Symbolic powers of ideals defining F-pure rings
joint work with Craig Huneke
Utah Commutative Algebra Seminar
April 21, 2017

Setup R RLR, containing a field

I radical ideal

$h =$ max height of an associated prime of I

n-th symbolic powers

P prime: $P^{(n)} = P^n R_P \cap R = \{f \in R : fs \in P^n, \text{ for some } s \notin P\}$

$=$ P-primary component of P^n $\rightsquigarrow$ $P^n = P^{(n)} \cap$ primary components associated to embedded primes

I radical: $\quad I^{(n)} = \bigcap_{P \in \mathrm{Ass}(R/I)} (I^n R_P \cap R)$

Note: $I^n \subseteq I^{(n)}$. In general, $I^n \neq I^{(n)}$.

If I is generated by a regular sequence, $I^n = I^{(n)}$.

Example 1: $I = (x,y) \cap (x,z) \cap (y,z) = (xy, xz, yz) \subseteq k[x,y,z]$.

$$I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \ni xyz$$

$xyz \notin I^2$, since $\alpha(I^2) \geq 4 \Rightarrow I^2 \neq I^{(2)}$. However, $I^{(3)} \subseteq I^2$.

Example 2: $P = \ker(k[x,y,z] \longrightarrow k[t^3, t^4, t^5])$ prime of height 2.

$$P^{(2)} \neq P^2. \quad \text{However, } P^{(3)} \subseteq P^2.$$

Containment Problem: When is $I^{(a)} \subseteq I^b$?

Theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002)

$$I^{(hn)} \subseteq I^n.$$

this does not necessarily provide a sharp answer to our question.

Example 1: $I = (x,y) \cap (x,z) \cap (y,z) \rightsquigarrow h = 2.$

theorem says $I^{(2n)} \subseteq I^n \Rightarrow I^{(4)} \subseteq I^2$. But $I^{(3)} \subseteq I^2$.

Question (Huneke, 2000) P prime of ht $P = 2$. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, ≤ 2006) $I^{(hn-h+1)} \subseteq I^n$

Note: When $h=2$, the Conjecture says $I^{(2n-1)} \subseteq I^n$

Facts: the conjecture holds for:

- Monomial ideals in any characteristic
- General points in $\mathbb{P}^2$ (Hattu) and $\mathbb{P}^3$ (Dumnicki)
- R/I F-pure, $h = 2$ (Hochster-Huneke)

Counterexample (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$$I = (x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3)) \subseteq \mathbb{C}[x,y,z]$$

$h = 2$, but $I^{(3)} \not\subseteq I^2$.

From now on: R of characteristic $p>0$

$S = R/I$ is F-pure if $M \otimes R \xrightarrow{1\otimes F} M \otimes R$ is injective $\forall M$ fg R-module.

or

$S = R/I$ F-finite, reduced is F-pure if the Frobenius map splits.

Examples of F-pure rings:

1) I squarefree monomial ideal $\Rightarrow R/I$ F-pure

2) Determinantal rings are F-pure:

X generic matrix: the k-algebra generated by the $t\times t$ minors of X is F-pure and $K[X]/I_t(X)$ is F-pure.

3) Veronese rings are F-pure.

4) "Nice" rings of invariants"

<u>Fedder's</u> <u>Criterion</u> (83) (R, m) RLR, char $p>0$, $I \subseteq R$ ideal.

R/I F-pure $\Longleftrightarrow$ $I^{[q]} : I \not\subseteq m^{[q]}$ $\quad \forall q = p^e \gg 0$

theorem (–, Huneke) R/I F-pure $\Rightarrow \forall n \geq 1$ $I^{(hn-h+1)} \subseteq I^n$.

<u>Idea of proof</u>: show that $(I^{[q]} : I) \subseteq (I^n : I^{(hn-h+1)})^{[q]}$, $q \gg 0$.

this uses several technical lemmas by Hochster-Huneke.

Example $I = \bigcap_{i \neq j} (x_i, x_j) = (x_1 \cdots \hat{x}_i \cdots x_d : 1 \leq i \leq d) \subseteq k[x_1, \dots, x_d]$

$h = 2 \Rightarrow I^{(2n-1)} \subseteq I^n$. Is $I^{(2n-2)} \subseteq I^n$?

$I^{(2n-2)} = \bigcap_{i \neq j} (x_i, x_j)^{2n-2} \ni (x_1 \cdots x_d)^{n-1}$

$\deg (x_1 \cdots x_d)^{n-1} = d(n-1) = dn - d < dn - n = n(d-1) \leq \deg$ element in I^n

$\therefore$ Our result is sharp.

Example $R = k[a, b, c, d] \longrightarrow k[s^3, s^2 t, st^2, t^3]$

P defining ideal: height 2, 3 generated prime.

$P^{(n)} = P^n \ \forall n$ (Huckaba - Huneke)

Example (Singh) $I = I_2\left(\begin{bmatrix} a^2 & b & d \\ c & a^2 & b^k - d \end{bmatrix}\right) \subseteq k[a, b, c, d]$

$I^n = I^{(n)}$ for $n = 2, 3, 4, \dots$

Can we specialize R/I more and get tighter containments?

R of char $p > 0$, reduced (no nilpotents), F-finite

R is strongly F-regular if for c not in a minimal prime of R

$\exists\, \phi : R^{1/q} \longrightarrow R \quad \phi(c^{1/q}) = 1$ for all/some/large enough $q = p^e$.

Glassbrenner's Criterion (96)

(R,m) RLR, char $p>0$, I radical ideal.

R/I strongly F-regular $\Leftrightarrow$ $c(I^{[q]}:I) \notin m^{[q]}$ $\forall q \gg 0$ $\forall c \notin$ min prime I

Examples: All of the above, except monomial ideals.

theorem (–, Huneke) R/I strongly F-regular $\Rightarrow I^{(hn-n+1)} \subseteq I^{n+1}$

or $I^{((h-1)n+1)} \subseteq I^{n+1}$ for all n.

Proof of theorem (sketch)

Claim $(I^d : I^{(d)})(I^{[q]} : I) \subseteq (I\, I^{(d-h+1)} : I^{(d)})^{[q]}$ $\forall q$. $\forall d \geq h-1$.

$(I^d : I^{(d)})$ always contains an element not in a minimal prime of I

Corollary P prime of height 2, R/P strongly F-regular

then $I^{(n)} = I^n$ $\forall n \geq 1$