particular that $I^{(4)} \subseteq I^2$, although we actually have $I^{(3)} \subseteq I^2$. A famous question of Huneke asks if this behavior is more general: given an unmixed height two reduced ideal I in a regular ring, is $I^{(3)} \subseteq I^2$? Brian Harbourne proposed the following generalization:

Conjecture 1 (Harbourne, 2006, first appeared in print in [HH13, TBS09])**.** *Let I be a radical ideal in a regular ring containing a field, and let h be the big height of I. Then for all $n \geqslant 1$,*

$$I^{(hn-h+1)} \subseteq I^n.$$

When $h = 2$, Harbourne asks if $I^{(2n-1)} \subseteq I^n$ for all $n \geqslant 2$, and in particular, if $I^{(3)} \subseteq I^2$. A counterexample to the containment $I^{(3)} \subseteq I^2$ was first found by Dumnicki, Szemberg and Tutaj-Gasińska in [DSTG13], and later extended to a larger class of ideals, known as the Fermat configurations, by Harbourne and Seceleanu in [HS15]. Other classes of counterexamples have since been found. However, these counterexamples correspond to very special configurations of points, and there is evidence to support the idea that the conjecture should hold for nice classes of ideals. In particular, there are no known counterexamples for prime ideals.

There is still a lot we do not know about the Containment Problem, and in particular, about Harbourne's Conjecture. Three main questions that remain open are:

Question 1. *When does Harbourne's Conjecture hold, meaning, for which classes of ideals I is $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$?*

Question 2. *Does Harbourne's Conjecture hold for all prime ideals?*

Question 3. *Does Harbourne's Conjecture hold* eventually*, meaning that given a radical ideal I, does $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$?*

In the next few subsections, I will talk about some of my own efforts to answer these questions, using techniques such as characteristic p methods, Rees algebras, linkage, and Homological Algebra.

1.3. **When does Harbourne's Conjecture hold?** There are various cases where Harbourne's conjecture is known to hold: if I is a monomial ideal (see [TBS09, Example 8.4.5]), if I corresponds to a general set of points in $\mathbb{P}^2$ ([BH10]) or $\mathbb{P}^3$ ([Dum15]), and if I corresponds to a star configuration of points ([HH13]). More recently, in [GH17], Craig Huneke and I proved Harbourne's Conjecture for ideals defining F-pure rings.

Theorem 1 (G–Huneke)**.** *Let R be a regular ring of characteristic $p > 0$ (respectively, essentially of finite type over a field of characteristic 0). Let I be an ideal in R with R/I F-pure (respectively, of dense F-pure type), and let h be the big height of I. Then for all $n \geqslant 1$, $I^{(hn-h+1)} \subseteq I^n$.*

This generalizes the result on squarefree monomial ideals, since these define F-pure rings. Veronese subrings of polynomial rings, locally acyclic cluster algebras and determinantal rings, or more generally certain ladder determinantal varieties, are also F-pure. Moreover, after reduction modulo p, rings of invariants of classical groups are F-pure.

Huneke and I also showed that we can obtain tighter containments under a stronger assumption:

Theorem 2 (G–Huneke)**.** *Let R be a regular ring of characteristic $p > 0$ (respectively, essentially of finite type over a field of characteristic 0), and let I be an ideal of height $h \geqslant 2$ such that R/I is strongly F-regular (respectively, has log-terminal singularities). Then $I^{(d)} \subseteq I I^{(d-h+1)}$ for all $d \geqslant h$. In particular, $I^{(n(h-1)+1)} \subseteq I^{n+1}$ for all $n \geqslant 1$.*

Note that this is the statement we obtain from Harbourne's Conjecture if we replace h by $h-1$. Strongly F-regular rings are always F-pure, and in fact all the F-pure examples above are strongly F-regular with the exception of rings defined by squarefree monomial ideals. As a corollary, we obtained the following result, which provides many nontrivial examples of ideals for which equality of symbolic and ordinary powers of ideals holds: if I is a height 2 prime such that R/I is strongly F-regular, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

While we have given examples showing that our results are sharp for the F-pure case, we do not know if they can be improved in the strongly F-regular case. I would like to find other classes of ideals verifying tighter containments such as the one we obtained in the strongly F-regular case.

Question 4. *What classes of radical ideals of big height h verify $I^{(n(h-1)+1)} \subseteq I^{n+1}$ for all $n \geqslant 1$? Is this containment best possible for ideals defining strongly F-regular rings?*

Question 5. *Are there other conditions on the singularities of R/I that allow us to answer the Containment Problem for I?*

1.4. **Does the conjecture hold for prime ideals?** There are no known prime counterexamples to $I^{(3)} \subseteq I^2$. In my PhD thesis, I studied the case of space monomial curves, which was the smallest open case:

Theorem 3 (G)**.** *Let k be a field of any characteristic except 2 or 3, and P be the defining ideal in $k[x, y, z]$ of the space monomial curve $k[t^a, t^b, t^c]$. Then $P^{(3)} \subseteq P^2$.*

In fact, building on work of Seceleanu [Sec15], I was able to prove that if I is the ideal $I_2(M)$ of maximal minors of a 2×3 matrix M in $k[x, y, z]$, $I^{(3)} \subseteq I^2$ holds as long as $M_{1,1} | M_{2,2}$ up to row and column operations. I would like to understand if this condition is necessary; the Fermat configurations of points seem to fail this condition in a minimal way. Moreover, the techniques involved can be applied to any containment $I^{(a)} \subseteq I^b$, and I was able to apply them to find sufficient conditions for $I^{(4)} \subseteq I^3$ in terms of the entries of the matrix M, and to show that $I^{(5)} \subseteq I^3$ for all space monomial curves.

It is natural to ask if there are similar statements for ideals of maximal minors of larger matrices.

Question 6. *If $I = I_n(M)$, where M is an $n \times (n+1)$ matrix in a regular ring containing a field, what are necessary and/or sufficient conditions involving the entries of M for $I^{(a)} \subseteq I^b$?*

1.5. **Does Harbourne's Conjecture hold eventually?** Even though the Fermat configurations of points fail the containment $I^{(3)} \subseteq I^2$, the containment $I^{(2n-1)} \subseteq I^n$ does hold for all $n \geqslant 3$, which follows from work in [DHN+15].

Conjecture 2. *Let I be a radical ideal in a regular ring, and let h be the big height of I. Then for all $n \gg 0$,*

$$I^{(hn-h+1)} \subseteq I^n.$$

There are no known counterexamples to this conjecture. One approach would be to prove the following stronger statement:

Question 7. *Let I be a radical ideal in a regular ring, and let h be the big height of I. If $I^{(hk-h+1)} \subseteq I^k$ for some k, does that imply that $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$?*

If the answer to this question is affirmative, the containment $I^{(3)} \subseteq I^2$ would imply Harbourne's Conjecture, and in particular my work on space monomial curves would be enough to show this class of ideals verifies Harbourne's Conjecture.

I have, however, been able to show a similar result holds under a slightly stronger assumption:

Theorem 4 (G)**.** *Let I be a radical ideal in a regular ring, and let h be the big height of I. If $I^{(hk-h)} \subseteq I^k$ for some k, then $I^{(hn-h)} \subseteq I^n$ for all $n \gg 0$.*

In particular, if I has big height 2, $I^{(4)} \subseteq I^3$ implies $I^{(2(n-1))} \subseteq I^n$ for all $n \gg 0$. Various classes of ideals verify this stronger containment; in particular, even though not all space monomial curves verify $I^{(4)} \subseteq I^3$, many do; using my sufficient conditions, I was able to show this stronger version of a stable Harbourne conjecture holds for certain space monomial curves: for example, $k[^3, t^4, t^5]$, or those such as in [SG94], even though their symbolic Rees algebras (see definition below) are not Noetherian in characteristic 0.

These asymptotic type containments can also be studied via the resurgence of Bocci and Harbourne [BH10]:

Definition 1 (Bocci-Harbourne)**.** *Let I be a radical ideal. The* **resurgence** *of I is*

$$\rho(I) := \sup \left\{ \frac{a}{b} \mid I^{(a)} \nsubseteq I^b \right\}.$$

Over a regular ring, $1 \leqslant \rho(I) \leqslant h$ for radical ideals of big height h. There are no known examples with $\rho(I) = h$, and in fact, if $\rho(I) < h$, then for all integers $C > 0$, $I^{(hn-C)} \subseteq I^n$ for all $n \gg 0$. I would like to determine if this is always the case.

Question 8. *Is there a radical ideal I with big height h such that $\rho(I) = h$?*

Note that $\rho(I) < h$ is only a sufficient condition for $I^{(hn-h)} \subseteq I^n$ for all $n \gg 0$, but not a necessary condition.

1.6. **What else can we say about the Containment Problem?** A key idea behind both my work and that of Seceleanu on the symbolic powers of the maximal minors of a 2×3 matrix is the fact that if I is a radical ideal of height $d-1$ in a Gorenstein local or polynomial ring of dimension d, $I^{(a)} \subseteq I^b$ if and only if the map $\mathrm{Ext}^{d-1}\left(I^b, R\right) \to \mathrm{Ext}^{d-1}\left(I^a, R\right)$ induced by the inclusion $I^a \subseteq I^b$ is the zero map. This allows us to convert a very difficult question about symbolic powers into a homological question involving only the ordinary powers of I. This turns out to be a very powerful tool, and I would like to find other similar homological criteria without assumptions on the height of I.

Question 9. *If I is a radical ideal in a regular ring, can the containment $I^{(a)} \subseteq I^b$ be described in terms of maps on* Ext*?*

Finding free resolutions in order to study this homological question involved studying the Rees Algebra $\oplus_{n \geqslant 0} I^n t^n$ of I. Another related interesting object of study is the graded algebra $\oplus_{n \geqslant 0} I^{(n)} t^n$, known as the **symbolic Rees Algebra** of I. A famous open question of Cowsik asks when this algebra is Noetherian; we might expect better containment results when this turns out to be true.

Question 10. *If I is a radical ideal in a regular ring, and the symbolic Rees algebra of I is Noetherian, what can we say about the Containment Problem for I?*

1.7. **Symbolic Powers in mixed Characteristic and p-derivations.** In a polynomial ring over a perfect field, the symbolic powers of a prime ideal can be described via differential operators [DDSG+17]: as mentioned before, the Zariski–Nagata Theorem states that the n-th symbolic power of a given prime ideal consists of the elements that vanish up to order n on the corresponding variety. However, this description fails in mixed characteristic, or even for a polynomial ring over $\mathbb{Z}$. Alessandro de Stefani, Jack Jeffries and I defined a new kind of differential powers over smooth $\mathbb{Z}$-algebras, using the p-derivations of Buium and Joyal [Joy85, Bui95], and proved that this new notion does coincide with the symbolic powers of prime ideals [DSGJ17].

Let R be essentially smooth over $\mathbb{Z}$ or a DVR. Given a prime ideal Q in R, we studied two different types of differential powers associated to Q. The first one is defined just in terms of differential operators, as in the classical sense of the Zariski–Nagata Theorem [DDSG+17]. More precisely, given an integer $n \geqslant 1$, the n-th (A-linear) differential power of Q is defined as

$$Q^{\langle n \rangle_A} = \left\{ f \in R \,|\, \partial(f) \in Q \text{ for all } \delta \in D^{n-1}_{R|A} \right\},$$

where $D^{n-1}_{R|A}$ are the A-linear differential operators on R of order at most $n-1$. If Q does not contain any prime integer, then $Q^{\langle n \rangle_A}$ characterizes symbolic powers for formally smooth algebras that are essentially of finite type over A.

Theorem 5 (De Stefani–G–Jeffries). *Let R be essentially smooth over $\mathbb{Z}$ or a DVR, and Q be a prime ideal of R such that $Q \cap A = (0)$. Then $Q^{(n)} = Q^{\langle n \rangle_A}$ for all $n \geqslant 1$.*

However, if the prime ideal Q contains a prime integer p, then differential powers are not sufficient to characterize symbolic powers. To overcome this issue, we use p-derivations [Joy85, Bui95], combined with the ordinary differential operators. Given a fixed p-derivation δ on R, we defined the n-th mixed differential power of I to be

$$I^{\langle n \rangle_{\mathrm{mix}}} := \{ f \in S \mid (\delta^s \circ \partial)(f) \in I \text{ for all } \partial \in D^t_{S|B} \text{ with } s + t \leqslant n-1 \}.$$

In principle, the mixed differential powers depend on the choice of the p-derivation δ. However, in our setting $Q^{\langle n \rangle_{\text{mix}}}$ is independent of the chosen p-derivation. The notion of mixed differential powers allows us to characterize symbolic powers of prime ideals that contain a given integer p.

Theorem 6 (De Stefani–G–Jeffries)**.** *Let p be a prime integer, and let $A = \mathbb{Z}$ or a DVR with uniformizer p and perfect residue field A/pA. Let R be an essentially smooth A-algebra that has a p-derivation δ, and consider a prime ideal Q of R that contains p. Then $Q^{(n)} = Q^{\langle n \rangle_{\text{mix}}}$ for all $n \geqslant 1$.*

Moreover, this allows us to show that $Q^{(n)} = \bigcap \mathfrak{m}^n$, where the intersection is taken over the maximal ideals $\mathfrak{m}$ that contain Q, similarly to the classical Zariski–Nagata theorem.

This project has shown us that p-derivations, an arithmetic tool, might be very useful in Commutative Algebra. As far as we know, this was the first application of p-derivations to Commutative Algebra, and we are working on applying this tool to other topics in the field.

2. Complexes, resolutions, and DG algebras

I am very interested in homological questions, especially questions related to free resolutions and DG algebras, and in applying homological methods to topics that do not appear homological in nature — such as symbolic powers. Here are some projects I have worked on that are of a more homological flavor.

2.1. Deviations of graded algebras.

The minimal free resolution of a module M encodes important information about the structure of the module. Given a finitely generated graded module M over a standard graded algebra $R = \oplus_{i \geqslant 0}$ over a field k with irrelevant maximal ideal $\mathfrak{m} = \oplus_{i>0} R_i$, a minimal free resolution of M is an exact complex

$$F_\bullet = \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

where each F_i is a free R-module, the maps d_i are graded homomorphisms of degree 0, and $\operatorname{im}(d_i) \subseteq \mathfrak{m}F_{i-1}$ for all i. A minimal free resolution for M is unique up to isomorphism of complexes of graded R-modules. The i-th Betti number of M is the integer $\beta_i^R(M)$ such that $F_i \cong R^{\beta_i^R(M)}$.

When $R = S/I$ for $S = k[T_1, \ldots, T_n]$ and $I \subseteq (T_1, \ldots, T_n)^2$ a homogeneous ideal, there are two especially interesting sets of Betti numbers one can study: the Betti numbers of R over S and of k with respect to R, given by

$$\beta_i^S(R) = \dim_k \operatorname{Tor}_i^S(R, k) \qquad \text{and} \qquad \beta_i^R(k) = \dim_k \operatorname{Tor}_i^R(k, k).$$

While a minimal free resolution of R over S must be finite, the sequence $\left\{\beta_i^R(k)\right\}$ is typically infinite, and we can study its behavior via the Poincaré series $P_k^R(t) = \sum_i \beta_i^R(k) t^i$ of k with respect to R. Much work has been devoted to studying the growth of Betti numbers for various classes of rings. By the Auslander-Buchsbaum-Serre Theorem, $P_k^R(t)$ is a polynomial if and only if R is regular. Moreover, the sequence $\{\beta_i^R(k)\}$ has polynomial growth if R is a complete intersection (cf. [Tat57]). Finally, if R is not a complete intersection, then the $\beta_i^R(k)$ have exponential growth (cf. [Avr83]).

Since $P_k^R(t)$ has integer coefficients and constant term equal to 1, there exist uniquely determined integers $\varepsilon_i = \varepsilon_i(R)$, called the **deviations** of R, such that the following infinite product expansion holds:

$$P_k^R(t) = \prod_{i=1}^{\infty} \frac{(1+t^{2i-1})^{\varepsilon_{2i-1}}}{(1-t^{2i})^{\varepsilon_{2i}}}.$$

Deviations play a crucial role in Avramov's proof that the complete intersection property localizes [Avr75]. The $\varepsilon_i^S(R)$ appear naturally as the number of generators of degree i in an acyclic closure of k over R, as the number of generators of degree $i-1$ in a minimal model of R over S, and as the ranks of the components of the homotopy Lie algebra of R. For more background, see [Avr98].

In [BDG+16], Adam Boocher, Alessio D'Alì, Jonathan Montaño, Alessio Sammartano and I proved that, for any term order $<$, the algebra presented by $\mathrm{in}_< I$ has larger deviations than R. Moreover, we showed the following theorem:

Theorem 7 (Boocher–D'Alì–G-Montaño–Sammartano)**.** *Let $S = k[T_1, \ldots, T_n]$, and consider a homogeneous ideal $I \subseteq (T_1, \ldots, T_n)^2$. Let L be the lex-segment ideal of I. Then, for every $i \geqslant 1$,*

$$\varepsilon_i(S/I) \leqslant \varepsilon_i(S/L).$$

This is a generalization of a result of Peeva [Pee96] which states that the lex-segment ideal attains the largest values of $P_k^R(t)$ among all I with the same Hilbert function. Peeva's theorem in turn relies on its analogue for $\beta_i^S(R)$, which is due to Bigatti, Hulett, and Pardue [Big93, Hul93, Par96]. Note that if I and J are ideals such that $\varepsilon_i(S/I) \leqslant \varepsilon_i(S/J)$ for all i, then $\beta_i(S/I) \leqslant \beta_i(S/J)$, so that we can recover the analogous result about Betti numbers from our result on deviations. On the other hand, larger Betti numbers do not necessarily imply larger deviations [BDG+16, Example 3.6].

We also showed that, for Golod rings and for certain Koszul algebras, the sequence of deviations is asymptotically equal to the sequence $\left\{\frac{b\rho^i}{i}\right\}_{i\geqslant 1}$ for some $\rho > 1, b \in \mathbb{N}$. Moreover, in a previous paper, [BDG+15], we studied the deviation sequence of certain edge ideals, and their Koszul homology $H^R = \mathrm{Tor}^S(R,k)$.

2.2. **Levels of Complexes.** The Improved New Intersection Theorem of Evans and Griffith [EG81, EG85] gives a lower bound on the length of a complex of finitely generated free modules. If $(R, \mathfrak{m})$ is a local ring of dimension d containing a field, and

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0$$

is a finite complex of finitely generated free modules such that $\mathrm{H}^i(F)$ has finite length for every $i \geqslant 1$ and $\mathrm{H}^0(F)$ has a minimal generator killed by $\mathfrak{m}$, then the length n of the complex is bounded below by d. Much work has been done towards generalizing and improving this result, some of it using levels of complexes.

Roughly, the projective level of a complex of R-modules F, denoted by $\mathrm{level}^{\mathcal{P}}(F)$, is the smallest number of steps needed to assemble some projective resolution of F from complexes of projective modules that have zero differentials. This is a special case of a general notion of level in triangulated categories, which was introduced and studied by Avramov, Buchweitz, Iyengar, and Miller in [ABIM10].

In [AGM+], Hannah Altmann, Jonathan Montaño, William Sanders, Than Vu and I showed that the projective level of a complex F can be bounded below in terms of the largest gap in the homology of F. Restricting to commutative Noetherian local rings, we applied this result to deduce a new version of The New Intersection Theorem, giving a lower bound to $\mathrm{level}^R(F)$, the smallest number of mapping cones needed to build F from finitely generated free R-modules. This result refines strong forms of the Improved New Intersection Theorem, due to Bruns and Herzog [BH93] and Iyengar [Iye99]. In particular, we do not need to assume that the ring contains a field, due to the recent proof of the existence of balanced big Cohen-Macaulay algebras in [And16].

Theorem 8 (Altmann–G–Montaño–Sanders–Vu). *Let R be a Noetherian local ring, and let*

$$F := 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

be a complex of finitely generated free R-modules such that $\mathrm{H}^i(F)$ has finite length for every $i \geqslant 1$. For any ideal I that annihilates a minimal generator of $\mathrm{H}^0(F)$, there is an inequality

$$\mathrm{level}^R(F) \geqslant \dim(R) - \dim(R/I) + 1.$$

References

[ABIM10] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, Srikanth B. Iyengar, and Claudia Miller. Homology of perfect complexes. *Advances in Mathematics*, 223(5):1731 – 1781, 2010.

[AGM+] Hannah Altmann, Eloísa Grifo, Jonathan Montaño, William Sanders, and Thanh Vu. Lower bounds on projective levels of complexes.

[And16] Yves André. La conjecture du facteur direct. *arXiv:1609.00345*, 2016.

[Avr75] Luchezar L. Avramov. Flat morphisms of complete intersections. *Dokl. Akad. Nauk SSSR*, 225(1):11–14, 1975.

[Avr83] Luchezar L Avramov. *Local algebra and rational homotopy*. Stockholms Universitet. Matematiska Institutionen, 1983.

[Avr98] Luchezar L Avramov. Infinite free resolutions. In *Six lectures on commutative algebra*, pages 1–118. Springer, 1998.

[BDG+15] Adam Boocher, Alessio D'Alì, Eloí sa Grifo, Jonathan Montaño, and Alessio Sammartano. Edge ideals and DG algebra resolutions. *Matematiche (Catania)*, 70(1):215–238, 2015.

[BDG+16] Adam Boocher, Alessio D'Alì, Eloísa Grifo, Jonathan Montaño, and Alessio Sammartano. On the growth of deviations. *Proceedings of the American Mathematical Society*, 144(12):5049–5060, 2016.

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[BH10] Cristiano Bocci and Brian Harbourne. Comparing powers and symbolic powers of ideals. *J. Algebraic Geom.*, 19(3):399–417, 2010.

[Big93] Anna Maria Bigatti. Upper bounds for the betti numbers of a given hilbert function. *Communications in Algebra*, 21(7):2317–2334, 1993.

[Bui95] Alexandru Buium. Differential characters of abelian varieties over p-adic fields. *Inventiones mathematicae*, 122(2):309–340, 1995.

[DDSG+17] Hailong Dao, Alessandro De Stefani, Eloísa Grifo, Craig Huneke, and Luis Núñez-Betancourt. Symbolic powers of ideals. 2017.

[DHN+15] M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg, and H. Tutaj-Gasińska. Resurgences for ideals of special point configurations in $\mathbf{P}^N$ coming from hyperplane arrangements. *J. Algebra*, 443:383–394, 2015.

[DSGJ17] Alessandro De Stefani, Eloísa Grifo, and Jack Jeffries. A Zariski-Nagata Theorem for smooth Z-algebras. *arXiv:1709.01049*, 2017.

[DSTG13] Marcin Dumnicki, Tomasz Szemberg, and Halszka Tutaj-Gasińska. Counterexamples to the $I^{(3)} \subseteq I^2$ containment. *Journal of Algebra*, 393:24–29, 2013.

[Dum15] Marcin Dumnicki. Containments of symbolic powers of ideals of generic points in $\mathbb{P}^3$. *Proc. Amer. Math. Soc.*, 143(2):513–530, 2015.

[EG81] E. Graham Evans and Phillip Griffith. The syzygy problem. *Ann. of Math. (2)*, 114(2):323–333, 1981.

[EG85] E. Graham Evans and Phillip Griffith. *Syzygies*, volume 106 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1985.

[ELS01] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. Uniform bounds and symbolic powers on smooth varieties. *Inventiones Math*, 144 (2):241–25, 2001.

[GH17] Eloísa Grifo and Craig Huneke. Symbolic powers of ideals defining F-pure and strongly F-regular rings, 2017.

[HH02] Melvin Hochster and Craig Huneke. Comparison of symbolic and ordinary powers of ideals. *Inventiones mathematicae*, 147(2):349–369, 2002.

[HH13] Brian Harbourne and Craig Huneke. Are symbolic powers highly evolved? *J. Ramanujan Math. Soc.*, 28A:247–266, 2013.

[HS15] Brian Harbourne and Alexandra Seceleanu. Containment counterexamples for ideals of various configurations of points in $\mathbf{P}^N$. *J. Pure Appl. Algebra*, 219(4):1062–1072, 2015.

[Hul93] Heather A Hulett. Maximum betti numbers of homogeneous ideals with a given hilbert function. *Communications in Algebra*, 21(7):2335–2350, 1993.

[Iye99] Srikanth Iyengar. Depth for complexes, and intersection theorems. *Mathematische Zeitschrift*, 230(3):545–567, 1999.

[Joy85] A. Joyal. δ-anneaux et vecteurs de Witt. *C.R. Acad. Sci. Canada*, VII(3):177–182, 1985.

[MS17] Linquan Ma and Karl Schwede. Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers. 2017.

[Nag62] Masayoshi Nagata. *Local rings.* Interscience, 1962.

[Par96] Keith Pardue. Deformation classes of graded modules and maximal betti numbers. *Illinois Journal of Mathematics*, 40(4):564–585, 1996.

[Pee96] Irena Peeva. 0-borel fixed ideals. *Journal of Algebra*, 184(3):945–984, 1996.

[Sec15] Alexandra Seceleanu. A homological criterion for the containment between symbolic and ordinary powers of some ideals of points in. *Journal of Pure and Applied Algebra*, 219(11):4857 – 4871, 2015.

[SG94] Kei-ichi Watanabe Shiro Goto, Koji Nishida. Non-cohen-macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question. *Proceedings of the American Mathematical Society*, 120(2), 1994.

[Tat57] John Tate. Homology of noetherian rings and local rings. *Illinois Journal of Mathematics*, 1(1):14–27, 1957.

[TBS09] B. Harbourne M. Kapustka A. Knutsen W. Syzdek T. Bauer, S. Di Rocco and T. Szemberg. A primer on Seshadri constants. *Contemporary Mathematics*, vol. 496:39–70, 2009.

[Zar49] Oscar Zariski. A fundamental lemma from the theory of holomorphic functions on an algebraic variety. *Ann. Mat. Pura Appl. (4)*, 29:187–198, 1949.