Problem Set 1

Problem 0. Install Macaulay2.[1]

Problem 1. Let $R = \mathbb{Q}[x,y,z]/(x^2, xy)$. Consider the bounded complex

$$C = \underset{3}{R} \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} \underset{2}{R^3} \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} \underset{1}{R^3} \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} \underset{0}{R}$$

Set C up in Macaulay2 and compute its homology. For which n is $\mathrm{H}_n(C) = 0$?

Problem 2. Let $C_n = \mathbb{Z}/8$ for all $n \geqslant 0$, and $C_n = 0$ for $n < 0$. Let

$$\begin{aligned} C_n &\xrightarrow{d_n} C_{n-1} \\ x &\longrightarrow 4x \end{aligned}$$

when $n > 0$, and otherwise let $d_n \colon C_n \longrightarrow C_{n-1}$ be the zero map.

a) Show that $(C_\bullet, d_\bullet)$ is a complex.

b) Compute its homology.

Problem 3 (The Five Lemma)**.** Consider the following commutative diagram of R-modules with exact rows:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "a"] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\ A \arrow[r] & B \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

Show that if a, b, d, and e are isomorphisms, then c is an isomorphism.

Problem 4. Let π denote the projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. Show that the chain map

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "0"] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism, but not a homotopy equivalence.

[1] You can only choose this problem if you did not do the corresponding problem in Commutative Algebra in the Winter.

Problem 5.

a) Show that in any category, every isomorphism is both an epi and a mono.

b) Show that the usual inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epi in the category **Ring**. This *should* feel weird: it says being epi and being surjective are *not* the same thing.

c) Show that the canonical projection $\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a mono in the category of divisible abelian groups.[2] Again, this is very strange: it says being monic and being injective are *not* the same thing.

Problem 6. Consider the category R**-mod** of R-modules and R-module homomorphisms. Show that the monomorphisms in R**-mod** are precisely the injective homomorphisms of R-modules, and that the epimorphisms in R**-mod** are precisely the surjective homomorphisms of R-modules.

Problem 7. Show that the bijection in the proof of the Yoneda Lemma is natural on both the object and the functor.

Problem 8.

a) The forgetful functor **Grp** $\longrightarrow$ **Set** is representable.

b) Given a ring R, the forgetful functor R**-mod** $\longrightarrow$ **Set** is representable.

[2] An abelian group A is divisible if for every $a \in A$ and every positive integer n there exists $b \in A$ such that $nb = a$.