Symbolic powers of ideals defining F-pure rings

AMS Sectional Meeting Ann Arbor 2018

Eloísa Grifo* (University of Michigan)
Linquan Ma (Purdue University)
Karl Schwede (University of Utah)

Background

Symbolic Power

The n-th **symbolic power** of a radical ideal I in a domain R is

$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(R/I)} (I^n R_P \cap R).$$

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.

(3) If I is generated by a regular sequence, then $I^n = I^{(n)}$ for all n.

How do symbolic powers compare to ordinary powers?

(1) $I^n \subseteq I^{(n)}$ for all $n \geqslant 1$.

(2) $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geqslant 1$.

(3) If I is generated by a regular sequence, then $I^n = I^{(n)}$ for all n.

(4) In general, $I^n \neq I^{(n)}$.

Containment Problem (Schenzel)

When is $I^{(b)} \subseteq I^a$?

Containment Problem (Schenzel)

When is $I^{(b)} \subseteq I^a$?

Does the question make sense?

For every a there exists a b such that $I^{(b)} \subseteq I^a$ if and only if the I-adic and I-symbolic topologies are equivalent.

Theorem (Swanson, 2000)

Let I be a radical ideal in a noetherian local ring. If the I-adic and I-symbolic topologies are equivalent, there exists a constant k such that $I^{(kn)} \subseteq I^n$ for all n.

Big height

The big height of an ideal I is the maximal height of an associated prime of I.

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal of big height h in a regular ring R. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Big height

The big height of an ideal I is the maximal height of an associated prime of I.

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal of big height h in a regular ring R. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

Big height

The big height of an ideal I is the maximal height of an associated prime of I.

Theorem (Ein-Lazersfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

Let I be a radical ideal of big height h in a regular ring R. Then for all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.

Example

$P \subseteq R = k[x, y, z]$ the defining ideal of $k[t^3, t^4, t^5]$.

$$h = 2 \Rightarrow P^{(2n)} \subseteq P^n \Rightarrow P^{(4)} \subseteq P^2.$$

In fact, $P^{(3)} \subseteq P^2$.

Question (Huneke, 2000)

Let P be a height 2 prime in a regular local ring. Is $P^{(3)} \subseteq P^2$?

Question (Huneke, 2000)

Let P be a height 2 prime in a regular local ring. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, $\leqslant$ 2008)

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Key point (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]} \subseteq I^q.$$

Notation: $I^{[q]} = (f^q \mid f \in I)$.

Key point (Hochster–Huneke)

Let I be a radical ideal of big height h in a regular ring of characteristic $p > 0$. Then for all $q = p^e$,

$$I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q.$$

Notation: $I^{[q]} = (f^q \mid f \in I)$.

Harbourne's Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Dumnicki, Szemberg, Tutaj-Gasińska, 2015

There exists a radical ideal in $\mathbb{C}[x,y,z]$ such that $I^{(3)} \nsubseteq I^2$:

$$I = \left(z(x^3 - y^3), x(y^3 - z^3), y(z^3 - x^3) \right).$$

Harbourne's Conjecture

Let I be a radical ideal in a regular ring, and let h be the maximal height of a minimal prime of I. For all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

When does Harbourne's Conjecture hold?

- For general points in $\mathbb{P}^2$ (Harbourne–Huneke), $\mathbb{P}^3$ (Dumnicki).
- If R/I is an F-pure ring (G–Huneke).
  Eg, when I is a squarefree monomial ideal, or when R/I is direct summand of a polynomial ring.

Theorem (G–Huneke, 2017)

Let R be a regular ring of prime characteristic p and I an ideal in R of big height h.
If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Theorem (G–Huneke, 2017)

Let R be an F-finite regular ring of prime characteristic p and I an ideal in R of big height h.
If R/I is strongly F-regular, then $I^{((h-1)(n-1)+1)} \subseteq I^n$ for all $n \geqslant 1$.

Corollary (G–Huneke, 2017)

Let R be an F-finite regular ring of prime characteristic p and I an ideal in R of big height 2.
If R/I is strongly F-regular, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

Towards non-regular rings

Want to prove

$I^{(a)} \subseteq I^b$

Strategy

- Assume $(R, \mathfrak{m})$ is a local ring.

Want to prove

$I^{(a)} \subseteq I^b$

Strategy

- Assume $(R, \mathfrak{m})$ is a local ring.
- $I^{(a)} \subseteq I^b \quad \text{iff} \quad (I^b : I^{(a)}) = R \quad \text{iff} \quad (I^b : I^{(a)}) \nsubseteq \mathfrak{m}.$

Want to prove

$I^{(a)} \subseteq I^b$

Strategy

- Assume $(R, \mathfrak{m})$ is a local ring.
- $I^{(a)} \subseteq I^b \quad \text{iff} \quad (I^b : I^{(a)}) = R \quad \text{iff} \quad (I^b : I^{(a)}) \nsubseteq \mathfrak{m}$.
- If $(I^b : I^{(a)}) \subseteq \mathfrak{m}$, then $(I^b : I^{(a)})^{[q]} \subseteq \mathfrak{m}^{[q]}$.

Want to prove

$I^{(a)} \subseteq I^b$

Strategy

- Assume $(R, \mathfrak{m})$ is a local ring.
- $I^{(a)} \subseteq I^b \quad$ iff $\quad (I^b : I^{(a)}) = R \quad$ iff $\quad (I^b : I^{(a)}) \nsubseteq \mathfrak{m}$.
- If $(I^b : I^{(a)}) \subseteq \mathfrak{m}$, then $(I^b : I^{(a)})^{[q]} \subseteq \mathfrak{m}^{[q]}$.
- Find an ideal J_q with the following properties:
  - $J_q \subseteq (I^b : I^{(a)})^{[q]}$ for all q large, and
  - $J_q \nsubseteq \mathfrak{m}^{[q]}$ if R/I is F-pure or strongly F-regular.

Theorem (G–Huneke, 2017)

Let R be a regular ring of prime characteristic p and I an ideal in R of big height h.
If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Theorem (Fedder's Criterion, 1984)

Let $(R, \mathfrak{m})$ be a RLR of prime characteristic p and I an ideal in R.
The ring R/I is F-pure if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

$$\left(I^{[q]} : I\right) \subseteq \left(I^n : I^{(hn-h+1)}\right)^{[q]} \qquad \text{for all } q = p^e \gg 0$$

Theorem (G–Huneke, 2017)

Let R be an F-finite regular ring of prime characteristic p and I an ideal in R of big height h. If R/I is strongly F-regular, then $I^{((h-1)(n-1)+1)} \subseteq I^n$ for all $n \geqslant 1$.

Theorem (Glassbrenner's Criterion, 1996)

Let $(R, \mathfrak{m})$ be an F-finite RLR of prime characteristic p. Given a radical ideal $I \subsetneq R$, R/I is strongly F-regular if and only if for each $c \in R$ not in any minimal prime of I,

$$c\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]} \text{ for all } q = p^e \gg 0.$$

$$\left(I^n : I^{(n)}\right)\left(I^{[q]} : I\right)^{[q]} \subseteq \left(I I^{(n-h+1)} : I^{(n)}\right)^{[q]} \text{ for all } q = p^e \gg 0$$

What we need

To extend this to a non-regular setting, we need some version of Fedder's Criterion and Glassbrenner's Criterion.

Theorem (G–Ma–Schwede)

Let $(R, \mathfrak{m})$ be an F-finite Gorenstein local ring of characteristic $p > 0$ and $Q \subseteq R$ be a radical ideal of big height h with finite projective dimension.

- *If R/Q is F-pure, then $Q^{(hn-h+1)} \subseteq Q^n$ for all $n \geqslant 1$.*
- *If R/Q is strongly F-regular, $Q^{((h-1)(n-1)+1)} \subseteq Q^n$ for $n \geqslant 1$.*

A Fedder-like criterion

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

A is F-split if $A \subseteq A^{1/p^e}$ splits for all e, meaning there exists a map $\phi \in \mathrm{Hom}_A(A^{1/p^e}, A)$ such that $\phi(1) = 1$.

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

A is F-split if $A \subseteq A^{1/p^e}$ splits for all e, meaning there exists a map $\phi \in \mathrm{Hom}_A(A^{1/p^e}, A)$ such that $\phi(1) = 1$.

$$\begin{tikzcd} \textcolor{magenta}{\text{free}}\ R^{1/p^e} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d] & R \arrow[d] \\ (R/I)^{1/p^e} \arrow[r, "\phi"] & R/I \end{tikzcd}$$

$$\mathrm{Hom}_{R/I}((R/I)^{1/p^e}, R/I) \cong \frac{\left(I^{[p^e]} : I\right)^{1/p^e}}{I^{[p^e]}}$$

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

A is F-split if $A \subseteq A^{1/p^e}$ splits for all e, meaning there exists a map $\phi \in \mathrm{Hom}_A(A^{1/p^e}, A)$ such that $\phi(1) = 1$.

$$\begin{tikzcd} {}^{\text{free}}R^{1/p^e} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d] & R \arrow[d] \\ (R/I)^{1/p^e} \arrow[r, "\phi"] & R/I \end{tikzcd}$$

$$\mathrm{Hom}_{R/I}((R/I)^{1/p^e}, R/I) \cong \frac{\left(I^{[p^e]} : I\right)^{1/p^e}}{I^{[p^e]}}$$

A map $\phi \in \mathrm{Hom}_{R/I}((R/I)^{1/p^e}, R/I)$ such that $\phi(1) = 1$ exists if and only if $\left(I^{[p^e]} : I\right) \nsubseteq \mathfrak{m}^{[p^e]}$.

Lemma

If $(R, \mathfrak{m})$ is an F-finite Gorenstein local ring and Q is an ideal of finite projective dimension, any $\phi \in \mathrm{Hom}_{R/Q}(F^e_*(R/Q), R/Q)$ lifts to a map $\widetilde{\phi} \in \mathrm{Hom}_R(F^e_* R, R)$.

A Fedder-like criterion

Let $(R, \mathfrak{m})$ be an F-finite Gorenstein local ring and let $Q \subseteq R$ be a radical ideal of finite projective dimension.

- If R/Q is F-pure, then $\Phi_e(F^e_*(I_e(Q) : Q)) = R$.
- If R/Q is strongly F-regular, then for any c not in a minimal prime of Q, $\Phi_e(F^e_*(c(I_e(Q) : Q))) = R$ for some e.

$$\begin{aligned} I_e(Q) &= \{r \in R : \varphi(F^e_* r) \in Q \text{ for all } \varphi \in \mathrm{Hom}_R(F^e_* R, R)\} \\ &= \{r \in R : \Phi_e(F^e_*(rR)) \subseteq Q\} \end{aligned}$$

$$\left(I_e(Q):Q\right) \subseteq \left(I_e(Q)\left(Q^{n-1}\right)^{[q]} : \left(Q^{(hn-h+1)}\right)^{[q]}\right)$$

$$\rightsquigarrow$$

$$\Phi_e\left(F^e_*\left(I_e(Q):Q\right)\right) \subseteq \Phi_e(F^e_*\left(\left(I_e(Q)\left(Q^{n-1}\right)^{[q]} : \left(Q^{(hn-h+1)}\right)^{[q]}\right)\right)$$

$$\rightsquigarrow$$

$$R \subseteq \left(QQ^{n-1} : Q^{(hn-h+1)}\right)$$

$$\rightsquigarrow$$

$$Q^{(hn-h+1)} \subseteq Q^n = QQ^{n-1}$$

Obrigada!

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

A is F-split if $A \subseteq A^{1/p^e}$ splits for all e, meaning there exists a map $\phi \in \mathrm{Hom}_A(A^{1/p^e}, A)$ such that $\phi(1) = 1$.

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

A is F-split if $A \subseteq A^{1/p^e}$ splits for all e, meaning there exists a map $\phi \in \operatorname{Hom}_A(A^{1/p^e}, A)$ such that $\phi(1) = 1$.

$$\begin{tikzcd} \text{free } R^{1/p^e} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d] & R \arrow[d] \\ (R/I)^{1/p^e} \arrow[r, "\phi"] & R/I \end{tikzcd}$$

$$\operatorname{Hom}_{R/I}((R/I)^{1/p^e}, R/I)$$

$$\cong$$

$$F^e_* \left(I^{[q]} : I\right) \cdot \Phi_e / I^{[p^e]}$$

Fedder's Criterion (1984)

If $(R, \mathfrak{m})$ is a F-finite RLR of prime characteristic p and I is an ideal in R, R/I is F-split if and only if $\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.

A is F-split if $A \subseteq A^{1/p^e}$ splits for all e, meaning there exists a map $\phi \in \mathrm{Hom}_A(A^{1/p^e}, A)$ such that $\phi(1) = 1$.

$$\begin{tikzcd} \text{free } R^{1/p^e} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d] & R \arrow[d] \\ (R/I)^{1/p^e} \arrow[r, "\phi"] & R/I \end{tikzcd}$$

$$\mathrm{Hom}_{R/I}((R/I)^{1/p^e}, R/I) \cong F_*^e \left(I^{[q]} : I\right) \cdot \Phi_e / I^{[p^e]}$$

If $\phi(1) = 1$ for some $\phi \in \mathrm{Hom}_{R/I}((R/I)^{1/p^e}, R/I)$, then $\phi \notin \mathfrak{m}\Phi_e$, and such ϕ exists if and only if $\left(I^{[p^e]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$.

Lemma

If $(R, \mathfrak{m})$ is an F-finite Gorenstein local ring and Q is an ideal of finite projective dimension, any $\phi \in \mathrm{Hom}_{R/Q}(F_*^e(R/Q), R/Q)$ lifts to a map $\widetilde{\phi} \in \mathrm{Hom}_R(F_*^e R, R)$.

Since R is Gorenstein, $\mathrm{Hom}_R(F_*^e R, R)$ is generated by one element, Φ_e. Each map $\Phi_e(F_*^e r \cdot -) \in \mathrm{Hom}_R(F_*^e R, R)$ induces a map in $\mathrm{Hom}_{R/Q}(F_*^e(R/Q), R/Q)$ if and only if it sends $F_*^e(Q)$ to Q, so $\Phi_e(F_*^e(rQ)) \subseteq Q$. In other words, $r \in (I_e(Q) : Q)$.

$$
\begin{aligned}
I_e(Q) &= \{r \in R : \varphi(F_*^e r) \in Q \text{ for all } \varphi \in \mathrm{Hom}_R(F_*^e R, R)\} \\
&= \{r \in R : \Phi_e(F_*^e(rR)) \subseteq Q\}
\end{aligned}
$$

Lemma

If $(R, \mathfrak{m})$ is an F-finite Gorenstein local ring and Q is an ideal of finite projective dimension, any $\phi \in \operatorname{Hom}_{R/Q}(F^e_*(R/Q), R/Q)$ lifts to a map $\widetilde{\phi} \in \operatorname{Hom}_R(F^e_* R, R)$.

$$\operatorname{Hom}_{R/Q}(F^e_*(R/Q), R/Q) \cong \frac{F^e_*\left(I_e(Q) : Q\right) \cdot \Phi_e}{F^e_*\left(I_e(Q)\right)}.$$

$$\begin{aligned} I_e(Q) &= \{r \in R : \varphi(F^e_* r) \in Q \text{ for all } \varphi \in \operatorname{Hom}_R(F^e_* R, R)\} \\ &= \{r \in R : \Phi_e(F^e_*(rR)) \subseteq Q\} \end{aligned}$$

A Fedder-like criterion

Let $(R, \mathfrak{m})$ be an F-finite Gorenstein local ring and let $Q \subseteq R$ be a radical ideal of finite projective dimension.

- If R/Q is F-pure, then $\Phi_e(F_*^e(I_e(Q) : Q)) = R$.
- If R/Q is strongly F-regular, then for any c not in a minimal prime of Q, $\Phi_e\left(F_*^e(c(I_e(Q) : Q))\right) = R$ for some e.

$$\begin{aligned} I_e(Q) &= \{r \in R : \varphi(F_*^e r) \in Q \text{ for all } \varphi \in \mathrm{Hom}_R(F_*^e R, R)\} \\ &= \{r \in R : \Phi_e(F_*^e(rR)) \subseteq Q\} \end{aligned}$$

$$\left(I_e(Q) : Q\right) \subseteq \left(I_e(Q)\left(Q^{n-1}\right)^{[q]} : \left(Q^{(hn-h+1)}\right)^{[q]}\right)$$

$$\rotatebox{-90}{$\rightsquigarrow$}$$

$$\varphi\left(F_*^e\left(I_e(Q) : Q\right)\right) \subseteq \varphi\left(F_*^e\left(\left(I_e(Q)\left(Q^{n-1}\right)^{[q]} : \left(Q^{(hn-h+1)}\right)^{[q]}\right)\right)\right)$$

$$\rotatebox{-90}{$\rightsquigarrow$}$$

$$R \subseteq \left(QQ^{n-1} : Q^{(hn-h+1)}\right)$$

$$\rotatebox{-90}{$\rightsquigarrow$}$$

$$Q^{(hn-h+1)} \subseteq Q^n = QQ^{n-1}$$