Problem Set 2

Instructions: You are welcome to work together with your classmates on all the problems, and I will be happy to give you hints or discuss the problems with you, but you should write up your solutions by yourself. You cannot use any resources besides me, your classmates, our course notes, and the Macaulay2 documentation.

Let R be a ring and M an R-module. The **Nagata idealization** of (R, M) is the ring $R \rtimes M$ defined as follows:

- as a set, $R \rtimes M = R \times M$;
- the addition is $(r, m) + (s, n) = (r + s, m + n)$;
- the multiplication is $(r, m)(s, n) = (rs, sm + rn)$.

Then $R \rtimes M$ with the operations specified about is a ring.

Note that R is a subring of $R \rtimes M$ (via the inclusion $r \mapsto (r, 0)$), and as an R-module, $R \rtimes M \cong R \oplus M$.

Problem 1. In this problem, we will construct an extension of rings $A \subseteq B \subseteq C$ such that $A \subseteq C$ is module-finite, but $A \subseteq B$ is not.

a) Can you find such an extension with A Noetherian?

b) Let R be a ring that is not Noetherian, and I an ideal that is not finitely generated. Show that $R \subseteq R \rtimes I \subseteq R \rtimes R$, that $R \subseteq R \rtimes R$ is module-finite, but $R \subseteq R \rtimes I$ is not.

Problem 2. Suppose that every ascending chain of prime ideals in R stabilizes. Must R be a noetherian ring?

Problem 3. Show that R is a Noetherian ring if and only if every prime ideal of R is finitely generated. Spoiler alert: there's a hint in the footnotes.[1]

[1] Hint: use Zorn's Lemma to show that if R is a ring that is not noetherian, the set of ideals that are not finitely generated has a maximal element; show that element is a prime ideal.