

Introduction to Modern Algebra II

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Chapter 1

Modules

Modules are a generalization of the concept of a vector space to any ring of scalars. But while vector spaces make for a great first example of modules, many of the basic facts we are used to from linear algebra are often a little more subtle over a general ring. These differences are features, not bugs. We will introduce modules, study some general linear algebra, and discuss the differences that make the general theory of modules richer and even more fun.

1.1 Basic assumptions

In this class, all rings have a multiplicative identity, written as 1 or 1_R if we want to emphasize that we are referring to the ring R . This is what some authors call *unital rings*; since for us all rings are unital, we will omit the adjective. Moreover, we will think of 1 as part of the structure of the ring, and thus require it be preserved by all natural constructions. As such, a subring S of R must share the same multiplicative identity with R , meaning $1_R = 1_S$. Moreover, any ring homomorphism must preserve the multiplicative identity. To clear any possible confusion, we include below the relevant definitions.

Definition 1.1. A **ring** is a set R equipped with two binary operations, $+$ and \cdot , satisfying:

- (1) $(R, +)$ is an abelian group with identity element denoted 0 or 0_R .
- (2) The operation \cdot is associative, so that (R, \cdot) is a semigroup.
- (3) For all $a, b, c \in R$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

- (4) there is a multiplicative identity, written as 1 or 1_R , such that $1 \cdot a = a = a \cdot 1$ for all $a \in R$.

To simplify notation, we will often drop the \cdot when writing the multiplication of two elements, so that ab will mean $a \cdot b$.

Definition 1.2. A ring R is a **commutative ring** if for all $a, b \in R$ we have $a \cdot b = b \cdot a$.

Definition 1.3. A ring R is a **division ring** if $1 \neq 0$ and $R \setminus \{0\}$ is a group under \cdot , so every nonzero $r \in R$ has a multiplicative inverse. A **field** is a commutative division ring.

Definition 1.4. A commutative ring R is a **domain**, sometimes called an **integral domain** if it has no zerodivisors: $ab = 0 \Rightarrow a = 0$ or $b = 0$.

For some familiar examples, $M_n(R)$ (the set of $n \times n$ matrices) is a ring with the usual addition and multiplication of matrices, \mathbb{Z} and \mathbb{Z}/n are commutative rings, \mathbb{C} and \mathbb{Q} are fields, and the real Hamiltonian quaternion ring \mathbb{H} is a division ring.

Definition 1.5. A **ring homomorphism** is a function $f: R \rightarrow S$ satisfying the following:

- $f(a + b) = f(a) + f(b)$ for all $a, b \in R$.
- $f(ab) = f(a)f(b)$ for all $a, b \in R$.
- $f(1_R) = 1_S$.

Under this definition, the map $f: \mathbb{R} \rightarrow M_2(\mathbb{R})$ sending $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ preserves addition and multiplication but not the multiplicative identities, and thus it is not a ring homomorphism.

Exercise 1. For any ring R , there exists a unique homomorphism $\mathbb{Z} \rightarrow R$.

Definition 1.6. A subset S of a ring R is a **subring** of R if it is a ring under the same addition and multiplication operations and $1_R = 1_S$.

So under this definition, $2\mathbb{Z}$, the set of even integers, is not a subring of \mathbb{Z} ; in fact, it is not even a ring, since it does not have a multiplicative identity!

Definition 1.7. Let R be a ring. A subset I of R is an **ideal** if:

- I is nonempty.
- $(I, +)$ is a subgroup of $(R, +)$.
- For every $a \in I$ and every $r \in R$, we have $ra \in I$ and $ar \in I$.

The final property is often called **absorption**. A **left ideal** satisfies only absorption on the left, meaning that we require only that $ra \in I$ for all $r \in R$ and $a \in I$. Similarly, a **right ideal** satisfies only absorption on the right, meaning that $ar \in I$ for all $r \in R$ and $a \in I$.

When R is a commutative ring, the left ideals, right ideals, and ideals over R are all the same. However, if R is not commutative, then these can be very different classes.

One key distinction between unital rings and nonunital rings is that if one requires every ring to have a 1, as we do, then the ideals and subrings of a ring R are very different creatures. In fact, the *only* subring of R that is also an ideal is R itself. The change lies in what constitutes a subring; notice that nothing has changed in the definition of ideal.

Remark 1.8. Every ring R has two **trivial ideals**: R itself and the zero ideal $(0) = \{0\}$.

A **nontrivial ideal** I of R is an ideal that $I \neq R$ and $I \neq (0)$. An ideal I of R is a **proper ideal** if $I \neq R$.

1.2 Modules: definition and examples

Definition 1.9. Let R be a ring with $1 \neq 0$. A **left R -module** is an abelian group $(M, +)$ together with an action $R \times M \rightarrow M$ of R on M , written as $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$ we have the following:

- $(r + s)m = rm + sm$,
- $(rs)m = r(sm)$,
- $r(m + n) = rm + rn$, and
- $1m = m$.

A **right R -module** is an abelian group $(M, +)$ together with an action of R on M , written as $M \times R \rightarrow M$, $(m, r) \mapsto mr$, such that for all $r, s \in R$ and $m, n \in M$ we have

- $m(r + s) = mr + ms$,
- $m(rs) = (mr)s$,
- $(m + n)r = mr + nr$, and
- $m1 = m$.

By default, we will be studying left R -modules. To make the writing less heavy, we will sometimes say **R -module** rather than left R -module whenever there is no ambiguity.

Remark 1.10. If R is a commutative ring, then any left R -module M may be regarded as a right R -module by setting $mr := rm$. Likewise, any right R -module may be regarded as a left R -module. Thus for commutative rings, we just refer to modules, and not left or right modules.

Lemma 1.11 (Arithmetic in modules). *Let R be a ring with $1_R \neq 0_R$ and M be an R -module. Then $0_R m = 0_M$ and $(-1_R)m = -m$ for all $m \in M$.*

Proof. Let $m \in M$. Then

$$0_R m = (0_R + 0_R)m = 0_R m + 0_R m.$$

Since M is an abelian group, the element $0_R m$ has an additive inverse, $-0_R m$, so adding it on both sides we see that

$$0_M = 0_R m.$$

Moreover,

$$m + (-1_R)m = 1_R m + (-1_R)m = (1_R - 1_R)m = 0_R m = 0_M,$$

so $(-1_R)m = -m$. □

Typically, one first encounters modules in an undergraduate linear algebra course: the vector spaces from linear algebra are modules over fields. Later we will see that vector spaces are much simpler modules than modules over other rings. So while one might take linear algebra and vector spaces as an inspiration for what to expect from a module, be warned that this perspective can often be deceiving.

Definition 1.12. Let F be a field. A **vector space** over F is an F -module.

We will see more about vector spaces soon. Note that many of the concepts we will introduce have special names in the case of vector spaces. Here are some other important examples:

Lemma 1.13. Let M be a set with a binary operation $+$. Then

- (1) M is an abelian group if and only if M is a \mathbb{Z} -module.
- (2) M is an abelian group such that $nm := \underbrace{m + \cdots + m}_{n \text{ times}} = 0_M$ for all $m \in M$ if and only if M has a \mathbb{Z}/n -module structure.

Proof. First, we show 1). If M is a \mathbb{Z} -module, then $(M, +)$ is an abelian group by definition of module. Conversely, if $(M, +)$ is an abelian group then there is a unique \mathbb{Z} -module structure on M given by the formulas below. The uniqueness of the \mathbb{Z} action follows from the identities below in which the right hand side is determined only by the abelian group structure of M . The various identities follow from the axioms of a module:

$$\begin{cases} i \cdot m = (\underbrace{1 + \cdots + 1}_i) \cdot m = \underbrace{1 \cdot m + \cdots + 1 \cdot m}_i = \underbrace{m + \cdots + m}_i & \text{if } i > 0 \\ 0 \cdot m = 0_M \\ i \cdot m = -(-i) \cdot m = -(\underbrace{m + \cdots + m}_{-i}) & \text{if } i < 0. \end{cases}$$

We leave it as an exercise to check that this \mathbb{Z} -action really satisfies the module axioms.

Now we show 2). If M is a \mathbb{Z}/n module, then $(M, +)$ is an abelian group by definition, and $nm = \underbrace{m + \cdots + m}_n = \underbrace{[1]_n \cdot m + \cdots + [1]_n \cdot m}_n = [0]_n m = 0_M$.

Conversely, there is a unique \mathbb{Z}/n -module structure on M given by the formulas below, which are analogous to the ones above:

$$\begin{cases} [i]_n \cdot m = (\underbrace{[1]_n + \cdots + [1]_n}_i) \cdot m = \underbrace{[1]_n \cdot m + \cdots + [1]_n \cdot m}_i = \underbrace{m + \cdots + m}_i & \text{if } i > 0 \\ 0 \cdot m = 0_M \\ [i]_n \cdot m = -(-[i]_n) \cdot m = -(\underbrace{m + \cdots + m}_{-i}) & \text{if } i < 0. \end{cases}$$

These formulas are well-defined, meaning they are independent of the choice of representative for $[i]_n$, because of the assumption that $nm = 0_M$. Again checking that this \mathbb{Z}/n -action really satisfies the module axioms is left as an exercise. \square

The proposition above says in particular that any group of the form

$$G = \mathbb{Z}^\ell \times \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_m$$

is a \mathbb{Z} -module, and if $\ell = 0, m \geq 1$ and $d_i \mid n$ for $1 \leq i \leq m$ then G is also a \mathbb{Z}/n -module. In particular, the Klein group is a $\mathbb{Z}/2$ -module.

In contrast to vector spaces, for M a module over a ring R , it can happen that $rm = 0$ for some $r \in R$ and $m \in M$ such that $r \neq 0_R$ and $m \neq 0_M$. For example, in the Klein group K_4 viewed as a \mathbb{Z} -module we have $2m = 0$ for all $m \in K_4$.

Example 1.14. (1) The trivial R -module is $0 = \{0\}$ with $r0 = 0$ for any $r \in R$.

- (2) If R is any ring, then R is a left and right R -module via the action of R on itself given by its internal multiplication.
- (3) If I is a left (respectively, right) ideal of a ring R then I is a left (respectively, right) R -module with respect to the action of R on I by internal multiplication.
- (4) If R is a subring of a ring S , then S is an R -module with respect to the action of R on S by internal multiplication in S .
- (5) If R is a commutative ring with $1 \neq 0$, then $R[x_1, \dots, x_n]$ is an R -module for any $n \geq 1$. This is a special case of (4).
- (6) If R is a commutative ring and G is a group, then $R[G]$ is an R -module. This is a special case of (4).
- (7) If R is a commutative ring, let $M_n(R)$ denote set of $n \times n$ matrices with entries in R . Then $M_n(R)$ is an R -module for $n \geq 1$, with the R -action given by multiplying all the entries of the given matrix by the given element of R .
- (8) The **free module** over R of rank n is the set

$$R^n = \left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mid r_i \in R, 1 \leq i \leq n \right\}$$

with componentwise addition and multiplication by elements of R , as follows:

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r'_1 \\ \vdots \\ r'_n \end{bmatrix} = \begin{bmatrix} r_1 + r'_1 \\ \vdots \\ r_n + r'_n \end{bmatrix} \quad \text{and} \quad r \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} rr_1 \\ \vdots \\ rr_n \end{bmatrix}.$$

We will often write the elements of R^n as n -tuples (r_1, \dots, r_n) instead. Notice that R is the free R -module of rank 1.

- (9) More generally, given a collection of R -modules $\{A_i\}$, the abelian group

$$\bigoplus_i A_i = \{(a_i)_i \mid a_i \in A_i, a_i = 0 \text{ for all } i \text{ but finitely many}\}$$

is an R -module with the R -action $r(a_i) := (ra_i)$.

1.3 Submodules and restriction of scalars

Definition 1.15. Let R be a ring and let M be a left R -module. An R -**submodule** of M is a subgroup N of M satisfying $rn \in N$ for all $r \in R$ and $n \in N$.

The submodules of an R -module M are precisely the subsets of M which are modules in their own right, via the same R -action as we are considering for M .

Exercise 2. Show that if N is a submodule of M , then N is an R -module via the restriction of the action of R on M to the subset N .

Example 1.16. Every R -module M has two **trivial submodules**: M itself and the **zero module** $0 = \{0_M\}$. A submodule N of M is **nontrivial** if $N \neq M$ and $N \neq 0$.

Lemma 1.17 (One-step test for submodules). *Let R be a ring with $1 \neq 0$ and let M be a left R -module. A nonempty subset N of M is an R -submodule of M if and only if $rn + n' \in N$ for all $r \in R$ and $n, n' \in N$.*

Proof. The One-step Test for subgroups says that if for all $n, n' \in N$ we have $n' - n \in N$, then N is a subgroup of M . By Lemma 1.11, by taking $r = -1$ we get $rn + n' = n' - n$, and by assumption this is an element of N . Therefore, N is a subgroup of M . As a consequence, $0_M \in N$. By taking $n' = 0_M$, we see that for all $n \in N$ and all $r \in R$ we have $rn = rn + n' \in N$, and thus we can now conclude that N is a submodule of M . \square

Example 1.18. Let R be a ring and let M be a subset of R . Then M is a left (respectively, right) R -submodule of R if and only if M is a left (respectively, right) ideal of R .

Exercise 3. Let R be a ring and let A and B be submodules of an R -module M . Then the **sum** of A and B ,

$$A + B := \{a + b \mid a \in A, b \in B\},$$

and $A \cap B$ are both R -submodules of M .

Exercise 4. Let R be a commutative ring with $1 \neq 0$, let I be an ideal of R and let M be an R -module. Show that

$$IM := \left\{ \sum_{k=1}^n j_k m_k \mid n \geq 0, j_k \in I, m_k \in M \text{ for } 1 \leq k \leq n \right\}$$

is a submodule of M .

Example 1.19. When R is a field, the submodules of a vector space V are precisely the subspaces of V . When $R = \mathbb{Z}$, then the class of R -modules is simply the class of all abelian groups, by Lemma 1.13. The submodules of a \mathbb{Z} -module M coincide with the subgroups of the abelian group M .

Definition 1.20. Let R be a ring with $1 \neq 0$ and let M be an R -module. Given elements $m_1, \dots, m_n \in M$, the **submodule generated by** m_1, \dots, m_n is the subset of M given by

$$Rm_1 + \dots + Rm_n := \{r_1 m_1 + \dots + r_n m_n \mid r_1, \dots, r_n \in R\}.$$

Exercise 5. Let R be a ring with $1 \neq 0$ and M be an R -module. Given $m_1, \dots, m_n \in M$, the submodule generated by m_1, \dots, m_n is indeed a submodule of M . Moreover, this is the smallest submodule of M that contains m_1, \dots, m_n , meaning that every submodule of M containing m_1, \dots, m_n must also contain $Rm_1 + \dots + Rm_n$.

Definition 1.21. Let R be a ring with $1 \neq 0$. An R -module M is **cyclic** if there exists an element $m \in M$ such that

$$M = Rm := \{rm \mid r \in R\}.$$

Given an R -module M , the ring R is often referred to as the **ring of scalars**, by analogy to the vector space case. Given an action of a ring of scalars on a module, we can sometimes produce an action of a different ring of scalars on the same set, producing a new module structure.

Lemma 1.22 (Restriction of scalars). *Let $\phi: R \rightarrow S$ be a ring homomorphism. Any left S -module M may be regarded via **restriction of scalars** as a left R -module with R -action defined by $rm := \phi(r)m$ for any $m \in M$. In particular, if R is a subring of a ring S , then any left S -module M may be regarded via restriction of scalars as a left R -module with R -action defined by the action of the elements of R viewed as elements of S .*

Proof. Let $r, s \in R$ and $m, n \in M$. One checks that the axioms in the definition of a module hold for the given action using properties of ring homomorphisms. For example:

$$(r + s)m = \phi(r + s)m = (\phi(r) + \phi(s))m = \phi(r)m + \phi(s)m = rm + sm.$$

The remaining properties are left as an exercise. \square

Note that the second module structure on M obtained via restriction of scalars is induced by the original module structure, so the two are related. In general, one can give different module structures on the same abelian group over different, possibly unrelated, rings.

Example 1.23. If I is an ideal of a ring R , applying restriction of scalars along the quotient homomorphism $q: R \rightarrow R/I$ tells us that any left R/I -module is also a left R -module. In particular, applying this to the R/I -module R/I makes R/I a left and right R -module by restriction of scalars along the quotient homomorphism. Thus the R -action on R/I is given by

$$r \cdot (a + I) := ra + I.$$

Example 1.24. Given any ring R there exists a unique ring homomorphism $\mathbb{Z} \rightarrow R$, by Exercise 1. Thus any R -module can be given the structure of a \mathbb{Z} -module by restriction of scalars along this unique map. Note also that a module over any ring is in particular an abelian group, so we can always regard any R -module as a \mathbb{Z} -module by forgetting the R -action and focusing only on the abelian group structure. These two constructions – the restriction of scalars to \mathbb{Z} and the *forgetful functor*¹ – actually coincide.

The next example explains why restriction of scalars is called a *restriction*.

Example 1.25. Let R be a subring of S , and let $i: R \rightarrow S$ be the inclusion map, which must by definition be a ring homomorphism. Applying restriction of scalars to an S -module M via i is the same as simply *restricting* our scalars to the elements of R .

¹This is a concrete abstract nonsense construction that we will discuss in Homological Algebra next Fall.

1.4 Module homomorphisms and isomorphisms

Definition 1.26. Given R -modules M and N , an R -**module homomorphism** from M to N is a function $f: M \rightarrow N$ such that for all $r \in R$ and $m, n \in M$ we have

- $f(m + n) = f(m) + f(n)$
- $f(rm) = rf(m)$.

Remark 1.27. The condition $f(m + n) = f(m) + f(n)$ says that f is a homomorphism of abelian groups, and the condition $f(rm) = rf(m)$ says that f is R -linear, meaning that it preserves the R -action. Since f is a homomorphism of abelian groups, it follows that $f(0) = 0$ must hold.

Definition 1.28. Let M and N be vector spaces over a field F . A **linear transformation** from M to N is an F -module homomorphism $M \rightarrow N$.

Example 1.29. Let R be a commutative ring and M be an R -module. For each $r \in R$, the multiplication map $\mu_r: M \rightarrow M$ given by $\mu_r(m) = rm$ is a homomorphism of R -modules: indeed, by the definition of R -module we have

$$\mu_r(m + n) = r(m + n) = rm + rn = \mu_r(m) + \mu_r(n),$$

and

$$\mu_r(sm) = r(sm) = (rs)m = (sr)m = s(rm) = s\mu_r(m).$$

Definition 1.30. An R -module homomorphism $h: M \rightarrow N$ is an R -**module isomorphism** if there is an R -module homomorphism $g: N \rightarrow M$ such that $h \circ g = \text{id}_N$ and $g \circ h = \text{id}_M$. We say M and N are **isomorphic**, denoted $M \cong N$, if there exists an isomorphism $M \rightarrow N$.

To check that an R -module homomorphism $f: M \rightarrow N$ is an isomorphism, it is sufficient to check that it is bijective.

Exercise 6. Let $f: M \rightarrow N$ be a homomorphism of R -modules. Show that if f is bijective, then its set-theoretic inverse $f^{-1}: N \rightarrow M$ is an R -module homomorphism. Therefore, every bijective homomorphism of R -modules is an isomorphism.

One should think of a module isomorphism as a relabelling of the names of the elements of the module. If two modules are isomorphic, that means that they are *essentially the same*, up to renaming the elements.

Definition 1.31. Let $f: M \rightarrow N$ be a homomorphism of R -modules. The **kernel** of f is

$$\ker(f) := \{m \in M \mid f(m) = 0\}.$$

The **image** of f , denoted $\text{im}(f)$ or $f(M)$, is

$$\text{im}(f) := \{f(m) \mid m \in M\}.$$

Exercise 7. Let R be a ring with $1 \neq 0$, let M be an R -module, and let N be an R -submodule of M . Then the inclusion map $i: N \rightarrow M$ is an R -module homomorphism.

Exercise 8. If $f: M \rightarrow N$ is an R -module homomorphism, then $\ker(f)$ is an R -submodule of M and $\text{im}(f)$ is an R -submodule of N .

Definition 1.32. Let R be a ring and let M and N be R -modules. Then $\text{Hom}_R(M, N)$ denotes the set of all R -module homomorphisms from M to N , and $\text{End}_R(M)$ denotes the set $\text{Hom}_R(M, M)$. We call $\text{End}(M)$ the **endomorphism ring** of M , and elements of $\text{End}(M)$ are called **endomorphisms** of M .

The endomorphism ring of an R -module M is called that because it *is* a ring, with multiplication given by composition of endomorphisms, 0 given by the zero map (the constant equal to 0), and 1 given by the identity map. However, two homomorphisms from M to N are not composable unless $M = N$, so $\text{Hom}_R(M, N)$ is not a ring.

When R is commutative, $\text{Hom}_R(M, N)$ is, however, an R -module; let us describe its R -module structure. Given $f, g \in \text{Hom}_R(M, N)$, $f + g$ is the map defined by

$$(f + g)(m) := f(m) + g(m),$$

and given $r \in R$ and $f \in \text{Hom}_R(M, N)$, $r \cdot f$ is the R -module homomorphism defined by

$$(r \cdot f)(m) := r \cdot f(m) = f(rm).$$

The zero element of $\text{Hom}_R(M, N)$ is the **zero map**, the constant equal to 0_N .

Lemma 1.33. *Let M and N be R -modules over a commutative ring R . Then the addition and multiplication by scalars defined above make $\text{Hom}_R(M, N)$ an R -module.*

Proof. There are many things to check, including:

- The addition and the R -action are both well-defined: given $f, g \in \text{Hom}_R(M, N)$ and $r \in R$, we always have $f + g, rf \in \text{Hom}_R(M, N)$.
- The axioms of an R -module are satisfied for $\text{Hom}_R(M, N)$.

We leave the details as exercises. □

We will see later that for an n -dimensional vector space V over a field F , there is an isomorphism of vector spaces $\text{End}_F(V) \cong M_n(F)$. This says that every linear transformation $T: V \rightarrow V$ corresponds to some $n \times n$ matrix. However, the story for general R -modules is a lot more complicated.

Lemma 1.34. *For any commutative ring R with $1 \neq 0$ and any R -module M there is an isomorphism of R -modules $\text{Hom}_R(R, M) \cong M$.*

Before we write a formal proof, it helps to think about *why* this theorem is true. What does it mean to give an R -module homomorphism $f: R \rightarrow M$? More precisely, what information do we need to determine such an f ? Do we need to be given the values of $f(r)$ for every $r \in R$? Since f is a homomorphism of R -modules, for any $r \in R$ we have

$$f(r) = f(r \cdot 1) = rf(1),$$

so the value of $f(1)$ *completely determines* which R -module homomorphism we are talking about. On the other hand, we can choose *any* $m \in M$ to be the image of 1, since thanks to the axioms for modules, the function

$$f(r) := rm$$

is a well-defined R -module homomorphism for any $m \in M$. In summary, to give an R -module homomorphism $R \rightarrow M$ is the same as choosing an element $m \in M$, and $\text{Hom}_R(R, M) \cong M$.

Proof. Let $f: M \rightarrow \text{Hom}_R(R, M)$ be given for each $m \in M$ by $f(m) = \phi_m$ where ϕ_m is the map defined by $\phi_m(r) = rm$ for all $r \in R$. Now we have many things to check:

- f is well-defined, meaning that for any $m \in M$, its image $f(m) = \phi_m$ is an element of $\text{Hom}_R(R, M)$, since

$$\phi_m(r_1 + r_2) = (r_1 + r_2)m = r_1m + r_2m = \phi_m(r_1) + \phi_m(r_2)$$

$$\phi_m(r_1r_2) = (r_1r_2)m = r_1(r_2m) = r_1\phi_m(r_2)$$

for all $r_1, r_2 \in R$.

- f is an R -module homomorphism, since

$$\phi_{m_1+m_2}(r) = r(m_1 + m_2) = rm_1 + rm_2 = \phi_{m_1}(r) + \phi_{m_2}(r)$$

$$\phi_{r'm}(r) = r(r'm) = (rr')m = r'(rm) = r'\phi_m(r)$$

- f is injective, since $\phi_m = \phi_{m'}$ implies in particular that $\phi_m(1_R) = \phi_{m'}(1_R)$, which by definition of ϕ_- means that $m = m'$.
- f is surjective, since for $\psi \in \text{Hom}_R(R, M)$ we have $\psi(r) = \psi(r1_R) = r\psi(1_R)$ for all $r \in R$, so $\psi = \phi_{\psi(1_R)}$.

This shows that f is an R -module isomorphism. □

Definition 1.35. Let R be a commutative ring with $1_R \neq 0_R$. An **R -algebra** is a ring A with $1_A \neq 0_A$ together with a ring homomorphism $f: R \rightarrow A$ such that $f(R)$ is contained in the center of A .

Given an R -algebra A , the R -algebra structure on A induces a natural R -module structure: given elements $r \in R$ and $a \in A$, the R -action is defined by

$$r \cdot a := f(r)a,$$

where the product on the right is the multiplication in A . Similarly, we get a natural right R -module structure on A , and since by definition $f(R)$ is contained in the center of A , we obtain what is called a *balanced bimodule* structure on A . We will discuss these further in Homological Algebra next Fall.

Example 1.36. Let R be a commutative ring with $1_R \neq 0_R$. The ring $R[x_1, \dots, x_n]$ together with the inclusion map $R \hookrightarrow R[x_1, \dots, x_n]$ is an R -algebra. More generally, any quotient of $R[x_1, \dots, x_n]$ is an R -algebra.

The ring of matrices $M_n(R)$ with the homomorphism $r \mapsto rI_n$ is also an R -algebra, as is the group ring $R[G]$ for any group G with the inclusion of R into $R[G]$ given by $r \mapsto re_G$.

Lemma 1.37. *Let R be a commutative ring with $1 \neq 0$ and let M be an R -module. Then $\text{End}_R(M)$ is an R -algebra, with addition and R -action defined as above, and multiplication defined by composition $(fg)(m) = f(g(m))$ for all $f, g \in \text{End}_R(M)$ and all $m \in M$.*

Proof. There are many things to check here, including that:

- The axioms of a (unital) ring are satisfied for $\text{End}_R(M)$.
- There is a ring homomorphism $f: R \rightarrow \text{End}_R(M)$ such that $f(1_R) = 1_{\text{End}_R(M)} = \text{id}_M$ and $f(R) \subseteq Z(\text{End}_R(M))$.

We will just check the last item and leave the others as exercises. Define $f: R \rightarrow \text{End}_R(M)$ by $f(r) = r \text{id}_M$. Notice that this is the map μ_r from Example 1.29. Then

$$f(r + s) = (r + s) \text{id}_M = r \text{id}_M + s \text{id}_M = f(r) + f(s)$$

and

$$f(rs) = (rs) \text{id}_M = (r \text{id}_M) \circ (s \text{id}_M) = f(r)f(s)$$

show that f is a ring homomorphism. Moreover, $\text{id}_M \in Z(\text{End}_R(M))$, and once can check easily that $\mu_r \in \text{End}_R(M)$: given any other $g \in \text{End}_R(M)$, and any $m \in M$, since g is R -linear we have

$$(g \circ \mu_r)(m) = g(\mu_r(m)) = g(rm) = rg(m) = (\mu_r \circ g)(m).$$

This shows that $f(R) \subseteq \text{End}_R(M)$. □

Remark 1.38. Let R be a commutative ring with $1 \neq 0$ and let M be an R -module. Then M is also an $\text{End}_R(M)$ -module with the action $\phi m = \phi(m)$ for any $\phi \in \text{End}_R(M)$, $m \in M$.

Definition 1.39. Let R be a ring, let M be an R -module, and let N be a submodule of M . The quotient module M/N is the quotient group M/N with R action defined by

$$r(m + N) := rm + N$$

for all $r \in R$ and $m + N \in M/N$.

Lemma 1.40. *Let R be a ring, let M be an R -module, and let N be a submodule of M . The quotient module M/N is an R -module, and the quotient map $q: M \rightarrow M/N$ is an R -module homomorphism with kernel $\ker(q) = N$.*

Proof. Among the many things to check here, we will only check the well-definedness of the R -action on M , and leave the others as exercises. To check well-definedness, consider $m + N = m' + N$. Then $m - m' \in N$, so $r(m - m') \in N$ by the definition of submodule. This gives that $rm - rm' \in N$, hence $rm + N = rm' + N$. □

Definition 1.41. Given an R -module M and a submodule N of M , the map $q: M \rightarrow M/N$ is the **canonical quotient map**, or simply the canonical map from M to N .

Example 1.42. If R is a field, quotient modules are the same thing as quotient vector spaces. When $R = \mathbb{Z}$, recall that \mathbb{Z} -modules are the same as abelian groups, by Lemma 1.13. Quotients of \mathbb{Z} -modules coincide with quotients of abelian groups.

Theorem 1.43. Let N be a submodule of M , let T be an R -module, and let $f: M \rightarrow T$ be an R -module homomorphism. If $N \subseteq \ker f$, then the function

$$\begin{aligned} M/N &\xrightarrow{\bar{f}} T \\ m + N &\longmapsto f(m) \end{aligned}$$

is a well-defined R -module homomorphism. In fact, $\bar{f}: M/N \rightarrow T$ is the unique R -module homomorphism such that $\bar{f} \circ q = f$, where $q: M \rightarrow M/N$ denotes the canonical map.

We can represent this in a more visual way by saying that \bar{f} is the unique R -module homomorphism that makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & T \\ & \searrow q & \nearrow \exists! \bar{f} \\ & M/N & \end{array}$$

commute.

Proof. By 817, we already know that \bar{f} is a well-defined homomorphism of groups under $+$ and that it is the unique one such that $\bar{f} \circ q = f$. It remains only to show \bar{f} is an R -linear map:

$$\bar{f}(r(m + N)) = \bar{f}(rm + N) = f(rm) = rf(m) = r\bar{f}(m + N).$$

where the third equation uses that f preserves scaling. □

Theorem 1.44 (First Isomorphism Theorem). Let N be an R -module and let $h: M \rightarrow N$ be an R -module homomorphism. Then $\ker(h)$ is a submodule of M and there is an R -module isomorphism $M/\ker(h) \cong \text{im}(h)$.

Proof. If we forget the multiplication by scalars in R , by the First Isomorphism Theorem for Groups, we know that there is an isomorphism of abelian groups under $+$, given by

$$\begin{aligned} \bar{h}: M/\ker(h) &\xrightarrow{\cong} \text{im}(h) \\ m + \ker(h) &\longmapsto h(m). \end{aligned}$$

It remains only to show this map preserves multiplication by scalars. And indeed:

$$\begin{aligned} \bar{h}(r(m + \ker(h))) &= \bar{h}(rm + \ker(h)) && \text{by definition of the } R\text{-action on } M/\ker(h) \\ &= h(rm) && \text{by definition of } \bar{h} \\ &= rh(m) && \text{since } h \text{ is an } R\text{-module homomorphism} \\ &= r\bar{h}(m + \ker(h)) && \text{by definition of } h. \end{aligned}$$

Theorem 1.45 (Second Isomorphism Theorem). *Let A and B be submodules of M , and let $A + B = \{a + b \mid a \in A, b \in B\}$. Then $A + B$ is a submodule of M , $A \cap B$ is a submodule of A , and there is an R -module isomorphism $(A + B)/B \cong A/(A \cap B)$.*

Proof. By Exercise 3, $A + B$ and $A \cap B$ are submodules of M . By the Second Isomorphism Theorem for Groups, there is an isomorphism of abelian groups

$$\begin{aligned} h: A/(A \cap B) &\xrightarrow{\cong} (A + B)/B \\ a + (A \cap B) &\longmapsto a + B \end{aligned}$$

It remains only to show h preserves multiplication by scalars:

$$h(r(a + (A \cap B))) = h(ra + A \cap B) = ra + B = r(a + B) = rh(a + (A \cap B)). \quad \square$$

Theorem 1.46 (Third Isomorphism Theorem). *Let A and B be submodules of M with $A \subseteq B$. Then there is an R -module isomorphism $(M/A)/(B/A) \cong M/B$.*

Proof. From 817, we know that B/A is a subgroup of M/A under $+$. Given $r \in R$ and $b + A \in B/A$ we have $r(b + A) = rb + A$ which belongs to B/A since $rb \in B$. This proves B/A is a submodule of M/A . By the Third Isomorphism Theorem for Groups, there is an isomorphism of abelian groups

$$\begin{aligned} (M/A)/(B/A) &\longrightarrow M/B \\ (m + A) + B/A &\longmapsto m + B \end{aligned}$$

and it remains only to show this map is R -linear:

$$\begin{aligned} h(r((m + A) + B/A)) &= h(r(m + A) + B/A) = h((rm + A) + B/A) \\ &= rm + B = r(m + B) \\ &= rh((m + A) + B/A). \end{aligned} \quad \square$$

Theorem 1.47 (Lattice Isomorphism Theorem). *Let R be a ring, let N be a R -submodule of an R -module M , and let $q: M \rightarrow M/N$ be the quotient map. Then the function*

$$\begin{aligned} \{R\text{-submodules of } M \text{ containing } N\} &\xrightarrow{\Psi} \{R\text{-submodules of } M/N\} \\ K &\longmapsto K/N \end{aligned}$$

is a bijection, with inverse defined by

$$\Psi^{-1}(T) := q^{-1}(T) = \{a \in M \mid a + N \in T\}$$

for each R -submodule T of M/N . Moreover, Ψ and Ψ^{-1} preserve sums and intersections of submodules.

Proof. From 817, we know there is a bijection between the set of subgroups of M and that contain N and subgroups of the quotient group M/N , given by the same map Ψ . We just need to prove that these maps send submodules to submodules. If K is a submodule of M containing N , then by the [Third Isomorphism Theorem](#) we know that K/N is a submodule of M/N . If T is a submodule of M/N , then $\pi^{-1}(T)$ is an abelian group, by 817. For $r \in R$ and $m \in \pi^{-1}(T)$, we have $\pi(m) \in T$, and hence $\pi(rm) = r\pi(m) \in T$ too, since T is a submodule. This proves $\pi^{-1}(T)$ is a submodule. \square

We come to a very important class of examples which will help us study linear transformations using module theory.

Lemma 1.48 ($F[x]$ -modules). *Let F be a field. There is a bijection*

$$\{V \text{ an } F[x]\text{-module}\} \longleftrightarrow \{V \text{ an } F\text{-vector space and } T \in \text{End}_F(V)\}.$$

Proof. If V is an $F[x]$ module then V is an F -vector space by restriction of scalars along the inclusion $F \hookrightarrow F[x]$. Let $T : V \rightarrow V$ be defined by $T(v) = xv$. To show that $T \in \text{End}_F(V)$, note that for any $c \in F$ and $v, v_1, v_2 \in V$ the axioms of the $F[x]$ -module give us

$$T(v_1 + v_2) = x(v_1 + v_2) = xv_1 + xv_2 = T(v_1) + T(v_2) \text{ and } T(cv) = x(cv) = c(xv).$$

Conversely, let V be an F -vector space and $T \in \text{End}_F(V)$. We claim that the action of $F[x]$ on V given by

$$f(x)v = (f(T))(v)$$

satisfies the axioms for a module (exercise!). Alternatively, we can explain this module structure in a more conceptual way, as follows. Consider the evaluation homomorphism $\varphi : F[x] \rightarrow \text{End}_F(V)$, $\varphi(f(x)) = f(T)$. Since V is an $\text{End}_F(V)$ -module by Remark 1.38, then V is also an $F[x]$ -module by restriction of scalars along ϕ ; the $F[x]$ action is the one we described above:

$$f(x)v = \varphi(f)(v) = (f(T))(v)$$

Finally, one can check that the two constructions above are inverse to each other. \square

Notation 1.49. We shall denote the $F[x]$ -module structure on an F -vector space V induced by $T \in \text{End}_F(V)$ by V_T .

Example 1.50. The proposition above says that if we fix an F -vector space V then any linear transformation T gives a different $F[x]$ module structure on V . For example,

- for $T = 0$ the $F[x]$ module V_0 carries an action given by scaling by the constant coefficient of f , that is if $f(x) = a^n x^n + \cdots + a_0$ then

$$f(x)v = (f(0))v = a_0 v \text{ for all } f \in F[x].$$

- for T the “shift operator” that takes $T(e_i) = e_{i-1}$, where e_i is the i -th standard basis

vector, the $F[x]$ module V_T has the action x^m

$$\begin{bmatrix} v_1 \\ \vdots \\ v_{n-m} \\ v_{n-m+1} \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_{m+1} \\ \vdots \\ v_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

1.5 Module generators, bases and free modules

Definition 1.51. Let M be an R -module. A **linear combination** of finitely many elements a_1, \dots, a_n of M is an element of M of the form $r_1 a_1 + \dots + r_n a_n$ for some $r_1, \dots, r_n \in R$.

Definition 1.52. Let R be a ring with $1 \neq 0$ and let M be an R -module. For a subset A of M , the submodule of M **generated by** A is

$$RA := \{r_1 a_1 + \dots + r_n a_n \mid n \geq 0, r_i \in R, a_i \in A\}.$$

We M is **generated by** A if $M = RA$. If M is an F -vector space, we say that M is **spanned** by a set A instead of generated by A .

A module M is **finitely generated** if there is a finite subset A of M that generates M . If $A = \{a\}$ has a single element, the module $RA = Ra$ is called *cyclic*.

Exercise 9. Let M be an R -module and let $A \subseteq M$. Then RA is the smallest submodule of M containing A , that is

$$RA = \bigcap_{A \subseteq N, N \text{ submodule of } M} N.$$

Exercise 10. Being finitely generated and being cyclic are R -module isomorphism invariants.

Example 1.53. Let R be a ring with $1 \neq 0$.

- (1) $R = R1$ is cyclic.
- (2) $R \oplus R$ is generated by $\{(1, 0), (0, 1)\}$.
- (3) $R[x]$ is generated as an R -module by the set $\{1, x, x^2, \dots, x^n, \dots\}$ of monic monomials in the variable x .
- (4) Let $M = \mathbb{Z}[x, y]$. M is generated by
 - $\{1, x, y\}$ as a ring,
 - $\{1, y, y^2, \dots, y^n, \dots\}$ as an $\mathbb{Z}[x]$ -module, and
 - $\{x^i y^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$ as a group (\mathbb{Z} -module).

Lemma 1.54. Let R be a ring with $1 \neq 0$, let M be an R -module, and let N be an R -submodule of M .

- (1) If M is finitely generated as an R -module, then so is M/N .
- (2) If N and M/N are finitely generated as R -modules, then so is M .

Proof. The proof of (2) will be a problem set question. To show (1), note that if $M = RA$ then $M/N = R\bar{A}$, where $\bar{A} = \{a + N \mid a \in A\}$. \square

Definition 1.55. Let M be an R -module and let A be a subset of M . The set A is **linearly independent** if whenever $r_1, \dots, r_n \in R$ and a_1, \dots, a_n are distinct elements of A satisfying $r_1 a_1 + \dots + r_n a_n = 0$, then $r_1 = \dots = r_n = 0$. Otherwise A is **linearly dependent**.

Definition 1.56. A subset A of an R -module M is a **basis** of M if A is linearly independent and generates M . An R -module M is a **free** R -module if M has a basis.

We will later see that over a field, every module is free. However, when R is not a field, there are R -modules that are not free; in fact, *most* modules are not free.

Example 1.57. Here are some examples of free modules:

- (1) If we think of R as a module over itself, it is free with basis $\{1\}$.
- (2) The module $R \oplus R$ is free with basis $\{(1, 0), (0, 1)\}$.
- (3) The R -module $R[x]$ is free, and $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis.
- (4) Let $M = \mathbb{Z}[x, y]$. Then $\{1, y, y^2, \dots, y^n, \dots\}$ is a basis for the $\mathbb{Z}[x]$ -module M , and $\{x^i y^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$ is a basis for the \mathbb{Z} -module M .

Example 1.58. $\mathbb{Z}/2$ is not a free \mathbb{Z} -module. Indeed suppose that A is a basis for $\mathbb{Z}/2$ and $a \in A$. Then $2a = 0$ so A cannot be linearly independent, a contradiction.

Lemma 1.59. *If A is a basis of M then every nonzero element $0 \neq m \in M$ can be written uniquely as $m = r_1 a_1 + \dots + r_n a_n$ with a_i distinct elements of A and $r_i \neq 0$.*

Proof. Suppose that if $m \neq 0$ and A_1, A_2 are finite subsets of A such that

$$m = \sum_{a \in A_1} r_a a = \sum_{b \in A_2} s_b b$$

for some $r_a, s_b \in R$. Then

$$\sum_{a \in A_1 \cap A_2} (r_a - s_a) a + \sum_{a \in A_1 \setminus A_2} r_a a - \sum_{a \in A_2 \setminus A_1} s_a a = 0.$$

Since A is a linearly independent set, we conclude that $r_a = s_a$ for $a \in A_1 \cap A_2$, $r_a = 0_R$ for $a \in A_1 \setminus A_2$, and $s_a = 0_R$ for $a \in A_2 \setminus A_1$. Set

$$B := \{a \in A_1 \cap A_2 \mid r_a \neq 0_R\}.$$

Then

$$m = \sum_{a \in B} r_a a$$

is the unique way of writing m as a linear combination of elements of A with nonzero coefficients. \square

Theorem 1.60. *Let R be a ring, M be a free R -module with basis B , N be any R -module, and let $j : B \rightarrow N$ be any function. Then there is a unique R -module homomorphism $h : M \rightarrow N$ such that $h(b) = j(b)$ for all $b \in B$.*

Proof. We have two things to prove: existence and uniqueness.

Existence: By Lemma 1.59, any $0 \neq m \in M$ can be written uniquely as

$$m = r_1 b_1 + \cdots + r_n b_n$$

with $b_i \in B$ distinct and $0 \neq r_i \in R$. Define $h: M \rightarrow N$ by

$$\begin{cases} h(r_1 b_1 + \cdots + r_n b_n) = r_1 j(b_1) + \cdots + r_n j(b_n) & \text{if } r_1 b_1 + \cdots + r_n b_n \neq 0 \\ h(0_M) = 0_N \end{cases}$$

One can check that this satisfies the conditions to be an R -module homomorphism (exercise!).

Uniqueness: Let $h: M \rightarrow N$ be an R -module homomorphism such that $h(b_i) = j(b_i)$. Then in particular $h: (M, +) \rightarrow (N, +)$ is a group homomorphism and therefore $h(0_M) = 0_N$ by properties of group homomorphisms. Furthermore, if $m = r_1 b_1 + \cdots + r_n b_n$ then

$$h(m) = h(r_1 b_1 + \cdots + r_n b_n) = r_1 h(b_1) + \cdots + r_n h(b_n) = r_1 j(b_1) + \cdots + r_n j(b_n)$$

by the definition of homomorphism, and because $h(b_i) = j(b_i)$. \square

Corollary 1.61. *If A and B are sets of the same cardinality, and fix a bijection $j: A \rightarrow B$. If M and N are free R -modules with bases A and B respectively, then there is an isomorphism of R -modules $M \cong N$.*

Proof. Let $g: M \rightarrow N$ and $h: N \rightarrow M$ be the module homomorphisms induced by the bijection $j: A \rightarrow B$ and its inverse $j^{-1}: B \rightarrow A$, which exist by Theorem 1.60. We will show that h and g are inverse homomorphisms. First, note that $g \circ h: N \rightarrow N$ is an R -module homomorphism and $(g \circ h)(b) = g(j^{-1}(b)) = j(j^{-1}(b)) = b$ for every $b \in B$. Since the identity map id_N is an R -module homomorphism and $\text{id}_N(b) = b$ for every $b \in B$, by the uniqueness in Theorem 1.60 we have $g \circ h = \text{id}_N$. Similarly, one shows that $h \circ g = \text{id}_M$. \square

The corollary gives that, up to isomorphism, there is only one free module with basis A , provided such a module exists. But does a free module generated by a given set A exist? It turns out it does.

Definition 1.62. Let R be a ring and let A be a set. The free R -module generated by A , denoted $F_R(A)$ is the set of formal sums

$$\begin{aligned} F_R(A) &= \{r_1 a_1 + \cdots + r_n a_n \mid n \geq 0, r_i \in R, a_i \in A\} \\ &= \left\{ \sum_{a \in A} r_a a \mid r_a \in R, r_a = 0 \text{ for all but finitely many } a \right\}, \end{aligned}$$

with addition defined by

$$\left(\sum_{a \in A} r_a a \right) + \left(\sum_{a \in A} s_a a \right) = \sum_{a \in A} (r_a + s_a) a$$

and R -action defined by

$$r \left(\sum_{a \in A} r_a a \right) = \sum_{a \in A} (r r_a) a.$$

Exercise 11. This construction $F_R(A)$ results in an R -module, which is free with basis A , and $F_R(A) \cong \bigoplus_{a \in A} R$.

Theorem 1.63 (Uniqueness of rank over commutative rings). *Let R be a commutative ring with $1 \neq 0$ and let M be a free R -module. If A and B are both bases for M , then A and B have the same cardinality, meaning that there exists a bijection $A \rightarrow B$.*

Proof. You will show this in the next problem set (at least in the case where M has a finite basis). \square

Definition 1.64. Let R be a commutative ring with $1 \neq 0$ and let M be a free R -module. The **rank** of M is the cardinality of any basis of M .

Example 1.65. Let R be a commutative ring with $1 \neq 0$. The rank of R^n is n . Note that by Corollary 1.61, any free R -module of rank n must be isomorphic to R^n .

Earlier, we described the R -module structure on the direct sum of R -modules; this is how we construct R^n , by taking the direct sum of n copies of the R -module R . This construction can also be described as the direct product of n copies of R . However, the direct sum and direct product are two different constructions.

Definition 1.66. Let R be a ring. Let $\{M_a\}_{a \in J}$ be a collection of R -modules. The **direct product** of the R -modules M_a is the Cartesian product

$$\prod_{a \in J} M_a := \{(m_a)_{a \in J} \mid m_a \in M_a\}$$

with addition defined by

$$(m_a)_{a \in J} + (n_a)_{a \in J} := (m_a + n_a)_{a \in J}$$

and R -action defined by

$$r(m_a)_{a \in J} = (rm_a)_{a \in J}.$$

The **direct sum** of the R -modules M_a is the R -submodule $\bigoplus_{a \in J} M_a$ of the direct product $\prod_{a \in J} M_a$ given by

$$\bigoplus_{a \in J} M_a = \{(m_a)_{a \in J} \mid m_a = 0 \text{ for all but finitely many } a\}.$$

Exercise 12. The direct sum and the direct product of an arbitrary family of R -modules are R -modules.

Example 1.67. Suppose that $|A| = n < \infty$. Let M_1, \dots, M_n be R -modules. The direct product module $M_1 \times \dots \times M_n$ is the abelian group $M_1 \times \dots \times M_n$ with ring action given by $r(m_1, \dots, m_n) = (rm_1, \dots, rm_n)$ for all $r \in R$ and $m_i \in M_i$. Comparing the definitions we see that

$$M_1 \times \dots \times M_n = M_1 \oplus \dots \oplus M_n.$$

If $M_i = R$ for $1 \leq i \leq n$, then we denote $R^n = \underbrace{R \times \dots \times R}_n = \underbrace{R \oplus \dots \oplus R}_n$.

It is useful to talk about maps from the factors/summands to the direct product/ direct sum and conversely.

Definition 1.68. For $i \in J$ the *inclusion of the i -th factor* into a direct product or direct sum is the map

$$\iota_i: M_i \rightarrow \prod_{a \in J} M_a \text{ or } \iota_i: M_i \rightarrow \bigoplus_{a \in J} M_a, \iota_i(m) = (m_a)_{a \in J}, \text{ where } m_a = \begin{cases} m & \text{if } a = i \\ 0 & \text{if } a \neq i \end{cases}.$$

For $i \in J$ the i -th *projection map* from a direct product or a direct sum module is

$$\pi_i: \prod_{a \in J} M_a \rightarrow M_i \text{ or } \pi_i: \bigoplus_{a \in J} M_a \rightarrow M_i, \pi_i((m_a)_{a \in J}) = m_i.$$

Lemma 1.69. *Projections from direct products or sums of R -module, inclusions into direct products or sums of R -modules, and products of R -module homomorphisms are R -module homomorphisms. Furthermore, inclusions are injective, projections are surjective, and*

$$\pi_i \circ \iota_i = \text{id}_{M_i}.$$

Also, $\iota_i(M_i)$ is an R -submodule of the direct product/sum which is isomorphic to M_i .

Note, however, that $\iota_i \circ \pi_i \neq \text{id}$.

1.6 Every module is a quotient of a free module

Lemma 1.70. *Given any ring R with $1 \neq 0$, any direct sum of copies of R is always a free R -module.*

Proof. Suppose that $F = \bigoplus_{i \in \Lambda} R$ is a direct sum of copies of R indexed by some set Λ . The tuples

$$e_i = (a_j)_{j \in \Lambda} \text{ with } a_j = 0 \text{ for all } j \neq i \text{ and } a_i = 1$$

generate F , since we can write any element as

$$(c_i)_{i \in \Lambda} = \sum_{i \in \Lambda} c_i e_i.$$

Notice that by definition $c_i \neq 0$ for only finitely many i , so the sum on the right has finitely many nonzero terms. Moreover, the e_i are linearly independent, and thus they form a basis for F . \square

We will show in the next chapter that every when R is a field, every R -module is free. In contrast, we will also see that if R is a commutative ring that is not a field, there always exists an R -module that is not free – in fact, given a ring R that is not a field, one can give a very concrete recipe for building nonfree modules.

However, even though not all modules are free, what is true is that every R -module can be written as a quotient of a free module, as follows. Given a module M , first take a generating

set for M , say $\Gamma = \{m_i\}_{i \in \Lambda}$. Notice that a generating set always exists: for example, we can take $\Gamma = M$, though of course that is a bit of an overkill, since it's quite likely that some elements can be obtained from linear combinations of others.

Next, we construct a free module on the set Λ ; more precisely, we take a free module on as many generators as generators for M that we picked. Now the map

$$\begin{aligned} \bigoplus_{i \in \Lambda} R &\xrightarrow{\pi} M \\ (r_i) &\longmapsto \sum_{i \in \Lambda} r_i m_i. \end{aligned}$$

Notice this map actually makes sense: the tuples (r_i) have only finitely many nonzero entries, and thus $\sum_{i \in \Lambda} r_i m_i$ is a (finite) linear combination of our chosen generators. Moreover, since we chose the m_i to be generators for M , this map π is surjective. It is also easy to check that it is an R -module homomorphism: in fact, this is the R -module homomorphism we would get from Theorem 1.60 by setting $e_i \mapsto m_i$.

By the [First Isomorphism Theorem](#),

$$M \cong \bigoplus_{i \in \Lambda} R / \ker \pi.$$

This shows the following:

Theorem 1.71. *Every R -module is a quotient of a free R -module.*

Notice that the map π we constructed above depends on a choice of generating set for M . Given the map π corresponding to the set of generators $\Gamma = \{m_i\}$, each element in $\ker(\pi)$ is a **relation** among the generators for M : the tuple (r_i) is a relation for the generators $\{m_i\}$ if

$$\sum_{i \in \Lambda} r_i m_i = 0.$$

A nonzero relation among the m_i tells us that the set $\{m_i\}$ is linearly dependent. Thus we see that

$$\pi \text{ is injective} \iff \{m_i\} \text{ is linearly independent} \iff \{m_i\} \text{ is a basis for } M.$$

In particular, the existence of such a map π that is injective is equivalent to M being free. Since π is always surjective (as long as $\{m_i\}$ forms a generating set for M , we can now rephrase this as

$$\pi \text{ is an isomorphism} \iff \{m_i\} \text{ is a basis for } M.$$

The module M is free if and only if we can find a basis for M , thus M if M is free then M is isomorphic to a direct sum of copies of R . Since we have already shown that a direct sum of copies of R is free, we conclude the following:

Theorem 1.72. *An R -module is free if and only if it is isomorphic to a direct sum of copies of R .*

Chapter 2

Vector spaces and linear transformations

2.1 Classification of vector spaces and dimension

Recall that for a subset A of an F -vector space V , the **span** of A , denoted $\text{span}(A)$, is the subspace generated by A :

$$\text{span}(A) := \left\{ \sum_{i=1}^n c_i a_i \mid n \geq 0, c_i \in F, a_i \in A \right\}.$$

Lemma 2.1. *Suppose I is a linearly independent subset of an F -vector space V and $v \in V \setminus \text{span}(I)$, then $I \cup \{v\}$ is also linearly independent.*

Proof. Let w_1, \dots, w_n be any list of distinct elements of $I \cup \{v\}$ and suppose that $\sum_i c_i w_i = 0$ for some $c_i \in F$. If none of the w_i 's is equal to v , then $c_i = 0$ for all i , since I is linearly independent. Without loss of generality, say $w_1 = v$. If $c_1 = 0$ then $c_i = 0$ for all i by the same reasoning as in the previous case. If $c_1 \neq 0$, then

$$v = \sum_{i \geq 2} \frac{c_i}{c_1} w_i \in \text{span}(I),$$

contrary to assumption. This proves that $I \cup \{v\}$ is a linearly independent set. \square

To prove that every vector space has a basis, we will need to use Zorn's Lemma. Before we recall what Zorn's Lemma says, let's recall some notation:

Definition 2.2. A **poset** is a set S with an order relation \leq such that for all elements $x, y, z \in S$ we have

- $x \leq x$,
- if $x \leq y$ and $y \leq z$ then $x \leq z$, and
- if $x \leq y$ and $y \leq x$ then $x = y$.

A **totally ordered** set is a poset (T, \leq) such that for all $x, y \in T$ either $x \leq y$ or $y \leq x$.

Example 2.3. Given a set X , the collection $\mathcal{P}(X)$ of all subsets of X forms a poset with \leq defined to be set containment \subseteq . Unless X is empty or a singleton, the poset $\mathcal{P}(X)$ is not totally ordered.

Definition 2.4. Let (\mathcal{A}, \leq) be a **poset**, meaning that \mathcal{A} is a set with a partial order \leq . A subset \mathcal{B} of \mathcal{A} is **totally ordered** if for all $b, b' \in \mathcal{B}$ either $b \leq b'$ or $b' \leq b$; a totally ordered subset of \mathcal{A} is sometimes called a **chain**. We say a subset \mathcal{B} of \mathcal{A} has an **upper bound** in \mathcal{A} if there exists an element $u_B \in \mathcal{A}$ such that $b \leq u_B$ for all $b \in \mathcal{B}$. We say \mathcal{A} has a **maximal element** if there exists $m \in \mathcal{A}$ such that whenever $x \in \mathcal{A}$ and $m \leq x$ then $m = x$.

Axiom 2.5 (Zorn's Lemma). If \mathcal{A} is a nonempty poset such that every totally ordered subset $\mathcal{B} \subseteq \mathcal{A}$ has an upper bound in \mathcal{A} , then there is a maximal element $m \in \mathcal{A}$.

Some mathematicians refuse to accept Zorn's Lemma into their axiom system. We will at least pretend to be mathematicians who do. Fun fact: Theorem 2.6 is actually equivalent to the Axiom of Choice, meaning that if one replaces the Axiom of Choice in the ZFC axioms for set theory by Theorem 2.6, that does not change set theory – and one would then be able to deduce the Axiom of Choice.

If we accept Zorn's Lemma, we can now show that every vector space has a basis.

Theorem 2.6 (Every vector space has a basis). *Let V be an F -vector space and assume $I \subseteq S \subseteq V$ are subsets such that I is linearly independent and S spans V . Then there is a subset B with $I \subseteq B \subseteq S$ such that B is a basis.*

Before we prove this theorem, note that a corollary of Theorem 2.6 is that every vector space has a basis; in particular, this says that every module over a field is free!

Corollary 2.7. *Every vector space V has a basis. Moreover, every linearly independent subset of V is contained in some basis, and every set of vectors that spans V contains some basis.*

Proof. For this first part, apply the theorem with $I = \emptyset$ and $S = V$. For the second and third, use I arbitrary and $S = V$ and $I = \emptyset$ and S arbitrary, respectively. \square

Example 2.8. \mathbb{R} has a basis as a \mathbb{Q} -vector space; just don't ask me what it looks like.

We will not prove Theorem 2.6. But before we give a formal proof, let's first give a heuristic proof. To so that, start with I . If $\text{span}(I) = V$, then $B = I$ does the job. If not, then since $\text{span}(S) = V$, there must be a $v \in S \setminus \text{span}(I)$. Let $I' := I \cup \{v\}$. Then $I' \subseteq S$ and, by Lemma 2.1, I' is linearly independent. If $\text{span}(I') = V$, we have found our B , and if not we construct I'' from I' just as we constructed I' from I . At this point we would like to say that this process cannot go on for ever, and this is more-or-less true. But at least in an infinite dimensional setting, we need to use Zorn's Lemma to complete the proof rigorously.

Proof of Theorem 2.6. Let \mathcal{P} denote the collection of all subsets X of V such that $I \subseteq X \subseteq S$ and X is linearly independent. We make \mathcal{P} into a poset by the order relation given by set containment \subseteq . We note that \mathcal{P} is not empty since, for example $I \in \mathcal{P}$.

Let \mathcal{T} be any nonempty chain in \mathcal{P} . Let $Z = \bigcup_{Y \in \mathcal{T}} Y$. We claim $Z \in \mathcal{P}$. Given $z_1, \dots, z_m \in Z$, for each i we have $z_i \in Y_i$ for some $Y_i \in \mathcal{T}$. Since \mathcal{T} is totally ordered, one of Y_1, \dots, Y_m contains all the others and hence contains all the z_i 's. Since Y_i is linearly independent, this shows z_1, \dots, z_m are linearly independent. Thus Z is linearly independent. Since \mathcal{T} is non-empty, $Z \supseteq I$ and hence $Z \in \mathcal{P}$. It is an upper bound for \mathcal{T} by construction.

By Zorn's Lemma, \mathcal{P} has a maximal element B , which we claim is a basis for V . Note that B is linearly independent and $I \subseteq B \subseteq S$ by construction. We need to show that it spans V . Suppose not. Since S spans V , if $S \subseteq \text{span}(B)$, then $\text{span}(B)$ would have to be all of V . So, there is at least one $v \in S$ such that $v \notin \text{span}(B)$, and set $X := B \cup \{v\}$. Clearly, $I \subset X \subseteq S$ and, by Lemma 2.1, X is linearly independent. This shows that X is an element of \mathcal{P} that is strictly bigger than B , contrary to the maximality of B . \square

Corollary 2.9. *Suppose F is a field and W is a subspace of the F -vector space V . Then every basis of W extends to a basis of V , that is, if B is a basis of W then there exists a basis \tilde{B} of V such that B is a subset of \tilde{B} .*

Proof. Apply Corollary 2.7 with $B = I$ and $S = V$. Since B is a basis of W , B is linearly independent, and B remains linearly independent when regarded as a subset of V . \square

Remark 2.10. It is *not* true that, with the notation of the previous Corollary, if \tilde{B} is a basis of V then there exists a basis B of W such that B is a subset of \tilde{B} . For instance, take $F = \mathbb{R}$, $V = \mathbb{R}^2$, $\tilde{B} = \{(1, 0), (0, 1)\}$ and W the subspace spanned by $(1, 1)$.

Definition 2.11. A vector space is **finite dimensional** if there is spanned by a finite subset.

Thanks to Theorem 2.6, this is equivalent to the property that it has a finite basis. In the language of modules, a finite dimensional vector space is just a finitely generated F -module.

The following is an essential property of vector spaces that eventually will allow us to compare bases in terms of size.

Lemma 2.12 (Exchange Property). *Let B be a basis for the vector space V and consider any finite set of linearly independent vectors $C = \{c_1, \dots, c_m\}$ in V . Then there are distinct vectors b_1, \dots, b_m in B such that $(B \setminus \{b_1, \dots, b_m\}) \cup C$ is also a basis for V .*

Proof. Using induction on k , we will show that for each k with $0 \leq k \leq m$ there are distinct vectors b_1, \dots, b_k in B such that $(B \setminus \{b_1, \dots, b_k\}) \cup \{c_1, \dots, c_k\}$ is also a basis of V . In the base case, $k = 0$, there is nothing to show. The terminal case, $k = m$, gives us the desired statement.

For the inductive step, assume $B' = (B \setminus \{b_1, \dots, b_k\}) \cup \{c_1, \dots, c_k\}$ is also a basis of V . Since $c_{k+1} \in V$, we can write

$$c_{k+1} = \sum_{i=1}^n \lambda_i b_i + \sum_{i=1}^k \mu_i c_i$$

for some scalars $\lambda_i, \mu_i \in F$ and some elements $b_i \in B \setminus \{b_1, \dots, b_k\}$. Note that since C is linearly independent, at least one of the scalars λ_i is nonzero. Let i_0 be such that $\lambda_{i_0} \neq 0$, and notice that solving for b_{i_0} from the displayed equation gives that $b_{i_0} \in \text{span}(B'')$ where $B'' = (B' \setminus \{b_{i_0}\}) \cup \{c_{k+1}\}$. Now we can “replace” b_{i_0} by c_k , since the previous statement implies $\text{span}(B'') = \text{span}(B') = V$ and moreover B'' is linearly independent since otherwise B' would be linearly dependent. \square

Next, we will show that all bases of the same vector space have the same cardinality. We will only prove this under the assumption that V is finite dimensional, though it is true even if V has infinite dimension.

Theorem 2.13 (Dimension Theorem). *Any two bases of the same vector space have the same cardinality.*

Proof of the finite dimensional case. Suppose V is finite dimensional. Then it has a finite basis B . Let B' be any other basis, and note that we cannot yet assume B' is necessarily finite. Let $\{c_1, \dots, c_m\}$ be any m -element subset of B' for any m . An immediate consequence of Lemma 2.12 is that $m \leq |B|$, since otherwise we could not find m distinct elements of B to replace the c_i 's by. Since every finite subset of B' has cardinality no larger than $|B|$, this proves that B' is finite and $|B'| \leq |B|$. By symmetry, we obtain $|B| \leq |B'|$ too, hence equality follows. \square

Definition 2.14. The **dimension** of a vector space V , denoted $\dim_F(V)$ or $\dim(V)$, is the cardinality of any of its bases.

Example 2.15. $\dim_F(F^n) = |\{e_1, e_2, \dots, e_n\}| = n$.

Theorem 2.16 (Classification of finitely generated vector spaces). *Let F be a field.*

- (1) *Every finitely generated vector space over F is isomorphic to F^n for $n = \dim_F(V)$.*
- (2) *For any $m, n \in \mathbb{Z}_{\geq 0}$, $F^m \cong F^n$ if and only if $m = n$.*

Proof. To show (1), let V be a finite dimensional F -vector space. Then F has a finite spanning set S and by Theorem 2.6 there is a basis $B \subseteq S$ for V . Notice that B is necessarily finite and $V = FB$. Set $|B| = n$ and $B = \{b_1, \dots, b_n\}$. By Theorem 1.60, there is a linear transformation $f: F^n \rightarrow V$ such that $f(e_i) = b_i$ as well as a linear transformation $g: V \rightarrow F^n$ such that $g(b_i) = e_i$. Then both $f \circ g: V \rightarrow V$ and $g \circ f: F^n \rightarrow F^n$ are linear transformation which agree with the identity map on a basis. Hence by the uniqueness part of Theorem 1.60 we have $f \circ g = \text{id}_V$ and $g \circ f = \text{id}_{F^n}$. Therefore, these maps are the desired isomorphisms.

To show (2), let $\varphi: F^m \cong F^n$ be a vector space isomorphism and let B be a basis of F^m . We claim that $\varphi(B)$ is a basis for F^n . Indeed, if

$$\sum_{i=1}^m c_i \varphi(b_i) = 0 \quad \text{then} \quad \varphi\left(\sum_{i=1}^m c_i b_i\right) = 0, \quad \text{so} \quad \sum_{i=1}^m c_i b_i = 0$$

since φ is injective. But B is linearly independent, so we must have $c_i = 0$ for all $1 \leq i \leq m$. If $v \in F^n$, then since B spans F^m we have

$$\varphi^{-1}(v) = \sum_{i=1}^m c_i b_i$$

for some c_i . Thus

$$v = \sum_{i=1}^m c_i \varphi(b_i),$$

which shows $\varphi(B)$ spans F^n . By the [Dimension Theorem](#), we have

$$\dim_F(F^n) = n = |\varphi(B)| = |B| = m. \quad \square$$

Remark 2.17.

- (1) The same proof as in part (1) of Theorem 2.16 above shows that every finitely generated free R -module is isomorphic to R^n for some $n \geq 0$.
- (2) Part (2) of the [Classification Theorem](#) can be extended to modules over commutative rings as stated in Theorem 1.63; this is a problem in Problem Set 3.
- (3) The [Classification Theorem](#) yields that dimension is an isomorphism invariant.

Corollary 2.18. *Two finite dimensional vector spaces V and V' over the same field F are isomorphic if and only if $\dim_F(V) = \dim_F(V')$.*

Proof. By Theorem 2.16, V and V' are both of the form $V \cong F^m$ and $V' \cong F^n$, while $F^m \cong F^n$ if and only if $m = n$. \square

A word on infinite-dimensional vector spaces.

Example 2.19. Consider the vector space $F[x]$. This cannot be a finite dimensional vector space. For instance, if $\{f_1, \dots, f_n\}$ were a basis, then setting

$$M = \max_{1 \leq j \leq n} \{\deg(f_j)\}$$

we see that the element x^{M+1} is not in the span of $\{f_1, \dots, f_n\}$. We can find a basis for this space though. Consider the collection $B = \{1, x, x^2, \dots\}$. This set is linearly independent and spans $F[x]$, thus it forms a basis for $F[x]$. This basis is *countable*, so $\dim_F(F[x]) = \aleph_0 = |\mathbb{N}|$.

Example 2.20. Consider the real vector space

$$V := \mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots.$$

This space can be identified with sequences $\{a_n\}$ of real numbers. One might be interested in a basis for this vector space. At first glance, the most obvious choice for a basis would be $E = \{e_1, e_2, \dots\}$. It turns out that E is the basis for the direct sum $\bigoplus_{i \in \mathbb{N}} \mathbb{R}$. However, it is immediate that this set does not span V , as $v = (1, 1, \dots)$ can not be represented as a finite linear combination of these elements. Since v is not in $\text{span}(E)$, then by Lemma 2.1 we know that $E \cup \{v\}$ is a linearly independent set. However, this new set $E \cup \{v\}$ does not span BV either, as $(1, 2, 3, 4, \dots)$ is not in the span of $E \cup \{v\}$. We know that V has a basis, but it can be shown that no countable collection of vectors forms a basis for this space, and in fact $\dim_{\mathbb{R}}(\mathbb{R}^{\mathbb{N}}) = |\mathbb{R}|$.

We now deduce some formulas that relate the dimensions of various vector spaces.

Theorem 2.21. *Let W be a subspace of a vector space V . Then*

$$\dim(V) = \dim(W) + \dim(V/W).$$

Here the dimension of a vector space is understood to be either a nonnegative integer or ∞ , and the arithmetic of the formula is understood to follow the rules $n + \infty = \infty = \infty + \infty$ for any $n \in \mathbb{Z}_{\geq 0}$. We leave the proof for Problem Set 4.

Example 2.22. Consider the vector space $V = \mathbb{R}^2$ and its subspace $W = \text{span}\{e_1\}$. Then the quotient vector space V/W is, by definition,

$$V/W = \{(x, y) + W \mid (x, y) \in \mathbb{R}^2\}.$$

Looking at each coset we see that

$$(x, y) + W = (x, y) + \text{span}\{e_1\} = \{(x, y) + (a, 0) \mid a \in \mathbb{R}\} = \{(t, y) \mid t \in \mathbb{R}\},$$

so $(x, y) + W$ is geometrically a line parallel to the x -axis and having the y -intercept y . It is intuitively natural to identify such a line with its intercept, which gives a map

$$V/W \rightarrow \text{span}\{e_2\} \quad (x, y) + W \mapsto (0, y).$$

It turns out that this map is a vector space isomorphism, hence

$$\dim(V/W) = \dim(\text{span}\{e_2\}) = 1$$

and we can check that

$$\dim(W) + \dim(V/W) = 1 + 1 = 2 = \dim(V).$$

If V and W are both infinite dimensional vector spaces, it can happen that V/W is finite dimensional but also that it is infinite dimensional.

Example 2.23. Let $V = F[x]$, which we saw in Example 2.19 is an infinite dimensional vector space over F . Fix a polynomial f with $\deg(f) = d$, and note that the ideal (f) of $F[x]$ generated by f is also an F -vector subspace of $F[x]$ via restriction of scalars. We will show later that $\dim(F[x]/(f)) = d$. In contrast, the subspace E of all even degree polynomials in $F[x]$ together with the zero polynomial, then $\dim(F[x]/E) = \infty$.

Definition 2.24. Let $T: V \rightarrow W$ be a linear transformation. The **nullspace** of T is $\ker(T)$. The **rank** of T is $\dim(\text{im}(T))$.

Corollary 2.25 (Rank-Nullity Theorem). *Let $f: V \rightarrow W$ be a linear transformation. Then*

$$\dim(\ker(f)) + \dim(\text{im}(f)) = \dim(V).$$

Proof. By the [First Isomorphism Theorem for modules](#) we have $V/\ker(f) \cong \text{im}(f)$, thus

$$\dim(V/\ker(f)) = \dim(\text{im}(f)).$$

By Theorem 2.21, we have

$$\dim(V) = \dim(\ker(f)) + \dim(V/\ker(f)).$$

Thus

$$\dim(V) = \dim(\ker(f)) + \dim(V/\ker(f)) = \dim(\ker(f)) + \dim(\text{im}(f)).$$

□

2.2 Linear transformations and homomorphisms between free modules

Exercise 13. If W is a free R -module with basis $C = \{c_1, \dots, c_m\}$ and $w \in W$, then w can be written *uniquely* as $w = \sum_{j=1}^m a_j c_j$ with $a_1, \dots, a_m \in R$.

Definition 2.26 (The matrix of a homomorphism between free modules). Let R be a commutative ring with $1 \neq 0$. Let V be a finitely generated free R -module of rank n , and let W be a finitely generated free R -module of rank m . Let $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_m\}$ be *ordered* bases of V, W . Given an R -module homomorphism $f : V \rightarrow W$, we define elements $a_{ij} \in R$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ by the formulas

$$f(b_j) = \sum_{i=1}^m a_{ij} c_i. \quad (2.2.1)$$

The matrix

$$[f]_B^C = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

is said to **represent** the homomorphism f with respect to the bases B and C .

Remark 2.27. By Exercise 13, the coefficients $a_{j,i}$ in equation 2.2.1 are uniquely determined by the $f(b_i)$ and the elements of C . The coefficients $a_{j,i}$ corresponding to $f(b_i)$ form the i th column of $[f]_B^C$. Note that $[f]_B^C$ is an $m \times n$ matrix with entries in R .

Definition 2.28. Let V and W be finite F -vector spaces of dimension n and m with ordered bases B and C respectively and let $f : V \rightarrow W$ be a linear transformation. The matrix $[f]_B^C$ is called the **matrix of the linear transformation** f with respect to the bases B and C .

Example 2.29. If $\text{id}_V : V \rightarrow V$ is the identity automorphism of an n -dimensional free R -module V , then for any basis B of V we have $\text{id}_V(b_i) = b_i$ for all i and hence

$$[\text{id}_V]_B^B = I_n.$$

Example 2.30. Let P_3 denote the the F -vector space of polynomials of degree at most 3 (including the zero polynomial) and consider the linear transformation $d : P_3 \rightarrow P_3$ given by taking the derivative $d(f) = f'$. Let $B = \{1, x, x^2, x^3\}$. Then

$$[d]_B^B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2.31. Let F be a field and consider a linear transformation $f : V \rightarrow W$, where $V = F^n$ and $W = F^m$. Consider also the standard ordered bases B and C , i.e. $b_i = e_i \in V$ and $c_i = e_i \in W$. Then for any

$$v = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix} = \sum_i l_i b_i$$

in V we have

$$f\left(\sum_i l_i b_i\right) = \sum_i l_i f(b_i).$$

Each $f(b_i)$ can be written uniquely as a linear combination of the c_j 's as in (2.2.1):

$$f(b_i) = \sum_j a_{j,i} c_j.$$

Then we get

$$f(v) = \sum_i l_i \left(\sum_j a_{j,i} c_j \right) = \sum_j \left(\sum_i a_{j,i} l_i \right) c_j.$$

In other words, we have

$$f(v) = \begin{bmatrix} \sum_i a_{1,i} l_i \\ \vdots \\ \sum_i a_{m,i} l_i \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix} = [f]_B^C \cdot v.$$

Then for any

$$v = \sum_i l_i b_i$$

in V we have

$$f\left(\sum_i l_i b_i\right) = \sum_i l_i f(b_i).$$

Each $f(b_i)$ is uniquely expressible as a linear combination of the c_j 's, say

$$f(b_i) = \sum_j a_{j,i} c_j.$$

Then we get

$$f(v) = \sum_i l_i \left(\sum_j a_{j,i} c_j \right) = \sum_j \left(\sum_i a_{j,i} l_i \right) c_j.$$

In other words, we have

$$f(v) = [f]_B^C \cdot v$$

where

$$[f]_B^C = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

and $[f]_B^C \cdot v$ denote the usual rule for matrix multiplication.

This says that any linear transformation $f : F^n \rightarrow F^m$ is given by multiplication by a matrix, since we noticed above that $f(v) = [f]_B^C \cdot v$. The same type of statement holds for free modules over commutative rings, and we will show it below in Theorem 2.32.

Theorem 2.32. *Let R be a commutative ring with $1 \neq 0$. Let V and W be finitely generated free R -modules of ranks n and m respectively. Fixing ordered bases B for V and C for W gives an isomorphism of R -modules*

$$\text{Hom}_R(V, W) \cong M_{m,n}(R) \quad f \mapsto [f]_B^C.$$

If $V = W$, so that in particular $m = n$, and $B = C$, then the above map is an R -algebra isomorphism $\text{End}_R(V) \cong M_n(R)$.

Proof. Let $\varphi : \text{Hom}_R(V, W) \rightarrow M_{m,n}(R)$ be defined by $\varphi(f) = [f]_B^C$. One needs to check that $[f + g]_B^C = [f]_B^C + [g]_B^C$ and $[\lambda f]_B^C = \lambda [f]_B^C$ for any $f, g \in \text{Hom}_R(V, W)$ and $\lambda \in R$. Let $A = [f]_B^C$ and $A' = [g]_B^C$. Then

$$(f + g)(b_i) = f(b_i) + g(b_i) = \sum_j a_{j,i} c_j + \sum_j a'_{j,i} c_j = \sum_j (a_{j,i} + a'_{j,i}) c_j$$

gives $[f + g]_B^C = A + A'$ and

$$(\lambda f)(b_i) = \lambda \left(\sum_j a_{j,i} c_j \right) = \sum_j (\lambda a_{j,i}) c_j$$

gives $[\lambda f]_B^C = \lambda A$. We leave the proof that for $f, g \in \text{End}_R(V)$ we have $[f \circ g]_B^B = [f]_B^B [g]_B^B$ as an exercise. \square

Remark 2.33. For the case where $V = R^n$, the proof we gave for Theorem 2.32 would be the following:

$$(f \circ g)(v) = f(g(v)) = f([g]_B^B v) = [f]_B^B ([g]_B^B v) = ([f]_B^B [g]_B^B) v,$$

so $[f \circ g]_B^B = [f]_B^B [g]_B^B$.

Corollary 2.34. *For any field F and finite F -vector spaces V, W of dimension n and m respectively, $\dim(\text{Hom}_F(V, W)) = mn$.*

Proof. The isomorphism $\text{Hom}_F(V, W) \cong M_{m,n}(F)$ gives

$$\dim_F(\text{Hom}_F(V, W)) = \dim_F(M_{m,n}(F)) = mn.$$

\square

2.3 Change of basis

Definition 2.35. Let V be a finitely generated free module over a commutative ring R , and let B and B' be bases of V . Let id_V be the identity map on V . Then $[\text{id}_V]_B^{B'}$ is an invertible matrix called the *change of basis* matrix from B to B' .

In the proof of Theorem 2.40 we will show that $[\text{id}_V]_B^{B'}$ is always invertible and its inverse is $([\text{id}_V]_B^{B'})^{-1} = [\text{id}_V]_{B'}^B$.

Example 2.36. Consider $V = P_2$, let $B = \{1, x, x^2\}$ and $B' = \{1, x - 2, (x - 2)^2\}$ be bases of V . We calculate the change of basis matrix. We have

$$\begin{aligned}\text{id}_V(1) &= 1, \\ \text{id}_V(x) &= 2 \cdot 1 + 1 \cdot (x - 2), \\ \text{id}_V(x^2) &= 4 \cdot 1 + 4 \cdot (x - 2) + 1 \cdot (x - 2)^2.\end{aligned}$$

Thus, the change of basis matrix is given by $[\text{id}_V]_B^{B'} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

We now record a general result regarding the matrix representing the composition of two linear transformations, which is a bit more general than that stated at the end of the proof of Proposition 2.32.

Lemma 2.37. If V, W, U are finitely generated free R -modules spaces with ordered bases B, C, D , and if $f: V \rightarrow W$ and $g: W \rightarrow U$ are R -module homomorphisms, then

$$[g \circ f]_D^B = [g]_D^C \cdot [f]_C^B.$$

Definition 2.38. Let V be a finitely generated free module over a commutative ring R . Two R -module homomorphisms $f, g: V \rightarrow V$ are **similar** if there is a linear transformation $h: V \rightarrow V$ such that $g = h \circ f \circ h^{-1}$. Two $n \times n$ matrices A and B with entries in R are **similar** if there is an invertible $n \times n$ matrix P such that $B = PAP^{-1}$.

Remark 2.39. For elements $A, B \in \text{GL}_n(R)$, the notions of similar and conjugate are the same.

Theorem 2.40. Let V, W be finitely generated free modules over a commutative ring R , let B and B' be bases of V , let C and C' be bases of W , and let $f: V \rightarrow W$ be a homomorphism. Then

$$[f]_{B'}^{C'} = [\text{id}_W]_C^{C'} [f]_B^C [\text{id}_V]_{B'}^B \quad (2.3.1)$$

In particular, if $g: V \rightarrow V$ is an R -module homomorphism, then $[g]_B^B$ and $[g]_{B'}^{B'}$ are similar.

Proof. From Lemma 2.37, since $f = \text{id}_W \circ f \circ \text{id}_V$, we have

$$[f]_{B'}^{C'} = [\text{id}_W]_C^{C'} [f]_B^C [\text{id}_V]_{B'}^B.$$

Setting $V = W$, $B = C$, $B' = C'$ and $f = \text{id}_V$ in (2.3.1) we have $[\text{id}_V]_{B'}^{B'} = [\text{id}_V]_B^{B'} [\text{id}_V]_B^B [\text{id}_V]_{B'}^B$. Notice that $[\text{id}_V]_B^B = [\text{id}_V]_{B'}^{B'} = I$ is the identity matrix, so the previous formula gives $I = [\text{id}_V]_B^{B'} I [\text{id}_V]_{B'}^B$. Setting $P = [\text{id}_V]_B^{B'}$, we notice that the previous identity gives $P^{-1} = [\text{id}_V]_{B'}^B$.

Now set $V = W$, $B = C$, $B' = C'$ and $f = g$ in (2.3.1) to obtain

$$[g]_{B'}^{B'} = [\text{id}_V]_B^{B'} [g]_B^B [\text{id}_V]_{B'}^B = P [g]_B^B P^{-1}.$$

□

We now come to special changes of basis and their matrices called elementary matrices:

Definition 2.41. Let R be a commutative ring with $1 \neq 0$, let M be a free R -module of finite rank n , and let $B = \{b_1, \dots, b_n\}$ be an ordered basis for M . An **elementary basis change operation** on the basis B is one of the following three types of operations:

1. Replacing b_i by $b_i + rb_j$ for some $i \neq j$ and some $r \in R$,
2. Replacing b_i by ub_i for some i and some unit u of R ,
3. Swapping the indices of b_i and b_j for some $i \neq j$.

Definition 2.42. Let R be a commutative ring with $1 \neq 0$. An **elementary row operation** on a matrix $A \in M_{m,n}(R)$ is one of the following three types of operations:

1. Adding an element of R times a row of A to a different row of A .
2. Multiplying a row of A by a unit of R .
3. Interchanging two rows of A .

Definition 2.43. Let R be a commutative ring with $1 \neq 0$. An **elementary matrix** over R is an $n \times n$ matrix obtained from I_n by applying a single elementary row operation:

1. For $r \in R$ and $1 \leq i, j \leq n$ with $i \neq j$, let $E_{i,j}(r)$ be the matrix with 1s on the diagonal, r in the (i, j) position, and 0 everywhere else.
2. For $u \in R^\times$ and $1 \leq i \leq n$ let $E_i(u)$ denote the matrix with (i, i) entry u , (j, j) entry 1 for all $j \neq i$, and 0 everywhere else.
3. For $1 \leq i, j \leq n$ with $i \neq j$, let $E_{(i,j)}$ denote the matrix with 1 in the (i, j) and (j, i) positions and in the (l, l) positions for all $l \notin \{i, j\}$, and 0 in all other entries.

Remark 2.44. Let E be an $n \times n$ elementary matrix.

- E is the change of basis matrix $[\text{id}_V]_{B'}^B$, where B is any basis of V and B' is the basis obtained from B by the corresponding elementary basis change operation.
- If $A \in M_{n,q}(R)$, then the product matrix EA is the result of performing the corresponding elementary row operation on A .
- If $B \in M_{m,n}(R)$, then the product matrix BE is the result of performing the corresponding elementary column operation on B .

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