

Problem Set 5

To solve these problems, you are not allowed to use any additional Macaulay2 packages besides the **Complexes** package and the ones that are automatically loaded with Macaulay2.

Definition (Jacobian ideal). *Let k be a field and $R = k[x_1, \dots, x_n]/I$, where $I = (f_1, \dots, f_r)$ has pure height h . The Jacobian matrix of R is the matrix given by*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}.$$

The jacobian ideal of R is the ideal generated by the h -minors of the Jacobian matrix.

Turns out the Jacobian ideal is indeed well-defined – meaning, our definition does not depend on the choice of presentation for R – and that it determines the nonsmooth locus of R . For proofs of these well-known facts, see Section 16.6 of Eisenbud’s *Commutative Algebra with a view towards algebraic geometry*.

Theorem (Jacobian criterion). *Let $R = k[x_1, \dots, x_n]/I$ with k a perfect field, and assume that I has pure height h . The Jacobian ideal J defines the nonsmooth locus of R : a prime P in $k[x_1, \dots, x_n]$ contains J if and only if R_P is not a regular ring.*

Problem 1. Let (R, \mathfrak{m}) be a regular local ring and I be an ideal in R .

a) Show that there is a short exact sequence

$$0 \longrightarrow \frac{I + \mathfrak{m}^2}{\mathfrak{m}^2} \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2 + I} \longrightarrow 0$$

and conclude that $\dim_k \left(\frac{I + \mathfrak{m}^2}{\mathfrak{m}^2} \right) = \text{embdim}(R) - \text{embdim}(R/I)$.

b) Show that if R/I is regular, then there exists a minimal set of generators x_1, \dots, x_d for \mathfrak{m} such that $I = (x_1, \dots, x_n)$ for some n .

c) Conclude that if I is not generated by a regular sequence, then R/I is not regular.

Problem 2. Let I be a radical ideal in $R = k[x_1, \dots, x_n]$, where k is a perfect field, and let J be the Jacobian ideal of I . Prove that $I^{(n)} = (I^n : J^\infty)$ for all $n \geq 1$.

Problem 3. Let k be a field, $R = k[x, y, z]$, and $I = (xy, xz, yz)$. Find the Jacobian ideal J of I , and show directly that $I^{(n)} = (I^n : J^\infty)$ for all $n \geq 1$ without using Problem 2.

Problem 4. For each of the following ideals I , find an **element** t such that $I^{(n)} = (I^n : t^\infty)$ for all $n \geq 1$.

a) $I = (xz, xw, yz, yw)$ in $R = k[x, y, z, w]$, where k is any field.

b) I is the defining ideal of the second Veronese in 3 variables $\mathbb{Q}[x, y, z]^{(2)}$.

Problem 5. Let k be a field, $R = k[x, y, z]$, and $I = (xy, xz, yz)$.

- a) Show that $I^{(2n)} = (I^{(2)})^n$ for all $n \geq 1$.
- b) Show that $I^{(2n+1)} = I (I^{(2)})^n$ for all $n \geq 1$.
- c) Show that the symbolic Rees algebra of I is a noetherian ring.

Problem 6. Let $R \rightarrow S$ be a flat ring homomorphism.

- a) Show that for every ideal I in R and every $x \in R$, $(I :_R x)S = (IS :_S x)$.

Hint: what is the kernel of $R/I \xrightarrow{x} R/I$?

- b) Show that I and J are ideals in R , then $IS \cap JS = (I \cap J)S$.

Hint: what is the kernel of the canonical map $R \rightarrow R/I \oplus R/J$?

- c) Show that if I and J are ideals in R with J finitely generated, then $(I :_R J)S = (IS :_S JS)$.

Problem 7. Let I be a squarefree monomial ideal in $R = k[x_1, \dots, x_d]$, where k is a perfect field of prime characteristic p .

- a) Show that R/I is F -pure.
- b) Show that R/I is strongly F -regular if and only if I is generated by variables.