Bayesian methods for the control-based continuation of multiple-timescale systems

Mark Blyth

Plan de jour

- CBC maths
- Surrogate modelling
- Novel discretisations

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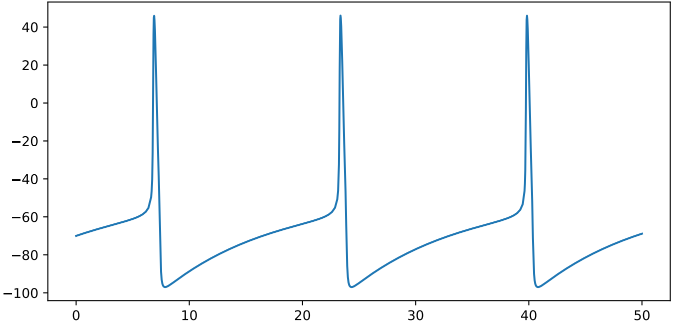
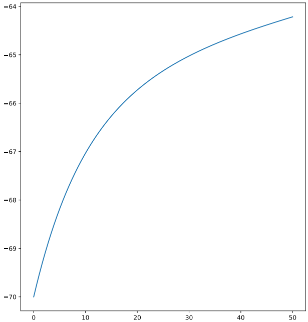
What is CBC?

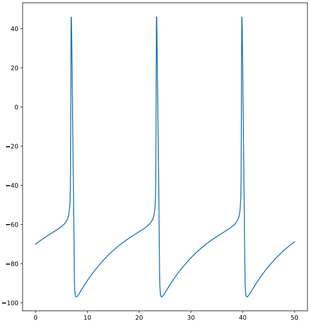
Dynamics are ‘what something does’



What is CBC?

A bifurcation is a change in dynamics





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1. Find a feature

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Bifurcation analysis:

1. Find a feature
2. Change a parameter slightly
3. Find where the feature moved to
4. Bifurcations occur when features change, appear, or disappear

What is CBC?

- Numerical continuation:
  - Features x defined given by $f(x, \lambda) = 0$
  - Change λ, see how x changes

George Box

All models are wrong, but some are useful

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4. Find how noninvasive $u^*(t)$ changed
   - Tracks system features, bifurcations without ever needing a model

CBC

Control-based continuation

A model-free bifurcation analysis method. Uses a controller to stabilise a system, and continuation to track features.

My project: use CBC to analyse the bifurcations that make neurons fire

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  - Basically, trying to solve for a function
- Discretisation lets us approximately solve the problem by solving a finite set of equations

What is discretisation?

Goal: solve $I\left[u^{*}\right]=0$

1. Translate problem to system of vector-valued equations
2. Solve system numerically
3. Translate solution back to a continuous function

Translation between continuous and vector-valued systems is discretisation

What is discretisation?

Definition (Discretisation)

The act of representing a continuous signal by a discrete counterpart

We want a discretisation that

- Has minimal discretisation error
- Is low-dimensional

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- This is how we track dynamical features

Issues with discretisation

- Solving the discretised system takes a long time when it is high-dimensional
- Neuron signals require lots of Fourier harmonics to discretise
- Higher-order harmonics are harder to get *[Nyquist cap]* and less accurate *[SNR]*

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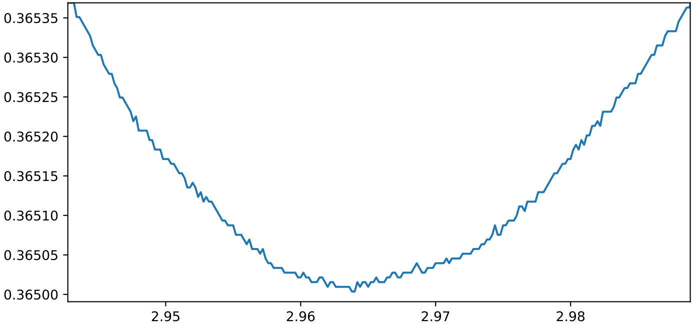
- Record experimental data
- Fit a surrogate model
- Perform analysis on model

Why surrogates?

Real data are noisy



[Thanks to LR for the data]

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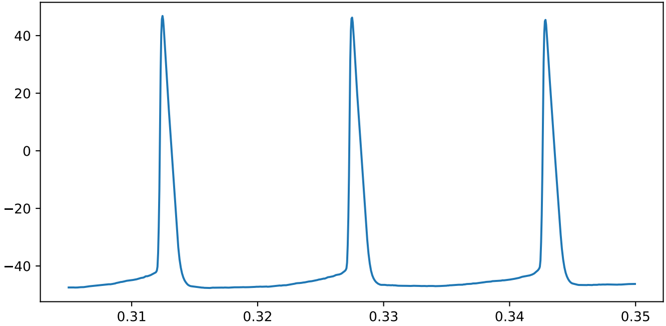
Real data are ‘fast’



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- A good surrogate lets us remove noise in a statistically optimal way
  - Less noise = better discretisation

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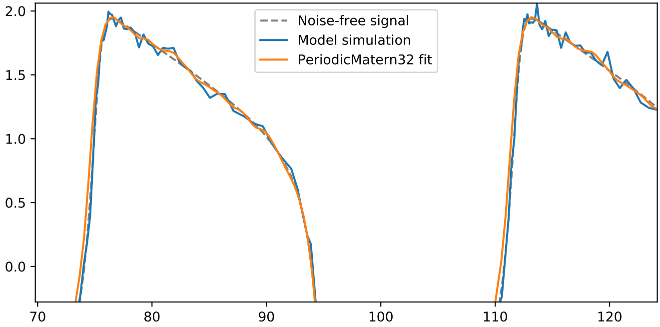
- This is Gaussian process regression!

Gaussian process regression surrogates

Build a statistically optimal regression model from noisy observations



GPR results

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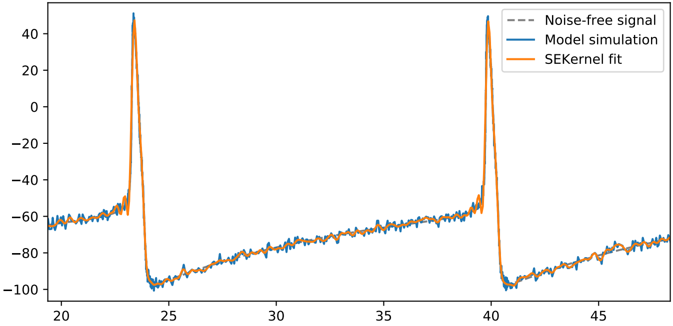
- Stationary covariance = poorly encoded beliefs = low belief in posterior
  - Bayes with bad priors = bad results!

GPR results


Noise-free signal
Model simulation
SEKernel fit
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−20
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35
40
45

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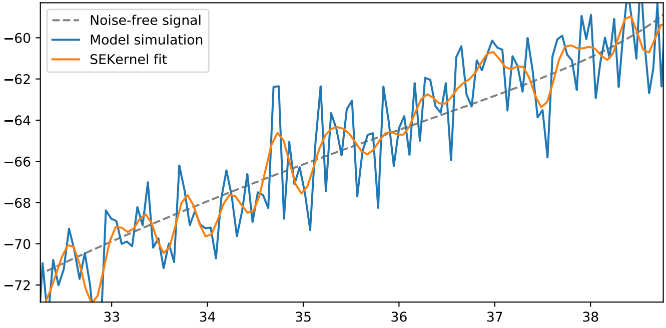
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36
37
38

GPR results

- Stationary GPR, non-stationary data = overly flexible models
- Non-stationary would fix this
- Non-stationary GPR is hard!

Splines

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- Choose polynomials so that the function is smooth
- Finite, low degree-of-freedom, forcibly averages out noise

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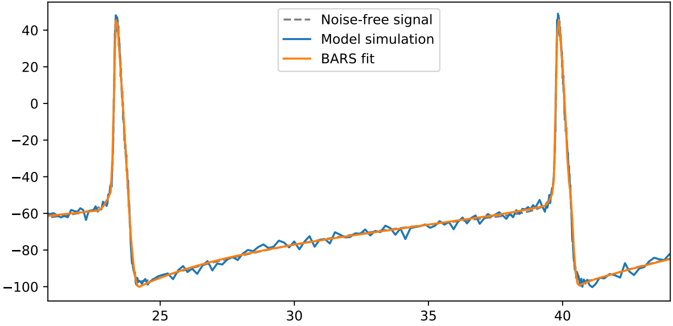
- This is Bayesian free-knot splines

Splines as a surrogate

Result 1: splines outperform stationary GPR as neuronal data surrogate



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- Issue: too many coefficients are needed to discretise the signal
  - Too many = too slow
- We can reconstruct signal from splines models
  - Is this a discretisation?

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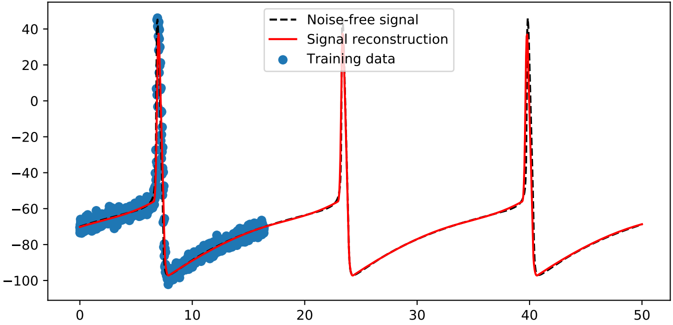
  - Result 2: probably…

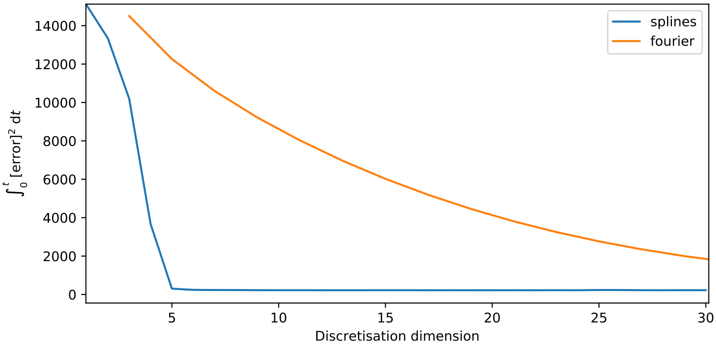
Spline discretisation

8-dimensional discretisation; but does it work with continuation?



Splines vs Fourier

Hodgkin-Huxley neuron; error decays *significantly* faster with splines



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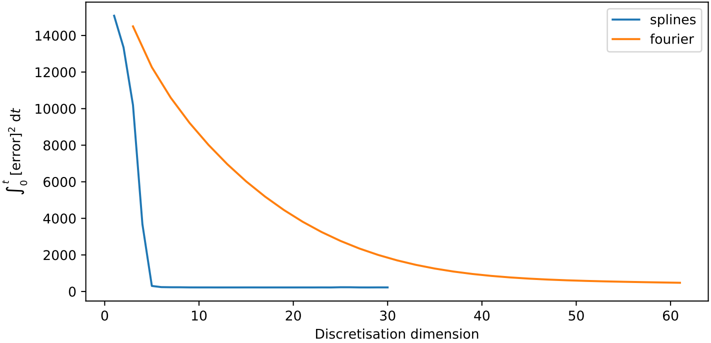

Where next?

- Test the robustness
- See if the discretisation breaks down with stochastic models
  - It probably will
- Test the discretisation with continuation
  - Splines discretisation is still only a local model
  - Need to ensure it can predict signals at other parameter values


That's all Folks