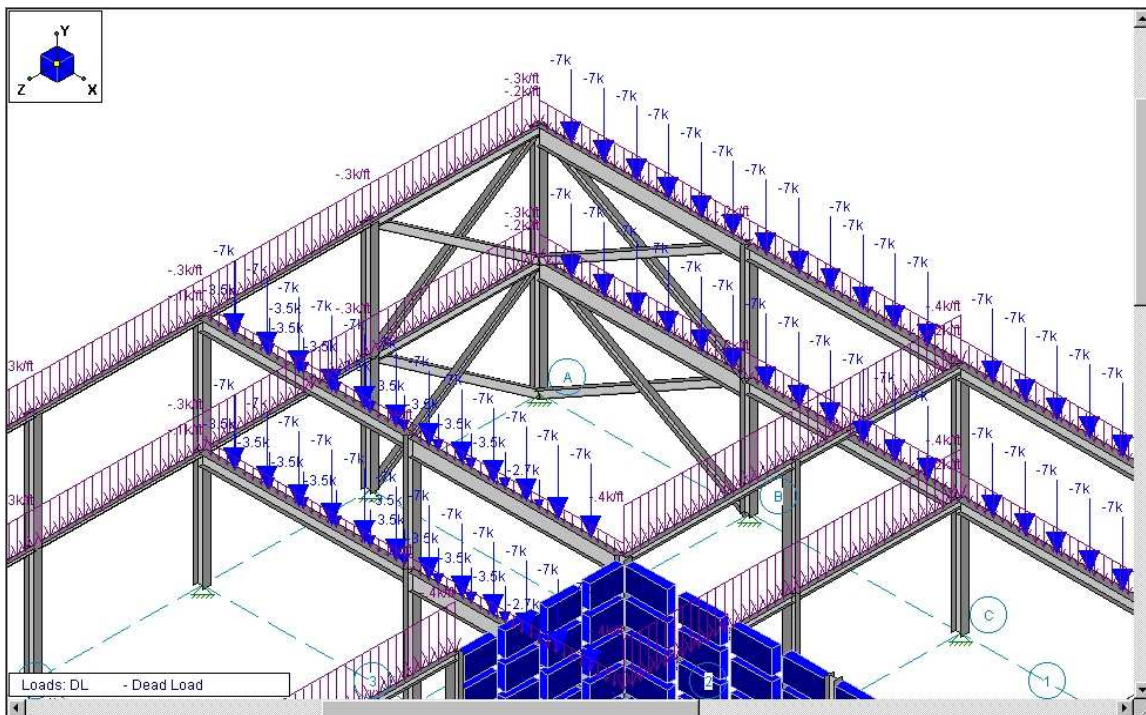


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Lecture Notes in:

FINITE ELEMENT I

Framed Structures



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Contents

1	INTRODUCTION	17
1.1	Why Matrix Structural Analysis?	17
1.2	Overview of Structural Analysis	18
1.3	Structural Idealization	19
1.3.1	Structural Discretization	20
1.3.2	Coordinate Systems	21
1.3.3	Sign Convention	21
1.4	Degrees of Freedom	22
1.5	Course Organization	25
I	Matrix Structural Analysis of Framed Structures	27
2	ELEMENT STIFFNESS MATRIX	29
2.1	Introduction	29
2.2	Influence Coefficients	29
2.3	Flexibility Matrix (Review)	30
2.4	Stiffness Coefficients	31
2.5	Force-Displacement Relations	32
2.5.1	Axial Deformations	32
2.5.2	Flexural Deformation	32
2.5.3	Torsional Deformations	35
2.5.4	Shear Deformation	36
2.6	Putting it All Together, $[k]$	39
2.6.1	Truss Element	39
2.6.2	Beam Element	40
2.6.2.1	Euler-Bernoulli	40
2.6.2.2	Timoshenko Beam	40
2.6.3	2D Frame Element	42
2.6.4	Grid Element	42
2.6.5	3D Frame Element	43
2.7	Remarks on Element Stiffness Matrices	43
2.8	Homework	45
3	STIFFNESS METHOD; Part I: ORTHOGONAL STRUCTURES	47
3.1	Introduction	47
3.2	The Stiffness Method	47
3.3	Examples	49
E 3-1	Beam	49
E 3-2	Frame	51
E 3-3	Grid	54
3.4	Observations	56

5.6.2.11	Nodal Displacements	111
5.6.2.12	Reactions	111
5.6.2.13	Internal Forces	112
5.6.2.14	Sample Output File	113
5.7	Homework	115
6	EQUATIONS OF STATICS and KINEMATICS	117
6.1	Statics Matrix $[B]$	117
6.1.1	Statically Determinate	117
E 6-1	Statically Determinate Truss Statics Matrix	117
E 6-2	Beam Statics Matrix	119
6.1.2	Statically Indeterminate	120
E 6-3	Statically Indeterminate Truss Statics Matrix	120
E 6-4	Selection of Redundant Forces	122
6.1.3	Kinematic Instability	124
6.2	Kinematics Matrix $[A]$	124
E 6-5	Kinematics Matrix of a Truss	125
6.3	Statics-Kinematics Matrix Relationship	125
6.3.1	Statically Determinate	126
6.3.2	Statically Indeterminate	126
6.4	Kinematic Relations through Inverse of Statics Matrix	127
6.5	‡ Congruent Transformation Approach to $[K]$	127
E 6-6	Congruent Transformation	128
E 6-7	Congruent Transformation of a Frame	130
7	FLEXIBILITY METHOD	133
7.1	Introduction	133
7.2	Flexibility Matrix	133
7.2.1	Solution of Redundant Forces	134
7.2.2	Solution of Internal Forces and Reactions	135
7.2.3	Solution of Joint Displacements	135
E 7-1	Flexibility Method	135
7.3	Stiffness Flexibility Relations	137
7.3.1	From Stiffness to Flexibility	137
E 7-2	Flexibility Matrix	137
7.3.2	From Flexibility to Stiffness	138
E 7-3	Flexibility to Stiffness	139
7.4	Stiffness Matrix of a Curved Element	140
7.5	Duality between the Flexibility and the Stiffness Methods	141
7.6	Homework	142
8	SPECIAL ANALYSIS PROCEDURES	143
8.1	Semi-Rigid Beams	143
8.2	Nonuniform Torsion	145
8.3	Inclined Supports	145
8.4	Condensation	145
8.5	Substructuring	145
8.6	Reanalysis	145
8.7	Constraints	145

E 10-8 Tapered Beam; Fourier Series	186
10.4 † Complementary Potential Energy	188
10.4.1 Derivation	188
10.4.2 Castigliano's Second Theorem	188
E 10-9 Cantilivered beam	189
10.4.2.1 Distributed Loads	189
E 10-10 Deflection of a Uniformly loaded Beam using Castigliano's second Theorem	189
10.5 Comparison of Alternate Approximate Solutions	190
E 10-11 Comparison of MPE Solutions	190
10.6 Summary	191
10.7 Variational Calculus; Preliminaries	191
10.7.1 Euler Equation	194
10.7.2 Boundary Conditions	197
E 10-12 Extension of a Bar	198
E 10-13 Flexure of a Beam	199
10.8 Homework	201
11 INTERPOLATION FUNCTIONS	203
11.1 Introduction	203
11.2 Shape Functions	203
11.2.1 Axial/Torsional	204
11.2.2 Generalization	205
11.2.3 Flexural	205
11.2.4 Constant Strain Triangle Element	207
11.3 Interpolation Functions	209
11.3.1 C^0 : Lagrangian Interpolation Functions	209
11.3.1.1 Constant Strain Quadrilateral Element	210
11.3.1.2 Solid Rectangular Trilinear Element	211
11.3.2 C^1 : Hermitian Interpolation Functions	211
11.4 Interpretation of Shape Functions in Terms of Polynomial Series	212
12 FINITE ELEMENT FORMULATION	213
12.1 Strain Displacement Relations	213
12.1.1 Axial Members	213
12.1.2 Flexural Members	214
12.2 Virtual Displacement and Strains	214
12.3 Element Stiffness Matrix Formulation	214
12.3.1 Stress Recovery	216
13 SOME FINITE ELEMENTS	217
13.1 Introduction	217
13.2 Truss Element	217
13.3 Flexural Element	218
13.4 Triangular Element	218
13.4.1 Strain-Displacement Relations	218
13.4.2 Stiffness Matrix	219
13.4.3 Internal Stresses	220
13.4.4 Observations	220
13.5 Quadrilateral Element	220
13.6 Homework	221

E B-3 Example	269
B.2.3 Cholesky's Decomposition	270
E B-4 Cholesky's Decomposition	270
B.2.4 Pivoting	271
B.3 Indirect Methods	271
B.3.1 Gauss Seidel	272
B.4 Ill Conditioning	272
B.4.1 Condition Number	272
B.4.2 Pre Conditioning	273
B.4.3 Residual and Iterative Improvements	273
C TENSOR NOTATION	275
C.1 Engineering Notation	275
C.2 Dyadic/Vector Notation	275
C.3 Indicial/Tensorial Notation	276
D INTEGRAL THEOREMS	279
D.1 Integration by Parts	279
D.2 Green-Gradient Theorem	279
D.3 Gauss-Divergence Theorem	279

List of Figures

1.1	Global Coordinate System	21
1.2	Local Coordinate Systems	21
1.3	Sign Convention, Design and Analysis	22
1.4	Total Degrees of Freedom for various Type of Elements	22
1.5	Independent Displacements	23
1.6	Examples of Global Degrees of Freedom	24
1.7	Organization of the Course	26
2.1	Example for Flexibility Method	30
2.2	Definition of Element Stiffness Coefficients	33
2.3	Stiffness Coefficients for One Dimensional Elements	34
2.4	Flexural Problem Formulation	34
2.5	Torsion Rotation Relations	35
2.6	Deformation of an Infinitesimal Element Due to Shear	37
2.7	Effect of Flexure and Shear Deformation on Translation at One End	38
2.8	Effect of Flexure and Shear Deformation on Rotation at One End	39
2.9	Coordinate System for Element Stiffness Matrices	40
3.1	Problem with 2 Global d.o.f. θ_1 and θ_2	48
3.2	Frame Example (correct K_{23})	51
3.3	Grid Example	54
4.1	Arbitrary 3D Vector Transformation	60
4.2	3D Vector Transformation	61
4.3	2D Frame Element Rotation with respect to Z (or z)	62
4.4	Grid Element Rotation	63
4.5	2D Truss Rotation	64
4.6	Reduced 3D Rotation	65
4.7	Special Case of 3D Transformation for Vertical Members	66
4.8	Complex 3D Rotation	67
5.1	Frame Example	69
5.2	Example for $[\mathbf{ID}]$ Matrix Determination	72
5.3	Simple Frame Analyzed with the MATLAB Code	73
5.4	75
5.5	Simple Frame Analyzed with the MATLAB Code	80
5.6	Stiffness Analysis of one Element Structure	82
5.7	Example of Bandwidth	86
5.8	Numbering Schemes for Simple Structure	87
5.9	Program Flowchart	91
5.10	Program's Tree Structure	92
5.11	Flowchart for the Skyline Height Determination	94
5.12	Flowchart for the Global Stiffness Matrix Assembly	96

List of Tables

1.1	Example of Nodal Definition	20
1.2	Example of Element Definition	20
1.3	Example of Group Number	21
1.4	Degrees of Freedom of Different Structure Types Systems	23
2.1	Examples of Influence Coefficients	29
4.1	3D Transformations of Linear Elements	65
6.1	Internal Element Force Definition for the Statics Matrix	118
6.2	Conditions for Static Determinacy, and Kinematic Instability	124
10.1	Possible Combinations of Real and Hypothetical Formulations	167
10.2	Comparison of 2 Alternative Approximate Solutions	191
10.3	Summary of Variational Terms Associated with One Dimensional Elements	194
10.4	Essential and Natural Boundary Conditions	198
11.1	Characteristics of Beam Element Shape Functions	208
11.2	Interpretation of Shape Functions in Terms of Polynomial Series (1D & 2D)	212
11.3	Polynomial Terms in Various Element Formulations (1D & 2D)	212

NOTATION

\mathbf{a}	Vector of coefficients in assumed displacement field
A	Area
\mathcal{A}	Kinematics Matrix
\mathbf{b}	Body force vector
\mathcal{B}	Statics Matrix, relating external nodal forces to internal forces
$[\mathbf{B}']$	Statics Matrix relating nodal load to internal forces $\mathbf{p} = [\mathbf{B}']\mathbf{P}$
$[\mathbf{B}]$	Matrix relating assumed displacement fields parameters to joint displacements
C	Cosine
$[C1 C2]$	Matrices derived from the statics matrix
$\{\mathbf{d}\}$	Element flexibility matrix (lc)
$\{d_c\}$	
$[\mathbf{D}]$	Structure flexibility matrix (GC)
E	Elastic Modulus
$[\mathbf{E}]$	Matrix of elastic constants (Constitutive Matrix)
$\{\mathbf{F}\}$	Unknown element forces and unknown support reactions
$\{\mathbf{F}_0\}$	Nonredundant element forces (lc)
$\{\mathbf{F}_x\}$	Redundant element forces (lc)
$\{\mathbf{F}_e\}$	Element forces (lc)
$\{\bar{\mathbf{F}}^0\}$	Nodal initial forces
$\{\bar{\mathbf{F}}^e\}$	Nodal energy equivalent forces
$\{\bar{\mathbf{F}}\}$	Externally applied nodal forces
FEA	Fixed end actions of a restrained member
G	Shear modulus
I	Moment of inertia
$[\mathcal{L}]$	Matrix relating the assumed displacement field parameters to joint displacements
$[\mathbf{I}]$	Identity matrix
$[\mathbf{ID}]$	Matrix relating nodal dof to structure dof
J	St Venant's torsional constant
$[\mathbf{k}]$	Element stiffness matrix (lc)
$[\mathbf{p}]$	Matrix of coefficients of a polynomial series
$[\mathbf{k}_g]$	Geometric element stiffness matrix (lc)
$[\mathbf{k}_r]$	Rotational stiffness matrix ($[\mathbf{d}]$ inverse)
$[\mathbf{K}]$	Structure stiffness matrix (GC)
$[\mathbf{K}_g]$	Structure's geometric stiffness matrix (GC)
L	Length
\mathbf{L}	Linear differential operator relating displacement to strains
l_{ij}	Direction cosine of rotated axis i with respect to original axis j
$\{LM\}$	structure dof of nodes connected to a given element
$\{\mathbf{N}\}$	Shape functions
$\{\mathbf{p}\}$	Element nodal forces = F (lc)
$\{\mathbf{P}\}$	Structure nodal forces (GC)
P, V, M, T	Internal forces acting on a beam column (axial, shear, moment, torsion)
\mathbf{R}	Structure reactions (GC)
S	Sine
\mathbf{t}	Traction vector
$\hat{\mathbf{t}}$	Specified tractions along Γ_t
\mathbf{u}	Displacement vector
\tilde{u}	Neighbour function to $u(x)$
$\hat{\mathbf{u}}(x)$	Specified displacements along Γ_u
u, v, w	Translational displacements along the x, y, and z directions
U	Strain energy

Chapter 1

INTRODUCTION

1.1 Why Matrix Structural Analysis?

¹ In most Civil engineering curriculum, students are required to take courses in: Statics, Strength of Materials, Basic Structural Analysis. This last course is a fundamental one which introduces basic structural analysis (determination of reactions, deflections, and internal forces) of both statically determinate and indeterminate structures.

² Also Energy methods are introduced, and most if not all examples are two dimensional. Since the emphasis is on hand solution, very seldom are three dimensional structures analyzed. The methods covered, for the most part lend themselves for “back of the envelope” solutions and not necessarily for computer implementation.

³ Those students who want to pursue a specialization in structural engineering/mechanics, do take more advanced courses such as Matrix Structural Analysis and/or Finite Element Analysis.

⁴ Matrix Structural Analysis, or Advanced Structural Analysis, or Introduction to Structural Engineering Finite Element, builds on the introductory analysis course to focus on those methods which lend themselves to computer implementation. In doing so, we will place equal emphasis on both two and three dimensional structures, and develop a thorough understanding of computer aided analysis of structures.

⁵ This is essential, as in practice most, if not all, structural analysis are done by the computer and it is imperative that as structural engineers you understand what is inside those “black boxes”, develop enough self assurance to be capable of opening them and modify them to perform certain specific tasks, and most importantly to understand their limitations.

⁶ With the recently placed emphasis on the finite element method in most graduate schools, many students have been tempted to skip a course such as this one and rush into a finite element one. Hence it is important that you understand the connection and role of those two courses. The Finite Element Method addresses the analysis of two or three dimensional continuum. As such, the primary unknowns is \mathbf{u} the nodal displacements, and internal “forces” are usually restricted to stress $\boldsymbol{\sigma}$. The only analogous one dimensional structure is the truss.

⁷ Whereas two and three dimensional continuum are essential in civil engineering to model structures such as dams, shells, and foundation, the majority of Civil engineering structures are constituted by “rod” one-dimensional elements such as beams, girders, or columns. For those elements, “displacements” and internal “forces” are somehow more complex than those encountered in continuum finite elements.

⁸ Hence, contrarily to continuum finite element where displacement is mostly synonymous with translation, in one dimensional elements, and depending on the type of structure, generalized displacements may include translation, and/or flexural and/or torsional rotation. Similarly, “internal forces” are not stresses, but rather axial and shear forces, and/or flexural or torsional moments. Those concepts are far

3. Type of solution:

- (a) Continuum, analytical, Partial Differential Equation
- (b) Discrete, numerical, Finite Element, Finite Difference, Boundary Element

¹³ Structural design must satisfy:

- 1. Strength ($\sigma < \sigma_f$)
- 2. Stiffness ("small" deformations)
- 3. Stability (buckling, cracking)

¹⁴ Structural analysis must satisfy

- 1. Statics (equilibrium)
- 2. Mechanics (stress-strain or force displacement relations)
- 3. Kinematics (compatibility of displacement)

1.3 Structural Idealization

¹⁵ Prior to analysis, a structure must be idealized for a suitable mathematical representation. Since it is practically impossible (and most often unnecessary) to model every single detail, assumptions must be made. Hence, structural idealization is as much an art as a science. *Some* of the questions confronting the analyst include:

- 1. Two dimensional versus three dimensional; Should we model a single bay of a building, or the entire structure?
- 2. Frame or truss, can we neglect flexural stiffness?
- 3. Rigid or semi-rigid connections (most important in steel structures)
- 4. Rigid supports or elastic foundations (are the foundations over solid rock, or over clay which may consolidate over time)
- 5. Include or not secondary members (such as diagonal braces in a three dimensional analysis).
- 6. Include or not axial deformation (can we neglect the axial stiffness of a beam in a building?)
- 7. Cross sectional properties (what is the moment of inertia of a reinforced concrete beam?)
- 8. Neglect or not haunches (those are usually present in zones of high negative moments)
- 9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
- 10. Small or large deformations (In the analysis of a high rise building subjected to wind load, the moments should be amplified by the product of the axial load times the lateral deformation, $P - \Delta$ effects).
- 11. Time dependent effects (such as creep, which is extremely important in prestressed concrete, or cable stayed concrete bridges).
- 12. Partial collapse or local yielding (would the failure of a single element trigger the failure of the entire structure?).

Group	Element	Material
No.	Type	Group
1	1	1
2	2	1
3	1	2

Table 1.3: Example of Group Number

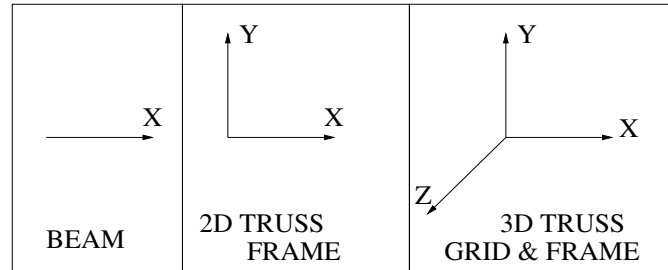


Figure 1.1: Global Coordinate System

1.3.2 Coordinate Systems

²² We should differentiate between 2 coordinate systems:

Global: to describe the structure nodal coordinates. This system can be arbitrarily selected provided it is a Right Hand Side (RHS) one, and we will associate with it upper case axis labels, X, Y, Z , Fig. 1.1 or 1,2,3 (running indices within a computer program).

Local: system is associated with each element and is used to describe the element internal forces. We will associate with it lower case axis labels, x, y, z (or 1,2,3), Fig. 1.2.

²³ The x -axis is assumed to be along the member, and the direction is chosen such that it points from the 1st node to the 2nd node, Fig. 1.2.

²⁴ Two dimensional structures will be defined in the X-Y plane.

1.3.3 Sign Convention

²⁵ The sign convention in structural analysis is completely different than the one previously adopted in structural analysis/design, Fig. 1.3 (where we focused mostly on flexure and defined a positive moment as one causing “tension below”. This would be awkward to program!).

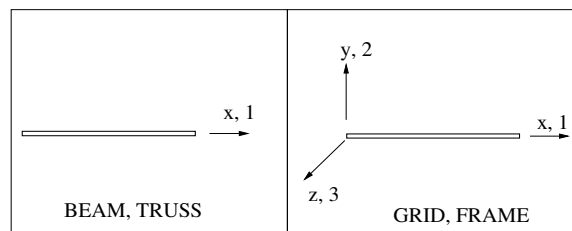


Figure 1.2: Local Coordinate Systems

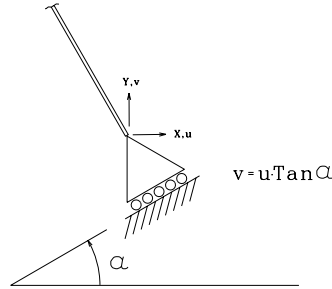


Figure 1.5: Independent Displacements

various types of structures made up of one dimensional rod elements, Table 1.4.

³⁴ This table shows the degree of freedoms and the corresponding generalized forces.

Type		Node 1	Node 2	$[k]$	$[K]$
				(Local)	(Global)
1 Dimensional					
Beam	$\{P\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\delta\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{P\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\delta\}$	u_1	u_2		
Frame	$\{P\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\delta\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{P\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\delta\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{P\}$	$F_{x1},$	F_{x2}	2×2	6×6
	$\{\delta\}$	$u_1,$	u_2		
Frame	$\{P\}$	$F_{x1}, F_{y2}, F_{y3},$ $T_{x4} M_{y5}, M_{z6}$	$F_{x7}, F_{y8}, F_{y9},$ $T_{x10} M_{y11}, M_{z12}$	12×12	12×12
	$\{\delta\}$	$u_1, v_2, w_3,$ $\theta_4, \theta_5 \theta_6$	$u_7, v_8, w_9,$ $\theta_{10}, \theta_{11} \theta_{12}$		

Table 1.4: Degrees of Freedom of Different Structure Types Systems

³⁵ We should distinguish between local and global d.o.f.'s. The numbering scheme follows the following simple rules:

Local: d.o.f. for a given element: Start with the first node, number the local d.o.f. in the same order as the subscripts of the relevant local coordinate system, and repeat for the second node.

Global: d.o.f. for the entire structure: Starting with the 1st node, number all the unrestrained global d.o.f.'s, and then move to the next one until all global d.o.f have been numbered, Fig. 1.6.

1.5 Course Organization

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Part I

Matrix Structural Analysis of Framed Structures

Chapter 2

ELEMENT STIFFNESS MATRIX

2.1 Introduction

¹ In this chapter, we shall derive the element stiffness matrix $[\mathbf{k}]$ of various one dimensional elements. Only after this important step is well understood, we could expand the theory and introduce the structure stiffness matrix $[\mathbf{K}]$ in its global coordinate system.

² As will be seen later, there are two fundamentally different approaches to derive the stiffness matrix of one dimensional element. The first one, which will be used in this chapter, is based on classical methods of structural analysis (such as moment area or virtual force method). Thus, in deriving the element stiffness matrix, we will be reviewing concepts earlier seen.

³ The other approach, based on energy consideration through the use of assumed shape functions, will be examined in chapter 12. This second approach, exclusively used in the finite element method, will also be extended to two and three dimensional continuum elements.

2.2 Influence Coefficients

⁴ In structural analysis an influence coefficient C_{ij} can be defined as the effect on d.o.f. i due to a unit action at d.o.f. j for an individual element or a whole structure. Examples of Influence Coefficients are shown in Table 2.1.

	Unit Action	Effect on
Influence Line	Load	Shear
Influence Line	Load	Moment
Influence Line	Load	Deflection
Flexibility Coefficient	Load	Displacement
Stiffness Coefficient	Displacement	Load

Table 2.1: Examples of Influence Coefficients

⁵ It should be recalled that influence lines are associated with the analysis of structures subjected to moving loads (such as bridges), and that the flexibility and stiffness coefficients are components of matrices used in structural analysis.

Virtual Force:

$$\left. \begin{aligned} \delta U &= \int \delta \bar{\sigma}_x \varepsilon_x \, \text{dvol} \\ \delta \bar{\sigma}_x &= \frac{\bar{M}_x y}{I} \\ \varepsilon_x &= \frac{\sigma_x}{E} = \frac{My}{EI} \\ \int y^2 \, \text{dA} &= I \\ \text{dvol} &= \text{dA} \, \text{d}x \\ \delta W &= \delta \bar{P} \Delta \\ \delta U &= \delta W \end{aligned} \right\} \delta U = \int_0^L \delta \bar{M} \frac{M}{EI} \, \text{d}x \left\{ \int_0^L \delta \bar{M} \frac{M}{EI} \, \text{d}x = \delta \bar{P} \Delta \quad (2.3)$$

Hence:

$$EI \underbrace{1}_{\delta \bar{M}} \underbrace{d_{11}}_{\Delta} = \underbrace{\int_0^L \left(1 - \frac{x}{L}\right)^2 \, \text{d}x}_{\delta \bar{M} \cdot M} = \frac{L}{3} \quad (2.4)$$

Similarly, we would obtain:

$$EI d_{22} = \int_0^L \left(\frac{x}{L}\right)^2 \, \text{d}x = \frac{L}{3} \quad (2.5\text{-a})$$

$$EI d_{12} = \int_0^L \left(1 - \frac{x}{L}\right) \frac{x}{L} \, \text{d}x = -\frac{L}{6} = EI d_{21} \quad (2.5\text{-b})$$

⁷ Those results can be summarized in a matrix form as:

$$[\mathbf{d}] = \frac{L}{6EI_z} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad (2.6)$$

⁸ The flexibility method will be covered in more detailed, in chapter 7.

2.4 Stiffness Coefficients

⁹ In the flexibility method, we have applied a unit force at a time and determined all the induced displacements in the statically determinate structure.

¹⁰ In the stiffness method, we

1. Constrain all the degrees of freedom
2. Apply a unit displacement at each d.o.f. (while restraining all others to be zero)
3. Determine the reactions associated with all the d.o.f.

$$\{\mathbf{p}\} = [\mathbf{k}]\{\boldsymbol{\delta}\} \quad (2.7)$$

¹¹ Hence k_{ij} will correspond to the reaction at dof i due to a unit deformation (translation or rotation) at dof j , Fig. 2.2.

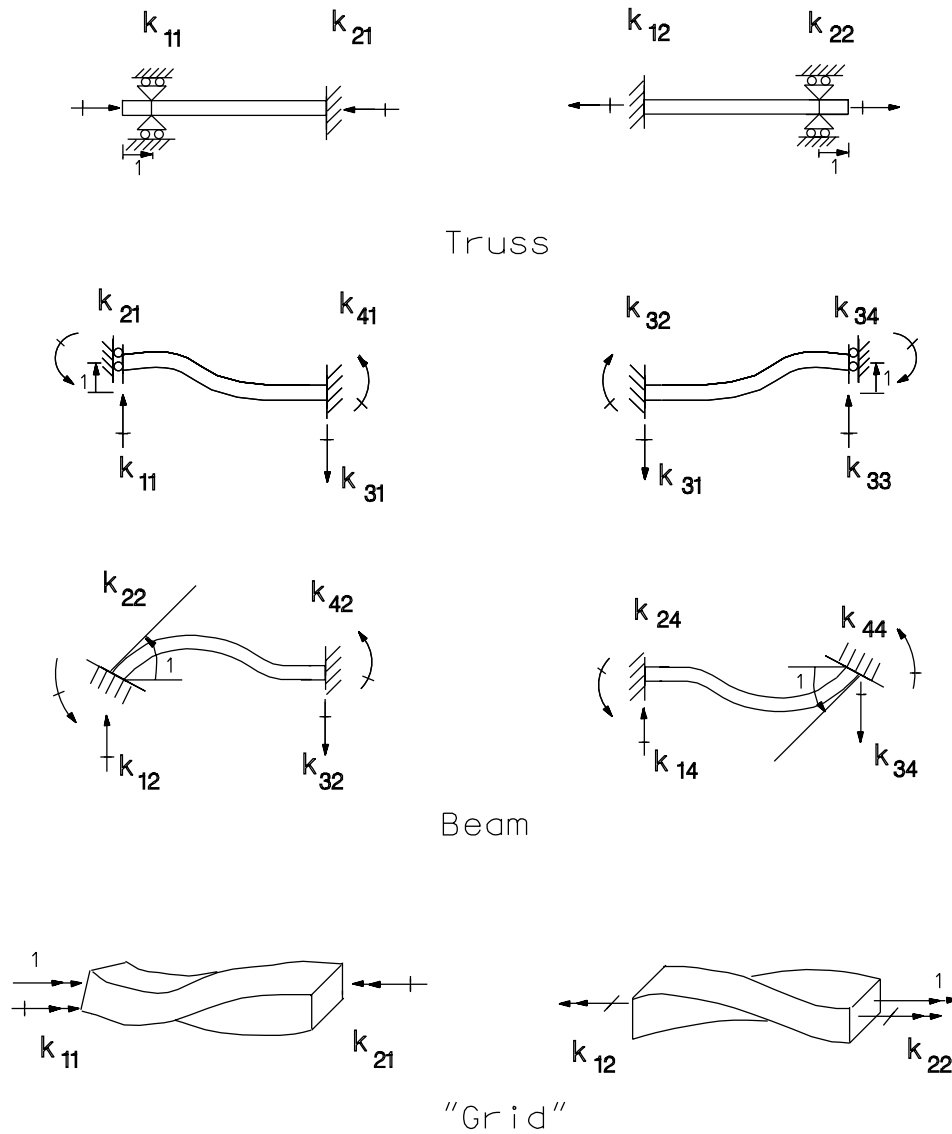


Figure 2.2: Definition of Element Stiffness Coefficients

Upon substitution, the grid element stiffness matrix is given by

$$[\mathbf{k}^g] = \begin{bmatrix} T_{1x} & V_{1y} & M_{1z} & T_{2x} & V_{2y} & M_{2z} \end{bmatrix} \begin{bmatrix} \alpha_{1x} & u_{1y} & \beta_{1z} & \alpha_{2x} & u_{2y} & \beta_{2z} \\ \frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{GI_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \quad (2.46)$$

⁵² Note that if shear deformations must be accounted for, the entries corresponding to shear and flexure must be modified in accordance with Eq. 2.42

2.6.5 3D Frame Element

$$[\mathbf{k}^{3dfr}] = \begin{bmatrix} P_{x1} & V_{y1} & V_{z1} & T_{x1} & M_{y1} & M_{z1} & P_{x2} & V_{y2} & V_{z2} & T_{x2} & M_{y2} & M_{z2} \end{bmatrix} \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\ k_{11}^t & 0 & 0 & 0 & 0 & 0 & k_{21}^t & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{11}^b & 0 & 0 & 0 & k_{12}^b & 0 & k_{13}^b & 0 & 0 & 0 & k_{14}^b \\ 0 & 0 & k_{11}^b & 0 & -k_{12}^b & 0 & 0 & 0 & k_{13}^b & 0 & -k_{14}^b & 0 \\ 0 & 0 & 0 & k_{11}^g & 0 & 0 & 0 & 0 & 0 & k_{12}^g & 0 & 0 \\ 0 & 0 & k_{32}^b & 0 & k_{22}^b & 0 & 0 & 0 & k_{12}^b & 0 & k_{24}^b & 0 \\ 0 & k_{21}^b & 0 & 0 & 0 & k_{22}^b & 0 & -k_{12}^b & 0 & 0 & 0 & k_{24}^b \\ k_{21}^t & 0 & 0 & 0 & 0 & 0 & k_{22}^t & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{13}^b & 0 & 0 & 0 & k_{14}^b & 0 & k_{33}^b & 0 & 0 & 0 & k_{34}^b \\ 0 & 0 & k_{13}^b & 0 & -k_{14}^b & 0 & 0 & 0 & k_{33}^b & 0 & -k_{34}^b & 0 \\ 0 & 0 & 0 & k_{12}^g & 0 & 0 & 0 & 0 & 0 & k_{22}^g & 0 & 0 \\ 0 & 0 & k_{12}^b & 0 & k_{24}^b & 0 & 0 & 0 & -k_{43}^b & 0 & k_{44}^b & 0 \\ 0 & -k_{12}^b & 0 & 0 & 0 & k_{24}^b & 0 & k_{43}^b & 0 & 0 & 0 & k_{44}^b \end{bmatrix} \quad (2.47)$$

For $[\mathbf{k}_{11}^{3D}]$ and with we obtain:

$$[\mathbf{k}^{3dfr}] = \begin{bmatrix} P_{x1} & V_{y1} & V_{z1} & T_{x1} & M_{y1} & M_{z1} & P_{x2} & V_{y2} & V_{z2} & T_{x2} & M_{y2} & M_{z2} \end{bmatrix} \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\ \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\ -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\ 0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix} \quad (2.48)$$

⁵³ Note that if shear deformations must be accounted for, the entries corresponding to shear and flexure must be modified in accordance with Eq. 2.42

2.7 Remarks on Element Stiffness Matrices

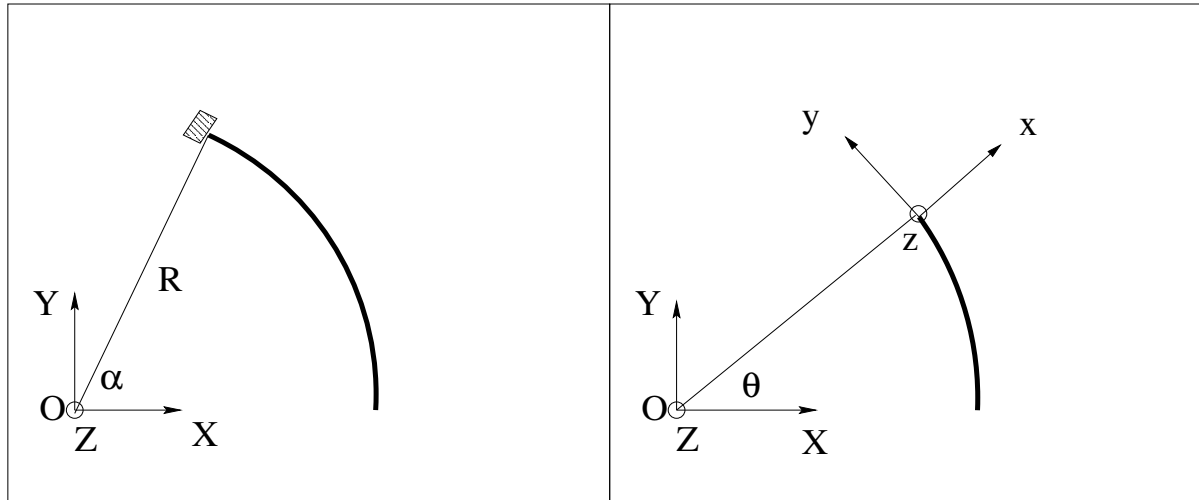
Singularity: All the derived stiffness matrices are singular, that is there is at least one row and one column which is a linear combination of others. For example in the beam element, row 4 = -row

2.8 Homework

Using the virtual force method, derive the flexibility matrix of a semi-circular box-girder of radius R and angle α in terms of shear, axial force, and moment.

The arch is clamped at one end, and free at the other.

Note: In a later assignment, you will combine the flexibility matrix with equilibrium relations to derive the element stiffness matrix.



Chapter 3

STIFFNESS METHOD; Part I: ORTHOGONAL STRUCTURES

3.1 Introduction

- ¹ In the previous chapter we have first derived displacement force relations for different types of rod elements, and then used those relations to define element stiffness matrices in local coordinates.
- ² In this chapter, we seek to perform similar operations, but for an orthogonal structure in global coordinates.
- ³ In the previous chapter our starting point was basic displacement-force relations resulting in element stiffness matrices $[\mathbf{k}]$.
- ⁴ In this chapter, our starting point are those same element stiffness matrices $[\mathbf{k}]$, and our objective is to determine the structure stiffness matrix $[\mathbf{K}]$, which when inverted, would yield the nodal displacements.
- ⁵ The element stiffness matrices were derived for fully restrained elements.
- ⁶ This chapter will be restricted to orthogonal structures, and generalization will be discussed later. The stiffness matrices will be restricted to the unrestrained degrees of freedom.
- ⁷ From these examples, the interrelationships between structure stiffness matrix, nodal displacements, and fixed end actions will become apparent. Then the method will be generalized in chapter 5 to describe an algorithm which can automate the assembly of the structure global stiffness matrix in terms of the one of its individual elements.

3.2 The Stiffness Method

- ⁸ As a “vehicle” for the introduction to the stiffness method let us consider the problem in Fig 3.1-a, and recognize that there are only two unknown displacements, or more precisely, two global d.o.f: θ_1 and θ_2 .
- ⁹ If we were to analyse this problem by the force (or flexibility) method, then
 1. We make the structure statically determinate by removing *arbitrarily* two reactions (as long as the structure remains stable), and the beam is now statically determinate.
 2. Assuming that we remove the two roller supports, then we determine the corresponding deflections due to the actual load (Δ_B and Δ_C).

Chapter 4

TRANSFORMATION MATRICES

4.1 Preliminaries

4.1.1 $[\mathbf{k}^e]$ $[\mathbf{K}^e]$ Relation

¹ In the previous chapter, in which we focused on orthogonal structures, the assembly of the structure's stiffness matrix $[\mathbf{K}^e]$ in terms of the element stiffness matrices was relatively straight-forward.

² The determination of the element stiffness matrix in global coordinates, from the element stiffness matrix in local coordinates requires the introduction of a transformation.

³ This chapter will examine the 2D and 3D transformations required to obtain an element stiffness matrix in global coordinate system prior to assembly (as discussed in the next chapter).

⁴ Recalling that

$$\{\mathbf{p}\} = [\mathbf{k}^{(e)}]\{\boldsymbol{\delta}\} \quad (4.1)$$

$$\{\mathbf{P}\} = [\mathbf{K}^{(e)}]\{\boldsymbol{\Delta}\} \quad (4.2)$$

⁵ Let us define a transformation matrix $[\boldsymbol{\Gamma}^{(e)}]$ such that:

$$\{\boldsymbol{\delta}\} = [\boldsymbol{\Gamma}^{(e)}]\{\boldsymbol{\Delta}\} \quad (4.3)$$

$$\{\mathbf{p}\} = [\boldsymbol{\Gamma}^{(e)}]\{\mathbf{P}\} \quad (4.4)$$

Note that we use the same matrix $\boldsymbol{\Gamma}^{(e)}$ since both $\{\boldsymbol{\delta}\}$ and $\{\mathbf{p}\}$ are vector quantities (or tensors of order one).

⁶ Substituting Eqn. 4.3 and Eqn. 4.4 into Eqn. 4.1 we obtain

$$[\boldsymbol{\Gamma}^{(e)}]\{\mathbf{P}\} = [\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]\{\boldsymbol{\Delta}\} \quad (4.5)$$

premultiplying by $[\boldsymbol{\Gamma}^{(e)}]^{-1}$

$$\{\mathbf{P}\} = [\boldsymbol{\Gamma}^{(e)}]^{-1}[\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]\{\boldsymbol{\Delta}\} \quad (4.6)$$

⁷ But since the rotation matrix is orthogonal, we have $[\boldsymbol{\Gamma}^{(e)}]^{-1} = [\boldsymbol{\Gamma}^{(e)}]^T$ and

$$\{\mathbf{P}\} = \underbrace{[\boldsymbol{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]}_{[\mathbf{K}^{(e)}]}\{\boldsymbol{\Delta}\} \quad (4.7)$$

$$\boxed{[\mathbf{K}^{(e)}] = [\boldsymbol{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]} \quad (4.8)$$

which is the general relationship between element stiffness matrix in local and global coordinates.

where l_{ij} is the direction cosine of axis i with respect to axis j , and thus the rows of the matrix correspond to the rotated vectors with respect to the original ones corresponding to the columns.

¹¹ With respect to Fig. 4.2, $l_{xX} = \cos \alpha$; $l_{xY} = \cos \beta$, and $l_{xZ} = \cos \gamma$ or

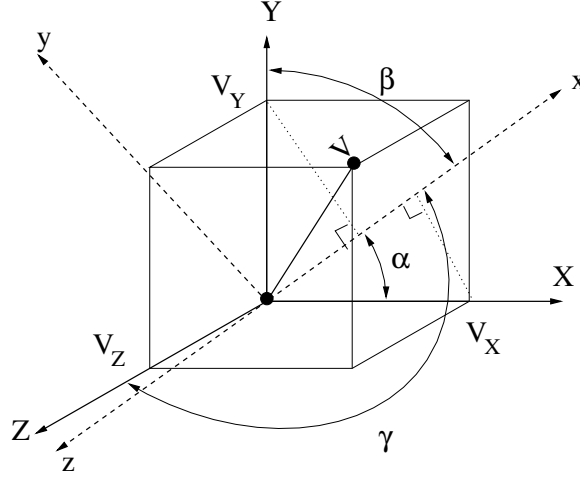


Figure 4.2: 3D Vector Transformation

$$V_x = V_X l_{xX} + V_Y l_{xY} + V_Z l_{xZ} \quad (4.14-a)$$

$$= V_X \cos \alpha + V_Y \cos \beta + V_Z \cos \gamma \quad (4.14-b)$$

¹² Direction cosines are unit orthogonal vectors satisfying the following relations:

$$\sum_{j=1}^3 l_{ij} l_{ij} = 1 \quad i = 1, 2, 3 \quad (4.15)$$

i.e:

$$l_{11}^2 + l_{12}^2 + l_{13}^2 = 1 \quad \text{or} \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 = \delta_{11} \quad (4.16)$$

and

$$\sum_{j=1}^3 l_{ij} l_{kj} = 0 \quad \begin{cases} i = 1, 2, 3 \\ k = 1, 2, 3 \\ i \neq k \end{cases} \quad (4.17-a)$$

$$l_{11} l_{21} + l_{12} l_{22} + l_{13} l_{23} = 0 = \delta_{12} \quad (4.17-b)$$

¹³ By direct multiplication of $[\gamma]^T$ and $[\gamma]$ it can be shown that: $[\gamma]^T [\gamma] = [\mathbf{I}] \Rightarrow [\gamma]^T = [\gamma]^{-1} \Rightarrow [\gamma]$ is an orthogonal matrix.

¹⁴ The reverse transformation (from local to global) would be

$$\{\mathbf{V}\} = [\gamma]^T \{\mathbf{v}\} \quad (4.18)$$

or

$$\begin{Bmatrix} V_X \\ V_Y \\ V_Z \end{Bmatrix} = \underbrace{\begin{bmatrix} l_{xX} & l_{yX} & l_{zX} \\ l_{xY} & l_{yY} & l_{zY} \\ l_{xZ} & l_{yZ} & l_{zZ} \end{bmatrix}}_{[\gamma]^{-1}=[\gamma]^T} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \quad (4.19)$$

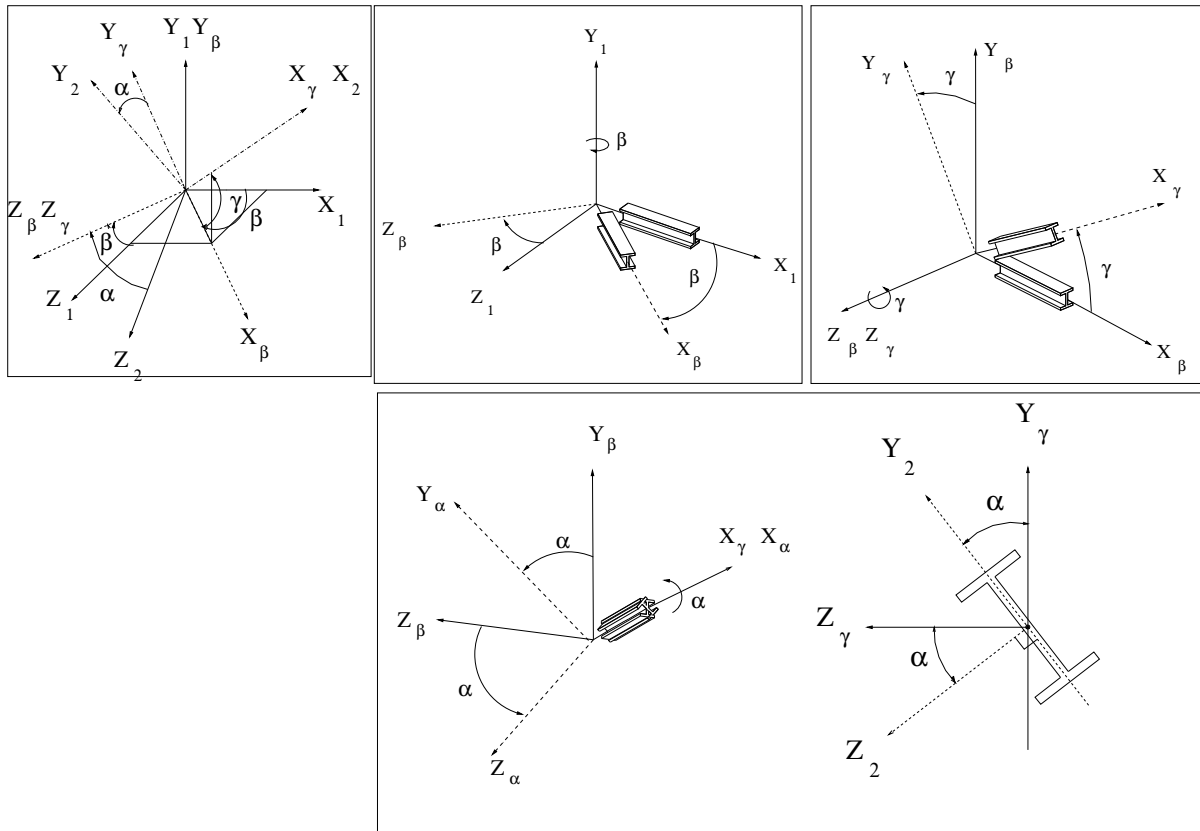


Figure 4.8: Complex 3D Rotation

Chapter 5

STIFFNESS METHOD; Part II

5.1 Direct Stiffness Method

5.1.1 Global Stiffness Matrix

¹ The physical interpretation of the global stiffness matrix \mathbf{K} is analogous to the one of the element, i.e. If all degrees of freedom are restrained, then K_{ij} corresponds to the force along global degree of freedom i due to a unit positive displacement (or rotation) along global degree of freedom j .

² For instance, with reference to Fig. 5.1, we have three global degrees of freedom, Δ_1 , Δ_2 , and θ_3 . and

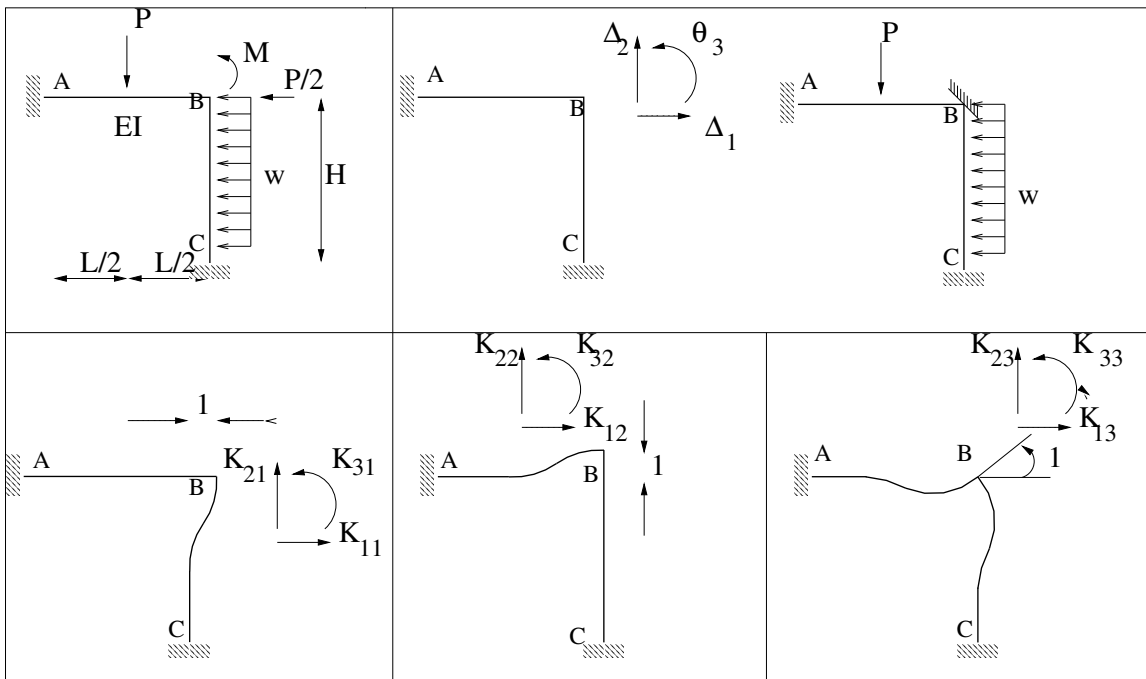


Figure 5.1: Frame Example

at the element level where $p_{int}^{(e)}$ is the six by six array of internal forces, $\mathbf{k}^{(e)}$ the element stiffness matrix in local coordinate systems, and $\delta^{(e)}$ is the vector of nodal displacements in local coordinate system. Note that this last array is obtained by first identifying the displacements in global coordinate system, and then premultiplying it by the transformation matrix to obtain the displacements in local coordinate system.

5.2 Logistics

5.2.1 Boundary Conditions, [ID] Matrix

¹⁰ Because of the boundary condition restraints, the total structure number of *active* degrees of freedom (i.e unconstrained) will be less than the number of nodes times the number of degrees of freedom per node.

¹¹ To obtain the global degree of freedom for a given node, we need to define an [ID] matrix such that:

ID has dimensions $l \times k$ where l is the number of degree of freedom per node, and k is the number of nodes).

ID matrix is initialized to zero.

1. At input stage read ID(idof, inod) of each degree of freedom for every node such that:

$$\text{ID}(\text{idof}, \text{inod}) = \begin{cases} 0 & \text{if unrestrained d.o.f.} \\ 1 & \text{if restrained d.o.f.} \end{cases} \quad (5.6)$$

2. After all the node boundary conditions have been read, assign incrementally equation numbers

- (a) First to all the active dof
- (b) Then to the other (restrained) dof, starting with -1.

Note that the total number of dof will be equal to the number of nodes times the number of dof/node NEQA.

3. The largest *positive* global degree of freedom number will be equal to NEQ (Number Of Equations), which is the size of the square matrix which will have to be decomposed.

¹² For example, for the frame shown in Fig. 5.2:

1. The input data file may contain:

Node No.	[ID] ^T
1	0 0 0
2	1 1 0
3	0 0 0
4	1 0 0

2. At this stage, the [ID] matrix is equal to:

$$\mathbf{ID} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.7)$$

3. After we determined the equation numbers, we would have:

$$\mathbf{ID} = \begin{bmatrix} 1 & -10 & 5 & -12 \\ 2 & -11 & 6 & 8 \\ 3 & 4 & 7 & 9 \end{bmatrix} \quad (5.8)$$

$[\mathbf{K}^S]_{(NEQA \times NEQA)}$ to zero, and then loop through all the elements, and then through each entry of the respective element stiffness matrix $K_{ij}^{(e)}$.

20 The assignment of the element stiffness matrix term $K_{ij}^{(e)}$ (note that e , i , and j are all known since we are looping on e from 1 to the number of elements, and then looping on the rows and columns of the element stiffness matrix i, j) into the global stiffness matrix K_{kl}^S is made through the LM vector (note that it is k and l which must be determined).

21 Since the global stiffness matrix is also symmetric, we would need to only assemble one side of it, usually the upper one.

22 Contrarily to the previous method, we will assemble the full *augmented* stiffness matrix.

■ Example 5-1: Assembly of the Global Stiffness Matrix

As an example, let us consider the frame shown in Fig. 5.3.

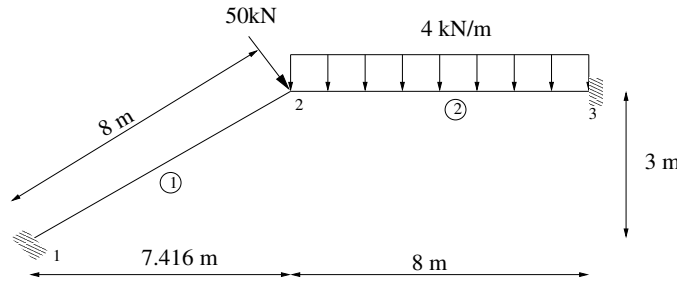


Figure 5.3: Simple Frame Analyzed with the MATLAB Code

The ID matrix is initially set to:

$$[ID] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad (5.9)$$

We then modify it to generate the global degrees of freedom of each node:

$$[ID] = \begin{bmatrix} -4 & 1 & -7 \\ -5 & 2 & -8 \\ -6 & 3 & -9 \end{bmatrix} \quad (5.10)$$

Finally the LM vectors for the two elements (assuming that Element 1 is defined from node 1 to node 2, and element 2 from node 2 to node 3):

$$[LM] = \left[\begin{array}{ccc|ccc} -4 & -5 & -6 & 1 & 2 & 3 \\ 1 & 2 & 3 & -7 & -8 & -9 \end{array} \right] \quad (5.11)$$

Let us simplify the operation by designating the element stiffness matrices in global coordinates as follows:

$$K^{(1)} = \begin{matrix} & \begin{matrix} -4 & -5 & -6 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} -4 \\ -5 \\ -6 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \end{matrix} \quad (5.12-a)$$

5. Backsubstitute and obtain nodal displacements in global coordinate system.
6. Solve for the reactions, Eq. 5.3.
7. For each element, transform its nodal displacement from global to local coordinates $\{\delta\} = [\Gamma^{(e)}]\{\Delta\}$, and determine the internal forces $[p] = [k]\{\delta\}$.

²⁴ Some of the prescribed steps are further discussed in the next sections.

■ Example 5-2: Direct Stiffness Analysis of a Truss

Using the direct stiffness method, analyze the truss shown in Fig. 5.4.

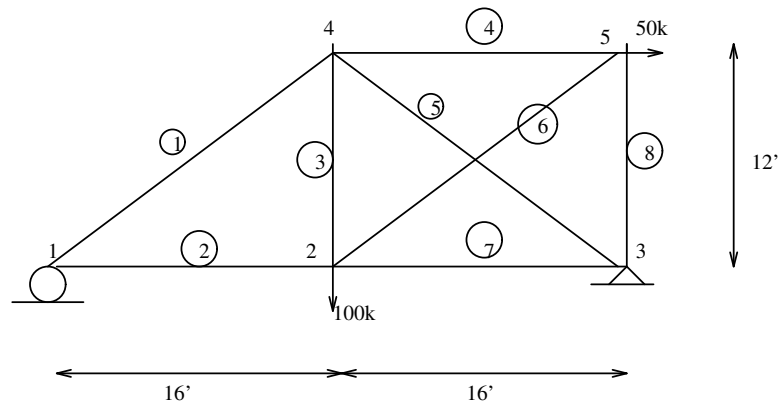


Figure 5.4:

Solution:

1. Determine the structure ID matrix

Node #	Bound. Cond.	
	X	Y
1	0	1
2	0	0
3	1	1
4	0	0
5	0	0

$$ID = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (5.14-a)$$

$$= \begin{matrix} & \text{Node} & 1 & 2 & 3 & 4 & 5 \\ \begin{bmatrix} 1 & 2 & -9 & 4 & 6 \\ -8 & 3 & -10 & 5 & 7 \end{bmatrix} & & & & & & \end{matrix} \quad (5.14-b)$$

2. The LM vector of each element is evaluated next

$$[LM]_1 = [1 \quad -8 \quad 4 \quad 5] \quad (5.15-a)$$

$$[LM]_2 = [1 \quad -8 \quad 2 \quad 3] \quad (5.15-b)$$

$$[LM]_3 = [2 \quad 3 \quad 4 \quad 5] \quad (5.15-c)$$

$$[LM]_4 = [4 \quad 5 \quad 6 \quad 7] \quad (5.15-d)$$

```

% FEA_Rest for all the restrained nodes
FEA_Rest=[0,0,0,FEA(4:6,2)'];
% Assemble the load vector for the unrestrained node
P(1)=50*3/8;P(2)=-50*7.416/8-FEA(2,2);P(3)=-FEA(3,2);
% Solve for the Displacements in meters and radians
Displacements=inv(Ktt)*P'
% Extract Kut
Kut=Kaug(4:9,1:3);
% Compute the Reactions and do not forget to add fixed end actions
Reactions=Kut*Displacements+FEA_Rest'
% Solve for the internal forces and do not forget to include the fixed end actions
dis_global(:,1)=[0,0,0,Displacements(1:3)'];
dis_global(:,2)=[Displacements(1:3)',0,0,0]; for elem=1:2
    dis_local=Gamma(:,elem)*dis_global(:,elem)';
    int_forces=k(:,elem)*dis_local+fea(1:6,elem)
end

function [k,K,Gamma]=stiff(EE,II,A,i,j)
% Determine the length
L=sqrt((j(2)-i(2))^2+(j(1)-i(1))^2);
% Compute the angle theta (careful with vertical members!)
if(j(1)-i(1))~=0
    alpha=atan((j(2)-i(2))/(j(1)-i(1)));
else
    alpha=-pi/2;
end
% form rotation matrix Gamma
Gamma=[ cos(alpha)  sin(alpha)  0  0  0  0; -sin(alpha)
cos(alpha)  0  0  0  0; 0  0  0  0  0  1  0  0
0; 0  0  0  cos(alpha) sin(alpha) 0; 0  0  0
0 -sin(alpha) cos(alpha) 0; 0  0  0  0  0  0
1];
% form element stiffness matrix in local coordinate system
EI=EE*II; EA=EE*A; k=[EA/L, 0, 0, -EA/L, 0, 0;
0, 12*EI/L^3, 6*EI/L^2, 0, -12*EI/L^3, 6*EI/L^2;
0, 6*EI/L^2, 4*EI/L, 0, -6*EI/L^2, 2*EI/L;
-EA/L, 0, 0, EA/L, 0, 0;
0, -12*EI/L^3, -6*EI/L^2, 0, 12*EI/L^3, -6*EI/L^2;
0, 6*EI/L^2, 2*EI/L, 0, -6*EI/L^2, 4*EI/L];
% Element stiffness matrix in global coordinate system
K=Gamma'*k*Gamma;

```

This simple proigram will produce the following results:

Displacements =

```

0.0010
-0.0050
-0.0005

```

Reactions =

```

130.4973
55.6766
13.3742

```

```

    react3

    % CALCULATE THE INTERNAL FORCES FOR EACH ELEMENT

    intern3

    %    END LOOP FOR EACH LOAD CASE

end

%    DRAW THE STRUCTURE, IF USER HAS REQUESTED (DRAWFLAG=1)
%    CALL SCRIPTFILE DRAW.M

draw

st=fclose('all');
%    END OF MAIN PROGRAM (CASAP.M)
disp('Program completed! - See "casap.out" for complete output');

```

5.6.2.2 Assembly of ID Matrix

```

%*****
%SCRIPTFILE NAME:  IDRASMBL.M
%
%MAIN FILE      :   CASAP
%
%Description    :   This file re-assembles the ID matrix such that the restrained
%                   degrees of freedom are given negative values and the unrestrained
%                   degrees of freedom are given incremental values beginning with one
%                   and ending with the total number of unrestrained degrees of freedom.
%
%
%*****

%    TAKE CARE OF SOME INITIAL BUSINESS: TRANSPOSE THE PNODS ARRAY

Pnods=Pnods.';

%    SET THE COUNTER TO ZERO

count=1;
negcount=-1;

%    REASSEMBLE THE ID MATRIX

if istrtp==3
    ndofpn=3;
    nterm=6;
else
    error('Incorrect structure type specified')
end

%    SET THE ORIGINAL ID MATRIX TO TEMP MATRIX

orig_ID=ID;

%    REASSEMBLE THE ID MATRIX, SUBSTITUTING RESTRAINED DEGREES OF FREEDOM WITH NEGATIVES,
%    AND NUMBERING GLOBAL DEGREES OF FREEDOM

for inode=1:npoin
    XXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXX
%    END OF IDRASMBL.M SCRIPTFILE

```

5.6.2.3 Element Nodal Coordinates

```

%*****

```

the unknown forces and reactions can be determined through inversion of $[\mathcal{B}]$:

$$\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{Bmatrix}}_{\{\mathbf{F}\}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1 \end{bmatrix}}_{[\mathcal{B}]^{-1}} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\{\mathbf{P}\}} = \underbrace{\begin{Bmatrix} P_{y2} \\ -P_{x2} \\ \frac{P_{x2}}{C} \\ -\frac{S}{C}P_{x2} \\ P_{x2} \\ \frac{S}{C}P_{x2} + P_{y2} \\ 0 \\ -\frac{S}{C}P_{x2} \end{Bmatrix}}_{\{\mathbf{F}\}} \quad (6.3)$$

We observe that the matrix $[\mathcal{B}]$ is totally independent of the external load, and once inverted can be used for multiple load cases with minimal computational efforts. ■

■ Example 6-2: Beam Statics Matrix

Considering the beam shown in Fig. 6.2, we have 3 elements, each with 2 internal unknowns (v and m) plus two unknown reactions, for a total of 8 unknowns. To solve for those unknowns we have 2 equations of equilibrium at each of the 4 nodes. Note that in this problem we have selected as primary unknowns the shear and moment at the right end of each element. The left components can be recovered from equilibrium. From equilibrium we thus have:

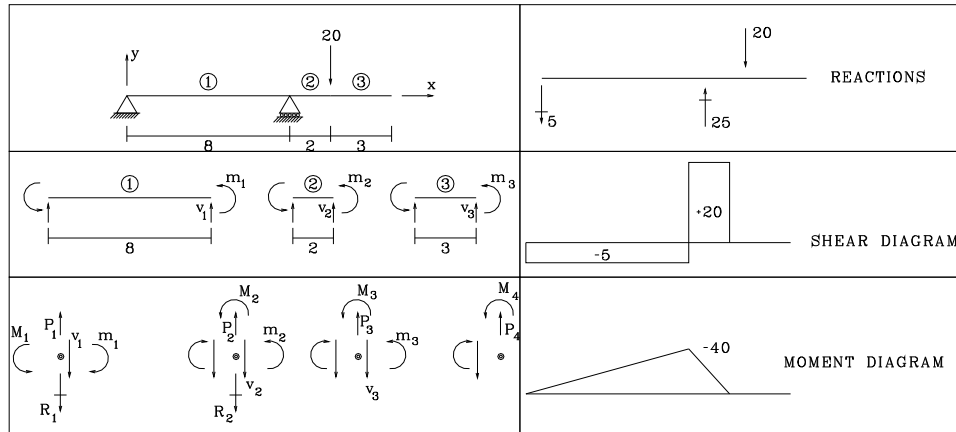
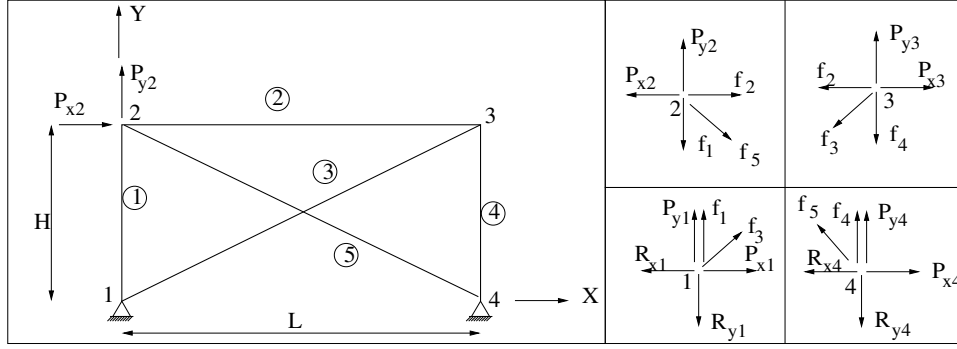


Figure 6.2: Example of $[\mathcal{B}]$ Matrix for a Statically Determinate Beam

$$\underbrace{\begin{Bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \\ P_3 \\ M_3 \\ P_4 \\ M_4 \end{Bmatrix}}_{\{\mathbf{P}\}} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{[\mathcal{B}]} \underbrace{\begin{Bmatrix} v_1 \\ m_1 \\ v_2 \\ m_2 \\ v_3 \\ m_3 \\ R_1 \\ R_2 \end{Bmatrix}}_{\{\mathbf{F}\}} \quad (6.4)$$

Figure 6.3: Example of $[\mathcal{B}]$ Matrix for a Statically Indeterminate Truss

We can solve for the internal forces in terms of the (still unknown) redundant force

$$\{\mathbf{F}_0\} = [\mathcal{B}_0]^{-1} \{\mathbf{P}\} - [\mathcal{B}_0]^{-1} [\mathcal{B}_x] \{\mathbf{F}_x\} \quad (6.10-a)$$

$$\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{Bmatrix}}_{\{\mathbf{F}_0\}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1 \end{bmatrix}}_{[\mathcal{B}_0]^{-1} \equiv [\mathcal{C}_1]} \left[\underbrace{\begin{Bmatrix} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\{\mathbf{P}\}} - \underbrace{\begin{Bmatrix} 0 \\ 0 \\ -C \\ S \\ 0 \\ 0 \\ C \\ -S \end{Bmatrix}}_{[\mathcal{B}_x]} \underbrace{\begin{Bmatrix} F_5 \\ \mathbf{F}_x \end{Bmatrix}}_{\{\mathbf{F}_x\}} \right] \quad (6.10-b)$$

Or using the following relations $[\mathcal{B}_0]^{-1} \equiv [\mathcal{C}_1]$ and $-[\mathcal{B}_0]^{-1} [\mathcal{B}_x] \equiv [\mathcal{C}_2]$ we obtain

$$\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{Bmatrix}}_{\{\mathbf{F}_0\}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1 \end{bmatrix}}_{[\mathcal{C}_1]} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\{\mathbf{P}\}} + \underbrace{\begin{Bmatrix} -S \\ -C \\ 1 \\ -S \\ C \\ 0 \\ -C \\ 0 \end{Bmatrix}}_{[\mathcal{C}_2]} \underbrace{\begin{Bmatrix} F_5 \\ \mathbf{F}_x \end{Bmatrix}}_{\{\mathbf{F}_x\}} \quad (6.11)$$

Note, that this equation is not sufficient to solve for the unknown forces, as $\{\mathbf{F}_x\}$ must be obtained through force displacement relations ($[\mathbf{D}]$ or $[\mathbf{K}]$). ■

10 *ddag* Whereas the identification of redundant forces was done by mere inspection of the structure in hand based analysis of structure, this identification process can be automated.

11 Starting with

$$\begin{aligned} \{\mathbf{P}\}_{2n \times 1} &= [\mathcal{B}]_{2n \times (2n+r)} \{\mathbf{F}\}_{(2n+r) \times 1} \\ [\mathcal{B}]_{2n \times (2n+r)} \{\mathbf{F}\}_{2n+r \times 1} - [\mathbf{I}]_{2n \times 2n} \{\mathbf{P}\}_{2n \times 1} &= \{\mathbf{0}\} \\ [\mathcal{B} \mid -\mathbf{I}]_{2n \times (4n+r)} \left\{ \begin{Bmatrix} \mathbf{F} \\ \mathbf{P} \end{Bmatrix} \right\} &= \{\mathbf{0}\} \end{aligned} \quad (6.12-a)$$

2. Interchange columns

$$\begin{array}{c}
 R_{x1} \quad R_{y1} \quad F_2 \quad F_1 \quad F_3 \quad F_4 \quad F_5 \quad R_{x4} \quad R_{y4} \quad P_{x2} \quad P_{y2} \\
 \begin{array}{l} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \left[\begin{array}{cccccccccccc}
 1 & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & -1 & -S & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & -C & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & S & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 1 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & S & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & -S & 0 & 1 & 0 & 0 & 0
 \end{array} \right]
 \end{array} \quad (6.17)$$

3. Operate as indicates

$$\begin{array}{c}
 R_{x1} \quad R_{y1} \quad F_2 \quad F_1 \quad F_3 \quad F_4 \quad F_5 \quad R_{x4} \quad R_{y4} \quad P_{x2} \quad P_{y2} \\
 \begin{array}{l} A \\ B' = B + DB \\ C \\ D \\ E' = \frac{E+C}{C} \\ F \\ G \\ H \end{array} \left[\begin{array}{cccccccccccc}
 1 & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & -S & 0 & S & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & -C & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & S & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1/C & 0 & 0 \\
 0 & 0 & 0 & 0 & S & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & -S & 0 & 1 & 0 & 0 & 0
 \end{array} \right]
 \end{array} \quad (6.18)$$

4. Operate as indicated

$$\begin{array}{c}
 R_{x1} \quad R_{y1} \quad F_2 \quad F_1 \quad F_3 \quad F_4 \quad F_5 \quad R_{x4} \quad R_{y4} \quad P_{x2} \quad P_{y2} \\
 \begin{array}{l} A' = A + CE' \\ B'' = B' + SE' \\ C' = -C \\ D \\ E' \\ F' = F - SE' \\ G \\ H' = H + F' \end{array} \left[\begin{array}{cccccccccccc}
 1 & 0 & 0 & 0 & 0 & 0 & -C & 0 & 0 & -1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S/C & -1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & C & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & S & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1/C & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & S & 0 & 0 & S/C & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & S/C & 0 & 0
 \end{array} \right]
 \end{array} \quad (6.19)$$

5. Interchange columns and observe that F_5 is the selected redundant.

$$\begin{array}{c}
 R_{x1} \quad R_{y1} \quad F_2 \quad F_1 \quad F_3 \quad F_4 \quad R_{x4} \quad R_{y4} \quad F_5 \quad P_{x2} \quad P_{y2} \\
 \begin{array}{l} A' \\ B'' \\ C' \\ D \\ E' \\ F' \\ G \\ H' \end{array} \left[\begin{array}{ccccccccccc}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C & -1 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S/C & -1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & S & 0 & -1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1/C & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & S & S/C & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & C & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & S/C & 0
 \end{array} \right]
 \end{array} \quad (6.20)$$

¹⁷ $[\mathcal{A}]$ is a rectangular matrix which number of rows is equal to the number of the element internal displacements, and the number of columns is equal to the number of nodal displacements. Contrarily to the rotation matrix introduced earlier and which transforms the displacements from global to local coordinate for one *single* element, the kinematics matrix applies to the entire structure.

¹⁸ It can be easily shown that for trusses (which corresponds to shortening or elongation of the member):

$$\Upsilon^e = (u_2 - u_1) \cos \alpha + (v_2 - v_1) \sin \alpha \quad (6.23)$$

where α is the angle between the element and the X axis. whereas for flexural members:

$$v_{21} = v_2 - v_1 - \theta_{z1}L \quad (6.24\text{-a})$$

$$\theta_{z21} = \theta_{z2} - \theta_{z1} \quad (6.24\text{-b})$$

■ Example 6-5: Kinematics Matrix of a Truss

Considering again the statically indeterminate truss of the previous example, the kinematic matrix will be given by:

$$\left\{ \begin{array}{c} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \Delta_5^e \\ \hline u_1 \\ v_1 \\ u_4 \\ v_4 \end{array} \right\} = \underbrace{\left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -C & -S & 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -C & S & 0 & 0 & C & -S \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]}_{[\mathcal{A}]} \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \right\} \quad (6.25)$$

Applying the constraints: $u_1 = 0$; $v_1 = 0$; $u_4 = 0$; and $v_4 = 0$ we obtain:

$$\left\{ \begin{array}{c} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \Delta_5^e \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} = \underbrace{\left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -C & -S & 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -C & S & 0 & 0 & C & -S \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]}_{[\mathcal{A}]} \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \right\} \quad (6.26)$$

We should observe that $[\mathcal{A}]$ is indeed the transpose of the $[\mathcal{B}]$ matrix in Eq. 6.15 ■

6.3 Statics-Kinematics Matrix Relationship

¹⁹ Having defined both the statics $[\mathcal{B}]$ and kinematics $[\mathcal{A}]$ matrices, it is intuitive that those two matrices must be related. In this section we seek to determine this relationship for both the statically determinate and statically indeterminate cases.

6.4 Kinematic Relations through Inverse of Statics Matrix

We now seek to derive some additional relations between the displacements through the inverse of the statics matrix. Those relations will be used later in the flexibility methods, and have no immediate applications.

Rewriting Eq. 6.33 as

$$\{\Delta\} = [\mathcal{A}_0]^{-1} \{\Upsilon_0\} = \left[[\mathcal{B}_0]^t \right]^{-1} \{\Upsilon_0\} = \left[[\mathcal{B}_0]^{-1} \right]^t \{\Upsilon_0\} \quad (6.36)$$

we can solve for $\{\mathbf{F}_0\}$ from Eq. 6.7

$$\{\mathbf{F}_0\} = \underbrace{[\mathcal{B}_0]^{-1}}_{[C_1]} \{\mathbf{P}\} - \underbrace{[\mathcal{B}_0]^{-1} [\mathcal{B}_x]}_{[C_2]} \{\mathbf{F}_x\} \quad (6.37)$$

Combining this equation with $[\mathcal{B}_0]^{-1} = [C_1]$ from Eq. 6.37, and with Eq. 6.36 we obtain

$$\boxed{\{\Delta\} = [C_1]^t \{\Upsilon_0\}} \quad (6.38)$$

Similarly, we can revisit Eq. 6.33 and write

$$\{\Upsilon_x\} = [\mathcal{A}_x] \{\Delta\} \quad (6.39)$$

When the previous equation is combined with the rightmost side of Eq. 6.36 and 6.35 we obtain

$$\{\Upsilon_x\} = [\mathcal{B}_x]^t \left[[\mathcal{B}_0]^{-1} \right]^t \{\Upsilon_0\} \quad (6.40)$$

Thus, with $[\mathcal{B}_0]^{-1} [\mathcal{B}_x] = -[C_2]$ from Eq. 6.37

$$\boxed{\{\Upsilon_x\} = -[C_2]^t \{\Upsilon_0\}} \quad (6.41)$$

This equation relates the unknown relative displacements to the relative known ones.

6.5 ‡ Congruent Transformation Approach to $[\mathbf{K}]$

Note: This section is largely based on section 3.3 of Gallagher, *Finite Element Analysis*, Prentice Hall.

For an arbitrary structure composed of n elements, we can define the *unconnected* nodal load and displacement vectors in global coordinate as

$$\{\mathbf{P}^e\} = \begin{bmatrix} [\mathbf{P}^1] & [\mathbf{P}^2] & \dots & [\mathbf{P}^n] \end{bmatrix}^T \quad (6.42-a)$$

$$\{\Upsilon^e\} = \begin{bmatrix} [\Upsilon^1] & [\Upsilon^2] & \dots & [\Upsilon^n] \end{bmatrix}^T \quad (6.42-b)$$

where $\{\mathbf{P}^i\}$ and $\{\Delta^i\}$ are the nodal load and displacements arrays of element i . The size of each submatrix (or more precisely of each subarray) is equal to the total number of d.o.f. in global coordinate for element i .

Similarly, we can define the unconnected (or unassembled) global stiffness matrix of the structure as $[\mathbf{K}^e]$:

$$\{\mathbf{F}\} = [\mathbf{K}^e] \{\Upsilon\} \quad (6.43-a)$$

$$[\mathbf{K}^e] = \begin{bmatrix} [\mathbf{K}^1] & & & \\ & [\mathbf{K}^2] & & \\ & & [\mathbf{K}^3] & \\ & & & \ddots \\ & & & & [\mathbf{K}^n] \end{bmatrix} \quad (6.43-b)$$

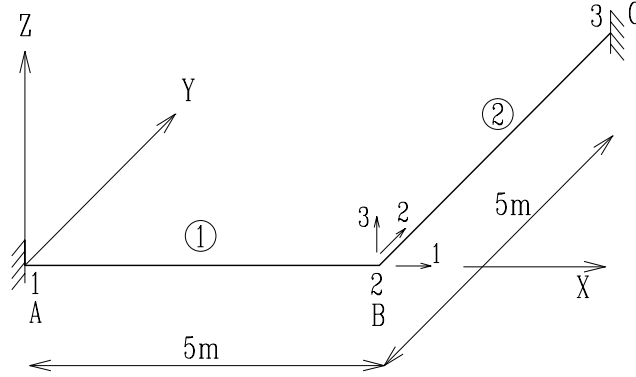


Figure 6.4: Example 1, Congruent Transfer

We shall determine the global stiffness matrix using the two approaches:

Direct Stiffness

$$[\mathbf{ID}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix} \quad (6.47-a)$$

$$\{\mathbf{LM}^1\} = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}^T \quad (6.47-b)$$

$$\{\mathbf{LM}^2\} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T \quad (6.47-c)$$

$$\begin{aligned} [\mathbf{K}] &= \begin{bmatrix} (7.692 + 1 \times 10^5) & (0. + 0.) & (18.75 + 0.) \\ (0. + 0.) & (.4 \times 10^5 + 14.423) & (12. + 0.) \\ (0. + 18.75) & (12. + 0.) & (.0048 + .00469) \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 10^5 & 0. & 18.75 \\ 0. & .4 \times 10^5 & 12. \\ 18.75 & 12. & .00949 \end{bmatrix} \end{aligned} \quad (6.48-a)$$

Congruent Transformation

1. The unassembled stiffness matrix $[\mathbf{K}^e]$, for node 2, is given by:

$$\begin{aligned} \{\mathbf{F}\} &= [\mathbf{K}^e] \{\mathbf{Y}\} \quad (6.49-a) \\ \left\{ \begin{array}{c} M_x^1 \\ M_y^1 \\ F_z^1 \\ M_x^2 \\ M_y^2 \\ F_z^2 \end{array} \right\} &= \left[\begin{array}{ccc|ccc} 7.692 & 0 & 0 & & & \\ & .4 \times 10^5 & 12. & & & \\ \text{sym} & & .0048 & & & \\ \hline & & & 1 \times 10^5 & 0. & -18.75 \\ & & & & 14.423 & 0. \\ & 0. & & \text{sym} & & .00469 \end{array} \right] \left\{ \begin{array}{c} \Theta_x^1 \\ \Theta_y^1 \\ W_z^1 \\ \Theta_x^2 \\ \Theta_y^2 \\ W_z^2 \end{array} \right\} \quad (6.49-b) \end{aligned}$$

element 1
element 2

Note that the B.C. are implicitly accounted for by ignoring the restrained d.o.f. however the connectivity of the elements is not reflected by this matrix.

2. The kinematics matrix is given by:

$$\{\mathbf{Y}\} = [\mathcal{A}] \{\mathbf{\Delta}\} \quad (6.50-a)$$

$$[\mathbf{\Gamma}]_{BC} = [\mathbf{I}] \quad (6.52-b)$$

The element stiffness matrices in global coordinates will then be given by:

$$[\mathbf{K}]_{AB} = [\mathbf{\Gamma}]_{AB}^T [\mathbf{k}]_{AB} [\mathbf{\Gamma}]_{AB} \quad (6.53-a)$$

$$= 200 \begin{bmatrix} .645 & .259 & -7.031 & -.645 & -.259 & -7.031 \\ & .109 & 17.381 & -.259 & -.109 & 17.381 \\ & & 1 \times 10^5 & 7.031 & -17.381 & .5 \times 10^5 \\ & & & .645 & .259 & 7.031 \\ \text{sym} & & & & .109 & -17.381 \\ & & & & & 1 \times 10^5 \end{bmatrix} \quad (6.53-b)$$

and $[\mathbf{K}]_{BC} = [\mathbf{k}]_{BC}$

Direct Stiffness: We can readily assemble the global stiffness matrix:

$$[\mathbf{K}] = 200 \begin{bmatrix} (.645 + .75) & (.259 + 0.) & (7.031 + 0.) \\ & (.109 + .00469) & (-17.38 + 18.75) \\ \text{sym} & & (1 + 1) \times 10^5 \end{bmatrix} \quad (6.54-a)$$

$$= 200 \begin{bmatrix} 1.395 & .259 & 7.031 \\ & .1137 & 1.37 \\ \text{sym} & & 2 \times 10^5 \end{bmatrix} \quad (6.54-b)$$

Congruent Transformation, global axis, Boolean $[\mathcal{A}]$

1. We start with the unconnected global stiffness matrix in global coordinate system:

$$\{\mathbf{F}\} = [\mathbf{K}^e] \{\mathbf{\Upsilon}\} \quad (6.55-a)$$

$$\left\{ \begin{array}{c} P_X^1 \\ P_Y^1 \\ M_Z^1 \\ P_X^2 \\ P_Y^2 \\ M_Z^2 \end{array} \right\} = 200 \left[\begin{array}{ccc|ccc} .645 & .259 & 7.031 & & & \\ & .109 & -17.381 & & 0 & \\ \text{sym} & & 1 \times 10^5 & & & \\ \hline & & & .75 & 0. & 0. \\ 0 & & & & .00469 & 18.75 \\ \text{sym} & & & & & 1 \times 10^5 \end{array} \right] \left\{ \begin{array}{c} U_X^1 \\ V_Y^1 \\ \Theta_Z^1 \\ U_X^2 \\ V_Y^2 \\ \Theta_Z^2 \end{array} \right\} \quad (6.55-b)$$

2. Next we determine the kinematics matrix \mathcal{A} :

$$\{\mathbf{\Upsilon}\} = [\mathcal{A}] \{\mathbf{\Delta}\} \quad (6.56-a)$$

$$\left\{ \begin{array}{c} u^1 \\ v^1 \\ \theta^1 \\ u^2 \\ v^2 \\ \theta^2 \end{array} \right\} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} u \\ v \\ \theta \end{array} \right\} \quad (6.56-b)$$

3. Finally, if we take the product $\underbrace{[\mathcal{A}]^T}_{3 \times 6} \underbrace{[\mathbf{K}^e]}_{6 \times 6} \underbrace{[\mathcal{A}]}_{6 \times 3}$ we obtain the structure global stiffness matrix

$[\mathbf{K}]$ in Eq. 6.54-b

Congruent Transformation (local axis):

1. Unconnected stiffness matrix in local coordinates:

$$\{\mathbf{p}^e\} = [\mathbf{k}^e] \{\mathbf{\delta}^e\} \quad (6.57-a)$$

$$\left\{ \begin{array}{c} P_X^1 \\ P_Y^1 \\ M_Z^1 \\ P_X^2 \\ P_Y^2 \\ M_Z^2 \end{array} \right\} = 200 \left[\begin{array}{ccc|ccc} .75 & 0. & 0. & & & \\ & .00469 & -18.75 & & & \\ \text{sym} & & 1 \times 10^5 & & & \\ \hline & & & .75 & 0. & 0. \\ & & & & .00469 & 18.75 \\ \text{sym} & & & & & 1 \times 10^5 \end{array} \right] \left\{ \begin{array}{c} u_x^1 \\ v_y^1 \\ \theta_z^1 \\ u_x^2 \\ v_y^2 \\ \theta_z^2 \end{array} \right\} \quad (6.57-b)$$

Chapter 7

FLEXIBILITY METHOD

7.1 Introduction

¹ Recall the definition of the flexibility matrix

$$\{\mathbf{\Upsilon}\} \equiv [\mathbf{d}]\{\mathbf{p}\} \quad (7.1)$$

where $\{\mathbf{\Upsilon}\}$, $[\mathbf{d}]$, and $\{\mathbf{p}\}$ are the element relative displacements, element flexibility matrix, and forces at the element degrees of freedom free to displace.

² As with the congruent approach for the stiffness matrix, we define:

$$\{\mathbf{F}^{(e)}\} = \begin{bmatrix} [\mathbf{F}^{(1)}] & [\mathbf{F}^{(2)}] & \dots & [\mathbf{F}^{(n)}] \end{bmatrix}^T \quad (7.2-a)$$

$$\{\mathbf{\Upsilon}^{(e)}\} = \begin{bmatrix} [\mathbf{\Upsilon}^{(1)}] & [\mathbf{\Upsilon}^{(2)}] & \dots & [\mathbf{\Upsilon}^{(n)}] \end{bmatrix}^T \quad (7.2-b)$$

for n elements, and where $\{\mathbf{F}^i\}$ and $\{\mathbf{\Upsilon}^i\}$ are the nodal load and displacements vectors for element i . The size of these vectors is equal to the total number of global dof for element i .

³ Denoting by $\{\mathbf{R}\}$ the reaction vector, and by $\{\mathbf{\Upsilon}^R\}$ the corresponding displacements, we define the unassembled structure flexibility matrix as:

$$\left\{ \frac{\mathbf{\Upsilon}^{(e)}}{\mathbf{\Upsilon}^R} \right\} = \left[\begin{array}{c|c} [\mathbf{d}^{(e)}] & [\mathbf{0}] \\ \hline [\mathbf{0}] & [\mathbf{0}] \end{array} \right] \left\{ \frac{\mathbf{F}^{(e)}}{\mathbf{R}} \right\} \quad (7.3)$$

where $[\mathbf{d}^{(e)}]$ is the *unassembled global flexibility matrix*.

⁴ In its present form, Eq. 7.3 is of no help as the element forces $\{\mathbf{F}^{(e)}\}$ and reactions $\{\mathbf{R}\}$ are not yet known.

7.2 Flexibility Matrix

⁵ We recall from Sect. 6.1.2 that we can automatically identify the redundant forces $[\mathbf{F}_x]$ and rewrite Eq. 7.3 as:

$$\left\{ \frac{\mathbf{\Upsilon}_0}{\mathbf{\Upsilon}_x} \right\} = \left[\begin{array}{c|c} \mathbf{d}_{00}^{(e)} & [\mathbf{0}] \\ \hline [\mathbf{0}] & \mathbf{d}_{xx}^{(e)} \end{array} \right] \left\{ \frac{\mathbf{F}_0}{\mathbf{F}_x} \right\} \quad (7.4)$$

7.2.2 Solution of Internal Forces and Reactions

¹² The internal forces and reactions can in turn be obtained through Eq. 6.14:

$$\{\mathbf{F}_0\} = [\mathcal{C}_1] \{\mathbf{P}\} + [\mathcal{C}_2] \{\mathbf{F}_x\} \quad (7.12)$$

which is combined with Eq. 7.11 to yield:

$$\{\mathbf{F}_0\} = \left[[\mathcal{C}_1] - [\mathcal{C}_2] [\mathbf{D}_{xx}]^{-1} [\mathbf{D}_{xp}] \right] \{\mathbf{P}\} \quad (7.13)$$

7.2.3 Solution of Joint Displacements

¹³ Joint displacements are in turn obtained by considering the top partition of Eq. 7.9:

$$\{\Delta_p\} = \underbrace{\left[[\mathbf{D}_{pp}] - [\mathbf{D}_{px}] [\mathbf{D}_{xx}]^{-1} [\mathbf{D}_{xp}] \right]}_{[\mathbf{D}]} \{\mathbf{P}\} \quad (7.14)$$

This equation should again be compared with Eq. 5.4.

■ Example 7-1: Flexibility Method

Solve for the internal forces and displacements of joint 2 of the truss in example 6.1.2. Let $H = 0.75L$ and assign area A to members 3 and 5, and $0.5A$ to members 1, 2, and 4. Let f_5 be the redundant force, and use the $[\mathcal{C}_1]$ and $[\mathcal{C}_2]$ matrices previously derived.

Solution:

$$C = \frac{L}{\sqrt{L^2 + H^2}} = 0.8 \text{ and } S = \frac{H}{\sqrt{L^2 + H^2}} = 0.6$$

From Eq. 7.4 we obtain

$$\left\{ \begin{array}{c} \Upsilon_0 \\ \Upsilon_x \end{array} \right\} = \left[\begin{array}{c|c} \mathbf{d}_{00}^{(e)} & 0 \\ \hline 0 & \mathbf{d}_{xx}^{(e)} \end{array} \right] \left\{ \begin{array}{c} \mathbf{F}_0 \\ \mathbf{F}_x \end{array} \right\} \quad (7.15\text{-a})$$

$$\left\{ \begin{array}{c} u_1 \\ v_1 \\ \Delta_2^{(e)} \\ \Delta_1^{(e)} \\ \Delta_3^{(e)} \\ \Delta_4^{(e)} \\ u_4 \\ v_4 \\ \Delta_5^{(e)} \end{array} \right\} = \frac{L}{AE} \left[\begin{array}{cccccc|c} 0 & & & & & & \\ & 0 & & & & & \\ & & 2 & & & & \\ & & & 1.5 & & & \\ & & & & 1.25 & & \\ & & & & & 1.5 & \\ & & & & & & 0 \\ & & & & & & & 0 \\ \hline & & & & & & & & 1.25 \end{array} \right] \left\{ \begin{array}{c} R_{x1} \\ R_{y1} \\ f_2 \\ f_1 \\ f_3 \\ f_4 \\ R_{x4} \\ R_{y4} \\ f_5 \end{array} \right\} \quad (7.15\text{-b})$$

From Example 6.1.2 we have

$$[\mathcal{C}_1] = \left[\begin{array}{cc} 1 & 0 \\ S/C & 1 \\ -1 & 0 \\ 0 & 1 \\ 1/C & 0 \\ -S/C & 0 \\ 0 & 0 \\ -S/C & 0 \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0.75 & 1 \\ -1 & 0 \\ 0 & 1 \\ 1.25 & 0 \\ -0.75 & 0 \\ 0 & 0 \\ -0.75 & 0 \end{array} \right] \quad [\mathcal{C}_2] = \left\{ \begin{array}{c} C \\ 0 \\ -C \\ -S \\ 1 \\ -S \\ -C \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} 0.8 \\ 0 \\ -0.8 \\ -0.6 \\ 1 \\ -0.6 \\ -0.8 \\ 0 \end{array} \right\} \quad (7.16)$$

Chapter 8

SPECIAL ANALYSIS PROCEDURES

To be edited

8.1 Semi-Rigid Beams

¹ Often times, a beam does not have either a flexible or rigid connection, but rather a semi-rigid one (such as in type II steel connection), Fig. 8.1

² The stiffness relationship for the beam element (with shear deformation) was previously derived in Eq. 2.42

$$M_1 = \frac{(4 + \Phi_y)EI_z}{(1 + \Phi_y)L} \phi_1 + \frac{(2 - \Phi_y)EI_z}{L(1 + \Phi_y)} \phi_2 \quad (8.1-a)$$

$$M_2 = \frac{(2 - \Phi_y)EI_z}{L(1 + \Phi_y)} \phi_1 + \frac{(4 + \Phi_y)EI_z}{L(1 + \Phi_y)} \phi_2 \quad (8.1-b)$$

³ If we now account for the springs at both ends, Fig. 8.2, with stiffnesses k_1^s and k_2^s , then

$$M_1 = k_1^s(\theta_1 - \phi_1) \text{ and } M_2 = k_2^s(\theta_2 - \phi_2) \quad (8.2)$$

The stiffness relationship between M_1 and M_2 and the joint rotations θ_1 and θ_2 is obtained by eliminating ϕ_1 and ϕ_2 from above, yielding

$$M_1 = \frac{EI}{L}(S_{11}\theta_1 + S_{12}\theta_2) \text{ and } M_2 = \frac{EI}{L}(S_{21}\theta_1 + S_{22}\theta_2)$$

(8.3)

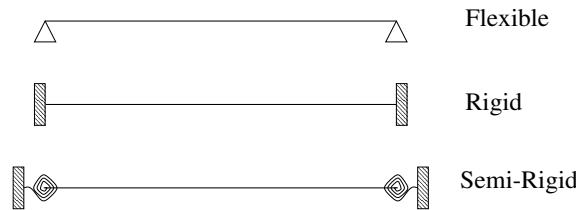


Figure 8.1: Flexible, Rigid, and Semi-Rigid Beams

Draft

Part II

Introduction to Finite Elements

Chapter 9

REVIEW OF ELASTICITY

9.1 Stress

¹ A **stress**, Fig 9.1 is a second order cartesian tensor, σ_{ij} where the 1st subscript (i) refers to the

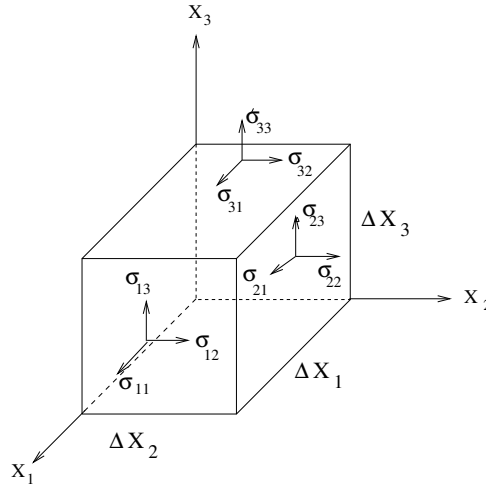


Figure 9.1: Stress Components on an Infinitesimal Element

direction of outward facing normal, and the second one (j) to the direction of component force.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \left\{ \begin{matrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{matrix} \right\} \quad (9.1)$$

² In fact the nine rectangular components σ_{ij} of $\boldsymbol{\sigma}$ turn out to be the three sets of three vector components $(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $(\sigma_{21}, \sigma_{22}, \sigma_{23})$, $(\sigma_{31}, \sigma_{32}, \sigma_{33})$ which correspond to the three tractions $\mathbf{t}_1, \mathbf{t}_2$ and \mathbf{t}_3 which are acting on the x_1, x_2 and x_3 faces (It should be noted that those tractions are not necessarily normal to the faces, and they can be decomposed into a normal and shear traction if need be). In other words, stresses are nothing else than the components of tractions (stress vector), Fig. 9.2.

³ The state of stress at a point cannot be specified entirely by a single vector with three components; it requires the second-order tensor with all nine components.

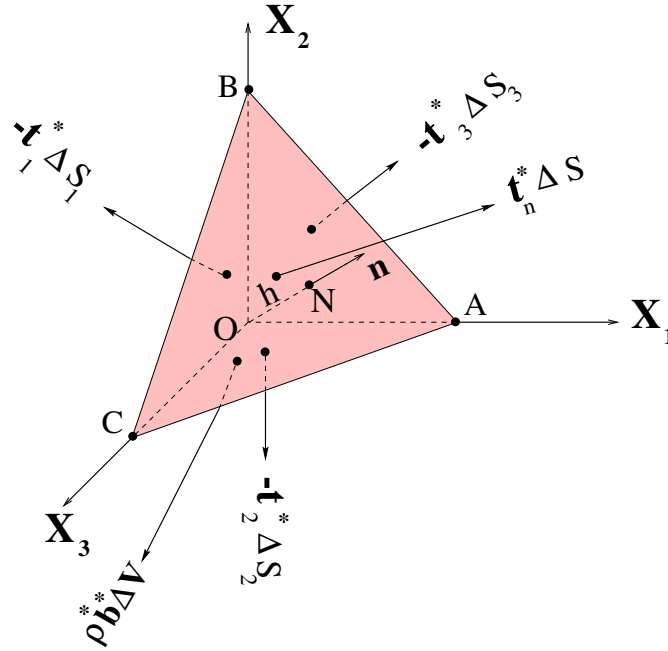


Figure 9.3: Stress Traction Relations

We seek to determine the traction (or stress vector) \mathbf{t} passing through P and parallel to the plane ABC where $A(4, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 6)$. **Solution:**

The vector normal to the plane can be found by taking the cross products of vectors AB and AC :

$$\mathbf{N} = AB \times AC = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{vmatrix} \quad (9.6-a)$$

$$= 12\mathbf{e}_1 + 24\mathbf{e}_2 + 8\mathbf{e}_3 \quad (9.6-b)$$

The unit normal of N is given by

$$\mathbf{n} = \frac{3}{7}\mathbf{e}_1 + \frac{6}{7}\mathbf{e}_2 + \frac{2}{7}\mathbf{e}_3 \quad (9.7)$$

Hence the stress vector (traction) will be

$$\begin{bmatrix} \frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{7} & \frac{5}{7} & \frac{10}{7} \end{bmatrix} \quad (9.8)$$

and thus $\mathbf{t} = -\frac{9}{7}\mathbf{e}_1 + \frac{5}{7}\mathbf{e}_2 + \frac{10}{7}\mathbf{e}_3$ ■

9.2 Strain

⁷ Given the displacement u_i of a point, the strain ε_{ij} is defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (9.9)$$

Chapter 10

VARIATIONAL AND ENERGY METHODS

10.1 Work, Energy & Potentials; Definitions

10.1.1 Introduction

¹ Work is defined as the product of a force and displacement

$$W \stackrel{\text{def}}{=} \int_a^b \mathbf{F} \cdot d\mathbf{s} \quad (10.1-a)$$

$$dW = F_x dx + F_y dy \quad (10.1-b)$$

² Energy is a quantity representing the ability or capacity to perform work.

³ The change in energy is proportional to the amount of work performed. Since only the change of energy is involved, any datum can be used as a basis for measure of energy. Hence energy is neither created nor consumed.

⁴ The first law of thermodynamics states

The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time:

$$\boxed{\frac{d}{dt}(K + U) = W_e + H} \quad (10.2)$$

where K is the kinetic energy, U the internal strain energy, W the external work, and H the heat input to the system.

⁵ For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to:

$$\boxed{W_e = U} \quad (10.3)$$

¹² Considering uniaxial stresses, in the absence of initial strains and stresses, and for linear elastic systems, Eq. 10.9 reduces to

$$U = \frac{1}{2} \int_{\Omega} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\Omega \quad (10.10)$$

¹³ When this relation is applied to various one dimensional structural elements it leads to

Axial Members:

$$\left. \begin{aligned} U &= \int_{\Omega} \frac{\varepsilon \sigma}{2} d\Omega \\ \sigma &= \frac{P}{A} \\ \varepsilon &= \frac{\Delta L}{L} \\ d\Omega &= A dx \end{aligned} \right\} \boxed{U = \frac{1}{2} \int_0^L \frac{P^2}{AE} dx} \quad (10.11)$$

Torsional Members:

$$\left. \begin{aligned} U &= \frac{1}{2} \int_{\Omega} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\Omega \\ U &= \frac{1}{2} \int_{\Omega} \gamma_{xy} \underbrace{G\tau_{xy}}_{\gamma_{xy}} d\Omega \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \tau_{xy} &= \frac{T r}{J} \\ d\Omega &= r d\theta dr dx \\ J &= \int_0^r \int_0^{2\pi} r^2 d\theta dr \end{aligned} \right\} \boxed{U = \frac{1}{2} \int_0^L \frac{T^2}{GJ} dx} \quad (10.12)$$

Flexural Members:

$$\left. \begin{aligned} U &= \frac{1}{2} \int_{\Omega} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\Omega \\ \sigma_x &= \frac{M_z y}{I_z} \\ \varepsilon &= \frac{M_z y}{EI_z} \\ d\Omega &= dA dx \\ \int_A y^2 dA &= I_z \end{aligned} \right\} \boxed{U = \frac{1}{2} \int_0^L \frac{M^2}{EI_z} dx} \quad (10.13)$$

10.1.2.1 Internal Work versus Strain Energy

¹⁴ During strain increment, the work done by internal forces in a differential element will be the negative of that performed by the stresses acting upon it.

$$W_i = - \int_{\Omega} \sigma d\epsilon d\Omega \quad (10.14)$$

¹⁵ If the strained elastic solid were permitted to slowly return to their unstrained state, then the solid would return the work performed by the external forces. This is due to the release of strain energy stored in the solid.

It can be shown that in the principle of virtual displacements, the Euler equations are the equilibrium equations, whereas in the principle of virtual forces, they are the compatibility equations.

Euler equations are differential equations which can not always be solved by exact methods. An alternative method consists in bypassing the Euler equations and go directly to the variational statement of the problem to the solution of the Euler equations.

Finite Element formulation are based on the weak form, whereas the formulation of Finite Differences are based on the strong form.

We still have to define $\delta\Pi$. The first variation of a functional expression is

$$\left. \begin{aligned} \delta F &= \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \\ \delta \Pi &= \int_a^b \delta F dx \end{aligned} \right\} \delta \Pi = \int_a^b \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \right) dx \quad (10.189)$$

As above, integration by parts of the second term yields

$$\boxed{\delta \Pi = \int_a^b \delta u \left(\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} \right) dx} \quad (10.190)$$

We have just shown that finding the stationary value of Π by setting $\delta\Pi = 0$ is equivalent to finding the extremal value of Π by setting $\left. \frac{d\Pi(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0}$ equal to zero.

Similarly, it can be shown that as with second derivatives in calculus, the second variation $\delta^2\Pi$ can be used to characterize the extremum as either a minimum or maximum.

10.7.2 Boundary Conditions

Revisiting the second part of Eq. 10.184, we had

$$\underbrace{\underbrace{\eta(x)}_{\text{Ess.}} \underbrace{\frac{\partial F}{\partial u'} \Big|_a^b}_{\text{Nat.}}}_{\text{Boundary Cond.}} = 0 \quad (10.191)$$

This can be achieved through the following combinations

$$\eta(a) = 0 \quad \text{and} \quad \eta(b) = 0 \quad (10.192\text{-a})$$

$$\eta(a) = 0 \quad \text{and} \quad \frac{\partial F}{\partial u'}(b) = 0 \quad (10.192\text{-b})$$

$$\frac{\partial F}{\partial u'}(a) = 0 \quad \text{and} \quad \eta(b) = 0 \quad (10.192\text{-c})$$

$$\frac{\partial F}{\partial u'}(a) = 0 \quad \text{and} \quad \frac{\partial F}{\partial u'}(b) = 0 \quad (10.192\text{-d})$$

Generalizing, for a problem with, one field variable, in which the highest derivative in the governing differential equation is of order $2m$ (or simply m in the corresponding functional), then we have

Essential (or forced, or geometric) boundary conditions, (because it was essential for the derivation of the Euler equation) if $\eta(a)$ or $\eta(b) = 0$. Essential boundary conditions, involve derivatives of order zero (the field variable itself) through $m-1$. Trial displacement functions are explicitly required to satisfy this B.C. Mathematically, this corresponds to *Dirichlet boundary-value problems*.

Euler Equation:

$$-\frac{d}{dx} \left(EA \frac{du}{dx} \right) = 0 \quad 0 < x < L \quad (10.196)$$

Natural Boundary Condition:

$$EA \frac{du}{dx} - P = 0 \quad \text{at } x = L \quad (10.197)$$

Solution II We have

$$F(x, u, u') = \frac{EA}{2} \left(\frac{du}{dx} \right)^2 \quad (10.198)$$

(note that since P is an applied load at the end of the member, it does not appear as part of $F(x, u, u')$) To evaluate the Euler Equation from Eq. 10.186, we evaluate

$$\frac{\partial F}{\partial u} = 0 \quad \& \quad \frac{\partial F}{\partial u'} = EAu' \quad (10.199-a)$$

Thus, substituting, we obtain

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \Rightarrow -\frac{d}{dx} (EAu') = 0 \quad \text{Euler Equation} \quad (10.200-a)$$

$$EA \frac{du}{dx} = 0 \quad \text{B.C.} \quad (10.200-b)$$

■

■ Example 10-13: Flexure of a Beam

The total potential energy of a beam supporting a uniform load p is given by

$$\Pi = \int_0^L \left(\frac{1}{2} M \kappa - pw \right) dx = \int_0^L \underbrace{\left(\frac{1}{2} (EIw'')w'' - pw \right)}_F dx \quad (10.201)$$

Derive the first variational of Π .

Solution:

Extending Eq. 10.189, and integrating by part twice

$$\delta \Pi = \int_0^L \delta F dx = \int_0^L \left(\frac{\partial F}{\partial w''} \delta w'' + \frac{\partial F}{\partial w} \delta w \right) dx \quad (10.202-a)$$

$$= \int_0^L (EIw'' \delta w'' - p \delta w) dx \quad (10.202-b)$$

$$= (EIw'' \delta w') \Big|_0^L - \int_0^L [(EIw'')' \delta w' - p \delta w] dx \quad (10.202-c)$$

$$= \underbrace{\underbrace{(EIw'' \delta w') \Big|_0^L}_{\text{Nat.}} - \underbrace{[(EIw'')' \delta w] \Big|_0^L}_{\text{Ess.}}}_{\text{BC}} + \underbrace{\int_0^L [(EIw'')'' + p] \delta w dx}_{\text{Euler Eq.}} = 0 \quad (10.202-d)$$

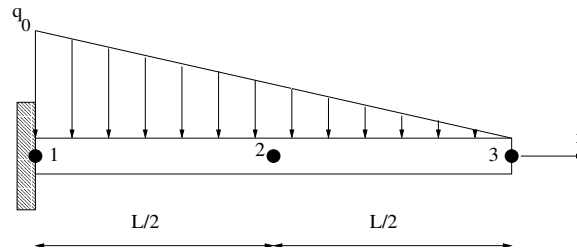
Or

$$(EIw'')'' = -p \quad \text{for all } x$$

which is the governing differential equation of beams and

10.8 Homework

For the cantilivered beam shown below:



subjected to a linearly varying vertical load of magnitude $q = q_0(1 - \frac{x}{L})$:

1. By means of the principle of virtual displacements, reanalyze the previous problem using the following expression for both the virtual and actual systems:

$$v = (1 - \cos \frac{\pi x}{2L})a_1 + (1 - \cos \frac{3\pi x}{2L})a_2 \quad (10.203)$$

and:

- (a) Determine the vertical displacement of node 3.
- (b) On the basis of your results, draw the shear and moment diagrams.
- (c) Compute the total potential energy of the system.
- (d) Compare your results with those of problem 1 (exact solution) and *Discuss*.

Note: You should first fit the coefficients a_1 and a_2 to the displacements of points 2 and 3, v_2 and v_3 .

2. Using the principle of virtual force, analyse the cantilivered beam (using a virtual unit point load), and:
 - (a) Determine the vertical displacement of point 3.
 - (b) Draw the shear and moment diagram.
 - (c) Compute the total potential energy.
3. Analyse using the principle of minimum total potential energy.
4. Analyse this problem by the Rayleigh-Ritz method.
5. Analyse using the principle of minimum complementary total potential energy.
6. Analyse this problem using Castigliano's second theorem
7. Discuss your results

You are strongly advised to use Mathematica.

Chapter 11

INTERPOLATION FUNCTIONS

11.1 Introduction

¹ Application of the Principle of Virtual Displacement requires an *assumed* displacement field. This displacement field can be approximated by *interpolation functions* written in terms of:

1. Unknown polynomial coefficients, most appropriate for continuous systems, and the Rayleigh-Ritz method

$$y = a_1 + a_2x + a_3x^2 + a_4x^3 \quad (11.1)$$

A major drawback of this approach, is that the coefficients have no physical meaning.

2. Unknown nodal deformations, most appropriate for discrete systems and Potential Energy based formulations

$$y = \Delta = N_1\bar{\Delta}_1 + N_2\bar{\Delta}_2 + \dots + N_n\bar{\Delta}_n \quad (11.2)$$

² For simple problems *both* Eqn. 11.1 and Eqn. 11.2 can readily provide the *exact* solutions of the governing differential equation (such as $\frac{d^4y}{dx^4} = \frac{q}{EI}$ for flexure), but for more complex ones, one must use an *approximate* one.

11.2 Shape Functions

³ For an element (finite or otherwise), we can write an expression for the generalized displacement (translation/rotation), Δ at any point in terms of all its known nodal ones, $\bar{\Delta}$.

$$\Delta = \sum_{i=1}^n N_i(x)\bar{\Delta}_i = [\mathbf{N}(x)]\{\bar{\Delta}\} \quad (11.3)$$

where:

1. $\bar{\Delta}_i$ is the (generalized) nodal displacement corresponding to d.o.f i
2. N_i is an interpolation function, or *shape function* which has the following characteristics:
 - (a) $N_i = 1$ at node i
 - (b) $N_i = 0$ at node j where $i \neq j$.
3. Summation of N at any point is equal to unity $\sum N = 1$.
4. \mathbf{N} can be derived on the bases of:
 - (a) Assumed deformation state defined in terms of polynomial series.
 - (b) Interpolation function (Lagrangian or Hermitian).

¹² Substituting and rearranging those expressions into Eq. 11.5 we obtain

$$u = \left(\frac{\bar{u}_2}{L} - \frac{\bar{u}_1}{L} \right) x + \bar{u}_1 \quad (11.8-a)$$

$$= \underbrace{\left(1 - \frac{x}{L} \right)}_{N_1} \bar{u}_1 + \underbrace{\frac{x}{L}}_{N_2} \bar{u}_2 \quad (11.8-b)$$

or:

$$\boxed{\begin{matrix} N_1 = 1 - \frac{x}{L} \\ N_2 = \frac{x}{L} \end{matrix}} \quad (11.9)$$

11.2.2 Generalization

¹³ The previous derivation can be generalized by writing:

$$u = a_1 x + a_2 = \underbrace{\begin{bmatrix} x & 1 \end{bmatrix}}_{[\mathbf{p}]} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{\{\mathbf{a}\}} \quad (11.10)$$

where $[\mathbf{p}]$ corresponds to the polynomial approximation, and $\{\mathbf{a}\}$ is the coefficient vector.

¹⁴ We next apply the boundary conditions:

$$\underbrace{\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}}_{\{\bar{\Delta}\}} = \underbrace{\begin{bmatrix} 0 & 1 \\ L & 1 \end{bmatrix}}_{[\mathcal{L}]} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{\{\mathbf{a}\}} \quad (11.11)$$

following inversion of $[\mathcal{L}]$, this leads to

$$\underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L} \begin{bmatrix} -1 & 1 \\ L & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}}_{\{\bar{\Delta}\}} \quad (11.12)$$

¹⁵ Substituting this last equation into Eq. 11.10, we obtain:

$$u = \underbrace{\begin{bmatrix} \left(1 - \frac{x}{L} \right) & \frac{x}{L} \end{bmatrix}}_{\underbrace{[\mathbf{p}][\mathcal{L}]^{-1}}_{[\mathbf{N}]}} \underbrace{\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}}_{\{\bar{\Delta}\}} \quad (11.13)$$

¹⁶ Hence, the shape functions $[\mathbf{N}]$ can be directly obtained from

$$\boxed{[\mathbf{N}] = [\mathbf{p}][\mathcal{L}]^{-1}} \quad (11.14)$$

11.2.3 Flexural

¹⁷ With reference to Fig. 11.2. We have 4 d.o.f.'s, $\{\bar{\Delta}\}_{4 \times 1}$: and hence will need 4 shape functions, N_1 to N_4 , and those will be obtained through 4 boundary conditions. Therefore we need to assume a polynomial approximation for displacements of degree 3.

$$v = a_1 x^3 + a_2 x^2 + a_3 x + a_4 \quad (11.15-a)$$

$$\theta = \frac{dv}{dx} = 3a_1 x^2 + 2a_2 x + a_3 \quad (11.15-b)$$

Chapter 12

FINITE ELEMENT FORMULATION

¹ Having introduced the virtual displacement method in chapter 10, the shape functions in chapter 11, and finally having reviewed the basic equations of elasticity in chapter 9, we shall present a general energy based formulation of the element stiffness matrix in this chapter.

² Whereas chapter 2 derived the stiffness matrices of one dimensional rod elements, the approach used could not be generalized to general finite element. Alternatively, the derivation of this chapter will be applicable to both one dimensional rod elements or continuum (2D or 3D) elements.

³ It is important to note that whereas the previously presented method to derive the stiffness matrix yielded an exact solution, it can not be generalized to continuum (2D/3D elements). On the other hands, the method presented here is an *approximate* method, which happens to result in an exact stiffness matrix for flexural one dimensional elements. Despite its approximation, this so-called finite element method will yield excellent results if enough elements are used.

12.1 Strain Displacement Relations

⁴ The displacement Δ at any point inside an element can be written in terms of the shape functions $[N]$ and the nodal displacements $\{\bar{\Delta}\}$

$$\Delta(x) = [N(x)]\{\bar{\Delta}\} \quad (12.1)$$

The strain is then defined as:

$$\varepsilon(x) \stackrel{\text{def}}{=} [B(x)]\{\bar{\Delta}\} \quad (12.2)$$

where $[B]$ is the matrix which relates nodal displacements to strain field and is clearly expressed in terms of derivatives of N .

12.1.1 Axial Members

$$u(x) = \underbrace{\left[\underbrace{\left(1 - \frac{x}{L}\right)}_{N_1} \quad \underbrace{\frac{x}{L}}_{N_2} \right]}_{[N]} \underbrace{\left\{ \begin{matrix} \bar{u}_1 \\ \bar{u}_2 \end{matrix} \right\}}_{\{\bar{\Delta}\}} \quad (12.3-a)$$

10 Let us now apply the principle of virtual displacement and restate some known relations:

$$\delta U = \delta W \quad (12.9-a)$$

$$\delta U = \int_{\Omega} [\delta \epsilon] \{\sigma\} d\Omega \quad (12.9-b)$$

$$\{\sigma\} = [D]\{\epsilon\} - [D]\{\epsilon^0\} \quad (12.9-c)$$

$$\{\epsilon\} = [B]\{\bar{\Delta}\} \quad (12.9-d)$$

$$\{\delta \epsilon\} = [B]\{\delta \bar{\Delta}\} \quad (12.9-e)$$

$$[\delta \epsilon] = [\delta \bar{\Delta}][B]^T \quad (12.9-f)$$

11 Combining Eqns. 12.9-a, 12.9-b, 12.9-c, 12.9-f, and 12.9-d, the internal virtual strain energy is given by:

$$\begin{aligned} \delta U &= \int_{\Omega} \underbrace{[\delta \bar{\Delta}][B]^T}_{\{\delta \epsilon\}} \underbrace{[D][B]\{\bar{\Delta}\}}_{\{\sigma\}} d\Omega - \int_{\Omega} \underbrace{[\delta \bar{\Delta}][B]^T}_{\{\delta \epsilon\}} \underbrace{[D]\{\epsilon^0\}}_{\{\sigma^0\}} d\Omega \\ &= [\delta \bar{\Delta}] \int_{\Omega} [B]^T [D] [B] d\Omega \{\bar{\Delta}\} - [\delta \bar{\Delta}] \int_{\Omega} [B]^T [D] \{\epsilon^0\} d\Omega \end{aligned} \quad (12.10)$$

12 The virtual external work in turn is given by:

$$\delta W = \underbrace{[\delta \bar{\Delta}]}_{\text{Virt. Nodal Displ.}} \underbrace{\{\bar{F}\}}_{\text{Nodal Force}} + \int_l [\delta \Delta] q(x) dx \quad (12.11)$$

13 combining this equation with:

$$\{\delta \Delta\} = [N]\{\delta \bar{\Delta}\} \quad (12.12)$$

yields:

$$\delta W = [\delta \bar{\Delta}] \{\bar{F}\} + [\delta \bar{\Delta}] \int_0^l [N]^T q(x) dx \quad (12.13)$$

14 Equating the internal strain energy Eqn. 12.10 with the external work Eqn. 12.13, we obtain:

$$\begin{aligned} &\underbrace{[\delta \bar{\Delta}] \int_{\Omega} \underbrace{[B]^T [D] [B] d\Omega}_{[k]} \{\bar{\Delta}\} - [\delta \bar{\Delta}] \int_{\Omega} \underbrace{[B]^T [D] \{\epsilon^0\}}_{\{\bar{F}^0\}} d\Omega}_{\delta U} = \\ &\underbrace{[\delta \bar{\Delta}] \{\bar{F}\} + [\delta \bar{\Delta}] \int_0^l \underbrace{[N]^T q(x) dx}_{\{\bar{F}^e\}}}_{\delta W} \end{aligned} \quad (12.14)$$

15 Cancelling out the $[\delta \bar{\Delta}]$ term, this is the same equation of equilibrium as the one written earlier on. It relates the (unknown) nodal displacement $\{\bar{\Delta}\}$, the structure stiffness matrix $[k]$, the external nodal force vector $\{\bar{F}\}$, the distributed element force $\{\bar{F}^e\}$, and the vector of initial displacement.

16 From this relation we define:

The element stiffness matrix:

$$\boxed{[k] = \int_{\Omega} [B]^T [D] [B] d\Omega} \quad (12.15)$$

Chapter 13

SOME FINITE ELEMENTS

13.1 Introduction

¹ Having first introduced the method of virtual displacements in Chapter 10, than the shape functions $[N]$ (Chapter 11) which relate internal to external nodal displacements, than the basic equations of elasticity (Chapter 9) which defined the $[D]$ matrix, and finally having applied the virtual displacement method to finite element in chapter 12, we now revisit some one dimensional element whose stiffness matrix was earlier derived, and derive the stiffness matrices of additional two dimensional finite elements.

13.2 Truss Element

² The shape functions of the truss element were derived in Eq. 11.9:

$$\begin{aligned} N_1 &= 1 - \frac{x}{L} \\ N_2 &= \frac{x}{L} \end{aligned}$$

³ The corresponding strain displacement relation $[B]$ is given by:

$$\begin{aligned} \varepsilon_{xx} &= \frac{du}{dx} \\ &= \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right] \\ &= \underbrace{\left[-\frac{1}{L} \quad \frac{1}{L} \right]}_{[B]} \end{aligned} \quad (13.2-a)$$

⁴ For the truss element, the constitutive matrix $[D]$ reduces to the scalar E ; Hence, substituting into Eq. 12.15 $[k] = \int_{\Omega} [B]^T [D] [B] d\Omega$ and with $d\Omega = A dx$ for element with constant cross sectional area we obtain:

$$[k] = A \int_0^L \left\{ -\frac{1}{L} \quad \frac{1}{L} \right\} \cdot E \cdot \left[-\frac{1}{L} \quad \frac{1}{L} \right] dx \quad (13.3)$$

$$\begin{aligned} [k] &= \frac{AE}{L^2} \int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \\ &= \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \quad (13.4)$$

Chapter 14

DYNAMIC ANALYSIS

14.1 Introduction

¹ When the frequency of excitation of a structure is less than about a third of its lowest natural frequency of vibration, then we can neglect inertia effects and treat the problem as a quasi-static one.

² If the structure is subjected to an impact load, than one must be primarily concerned with (stress) wave propagation. In such a problem, we often have high frequencies and the duration of the dynamic analysis is about the time it takes for the wave to travel across the structure.

³ If inertia forces are present, then we are confronted with a dynamic problem and can analyse it through any one of the following solution procedures:

1. Natural frequencies and mode shapes (only linear elastic systems)
2. Time history analysis through modal analysis (again linear elastic), or direct integration.

⁴ Methods of structural dynamics are essentially independent of finite element analysis as these methods presume that we already have the stiffness, mass, and damping matrices. Those matrices may be obtained from a single degree of freedom system, from an idealization/simplification of a frame structure, or from a very complex finite element mesh. The time history analysis procedure remains the same.

14.2 Variational Formulation

⁵ In a general three-dimensional continuum, the equations of motion of an elementary volume Ω without damping is

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{m} \ddot{\mathbf{u}} \quad (14.1)$$

where \mathbf{m} is the mass density matrix equal to $\rho \mathbf{I}$, and \mathbf{b} is the vector of body forces. The Differential operator \mathbf{L} is

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \quad (14.2)$$

⁶ For linear elastic material

$$\boldsymbol{\sigma} = \mathbf{D}_e \boldsymbol{\epsilon} \quad (14.3)$$

Since this identity must hold for any admissible $\delta \bar{\mathbf{u}}$, we conclude that

$$\mathbf{M} \ddot{\bar{\mathbf{u}}}^t = \mathbf{f}_e^t - \mathbf{f}_i^t \quad (14.12)$$

Which represents the semi-discrete linear equation of motion in the explicit time integration in terms of the mass matrix

$$\mathbf{M} = \int_{\Omega} \mathbf{N}^T \mathbf{m} \mathbf{N} d\Omega \quad (14.13)$$

the vector of external forces

$$\mathbf{f}_e^t = \int_{\Omega} \mathbf{N}^T \mathbf{b}^t d\Omega + \int_{\Gamma} \mathbf{N}^T \mathbf{t}^t d\Gamma \quad (14.14)$$

and the vector of internal forces

$$\mathbf{f}_i^t = \int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma}^t d\Omega \quad (14.15)$$

But since $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{B}\mathbf{u}$, then

$$\mathbf{f}_i^t = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \mathbf{u} \quad (14.16)$$

¹⁰ It should be noted that Eq. 14.13 defines the *consistent mass matrix*, which is a fully populated matrix. Alternatively, a *lumped mass matrix* can be defined, resulting only in a diagonal matrix. For an axial element, the lumped mass matrix is obtained by placing half of the total element mass as a particle at each node.

¹¹ For linear elastic systems, and if damping was considered

$$\mathbf{M} \ddot{\bar{\mathbf{u}}}^t + \mathbf{C} \dot{\bar{\mathbf{u}}}^t + \mathbf{K} \bar{\mathbf{u}} = \mathbf{f}_e^t \quad (14.17)$$

the damping matrix matrix is often expressed in terms of the mass and stiffness matrix, called Rayleigh damping, as

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \quad (14.18)$$

14.2.2 Implicit Time Integration

14.2.2.1 Linear Case

¹² For implicit time integrate we consider the virtual work equation at time $t + \Delta t$ (instead of t in the explicit method), thus Eq. 14.8 transforms into

$$\int_{\Omega} \delta \mathbf{u}^T \mathbf{m} d\Omega \ddot{\bar{\mathbf{u}}}^{t+\Delta t} + \int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}^{t+\Delta t} d\Omega = \int_{\Gamma} \delta \mathbf{u}^T \mathbf{t}^{t+\Delta t} d\Gamma + \int_{\Omega} \delta \mathbf{u}^T \mathbf{b}^{t+\Delta t} d\Omega \quad (14.19)$$

¹³ For linear problems

$$\boldsymbol{\sigma}^{t+\Delta t} = \mathbf{D}_e \mathbf{B} \bar{\mathbf{u}}^{t+\Delta t} \quad (14.20)$$

²⁰ In practical finite element analysis, we are therefore mainly interested in a few effective methods, namely direct integration and mode superposition. These two techniques are closely related and the choice for one method or the other is determined only by their numerical effectiveness. In the present implementation the direct integration method is employed.

²¹ In the direct integration method Eq. ?? are integrated using a numerical step by step procedure, the term “direct” meaning that prior to the numerical integration, no transformation of the equations into a different form is carried out.

14.3.1 Explicit Time Integration

²² For the explicit time integration, we adopt at starting point a slight variation of the differential equation previously derived, one which includes the effects of damping.

$$\boxed{\underbrace{\mathbf{M}\ddot{\mathbf{u}}}_{\mathbf{F}_I(t)} + \underbrace{\mathbf{C}\dot{\mathbf{u}}}_{\mathbf{F}_D(t)} + \underbrace{\mathbf{K}\mathbf{u}}_{\mathbf{F}_e(t)} = \underbrace{\mathbf{f}_e}_{\mathbf{f}_e(t)}} \quad (14.29)$$

where \mathbf{M} , \mathbf{C} and \mathbf{K} are the mass, damping and stiffness matrices, \mathbf{f}_e is the external load vector and $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$ and \mathbf{u} are the acceleration, velocity and displacement vectors of the finite element assemblage.

²³ An alternative way to consider this equation, is to examine the equation of statics at time t , where $\mathbf{F}_I(t)$, $\mathbf{F}_D(t)$, $\mathbf{F}_i(t)$, $\mathbf{F}_e(t)$ are the inertia, damping, internal and external elastic forces respectively.

14.3.1.1 Linear Systems

²⁴ For the semi-discretized equation of motion, we adopt the *central difference method*. This method is based on a finite difference approximation of the time derivatives of displacement (velocity and acceleration). Assuming a linear change in displacement over each time step, for the velocity and the acceleration at time t :

$$\dot{\mathbf{u}}^t = \frac{\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t-\Delta t}}{2\Delta t} \quad (14.30\text{-a})$$

$$\ddot{\mathbf{u}}^t = \frac{\mathbf{u}^{t-\Delta t} - 2\mathbf{u}^t + \mathbf{u}^{t+\Delta t}}{\Delta t^2} \quad (14.30\text{-b})$$

where Δt is the time step.

²⁵ Substituting into Eq. 14.17 ($\mathbf{M}\ddot{\mathbf{u}}^t + \mathbf{C}\dot{\mathbf{u}}^t + \mathbf{K}_e\mathbf{u} = \mathbf{f}_e^t$) we obtain

$$\boxed{\underbrace{\left(\underbrace{\frac{1}{\Delta t^2}\mathbf{M}}_{a_0} + \underbrace{\frac{1}{2\Delta t}\mathbf{C}}_{a_1}\right)}_{\hat{\mathbf{M}}} \mathbf{u}^{t+\Delta t} = \mathbf{f}_e^t - \underbrace{\left(\mathbf{K} - \underbrace{\frac{2}{\Delta t^2}\mathbf{M}}_{a_2}\right)}_{\hat{\mathbf{f}}_e^t} \mathbf{u}^t - \underbrace{\left(\underbrace{\frac{1}{\Delta t^2}\mathbf{M}}_{a_0} - \underbrace{\frac{1}{2\Delta t}\mathbf{C}}_{a_1}\right)}_{\hat{\mathbf{f}}_e^t} \mathbf{u}^{t-\Delta t}} \quad (14.31)$$

from where we can solve for $\mathbf{u}^{t+\Delta t}$.

²⁶ It should be noted that:

1. Solution is based on the equilibrium condition at time t
2. No factorization of the stiffness matrix \mathbf{K} is necessary (i.e. no \mathbf{K}^{-1} term appears in the equation).
3. To initiate the method, we should determine the displacements, velocity and acceleration at time $-\Delta t$.

²⁷ Algorithm:

³¹ Since no structural stiffness matrix needs to be assembled, very large problems can be effectively solved. This includes quasi-static problems (linear and non-linear) where the load is applied incrementally in time. Codes such as DYNA or ABAQUS/EXPLICIT exploit this feature to perform complex analyses such as metal forming, penetration, crash worthiness, etc...

³² A major disadvantage of the explicit method is that it converges only if

$$\Delta t < \Delta t_{cr} = \frac{2}{\omega_{max}} \quad (14.41)$$

where ω_{max} is the highest natural frequency (or smallest wave length) of the structure. Recall that

$$\omega = \frac{2\pi}{T} \quad (14.42)$$

If this condition is violated, numerical instability occurs.

³³ Hence, an eigenvalue analysis of the complete system must first be undertaken. But since the global structural stiffness matrix is not necessarily assembled, an upper bound estimate of ω_{max} is obtained from

$$\omega_{max} < \omega_{max}^{(e)} \quad (14.43)$$

where $\omega_{max}^{(e)}$ is the maximum frequency of a single finite element (usually the smallest one). It can in turn be determined from

$$\det(\mathbf{K}^{(e)} - (\omega_{max}^e)^2 \mathbf{M}^e) = 0 \quad (14.44)$$

in nonlinear problems, \mathbf{K} is the tangent stiffness.

³⁴ Note that the determination of the eigenvalues (which correspond to ω^2) of the simple problem without damping $\mathbf{K} - \omega^2 \mathbf{M}$ can be solved by either expanding the determinant of the resulting matrix, or simply determining the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$.

³⁵ Alternatively,

$$\omega_{max}^e = \frac{2c}{L} \quad (14.45)$$

where c is the acoustic wave speed $c = \sqrt{E/\rho}$ and L is a representative length of element (e). Essentially, this means that Δt must be small enough so that information does not propagate across more than one element per time step.

■ Example 14-1: MATLAB Code for Explicit Time Integration

```
%=====
%I N I T I A L I Z A T I O N
%=====
% Initialize the matrices
M=[2 0;0 1]; K=[6 -2;-2 4];C=[0 0;0 0];P=[0 ; 10];
% determine the eigenvalues, minimum frequency, and delta t critical
omega2=eig(inv(M)*K);
Tcrit=min(2*pi./sqrt(omega2));
Delta_t=Tcrit/10;
% Initialize the displacement and velocity vectors at time = 0;
u_t(1:2,:)=0;
du_t(1:2,:)=0;
Phat_t(1:2,:)=0;
% solve for the initial acceleration at time 0
ddu_t(1:2,:)=inv(M)*(P- K*u_t);
```


Chapter 15

GEOMETRIC NONLINEARITY

15.1 Introduction

¹ With reference to Fig. 15.1, we distinguish different levels of analysis:

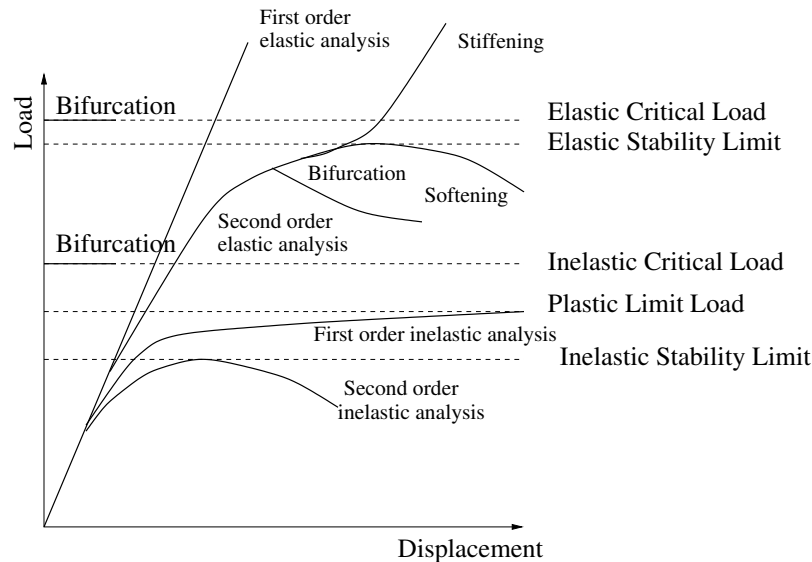


Figure 15.1: Level of Analysis

First Level elastic which excludes any nonlinearities. This is usually acceptable for service loads.

Elastic Critical load is usually determined from an **eigenvalue** analysis resulting in the buckling load.

Second-order elastic accounts for the effects of finite deformation and displacements, equilibrium equations are written in terms of the geometry of the deformed shape, does not account for material non-linearities, may be able to detect bifurcation and or increased stiffness (when a member is subjected to a tensile axial load).

First-order inelastic equilibrium equations written in terms of the geometry of the undeformed structure, accounts for material non-linearity.

Second-order inelastic equations of equilibrium written in terms of the geometry of the deformed shape, can account for both geometric and material nonlinearities. Most suitable to determine failure or ultimate loads.

Thus buckling will occur if $\frac{P}{EI} = \left(\frac{n\pi}{L}\right)^2$ or

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad (15.7)$$

The fundamental buckling mode, i.e. a single curvature deflection, will occur for $n = 1$; Thus Euler critical load for a pinned column is

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad (15.7)$$

The corresponding critical stress is

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} \quad (15.8)$$

where $I = Ar^2$.

Note that buckling will take place with respect to the weakest of the two axis.

15.1.1.2 Higher Order Differential Equation; Essential and Natural B.C.

In the preceding approach, the buckling loads were obtained for a column with specified boundary conditons. A second order differential equation, valid specifically for the member being analyzed was used.

In the next approach, we derive a single fourth order equation which will be applicable to any column regardless of the boundary conditions.

Considering a beam-column subjected to axial and shear forces as well as a moment, Fig. 15.3, taking the moment about i for the beam segment and assuming the angle $\frac{dv}{dx}$ between the axis of the beam and

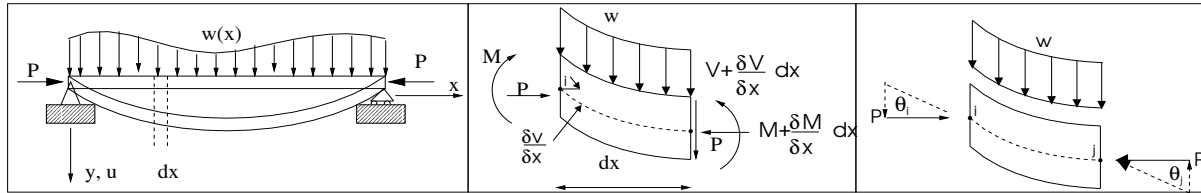


Figure 15.3: Simply Supported Beam Column; Differential Segment; Effect of Axial Force P

the horizontal axis is small, leads to

$$M - \left(M + \frac{dM}{dx} dx\right) + w \frac{(dx)^2}{2} + \left(V + \frac{dV}{dx} dx\right) dx - P \left(\frac{dv}{dx}\right) dx = 0 \quad (15.9)$$

Neglecting the terms in dx^2 which are small, and then differentiating each term with respect to x , we obtain

$$\frac{d^2 M}{dx^2} - \frac{dV}{dx} - P \frac{d^2 v}{dx^2} = 0 \quad (15.10)$$

However, considering equilibrium in the y direction gives

$$\frac{dV}{dx} = -w \quad (15.11)$$

obtain

$$\mathbf{K}_e = \frac{EI}{L^3} \begin{bmatrix} 1 & 4 & 3 & 5 & 6 \\ \frac{AL^2}{I} & -\frac{AL^2}{I} & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 2\frac{AL^2}{I} & 0 & 0 & 0 \\ 0 & 0 & 4L^2 & -6L & 2L^2 \\ 0 & 0 & -6L & 24 & 0 \\ 0 & 0 & 2L^2 & 0 & 8L^2 \end{bmatrix} \quad (15.54)$$

and

$$\mathbf{K}_g = \frac{-P}{L} \begin{bmatrix} 1 & 4 & 3 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{15}L^2 & \frac{-L}{10} & \frac{-L^2}{30} \\ 0 & 0 & \frac{-L}{10} & \frac{12}{5} & 0 \\ 0 & 0 & \frac{-L^2}{30} & 0 & \frac{4}{15}L^2 \end{bmatrix} \quad (15.55)$$

4. Noting that in this case $\mathbf{K}_g^* = \mathbf{K}_g$ for $P = 1$, the determinant $|\mathbf{K}_e + \lambda\mathbf{K}_g^*| = 0$ leads to

$$\begin{vmatrix} 1 & 4 & 3 & 5 & 6 \\ \frac{AL^2}{I} & -\frac{AL^2}{I} & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 2\frac{AL^2}{I} & 0 & 0 & 0 \\ 0 & 0 & 4L^2 - \frac{2}{15}\frac{\lambda L^4}{EI} & -6L + \frac{1}{10}\frac{\lambda L^3}{EI} & 2L^2 + \frac{1}{30}\frac{\lambda L^4}{EI} \\ 0 & 0 & -6L + \frac{1}{10}\frac{\lambda L^3}{EI} & 24 - \frac{12}{5}\frac{\lambda L^2}{EI} & 0 \\ 0 & 0 & 2L^2 + \frac{1}{30}\frac{\lambda L^4}{EI} & 0 & 8L^2 - \frac{4}{15}\frac{\lambda L^4}{EI} \end{vmatrix} = 0 \quad (15.56)$$

5. Introducing $\phi = \frac{AL^2}{I}$ and $\mu = \frac{\lambda L^2}{EI}$, the determinant becomes

$$\begin{vmatrix} 1 & 4 & 3 & 5 & 6 \\ \phi & -\phi & 0 & 0 & 0 \\ -\phi & 2\phi & 0 & 0 & 0 \\ 0 & 0 & 2(2 - \frac{\mu}{15}) & -6L + \frac{\mu}{10} & 2 + \frac{\mu}{30} \\ 0 & 0 & -6L + \frac{\mu}{10} & 12(2 - \frac{\mu}{5}) & 0 \\ 0 & 0 & 2 + \frac{\mu}{30} & 0 & 4(2 - \frac{\mu}{15}) \end{vmatrix} = 0 \quad (15.57)$$

6. Expanding the determinant, we obtain the cubic equation in μ

$$3\mu^3 - 220\mu^2 + 3,840\mu - 14,400 = 0 \quad (15.58)$$

and the lowest root of this equation is $\mu = 5.1772$.

7. We note that from Eq. 15.21, the exact solution for a column of length L was

$$P_{cr} = \frac{(4.4934)^2}{l^2} EI = \frac{(4.4934)^2}{(2L)^2} EI = \boxed{5.0477 \frac{EI}{L^2}} \quad (15.59)$$

and thus, the numerical value is about 2.6 percent higher than the exact one.

51 The *mathematica* code for this operation is:

```
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:=
{e a/l , 0 , 0 , -e a/l , 0 , 0 },
{0 , 12 e i/l^3 , 6 e i/l^2 , 0 , -12 e i/l^3 , 6 e i/l^2 },
{0 , 6 e i/l^2 , 4 e i/l , 0 , -6 e i/l^2 , 2 e i/l },
{-e a/l , 0 , 0 , e a/l , 0 , 0 },
{0 , -12 e i/l^3 , -6 e i/l^2 , 0 , 12 e i/l^3 , -6 e i/l^2 },
{0 , 6 e i/l^2 , 2 e i/l , 0 , -6 e i/l^2 , 4 e i/l }
```

The element stiffness matrices are given by

$$\mathbf{k}_e^1 = \begin{bmatrix} u_1 & \theta_2 & 0 & 0 \\ 20 & 1,208 & \cdots & \cdots \\ 1,208 & 96,667 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (15.60\text{-a})$$

$$\mathbf{k}_g^1 = -P \begin{bmatrix} u_1 & \theta_2 & 0 & 0 \\ 0.01 & 0.10 & \cdots & \cdots \\ 0.10 & 16.00 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (15.60\text{-b})$$

$$\mathbf{k}_e^2 = \begin{bmatrix} 0 & \theta_2 & 0 & \theta_3 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 128,890 & \cdots & 64,440 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 64,440 & \cdots & 128,890 \end{bmatrix} \quad (15.60\text{-c})$$

$$\mathbf{k}_e^3 = \begin{bmatrix} u_1 & \theta_3 & 0 & 0 \\ 47 & 1,678 & \cdots & \cdots \\ 1,678 & 80,556 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (15.60\text{-d})$$

$$\mathbf{k}_g^3 = -P \begin{bmatrix} u_1 & \theta_3 & 0 & 0 \\ 0.01667 & 0.1 & \cdots & \cdots \\ 0.1 & 9.6 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (15.60\text{-e})$$

The global equilibrium relation can now be written as

$$(\mathbf{K}_e - P\mathbf{K}_g) \boldsymbol{\delta} = 0 \quad (15.61)$$

$$\begin{vmatrix} u_1 & \theta_2 & \theta_3 \\ (66.75) - P(0.026666) & (1,208.33) - P(0.1) & (1,678.24) - P(0.1) \\ (1,208.33) - P(0.1) & (225,556.) - P(16.) & (64,444.4) - P(0) \\ (1,678.24) - P(0.1) & (64,444.) - P(0) & (209,444.) - P(9.6) \end{vmatrix} = 0 \quad (15.62)$$

The smallest buckling load amplification factor λ is thus equal to 2,017 kips.

```
(* Initialize constants *)
a1=0
a2=0
a3=0
i1=100
i2=200
i3=50
l1=10 12
l2=15 12
l3=6 12
e1=29000
e2=e1
e3=e1
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:=
{e a/l , 0 , 0 , -e a/l , 0 , 0 },
{0 , 12 e i/l^3 , 6 e i/l^2 , 0 , -12 e i/l^3 , 6 e i/l^2 },
{0 , 6 e i/l^2 , 4 e i/l , 0 , -6 e i/l^2 , 2 e i/l },
{-e a/l , 0 , 0 , e a/l , 0 , 0 },
```

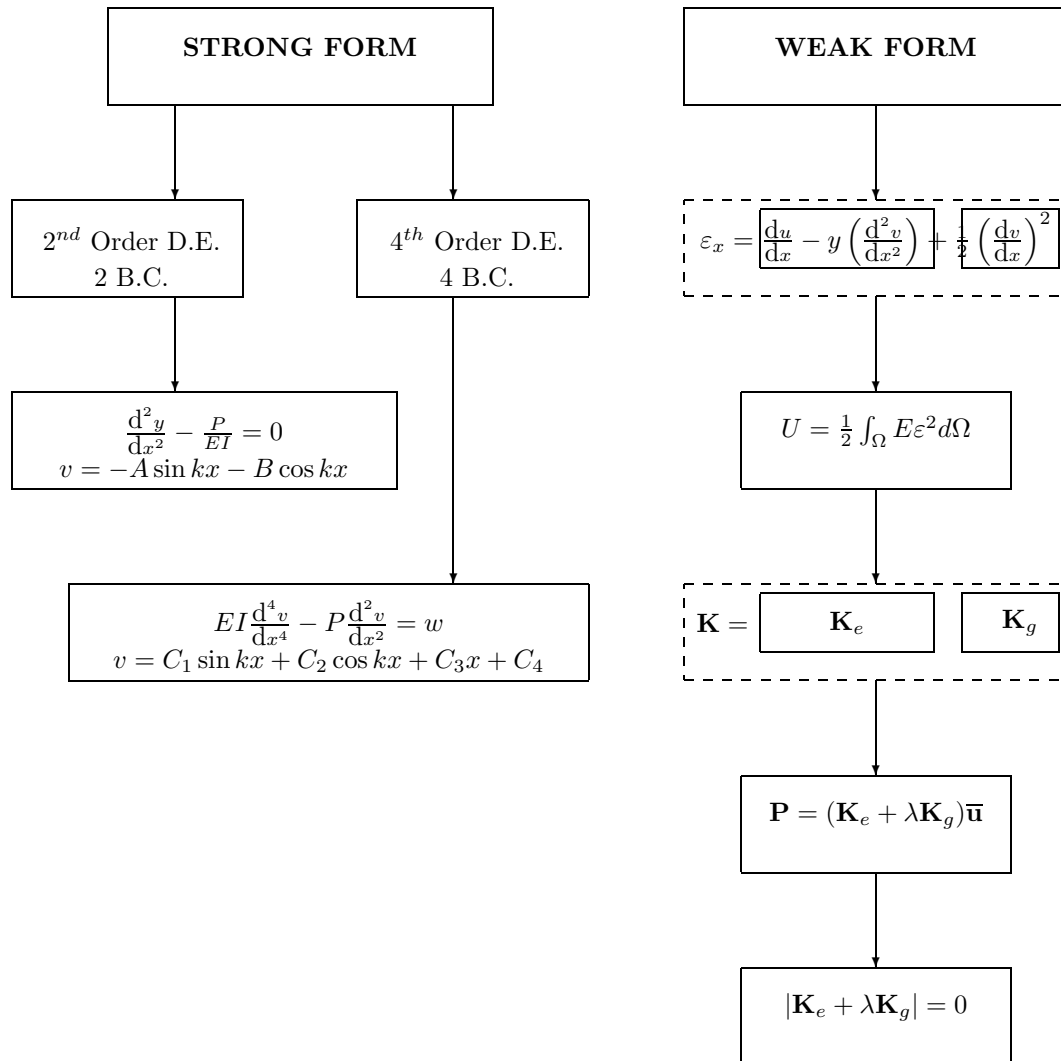


Figure 15.5: Summary of Stability Solutions

Chapter 16

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Appendix A

REVIEW of MATRIX ALGEBRA

Because of the discretization of the structure into a finite number of nodes, its solution will always lead to a matrix formulation. This matrix representation will be exploited by the computer ability to operate on vectors and matrices. Hence, it is essential that we do get a thorough understanding of basic concepts of matrix algebra.

A.1 Definitions

Matrix:

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1j} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2j} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{ij} & \dots & A_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mj} & \dots & A_{mn} \end{bmatrix} \quad (\text{A.1})$$

We would indicate the size of the matrix as $[\mathbf{A}]_{m \times n}$, and refer to an individual term of the matrix as A_{ij} . Note that matrices, and vectors are usually boldfaced when typeset, or with a tilde when handwritten \tilde{A} .

Vectors: are one column matrices:

$$\{\mathbf{X}\} = \left\{ \begin{matrix} B_1 \\ B_2 \\ \vdots \\ B_i \\ \vdots \\ B_m \end{matrix} \right\} \quad (\text{A.2})$$

A row vector would be

$$[\mathbf{C}] = [B_1 \quad B_2 \quad \dots \quad B_i \quad \dots \quad B_m] \quad (\text{A.3})$$

Note that scalars, vectors, and matrices are tensors of order 0, 1, and 2 respectively.

Square matrix: are matrices with equal number of rows and columns. $[\mathbf{A}]_{m \times m}$

Symmetry: $A_{ij} = A_{ji}$

Identity matrix: is a square matrix with all its entries equal to zero except the diagonal terms which are equal to one. It is often denoted as $[\mathbf{I}]$, and

$$I_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \quad (\text{A.4})$$

Scalar Multiplication:

$$\begin{aligned} [\mathbf{B}] &= k \cdot [\mathbf{A}] \\ B_{ij} &= kA_{ij} \end{aligned} \quad (1.11-a)$$

Matrix Multiplication: of two matrices is possible if the number of columns of the first one is equal to the number of rows of the second.

$$\begin{aligned} [\mathbf{A}]_{m \times n} &= [\mathbf{B}]_{m \times p} \cdot [\mathbf{C}]_{p \times n} \\ A_{ij} &= \underbrace{[\mathbf{B}_i]_{1 \times p} \cdot \{\mathbf{C}_j\}_{p \times 1}}_{1 \times 1} \end{aligned} \quad (1.12-a)$$

$$= \sum_{r=1}^p B_{ir} C_{rj} \quad (1.12-b)$$

$$(1.12-c)$$

Some important properties of matrix products include:

Associative: $[\mathbf{A}](\mathbf{B}[\mathbf{C}]) = ([\mathbf{A}][\mathbf{B}])[\mathbf{C}]$

Distributive: $[\mathbf{A}](\mathbf{B} + \mathbf{C}) = [\mathbf{A}][\mathbf{B}] + [\mathbf{A}][\mathbf{C}]$

Non-Commutativity: $[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}]$

A.3 Determinants

The Determinant of a matrix $[\mathbf{A}]_{n \times n}$, denoted as $\det \mathbf{A}$ or $|\mathbf{A}|$, is recursively defined as

$$\det \mathbf{A} = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j} \quad (A.13)$$

Where \mathbf{A}_{1j} is the $(n-1) \times (n-1)$ matrix obtained by eliminating the i th row and the j th column of matrix \mathbf{A} . For a 2×2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (A.14)$$

For a 3×3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (1.15-a)$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) \quad (1.15-b)$$

$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22}) \quad (1.15-c)$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \quad (1.15-d)$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \quad (1.15-e)$$

Can you write a computer program to compute the determinant of an $n \times n$ matrix?

We note that an $n \times n$ matrix would have a determinant which contains $n!$ terms each one involving n multiplications. Hence if $n = 10$ there would be $10! = 3,628,800$ terms, each one involving 9 multiplications hence over 30 million floating operations should be performed in order to evaluate the determinant.

This is why it is impractical to use Cramer's rule to solve a system of linear equations.

Some important properties of determinants:

A.6 Eigenvalues and Eigenvectors

A special form of the system of linear equation

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{Bmatrix} \quad (1.21)$$

is one in which the right hand side is a multiple of the solution:

$$[\mathbf{A}] \{\mathbf{x}\} = \lambda \{\mathbf{x}\} \quad (1.22)$$

which can be rewritten as

$$[\mathbf{A} - \lambda \mathbf{I}] \{\mathbf{x}\} = \mathbf{0} \quad (1.23)$$

A nontrivial solution to this system of equations is possible if and only if $[\mathbf{A} - \lambda \mathbf{I}]$ is singular or

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (1.24)$$

or

$$[\mathbf{A}] = \begin{vmatrix} A_{11} - \lambda & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} - \lambda & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{nn} - \lambda \end{vmatrix} = 0 \quad (1.25)$$

When the determinant is expanded, we obtain an n^{th} order polynomial in terms of λ which is known as the characteristic equation of $[\mathbf{A}]$. The n solutions (which can be real or complex) are the eigenvalues of $[\mathbf{A}]$, and each one of them λ_i satisfies

$$[\mathbf{A}] \{\mathbf{x}_i\} = \lambda_i \{\mathbf{x}_i\} \quad (1.26)$$

where $\{\mathbf{x}_i\}$ is a corresponding eigenvector.

It can be shown that:

1. The n eigenvalues of real symmetric matrices of rank n are all real.
2. The eigenvectors are orthogonal and form an orthogonal basis in E_n .

Eigenvalues and eigenvectors are used in stability (buckling) analysis, dynamic analysis, and to assess the performance of finite element formulations.

Appendix B

SOLUTIONS OF LINEAR EQUATIONS

Note this chapter is incomplete

B.1 Introduction

¹ Given a system of linear equations $[\mathbf{A}]_{n \times n} \{\mathbf{x}\} = \{\mathbf{b}\}$ (which may result from the direct stiffness method), we seek to solve for $\{\mathbf{x}\}$. *Symbolically* this operation is represented by: $\{\mathbf{x}\} = [\mathbf{A}]^{-1} \{\mathbf{b}\}$

² There are two approaches for this operation:

Direct inversion using Cramer's rule where $[\mathbf{A}]^{-1} = \frac{adj[\mathbf{A}]}{[\mathbf{A}]}$. However, this approach is computationally very inefficient for $n \geq 3$ as it requires evaluation of n high order determinants.

Decomposition: where in the most general case we seek to decompose $[\mathbf{A}]$ into $[\mathbf{A}] = [\mathbf{L}][\mathbf{D}][\mathbf{U}]$ and where:

- $[\mathbf{L}]$ lower triangle matrix
- $[\mathbf{D}]$ diagonal matrix
- $[\mathbf{U}]$ upper triangle matrix

There are two classes of solutions

Direct Method: characterized by known, finite number of operations required to achieve the decomposition yielding exact results.

Indirect methods: or iterative decomposition technique, with no *a-priori* knowledge of the number of operations required yielding an approximate solution with user defined level of accuracy.

B.2 Direct Methods

B.2.1 Gauss, and Gauss-Jordan Elimination

³ Given $[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$, we seek to transform this equation into

1. **Gaus Elimination:** $[\mathbf{U}]\{\mathbf{x}\} = \{\mathbf{y}\}$ where $[\mathbf{U}]$ is an upper triangle, and then backsubstitute from the bottom up to solve for the unknowns. Note that in this case we operate on both $[\mathbf{A}]$ & $\{\mathbf{b}\}$, yielding $\{\mathbf{x}\}$.
2. **Gauss-Jordan Elimination:** is similar to the Gaus Elimination, however rather than transforming the $[\mathbf{A}]$ matrix into an upper diagonal one, we transform $[\mathbf{A}|\mathbf{I}]$ into $[\mathbf{I}|\mathbf{A}^{-1}]$. Thus no backsubstitution is needed and the matrix inverse can be explicitly obtained.

2. Elimination of the first column:

- (a) row 1=0.1(row 1)
- (b) row 2=(row 2)+20(new row 1)
- (c) row 3=(row 3) -5(new row 1)

$$\left[\begin{array}{ccc|ccc} 1 & 0.1 & -0.5 & 0.1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & 2.5 & 7.5 & -0.5 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} 0.1 \\ 4 \\ 5.5 \end{array} \right\} \quad (2.6-a)$$

3. Elimination of second column

- (a) row 2=0.2(row 2)
- (b) row 1=(row 1)-0.1(new row 2)
- (c) row 3=(row 3) -2.5(new row 2)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -0.7 & 0.06 & -0.02 & 0 \\ 0 & 1 & 2 & 0.4 & 0.2 & 0 \\ 0 & 0 & 2.5 & -1.5 & -0.5 & 1 \end{array} \right] \left\{ \begin{array}{c} 0.02 \\ 0.8 \\ 3.5 \end{array} \right\} \quad (2.7-a)$$

4. Elimination of the third column

- (a) row 3=0.4(row 3)
- (b) row 1=(row 1)+0.7(new row 3)
- (c) row 2=(row 2)-2(new row 3)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.36 & -0.16 & 0.28 \\ 0 & 1 & 0 & 1.6 & 0.6 & -0.8 \\ 0 & 0 & 1 & -0.6 & -0.2 & 0.4 \end{array} \right] \underbrace{\left\{ \begin{array}{c} 1 \\ -2 \\ 1.4 \end{array} \right\}}_{\{\mathbf{x}\}} \quad (2.8-a)$$

This last equation is $[\mathbf{I}|\mathbf{A}^{-1}]$ ■

B.2.1.1 Algorithm

4 Based on the preceding numerical examples, we define a two step algorithm for the Gaussian elimination.

5 Defining a_{ij}^k to be the coefficient of the i^{th} row & j^{th} column at the k^{th} reduction step with $i \geq k$ & $j \geq k$:

Reduction:

$$\boxed{\begin{aligned} a_{ik}^{k+1} &= 0 & k < i \leq n \\ a_{ij}^{k+1} &= a_{ij}^k - \frac{a_{ik}^k a_{kj}^k}{a_{kk}^k} & k < i \leq n; \quad k < j \leq n \\ b_{ij}^{k+1} &= b_{ij}^k - \frac{a_{ik}^k b_{kj}^k}{a_{kk}^k} & k < i \leq n; \quad 1 < j \leq m \end{aligned}} \quad (2.9)$$

Backsubstitution:

$$\boxed{x_{ij} = \frac{b_{ij}^i - \sum_{k=i+1}^n a_{ik}^i x_{kj}}{a_{ii}^i}} \quad (2.10)$$

Note that Gauss-Jordan produces both the solution of the equations as well as the inverse of the original matrix. However, if the inverse is not desired it requires three times (N^3) more operations than Gauss or LU decomposition ($\frac{N^3}{3}$).

4. Take row by row or column by column

$$\boxed{\begin{aligned} l_{ij} &= \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}}{u_{jj}} & i > j \\ u_{ij} &= a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} & i \leq j \\ l_{ii} &= 1 \end{aligned}} \quad (2.16)$$

Note:

1. Computed elements l_{ij} or u_{ij} may always overwrite corresponding element a_{ij}
2. If $[\mathbf{A}]$ is symmetric $[\mathbf{L}]^T \neq [\mathbf{U}]$, symmetry is destroyed in $[\mathbf{A}]^F$

For symmetric matrices, LU decomposition reduces to:

$$\boxed{\begin{aligned} u_{ij} &= a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} & i \leq j \\ l_{ii} &= 1 \\ l_{ij} &= \frac{u_{ji}}{u_{jj}} \end{aligned}} \quad (2.17)$$

■ Example B-3: Example

Given:

$$\mathbf{A} = \begin{bmatrix} 7 & 9 & -1 & 2 \\ 4 & -5 & 2 & -7 \\ 1 & 6 & -3 & -4 \\ 3 & -2 & -1 & -5 \end{bmatrix} \quad (2.18\text{-a})$$

Solution:

Following the above procedure, it can be decomposed into:

Row 1: $u_{11} = a_{11} = 7$; $u_{12} = a_{12} = 9$; $u_{13} = a_{13} = -1$; $u_{14} = a_{14} = 2$

Row 2:

$$\begin{aligned} l_{21} &= \frac{a_{21}}{u_{11}} = \frac{4}{7} \\ u_{22} &= a_{22} - l_{21}u_{12} = -5 - 4\frac{9}{7} = -10.1429 \\ u_{23} &= a_{23} - l_{21}u_{13} = 2 + 4\frac{1}{7} = 2.5714 \\ u_{24} &= a_{24} - l_{21}u_{14} = -7 - 4\frac{2}{7} = -10.1429 \end{aligned}$$

Row 3:

$$\begin{aligned} l_{31} &= \frac{a_{31}}{u_{11}} = \frac{1}{7} \\ l_{32} &= \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{6 - (0.1429)(9)}{-10.1429} = -0.4647 \\ u_{33} &= a_{33} - l_{31}u_{13} - l_{32}u_{23} = -3 - (0.1429)(-1) - (-0.4647)(2.5714) = -1.6622 \\ u_{34} &= a_{34} - l_{31}u_{14} - l_{32}u_{24} = -4 - (0.1429)(2) - (-0.4647)(-10.1429) = -8.0698 \end{aligned}$$

Row 4:

$$\begin{aligned} l_{41} &= \frac{a_{41}}{u_{11}} = \frac{3}{7} \\ l_{42} &= \frac{a_{42} - l_{41}u_{12}}{u_{22}} = \frac{-2 - (0.4286)(9)}{-10.1429} = -0.5775 \\ l_{43} &= \frac{a_{43} - l_{41}u_{13} - l_{42}u_{23}}{u_{33}} = \frac{-1 - (0.4286)(-1) - (-0.5775)(2.5714)}{-1.6622} = 1.2371 \\ u_{44} &= a_{44} - l_{41}u_{14} - l_{42}u_{24} - l_{43}u_{34} = -5 - (0.4286)(2) - (-0.5775)(-10.1429) - (1.2371)(-8.0698) = 8.8285 \end{aligned}$$

Given:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 10 & 4 \\ 6 & 13 & 13 & 6 \\ 10 & 13 & 27 & 2 \\ 4 & 6 & 2 & 72 \end{bmatrix} \quad (2.28-a)$$

Solution:

Column 1:

$$\begin{aligned} l_{11} &= \sqrt{a_{11}} = \sqrt{4} = 2 \\ l_{21} &= \frac{a_{21}}{l_{11}} = \frac{6}{2} = 3 \\ l_{31} &= \frac{a_{31}}{l_{11}} = \frac{10}{2} = 5 \\ l_{41} &= \frac{a_{41}}{l_{11}} = \frac{4}{2} = 2 \end{aligned}$$

Column 2:

$$\begin{aligned} l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{13 - 3^2} = 2 \\ l_{32} &= \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{13 - (5)(3)}{2} = -1 \\ l_{42} &= \frac{a_{42} - l_{41}l_{21}}{l_{22}} = \frac{6 - (2)(3)}{2} = 0 \end{aligned}$$

Column 3:

$$\begin{aligned} l_{33} &= \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{27 - 5^2 - (-1)^2} = 1 \\ l_{43} &= \frac{a_{43} - l_{41}l_{31} - l_{42}l_{32}}{l_{33}} = \frac{2 - (2)(5) - (0)(-1)}{1} = -8 \end{aligned}$$

Column 4:

$$l_{44} = \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2} = \sqrt{72 - (2)^2 - (0)^2 - (-8)^2} = 2$$

or

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 5 & -1 & 1 & 0 \\ 2 & 0 & -8 & 2 \end{bmatrix}}_{[\mathbf{L}]} \underbrace{\begin{bmatrix} 2 & 3 & 5 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{[\mathbf{U}]} \quad (2.33-a)$$

[A]

■

B.2.4 Pivoting

B.3 Indirect Methods

¹⁶ Iterative methods are most suited for

1. Very large systems of equation $n > 10$, or 100,000
2. systems with a known “guess” of the solution

¹⁷ The most popular method is the Gauss Seidel.

B.4.2 Pre Conditioning

²² If a matrix $[\mathbf{K}]$ has an unacceptably high condition number, it can be *preconditioned* through a congruent operation:

$$[\mathbf{K}'] = [\mathbf{D}_1][\mathbf{K}][\mathbf{D}_2] \quad (2.41)$$

However there are no general rules for selecting $[\mathbf{D}_1]$ and $[\mathbf{D}_2]$.

B.4.3 Residual and Iterative Improvements

Appendix C

TENSOR NOTATION

NEEDS SOME EDITING

¹ Equations of elasticity are expressed in terms of tensors, where

- A tensor is a physical quantity, independent of any particular coordinate system yet specified most conveniently by referring to an appropriate system of coordinates.
- A tensor is classified by the rank or order
- A Tensor of order zero is specified in any coordinate system by one coordinate and is a scalar.
- A tensor of order one has three coordinate components in space, hence it is a vector.
- In general 3-D space the number of components of a tensor is 3^n where n is the order of the tensor.

² For example, force and a stress are tensors of order 1 and 2 respectively.

³ To express tensors, there are three distinct notations which can be used: 1) Engineering; 2) indicial; or 3) Dyadic.

⁴ Whereas the Engineering notation may be the simplest and most intuitive one, it often leads to long and repetitive equations. Alternatively, the tensor and the dyadic form will lead to shorter and more compact forms.

C.1 Engineering Notation

In the engineering notation, we carry on the various subscript(s) associated with each coordinate axis, for example σ_{xx}, σ_{xy} .

C.2 Dyadic/Vector Notation

⁵ Uses bold face characters for tensors of order one and higher, $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$. This notation is independent of coordinate systems.

⁶ Since scalar operations are in general not applicable to vectors, we define

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad (3.1-a)$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad (3.1-b)$$

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \quad (3.1-c)$$

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}| |\mathbf{B}| \cos(\mathbf{A}, \mathbf{B}) \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned} \quad (3.1-d)$$

this simple compacted equation (expressed as $\mathbf{x} = \mathbf{c}\mathbf{z}$ in dyadic notation), when expanded would yield:

$$\begin{aligned}x_1 &= c_{11}z_1 + c_{12}z_2 + c_{13}z_3 \\x_2 &= c_{21}z_1 + c_{22}z_2 + c_{23}z_3 \\x_3 &= c_{31}z_1 + c_{32}z_2 + c_{33}z_3\end{aligned}\tag{3.8-a}$$

Similarly:

$$A_{ij} = B_{ip}C_{jq}D_{pq}\tag{3.9}$$

$$\begin{aligned}A_{11} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\A_{12} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\A_{21} &= B_{21}C_{11}D_{11} + B_{21}C_{12}D_{12} + B_{22}C_{11}D_{21} + B_{22}C_{12}D_{22} \\A_{22} &= B_{21}C_{21}D_{11} + B_{21}C_{22}D_{12} + B_{22}C_{21}D_{21} + B_{22}C_{22}D_{22}\end{aligned}\tag{3.10-a}$$

Appendix D

INTEGRAL THEOREMS

¹ Some useful integral theorems are presented here without proofs. Schey's textbook *div grad curl and all that* provides an excellent informal presentation of related material.

D.1 Integration by Parts

The integration by part formula is

$$\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)dx \quad (4.1)$$

or

$$\int_a^b u dv = uv|_a^b - \int_a^b v du \quad (4.2)$$

D.2 Green-Gradient Theorem

Green's theorem is

$$\oint (Rdx + Sdy) = \int_{\Gamma} \left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y} \right) dx dy \quad (4.3)$$

D.3 Gauss-Divergence Theorem

² The general form of the Gauss' integral theorem is

$$\int_{\Gamma} \mathbf{v} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \text{div} \mathbf{v} d\Omega \quad (4.4)$$

or

$$\int_{\Gamma} v_i n_i d\Gamma = \int_{\Omega} v_{i,i} d\Omega \quad (4.5)$$