Q1) (a) The rule for differentiating a quotient of functions is given by:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad (*)$$

In the case $f(x) = \dfrac{x}{x+e^{-x}}$ we have that

$u = x$ ($\Rightarrow u' = 1$) and $v = x + e^{-x}$ ($\Rightarrow v' = 1 - e^{-x}$).

Putting these values into eq^n (*) we see that:

$$f'(x) = \frac{(x)'(x+e^{-x}) - (x)(x+e^{-x})'}{(x+e^{-x})^2}$$

$$= \frac{1(x+e^{-x}) - x(1-e^{-x})}{(x+e^{-x})^2}$$

$$= \frac{e^{-x}(1+x)}{(x+e^{-x})^2} \qquad \text{(correct answer)}$$

The problem in the answer on the sheet is that the student has wrongly interpreted the quotient rule as:

$$\left(\frac{u}{v}\right)' = \frac{v'u - u'v}{v^2}$$] This is incorrect

and this accounts for why the sign of the answer on the sheet is incorrect.

Q1) (b) The rule for integration by parts is:

$$\int_a^b u v' \, dx = [uv]_a^b - \int_a^b u' v \, dx \qquad (*)$$

for functions u, v of x.

In the evaluation of the integral $\int_0^{\pi/4} x^2 \sin(2x)\, dx$ the u & v functions are:

$$u = x^2 \ (\Rightarrow u' = 2x) \quad \text{and} \quad v' = \sin(2x) \ (\Rightarrow v = -\tfrac{1}{2}\cos(2x)).$$

Applying these functions in $(*)$:

$$\int_0^{\pi/4} x^2 \sin(2x)\, dx = \left[x^2\left(-\tfrac{1}{2}\cos 2x\right)\right]_0^{\pi/4} + \int_0^{\pi/4} x \cos(2x)\, dx.$$

Here we see the first difference to what is in the script. $\left[2x \cdot -\tfrac{1}{2}\cos(2x)\right]_0^{\pi/4}$ is incorrect, it should be $\left[x^2 \cdot \left(-\tfrac{1}{2}\cos 2x\right)\right]_0^{\pi/4}$.
However, both evaluate to zero.

Now: $$\int_0^{\pi/4} x \cos 2x \, dx = \left[x \cdot \tfrac{1}{2}\sin(2x)\right]_0^{\pi/4} - \tfrac{1}{2}\int_0^{\pi/4} \sin(2x)\, dx$$ integration by parts again

Here, again, we see a difference in sign before the last integral!
Evaluating:

$$\int_0^{\pi/4} x \cos(2x)\, dx = \frac{\pi}{8} - \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}.$$

So, the correct solution is:

$$\int_0^{\pi/4} x^2 \sin(2x)\, dx = \frac{\pi}{8} - \frac{1}{4}$$

Q2) $f(x,y,z) = \log[g(x,y)] + \log[h(y,z)]$.

(a) The gradient is defined to be:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)^T.$$

Now: $\frac{\partial f}{\partial x} = \frac{1}{g}\frac{\partial g}{\partial x}$; $\frac{\partial f}{\partial y} = \frac{1}{g}\frac{\partial g}{\partial y} + \frac{1}{h}\frac{\partial h}{\partial y}$; $\frac{\partial f}{\partial z} = \frac{1}{h}\frac{\partial h}{\partial z}$.

$$\therefore \quad \nabla f = \left(\frac{1}{g}\frac{\partial g}{\partial x}, \frac{1}{g}\frac{\partial g}{\partial y} + \frac{1}{h}\frac{\partial h}{\partial y}, \frac{1}{h}\frac{\partial h}{\partial z} \right)^T$$

(b) The Hessian is defined to be:

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

<u>Note</u>: This is a symmetric matrix as: $\frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial b \partial a}$, $a, b \in \{x, y, z\}$.

To determine $\nabla^2 f$, given the above, we now only need to evaluate all the 2^{nd} order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left\{\frac{1}{g}\frac{\partial g}{\partial x}\right\} = \frac{-1}{g^2}\left(\frac{\partial g}{\partial x}\right)^2 + \frac{1}{g}\frac{\partial^2 g}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{g}\frac{\partial g}{\partial y} + \frac{1}{h}\frac{\partial h}{\partial y}\right\} = \frac{-1}{g^2}\left(\frac{\partial g}{\partial y}\right)^2 + \frac{1}{g}\frac{\partial^2 g}{\partial y^2} - \frac{1}{h^2}\left(\frac{\partial h}{\partial y}\right)^2 + \frac{1}{h}\frac{\partial^2 h}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left\{\frac{1}{h}\frac{\partial h}{\partial z}\right\} = -\frac{1}{h^2}\left(\frac{\partial h}{\partial z}\right)^2 + \frac{1}{h}\frac{\partial^2 h}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{g}\frac{\partial g}{\partial x}\right\} = \frac{-1}{g^2}\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right) + \frac{1}{g}\frac{\partial^2 g}{\partial y \partial x} \qquad \left[= \frac{\partial^2 f}{\partial x \partial y}\right]$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{h}\frac{\partial h}{\partial z}\right\} = -\frac{1}{h^2}\left(\frac{\partial h}{\partial y}\right)\left(\frac{\partial h}{\partial z}\right) + \frac{1}{h}\frac{\partial^2 h}{\partial y \partial z} \qquad \left[= \frac{\partial^2 f}{\partial z \partial y}\right]$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial z}\left\{\frac{1}{g}\frac{\partial g}{\partial x}\right\} = 0 \qquad \left[= \frac{\partial^2 f}{\partial x \partial z}\right].$$

Q3) (a) First we create a matrix (B) that contain the given basis vectors for the subspace:

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Then the projection matrix, P, is given by:

$$P = B[B^TB]^{-1}B^T$$

Now:

$$B^TB = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 5 \\ 4 & 6 & 3 \\ 5 & 3 & 14 \end{pmatrix}. \qquad (*).$$

To determine the inverse of (B^TB) (ie $(B^TB)^{-1}$) we must first calculate the cofactors of B^TB:

$$C_{11} = \begin{vmatrix} 6 & 3 \\ 3 & 14 \end{vmatrix} = 75 \qquad C_{12} = -\begin{vmatrix} 4 & 3 \\ 5 & 14 \end{vmatrix} = -41 \qquad C_{13} = \begin{vmatrix} 4 & 6 \\ 5 & 3 \end{vmatrix} = -18$$

$$C_{21} = -\begin{vmatrix} 4 & 5 \\ 3 & 14 \end{vmatrix} = -41 \qquad C_{22} = \begin{vmatrix} 6 & 5 \\ 5 & 14 \end{vmatrix} = 59 \qquad C_{23} = -\begin{vmatrix} 6 & 4 \\ 5 & 3 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} 4 & 5 \\ 6 & 3 \end{vmatrix} = -18 \qquad C_{32} = -\begin{vmatrix} 6 & 5 \\ 4 & 3 \end{vmatrix} = 2 \qquad C_{33} = \begin{vmatrix} 6 & 4 \\ 4 & 6 \end{vmatrix} = 20.$$

To get the determinant of B^TB we expand along the 1^{st} row of the matrix (see (*)).

$$\det(B^TB) = 6C_{11} + 4C_{12} + 5C_{13} = 6\times 75 - 4\times 41 - 5\times 18 = 196.$$

Now:

$$(B^TB)^{-1}_{ij} = \frac{1}{\det(B^TB)} C_{ji}.$$

Hence:

$$(B^TB)^{-1} = \frac{1}{196}\begin{vmatrix} 75 & -41 & -18 \\ -41 & 59 & 2 \\ -18 & 2 & 20 \end{vmatrix}$$

$$\therefore \quad P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}\left[\frac{1}{196}\begin{pmatrix} 75 & -41 & -18 \\ -41 & 59 & 2 \\ -18 & 2 & 20 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.75 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.75 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.75 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.75 \end{pmatrix}.$$

Given the vector $v = (1,2,1,3)^T$ we can calculate its projection onto the subspace spanned by the vectors in the matrix B as:

$$Pv = \begin{pmatrix} 0.75 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.75 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.75 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.75 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.75 \\ 2.25 \\ 1.25 \\ 2.75 \end{pmatrix}$$

(b) Given the vectors:

$v_1 = (1,0,2,1)^T$; $v_2 = (2,1,1,0)^T$, $v_3 = (0,2,1,3)^T$ & $v_4 = (2,1,1,2)^T$

we determine all inner products:

$\langle v_1, v_1 \rangle = 6$, $\langle v_1, v_2 \rangle = 4$, $\langle v_1, v_3 \rangle = 5$, $\langle v_1, v_4 \rangle = 6$

$\langle v_2, v_1 \rangle = 4$, $\langle v_2, v_2 \rangle = 6$, $\langle v_2, v_3 \rangle = 3$, $\langle v_2, v_4 \rangle = 6$

$\langle v_3, v_1 \rangle = 5$, $\langle v_3, v_2 \rangle = 3$, $\langle v_3, v_3 \rangle = 14$, $\langle v_3, v_4 \rangle = 9$

$\langle v_4, v_1 \rangle = 6$, $\langle v_4, v_2 \rangle = 6$, $\langle v_4, v_3 \rangle = 9$, $\langle v_4, v_4 \rangle = 10$

The Gram matrix will be given by:

$$G_{ij} = \langle v_i, v_j \rangle.$$

$$\therefore G = \begin{pmatrix} 6 & 4 & 5 & 6 \\ 4 & 6 & 3 & 6 \\ 5 & 3 & 14 & 9 \\ 6 & 6 & 9 & 10 \end{pmatrix}.$$

The 4 given vectors are all linearly independent if the rank of the Gram matrix is also 4.

As the Gram matrix is 4×4 this is the same as saying that it is invertible, which is the same as saying its determinant is non-zero.

So, let's check:

$$|G| = 6\begin{vmatrix} 6 & 3 & 6 \\ 3 & 14 & 9 \\ 6 & 9 & 10 \end{vmatrix} - 4\begin{vmatrix} 4 & 3 & 6 \\ 5 & 14 & 9 \\ 6 & 9 & 10 \end{vmatrix} + 5\begin{vmatrix} 4 & 6 & 6 \\ 5 & 3 & 9 \\ 6 & 6 & 10 \end{vmatrix} - 6\begin{vmatrix} 4 & 6 & 3 \\ 5 & 3 & 14 \\ 6 & 6 & 9 \end{vmatrix}.$$

after expanding cofactors along the first row.

Now:

$$\begin{vmatrix} 6 & 3 & 6 \\ 3 & 14 & 9 \\ 6 & 9 & 10 \end{vmatrix} = 6\begin{vmatrix} 14 & 9 \\ 9 & 10 \end{vmatrix} - 3\begin{vmatrix} 3 & 9 \\ 6 & 10 \end{vmatrix} + 6\begin{vmatrix} 3 & 14 \\ 6 & 9 \end{vmatrix} = 6 \times 59 - 3 \times (-24) + 6 \times (-57) = 84$$

$$\begin{vmatrix} 4 & 3 & 6 \\ 5 & 14 & 9 \\ 6 & 9 & 10 \end{vmatrix} = 4\begin{vmatrix} 14 & 9 \\ 9 & 10 \end{vmatrix} - 3\begin{vmatrix} 5 & \cancel{14}9 \\ 6 & 10 \end{vmatrix} + 6\begin{vmatrix} 5 & 14 \\ 6 & 9 \end{vmatrix} = 4 \times 59 - 3 \times (\overset{-4}{\cancel{-24}}) + 6 \times (-39) = 14$$

$$\begin{vmatrix} 4 & 6 & 6 \\ 5 & 3 & 9 \\ 6 & 6 & 10 \end{vmatrix} = 4\begin{vmatrix} 3 & 9 \\ 6 & 10 \end{vmatrix} - 6\begin{vmatrix} 5 & 9 \\ 6 & 10 \end{vmatrix} + 6\begin{vmatrix} 5 & 3 \\ 6 & 6 \end{vmatrix} = 4 \times (-24) - 6(-4) + 6 \times (12) = 0$$

$$\begin{vmatrix} 4 & 6 & 3 \\ 5 & 3 & 14 \\ 6 & 6 & 9 \end{vmatrix} = 4\begin{vmatrix} 3 & 14 \\ 6 & 9 \end{vmatrix} - 6\begin{vmatrix} 5 & 14 \\ 6 & 9 \end{vmatrix} + 3\begin{vmatrix} 5 & 3 \\ 6 & 6 \end{vmatrix} = 4 \times -57 - 6 \times -39 + 3 \times 12 = 42$$

$\therefore$

$$|G| = (6 \times 84) - (4 \times 14) + (5 \times 0) - (6 \times 42) = 196$$

So, G is non-singular $\Rightarrow G$ is of rank $4 \Rightarrow \{v_1, v_2, v_3, v_4\}$ are linearly independent.