Symbolic Powers

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This is a preliminary version of the notes for a lectures series at the summer school BRIDGES, hosted by the University of Utah in July 2021 and aimed at advanced undergraduates and graduate students in their first few years. These notes are very loosely based on two previous lecture series, both aimed at advanced commutative algebra students: a lecture series at the *Escuela de Outoño en Álgebra Conmutativa* (Fall School in Commutative Algebra) in November 2019 at CIMAT, in Guanajuato, Mexico, which I delivered in spanish, and a lecture series at the RTG Advanced Summer Mini-course in Commutative Algebra at the University of Utah, in May 2018. Both those previous notes contain a lot more material, but less detail in the early sections. While the present notes do not assume previous knowledge in commutative algebra, the others do assume at least a first course.

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These notes are in no way comprehensive, but more about symbolic powers can be found in the references — for example, see the surveys [DDSG+18] and [SS17]. We assume only a first course in algebra, though we will need some fundamental concepts and results in commutative algebra that one would learn in a first course, which we will introduce quickly without detail. The details can be found in any standard commutative algebra reference book, such as [AM69], [Mat80], or [BH93], or my commutative algebra class notes.

1 Day 1: Algebra

Our goal is to study symbolic powers. Besides being an interesting subject in its own right, symbolic powers appear as auxiliary tools in several important results in commutative algebra, such as Krull's Principal Ideal Theorem or the Hartshorne—Lichtenbaum Vanishing Theorem on Local Cohomology. Symbolic powers arise naturally from the theory of primary decomposition, and thanks to the Zariski–Nagata theorem, they also contain geometric information. We will start our story from the algebraic perspective.

1.1 Primary decomposition and associated primes

Noetherian rings are commutative algebraists' favorite rings. Many rings of interest are noetherian, including every quotient of a polynomial ring in finitely many variables over a field. Assuming the ambient ring is noetherian is considered a mild assumption in commutative algebra, and from now on we will always assume our rings are noetherian, as one often does.

Definition 1.1. Let R be a ring. We say R is **noetherian** if every ideal in R is finitely generated.

It is most common to first define noetherianity via an equivalent condition called the ascending chain condition.

Exercise 1. Show that the following are equivalent:

1) Every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes: there is some N for which $I_n = I_{n+1}$ for all $n \geqslant N$.

2) Every nonempty family of ideals has a maximal element (under $\subseteq$).

3) Every ideal of R is finitely generated.

One of the fundamental classical results in commutative algebra is the fact that every ideal in any noetherian ring has a primary decomposition. This can be thought of as a generalization of the Fundamental Theorem of Arithmetic:

Theorem 1.2. Given any positive integer n, there exist distinct primes $p_1, \ldots, p_k$ and integers $a_1, \ldots, a_k \geqslant 1$ such that

$$n = (p_1)^{a_1} \cdots (p_k)^{a_k}.$$

Moreover, this decomposition is unique up to the order of the factors.

We will soon discover that such a product *is* a primary decomposition, perhaps after some light rewriting. But before we get to the *what* and the *how* of primary decomposition, it is worth discussing the *why*. If we wanted to extend the Fundamental Theorem of Arithmetic to other rings, our first attempt might involve irreducible elements. Unfortunately — or fortunately, since after all, it makes the story more interesting — we don't have to go far to find rings where we *cannot* write elements as a unique product of irreducibles up to multiplication by a unit.

Example 1.3. In $\mathbb{Z}[\sqrt{-5}]$,

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two *different* ways to write 6 as a product of irreducible elements. In fact, we cannot obtain 2 nor 3 by multiplying $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$ by a unit.

Instead of writing *elements* as products of irreducibles, we will write *ideals* in terms of *primary ideals*.

Definition 1.4. We say that an ideal is **primary** if

$$xy \in I \implies x \in I \text{ or } y^n \in I \text{ for some } n \geqslant 1.$$

In fact, we can write this definition in a slightly fancier looking fashion using the concept of radical of an ideal.

Definition 1.5. The **radical** of an ideal I in a ring R is the ideal

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n\}.$$

An ideal I is a **radical ideal** if $I = \sqrt{I}$.

Exercise 2. Given an ideal I in any ring R, its radical $\sqrt{I}$ is also an ideal.

Example 1.6. Prime ideals are radical.

So now we can say an ideal I is **primary** if

$$xy \in I \implies x \in I \text{ or } y \in \sqrt{I}.$$

Remark 1.7. From the definition, it follows that the radical of a primary ideal is always a prime ideal. If the radical of a primary ideal Q is the prime ideal P, we say that Q is P**-primary**.

More generally, one can show that the radical of any ideal I is always an intersection of primes:

$$\sqrt{I} = \bigcap_{\substack{P \supseteq I \\ P \text{ is prime}}} P.$$

This intersection is usually quite massive, but we can improve it considerably by noting that if $P \subseteq Q$ are both primes containing I, then Q can be deleted from this intersection.

Definition 1.8. A prime P is a **minimal prime** of an ideal I is $P \supseteq I$ and P is minimal with respect to this property, meaning that if Q is a prime ideal and $I \subseteq Q \subseteq P$, we must have $Q = P$. We denote the set of minimal primes of I by $\operatorname{Min}(I)$.

Over a noetherian ring, every ideal has only finitely many minimal primes. Therefore, any radical ideal is actually the intersection of finitely many primes.

$$\sqrt{I} = \bigcap_{\substack{P \supseteq I \\ P \text{ is prime}}} P = \bigcap_{P \in \operatorname{Min}(I)} P.$$

Exercise 3. If the radical of an ideal I is maximal, then I is primary.

Note, however, that not all ideals with a prime radical are primary, as we will see in Example 1.19.

Exercise 4. Show that the primary ideals in $\mathbb{Z}$ are precisely the ideals of the form (p^n) for some $n > 1$ and some prime p.

This example can be a bit misleading, as it makes it semm like primary ideals are simply powers of a prime ideals — they are not! We will discuss this in more detail more soon.

Definition 1.9 (Irredundant Primary Decomposition)**.** A **primary decomposition** of the ideal I consists of primary ideals $Q_1, \ldots, Q_n$ such that $I = Q_1 \cap \cdots \cap Q_n$. An **irredundant** primary decomposition of I is one such that no Q_i can be omitted, and such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$.

Given a primary decomposition for I, we can always slightly modify it into an irredundant primary decomposition, by deleting unnecessary components and intersecting primary ideals with the same radical.

Exercise 5. Show that a finite intersection of P-primary ideals is a P-primary ideal.

As advertised, primary decompositions always exist, modulo a reasonable restriction: we need our ring to be noetherian.

Theorem 1.10 (Lasker [Las05], Noether [Noe21])**.** Every ideal in a noetherian ring has a primary decomposition.

Example 1.11. Here are some examples of primary decompositions:

a) The ideals in $\mathbb{Z}$ are all principal. Given any integer n, if we write n as a product of powers of distinct primes, say $n = p_1^{a_1} \cdots p_k^{a_k}$, then
$$(n) = (p_1^{a_1}) \cap \cdots \cap (p_k^{a_k})$$
is an irredundant primary decomposition for the ideal (n).

b) Whenever I is a radical ideal, we mentioned above that I coincides with the intersection of its minimal primes; there are finitely many such primes as long as our ambient ring is noetherian. Since prime ideals are primary, writing I as the intersection of its minimal primes gives a primary decomposition for I.

c) The ideal (xy, xz, yz) in $\mathbb{C}[x, y, z]$ is radical, so we just need to find its minimal primes. One can check that the decomposition is $(xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$. More generally, the monomial ideals that are radical are precisely those that are squarefree, and the primary components of a monomial ideal are also monomial.

d) Primary decompositions, even irredundant ones, are not unique. For example, over any field k, the ideal (x^2, xy) in $k[x, y]$ has infinitely many irredundant primary decompositions: given any $n \geqslant 1$, we have $(x^2, xy) = (x) \cap (x^2, xy, y^n)$. One thing all of these have in common is the radicals of the primary components: they are always (x) and (x, y). Notice also that the component with radical (x) also stays the same in all the examples, which is not a coincidence.

What information can we extract from a primary decomposition? Is there any sense in which primary decompositions are unique? What primes can appear as radicals of the primary components of I? Let's start with the last question: the prime ideals that appear are indeed interesting.

Definition 1.12 (Associated Prime)**.** Let M be an R-module. A prime ideal P is an **associated prime** of M if the following equivalent conditions hold:

(a) There exists a non-zero element $m \in M$ such that $P = \operatorname{ann}_R(m) := \{r \in R \mid rm = 0\}$.

(b) There is an inclusion of R/P into M.

If I is an ideal of R, we refer to an associated prime of the R-module R/I as simply an associated prime of I. We will denote the set of associated primes of I by $\operatorname{Ass}(R/I)$.

Remark 1.13. If you're not used to thinking about modules, the annihilator $\operatorname{ann}_R(m)$ of an element $m \in M$ is simply the set of elements in R that kill m, meaning those $r \in R$ with $rm = 0$. Requiring that P is the annihilator of some element in R/I is asking for some element $a + I$ in R/I such that

$$P = \{r \in R \mid r(a + I) = 0 \text{ in } R/I\},$$

which we can rephrase as saying that there exists some $a \in R$ such that

$$P = \{r \in R \mid ra \in I\}.$$

The annihilators of elements in R/I (or any module, for that matter) are not necessarily prime ideals, but those that actually are primes are especially interesting.

The philosophy is that the set of associated primes of an ideal or module should give us lots of information about the ideal or module in question. Over a noetherian ring, the set of associated primes of an ideal $I \neq 0$ is always non-empty and finite. Moreover, $\operatorname{Ass}(R/I) \subseteq \operatorname{Supp}(R/I)$, where $\operatorname{Supp}(M)$ denotes the support of the module M, meaning the set of primes p such that $M_p \neq 0$. In fact, the minimal primes of the support of R/I coincide with the minimal associated primes of I, which are precisely all the minimal primes of I.

As stated above, every minimal prime over I is actually an associated prime of I. If P is associated to I but not a minimal prime of I, we say that P is an **embedded prime** of I.

Exercise 6. Show that Q is P-primary if and only if $\operatorname{Ass}(R/Q) = \{P\}$.

Given an ideal I, we will be interested not only in its associated primes, but also in the associated primes of its powers. Fortunately, the set of prime ideals that are associated to some power of I is finite, a result first proved by Ratliff [Rat76] and then extended by Brodmann [Bro79].

Theorem 1.14 (Brodmann, 1979)**.** Let R be a noetherian domain and $I \neq 0$ an ideal in R. For n sufficiently large, $\operatorname{Ass}(R/I^n)$ is independent of n. In particular,

$$\bigcup_{n \geqslant 1} \operatorname{Ass}(R/I^n)$$

is a finite set.

Turns out that the associated primes of I are precisely the prime ideals that appear when we take an irredundant primary decomposition of I.

Theorem 1.15 (Uniqueness theorems)**.** Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of I, where Q_i is a P_i-primary ideal for each i. Then

$$\operatorname{Ass}(R/I) = \{P_1, \ldots, P_n\}.$$

Moreover, if P_i is minimal in $\operatorname{Ass}(R/I)$, then Q_i is unique, and given by

$$Q_i = IR_{P_i} \cap R,$$

where $- \cap R$ denotes the pre-image in R via the natural map $R \longrightarrow R_P$.

The notation R_P above refers to the localization of R at P. This is a standard construction in commutative algebra: we create a new ring from R by adding in inverses to all the elements outside of P, which results in a local ring with maximal ideal given by the image of P. Informally, think of R_P as zooming in at P. Formally, the elements in R_P are equivalence classes with representatives $\frac{r}{s}$, where $r, s \in R$ and $s \notin P$, with

$$\frac{r}{s} = \frac{u}{t} \text{ whenever } v(rt - su) \text{ for some } v \notin P.$$

When R is a domain, we can drop the v above, and simply ask if $rt = su$. There is a natural ring homomorphism $R \to R_P$ sending each $r \in R$ to $\frac{r}{1}$, and the notation IR_P above refers to the image of I under this map, meaning the ideal generated by all $\frac{a}{1}$ with $a \in I$. So whenever P is a minimal prime of I, Theorem 1.15 says that the P-primary component in any irredundant primary decomposition of I is

$$IR_P \cap R = \{r \in R \mid sr \in I \text{ for some } s \notin P\}.$$

Theorem 1.15 does not say anything about the embedded components of I. Indeed, the primary components corresponding to embedded primes are not necessarily unique.

Example 1.16. Let's look back at our last example, the ideal $I = (x^2, xy)$ in $k[x, y]$. All its irredundant primary decompositions must have precisely two components, one for each associated prime of I, which are (x) and (x, y). The minimal component is always (x) itself, since when we localize at (x), y becomes invertible and $I_{(x)} = (x)_{(x)}$. The other component is (x, y)-primary, since (x, y) is the only embedded prime of I. That embedded component, as we saw before, can now take many forms, such as (x^2, xy, y^n) for any n.

1.2 Symbolic powers: definition and basic properties

We are interested in studying the primary decompositions of powers of ideals.

Definition 1.17. Given an ideal I, the n**th power** of I is the ideal

$$I^n := (f_1 \cdots f_n \mid f_i \in I).$$

It is not difficult to show that given a set of generators for I, say $I = (g_1, \ldots, g_k)$, the nth power of I is generated by the n-fold products of the g_i.

Example 1.18. Let R be any ring and $f, g \in R$. Then $(f, g)^2 = (f^2, fg, g^2)$.

Exercise 7. Show that $\sqrt{I^n} = \sqrt{I}$ for any ideal I. In particular, if P is prime, then $\sqrt{P^n} = P$.

Exercise 8. Show that if P is a prime ideal, $\operatorname{Min}(P^n) = \{P\}$ for any $n \geqslant 1$.

Say we start with a prime ideal P. The powers of P are not necessarily primary; in fact, they usually are *not* primary.

Example 1.19. Consider a field k and an integer $n > 1$ and let $R = k[x, y, z]/(xy - z^n)$. The prime ideal $P = (x, z)$ in R satisfies

$$xy = z^n \in P^n, x \notin P^n \text{ and } y \notin \sqrt{P^n} = P.$$

This shows that P^n is not a primary ideal, even though its radical is the prime P.

Still, we can write a primary decomposition for P^n. Whenever P^n is not primary, it must necessarily have embedded components. We are, however, more interested in the minimal component of P^n.

Definition 1.20. Let P be a prime ideal. The **nth symbolic power** of P is the ideal

$$P^{(n)} := \{r \in R \mid sr \in P^n \text{ for some } s \notin P\}.$$

The following equivalent properties completely characterize the symbolic powers of a prime P:

- The n-th symbolic power of a P is the unique P-primary component in an irredundant primary decomposition of P^n, by Theorem 1.15.
- The n-th symbolic power of P is the smallest P-primary ideal containing P^n.

The equality $P^{(n)} = P^n$ is thus equivalent to the condition that P^n is a primary ideal.

Exercise 9. Show that if P is prime, $P^{(n)}$ is the smallest P-primary ideal containing P^n.

Exercise 10. Show that if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}^n = \mathfrak{m}^{(n)}$ for all n.

We can define symbolic powers much more generally. For technical reasons, we will focus only on radical ideals. As we will see on Day 2, this is a reasonable assumption, and to some extent the only situation that matters from a geometric point of view.

Definition 1.21 (Symbolic Powers)**.** Let I be a radical ideal in a a noetherian ring R. The **n-th symbolic power** of I is the ideal

$$I^{(n)} = \bigcap_{P \in \operatorname{Min}(I)} \left(I^n R_P \cap R \right).$$

So say that I is the radical ideal

$$I = P_1 \cap \cdots \cap P_k,$$

where $P_1, \ldots, P_k$ are the minimal primes of I. In fact, this is the unique irredundant primary decomposition of I. Now for elementary reasons — a simple exercise if you know the basics of localization — we have $IR_{P_i} = P_i R_{P_i}$ for each i, so our definition of symbolic powers can be rewritten as

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_k^{(n)}.$$

Remark 1.22. In the definition above, the assumption that I is radical implies in particular that it has no embedded primes, meaning that $\mathrm{Ass}(R/I) = \mathrm{Min}(I)$. We can use the exact same definition for any ideal with no embedded primes. However, when I does have embedded primes, we do have two distinct possible definitions for symbolic powers, given by intersecting $I^n R_P \cap R$ with P ranging over $\mathrm{Ass}(I)$ or $\mathrm{Min}(I)$. We will focus on ideals with no embedded primes, so this distinction is not relevant.

Both definitions have advantages. When we take P ranging over $\mathrm{Ass}(I)$, we get $I^{(1)} = I$, while taking P ranging over $\mathrm{Min}(I)$ means that $I^{(n)}$ coincides with the intersection of the primary components of I^n corresponding to its minimal primes.

Be warned that we will assume that I is radical throughout.

Exercise 11. Let I be a radical ideal in a noetherian ring R. Show the following basic properties hold:

(a) $I^{(1)} = I$.

(b) For all $n \geqslant 1$, $I^n \subseteq I^{(n)}$.

(c) $I^a \subseteq I^{(b)}$ if and only if $a \geqslant b$.

(d) If $a \geqslant b$, then $I^{(a)} \subseteq I^{(b)}$.

(e) For all $a, b \geqslant 1$, $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$.

(f) $I^n = I^{(n)}$ if and only if I^n has no embedded primes.

As (f) suggests, even if I has no embedded primes, I^n may still have some embedded primes, and in particular the converse containments to (b) and (d) do not hold in general. In particular, the symbolic powers of a prime ideal are not, in general, trivial:

Example 1.23. In Example 1.19 we considered $R = k[x, y, z]/(xy - z^n)$ and the prime ideal $P = (x, y)$. Rephrasing the same computation we did in Example 1.19, we can see that $P^n \neq P^{(n)}$. We have $xy = z^n \in P^n$ but $y \notin P$, so $x \in P^{(n)}$. However, $x \notin P^n$, so in fact $P^n \subsetneq P^{(n)}$.

The equality of ordinary and symbolic powers did not fail in this example because the ring is singular: this phenomenon happens even over a nice, well-behaved, polynomial ring.

Exercise 12. Consider the ideal $I = I_2(X)$ of 2×2 minors of a generic 3×3 matrix

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}$$

in the polynomial ring $R = k[X] = k\left[x_{i,j} \,|\, 1 \leqslant i, j \leqslant 3\right]$ generated by the variables in X over a field k. Show that $g = \det X \in P^{(2)}$, while $g \notin P^2$.

2 Day 2: Geometry

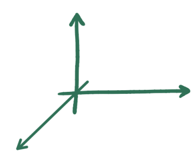
2.1 Nullstellensatz: a dictionary between algebra and geometry

One motivation to study symbolic powers is that they carry geometrical information. But first, we need to take a step back and talk about the classical dictionary between algebra and geometry given by Hilbert's Nullstellensatz.

Colloquially, we often identify systems of polynomial equations with their solution sets. We see the system of equations

$$\begin{cases} xy = 0 \\ yz = 0 \\ xz = 0 \end{cases}$$

and immediately think of the three coordinate lines



and vice-versa. This idea that we are so used to can be formalized, and it is the basis for the classical connection between commutative algebra and algebraic geometry. On the algebraic side, we study systems of equations — ideals! — while on the geometry side, we study solution sets — varieties. At the end of the day, we are all studying the same thing, just from a different perspective.

Definition 2.1. Let k be a field. For a subset T of $k[x_1, \ldots, x_d]$, we define $\mathcal{V}(T) \subseteq k^d$ to be the set of common zeros or the **zero set** of the polynomials (equations) in T:

$$\mathcal{V}(T) = \{(a_1, \ldots, a_d) \in k^d \mid f(a_1, \ldots, a_d) = 0 \text{ for all } f \in T\}.$$

A subset of k^d of the form $\mathcal{V}(T)$ for some subset T is called an **algebraic set**, or an **affine algebraic variety**, which we will often shorten to **variety**. In summary, a variety is the set of common solutions of some (possibly infinite) collection of polynomial equations. A variety is **irreducible** if it cannot be written as the union of two proper subvarieties.

Note that some authors use the word *variety* to refer only to irreducible algebraic sets. Note also that the definitions given here are only completely standard when k is algebraically closed, so we will focus on the more familiar case of $\mathbb{C}$.

In a reverse way, can also consider the equations that a subset of affine space satisfies.

Definition 2.2. Given any subset X of $\mathbb{A}^d_k$ for a field k, define

$$\mathcal{I}(X) = \{g(x_1, \ldots, x_d) \in k[x_1, \ldots, x_d] \mid g(a_1, \ldots, a_d) = 0 \text{ for all } (a_1, \ldots, a_d) \in X\}.$$

Exercise 13. $\mathcal{I}(X)$ is an ideal in $k[x_1, \ldots, x_d]$ for any $X \subseteq k^d$.

Exercise 14. Here are some properties of the functions $\mathcal{V}$ and $\mathcal{I}$:

a) For any field, we have $\mathcal{V}(0) = k^d$ and $\mathcal{V}(1) = \emptyset$.

b) $\mathcal{I}(\emptyset) = (1) = k[x_1, \ldots, x_d]$ (the improper ideal).

c) $\mathcal{I}(k^d) = (0)$ if and only if k is infinite.

d) If $I \subseteq J \subseteq k[x_1, \ldots, x_d]$ then $\mathcal{V}(I) \supseteq \mathcal{V}(J)$.

e) Given any subsets $S \subseteq T$ of k^d, we always have $\mathcal{I}(S) \supseteq \mathcal{I}(T)$.

f) If $I = (T)$ is the ideal generated by the elements of $T \subseteq k[x_1, \ldots, x_d]$, then $\mathcal{V}(T) = \mathcal{V}(I)$.

Thanks to Exercise 14 f, we can focus on the solution sets of ideals, rather than of an arbitrary collection of equations.

Example 2.3. Let

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

be a 2×3 matrix of variables — we usually call these *generic* matrices — and let

$$R = k[X] = k\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Let $\Delta_1, \Delta_2, \Delta_3$ the 2×2-minors of X. Consider the ideal $I = (\Delta_1, \Delta_2, \Delta_3)$. Thinking of these generators as equations, a solution to the system corresponds to a choice of 2×3 matrix whose 2×2 minors all vanish — that is, a matrix of rank at most one. So $\mathcal{V}(I)$ is the set of rank at most one matrices. Note that $I \subseteq (x_1, x_2, x_3) =: J$, and $\mathcal{V}(J)$ is the set of 2×3 matrices with top row zero. The containment $\mathcal{V}(J) \subseteq \mathcal{V}(I)$ we obtain from $I \subseteq J$ translates to the fact that a 2×3 matrix with a zero row has rank at most 1.

Note also that the union and intersection of varieties is also a variety.

Exercise 15. Suppose that I and J are ideals in $k[x_1, \ldots, x_d]$.

a) $\mathcal{V}(I) \cap \mathcal{V}(J) = \mathcal{V}(I + J)$.

b) $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J) = \mathcal{V}(IJ)$.

However, in general $IJ \neq I \cap J$.

Exercise 16. Find ideals I and J in $k[x_1, \ldots, x_d]$ such that

Some theorems about ideals in a polynomial ring now get a new geometric meaning.

Theorem 2.4 (Hilbert's Basis Theorem)**.** Given any field k, $k[x_1, \ldots, x_d]$ is a Noetherian ring.

Hilbert's Basis Theorem says that every ideal in $k[x_1, \ldots, x_d]$ is finitely generated, so any system of equations in $k[x_1, \ldots, x_d]$ can be replaced with a system of *finitely many* equations.

Given any ideal I, we can now construct a variety $\mathcal{V}(I)$, and given a variety X we can construct a variety X we can construct an ideal $\mathcal{I}(X)$. Some varieties are easy to calculate: for example, we all know that the system of equations

$$\begin{cases} x_1 = a_1 \\ \quad\vdots \\ x_d = a_d \end{cases}$$

has exactly one solution, the point $(a_1, \ldots, a_d)$, which we can now write in a more sophisticated way as $\mathcal{V}(x_1 - a_1, \ldots, x_d - a_d) = \{(a_1, \ldots, a_d)\}$. The ideal $(x_1 - a_1, \ldots, x_d - a_d)$ can easily be shown to be maximal in $k[x_1, \ldots, x_d]$. What is more interesting is that over $\mathbb{C}$, or more generally any algebraically closed field, there are no other maximal ideals. This gives us a bijective correspondence between maximal ideals in $\mathbb{C}[x_1, \ldots, x_d]$ and points in $\mathbb{C}^d$.

Theorem 2.5 (Nullstellensatz)**.** There is a bijection

$$\begin{array}{ccc} \mathbb{C}^d & \longrightarrow & \{\text{maximal ideals } \mathfrak{m} \text{ of } \mathbb{C}[x_1, \ldots, x_d]\} \\ (a_1, \ldots, a_d) & \longmapsto & (x_1 - a_1, \ldots, x_d - a_d) \end{array}$$

The maps $\mathcal{I}$ and $\mathcal{V}$ give us this bijection between points and maximal ideals. So we can start from the solution set — a point — and recover an ideal that corresponds to it. What if we start with some non-maximal ideal I, and consider its solution set $\mathcal{V}(I)$ — can we recover I in some way? Not so fast: many ideals define the same solution set.

Example 2.6. In $R = \mathbb{C}[x]$, the ideals $I_n = (x^n)$, for any $n \geqslant 1$, all define the same solution set $\mathcal{V}(I_n) = \{0\}$. In particular, $\mathcal{I}$ and $\mathcal{V}$ are not inverse constructions: $\mathcal{I}(\mathcal{V}(x^n)) = (x)$ for all $n \geqslant 1$.

Roughly speaking, the issue that makes $\mathcal{I}$ and $\mathcal{V}$ fail to be bijections is the fact that if f^n vanishes at a particular point $a \in k^d$, then f must also vanish at a. We do get a true bijection if we restrict to radical ideals.

Theorem 2.7 (Strong Nullstellensatz)**.** There is an order-reversing bijection between the collection of subvarieties of $\mathbb{C}$ and the collection of radical ideals of $R = \mathbb{C}[x_1, \ldots, x_d]$:

$$\begin{array}{ccc} \{\text{subvarieties of } k^d\} & \longleftrightarrow & \{\text{radical ideals } I \subseteq R\} \\ X & \xmapsto{\mathcal{I}} & \{f \in R \mid X \subseteq \mathcal{V}(f)\} \\ \mathcal{Z}_k(I) & \xleftarrow{\mathcal{Z}} & I \end{array}$$

In particular, given ideals I and J, we have $\mathcal{V}(I) = \mathcal{V}(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

The study of varieties in $\mathbb{C}^d$ is therefore the study of radical ideals in $\mathbb{C}[x_1, \ldots, x_d]$. Under this bijection, irreducible varieties correspond to prime ideals.

Exercise 17. A variety $X \subseteq k^d$ is irreducible if and only if $\mathcal{I}(X)$ is prime.

Given a variety X, we can decompose it in irreducible components by writing it as a union $X = V_1 \cup \cdots \cup V_n$. We can do this decomposition algebraically, by considering the radical ideal $I = \mathcal{I}(X)$ and writing it as an intersection of its minimal primes. The fact that we mentioned before that every ideal has finitely many minimal primes now translates into saying that every variety can be written as a finite union of finitely many irreducible varieties. More precisely, we can write $\mathcal{I}(X)$ as a finite intersection of prime ideals, say

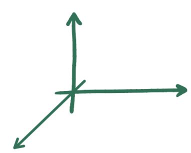
$$\mathcal{I}(X) = P_1 \cap \cdots \cap P_k,$$

and then

$$X = \mathcal{V}(P_1) \cup \cdots \cup \mathcal{V}(P_k)$$

is a decomposition of X into irreducible components.

Example 2.8. Formalizing the example we started from, the radical ideal $I = (xy, xz, yz)$ in $k[x, y, z]$ corresponds to the variety X given by the union of the three coordinate axes.



Each of these axes is a variety in its own right, corresponding to the ideals (x, y), (x, z) and (y, z). The three axes are the irreducible components of X. And indeed, (x, y), (x, z) and (y, z) are the three minimal primes of I, and

$$(xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z).$$

In summary, we have the following dictionary between varieties and ideals:

<u>Algebra</u>	$\longleftrightarrow$	<u>Geometry</u>
algebra of ideals	$\longleftrightarrow$	geometry of varieties
algebra of $R = k[x_1, \ldots, x_d]$	$\longleftrightarrow$	geometry of k^d
radical ideals	$\longleftrightarrow$	varieties
prime ideals	$\longleftrightarrow$	irreducible varieties
maximal ideals	$\longleftrightarrow$	points
(0)	$\longleftrightarrow$	variety k^d
$k[x_1, \ldots, x_d]$	$\longleftrightarrow$	variety $\varnothing$
$(x_1 - a_1, \ldots, x_d - a_d)$	$\longleftrightarrow$	point $\{(a_1, \ldots, a_d)\}$

$$\begin{array}{ccc} \text{smaller ideals} & \longleftrightarrow & \text{larger varieties} \\ \text{larger ideals} & \longleftrightarrow & \text{smaller varieties} \end{array}$$

Example 2.9. Consider the curve in $\mathbb{C}^3$ parametrized by (t^3, t^4, t^5), meaning

$$C = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\}.$$

This curve is a variety! It corresponds to the ideal of polynomials $f(x, y, z) \in R = \mathbb{C}[x, y, z]$ that vanish along C, which is to say, at every point in C. We can compute $\mathcal{I}(C)$ explicitly by considering the map

$$\begin{tikzcd} \mathbb{C}[x,y,z] \arrow[r, "\psi"] & \mathbb{C}[t] \\ x \arrow[r, mapsto] & t^3 \\ y \arrow[r, mapsto] & t^4 \\ z \arrow[r, mapsto] & t^5 \end{tikzcd}$$

and noting that the polynomials that vanish along C are precisely those in the kernel of ψ. So our curve corresponds to the prime ideal

$$P = \ker \psi = \left(x^2 y - z^2, xz - y^2, yz - x^3\right).$$

2.2 Back to symbolic powers with Zariski–Nagata

What does this old geometry story have to do with symbolic powers? First, it helps to think of Nullstellensatz in yet another format.

Theorem 2.10 (Nullstellensatz again)**.** Let I be a radical ideal in $\mathbb{C}[x_1, \ldots, x_d]$. Then

$$I = \bigcap_{(a_1,\ldots,a_d)\in\mathcal{V}(I)} (x_1 - a_1, \ldots, x_d - a_d) = \bigcap_{\substack{\mathfrak{m}\supseteq I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}.$$

This is just a fancier way of saying that I is the set of polynomials that vanish at every point in $\mathcal{V}(I)$, though by noting that the maximal ideals containing I are precisely the maximal ideals corresponding to points in $\mathcal{V}(I)$, we can avoid any mention to the variety corresponding to I, and write a statement that is purely algebraic.

While the elements in a radical ideal I are all the polynomials that vanish on $\mathcal{V}(I)$, not all those polynomials vanish along X *equally*; roughly speaking, some polynomials vanish more than others. You probably already have an intuitive idea of what this should mean: for example, the polynomial x^2 vanishes *more* at the point 0 than the polynomial x.

Theorem 2.11 (Zariski–Nagata)**.** Let I be a radical ideal in $\mathbb{C}[x_1, \ldots, x_d]$. Then

$$I^{(n)} = \bigcap_{\substack{\mathfrak{m}\supseteq I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}^n.$$

Think of this theorem as a higher order version of Nullstellensatz in its 2.10 format. Zariski–Nagata tells us that the polynomials in $I^{(n)}$ are those that vanish **to order** n along the variety that I defines.

Example 2.12. Let's get back to the curve in example Example 2.9, parametrized by (t^3, t^4, t^5) and corresponding to the ideal

$$P = \left(\underbrace{x^2y - z^2}_{f}, \underbrace{xz - y^2}_{g}, \underbrace{yz - x^3}_{h} \right)$$

in $R = \mathbb{C}[x, y, z]$. We will see that $P^{(2)} \neq P^2$, meaning that there are polynomials that vanish to order 2 along C that are not in P^2.

First, consider a non-standard grading on R under which the map ψ in Example 2.9 is a degree 0 map, and I is a homogeneous ideal: give x degree 3, y degree 4, and z degree 5. With this grading, $\deg(f) = 10$, $\deg(g) = 8$ and $\deg(h) = 9$, and the polynomial $fg - h^2$ is homogeneous of degree 18. Note that $fg - h^2 = xq$, for some q of degree $18 - 3 = 15$. Since $x \notin P$ and $fg - h^2 \in P^2$, we have $q \in P^{(2)}$. However, since all elements in P have degree at least 8, then all elements in P^2 must have degree at least 16, so that $q \notin P^2$. We conclude that $P^2 \neq P^{(2)}$.

This phenomenon that the symbolic powers of a radical ideal can be different from the powers is ubiquitous: in general, we should expect to find polynomials vanishing to order n on a variety X that are not in $\mathcal{I}(X)^n$. On the other hand, we do always have $I^n \subseteq I^{(n)}$. Another common phenomenon is that we can find elements vanishing to order n along X that live in interesting degrees — degrees that we would not find in $\mathcal{I}(X)^n$.

Example 2.13. Let's get back to the union of the three coordinate lines in Example 2.8, which corrresponds to the radical ideal

$$I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z).$$

To compute its symbolic powers, we compute the symbolic powers of each of its minimal primes. These turn out to be quite simple: since (x, y), (x, z), and (y, z) are each generated by some subset of the variables, their symbolic and ordinary powers actually coincide, as we will later discuss. Therefore,

$$I^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2.$$

Now note that $I^{(2)}$ contains the element xyz, of degree 3, but

$$I^2 = (x^2, y^2, z^2, xy, xz, yz)$$

is generated by elements of degree 4, so $xyz \notin I^2$. So once again, we found an ideal with $I^2 \subsetneq I^{(2)}$. In fact, one can show that 3 is the smallest degree of a polynomial vanishing to order 2 along the three coordinate lines.

3 Day 3: Open Problems

There are many innocent sounding questions about symbolic powers that turn out to be very difficult, and that are essentially wide open. We are going to discuss some of these questions, starting from sometimes untreatable, unreasonable questions, and distilling them into proper research questions that have sparked lots of active research.

The main obstruction to solving problems on symbolic powers is that actually computing examples can be extremely difficult — for starters, because finding a primary decomposition for a given ideal is a hard computational problem.

3.1 The (in)equality of symbolic and ordinary powers

While computing symbolic powers is difficult in general, computing powers is in contrast quite simple. So when is $I^{(n)} = I^n$? There are different ways we can make this question precise, starting with asking for sufficient conditions on I that guarantee equality of *all* symbolic and ordinary powers of I. The most elementary result on equality of symbolic and ordinary powers of ideals is about complete intersections.

Definition 3.1. Let R be a ring and M be an R-module. An element $r \in R$ is **regular** (or a nonzerodivisor) on an R-module M if $rm = 0$ for some $m \in M$ implies $m = 0$. More generally, a sequence of elements $x_1, \ldots, x_n$ is a **regular sequence on** M if

- $(x_1, \ldots, x_n)M \neq M$, and
- for each i, x_i is regular on $M/(x_1, \ldots, x_{i-1})M$.

In particular, elements $x_1, \ldots, x_n$ form a regular sequence on R if

- $(x_1, \ldots, x_n) \neq R$, and
- for each i, x_i is regular on $R/(x_1, \ldots, x_{i-1})M$.

If an ideal I is generated by a regular sequence, we say that I is a **complete intersection**.

Remark 3.2. Every element in the ideal $(x_1, \ldots, x_i)$ sends x_{i+1} inside $(x_1, \ldots, x_i)$. Saying that x_{i+1} is regular on $R/(x_1, \ldots, x_i)$ is equivalent to saying that there are no other elements sending x_{i+1} inside $(x_1, \ldots, x_i)$ but the trivial ones.

The example to keep in mind is an ideal generated by variables in a polynomial ring.

Example 3.3.

a) The variables $x_1, \ldots, x_n$ form a regular sequence on the polynomial ring $R = k[x_1, \ldots, x_n]$ in n variables over a field k. In fact, any ideal in R generated by a subset of the variables is a complete intersection.

b) Let k be a field and $R = k[x, y, z]$. The sequence xy, xz is not regular on R, since xz kills y in $R/(xy)$.

We can describe complete intersections from a more geometric perspective. Given a radical ideal I in $\mathbb{C}[x_1, \ldots, x_d]$, consider the codimension c of the corresponding variety $\mathcal{V}(I)$, that is,

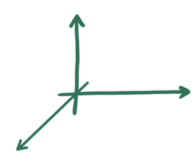
$$c = d - \text{dimension of } \mathcal{V}(I).$$

It turns out this number matches another algebraic invariant you may have heard of — the **height** of I, which is defined in terms of lengths of chains of prime ideals. We can compare c to the number of generators we need to define I. If I is generated by n elements, then there is a theorem that says that $c \leqslant n$. Equality happens precisely when I is a complete intersection.

Theorem 3.4. If I is a complete intersection in a Cohen-Macaulay ring, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

The fine print is not important here: our favorite rings, the polynomial rings over fields, are all Cohen-Macaulay.

Example 3.5. In our favorite example $I = (xy, xz, yz)$, the variety $\mathcal{V}(I)$ is the union of 3 lines, so it has dimension 1:



Therefore, the codimension of I is $3 - 1 = 2$. If you know a bit about dimension theory, you will quickly recognize that indeed the ideal I has height 2. On the other hand, I cannot be generated by less than the 3 generators we presented, so I is not a complete intersection. And indeed, we saw back in Example 2.13 that $I^{(2)} \neq I^2$.

Nevertheless, Theorem 3.4 is very far from an if and only if.

Example 3.6. The prime P defining the Veronese ring $k[s^2, st, t^2]$, that is, the kernel of the map

$$\begin{array}{ccc} k[a,b,c,d] & \longrightarrow & k[s^3, s^2t, st^2, t^3] \\ a & \longmapsto & s^3 \\ b & \longmapsto & s^2t \\ c & \longmapsto & st^2 \\ d & \longmapsto & t^3 \end{array}$$

is not a complete intersection: it has codimension $4 - 2 = 2$, but it is minimally generated by 3 elements,

$$P = (ad - bc, c^2 - bd, b^2 - ac).$$

Nevertheless, $P^{(n)} = P^n$ for all n, a fact which can be found in [HH92, Example 4.4], and that also follows as a special case of [Sch91] and [GH19, Corollary 4.3].

In general, the question of when the symbolic and ordinary powers of a given ideal coincide is very difficult. Hochster [Hoc73] gave conditions on I that are equivalent to $I^{(n)} = I^n$ for all $n \geqslant 1$ when I is prime, and those conditions were extended by Li and Swanson [LS06] to the case when I is a radical ideal. However, even thought these equivalent conditions hold over any noetherian ring, they are not easy to check in practice, nor to describe here.

Problem 1. Let $R = k[x_1, \dots, x_d]$. Can we completely describe the class of radical ideals I that satisfy $I^{(n)} = I^n$ for all $n \geqslant 1$? Is there an invariant d depending on the ring d or the ideal I such that $I^{(n)} = I^n$ for all $n \leqslant d$ (or for $n = d$) implies that $I^{(n)} = I^n$ for all $n \geqslant 1$?

There are some settings under which this is understood. The following is [Hun86, Corollary 2.5], which involves some technical terms that may be unfamiliar:

Theorem 3.7 (Huneke, 1986)**.** Let R be a regular local ring of dimension 3, and P a prime ideal in R of height 2. The following are equivalent:

(a) $P^{(n)} = P^n$ for all $n \geqslant 1$;

(b) $P^{(n)} = P^n$ for some $n \geqslant 2$;

(c) P is generated by a regular sequence.

As a corollary, if P is a homogeneous prime ideal in $k[x, y, z]$ of codimension 2, we have $P^{(n)} = P^n$ for all n exactly when P is a complete intersection, and $P^{(n)} \neq P^n$ for all $n \geqslant 2$ whenever P has at least 3 generators. In dimension higher than 3, we can find prime ideals that are not generated by regular sequences but whose symbolic and ordinary coincide nevertheless, such as Example 3.6.

There are some situations where we can guarantee $I^{(n)} = I^n$ always holds for small values of n, even if it fails for large n.

Theorem 3.8 (see Theorem 2.3 in [CFG+16], see also [Mor99, HU89])**.** Let $R = k[x_0, \dots, x_n]$ be a polynomial ring over a field k. Let I be a height 2 ideal in R such that R/I is Cohen-Macaulay and such that I_P is generated by a regular sequence for all primes $P \neq (x_0, \dots, x_n)$ containing I. Then $I^{(k)} = I^k$ for all $k < n$ regardless of the minimal number of generators of I. Moreover, the following statements are equivalent:

(a) $I^{(k)} = I^k$ for all $k \geqslant 1$;

(b) $I^{(n)} = I^n$;

(c) I is generated by at most n elements.

Remark 3.9. Notice that if P is a height 2 prime ideal in a polynomial ring in 3 variables, meaning that $n = 2$ in the statement of Theorem 3.8, then the conclusions of Theorems 3.7 and 3.8 coincide, although Theorem 3.7 also adds the equivalence with condition

(d) $I^{(k)} = I^k$ for some $k \geqslant 2$;

This suggests that Theorem 3.8 might hold if we add condition (d) to the equivalences stated.

Often in commutative algebra, when one studies monomial ideals — ideals generated by monomials, such as our old friend (xy, xz, yz) — things tend to get a bit more treatable, since we can often apply combinatorial techniques. Things get even better when I is generated by monomials given by products of distinct variables, in which case we say I is a **squarefree monomial ideal**. However, there is still no characterization of the squarefree monomial ideals that satisfy $I^{(n)} = I^n$ for all $n \geqslant 1$. It is conjectured that this condition is equivalent to I being packed.

Definition 3.10 (König ideal)**.** Let I be a squarefree monomial ideal of height c in a polynomial ring over a field. We say that I **könig** if I contains a regular sequence of monomials of length c.

Despite the fact that all squarefree monomial ideals do contain a regular sequence of length equal to their height, not all squarefree monomial ideals are könig.

Exercise 18. Show that (xy, xz, yz) is not könig.

Definition 3.11 (Packed ideal)**.** A squarefree monomial ideal of height c is said to be **packed** if every ideal obtained from I by setting any number of variables equal to 0 or 1 is könig.

Exercise 19. Give an example of an ideal that is packed and of one that is not packed.

The following is a restatement by Gitler, Valencia, and Villarreal in the setting of symbolic powers of a conjecture of Conforti and Cornuéjols about max-cut min-flow properties.

Conjecture 3.12 (Packing Problem)**.** Let I be a squarefree monomial ideal in a polynomial ring over a field k. The symbolic and ordinary powers of I coincide if and only if I is packed.

The Packing Problem has been solved for the case when I is the edge ideal of a graph [GVV07].

Definition 3.13 (Edge ideal)**.** Let G be a simple graph on n vertices $\{v_1, \ldots, v_n\}$. Given a field k, the *edge ideal* of G in $k[x_1, \ldots, x_n]$ is the ideal

$$I = \left(x_i x_j \;\middle|\; \text{if there is an edge between the vertices } v_i \text{ and } v_j\right).$$

Theorem 3.14 (Gitler–Valencia–Villareal, [GVV07])**.** Let I be the edge ideal of a graph G. The following are equivalent:

(a) G is a bipartite graph;

(b) $I^{(n)} = I^n$ for all $n \geqslant 1$;

(c) I is packed.

While the more general version of the Packing Problem is still open, the question of whether it is sufficient to test $I^{(n)} = I^n$ only for finitely many values of n is settled for monomial ideals.

Theorem 3.15 (Núñez Betancourt – Montaño [MnNnB19])**.** Let I be a squarefree monomial ideal generated by μ elements. If $I^{(n)} = I^n$ for all $n \leqslant \frac{\mu}{2}$, then $I^{(n)} = I^n$ for all $n \geqslant 1$.

3.2 What is the degree of an element in $I^{(n)}$?

When I is a homogeneous ideal in a graded ring, the symbolic powers of I are also homogeneous ideals. We have used this idea informally before a few times, when we showed that certain elements could not possibly be in a given symbolic power because of degree reasons. Both in Example 2.12 and Example 2.13, we found elements in a symbolic power that lived in an unexpectedly small degree. It is then natural to ask what is the minimal degree of an element in $I^{(n)}$ for each n.

The situation in $\mathbb{C}[x_1, \ldots, x_d]$ is already interesting enough. When we talk about homogeneous ideals in $\mathbb{C}[x_1, \ldots, x_d]$, we often think of a slightly different geometric perspective than the one we described before. Rather than thinking of the affine space $\mathbb{C}^d$, we consider the projective space $\mathbb{P}^d_{\mathbb{C}}$, which we may abbreviate to $\mathbb{P}^d$, which we obtain from $\mathbb{C}^{d+1} \setminus \{0\}$ by identifying all points in the same line through the origin. More precisely, we identify the point $(a_0, \ldots, a_d) \in \mathbb{C}^{d+1} \setminus \{0\}$ with any point of the form $(\lambda a_0, \ldots, \lambda a_d)$ for some $\lambda \neq 0$, and represent the equivalence class of $(a_0, \ldots, a_d)$ by $[a_0 : \cdots : a_d]$.

There is also a notion of projective varieties, which now require us to consider homogeneous polynomials in $\mathbb{C}[x_1, \ldots, x_d]$. A **projective variety** is any set of points in $\mathbb{P}^d$ that contains precisely the solutions to some system of **homogeneous** polynomials in $\mathbb{C}[x_1, \ldots, x_d]$. A polynomial f is **homogeneous** if it can be written as a sum of monomials all in the same degree n, or equivalently, that satisfy

$$f(\lambda x_1, \ldots, \lambda x_d) = \lambda^n f(x_1, \ldots, x_d)$$

for any λ. An ideal generated by homogeneous polynomials is called a **homogeneous ideal**. There is a bijection

$$\begin{array}{ccc} \{\text{subvarieties of } \mathbb{P}^d\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{homogenous radical ideals} \\ I \neq (x_0, \ldots, x_d) \text{ in } \mathbb{C}[x_0, \ldots, x_d] \end{array} \right\}. \\ & & \\ \{(a_0, \ldots, a_d)\} & \longleftrightarrow & (a_i x_j - a_j x_i \mid 0 \leqslant i, j \leqslant d) \end{array}$$

A lot of the rules we saw in the affine case also apply here; for example, a finite set of points will correspond to the intersection of the corresponding ideals.

Definition 3.16. Given a homogeneous ideal I in $k[x_1, \ldots, x_d]$, $\alpha(I)$ denotes the minimal degree of a homogeneous element in I.

We are interested in the values of $\alpha(I^{(n)})$, how $\alpha(I^{(n)})$ grows with n, and how small it can be. If I corresponds to a finite set of points in $\mathbb{P}^N$, $\alpha(I)$ is the smallest degree of a polynomial vanishing on each of our points, or equivalently, the smallest degree of a hypersurface passing through each of the given points, while $\alpha(I^{(n)})$ is the smallest degree of a hypersurface passing through each of the given points with multiplicity n.

Conjecture 3.17 (Nagata [Nag65])**.** If I defines $n \geqslant 10$ general points in $\mathbb{P}^2_{\mathbb{C}}$,

$$\alpha(I^{(m)}) > m\sqrt{n}.$$

This question remains open except for some special cases.

Conjecture 3.18 (Chudnovsky, [Chu81]). Let X be a finite set of points in $\mathbb{P}^N$, and $I = I(X)$ be the corresponding radical ideal in $k[x_0, \dots, x_N]$. Then

$$\frac{\alpha(I^{(m)})}{m} \geqslant \frac{\alpha(I) + N - 1}{N}.$$

Turns out that the limit of the left hand side exists and equals the infimum on the same set. More precisely,

$$\hat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m} = \inf_m \frac{\alpha(I^{(m)})}{m}.$$

We can restate Chudnovsky's conjecture in terms of this constant $\hat{\alpha}$, known as the **Waldschmidt constant** of I. More precisely, Chudnovsky's conjecture asks if

$$\hat{\alpha}(I) \geqslant \frac{\alpha(I) + N - 1}{N}.$$

This conjecture has been shown for finite sets of very general points in $\mathbb{P}_k^N$ as long as k is an algebraically closed field [FMX18, Theorem 2.8], and also for sufficiently large sets of general points. These conditions refer to the fact that the conjecture holds over some reasonably large set. For each fixed s, there is a space called the Hilbert scheme that parametrizes all the sets of s many points in $\mathbb{P}^d$, with a certain topology; a statement holds for a **general** set of s points if there exists an open dense set in the Hilbert scheme of s points for which the statement holds.

Theorem 3.19 (Bisui–Grifo–Hà–Nguyễn, [BGHN20a]). Chudnovsky's Conjecture holds for a general set of $s \geqslant 4^N$ points in $\mathbb{P}^N$.

In fact, the theorem holds over any algebraically closed field of any characteristic. Chudnovsky's bound remains open for the case of an arbitrary set of finitely many points. One might wonder if we can extend this to any homogeneous ideal, perhaps by substituting N by a more appropriate invariant.[1] Such a result has been shown to hold for squarefree monomial ideals [BCG⁺16, Theorem 5.3] and other classes of ideals with nice properties [BGHN20b]. Chudnovsky's Conjecture is a special case of a more general conjecture by Demailly [Dem82].

3.3 The Eisenbud–Mazur Conjecture

While $I^{(2)} \subseteq I$ always holds, we may wonder whether $I^{(2)}$ may contain a minimal generator of I.

Conjecture 3.20 (Eisenbud–Mazur [EM97]). Let $(R, \mathfrak{m})$ be a localization of a polynomial ring over a field k of characteristic 0. If I is a radical ideal in R, then $I^{(2)} \subseteq \mathfrak{m}I$.

This fails in prime characteristic:

[1] The most appropriate invariant is likely to be something called the **big height**.

Example 3.21 (Eisenbud–Mazur [EM97]). Let p be a prime integer, and let I be the kernel of the map

$$\begin{aligned} \mathbb{F}_p[x_1, x_2, x_3, x_4] &\longrightarrow \mathbb{F}_p[t] \\ x_1 &\longmapsto t^{p^2} \\ x_2 &\longmapsto t^{p(p+1)} \\ x_3 &\longmapsto t^{p^2+p+1} \\ x_4 &\longmapsto t^{(p+1)^2}. \end{aligned}$$

Consider the polynomial $f = x_1^{p+1} x_2 - x_2^{p+1} - x_1 x_3^p + x_4^p \in I$. Note that f is a quasi-homogeneous polynomial, and in fact $f \in I^{(2)}$. To see that, consider

$$\begin{aligned} g_1 &= x_1^{p+1} - x_2^p \in I, \\ g_2 &= x_1 x_4 - x_2 x_3 \in I, \\ g_3 &= x_1^p x_2 - x_3^p \in I, \end{aligned}$$

and note that

$$x_1^p f = g_1 g_3 + g_2^p \in I^2.$$

We also claim that f is in fact a minimal generator of I, meaning $f \notin (x_1, x_2, x_3, x_4)I$. To do this, one can show that no element of I contains a term of the form x_4^a for any $1 \leqslant a < p$, and since I is generated by binomials, it suffices to show there is no element of the form $x_4^a - x_3^b x_2^c x_1^d$ in I. We leave this as an exercise.

The Eisenbud–Mazur Conjecture also fails if the ring is not regular. It is still open in most cases over fields of characteristic 0.

Exercise 20. Show the Eisenbud–Mazur conjecture for squarefree monomial ideals.

More generally, Eisenbud and Mazur showed that if I in a monomial ideal and P is a monomial prime containing I, then $I^{(d)} \subseteq PI^{(d-1)}$ for all $d \geqslant 1$ [EM97, Proposition 7]. They also show Conjecture 3.20 for licci ideals [EM97, Theorem 8] and quasi-homogeneous unmixed ideals in equicharacteristic 0 [EM97, Theorem 9]. For more on the status of this conjecture, see [DDSG+18, Section 2.3].

3.4 Symbolic Rees algebras

The symbolic powers of an ideal I form a graded family, meaning that $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all a and b. This simple property allows us to package them together in a single graded object, called the symbolic Rees algebra (or symbolic blowup) of I.

Definition 3.22 (Symbolic Rees algebra). Let R be a ring and I an ideal in R. The **symbolic Rees algebra** of I is the graded algebra

$$\mathcal{R}_s(I) := \bigoplus_{n \geqslant 0} I^{(n)} t^n \subseteq R[t],$$

where the indeterminate t is used to keep track of the grading.

This mimics the construction of the usual Rees algebra of I. But unlike the Rees algebra, the symbolic Rees algebra may fail to be a noetherian ring.

Exercise 21. Show that the symbolic Rees algebra of an ideal I in a ring R is a finitely generated R-algebra if and only if it is a noetherian ring.

The symbolic Rees algebra of I is noetherian, or equivalently finitely generated, if the symbolic powers of I can all be written in terms of a fixed finite set of symbolic powers. More precisely, $\mathcal{R}_s(I)$ if and only if there exists d such that every symbolic power of I can be written in terms of $I, I^{(2)}, \ldots, I^{(d)}$, meaning that

$$I^{(n)} = \sum_{a_1+2a_2\cdots+da_d=n} I^{a_1}\left(I^{(2)}\right)\cdots\left(I^{(d)}\right)$$

for all $n \geqslant 1$. Note that the product on the right is always contained in $I^{(n)}$, so we can think of the elements on the right side as expected elements of $I^{(n)}$. If the symbolic Rees algebra of I is *not* finitely generated, that means we will see completely unexpected elements in arbitrarily high symbolic powers of I.

Exercise 22. If the symbolic Rees algebra of an ideal I in a ring R is finitely generated, show that there exists k such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. The converse also holds as long as R is excellent.

Which ideals do have a noetherian symbolic Rees algebra? For example, the symbolic Rees algebra of a monomial ideal is noetherian [Lyu88, Proposition 1]. What is maybe more surprising is that symbolic Rees algebras are often not finitely generated. The first example of this is due to Rees [Ree58], and Roberts showed this can happen even when R is a regular ring [Rob85], building on Nagata's counterexample to Hilbert's 14th Problem [Nag65]. Roberts' example gave a negative answer to the following question of Cowsik:

Question 3.23 (Cowsik)**.** Let P be a prime ideal in a regular ring R. Is the symbolic Rees algebra of P always a noetherian ring, or equivalently, a finitely generated R-algebra?

Cowsik's motivation was a result of his [Cow84] showing that a positive answer would imply that all such primes are set-theoretic complete intersections, that is, complete intersections up to radical. Eliahou, Huckaba, Huneke, Vasconcelos and others proved various criteria that imply noetherianity. Space monomial curves (t^a, t^b, t^c), however, were known to be set-theoretic complete intersections [Bre79, Her80, Val81], and much work was devoted to studying their symbolic Rees algebras. Surprisingly, the answer to Cowsik's question is negative even for this class of primes, with the first non-noetherian example found in [GNW94]. In [Cut91], Cutkosky gives criteria for the symbolic Rees algebra of a space monomial curve to be noetherian, and in particular shows that the symbolic Rees algebra of $k[t^a, t^b, t^c]$ is noetherian when $(a+b+c)^2 > abc$. There is a vast body of literature on the case of ideals defining space monomial curves (t^a, t^b, t^c) alone [Cut91, Mor91, GNS91b, GM92, GNW94, GNS91a, HU90, Sri91].

3.5 The Containment Problem

If $I^n \neq I^{(n)}$, how different are they? Can we compare them in some reasonable way?

Problem 2 (Containment Problem)**.** Let R be a noetherian ring and I be an ideal in R. When is $I^{(a)} \subseteq I^b$?

This packages together a few different questions. First, it contains the equality problem, since the b-th symbolic and ordinary powers coincide if and only if $I^{(b)} \subseteq I^b$. When the answer is no, the containment problem is a way to measure how far the symbolic and ordinary powers are from each other. If we do have a particular answer to the containment problem, say $I^{(a)} \subseteq I^b$, and our ideal I is homogeneous, we automatically gain lower bounds for the degrees of the elements in $I^{(a)}$. Indeed, the containment $I^{(a)} \subseteq I^b$ implies

$$\alpha\left(I^{(a)}\right) \geqslant \alpha\left(I^b\right) = b\alpha(I).$$

In general, the Containment Problem can be quite difficult, although we can answer it completely if we have an explicit description of both the symbolic and ordinary powers of our ideal. Having an explicit description of symbolic powers, however, is fairly rare.

But does Question 2 always make sense? That is, given b, must there exist an a such that $I^{(a)} \subseteq I^b$? If so, then the two graded families of ideals $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal, and thus induce equivalent topologies. In 1985, Schenzel [Sch85] gave a characterization of when $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal. In particular, if I is a radical ideal in $R = k[x_1, \ldots, x_d]$, then $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are indeed cofinal. Schenzel's characterization did not, however, provide information on the relationship between a and b.

It was not until the late 90s that Irena Swanson showed that the I-adic and I-symbolic topologies are equivalent if and only if they are linearly equivalent.[2]

Theorem 3.24 (Swanson, 2000, [Swa00])**.** Let R be a noetherian ring, and I and J two ideals in R. The following are equivalent:

(i) $\{I^n : J^\infty\}$ is cofinal with $\{I^n\}$.

(ii) There exists an integer c such that $(I^{cn} : J^\infty) \subseteq I^n$ for all $n \geqslant 1$.

In particular, given a radical ideal in a regular ring — for example, over $k[x_1, \ldots, x_d]$ — there exists an integer c such that $I^{(cn)} \subseteq I^n$ for all $n \geqslant 1$. More surprisingly, over a regular ring this constant can be taken uniformly, meaning depending only on R.

Definition 3.25 (Big height)**.** Let I be an ideal with no embedded primes. The **big height**[3] of I is the maximal height of an associated prime of I. If the big height of I coincides with the height of I, meaning that all associated primes of I have the same height, we say that I has **pure height**.

So the big height of a radical ideal in $\mathbb{C}[x_1, \ldots, x_d]$ is the largest codimension of an irreducible component of the corresponding variety.

[2] Word of caution: the words *linearly equivalent* have been used in the past to refer to other condition. For example, Schenzel used this term to refer to $I^{(n+k)} \subseteq I^n$ for all $n \geqslant 1$ and some constant k.

[3] According to google, 6'2".

Theorem 3.26 (Ein–Lazarsfeld–Smith, Hochster–Huneke, Ma–Schwede [ELS01, HH02, MS18a])**.** Let R be a excellent regular ring and I a radical ideal in R. If h is the big height of I, then

$$I^{(hn)} \subseteq I^n$$

for all $n \geqslant 1$.

This theorem applies, for example, to polynomial rings over in finitely many variables over a field, $\mathbb{Z}$, or the p-adics.

Remark 3.27. The conclusion of Theorem 3.26 is equivalent to $I^{(n)} \subseteq I^{\left\lfloor \frac{n}{h} \right\rfloor}$ for all $n \geqslant 1$.

We cannot replace big height by height in Theorem 3.26.

Example 3.28. Consider the ideal

$$I = (x, y) \cap (y, z) \cap (x, z) \cap (a) = (xya, xza, yza) \subseteq k[x, y, z, a],$$

which has height 1 and big height 2. If we replaced big height by height in Theorem 3.26, we would have $I^{(n)} = I^n$ for all $n \geqslant 1$. However, similarly to Example 2.13, $I^{(2)} \neq I^2$. Indeed, note that

$$xyza^2 \in I^{(2)} = (x, y)^2 \cap (y, z)^2 \cap (x, z)^2 \cap (a)^2,$$

whereas all elements in I^2 must have degree at least 6.

Exercise 23. Given integers $c < h$, construct an ideal I with height c and big height h in a polynomial ring such that $I^{(cn)} \not\subseteq I^n$ for some n

Ein, Lazarsfeld, and Smith first proved Theorem 3.26 in the equicharacteristic 0 geometric case, using multiplier ideals, a tool from algebraic geometry. Hochster and Huneke then used reduction to characteristic p and tight closure techniques to prove the result in the equicharacteristic case, where our ring contains a field. Recently, Ma and Schwede built on ideas used in the recent proof of the Direct Summand Conjecture to define a mixed characteristic analogue of multiplier/test ideals, allowing them to deduce the mixed characteristic version of Theorem 3.26.

Given an ideal I and $t \geqslant 0$, the multiplier ideal $\mathcal{J}(R, I^t)$ measures the singularities of $V(I) \subseteq \operatorname{Spec}(R)$, scaled by t in a certain sense; we refer to [ELS01, MS18a] for the definition. The proof of Theorem 3.26 in the characteristic 0 case relies on a few key properties of multiplier ideals:

- $I \subseteq \mathcal{J}(R, I)$;
- For all $n \geqslant 1$, $\mathcal{J}\left(R, \left(P^{(nh)}\right)^{\frac{1}{n}}\right) \subseteq P$ whenever P is a prime of height h;
- For all integers $n \geqslant 1$, $\mathcal{J}\left(R, I^{tn}\right) \subseteq \mathcal{J}\left(R, I^t\right)^n$.

Then, given a prime ideal P of height h,

$$P^{(hn)} \subseteq \mathcal{J}\left(R, \left(P^{(nh)}\right)\right) \subseteq \mathcal{J}\left(R, \left(P^{(nh)}\right)^{\frac{1}{n}}\right)^n \subseteq P^n.$$

In characteristic p, a similar proof works, replacing multiplier ideals by test ideals.

Remark 3.29. As a corollary of Theorem 3.26, we obtain a uniform constant c as in Theorem 3.24. Indeed, the big height of any ideal in $k[x_1, \ldots, x_d]$ is at most d, so that $I^{(dn)} \subseteq I^n$ for all n. This constant can actually be improved to $d-1$, since the ordinary and symbolic powers of any maximal ideal coincide. This holds more generally over any regular ring of dimension d.

When we move away from polynomial rings to the non-regular setting, we do know that the two topologies are equivalent for all prime ideals in some interesting cases. However, the question of whether the constant h can be taken independently of the prime ideal considered is still open. Since this asks for a uniform comparison between the symbolic and adic topologies, rings with this property are said to satisfy the Uniform Symbolic Topologies Property.

Problem 3 (Uniform Symbolic Topologies)**.** Let R be a complete local domain. Is there a uniform constant h depending only on R such that

$$P^{(hn)} \subseteq P^n$$

for all primes P and all $n \geqslant 1$?

The answer is known to be yes in some special settings [HKV09, HKV15]. Finding effective bounds for what h might be can be even harder [Wal16, Wal18].

Example 3.30 (Carvajal-Rojas — Smolkin, 2020 [CRS20])**.** Let k be a field of characteristic p and consider $R = k[a, b, c, d]/(ad - bc)$. Then for all primes P in R, $P^{(2n)} \subseteq P^n$ for all $n \geqslant 1$.

Back in the case of polynomial rings, though, Theorem 3.26 is not the end of the story. If we want to solve the Containment Problem from a best possible perspective, we would want to find, for each a, the smallest b such that $I^{(b)} \subseteq I^a$.

Example 3.31. The ideal $I = (x, y) \cap (x, z) \cap (y, z)$ from Example 2.13 has big height 2, so that Theorem 3.26 implies that $I^{(2n)} \subseteq I^n$ for all $n \geqslant 1$. However, one can easily check that $I^{(3)} \subseteq I^2$, even though the theorem only guarantees $I^{(4)} \subseteq I^2$.

Question 3.32 (Huneke, 2000)**.** Let P be a prime ideal of height 2 in a regular local ring containing a field. Does the containment $P^{(3)} \subseteq P^2$ always hold?

This question remains open even in dimension 3. Harbourne proposed the following generalization of Question 3.32, which can be found in [HH13, BRH+09]:

Conjecture 3.33 (Harbourne, 2006)**.** Let I be a radical homogeneous ideal in $k[x_0, \ldots, x_d]$, and let h be the big height of I. Then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Remark 3.34. Equivalently, Harbourne's Conjecture asks if $I^{(n)} \subseteq I^{\lceil \frac{n}{h} \rceil}$ for all $n \geqslant 1$.

Remark 3.35. When $h = 2$, the conjecture asks that $I^{(2n-1)} \subseteq I^n$, and in particular that $I^{(3)} \subseteq I^2$.

There are many good reasons why Harbourne's Conjecture is reasonable. One of those reasons appears back in the work of Hochster and Huneke proving Theorem 3.26. With the appropriate tools, the characteristic p case of Theorem 3.26 for $n = p^e$ turns out to be a beautiful application of the Pigeonhole Principle, where we can even replace the power I^q of I by the qth **Frobenius power** of I, the ideal

$$I^{[q]} := \{f^q \mid f \in I\}.$$

Lemma 3.36 (Hochster–Huneke [HH02])**.** Suppose that I is a radical ideal of big height h in a regular ring R containing a field of prime characteristic p. For all $q = p^e$,

$$I^{(hq)} \subseteq I^{[q]}.$$

In fact, the same proof they give in their paper but using the full power of the Pidgenhole Principle can be used to show Harbourne's Conjecture 3.33 for powers of p, a fact first noted by Craig Huneke: that

$$I^{(hq-h+1)} \subseteq I^{[q]}$$

for all $q = p^e$. As an easy corollary, we obtain an affirmative answer to Huneke's Question 3.32 in characteristic 2, that is, $I^{(3)} \subseteq I^2$ always holds in characteristic 2.

Exercise 24. Suppose that $I = (f_1, \ldots, f_h)$ is a radical ideal in $k[x_1, \ldots, x_d]$, where k is a field of prime characteristic p. Show that for all $q = p^e$,

$$I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q.$$

Exercise 25. Let I be a squarefree monomial ideal over a field of any characteristic. Show that I satisfies Harbourne's Conjecture.

Hint: given a monomial ideal, we can take its bracket power, which is a sort of fake Frobenius power. The n-th bracket power of the monomial ideal I is the ideal

$$I^{[n]} = \left(f^n \,\middle|\, f \in I \text{ is a monomial}\right).$$

As the notation suggests, these behave a lot like the Frobenius powers.

However, despite all this evidence, Conjecture 3.33 turns out to be too general; it does not hold for all homogeneous radical ideals.

Example 3.37 (Fermat configurations of points)**.** Let $n \geqslant 3$ be an integer and consider a field k of characteristic not 2 such that k contains n distinct roots of unity. Let $R = k[x, y, z]$, and consider the ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)).$$

When $n = 3$, this corresponds to a configuration of 12 points in $\mathbb{P}^2$, as described in Figure 1. Over $\mathbb{P}^2(\mathbb{C})$, these 12 points are given by the 3 coordinate points plus the 9 points defined by the intersections of $y^3 - z^3$, $z^3 - x^3$ and $x^3 - y^3$.

The ideal I is radical and has pure height 2. However, $I^{(3)} \nsubseteq I^2$, since the element $f = (y^n - z^n)(z^n - x^n)(x^n - y^n) \in I^{(3)}$ but not in I^2. This can be shown via geometric

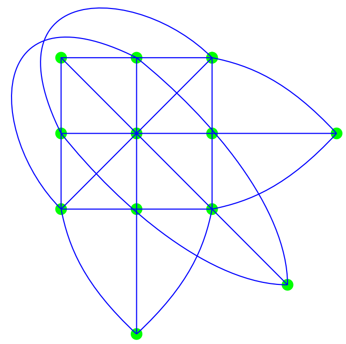


Figure 1: Fermat configuration of points when $n = 3$.

arguments, noting that f defines 9 lines, some three of which go through each of the 12 points.

This was first proved by Dumnicki, Szemberg e Tutaj-Gasińska [DSTG13] over $k = \mathbb{C}$, and then generalized in [HS15, Proposition 3.1] to any k and any n. Other extensions of this example can be found in [Dra17, MS18b].

Other configurations of points in $\mathbb{P}^2$ have been shown to produce ideals that fail the containment $I^{(3)} \subseteq I^2$, such as the Klein and Wiman configurations of points [Sec15], among others [DS21]. Drabkin and Seceleanu showed that symmetry plays an important role in these counterexamples, which in some way have *too much* symmetry. Given a configuration of points in $\mathbb{P}^k$ that produces an ideal I with $I^{(hn-h+1)} \nsubseteq I^n$, one can produce other counterexamples to the same type of containment by applying flat morphisms $\mathbb{P}^k \to \mathbb{P}^k$ [Ake17].

Example 3.38. Harbourne and Seceleanu [HS15] showed that $I^{(hn-h+1)} \subseteq I^n$ can fail for arbitrarily high values of n in characteristic $p > 0$. However, their counterexamples are constructed depending on n, meaning that given n, there exists an ideal I_n of pure height 2 (corresponding, once more, to a configuration of points in $\mathbb{P}^2$) which fails $I_n^{(hn-h+1)} \subseteq I_n^n$.

Nevertheless, Harbourne's Conjecture 3.33 is satisfied by many interesting classes of ideals:

- if I is a monomial ideal (which first appeared in [BRH$^+$09, Example 8.4.5]);
- if I corresponds to a generic set of points in $\mathbb{P}^2$ ([BH10]) or $\mathbb{P}^3$ ([Dum15]);
- if I corresponds to a star configuration of points ([HH13]),

among others. The conjecture also holds if R/I has nice enough singularities. More precisely, it holds if R/I is F-pure in prime characteristic p — some measure of niceness — or of dense F-pure type over a field of characteristic 0 [GH19]. This class includes all squarefree monomial ideals, and also ideals defining Veronese rings, generic determinantal rings, and more generally nice rings of invariants.

Every counterexample to Conjecture 3.33 known to date actually satisfies the following open conjecture:

Conjecture 3.39 (Stable Harbourne [Gri20]). If I is a radical ideal of big height h in a regular ring, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

We are asking if Harbourne's Conjecure holds for n large — where large enough should depend on I, as Harbourne and Seceleanu's examples suggest [HS15]. There are no counterexamples known to this conjecture, and the evidence supporting the conjecture keeps growing [Gri20, BGHN20a, GHM20a, GHM20b].

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