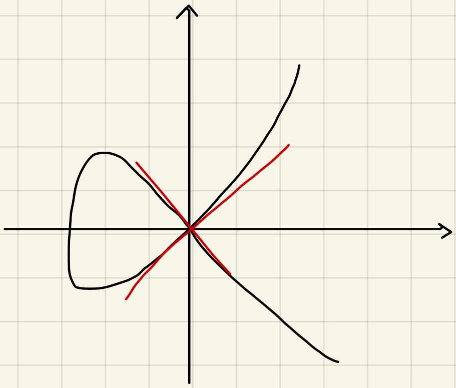
Rees algebras and free resolutions

02/05/21

Curve singularities and blowup



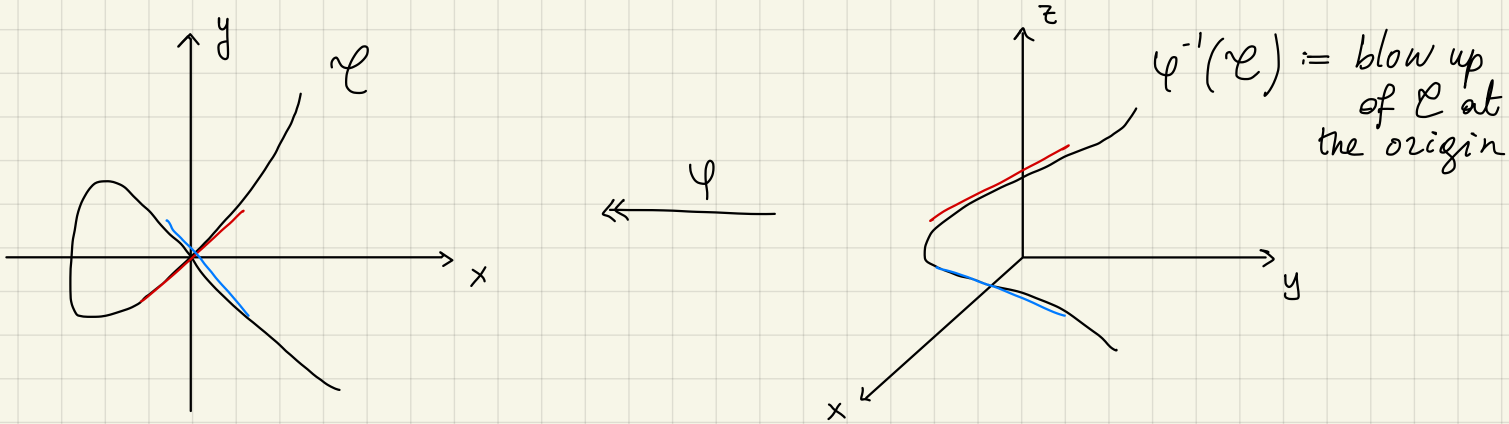
$\mathcal{C} = \{(x,y) \in \mathbb{A}^2 \mid y^2 - x^2(x+1) = 0\}$

$R = A(\mathcal{C}) = k[X,Y] / (Y^2 - X^2(X+1))$

Coord. ring of $\varphi^{-1}(\mathcal{C})$ is

$R[xt, yt] \subseteq R[t]$

blow up



$I = (x,y)$ ideal in R

$R[xt, yt] = R[It] \subseteq R[t]$

Rees algebra of I

More generally:

R Noetherian ring, $I = (f_1, \dots, f_n)$

$$\mathcal{R}(I) = \bigoplus_{j \geq 0} I^j \cong \bigoplus_{j \geq 0} I^j t^j = R[It] = R[f_1 t, \dots, f_n t] \subseteq R[t]$$

$I^0 = R \qquad I^j = \{ j\text{-fold products of } f_1, \dots, f_n \}$

Many numerical invariants of the ring R/I are described in terms of powers of I

$$\pi : R[T_1, \dots, T_n] \twoheadrightarrow R[It] \xrightarrow{\sim} \mathcal{R}(I)$$

$$T_i \longmapsto f_i t \longmapsto f_i$$

polynomial relations among the generators of I

$$J = \ker \pi = \{ F(T_1, \dots, T_n) \in R[T_1, \dots, T_n] : F(f_1, \dots, f_n) = 0 \}$$

defining ideal of $\mathcal{R}(I)$ or Rees ideal of I

$$J_1 = \{ a_1 T_1 + \dots + a_n T_n \mid a_i \in R, \ a_1 f_1 + \dots + a_n f_n = 0 \} = \mathcal{L}$$

elements in degree 1

R-relation among the generators of I

When $J = J_1 = \mathcal{L}$ (linear relations)

we say that I is an ideal of linear type

$\longrightarrow \mathcal{R}(I) = R[T_1 \dots T_n]/\mathcal{L} \rightarrow$ ideal of linear form

This is always true when I is a complete intersection (gen. by a regular sequence)

What if I is not of linear type?

$R = k[X_1, \dots, X_d]$, $I = (f_1, \dots, f_n)$

$R^s \xrightarrow{A} R^n \twoheadrightarrow I$

$\{e_i\} \mapsto \{f_i\}$

$A = n \times s$ matrix whose entries are <u>linear polynomials</u> in R

$[T_1, \dots, T_n]\, A = [\ell_1, \dots, \ell_s] = B\, [X_1 \dots X_d]$

ℓ_i: linear forms in the Ts with linear coeff in Xs

B → Jacobian dual of A

Always: $\mathcal{J} \supseteq \mathcal{L} + I_d(B)$

<u>Ex</u>: $R = k[X_1, X_2, X_3]$

$0 \to R^3 \xrightarrow{A} R^4 \to I \to 0$ where

$$A = \begin{bmatrix} X_1 & 0 & X_3 \\ X_2 & X_1 & 0 \\ X_3 & X_2 & X_1 \\ 0 & X_3 & X_2 \end{bmatrix}$$

linear in the x

$$[T_1, T_2, T_3, T_4]\, A = \begin{bmatrix} T_1 X_1 + T_2 X_2 + T_3 X_3 \\ T_2 X_1 + T_3 X_2 + T_4 X_3 \\ T_1 X_3 + T_3 X_1 + T_4 X_2 \end{bmatrix}^t$$

$$= \underbrace{\begin{bmatrix} T_1 & T_2 & T_3 \\ T_2 & T_3 & T_4 \\ T_3 & T_4 & T_1 \end{bmatrix}}_{B} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

linear in the Ts

In this case

$\mathcal{J} = \mathcal{L} + I_d(B)$

Thm [Morey-Ulrich, 1996]: $R = k[x_1, \dots, x_d]$, k an infinite field

$I = (f_1, \dots, f_n)$ R-ideal of codimension 2 with a linear presentation

$$0 \to R^{n-1} \xrightarrow{A} R^n \to I \to 0.$$

Suppose that for all $i \leq d-1$ $\operatorname{ht} I_{n-i}(A) \geq i+1$.

Then, $\mathcal{J} = \mathcal{L} + I_d(B)$.

$\longrightarrow$ Jacobian dual (Vasconcelos, 1981)

Relax assumptions:

possibly >1

- [Boswell-Mukundan, 2016]: A linear except one column of degree m, $\mu(I) = d+1$

min. # of gen.

Then $\mathcal{J} = \mathcal{L} + I_d(B_m)$

$\longrightarrow$ m^{th} iterated Jacobian dual

$\mu(I) = d+1$ is needed in order for $I^2 = I^{(2)}$

$m = 1 \longrightarrow B_m = B$

induction procedure to construct matrices B_i

- [Weaver, 2021]: $R = k[x_1, \dots, x_{d+1}]/(f)$, f irreducible homogeneous of degree m,

$\mu(I) = d+1$, A linear

$$\mathcal{J} = \overline{\mathcal{L}} + \overline{I_{d+1}(B_m)}$$

"modified" iterated Jac. dual.

lift the presentation matrix A to a matrix with coeff. in $k[x_1, \dots, x_{d+1}]$

What if $\operatorname{ht} I > 2$?

Def: $R = k[x_1, \dots, x_d]$, $I = (f_1, \dots, f_n)$.

Main idea: presentation is not enough

$$0 \to R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \xrightarrow{d_0} I \to 0$$

free resolution of I

find gen. of $\ker(d_0)$

Find gen of $\ker(d_1)$

- [Kustin-Polini-Ulrich, 2017] $R = k[x_1, \dots, x_d]$, k an infinite field,

  $I = (f_1, \dots, f_n)$ a licci ideal with a linear resolution

  all the maps are linear

  Suppose that for all $i \le d-1$ $\operatorname{ht} I_{n-i}(A) \geq i+1$.

  Then, $J = \sqrt{\mathcal{L} + I_d(B)}$.

Remember: $\sqrt{J} = \{ r \in R \mid r^m \in J \text{ for some } m \}$

The role of the special fiber ring

<u>Motivation:</u> $X \subseteq \mathbb{P}^n_k$, $k = \bar{k}$, $R = A(X)$

rational map : $X \overset{\Phi = [f_1 : \cdots : f_n]}{- - - - - \rightarrow} \mathbb{P}^{n-1}_k$, $\deg f_i = \delta \ \forall i$

$$\mathrm{graph}(\Phi) = \mathrm{BiProj}(\mathcal{R}(I)) \qquad I = (f_1, \dots, f_n)$$

$$\mathrm{im}(\Phi) = \mathrm{Proj}(\mathcal{F}(I))$$

$\longrightarrow$ fiber cone = special fiber of blow up over the unique closed point.

$$\mathcal{F}(I) \cong k[T_1, \dots, T_n] / I(X)$$

and $\quad I(X)\, R[T_1, \dots, T_n] + \mathcal{L} \subseteq \mathcal{J}$

An ideal I is of fiber type when equality holds

Def: $R = K[x_1, \dots, x_d]$, I an ideal minimally generated by monomials of the same degree.

- I is called polymatroidal if, given two minimal generators $x_1^{a_1} \cdots x_d^{a_d}$ and $x_1^{b_1} \cdots x_d^{b_d}$ with $a_i > b_i$ for some i, then there exists a j so that $a_j < b_j$ and $\left(\frac{x_j}{x_i}\right)\left(x_1^{a_1} \cdots x_d^{a_d}\right)$ is a minimal generator of I

- I is said to have the strong exchange property if, given two minimal generators $x_1^{a_1} \cdots x_d^{a_d}$ and $x_1^{b_1} \cdots x_d^{b_d}$ with $a_i > b_i$ & $a_j < b_j$ for some i and j, then $\left(\frac{x_j}{x_i}\right)\left(x_1^{a_1} \cdots x_d^{a_d}\right)$ is a minimal generator of I.

- [Herzog-Takayama, 2002]: Polymatroidal ideals have linear resolutions
- [Herzog-Hibi-Vladiou, 2005]: Polymatroidal ideals are of fiber type
- [Nicklasson, 2019]: If I has the strong exchange property, the non-linear relations for the Rees ideal are of $\underline{I}$-degree 2

Examples

1. Ideal gen. by all squarefree monomials of a given degree
2. Edge ideals $\longleftrightarrow$ squarefree monomial ideals of deg. 2.