Chapter 6: Indeterminate Structures – Direct Stiffness Method

1. Introduction

- *Force method* and *slope-deflection* method can be used, with hand calculation, for solving the indeterminate structures when the degree of static or kinematical indeterminacy is small.

- In this chapter, *direct stiffness method* (which is also called the *displacement method*) will be introduced that is a modern method for structural analysis. *Statically determinate and indeterminate problems can be solved in the same way*. The most important characteristic is the ability to automate the solution process so that implementation in a computer program is possible. Its methodology forms the backbone of the modern *finite element method*-based commercial programs that are used routinely to analyze a variety of structural systems.

2. Fundamentals of Matrix Algebra

- ***Definitions***:

  Matrix $A_{m\times n}$, Square matrix $A_{m\times m}$, Vector $a_{1\times n}$ (row vector) or $a_{m\times 1}$ (column vector).

- ***Operations***:

  Addition and subtraction:

  If $A_{m\times n} = B_{m\times n} \pm C_{m\times n}$ then $A_{ij} = B_{ij} \pm C_{ij}$ $(i = 1, 2, ..., m;\ j = 1, 2, ..., n)$.

  Multiplication:

  If $A_{m\times n} = B_{m\times k} C_{k\times n}$ then $A_{ij} = \sum_{l=1}^{k} B_{il} C_{lj}$.

  Transpose: $A^T_{n\times m}$.

Determinant: $\det(A_{m\times m})$.

Inverse: $A_{m\times m}^{-1}$.

Solution of linear algebraic equations:

$A_{m\times n}x_{n\times 1} = b_{m\times 1}$.

3. Basic Procedure of the Stiffness Method

Use a two-member spring example to illustrate these steps.

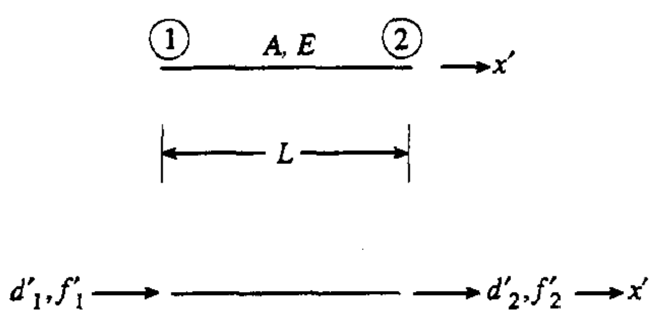
- ♦ Assign a coordinate system for the structure. Assign node numbers for the structure.
- ♦ Define degrees of freedom for the structure and assign numbers for them.
- ♦ Assign numbers to the members of the structure.
- ♦ Break-up the structure into smaller pieces called elements or members. Define element nodal displacements and forces.
- ♦ Identify boundary conditions for the structure in terms of displacements.
- ♦ Write compatibility conditions between the structural nodal displacements and element nodal displacements for each member.
- ♦ Identify external loads for each degree of freedom
- ♦ Write equilibrium equations for each element in terms of displacements. **kd** = **f**
- ♦ Combine elements equilibrium equations to form equilibrium equations for the entire structure. **KD** = **F**
- ♦ Apply boundary conditions to the system equilibrium equations and solve for the system nodal displacements.
- ♦ Finally, consider equilibrium equation for each element and solve for the element nodal forces.

4. Direct Stiffness Method for Truss Analysis

- A truss is a structural system that satisfies the following requirements:
  a. The members are straight, slender, and prismatic. The cross-sectional dimensions are small in comparison to the member lengths. The weights of the members are small compared to the applied loads and can be neglected. Also when constructing the truss model for analysis, we treat the members as a one-dimensional entity (having length and negligible cross-sectional dimensions).
  b. The joints are assumed to be frictionless pins (or internal hinges).
  c. The loads are applied only at the joints in the form of concentrated forces.

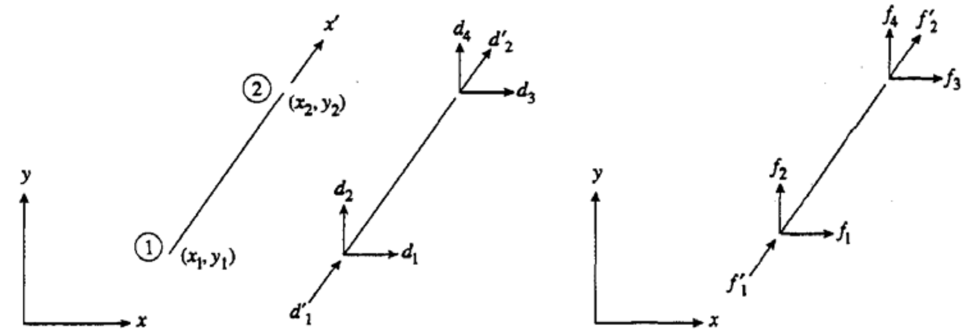
As a consequence of these assumptions, the members are two-force members, meaning that they carry only axial forces. In very many ways, a truss member is quite similar to the typical linear spring. Two nodes define a typical truss element.

- For a two-node truss element shown below, governing equations with respect to the local coordinate system x' can be expressed as:



$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1' \\ d_2' \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix} \quad \text{or} \quad \mathbf{k}'_{2\times 2}\boldsymbol{d}'_{2\times 1} = \boldsymbol{f}'_{2\times 1}.$$

- From the transformation matrix between the local and global coordinate systems shown below, the relationship between the local nodal displacements and global nodal displacements is derived as



$$\begin{Bmatrix} d'_1 \\ d'_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} \quad \text{or} \quad \boldsymbol{d}'_{2\times1} = \boldsymbol{T}_{2\times4}\boldsymbol{d}_{4\times1}.$$

The corresponding nodal force relationship reads

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{Bmatrix} f'_1 \\ f'_2 \end{Bmatrix} \quad \text{or} \quad \boldsymbol{f}_{4\times1} = \boldsymbol{T}^{\mathrm{T}}_{4\times2}\boldsymbol{f}'_{2\times1}.$$

- The global governing equations for a truss element are therefore written as

$$\frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \quad \text{or} \quad \mathbf{k}_{4\times4}\boldsymbol{d}_{4\times1} = \boldsymbol{f}_{4\times1}.$$

k_{ij} = force required in the direction of dof i to produce unit displacement along the dof j.

- The major steps in solving planar truss problems using the direct stiffness method:

Step 1: Select the problem units. Set up the coordinate system. Identify and label the nodes and the elements. For each element select a start node (node 1) and an end node (node2). We use an arrow along the member to indicate the direction from the start node to the end node. This establishes the local coordinate system for each element. Label the two global dof at each node starting at node 1 and proceeding sequentially.

Step 2: Construct the equilibrium-compatibility equations for a typical element.

Step 3: Using the problem data, construct the element equations from Step 2 for all the elements in the problem.

Step 4: Assemble the element equations into the system equations, $\mathbf{k}_{2j\times 2j}\boldsymbol{d}_{2j\times 1} = \boldsymbol{f}_{2j\times 1}$ where *j* is the number of joints in the truss.

Step 5: Impose the boundary conditions.

Step 6: Solve the system equations **KD = F** for the nodal displacements **D**.

Step 7: For each element using the nodal displacements, compute the element nodal forces.

- **Example 1**: The figure shows a planar truss. The material is steel with elastic modulus $E = 200\,GPa$ and the cross-sectional area of both members is $A = 0.01m^2$. Use the direct stiffness method to solve for nodal displacements and member forces. (Rajan's book page 351-353, Example 6.2.1)

- **Example 2**: The figure shows a planar truss. The material is steel with elastic modulus $E = 30\times10^6\,\mathrm{Psi}$ and the cross-sectional area of each members is $A = 1.2\mathrm{in}^2$. Use the direct stiffness method to solve for nodal displacements and member forces. (Rajan's book page 354-358, Example 5.2.5)

5. Direct Stiffness Method for Frame Analysis

- A planar frame is a structural system that satisfies the following requirements:
  a. The members are slender and prismatic. They can be straight or curved, vertical, horizontal, or inclined. The cross-sectional

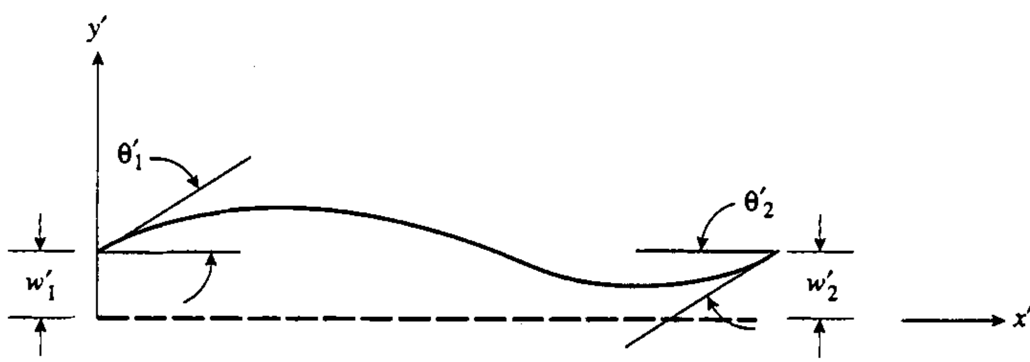
dimensions are small in comparison to the member lengths. Also when constructing the frame model, we treat the members as one-dimensional entities (having length and negligible cross-sectional dimensions).

b. The joints can be assumed to be rigid connection, frictionless pins (or internal hinges), or typical connections.

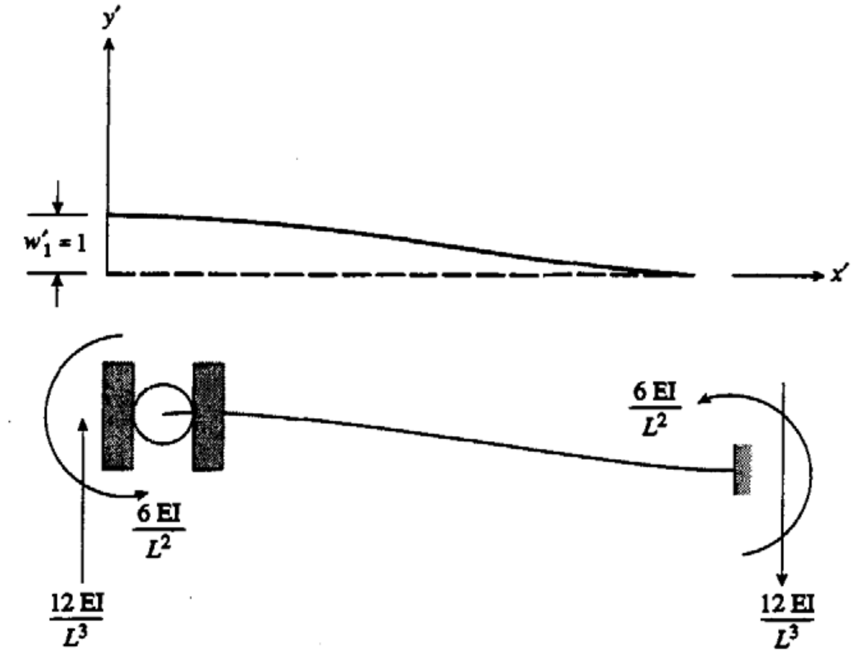
c. The loads can be concentrated forces or moments that act at joints or on the frame members, or distributed forces acting on the members.

In this section, however, we assume that the frame is made of straight members and that the connections are rigid. We develop the element capable of modeling a planar frame in two stages. In the first stage, the flexure effects (due to shear force and bending moments) will be considered. In the second stage, the axial effects will be considered. Using the superposition principle, we can then construct the behavior of a frame element. As before, the superposition is valid only if the displacements are small. In structural analysis terminology, members that are subjected primarily to flexural effects are said to be beams whereas members with combined axial-flexural effects are called beam-columns. By the end of this section we will have developed the element equations for the combined effects that can also be used to model pure beam behavior. To avoid construction and usage of several different terminology, we refer to this element simply as the **beam element**.

The planar beam element displacements (degrees of freedom) in the local coordinate system (x’-y’)



- For a two-node planar frame element shown below, governing equations with respect to the local coordinate system x' can be expressed as:



without axial forces

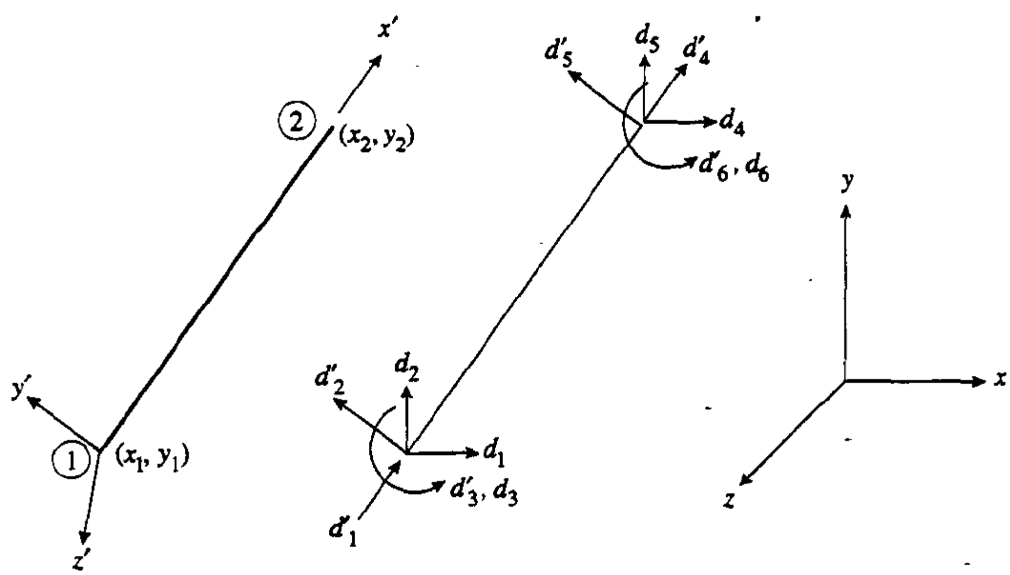
or $\boldsymbol{k'}_{4\times4}\boldsymbol{d'}_{4\times1} = \boldsymbol{f'}_{4\times1}$. $\boldsymbol{k'}_{4\times4} = \dfrac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

with axial forces

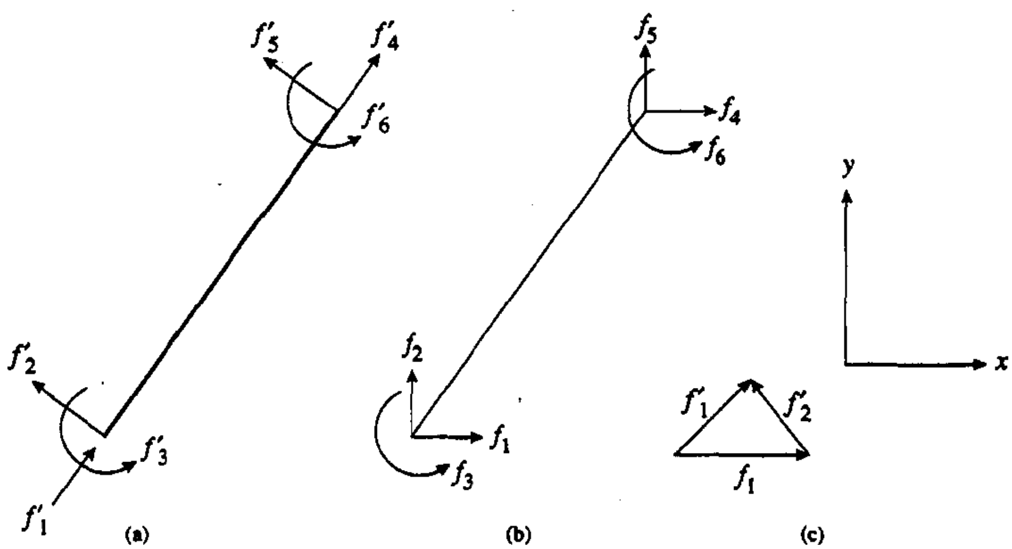
or $\boldsymbol{k'}_{6\times6}\boldsymbol{d'}_{6\times1} = \boldsymbol{f'}_{6\times1}$.

$$\boldsymbol{k'}_{6\times6} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

- From the transformation matrix between the local and global coordinate systems shown below, the relationship between the local nodal displacements and global nodal displacements is derived as



Beam element’s local and global coordinate systems and defrees of freedom. Z and z’ axes coincide and point out of the page.



Beam element's local and global nodal forces

$$\begin{Bmatrix} d'_1 \\ d'_2 \\ d'_3 \\ d'_4 \\ d'_5 \\ d'_6 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} \quad \text{or} \quad \boldsymbol{d}'_{6\times1} = \boldsymbol{T}_{6\times6}\boldsymbol{d}_{6\times1}.$$

The corresponding nodal force relationship reads

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{bmatrix} l & -m & 0 & 0 & 0 & 0 \\ m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & -m & 0 \\ 0 & 0 & 0 & m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f'_1 \\ f'_2 \\ f'_3 \\ f'_4 \\ f'_5 \\ f'_6 \end{Bmatrix} \quad \text{or} \quad \boldsymbol{f}_{6\times1} = \boldsymbol{T}^{\mathrm{T}}_{6\times6}\boldsymbol{f}'_{6\times1}.$$

- The global governing equations for a truss element are therefore written as

$$\boldsymbol{k}_{6\times6}\boldsymbol{d}_{6\times1} = \boldsymbol{f}_{6\times1}.$$

- **Element Loads**. All the loads on the elements must be transformed to equivalent loads at the node points. The equivalent force system (equivalent joint forces) is nothing but the *opposite of the fixed-end forces*.
- The major steps in solving any planar frame problem using the direct stiffness method:

Step 1: Select the problem units. Set up the coordinate system. Identify and label the nodes and the elements. For each element select a start node (node 1) and an end node (node 2). We use an arrow along the member to indicate the direction from the start node to the end node. This establishes the local coordinate system for each element. Label the three global dof at each node starting at node 1 and proceeding sequentially.

Step 2: Construct the equilibrium-compatibility equations for a typical element.

Step 3: Using the problem data, construct the element equations from Step 2 for all the elements in the problem**.** If there are element loads, compute the equivalent joint loads and transform them to the global coordinate system. Note that if there is more than one element load acting on an element, use linear superposition (algebraic sum) of all the element loads acting on that element.

Step 4: Assemble the element equations into the system equations.

Step 5: Impose the boundary conditions.

Step 6: Solve the system equations **KD = F** for the nodal displacements **D**.

Step 7: For each element using the nodal displacements, compute the element nodal forces.

- **Example 1**: The figure shows a continuous beam. The material is steel with elastic modulus $E = 200\,GPa$ and the cross-sectional properties are such that $A = 0.01m^2$, and $I = 0.0001m^4$. Use the direct stiffness method to solve for nodal displacements and member forces. (Rajan's book page 369-373, Example 6.2.5)

- **Example 2**: The figure shows a planar frame. The material is steel with elastic modulus $E = 200 GPa$ and the cross-sectional properties are such that $A = 0.01 m^2$, and $I = 0.0001 m^4$. Use the direct stiffness method to solve for nodal forces and support reactions. (Rajan's book page 373-377, Example 6.2.6)