

Periodic splines discretisation

Mark Blyth

Week's activities

- ✎ Finish manuscripts
 - ▶ Paper in progress
 - ▶ Proposed edits completed
 - ▶ Word-count down to 6050+400
 - ▶ Abstract done
- ✎ Read about numerical continuation discretisation
- ✎ Read up on non-Bayesian splines
- ✎ Started periodic splines discretisation

Conference paper goals

Two separate-but-related goals:

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 - ▶ Demonstrate an alternative [*lower-dimensional*] method
 - ▶ Use simple standard methods to do so

Surrogate modelling

This part is done, doesn't require much work

- ✂ Previous work on GPR, BARS
- ✂ Shows how to reconstruct a signal from noisy observations
 - ▶ Alternative to low-pass filter, that works on spiking signals
- ✂ Useful for applying Fourier to noisy signals
 - ▶ Removes noise
 - ▶ No Nyquist cap

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 - ✂ NODYCON discretisation method: [*non-Bayesian*] periodic splines
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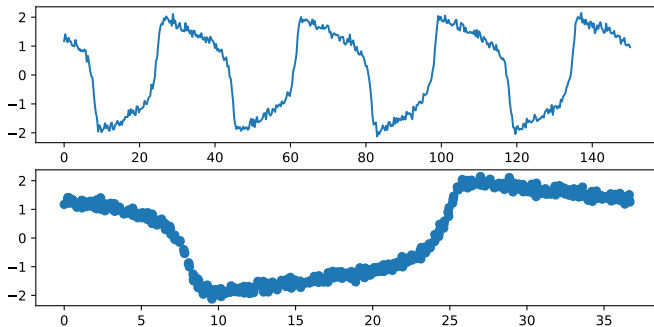
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 - ▶ Discretisation = BSpline coefficients

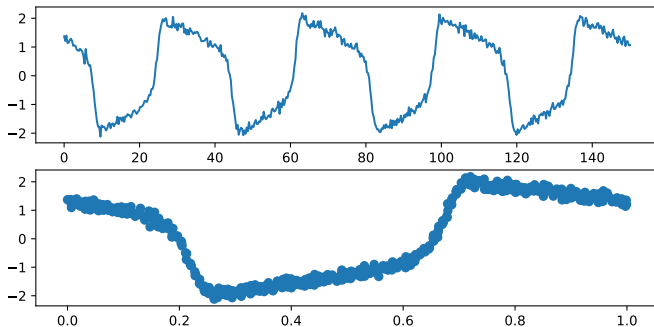
Period stacking

Uses ACF to estimate frequency, then NLS to refine estimate



Period stacking

Uses ACF to estimate frequency, then NLS to refine estimate; removes period from continuation scheme



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 - ▶ Standard methods don't give a minimal knotset
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- ✂ Turns out we can make a naive method, that works very well...

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 - ▶ Helps overcome the local minima issue

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This works surprisingly well:

- ✦ Few knots = model quick to fit, easier to optimise
- ✦ Nice result would be to analytically derive a LSQ fitting procedure

For comparison. . .

- ✦ Bayesian automatically selects required number of knots

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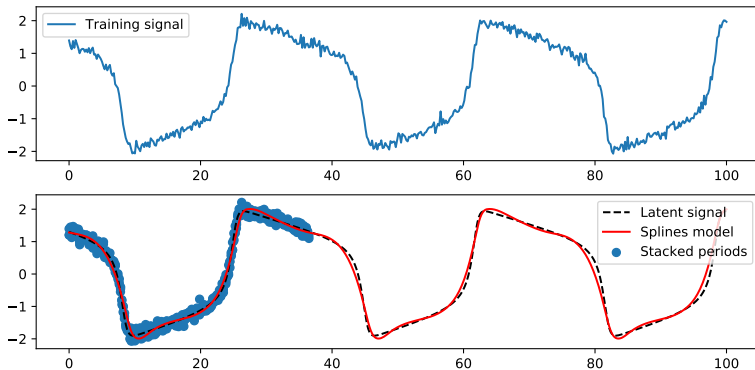
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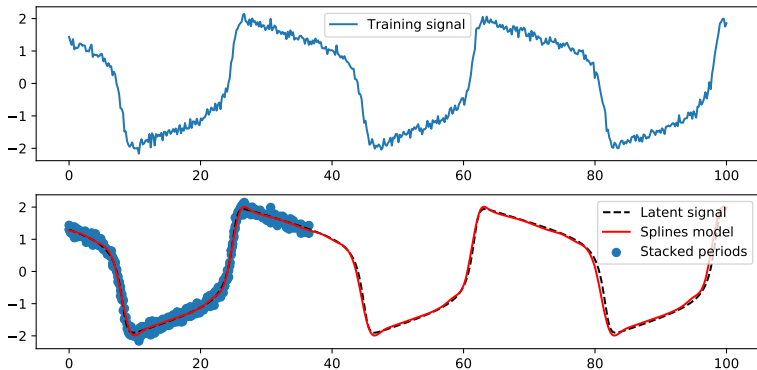
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- ✂ Bayesian could overcome the period estimation problem *[see later]*
- ✂ This method gets good results much more simply

Smoothness fit



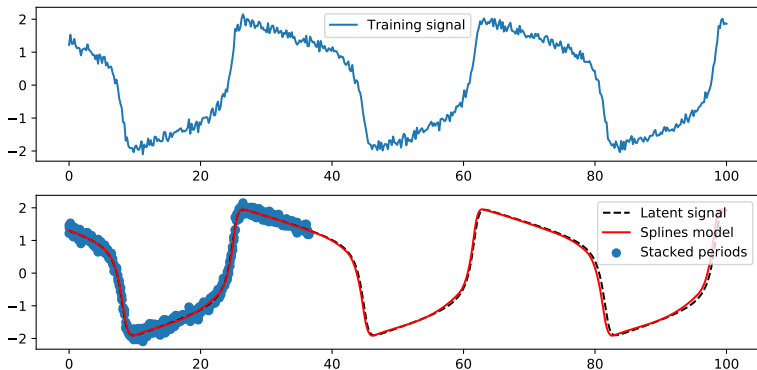
Smoothness = 10; 38-dimensional discretisation.

Smoothness fit



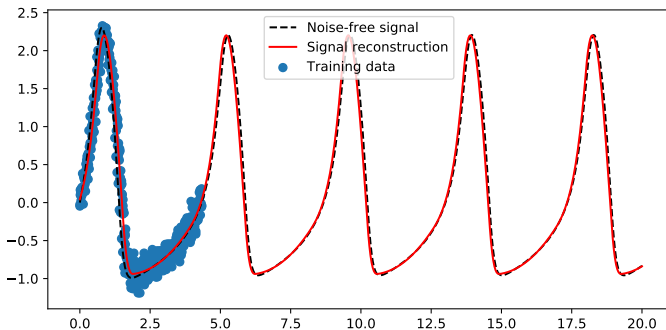
Smoothness = 5; 58-dimensional discretisation.

Optimizer fit



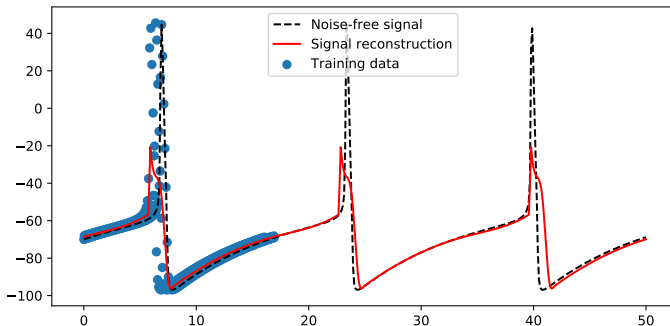
8 interior knots

Optimizer fit



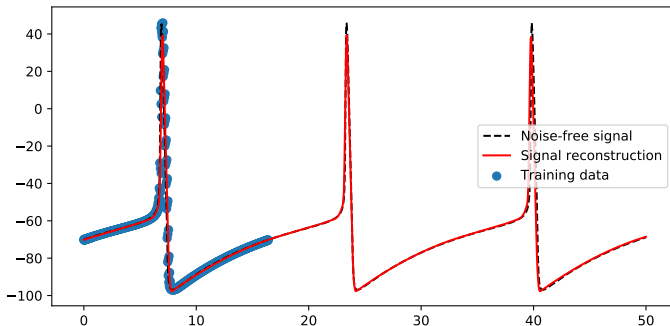
Also works on more neuron-like data

Optimizer fit



Too few datapoints across the spikes

Optimizer fit

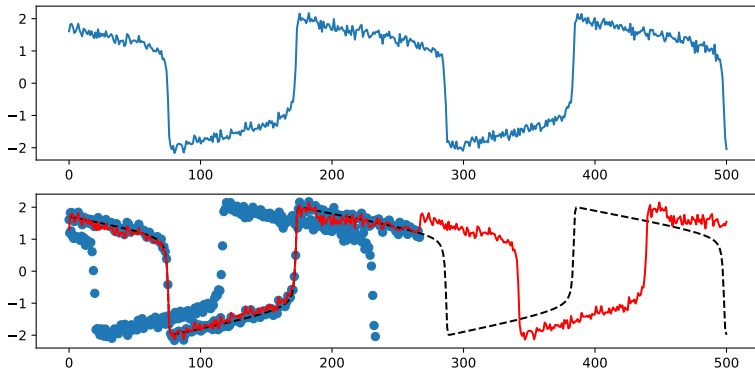


Much better!

Issue

- ✖ Inaccuracies in the period will add up to a big phase shift over time
- ✖ Bad period estimate can have disastrous results!

The period-estimation problem



The period-estimation problem

- ✖ Increasing the timescale separation ‘squares up’ the signal, but breaks F0 period estimation
- ✖ NLS F0 estimation also uses Fourier harmonics, so breaks on the same signals Fourier discretisation would break on *[only tried one test-case!]*
- ✖ Playing with the F0 estimation parameters / methods helps with this, but adds more mysterious hyperparameters
 - ▶ Bayesian methods also offer a way around this

Next steps

- ✦ Keep working on paper
- ✦ Compare reconstruction error for a given number of knots, for Fourier and splines
- ✦ Use discretisation in CBC
 - ▶ Treat knot positions as a fixed hyperparameter
 - ▶ BSpline coefficients become a signal discretisation
- ✦ Mini-review knot selection methods
 - ▶ Worth discussing the alternative methods in a paper