Deep learning: An Introduction for Applied Mathematicians

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Background

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- Take input vectors $\mathbf{x}_i$
- Take output vectors $\mathbf{y}_i$
- Fit some nonlinear model $\mathbf{y} = f(\mathbf{x})$

Drilling site example

- Input data: $\mathbf{x}_i = (u_i, v_i)$, oil well location
- Output data: $y_i \in \{0, 1\}$, success or failure
- Learn some nonlinear model mapping locations to drill successes

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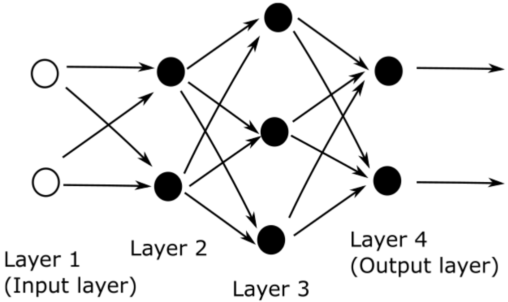
- Fit the model
  - Find appropriate values for each w, b

Why ‘neural network’?

- We can represent the equation nicely as a graph



- Information flows forward through the graph much like it does through the brain

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- Inputs can be efficiently computed with some linear algebra
- Much neater representation than 'stringed nonlinearities'

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- Having lots of W_k, b_k gives us a very general model
- We can fit the data by selecting W, b to minimise the error $\sum \|f(\mathbf{x}_i - \mathbf{y}_i\|^2$
- Universal approximator!

Network training

We now have a model; how do we fit it?

Definition (Gradient descent)

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- Flat bit means cost stops changing
  - Could be a good fit, could be a saddle or local minimum

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- How do we find $\frac{\partial \mathrm{cost}}{\partial p_i}$?

Backprop

Definition (Backpropagation)

A method for finding cost-function gradient, given

- a cost function
- a nonlinearity
- a network topology

Backprop is the core of NN training!

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- Can find cost function gradient by chain-ruling these all together
- Can sum the resulting gradient across the full minibatch

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  - Relate next layer's error to cost function gradient
- Propagates errors backward through the network

Convolutional neural networks

- Visual cortex has a ‘receptive field’
- CNN mirror this with local kernel transforms
- Convolutional layers automatically extract features
- Allows NNs to efficiently manipulate high-dimensional data

Practical aspects

Definition (Overfitting)

Representing the training data too closely, and losing the ability to generalise

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- We don't have to use residuals
  - Softmax-log-loss

The essence of ML

- Machine learning sounds flashy and cool; it's just big statistics
- Large-scale model definitions and cost-function-fitting

Section 8 discussion points

- Why use NNs?
  - When do other methods generalise better?

Section 8 discussion points

- How robust are the results?
  - Do small input changes matter much? Should they?

Section 8 discussion points

- What's a sensible nonlinearity?
  - Is there any reason to choose ReLU over sigmoids?

Section 8 discussion points

- What topology do we need?
  - How many hidden layers? How big?

Section 8 discussion points

- Can we regularise?
  - Reduce overfitting by penalising model complexity

Section 8 discussion points

- Explainability
  - *Why* should NNs give good results? Why do they give the results they do?

Discussion

- Why use NNs vs. another method?
- What topology do we need?
- What's a sensible nonlinearity?
- How robust are the results, and how much do we care?
- Can we regularise?
- Explainability – how much do we trust the results?
- How much can we actually learn from a black box?
- How much data is enough data?
- What are the applications to nonlinear dynamics?

Next paper

Someone to lead?

Suggestion: Heinonen, Markus, et al. "Learning unknown ODE models with Gaussian processes." arXiv preprint arXiv:1803.04303 (2018).