Symbolic powers in characteristic p
joint work with Craig Huneke
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R regular ring, containing a field.

$I \subseteq R$ radical ideal. $\qquad$ $Q \subseteq R$ prime ideal

$h = \max\{\operatorname{ht} P : P \in \operatorname{Ass}(R/I)\}$

n-th symbolic power of Q:

$$Q^{(n)} = Q^n R_Q \cap R = \{f \in I : sf \in Q^n \text{ for some } s \notin Q\}$$
$$= Q\text{-primary component in a primary decomposition of } Q^n$$

Recall: a power of a prime might not be primary

$$I^{(n)} = \bigcap_{P \in \operatorname{Ass}(R/I)} (I^n R_P \cap R) = \{f \in I : sf \in I^n, \ s \text{ not in any associated prime of } I\}$$

Facts:

1) $I^n \subseteq I^{(n)}$
2) $I^n \neq I^{(n)}$ in general $\to$ examples soon
3) If I is generated by a regular sequence in a CM ring, then $I^n = I^{(n)}$.
4) $I^{(n+1)} \subseteq I^{(n)}$
5) $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$

Example: $I = (x,y) \cap (x,z) \cap (x,z) = (xy, xz, yz) \subseteq k[x,y,z]$.

$$I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 \ni xyz$$

But $xyz \notin I^2 \Rightarrow I^2 \neq I^{(2)}$ $\qquad$ But $I^{(3)} \subseteq I^2$.

Comments to make here:

1) Characterizing the ideals I with $I^{(n)} = I^n$ is open in general.

2) Determining symbolic powers can be very hard

3) Even determining what degrees $I^{(n)}$ lives in is a difficult question.

$I^{(n)} \neq I^n$ even for prime ideals:

<u>Example</u>: (Macaulay) $R = K[x, y, z] \xrightarrow{\phi} K[t]$

$$x \mapsto t^3,\ y \mapsto t^4,\ z \mapsto t^5$$

$$P = \ker \phi = (\underbrace{xz - y^2}_{f}, \underbrace{x^3 - yz}_{g}, \underbrace{x^2y - z^2}_{h})$$

$P^{(2)} \neq P^2$: Can show that $fh - g^2 = xq \Rightarrow q \in P^{(2)}$, but $q \notin P^2$ for degree reasons

However, $P^{(3)} \subseteq P^2$.

Fun fact: We can consider maps like this one, where we take each variable to a different power of t, and study the family of height 2 primes we obtain this way. Some of them are complete intersections, so their symbolic powers are the usual powers, but for all the other ones we see the same behavior: $P^{(2)} \neq P^2$, but $P^{(3)} \subseteq P^2$.

this is the kind of containments we'd like to study:

<u>Question</u> When is $I^{(a)} \subseteq I^b$?

<u>Comments</u>:

1) the question makes sense: given b, there exists a.

this has been known since the 60's, by work of Schenzel.

<u>theorem</u> (Ein – Lazarsfeld – Smith, 2001, Hochster – Huneke, 2002)

$$I^{(hn)} \subseteq I^{n}$$

this is a more technical version of this theorem than what you may have seen before. We can always replace hn by a larger number (meaning a smaller symbolic power): so we could replace h with the dimension of the ring, and we would get a uniform containment result for all radical ideals in R.

We will talk more about this theorem later, but let's see what it says in our examples:

<u>Example</u> 1: $I = (x,y) \cap (x,z) \cap (y,z) \rightarrow h = 2$

theorem says: $I^{(2n)} \subseteq I^{n}$. In particular, $I^{(4)} \subseteq I^{2}$.

But actually, we can do better: $I^{(3)} \subseteq I^{2}$.

We also said that for those monomial curve examples, $P^{(3)} \subseteq P^{2}$.

<u>Question</u> (Huneke, 2000) If P is a prime of height 2, is $P^{(3)} \subseteq P^{2}$?

<u>Conjecture</u> (Harbourne, ≤ 2006) $I^{(hn-h+1)} \subseteq I^{n}$.

<u>Note</u>: Harbourne had examples with $I^{(hn-h)} \not\subseteq I^{n}$.

We will talk more about that tomorrow.

Cases where Harbourne's Conjecture holds:

→ Char 2

→ (Squarefree) monomial ideals

→ General points in $\mathbb{P}^2$ (Bocci–Harbourne) and $\mathbb{P}^3$ (Dumnicki)

→ In char $p>0$, if R/I is F-pure and $h=2$ (Hochster – Huneke)

(What they actually prove is that $I^{(hn-1)} \subseteq I^n$)

But we will talk about what an F-pure ring is tomorrow.

<u>Bad news</u>: <u>Example</u> (Dumnicki, Szemberg, Tutaj-Gasińska)

$I = (x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)) \subseteq \mathbb{C}[x,y,z]$

this is a radical ideal of pure height 2

But $I^{(3)} \subseteq I^2$.

⇝ May replace the cubes with any fixed $n \geq 3$ (Harbourne – Seceleanu)

and $\mathbb{C}$ by any field, as long as there are enough roots of unity.

⇝ there are other counterexamples in $\mathbb{P}^2$, all of special configurations of points.

⇝ there is a counterexample in $\mathbb{P}^3$ (arrangement of lines) (Malara–Szpond)

⇝ You can build other counterexamples out of the ones we have (Akesseh)

<u>Note</u> this is a very active area of research, new examples are still being found, and we are still trying to understand them

Good news: there are no counterexamples for primes nor higher containments. And tomorrow we will discuss more classes of ideals that do verify Harbourne's Conjecture.

Goal: Use characteristic p techniques to study symbolic powers (and in particular, to prove Harbourne's Conjecture in some cases)

Theorem (Ein - Lazarsfeld - Smith, 2001, Hochster-Huneke, 2002)

$$I^{(hn)} \subseteq I^n$$

Some history:

→ ELS proved the theorem over $\mathbb{C}$, using multiplier ideals.

→ HH extended the result to any regular ring containing a field.

Sketch of their proof:

0) Reduction to characteristic $p>0$:

the idea is as follows: we have some statement you want to show in equicharacteristic 0 (in this case, a containment). We can take our ideal I and look at in char $p>0$ (imagine replacing $\mathbb{Z}[x_1, \dots, x_n]$ by $\mathbb{Z}/p[x_1, \dots, x_n]$). The idea is to show that as you vary the characteristic, the statement we want may fail sometimes, but only for a finite number of characteristics. So if we prove our statement holds in char p, we are done.

1) Prove the statement in char $p>0$.

From now on:

R regular ring of char $p>0$

Now we gain a powerful tool: the Frobenius homomorphism.

$F(a) = a^p$ $\qquad (a+b)^p = a^p + b^p$

When we apply F to an ideal (as many times as we like), we get its Frobenius powers: given $q=p^e$, $I^{[q]} = (a^q : a \in I)$

We will show that if Q is a prime of height h, then

$$Q^{(hn)} \subseteq Q^n \quad \text{for all } n \geq 1.$$

Ingredients:

1) to show a containment of ideals $a \subseteq b$, we only need to show the containment holds locally – and its enough to show it holds after localizing at the associated primes of b.

In our case, the issue is precisely that we don't know what the associated primes of Q^n are (the difference between the symbolic vs ordinary powers of Q is the potentially bad primes of Q^n), so we need to replace Q^n by something unmixed.

2) $Q^{[q]}$ is unmixed! Q is the only associated prime of $Q^{[q]}$.

Here it is <u>crucial</u> that we are in a regular ring.

The Frobenius map is flat over regular rings, and that property actually characterizes regular rings.

$\leadsto$ We can show $Q^{(hq)} \subseteq Q^{[q]}$

<u>Proof</u>: Localize at Q. Now we live in a RLR of dim h, so our maximal ideal is $m = (x_1, \dots, x_h)$, and we want to show $m^{hq} \subseteq m^{[q]}$. the generators of m^{hq} look like

$x_1^{a_1} \cdots x_h^{a_h}$ where $a_1 + \dots + a_h = hq \Rightarrow a_i \geq q$ for some i.

(this is just the pidgenhole principle!)

In fact, if we count it carefully, we can show

$$\boxed{Q^{(h(q-1)+1)} \subseteq Q^{[q]}.}$$

this is Harbourne's Conjecture!

----- end of day 1 ----- If it's early:

3) to show $Q^{(hn)} \subseteq Q^n$, we use tight closure!

We are in a regular ring, so ideals are tightly closed.

<u>ETS</u>: Given $u \in Q^{(hn)}$, $cu^q \in (Q^n)^{[q]}$ for all $q \gg 0$ and some $c \neq 0$.

<u>Proof</u>: Write $q = an + r$, $0 \leq r < n$. then:

$u^a \in (Q^{(hn)})^a \subseteq Q^{(hna)} \Longrightarrow Q^{hn} u^a \subseteq Q^{hr} u^a \subseteq Q^{(hq)} \subseteq Q^{[q]}$

Take n powers: $Q^{hn^2} u^{an} \subseteq Q^{[q]} \overset{q \geq an}{\Longrightarrow} \underbrace{Q^{hn^2}}_{\exists c \neq 0 \text{ here!}} u^q \subseteq Q^{[q]}$

$\square$

Part II :

R regular ring of char $p>0$

I radical ideal in R

$h = \max\{ \operatorname{ht} P : P \in \operatorname{Ass}(R/I)\}$

Harbourne's Conjecture : $I^{(hn-h+1)} \subseteq I^n$

Recall: It holds for monomial ideals.

Luis' philosophy : If something holds for monomial ideals, it holds for ideals defining F-pure rings.

Goal: Study this conjecture in char $p>0$ (under Luis' philosophy)

$F(a) = a^p$ Frobenius map

We want to study ideals for which R/I verifies some condition:

F-pure: $S = R/I$ is F-pure if the Frobenius map is pure: given a fg R-module M, $S \otimes M \xrightarrow{F\otimes 1} S \otimes M$ is injective.

or

if S is F-finite, which means that S is a fg module over itself via the action of Frobenius, S is F-pure if and only if the Frobenius map splits (as a map of S-modules).

Examples of rings that are F-pure:

1) If I is a squarefree monomial ideal, then R/I is F-pure.

2) Consider a generic matrix (meaning, the entries are just variables) and take all minors of a fixed size.

⇝ the ideal they generate defines an F-pure ring

⇝ the algebra they generate is an F-pure ring

3) Veronese rings are F-pure.

4) Rings of invariants are F-pure, under some mild assumptions

Need: linearly reductive group acting on a polynomial ring over a field.

these include: finite groups, tori, GL_n, O_n

<u>Fedder's Criterion</u> (83) (R,m) RLR of char $p>0$, $I \subseteq R$ ideal

R/I F-pure $\Leftrightarrow (I^{[q]} : I) \not\subseteq m^{[q]}$ for all/some/large $q = p^e$.

this is the key ingredient to prove

<u>theorem</u> (–, Huneke) R/I F-pure $\Rightarrow$ I verifies Harbourne's Conjecture.

<u>Proof</u>: Main idea: $a \subseteq b \Leftrightarrow (b : a) = R$.

Will show: Given n fixed,

$$(I^{[q]} : I) \subseteq (I^n : I^{(hn-h+1)})^{[q]} \quad \text{for all } q = p^e \gg 0.$$

<u>Note</u>: this holds even if R/I is not F-pure.

Also, can reduce to the local case.

Once we show this, if $I^{(hn-h+1)} \not\subseteq I^n$, then $(I^n : I^{(hn-h+1)}) \subseteq m$, so

$(I^{[q]} : I) \subseteq (I^n : I^{(hn-h+1)})^{[q]} \subseteq m^{[q]} \longrightarrow R/I$ NOT F-pure.

— — — — right skip entire page, except for key ingredients — — —

<u>Recall</u>: R regular $\Leftrightarrow$ Frobenius is flat.

Consequence: $(I^n : I^{(hn-h+1)})^{[q]} = ((I^n)^{[q]} : (I^{(hn-h+1)})^{[q]})$

Want to show: given $u \in (I^{[q]} : I)$, $u(I^{(hn-h+1)})^{[q]} \subseteq (I^n)^{[q]}$.

Key ingredients:

<u>Lemma 1</u> $I^{(hq)} \subseteq I^{[q]}$ (we proved that yesterday!)

<u>theorem</u> (HH) $I^{(hs+ks)} \subseteq (I^{(k+1)})^s$ this is what they actually proved!

$$u(I^{(hn-h+1)})^{[q]} \subseteq u\underbrace{I^{(hn-h+1)}}_{\subseteq I}(I^{(hn-h+1)})^{q-1} \subseteq I^{[q]}(I^{(hn-h+1)})^{q-1}$$

<u>Want</u>: $\subseteq (I^n)^{[q]}$

Enough: $(I^{(hn-h+1)})^{q-1} \subseteq (I^{[q]})^{n-1}$

Lemma 1 $\Rightarrow (I^{(hq)})^{n-1} \subseteq (I^{[q]})^{n-1}$

theorem $\Rightarrow \underbrace{I^{(h(n-1)+(hq-1)(n-1))}}_{\circledast} \subseteq (I^{(hq)})^{n-1}$.

So in the end all we need is $(I^{(hn-h+1)})^{q-1} \subseteq I^{((hn-h+1)(q-1))} \subseteq \circledast$

But it's really enough to see that

$(hn-h+1)(q-1) \geq$ exponent of $\circledast$ when $q \gg 0$.

— — — —

Is the result sharp? Yes!

<u>Example</u>: $I = \bigcap_{i \neq j} (x_i, x_j) = (x_1 \cdots \hat{x_i} \cdots x_d : 1 \leq i \leq d) \subseteq k[x_1, \dots, x_d]$
(HH)

$h = 2$, theorem says $I^{(2n-1)} \subseteq I^n \quad \forall n \geq 1$

Is $I^{(2n-2)} \subseteq I^n$?

$$I^{(2n-2)} = \bigcap_{i \neq j} (x_i, x_j)^{2n-2} \ni (x_1 \cdots x_d)^{n-1}$$

However, I is generated in degree $d-1$, so if $n < d$,

elements in I^n have degree $n(d-1) = nd - n < nd - d = d(n-1)$

<u>Note</u> Can generalize this example to any h.

<u>Examples</u>:

1) $R = k[a,b,c,d] \xrightarrow{\phi} k[s^3, s^2t, st^2, t^3]$ (Veronese!)
$P = \ker \phi = (c^2 - bd, bc - ad, b^2 - ac) \Rightarrow P^{(n)} = P^n \quad \forall n \geq 1$

2) (Singh) $P = I_2 \begin{pmatrix} a^2 & b & d \\ c & a^2 & b^n d \end{pmatrix} \subseteq k[a,b,c,d]$ any n.
$P^{(n)} = P^n$.

Can we restrict the assumptions on I and get better containments?

$S = R/I$ F-finite reduced ring is strongly F-regular if given $f \in S$, $f \neq 0$, $\exists\, q = p^e$: $f^{1/q} S \longrightarrow S^{1/q}$ splits.

"there are lots of splittings, rather than just one"

<u>Glassbrenner's Criterion</u> (96) (R,m) F-finite RLR, char $p>0$
c not in a minimal prime of I

R/I strongly F-regular $\iff$ $c(I^{[q]}:I) \not\subseteq m^{[q]}$ for all/some/large $q=p^e$

<u>Examples</u> All of the above except monomial ideals.

<u>Theorem</u> (—, Huneke)

R/I strongly F-regular $\Rightarrow$ $I^{((h-1)n+1)} \subseteq I^{n+1}$

<u>Remark</u> this is Harbourne's Conjecture replacing h by $h-1$.

<u>Sketch of proof</u> Given d, show that for all $q=p^e$,

$$(I^d : I^{(d)})(I^{[q]}:I) \subseteq (I I^{(d-h+1)} : I^{(d)})^{[q]}$$

<u>Idea</u>: $(I^d : I^{(d)})$ <u>always</u> contains an element $\notin$ minimal prime of I

R/I strongly F-regular $\Rightarrow$ $I^{(d)} \subseteq I I^{(d-h+1)}$

Use induction.

Is this sharp? We don't know, but it is sharp when $h=2$:

<u>Corollary</u> $h=2 \Rightarrow I^{(n)} = I^n \quad \forall n \geq 1$.

Example (Determinantal ideals)

$R = K[X]$, where X is a generic $n \times n$ matrix

$I = I_t(X)$.

then $I^{(a)} \subseteq I^b$ if and only if $a \geq \frac{t(n-t+1)}{n} b$.

So if $n = 2t$, $I^{(a)} \subseteq I^b \iff a \geq \frac{t(t+1)}{2t} b = \frac{t+1}{2} b$.

In this case, ht $I = (n-t+1)^2$. If $n = 2t$, get $(t+1)^2$.

Char 0 results: Need R to be essentially of finite type over a field

1) Harbourne's Conjecture holds for R/I of dense F-pure type.

2) Harbourne -1 holds for R/I with log-terminal singularities