Final Exam practice

Here is a selection of some old qualifying exam problems to practice for the final exam.

Problem 1 (January 2014)**.** Let E be a subfield of $\mathbb{C}$ and assume that every element of E is a root of a polynomial of degree 10 in $\mathbb{Q}[x]$. Prove that $[E : \mathbb{Q}] \leqslant 10$.

Problem 2 (January 2016)**.** Let L be a finite Galois field extension of $\mathbb{Q}$. Let E and F be subfields of L such that $EF = L$, $E/\mathbb{Q}$ is Galois, and $E \cap F = \mathbb{Q}$. Prove that $[L : \mathbb{Q}] = [E : \mathbb{Q}][F : \mathbb{Q}]$.

Problem 3 (May 2022)**.** Let L be the splitting field of $x^4 - 2022$ over $\mathbb{Q}$. Prove there exists a unique intermediate field $Q \subseteq K \subseteq L$ such that $[K : Q] = 4$ and $Q \subseteq K$ is a Galois extension.

Problem 4. Let

$$A = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & 0 \\ 2 & 4 & 0 \end{pmatrix} \in \mathrm{M}_3(\mathbb{R}) \text{ and } B = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix} \in \mathrm{M}_3(\mathbb{R}).$$

For each of the matrices A and B, determine the following:

a) Find the rational canonical form for A and B.

b) Find the Jordan canonical form for A and B, if they exist.

c) Is A diagonalizable? Is B diagonalizable?

Problem 5 (May 2017)**.** Make $\mathbb{R}^3$ into an $\mathbb{R}[x]$-module as follows: given any $f(x) \in \mathbb{R}[x]$ and any $v \in \mathbb{R}^3$, let $f(x)v = Av$, where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

This makes $\mathbb{R}^3$ into an $\mathbb{R}[x]$-module isomorphic to $\mathbb{R}[x]^3/\operatorname{im}(t_A)$, where $t_A : \mathbb{R}[x]^3 \to \mathbb{R}[x]^3$ is given by $\varphi(v) = (Ix - A)v$. It turns out that this module is cyclic; find an explicit polynomial $p(x)$ such that $\mathbb{R}^3 \cong \mathbb{R}[x]/(p(x))$ as $\mathbb{R}[x]$-modules.

Problem 6. List all possible rational canonical forms over $\mathbb{Q}$ and Jordan canonical forms over $\mathbb{C}$ for 8×8 matrices with determinant 81 and minimal polynomial $(x - 3)^2(x^2 + 1)$. Carefully justify.

Problem 7.

a) Consider the $\mathbb{Q}[x]$-module

$$M = \frac{\mathbb{Q}[x]}{(x^4 - 1)} \oplus \frac{\mathbb{Q}[x]}{(x^2(x - 1))}.$$

Let V be the vector space obtained from M by restriction of scalars along the obvious inclusion $\mathbb{Q} \subseteq \mathbb{Q}[x]$, and let $t\colon V \to V$ be the linear transformation given by multiplication by x. Find, with justification, the rational canonical form of t.

b) Consider the $\mathbb{C}[x]$-module

$$N = \frac{\mathbb{C}[x]}{(x^4 - 1)} \oplus \frac{\mathbb{C}[x]}{(x^2(x - 1))}.$$

Let W be the vector space obtained from N by restriction of scalars along the obvious inclusion $\mathbb{C} \subseteq \mathbb{C}[x]$, and let $t\colon W \to W$ be the linear transformation given by multiplication by x. Find, with justification, the rational canonical form of t.