

## Problem Set 5

**Problem 1.** Let  $R$  be a domain and  $Q$  be its fraction field. Let  $T(-)$  denote the torsion functor we introduced in Problem Set 3.

- a) Show that  $T(M) = \text{Tor}_1^R(M, Q/R)$ .<sup>1</sup>  
 b) Show that for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of  $R$ -modules gives rise to an exact sequence<sup>2</sup>

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow (Q/R) \otimes_R A \longrightarrow (Q/R) \otimes_R B \longrightarrow (Q/R) \otimes_R C \longrightarrow 0.$$

- c) Show that the right derived functors of  $T$  are  $R^1T = (Q/R) \otimes_R -$  and  $R^iT = 0$  for all  $i \leq 2$ .

**Problem 2.** Let  $I$  be an ideal in  $R$ . Show that

$$\text{Ext}_R^n(I, M) \cong \text{Ext}_R^{n+1}(R/I, M)$$

for all  $n \geq 1$  and all  $R$ -modules  $M$ .

**Problem 3.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring. Let  $r \in R$  and  $M$  and  $N$  be finitely generated  $R$ -modules.

- a) Show that the map  $\text{Ext}_R^i(M, N) \rightarrow \text{Ext}_R^i(M, N)$  induced by  $M \xrightarrow{r} M$  is the map given by multiplication by  $r$ .  
 b) Show that if  $r$  is regular on  $M$  and  $\text{Ext}_R^i(M/rM, N) = 0$  for  $i \gg 0$ , then  $\text{Ext}_R^i(M, N) = 0$  for  $i \gg 0$ .

**Problem 4.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring.

- a) Show that for every short exact sequence  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  of  $R$ -modules,

$$\text{depth}(A) \geq \min\{\text{depth}(B), \text{depth}(C) + 1\}.$$

- b) Given any finitely generated  $R$ -module  $M$  over a Cohen-Macaulay ring  $R$ , show that there exists  $n \geq 1$  such that either  $\text{pdim}(M) < n$  or  $\text{depth}(\Omega_n M) = \dim(R)$ .

**Problem 5.** Here are two fun, unrelated, but easy problems about regular rings.

- a) Show that every principal ideal domain is a regular ring.  
 b) Solve the Localization Problem that baffled mathematicians for decades: if  $R$  is a regular local ring, then  $R_P$  is a regular local ring for every prime  $P$ .

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<sup>1</sup>Hint: you want to look at some long exact sequence for  $\text{Tor}$ .

<sup>2</sup>Hint: apply the Snake Lemma to some nice diagram.

**Problem 7.** Consider the ring  $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$  and the 2-generated  $R$ -module  $M = Rf + Rg$ , where the generators  $f, g$  satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

Let  $P$  be the ideal in  $S = \mathbb{Q}[x, y, z]$  defining the curve  $\{(t^{13}, t^{42}, t^{73}) \mid t \in \mathbb{Q}\}$ .

To solve this problem, you are not allowed to use any additional Macaulay2 packages besides the **Complexes** package and the ones that are automatically loaded with Macaulay2.

- a) Find  $\text{pdim}_S(S/P)$  and  $\text{depth}(S/P)$ .
- b) Is  $P$  generated by a regular sequence?
- c) Find  $\text{pdim}_R(M)$  and  $\text{depth}(M)$ .
- d) Is  $R$  a regular ring? Is it Cohen-Macaulay?