Symbolic powers of ideals defining F-pure rings

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April 6, 2017

Symbolic Power

The n-th symbolic power of an ideal I in R is given by

$$I^{(n)} = \bigcap_{P \in \operatorname{Ass}(R/I)} I^n R_P \cap R.$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) If I is generated by a regular sequence in a Cohen-Macaulay ring, then $I^n = I^{(n)}$.

(3) In general, $I^n \neq I^{(n)}$.

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Example

$I = (x, y) \cap (y, z) \cap (x, z) = (xy, xz, yz)$ in $R = \mathbb{C}[x, y, z]$.

$$I^{(2)} = (x, y)^2 \cap (y, z)^2 \cap (x, z)^2 \ni xyz.$$

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$$I^{(2)} \neq I^2.$$

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Main question

When is $I^{(b)} \subseteq I^a$?

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002)

Let I be a radical ideal in a regular ring containing a field, R, and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$,

$$I^{(hn)} \subseteq I^n.$$

Example

$I = (x, y) \cap (y, z) \cap (x, z) = (xy, xz, yz)$ in $R = \mathbb{C}[x, y, z]$.

$$h = 2 \Rightarrow I^{(2n)} \subseteq I^n \Rightarrow I^{(4)} \subseteq I^2.$$

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Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in $k[\mathbb{P}^n]$, h the maximal height of a minimal prime of I. For all $n \geqslant 1$,

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Harbourne's Conjecture holds

- For arbitrary ideals in characteristic 2. (Huneke)
- For monomial ideals in arbitrary characteristic.
- For general points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).
- If R/I is F-pure and $h = 2$ (Hochster-Huneke).

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Counterexample (Dumnicki, Szemberg, Tutaj-Gasińska, 2015)

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \not\subseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

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Is all lost?

- The conjecture could still hold for n large.
- There are no known counterexamples for prime ideals.

Definition (F-pure ring)

Let A be an F-finite ring of characteristic $p > 0$. We say that A is *F-pure* if the Frobenius map splits as a map of A-modules.

Facts about F-pure rings

- Regular rings are F-pure
- Squarefree monomial ideals define F-pure rings

Theorem (–, Huneke)

Let R be a regular ring of characteristic $p > 0$. Let I be an ideal in R with R/I F-pure, and let h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Harbourne's Conjecture holds whenever R/I is F-pure.

Definition (Strongly F-regular ring)

An F-finite reduced ring A is *strongly F-regular* if given any $f \in A$, $f \neq 0$, there exists $q = p^e$ such that the inclusion $f^{1/q}A \longrightarrow A^{1/q}$ splits.

Example

- Veronese subrings of polynomial rings are strongly F-regular.
- Determinantal rings are strongly F-regular.

Theorem (–, Huneke)

Let R be a regular ring of characteristic $p > 0$. Let I be an ideal such that R/I is strongly F-regular, and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$,

$$I^{((h-1)n+1)} \subseteq I^{n+1}.$$

This is Harbourne's Conjecture replacing h by $h - 1$.

Corollary (–, Huneke)

Let R be a regular ring of characteristic $p > 0$. Let P be a prime of height 2 in R such that R/P is strongly F-regular. Then all powers of P are unmixed, that is, for all $n \geqslant 1$,

$$P^n = P^{(n)}.$$

Thank you!

Theorem (Fedder's Criterion)

Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Given an ideal I in R, R/I is F-pure if and only if for all $q = p^e \gg 0$,

$$\left(I^{[q]} : I\right) \not\subseteq \mathfrak{m}^{[q]}.$$

Theorem (Glassbrenner's Criterion for strong F-regularity)

Let $(R, \mathfrak{m})$ be an F-finite regular local ring of prime characteristic p. Given a proper radical ideal I of R, R/I is strongly F-regular if and only if for each element $c \in R$ not in any minimal prime of I,

$$c\left(I^{[p^e]} : I\right) \not\subseteq \mathfrak{m}^{[p^e]}$$

for all $e \gg 0$.