Periodic splines discretisation

Mark Blyth

Recent activities

- Emulation methods
- Discretisation methods
- Working on manuscripts
  - Numerical continuations paper
    - Word-count down to 6050+400
  - Emulators and discretisors abstract

Conference paper goals

Two separate-but-related goals:

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   - Demonstrate an alternative *[lower-dimensional]* method
   - Use simple standard methods to do so

Surrogate modelling

This part is done, doesn't require much work

- Shows how to reconstruct a signal from noisy observations
  - Replace real signal with noise-free surrogate model
  - Surrogate model gives a nice clean, interpolable *[no Nyquist cap]* signal
  - Alternative to low-pass filter, that works on spiking signals
- Useful for applying Fourier to noisy signals
- Tested methods: free-knot splines, Gaussian process regression

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- NODYCON discretisation method: *[non-Bayesian]* periodic splines

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Simple non-parametric method for periodic signals

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- To discretise, fit knots, b_i at the start; β_i become discretisation

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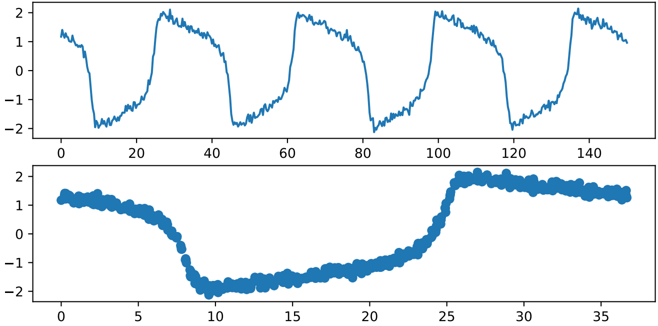
   - Discretisation = BSpline coefficients

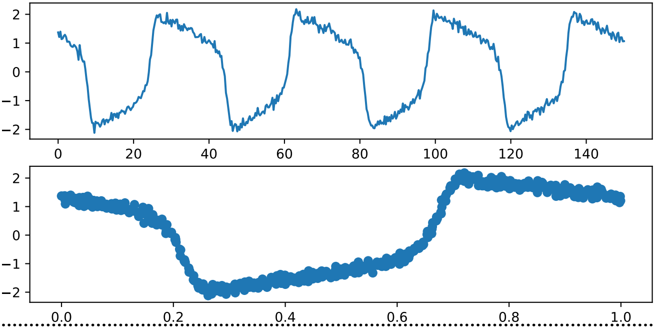
Period stacking

Uses ACF to estimate frequency, then NLS to refine estimate



Period stacking

Uses ACF to estimate frequency, then NLS to refine estimate; removes period from continuation scheme



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- Turns out we can make a naive method, that works very well…

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   - Helps overcome the local minima issue

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- Few knots = model quick to fit, easier to optimise
- Nice result would be to analytically derive a LSQ fitting procedure

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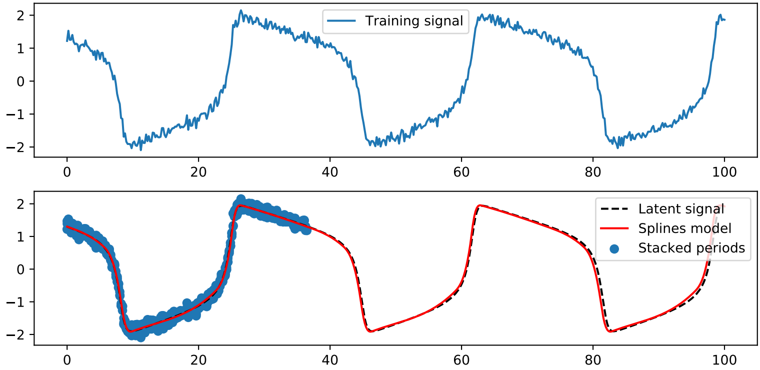
- Bayesian could overcome the period estimation problem *[see later]*
- This method gets good results much more simply

Optimizer fit



8 interior knots

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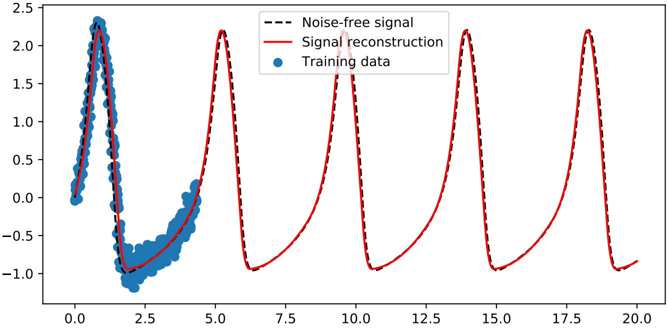

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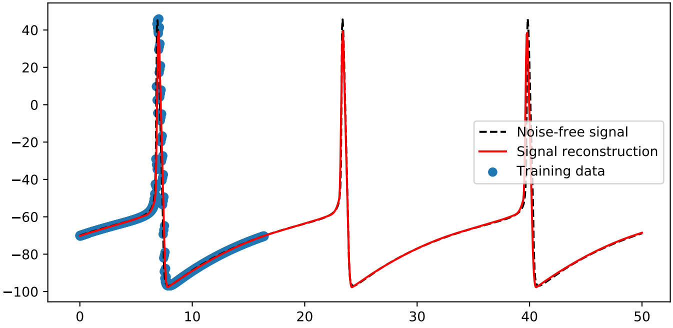

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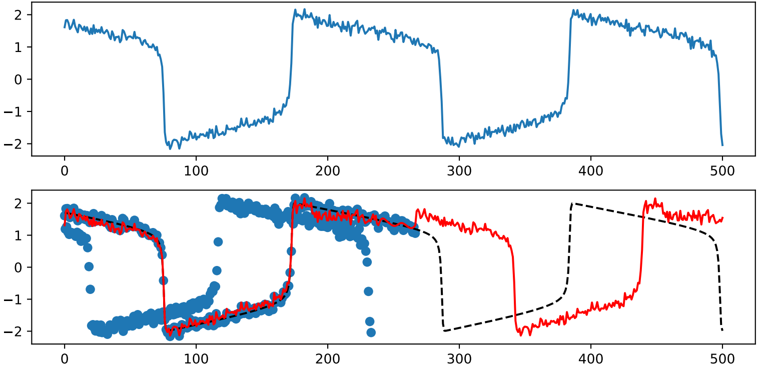
Issue

- Inaccuracies in the period will add up to a big phase shift over time
- Bad period estimate can have disastrous results!

The period-estimation problem



The period-estimation problem

- Increasing the timescale separation 'squares up' the signal, but breaks F0 period estimation
- NLS F0 estimation also uses Fourier harmonics, so breaks on the same signals Fourier discretisation would break on *[only tried one test-case!]*
- Playing with the F0 estimation parameters / methods helps with this, but adds more mysterious hyperparameters
  - Bayesian methods also offer a way around this

Next steps

- Keep working on paper
- Compare reconstruction error for a given number of knots, for Fourier and splines
- Use discretisation in CBC
  - Treat knot positions as a fixed hyperparameter
  - BSpline coefficients become a signal discretisation
- Mini-review knot selection methods
  - Worth discussing the alternative methods in a paper