# Problem Set 3

**Instructions:** For full credit, turn in 4 problems in a pdf file. You are welcome to work together with your classmates on all the problems, and I will be happy to give you hints or discuss the problems with you, but you should write up your solutions by yourself. You cannot use any resources besides me, your classmates, our course notes, and the Macaulay2 documentation.

**Problem 1.** Let $k$ be a field, $x$ an indeterminate, and $R := k[x^2, x^3] \subseteq S := k[x]$. Find an ideal $I$ in $R$ such that $IS \cap R \supsetneq I$.

**Problem 2.** Let $k$ be an infinite field, $m \geqslant 1$, and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring. Let $G = (k^\times)^m$ act on $R$ as follows:

$$\begin{aligned} (\lambda_1, \ldots, \lambda_m) \cdot c &= c && \text{for } c \in k, \\ (\lambda_1, \ldots, \lambda_m) \cdot x_i &= \lambda_1^{a_{1i}} \cdots \lambda_m^{a_{mi}} x_i && i = 1, \ldots, n \end{aligned}$$

for some $m \times n$ matrix of integers $A = [a_{ij}]$.

a) Show that $R^G$ has a $k$-vector space basis given by the set of monomials $x_1^{b_1} \cdots x_n^{b_n}$ such that, for $b = (b_1, \ldots, b_n)$, $Ab = 0$.

b) Consider the polynomial ring $R$ with a (nonstandard) $\mathbb{Z}^m$-grading given by setting

$$\deg(x_i) = (a_{1i}, \ldots, a_{mi})$$

for each $i$. Show that $R^G$ is the degree zero piece of $R$ under this grading.

c) Show that $R^G$ is a direct summand of $R$, and conclude that $R^G$ is a finitely generated $k$-algebra.

d) Let $R = k[x, y, z, w]$ and consider the $k$-linear action of $G = k^\times$ on $R$ given by

$$\lambda \cdot x = \lambda x \qquad \lambda \cdot y = \lambda y \qquad \lambda \cdot z = \lambda^{-1} z \qquad \lambda \cdot w = \lambda^{-1} w.$$

Find a finite set of generators for $R^G$ as a $k$-algebra.

**Problem 3.** Consider the inclusion map $k[xy, xz, yz] \subseteq k[x, y, z]$, where $k$ is a field. Is the induced map on Spec surjective? If not, give an explicit prime not in the image.

Let $R$ be a ring of prime characteristic $p$, meaning that the smallest $n > 0$ such that

$$\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0$$

is $n = p$. Then the map $F : R \to R$ given by $F(r) = r^p$ is a ring homomorphism, called the **Frobenius map**.

**Problem 4.** Let $R$ be a ring of prime characteristic $p$.

a) Show that the Frobenius map is injective if and only if $\sqrt{(0)} = (0)$.

b) Show that the $F(R) \subseteq R$ is module-finite if and only if it is algebra-finite.

c) Show that the map on spectra induced by the Frobenius map is the identity map.

A topological space is *irreducible* if it cannot be written as a union of two proper closed subsets.

**Problem 5.** Let $R$ be a ring and be an ideal in $I$. Show that $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime.