Symbolic powers and the Containment Problem

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Background

Symbolic Power

The n-th **symbolic power** of a radical ideal I in a regular ring R is

$$I^{(n)} \quad = \bigcap_{P \in \mathrm{Min}(R/I)} (I^n R_P \cap R).$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

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(4) In general, $I^n \neq I^{(n)}$.

Containment Problem (Schenzel)

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Does the question make sense?

For every a there exists a b such that $I^{(b)} \subseteq I^a$ if and only if the I-adic and I-symbolic topologies are equivalent.

Theorem (Swanson, 2000)

Let I be a radical ideal in a noetherian local ring. If the I-adic and I-symbolic topologies are equivalent, there exists a constant k such that $I^{(kn)} \subseteq I^n$ for all n.

Theorem (Huneke-Katz-Validashti, 2009)

Let R be a complete local domain, and I a radical ideal in R. The I-adic and I-symbolic topologies are equivalent.

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal in a regular ring R and h be the maximal height of a minimal prime of I. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

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Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

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In fact, $P^{(3)} \subseteq P^2$.

Question (Huneke, 2000)

Let P be a height 2 prime in a regular ring. Is $P^{(3)} \subseteq P^2$?

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Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Theorem (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]}.$$

Notation: $I^{[q]} = (f^q \mid f \in I)$.

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Harbourne's Conjecture

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Dumnicki, Szemberg, Tutaj-Gasińska, 2015

There exists a radical ideal in $\mathbb{C}[x, y, z]$ such that $I^{(3)} \not\subseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3)\right).$$

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When does Harbourne's Conjecture hold?

- For squarefree monomial ideals.
- For general points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki).
- For star configurations (Harbourne–Huneke).

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Let R be a regular ring of characteristic p, I a radical ideal in R and h the maximal height of a minimal prime of I.

Theorem (G–Huneke)

If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

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Theorem (G–Huneke)

If R/I is strongly F-regular, then $I^{((h-1)n-(h-1)+1)} \subseteq I^n$ for all n.

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Corollary

If R/I is strongly F-regular and $h = 2$, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

Does the conjecture hold eventually?

Evidence for the Stable Harbourne Conjecture

Let $a \geqslant 3$, k be a field, $R = k[x, y, z]$, and

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) \, .$$

This is a well-known counterexample to $I^{(3)} \subseteq I^2$. However, by work of Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, and Tutaj-Gasińska, we have

$$I^{(2n-1)} \subseteq I^n$$

for all $n \geqslant 3$.

Stable Harbourne Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \gg 0$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Question

If there exists a value of n such that

$$I^{(hn-h+1)} \subseteq I^n,$$

does that imply that

$$I^{(hm-h+1)} \subseteq I^m$$

for all $m \gg 0$?

Theorem (–)

Let I be a radical ideal in a regular ring containing a field, and let h be the maximal height of a minimal prime of I. If there exists a value of n such that

$$I^{(hn-h)} \subseteq I^n,$$

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Let I be a radical ideal in a regular ring containing a field, and let h be the maximal height of a minimal prime of I. If there exists a value of n such that

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Example

The defining ideal of the space monomial curve $k[t^3, t^4, t^5]$ in $k[x, y, z]$ verifies $P^{(2\times 3-2=4)} \subseteq P^3$, so $P^{(2m-2)} \subseteq P^m$ for all $m \gg 0$.

Huneke's question and prime ideals

Huneke's Question

If P is a prime of height 2 in a regular ring, is $P^{(3)} \subseteq P^2$?

Theorem (–)

Let k be a field of characteristic not 3, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2.$$

Monomial space curves

Let k be a field. The kernel of the map

$$k[x, y, z] \longrightarrow k[t^a, t^b, t^c] \subseteq k[t]$$

is a prime ideal of height 2, generated by the maximal minors of

$$\begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}.$$

Theorem (G–Huneke–Mukundan)

Let k be a field of characteristic not 3, and $I \subseteq k[x, y, z]$ be the height 2 ideal generated by the maximal minors of

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

If $I_1(M)$ is generated by 5 or less elements, then $I^{(3)} \subseteq I^2$.

Fermat configurations

Let $a \geqslant 3$, k be a field, $R = k[x, y, z]$, and

$$\begin{aligned} I &= (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) \\ &= I_2 \begin{pmatrix} x^{a-1} & y^{a-1} & z^{a-1} \\ yz & xz & xy \end{pmatrix} \end{aligned}$$

Fun fact: if we switch the order of the entries, we get an ideal I with $I^{(3)} \subseteq I^2$.

Theorem (–)

Let k be a field of characteristic not 2, 3 or 5, let a, b and c be integers, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(3)} \subseteq P^2 \text{ and } P^{(5)} \subseteq P^3.$$

Ingredients in all these proofs (by Alexandra Seceleanu)

- $I^{(a)} \subseteq I^b$ if and only if the map on Ext induced by $I^a \subseteq I^b$, $\mathrm{Ext}^2\left(I^b, R\right) \longrightarrow \mathrm{Ext}^2\left(I^a, R\right)$, is the 0 map.
- Use Rees Algebra techniques to find the resolutions of I^n.

Theorem (–)

Let k be a field of characteristic not 2 nor 3, $a \leqslant b \leqslant c$ integers, $a = 3$ or 4, and let P be the defining ideal of $k[t^a, t^b, t^c]$. Then

$$P^{(4)} \subseteq P^3.$$

As a consequence, $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Example

The defining ideal P of $k[t^9, t^{11}, t^{14}]$ has $P^{(4)} \not\subseteq P^3$, but according to Macaulay2 computations,

$$P^{(2\times 4-2=6)} \subseteq P^4,$$

so $P^{(2n-2)} \subseteq P^n$ for all $n \gg 0$.

Obrigada!