

Problem Set 5

Problem 1. Let R be a domain and Q be its fraction field. Let $T(-)$ denote the torsion functor we introduced in Problem Set 3.

- a) Show that $T(M) = \text{Tor}_1^R(M, Q/R)$.¹
- b) Show that for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of R -modules gives rise to an exact sequence²

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow (Q/R) \otimes_R A \longrightarrow (Q/R) \otimes_R B \longrightarrow (Q/R) \otimes_R C \longrightarrow 0.$$

- c) Show that the right derived functors of T are $R^1T = (Q/R) \otimes_R -$ and $R^iT = 0$ for all $i \leq 2$.

Problem 2. Let I be an ideal in R . Show that

$$\text{Ext}_R^n(I, M) \cong \text{Ext}_R^{n+1}(R/I, M)$$

for all $n \geq 1$ and all R -modules M .

Problem 3. Let (R, \mathfrak{m}) be a Noetherian local ring. Let $r \in R$ and M and N be finitely generated R -modules.

- a) Show that the map $\text{Ext}_R^i(M, N) \rightarrow \text{Ext}_R^i(M, N)$ induced by $M \xrightarrow{r} M$ is the map given by multiplication by r .
- b) Show that if r is regular on M and $\text{Ext}_R^i(M/rM, N) = 0$ for $i \gg 0$, then $\text{Ext}_R^i(M, N) = 0$ for $i \gg 0$.

Problem 4. Let (R, \mathfrak{m}) be a Noetherian local ring.

- a) Show that for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R -modules,

$$\text{depth}(A) \geq \min\{\text{depth}(B), \text{depth}(C) + 1\}.$$

- b) Given any finitely generated R -module M , show that there exists $n \geq 1$ such that either $\text{pdim}(M) < n$ or $\text{depth}(\Omega_n M) = \text{depth } R$.

¹Hint: you want to look at some long exact sequence for Tor .

²Hint: apply the Snake Lemma to some nice diagram.

Problem 7. Consider the ring $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$ and the 2-generated R -module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$yf - xg = 0 \quad bf - cg = 0 \quad cf - zg = 0.$$

Let P be the ideal in $S = \mathbb{Q}[x, y, z]$ defining the curve $\{(t^{13}, t^{42}, t^{73}) \mid t \in \mathbb{Q}\}$.

To solve this problem, you are not allowed to use any additional Macaulay2 packages besides the complexes package and the ones that are automatically loaded with Macaulay2.

- a) Find $\text{pdim}_S(S/P)$ and $\text{depth}(S/P)$.
- b) Find $\text{pdim}_R(M)$ and $\text{depth}(M)$.
- c) Is R a regular ring? Is it Cohen-Macaulay?
- d) Is a, b, c a regular sequence on R ?