Problem Set 4

Instructions: For full credit, turn in 5 problems in a pdf file and a .m2 file. You are welcome to work together with your classmates on all the problems, and I will be happy to give you hints or discuss the problems with you, but you should write up your solutions by yourself. You cannot use any resources besides me, your classmates, our course notes, and the Macaulay2 documentation.

A topological space X is **disconnected** if there exist disjoint, nonempty closed subsets Y and Z such that $X = Y \cup Z$. A topological space is **connected** if no such X and Y exist.

Let R be a ring. An element $e \in R$ is a **nontrivial idempotent** if $e^2 = e$ and $e \neq 0, 1$.

Problem 1. Let R be a ring.

a) Show that $\operatorname{Spec}(R)$ is connected if and only if R does not contain any nontrivial idempotent.

b) Show that if R is a local ring or a domain, then $\operatorname{Spec}(R)$ is connected.

Problem 2. Let R be a noetherian ring and I and J be ideals in R. Show that the following are equivalent:

(1) $I \subseteq J$

(2) $I_P \subseteq J_P$ for all $P \in \operatorname{Spec}(R)$.

(3) $I_P \subseteq J_P$ for all $P \in \operatorname{mSpec}(R)$.

Problem 3. Use Macaulay2 to help you answer the following questions. Consider the ideal I defining the $\mathbb{Q}$-algebra $R = \mathbb{Q}[t^3, t^{13}, t^{47}]$ as a quotient of a polynomial ring in three variables.

(a) Find the minimal number of generators of I.

(b) Find $\operatorname{Min}(I)$ and $\operatorname{Min}(I^2)$ with Macaulay2, and give a proof that Macaulay2 is correct.

(c) Is $\operatorname{Ass}(I) = \operatorname{Ass}(I^2)$?

(d) There are two more parts commented out about material we have yet to discuss.