

Problem Set 5

Problem 1. Let $R = \mathbb{Z}[\sqrt{-5}]$. While $6 \in R$ cannot be written as a unique product of irreducibles, we are going to show that the ideal $I = (6)$ does have a unique primary decomposition. Unfortunately, Macaulay2 cannot take primary decompositions over \mathbb{Z} , but this one we can do the old fashioned way.

- a) Prove that (2) is a primary ideal.
- b) Prove that (3) is *not* a primary ideal.
- c) Prove that $(3, 1 + \sqrt{-5})$ and $(3, 1 - \sqrt{-5})$ are both primary.
- d) Show that $(6) = (2) \cap (3, 1 + \sqrt{-5}) \cap (3, 1 - \sqrt{-5})$.
- e) Why is this primary decomposition unique?

Problem 2. Let R be a Noetherian ring. Let I be an ideal in R , and $x \in R$. The *saturation* of I with respect to x is the ideal

$$(I : x^\infty) := \bigcup_{n=1}^{\infty} (I : x^n).$$

- a) Let Q be a primary ideal. Show that

$$(Q : x^\infty) = \begin{cases} Q & \text{if } x \notin P \\ R & \text{if } x \in P \end{cases}.$$

- b) Show that $(I : x^\infty) = (I : x^n)$ for some n .
- c) Show that $(I \cap J : x^\infty) = (I : x^\infty) \cap (J : x^\infty)$ for any ideals I and J .
- d) Let $I = Q_1 \cap \cdots \cap Q_k$ be a primary decomposition, and $x \in R$. Show that

$$(I : x^\infty) = \bigcap_{x \notin P_i} Q_i.$$

Problem 3. Let I be a radical ideal in a Noetherian ring R . A beautiful theorem of Brodmann says that the set

$$\bigcup_{n \geq 1} \text{Ass}(R/I^n)$$

is finite. Show that there exists an element x such that:

- x is contained in every embedded prime of I^n for every n , and
- $x \notin P$ for all $P \in \text{Min}(I)$.

Conclude that there exists $x \in R$ such that $I^{(n)} = (I^n : x^\infty)$ for all $n \geq 1$.

Problem 4. Consider s points $P_1 = (a_{11}, \dots, a_{1d}), \dots, P_s = (a_{s1}, \dots, a_{sd})$ in \mathbb{A}^d , and let I be the corresponding radical ideal in $\mathbb{C}[x_1, \dots, x_d]$. Show that for all $n \geq 1$,

$$I^{(n)} = \bigcap_{i=1}^s (x_1 - a_{i1}, \dots, x_d - a_{id})^n.$$

Problem 5. Let R be any ring.

- a) Show that if \mathfrak{m} is any maximal ideal in R , then \mathfrak{m}^n is \mathfrak{m} -primary for any $n \geq 1$.
- b) If R is a domain, then

$$\bigcap_{n \geq 1} I^n = 0.$$

for any proper ideal I in R .¹

Problem 6. If (R, \mathfrak{m}) is a Noetherian local ring, show that M has finite length if and only if M is finitely generated and $\mathfrak{m}^n M = 0$ for some n .

Problem 7. Let k be a field, and $R = k[a, b, c, d]/(ad - bc)$. Find prime ideals P and Q in R such that $\text{ht}(P) + \text{ht}(Q) < \text{ht}(P + Q)$.

Problem 8.

- a) Find the height of $J = (ab, bc, cd, ad)$ in $k[a, b, c, d]$ over any field k , and the dimension of $k[a, b, c, d]/J$.
- b) Find the dimension of the ring S , where $S = \mathbb{Q}[x^3y^3, x^3y^2z, x^2z^3] \subseteq \mathbb{Q}[x, y, z]$.
- c) Let I be the defining ideal of the curve parametrized by (t^{13}, t^{42}, t^{73}) over \mathbb{Q} . Find the height of I , and notice that $\text{height}(I) < \mu(I)$.
- d) Let $R = \mathbb{Q}[x, y, z]$, and $I = (x^3, x^2y, x^2z, xyz)$. Find the dimension of R/I and the height of I .
- e) Find the dimension of the module I/I^2 , where $I = (xz)$ in $R = \mathbb{C}[x, y, z]/(xy, yz)$.

¹Hint: first, do the case where I is a maximal ideal in R . Be wise, localize!