Sequences of Symmetric Ideals

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Motivation

- In Algebraic Geometry and Commutative Algebra several objects of interest form sequences of (symmetric) objects, where the number of variables involved varies with the objects.
- Often some "limit behavior" is visible. Is this an accident or is there a systematic reason?

Example

Fix a field K and an integer $c \geq 2$ and consider the ideal I_n, where

- (explicitly) I_n is generated by the 2-minors of a generic $c \times n$ matrix $X_{c\times n}$;   or
- (implicitly)

$$I_n = \ker(K[X_{c\times n}] \to K[y_1, \dots, y_c, z_1, \dots, x_n],\ x_{i,j} \mapsto y_i z_j).$$

Note

$$\dim K[X_{c\times n}]/I_n = n + c - 1.$$

Powers of Ideals

Theorem

If $J \subseteq S = K[y_1, \ldots, y_m]$ is a homogeneous ideal, then:

- [Brodmann, 1979]
  $\mathrm{Ass}_S(S/J^n)$ *is a constant set for* $n \gg 0$.
- [Brodmann, 1979]
  $\mathrm{depth}(S/J^n)$ *is constant for* $n \gg 0$.
- [Cutkosky, Herzog,Trung 1999; Kodiyalam, 2000]
  $\mathrm{reg}(J^n)$ *is a linear function for* $n \gg 0$.

Key: Rees algebra $\oplus_{n \geq 0} J^n$ is noetherian.

Invariant Filtrations

Ascending sequence of compatible symmetric ideals $I_n \subset K[X_n]$:

$K[X_n] = K[x_j \mid 1 \leq j \leq n]$
$\mathrm{Sym}(n)$ acts on $K[X_n]$ via $\pi \cdot x_j = x_{\pi(j)}$

I_n is symmetric or $\mathrm{Sym}(n)$-invariant if $\pi \cdot f \in I_n$ whenever $\pi \in \mathrm{Sym}(n)$ and $f \in I_n$.

Compatible means, as subsets of $K[X_n]$, one has

$$\mathrm{Sym}(n)(I_m) \subset I_n \text{ whenever } m \leq n.$$

Invariant Filtrations

Example 1

$$I_n = \langle x_i x_j \mid 1 \leq i < j \leq n \rangle \subset K[X_n] = K[x_1, \ldots, x_n].$$

$$\begin{aligned} I_1 &= 0 \subset K[x_1], \\ I_2 &= \langle x_1 x_2 \rangle \subset K[x_1, x_2], \\ I_3 &= \langle x_1 x_2, x_1 x_3, x_2 x_3 \rangle \subset K[x_1, x_2, x_3]. \end{aligned}$$

Note, I_n is generated by $x_1 x_2$ up to symmetry, i.e. the $\mathrm{Sym}(n)$-orbit of $x_1 x_2$.
(E.g., $(1, 2, 3) \cdot x_1 x_2 = x_2 x_3$.)

Invariant Filtrations

Example 2

$(c = 3)$

$I_n \subset (K[X_n])^{\otimes 3} = K[x_{i,j} \mid 1 \le i \le 3,\ 1 \ \le j \le n]$,
generated by the 2-minors of a generic $3 \times n$ matrix.

$I_1 = 0 \subset (K[X_1]))^{\otimes 3}$,

$I_2 = \langle \det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix},\ \det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{3,1} & x_{3,2} \end{bmatrix},\ \det \begin{bmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{bmatrix} \rangle \subset (K[X_2])^{\otimes 3}$,

Note, I_n is generated up to symmetry (column-wise action) by three polynomials:

$$\det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix},\ \det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{3,1} & x_{3,2} \end{bmatrix},\ \det \begin{bmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{bmatrix}$$

In Algebraic Statistics, similar examples arise.

Invariant Filtrations

A Sym-invariant filtration $(I_n)_{n \in \mathbb{N}}$ is a sequence of ideals $I_n \subset (K[X_n])^{\otimes c} = K[X_{c \times n}] = K[x_{i,j} \mid 1 \leq i \leq c,\ 1 \leq j \leq n]$ satisfying

(i) Each I_n is invariant under the ("column-wise") action of $\mathrm{Sym}(n)$ given by $\pi \cdot x_{i,j} = x_{i,\pi(j)}$.

(ii) (ascending chain) $\langle I_n \cdot (K[X_{n+1}])^{\otimes c} \rangle \subset I_{n+1}$.

(iii) (compatible symmetry)

$$\mathrm{Sym}(n+1) \cdot \langle I_n \cdot K[X_{n+1}])^{\otimes c} \rangle \subset I_{n+1}.$$

Use natural embeddings

$K[X_n])^{\otimes c} \subset K[X_{n+1}])^{\otimes c}$ and $\mathrm{Sym}(n) \subset \mathrm{Sym}(n+1)$.

Stabilization

Informally:
If $\mathcal{I} = (I_n)_{n\in\mathbb{N}}$ is an invariant filtration, then the generators of the ideals I_n look alike eventually.

Officially:

Theorem (Aschenbrenner, Hillar, 2007; Hillar, Sullivant, 2012)

Every invariant filtration $\mathcal{I} = (I_n)_{n\geq 1}$ stabilizes, that is, there is an integer n_0 such that, as ideals of $K[X_n]$, one has

$$\langle \mathrm{Sym}(n) I_{n_0} \rangle_{K[X_n]} = I_n \text{ whenever } n \geq n_0.$$

The proof relies on the fact that

$$I = \bigcup_{n\geq 1} I_n \subseteq K[X_{i,j} : 1 \leq i \leq c,\ j \geq 1] = (K[X_{\mathbb{N}}])^{\otimes c}$$

is an ideal which is finitely generated up to symmetry.

Equivariant Hilbert series

Definition

For an invariant filtration $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$ of homogeneous ideals define its bigraded or equivariant Hilbert series as

$$H_{\mathcal{I}}(s,t) = \sum_{n \geq 0} H_{K[X_{c \times n}]/I_n}(t) \cdot s^n = \sum_{n \geq 0, j \geq 0} \dim_K [K[X_{c \times n}]/I_n]_j \cdot s^n t^j.$$

Equivariant Hilbert series

Theorem (N., Römer, 2015)

Each equivariant Hilbert series is a rational function in s and t of the form

$$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a \cdot \prod_{j=1}^{b}[(1-t)^{c_j} - s \cdot f_j(t)]},$$

where $a, b, c_j \in \mathbb{N}_0$ with $c_j \leq c$, $g(s,t) \in \mathbb{Z}[s,t]$, each $f_j(t) \in \mathbb{Z}[t]$, and $f_j(1) > 0$.

Remark

(i) Examples in [N., Römer, 2015] and in [N., Güntürkun, 2016] indicate that the description of the denominator is rather optimal.
(ii) Alternate proof of rationality by Krone, Leykin, and Snowden in 2016. No information about the denominator.
(iii) N., Maraj, 2019: Rationality results for some related algebras.

Krull dimension and Degree

Theorem (N., Römer, 2015)

There are integers A, B, M, L with $0 \leq A \leq c$, $M > 0$, and $L \geq 0$ such that, for all $n \gg 0$,

$$\dim K[X_n]/I_n = An + B$$

and the limit of $\frac{\deg I_n}{M^n \cdot n^L}$ as $n \to \infty$ exists and is equal to a positive rational number.

Example (N., Güntürkün, 2016)

For $I_n \subset K[X_n]$ generated by $\mathrm{Sym}(n)$-orbit of $x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$,

$$\lim_{n\to\infty} \sqrt[n]{\deg I_n} = \max\{a_1, ..., a_r\},$$

Castelnuovo-Mumford regularity

Conjecture (Le, N., Nguyen, Römer, 2018)

If $(I_n)_{n\geq 1}$ is any Sym-invariant filtration of homogeneous ideals, then $\operatorname{reg} I_n$ is eventually a linear function in n, that is,

$$\operatorname{reg} I_n = An + B$$

for some integers A, B whenever $n \gg 0$.

Evidence for the conjecture

Theorem (Le, N., Nguyen, Römer, 2018)

For any Sym-invariant filtration of homogeneous ideals, there are integers C, D such that

$$\operatorname{reg} I_n \leq Cn + D \text{ for all } n \gg 0$$

with an explicit formula for C.

Evidence for the conjecture

Theorem (Le, N., Nguyen, Römer, 2018)

If $(I_n)_{n\geq 1}$ is any Sym-invariant filtration of homogeneous ideals, then $\operatorname{reg} I_n$ is eventually a linear function in the following cases:

- $I_n \subseteq K[X_n])^{\otimes c}$ is generated by the orbit of one monomial.
- I_n is an Artinian ideal in $K[X_n])^{\otimes c}$ for some $n \geq 1$.
- $(c = 1)$ $I_n \subseteq K[X_n])$ is a squarefree monomial ideal.

Theorem (Murai, 2019; Raicu, 2019)

$(c = 1)$

If $(I_n)_{n\geq 1}$ is any Sym-invariant filtration of monomial ideals $I_n \subseteq K[X_n])$, then $\operatorname{reg} I_n$ is eventually a linear function.

Projective dimension

Conjecture (Le, N., Nguyen, Römer, 2018)

If $(I_n)_{n\geq 1}$ is any Sym-invariant chain of ideals., then $\operatorname{pd} R_n/I_n$ is eventually a linear function, that is,

$$\operatorname{pd} R_n/I_n = An + B$$

for some integers A, B whenever $n \gg 0$.

Evidence for the conjecture

- Conjecture is true if I_n is perfect for $n \gg 0$ since $\operatorname{codim} I_n$ is eventually a linear function.
- There are upper and lower linear bounds for $\operatorname{pd}(R_n/I_n)$:

$$cn \geq \operatorname{pd}(R_n/I_n) \geq \operatorname{codim} I_n.$$

- [Le, N., Nguyen, Römer, 2018]: Improved lower linear bound for filtrations of monomial ideals.

Theorem (Murai, 2019; Raicu, 2019)

($c = 1$)
If $(I_n)_{n\geq 1}$ is any Sym-invariant filtration of monomial ideals $I_n \subseteq K[X_n]$), then $\operatorname{pd} I_n$ is eventually a linear function.

Syzygies

Example 1 (continued)

$$I_n = \langle x_i x_j \mid 1 \leq i < j \leq n \rangle \subset K[X_n] = K[x_1, \ldots, x_n]$$

Its first syzygies have the form

$$\begin{aligned} x_{j+1} \cdot (x_i x_j) - x_j \cdot (x_i x_{j+1}) &= 0 \quad i < j \text{ and} \\ x_{i+1} \cdot (x_i x_j) - x_i \cdot (x_{i+1} x_j) &= 0 \quad i + 1 < j. \end{aligned}$$

Informally:
1 master generator, namely $x_1 x_2$
2 master first syzygies, namely $x_3 \cdot (x_1 x_2) - x_2 \cdot (x_1 x_3) = 0$ and
$x_2 \cdot (x_1 x_3) - x_1 \cdot (x_2 x_3) = 0$

Example 2 (continued)

I_n generated by 2-minors of a generic $3 \times n$ matrix.

First syzygies by repeating a column and taking the determinant: $j \in \{1, 2\}$

$$\left[\begin{array}{cc:c} x_{1,1} & x_{1,2} & x_{1,j} \\ x_{2,1} & x_{2,2} & x_{2,j} \\ x_{3,1} & x_{3,2} & x_{3,j} \end{array}\right].$$

$$x_{1,j} \cdot \det \begin{bmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{bmatrix} - x_{2,j} \cdot \det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} + x_{3,j} \cdot \det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{3,1} & x_{3,2} \end{bmatrix} = 0.$$

Informally: 2 master first syzygies.

Syzygies

Theorem (N., Römer, 2017) - Informal Version

Let $(I_n)_{n\in\mathbb{N}}$ be an invariant filtration of homogeneous ideals. Then, for every integer $p \geq 0$, the p-syzygies of the ideals I_n look alike eventually.

Slightly more precisely, for every integer $p \geq 0$, there are some integer n_p and finitely many master p-syzygies that generate all p-syzygies of I_n up to symmetry whenever $n \geq n_p$.

Categories of Finite Sets

Denote by FI the category whose objects are finite sets and whose morphisms are injections.

(Ordered version) The category OI is the subcategory of FI whose objects are totally ordered finite sets and whose morphisms are *order-preserving* injective maps

Notation: $[n] = \{1, 2, \ldots, n\}$. Thus, $[0] = \emptyset$.
$\mathbb{N}_0$ set of non-negative integers

Note: OI is equivalent to the category with objects $[n]$ for $n \in \mathbb{N}_0$ and morphisms being order-preserving injective maps $\varepsilon\colon [m] \to [n]$.

Algebras

K any commutative ring (with unity)

$K\text{-Alg}$ category of commutative, associative, unital K-algebras

Informally:
An FI-algebra over K is a sequence of compatible similar K-algebras $\mathbf{A}_n$.

Officially:
An FI-algebra over K is a covariant functor $\mathbf{A}\colon \text{FI} \to K\text{-Alg}$ with $\mathbf{A}(\emptyset) = K$.

Analogously, an OI-algebra over K is a covariant functor $\mathbf{A}\colon \text{OI} \to K\text{-Alg}$ with $\mathbf{A}(\emptyset) = K$.

Notation: $\mathbf{A}_m := \mathbf{A}([m])$

Polynomial Algebras

Example

Define an FI-algebra $\mathbf{P}$ by $\mathbf{P}_m = K[x_1, \dots, x_m]$
and, for $\varepsilon \in \mathrm{Hom}_{\mathrm{FI}}([m], [n])$, the homomorphism

$$\varepsilon^* \colon K[x_1, \dots, x_m] \to K[x_1, \dots, x_n] \quad \text{by } \varepsilon^*(x_i) = x_{\varepsilon(i)}.$$

More generally, for any $d \geq 0$, define a functor $\mathbf{X}^{\mathrm{FI},d} : \mathrm{FI} \to K\text{-Alg}$ by letting

$$\mathbf{X}_n^{\mathrm{FI},d} = K\left[x_\pi \ : \ \pi \in \mathrm{Hom}_{\mathrm{FI}}([d], [n])\right]$$

be the polynomial ring over K with variables x_π, and, for $\varepsilon \in \mathrm{Hom}_{\mathrm{FI}}([m], [n])$, by defining

$$\mathbf{X}^{\mathrm{FI},d}(\varepsilon) \colon \mathbf{X}_m^{\mathrm{FI},d} \to \mathbf{X}_n^{\mathrm{FI},d} \quad \text{by } \ x_\pi \mapsto x_{\varepsilon \circ \pi}.$$

Similarly, define an OI-algebra $\mathbf{X}^{\mathrm{OI},d} \colon \mathrm{OI} \to K\text{-Alg}$.
$d = 1$: $\mathbf{X}^{\mathrm{OI},1} = \mathbf{P}$ (but with fewer maps)
$d = 0$: $\mathbf{X}_n^{\mathrm{OI},0} = K$ for all n (constant coefficients)

Modules

$\mathbf{A}$ an FI-algebra over K

Informally:
An FI-module over $\mathbf{A}$ is a sequence of compatible similar $\mathbf{A}_n$-modules $\mathbf{M}_n$

Modules

Officially:

An FI-module over **A** is a covariant functor $\mathbf{M}\colon \mathrm{FI} \to K\text{-Mod}$ such that,

(1) for every $m \in \mathbb{N}_0$, the K-module $\mathbf{M}_m = \mathbf{M}([m])$ is also an $\mathbf{A}_m$-module, and

(2) for any morphism $\varepsilon\colon [m] \to [n]$ and any $a \in \mathbf{A}_m$, the following diagram is commutative

$$\begin{tikzcd}
\mathbf{M}_m \arrow[r, "\mathbf{M}(\varepsilon)"] \arrow[d, "\cdot a"'] & \mathbf{M}_n \arrow[d, "\cdot \mathbf{A}(\varepsilon)(a)"] \\
\mathbf{M}_m \arrow[r, "\mathbf{M}(\varepsilon)"] & \mathbf{M}_n,
\end{tikzcd}$$

where the vertical maps are given by multiplication by the indicated elements.

The morphisms of $\mathrm{FI}\text{-Mod}(\mathbf{A})$ are natural transformations

$F\colon \mathbf{M} \to \mathbf{N}$ such that, for every $m \in \mathbb{N}_0$, the map $\mathbf{M}_m \xrightarrow{F(m)} \mathbf{N}_m$ is an $\mathbf{A}_m$-module homomorphism.

Remark

(i) Analogously, one defines an OI-module over $\mathbf{A}$ as a functor $\mathbf{M}\colon \text{OI} \to K\text{-Mod}$.

(ii) Any Sym-invariant filtration (I_n) of ideals $I_n \subset K[x_1, \ldots, x_n]$ determines an ideal $\mathbf{I}$ of $\mathbf{X}^{\text{FI},1} = \mathbf{P}$.

(iii) Any ideal $J \subset K[x_1, \ldots, x_m]$ generates an ideal $\mathbf{I}$ of $\mathbf{X}^{\text{OI},1}$. For example, $J = (x_1 x_3) \subset K[x_1, x_2, x_3]$ generates $\mathbf{I}$ with

$$\mathbf{I}_n = (x_i x_j \mid 1 \leq i,\ i + 2 \leq j \leq n) \subset K[x_1, \ldots, x_n].$$

(iv) FI-modules over the algebra $\mathbf{X}^{\text{FI},0}$ with constant coefficients are called FI-modules and were studied by Church, Ellenberg, Farb, 2014, and many others.

(v) OI-modules over the algebra $\mathbf{X}^{\text{OI},0}$ with constant coefficients were first studied by Sam and Snowden, 2017.

Noetherian Modules

M is said to be finitely generated, if there exists a finite subset $G \subset \coprod_{n \geq 0} \mathbf{M}_n$ which is not contained in any proper submodule of **M**.

For example, the filtration in Example 1 corresponds to an ideal of $\mathbf{X}^{\mathrm{FI},1}$ generated by $x_1 x_2$.

An FI-module **M** over **A** is said to be noetherian if every FI-submodule of **M** is finitely generated.
An algebra **A** is noetherian if it is a noetherian FI-module over itself.

Analogously, one defines a noetherian OI-module and a noetherian OI-algebra.

Noetherian algebras and modules

Theorem (N., Römer, 2017)

The polynomial algebras $\mathbf{X}^{\mathrm{FI},d}$ and $\mathbf{X}^{\mathrm{OI},d}$ are noetherian if and only if $d \in \{0, 1\}$.

Theorem (N., Römer, 2017)

For every integer $c > 0$, one has:

(a) Every finitely generated FI-module over $(\mathbf{X}^{\mathrm{FI},1})^{\otimes c}$ is noetherian.

(b) Every finitely generated OI-module over $(\mathbf{X}^{\mathrm{OI},1})^{\otimes c}$ is noetherian.

Corollary

(a) Every finitely generated FI-module over $\mathbf{X}^{\mathrm{FI},0}$ is noetherian (Church, Ellenberg, Farb, Nagpal, 2014, 2015).

(b) Every finitely generated OI-module over $\mathbf{X}^{\mathrm{OI},0}$ is noetherian (Sam and Snowden, 2017).

Noetherian algebras and modules

Strategy of Proof:

- Use Gröbner basis theory, and so ordered versions.
- Higman's lemma about well-partial-orders, a generalization of Dickson's lemma.
- FI-results as consequence of OI-results.

Free Modules

$\mathbf{A}$ an FI-algebra over K.
For $d \geq 0$, define an FI-module $\mathbf{F}^{\mathrm{FI},d}$ over $\mathbf{A}$ by

$$\mathbf{F}_n^{\mathrm{FI},d} = \oplus_\pi \mathbf{A}_n e_\pi \cong (\mathbf{A}_n)^{\binom{n}{d} d!},$$

where the sum is taken over all $\pi \in \mathrm{Hom}_{\mathrm{FI}}([d],[n])$, and

$$\mathbf{F}^{\mathrm{FI},d}(\varepsilon) : \mathbf{F}_m^{\mathrm{FI},d} \to \mathbf{F}_n^{\mathrm{FI},d} \; ae_\pi \mapsto \varepsilon^*(a)e_{\varepsilon\circ\pi},$$

where $a \in \mathbf{A}_m$ and $\varepsilon : [m] \to [n]$ is a FI morphism.
A free FI-module over $\mathbf{A}$ is an FI-module that is isomorphic to a direct sum $\bigoplus_{\lambda \in \Lambda} \mathbf{F}^{\mathrm{FI},d_\lambda}$.

Similarly, define $\mathbf{F}^{\mathrm{OI},d}$ and a free OI-module over $\mathbf{A}$

Proposition

An FI-module $\mathbf{M}$ over $\mathbf{A}$ is finitely generated if and only if there is a surjection $\bigoplus_{i=1}^k \mathbf{F}^{\mathrm{FI},d_i} \to \mathbf{M}$ for some integers $d_i \geq 0$.

Resolutions

Theorem

Let $\mathbf{M}$ be a finitely generated FI-module (or OI-module, respectively) over $(\mathbf{X}^{\mathrm{FI},1})^{\otimes c}$ (or $(\mathbf{X}^{\mathrm{OI},1})^{\otimes c}$). There exists a projective resolution $\mathbf{F}_{\bullet}$ of $\mathbf{M}$

$$\cdots \to \mathbf{F}^{(1)} \to \mathbf{F}^{(0)} \to \mathbf{M} \to 0,$$

where every module $\mathbf{F}^{(p)}$ is a finitely generated free module.

Stabilization of Syzygies

Theorem (N., Römer, 2017) - Official Version

Let $\mathbf{M}$ be a finitely generated graded FI-module over $\mathbf{X} = (\mathbf{X}^{\mathrm{FI},1})^{\otimes c}$. Then, for any $p \in \mathbb{N}$,

$$\operatorname{Tor}_p^{\mathbf{X}}(\mathbf{M}, \mathbf{X}^{\mathrm{FI},0})$$

is a finitely generated graded FI-module over $\mathbf{X}$ and, for all n,

$$\operatorname{Tor}_p^{\mathbf{X}}(\mathbf{M}, \mathbf{X}^{\mathrm{FI},0})_n = \operatorname{Tor}_p^{\mathbf{X}_n}(\mathbf{M}_n, K)$$

Moreover, there are integers $j_0 < \cdots < j_t$ depending on p and $\mathbf{M}$ such that for every $n \gg 0$,

$$[\operatorname{Tor}_p^{\mathbf{X}_n}(\mathbf{M}_n, K)]_j \neq 0 \text{ if and only if } j \in \{j_0, \ldots, j_t\}.$$

Analogously, for an OI-module over $(\mathbf{X}^{\mathrm{OI},1})^{\otimes c}$

Stabilization of Syzygies

Uniform vanishing independent of p impossible.

Example: $M_n = (x_1^2, \ldots, x_n^2)$.

Summary

- Invariant filtrations arise in various contexts. There are instances of asymptotic stabilization.
- There is a need for developing new Commutative Algebra methods.
- The theory of FI- and OI-modules provides tools to study invariant filtrations.