Problem Set 4

Problem 1. Let $R = k[x, y]$, where k is a field, let $Q = \operatorname{frac}(R)$ be the fraction field of R. We are going to show that the R-module $M = Q/R$ is divisible but not injective.

a) Show that if $ax + by = 0$ for some $a, b \in R$, we must have $b \in (x)$.

b) Show that $x \mapsto \frac{1}{y}$ and $y \mapsto 0$ induces a well-defined R-module homomorphism $(x, y) \xrightarrow{f} Q/R$.

c) Show that M is a divisible R-module, but not injective.

Problem 2. Let R be a domain. Show that if R has a nonzero module M that is both injective and projective, then R must be a field.[1]

Problem 3. Let R be a Noetherian ring, M a finitely generated R-module, N an R-module, and W a multiplicatively closed subset of R. Show that there is an isomorphism

$$W^{-1} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N).$$

Clearly indicate where you are using the hypotheses that R is Noetherian and M is finitely generated, as they are necessary.[2]

Problem 4. Let $\mathcal{A}$ be an abelian category.

a) Show that $\ker(x \xrightarrow{0} y) = 1_x$, $\operatorname{coker}(x \xrightarrow{0} y) = 1_y$, and $\operatorname{im}(x \xrightarrow{0} y) = 0 \longrightarrow y$.

b) Show that f is a mono if and only if $fg = 0$ implies $g = 0$, and g is an epi if and only if $gf = 0$ implies $g = 0$.

c) Show that f is a mono if and only if $\ker f = 0$, and g is an epi if and only if $\operatorname{coker} g = 0$.

d) Show that $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if f is a mono.

e) Show that $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if g is an epi.

Problem 5. Consider an abelian category. If g is an epi and f is a mono, then $\ker(fg) = \ker g$, $\operatorname{coker}(fg) = \operatorname{coker} f$, and $\operatorname{im}(fg) = \operatorname{im} f = f$.

[1] Hint: show that any nonzero R-module homomorphism $M \longrightarrow R$ must be surjective, and then show that such a homomorphism must exist.

[2] Hint: start by noting that the obvious map $W^{-1} \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$ is natural on M and an isomorphism when $M = R^n$. Then apply appropriate functors to a presentation $R^m \longrightarrow R^n \longrightarrow M$ for M.

An R-module F is *faithfully flat* if F is flat and $F \otimes_R M \neq 0$ for every nonzero R-module M.

Problem 6. Give an example of a module that is flat but not faithfully flat. Show[3] that the following are equivalent:

a) F is faithfully flat.

b) F is flat and for every proper ideal I, $IF \neq F$.

c) F is flat and for every maximal ideal $\mathfrak{m}$, $\mathfrak{m}F \neq F$.

d) For every sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$, $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $F \otimes_R A \xrightarrow{1\otimes f} F \otimes_R B \xrightarrow{1\otimes g} F \otimes_R C$ is exact.

Problem 7. Consider the ring $R = \mathbb{Q}[x,y,z,a,b,c]/(xb - ac, yc - bz, xc - az)$, the ideal $I = (x,a)$ in R, $I = (x,a)$, and the 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

Let $S = \mathbb{Q}[x,y,z]$ and P be the ideal in R defining the curve $\{(t^{13}, t^{42}, t^{73}) \mid t \in \mathbb{Q}\}$.

a) Find the first 6 steps in the minimal free resolutions for R/I and N over R.

b) Apply $\operatorname{Hom}_R(-, N)$ to the portion of a minimal free resolution you found for R/I. Is this an exact complex? If not, in what homological degrees do we have non-trivial homology?

c) Find a minimal free resolution for P over S. Make sure your resolution *is* minimal!

[3] Hints:

- For $c) \implies a)$, for each R-module $M \neq 0$ consider some nonzero $m \in M$ and $I = \operatorname{ann} m$.
- For $a) \implies d)$, show that $\operatorname{im}(1_F \otimes f) = F \otimes_R \operatorname{im} f$ and $\ker(1_F \otimes f) = F \otimes_R \ker g$, and then consider the short exact sequence $0 \longrightarrow \operatorname{im} f \longrightarrow \ker g \longrightarrow \ker g / \operatorname{im} f \longrightarrow 0$.
- For $d) \implies a)$, show that for any R-module $M \neq 0$, the identity map on M induces a nonzero map $F \otimes_R M \longrightarrow R \times_R M$.