A story of Algebra vs Geometry

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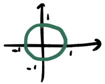
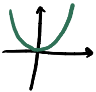
$$\text{Algebra} \longleftrightarrow \text{Geometry}$$

$$xy = 0$$

$$x^2 + y^2 - 1 = 0$$

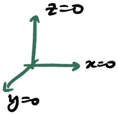


$$y - x^2 = 0$$



Algebra $\longleftrightarrow$ Geometry

$$\begin{cases} xy = 0 \\ xz = 0 \\ yz = 0 \end{cases}$$



$$\begin{cases} x_1 - a_1 = 0 \\ \quad\vdots \\ x_d - a_d = 0 \end{cases}$$

point $(a_1, \ldots, a_d)$
in d-dimensional space

systems of polynomial
equations in d variables

nice subsets of
d-dimensional space

Algebra	$\longleftrightarrow$	Geometry
systems of polynomial equations in d variables	$\longleftrightarrow$	nice subsets of d-dimensional space

Definition (Variety)

A *variety* is a subset $V \subseteq \mathbb{C}^d$ that consists of precisely all the common zeroes of a system of polynomial equations.

Theorem (Hilbert's Basis Theorem)

Every system of polynomial equations in d variables with coefficients in $\mathbb{R}$, $\mathbb{C}$, or more generally any field can be described by a finite number of equations.

Algebra $\longleftrightarrow$ Geometry

$f_1, f_2, \dots$ polynomials in d variables

$$\begin{cases} f_1 = 0 \\ f_2 = 0 \\ \quad\vdots \end{cases} \longmapsto$$

variety V in d-space all common zeros of all f_i

all polynomials f with $f(v) = 0$ for all $v \in V$

$$\Big\downarrow \text{Hilbert gives finitely many } f$$

$I = (f_1, \dots, f_n) := \{g_1 f_1 + \cdots + g_n f_n : g_i \text{ polynomial}\}$ <u>ideal</u>

Ideals

An **ideal** I of the **ring** of polynomials in d variables, $R = \mathbb{C}[x_1, \ldots, x_d]$, is a nice set of polynomials with good algebraic properties.

- $0 \in I$
- $a, b \in I \Rightarrow a + b \in I$
- $a \in I$ and $r \in R \Rightarrow ra \in I$

Algebra $\longleftrightarrow$ Geometry

algebra of ideals $\longleftrightarrow$ geometry of varieties

algebra of $R = \mathbb{C}[x_1, \ldots, x_d]$ $\longleftrightarrow$ geometry of $\mathbb{C}^d$

radical ideals
$(f^n \in I \Rightarrow f \in I)$ $\xleftrightarrow{1:1}$ varieties

Hilbert's Nullstellensatz

Algebra		Geometry
$(0) = \{0\}$	$\longleftrightarrow$	variety $\mathbb{C}^d$
$\mathbb{C}[x_1, \ldots, x_d]$	$\longleftrightarrow$	variety $\varnothing$
$(x_1 - a_1, \ldots, x_d - a_d)$	$\longleftrightarrow$	point $\{(a_1, \ldots, a_d)\}$
smaller ideals	$\longleftrightarrow$	larger varieties
prime ideals $(fg \in I \Rightarrow f \in I \text{ or } g \in I)$	$\longleftrightarrow$	irreducible varieties (not the union of smaller varieties)

variety V $\longleftrightarrow$ ideal of all polynomials f that vanish at every point $v \in V$

Question

How do we measure vanishing?

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Example

The polynomial $f = x^3$ vanishes *more* at the point 0 than the polynomial $g = x$.

Definition (Algebraic Powers)

For an ideal I, its nth power I^n is the ideal

$$I^n = (f_1 \cdots f_n : f_i \in I).$$

Example

In $\mathbb{C}[x, y]$, $(x, y)^2 = (x^2, xy, y^2)$.

Symbolic Powers

For an ideal I, its *nth symbolic power* $I^{(n)}$ can be defined via *primary decomposition*. Roughly speaking, primary decomposition is an ideal version of the fundamental theorem of algebra, which says (for $\mathbb{Z}$) that we can write things as products of primes.

Algebraic Powers

The algebraic powers I^n are very easy to describe algebraically, but have no clear geometric meaning.

Symbolic Powers

The symbolic powers $I^{(n)}$ are very hard to describe algebraically, even with a computer, but have a very important geometric meaning.

Theorem (Zariski–Nagata)

I ideal $\longleftrightarrow$ variety V

$$I^{(n)} = \{f \in I : f \text{ vanishes to order } n \text{ at every } v \in V\}$$

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3. If I is generated by a subset of the variables, then $I^n = I^{(n)}$ for all n.
4. In general, $I^n \neq I^{(n)}$.

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4. When is $I^{(a)} \subseteq I^b$? Compare symbolic and algebraic powers.

Thank you!

Symbolic Power

For a prime ideal P in $R = \mathbb{C}[x_1, \ldots, x_d]$, the n-th **symbolic power** of P is

$$P^{(n)} \quad = \{f \in R : sf \in P^n \text{ for some } s \notin P\}$$