Problem Set 4

Due Friday, December 1

Turn in **5** of the following problems. Slightly more challenging problems are indicated by ($\star$) .

Problem 1. Let $R = k[x, y]$, where k is a field, let $Q = \mathrm{frac}(R)$ be the fraction field of R. We are going to show that the R-module $M = Q/R$ is divisible but not injective.

a) Show[1] that if $ax + by = 0$ for some $a, b \in R$, we must have $b \in (x)$.

b) Show that $x \mapsto \frac{1}{y} + R$ and $y \mapsto 0$ induces a well-defined R-module homomorphism $(x, y) \xrightarrow{f} Q/R$.

c) Show that M is a divisible R-module, but not injective.

Problem 2. ($\star$) Let R be a domain. Show that if R has a nonzero module M that is both injective and projective, then R must be a field.[2]

An R-module F is *faithfully flat* if F is flat and $F \otimes_R M \neq 0$ for every nonzero R-module M.

Problem 3. ($\star$) Let R be a commutative ring. Show that the following are equivalent:

a) F is faithfully flat.

b) F is flat and for every proper ideal I, $IF \neq F$.

c) F is flat and for every maximal ideal $\mathfrak{m}$, $\mathfrak{m}F \neq F$.

d) The complex

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if

$$F \otimes_R A \xrightarrow{1 \otimes f} F \otimes_R B \xrightarrow{1 \otimes g} F \otimes_R C$$

is exact.

Problem 4. Let M be an R-module. Show that M is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for every R-module N.

Problem 5. Let R be a commutative ring and M and N be R-modules. Consider the R-module homomorphism $f : M \to M$ given by multiplication by a fixed element $r \in R$. Show that the map $\mathrm{Ext}^i(f, M) : \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$ induced by f is multiplication by r on $\mathrm{Ext}^i_R(M, N)$.

[1] If you know about regular sequences, this is easy to justify. But we aren't assuming anyone has seen regular sequences, so the challenge here is to give a clear, easy justification without invoking anything about regular sequences; though it's certainly ok to say the word regular.

[2] Hint: show that any nonzero R-module homomorphism $M \longrightarrow R$ must be surjective, and then show that such a homomorphism must exist.

Problem 6. Let $(R, \mathfrak{m})$ be a commutative local ring, and let M be a finitely presented R-module with minimal presentation

$$0 \longrightarrow K \longrightarrow R^n \xrightarrow{\pi} M \longrightarrow 0.$$

Note that the assumption here is that K is also a finitely generated module.

a) Show that if M is flat, then

$$0 \longrightarrow K \otimes_R R/\mathfrak{m} \longrightarrow R^n \otimes_R R/m \longrightarrow M \otimes_R R/m \longrightarrow 0$$

is exact.

b) Show that M is free $\iff$ M is projective $\iff$ M is flat.

Problem 7. $(\star)$ Let R be a domain and Q be its fraction field. Let T denote the torsion functor.

a) Show that $T(M) = \operatorname{Tor}_1^R(M, Q/R)$.

b) Show that for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of R-modules gives rise to an exact sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow (Q/R) \otimes_R A \longrightarrow (Q/R) \otimes_R B \longrightarrow (Q/R) \otimes_R C \longrightarrow 0.$$

c) Show that the right derived functors of T are $R^1 T = (Q/R) \otimes_R -$ and $R^i T = 0$ for all $i \geqslant 2$.

Problem 8. Let k be a field, $R = k[x, y]$, and $\mathfrak{m} = (x, y)$.

a) Show that

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

is a free resolution for $k = R/\mathfrak{m}$.

b) Compute $\operatorname{Tor}_i^R(k, k)$ for all i.

c) Show that

$$\operatorname{Tor}_1(\mathfrak{m}, k) \cong \operatorname{Tor}_2(k, k).$$