Deterministic continuation of stochastic metastable equilibria via Lyapunov equations and ellipsoids

Mark Blyth

Background

- Numerical continuation is a useful tool for deterministic systems
- Stochastic dynamics can't be studied with standard continuation
- This work tries to change that

Section 1: Intro

Deterministic systems:

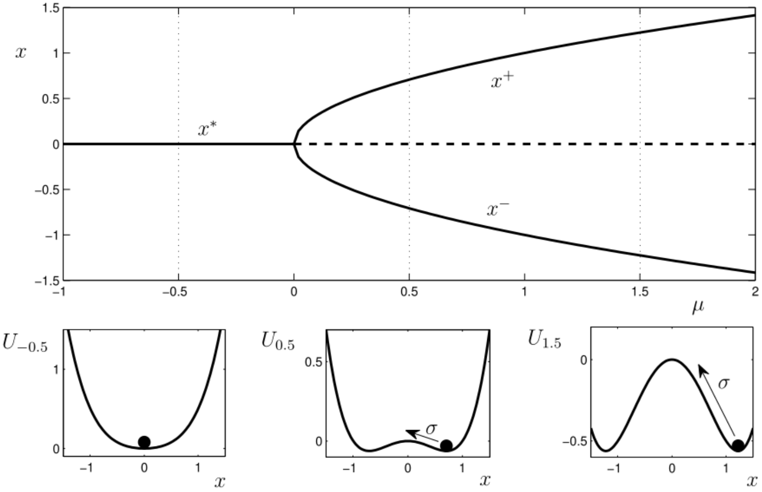
- Equilibrium = time-invariant solution
- Equilibrium state depends on parameters
- Continuation reveals that dependence

From deterministic to stochastic

- Very little work to extend continuation to SDEs
- How does small noise change deterministic results?
- We can extend numerical continuation to track local information about metastable equilibria

An analytical example

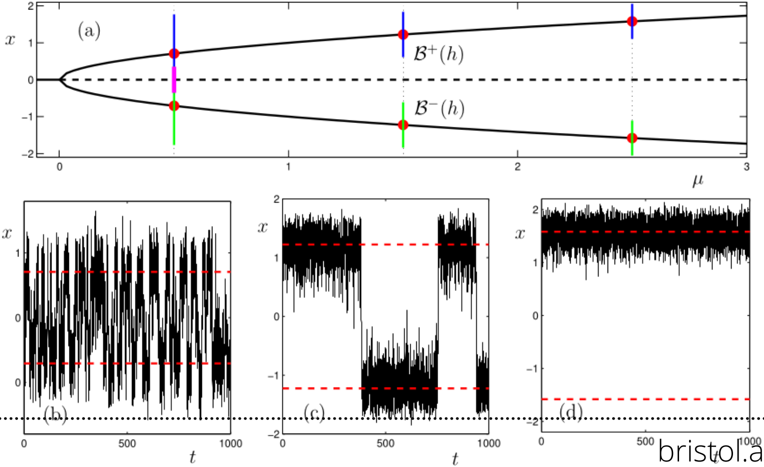
Consider a noise-corrupted pitchfork normal form



Noise-induced pitchfork dynamics

- Interesting noise-induced dynamics occur in the bistable region
- For any initial condition, we will almost surely visit visit both potential wells in finite time
- We seek a stronger result; what insights can we gain into the timescales of the stochastic transitions?

A graphic example



Studying stochastic transitions

- What methods can we use to study the high-density regions?
- Fokker-Planck equations are inefficient
- Can we find a more efficient way of defining them?

Continuation of metastable equilibria

Can we find a more efficient way of defining high-density regions?

- Linearise the system about each equilibrium point
- Calculate the variance of the resulting stochastic process
- Choose a ball around the deterministic equilibria, such that sample paths stay within it with high probability

Towards a stochastic continuation algorithm

We need three results to be able to use this definition in continuation

- Generalise variance ball construction to arbitrary-dimensional SDEs
- Efficiently compute the covariance matrix of the linearised SDE, at each point on the continuation curve
- Define test-functions for overlapping stochastic neighbourhoods

Problem 1: generalised variance balls

- We can linearise multidimensional processes easily
- The covariance matrix is then given by the solution to an ODE
- The time-invariant solution is a solution of a Lyapunov equation
- Ellipsoids are defined with their principle axes scaled according to the inverse covariance matrix

Problem 2: Solving the Lyapunov equation

We have established the covariance matrix is given by the solution to a Lyapunov equation. How do we efficiently solve it for a single equilibrium? And for a branch of equilibria?

- Solution methods are well-studied within control theory
- The continuation consideration adds several new aspects to the problem
- Covariance computation is actually fairly straightforward, with several methods available

Noise structure and degenerate ellipsoids

- If the covariance matrix is noninvertible, we can't define ellipsoids
- This can happen for certain system and noise structures
- Define density neighbourhood over the stochastic variables only

Problem 3: Ellipsoids and test functions

- Distance between two ellipsoids indicates the timescale of their stochastic transitions; how do we compute it?
- We choose a distance measure that doubles up as a test function
- The distance is given by the solution to an optimization problem

Algorithm summary

Initialization step:

- Find a stable equilibrium of the deterministic component of the system
- Compute the linearisation of the deterministic system at that equilibrium
- Set up the Lyapunov equation for covariances, and solve using Bartels-Stewart algorithm

Algorithm summary

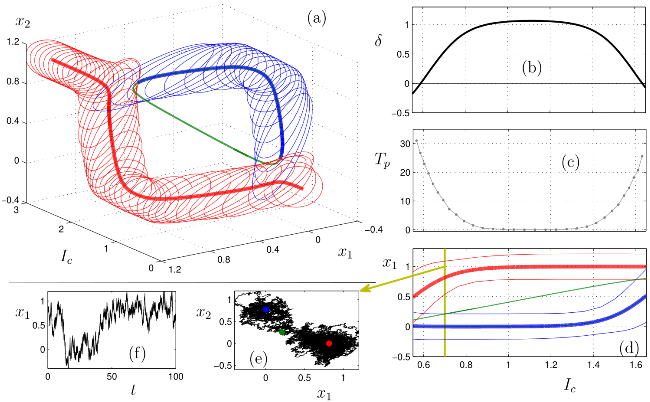
Iteratively…

- Take a predictor-corrector step, solving deterministic continuation equations at a new parameter value
- Iteratively solve the Lyapunov equation
- Construct a high-density ball, for some chosen confidence level
- Solve an optimization problem for the distances between each pair of balls

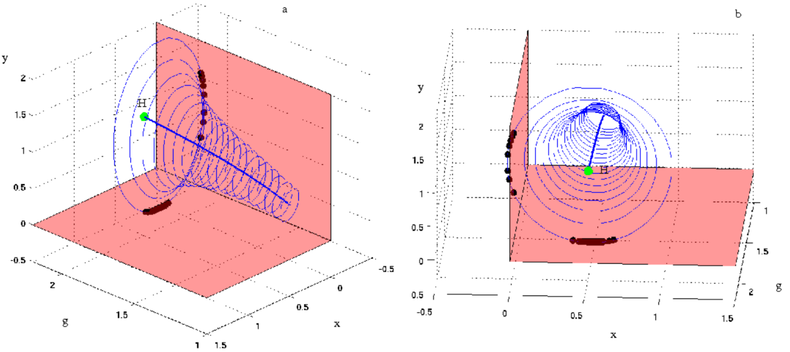
Outputs

- Deterministic equilibria
- Ellipsoids
- Mutual distances

Example results



Example results



Closing remark: a special case

- Ellipsoid separation is only a local heuristic for stochastic timescales
- What if we could incorporate global information into the continuation?
- Eyring-Kramer's law gives analytical switching rates in special cases