University of BRISTOL

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Collocation, collocation, collocation

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Week's work

- Collocation methods for standard continuation
- Annual review
- Brainwave: collocation methods for CBC

Annual review

Very positive! Main take-aways:

1. It's worth looking into how collocation could be used in CBC
2. Differentiation matrices, spectral, pseudospectral methods might let us find a Jacobian without finite differences
3. Next project should work towards some grand unifying goal
    - Physiologists have lots of good experimental techniques for studying neurons; a CBC project would need to find a physiologically useful question to answer
    - Alternatively, could explore the theoreticals of CBC – noise-robustness, efficiency, accuracy, higher-codimension continuation, etc., to work answer an experimental nonlinear dynamics question
    - Challenge is to find an overall question to answer that is useful its target community

Collocation

- Deriving collocation equations for generic boundary value problems
- Re-read Kuznetsov Elements; realised I'm making things harder than they need to be
- Wikipedia suggested scalar coefficients, separate BSpline basis functions for each dimension
- More standard method: vector coefficients, scalar basis funcs shared across dimensions
- Re-deriving the collocation equations with scalar basis funcs
- Realising CBC could use collocation
- Deriving CBC collocation equations

Discretisations

CBC discretisation:

1. Project the input signal onto a set of basis functions
2. Project the output signal onto a set of basis functions
3. Solve for equality between input and output coefficients

AUTOesque discretisation:

1. Construct a boundary-value problem for the periodic orbit
2. Construct a mesh across the domain of the independent variable
3. Construct an approximate BVP solution-form
4. Find the basis func. coeff's s.t. the BVP is solved exactly at the meshpoints

Generalisations

Collocation doesn't seem applicable to CBC

- It's a method for solving differential equations
- We don't have access to any differential equations, only the IO map

- Consider the CBC IO map
  - Maps some periodic control-target to some periodic system response
  - We can consider the controlled system as some nonlinear operator on the space of periodic functions
  - We seek a function that solves the operator's fixed-point problem

- Consider the boundary value problem
  - The ODE represents some (possibly nonlinear) differential operator
  - We seek a function that satisfies the differential operator, and some boundary conditions

- More abstract'ly: collocation helps solve an operator problem; CBC gives us an operator problem

Collocation for CBC

Choose some points along our control target; find coefficients s.t. those points remain unchanged by the IO map

- Let C_T be the space of T-periodic functions
- Let $N : C_T \to C_T$ be a nonlinear operator defined by the controlled system
- We seek $x(t)$ s.t. $N\left(x(Tt)\right) = x(Tt), t \in [0, 1]$
- Define a collocation mesh $\xi_1 = 0 < \xi_2 < \cdots < \xi_n = 1$
- Assume a solution of form $x(t) = \sum \beta_i B_i(t)$
- Mandate solution correctness at collocation points
  - $N\left(\sum \beta_i B_i(\xi_i)\right) = \sum \beta_i B_i(\xi_i), i \in \{1, 2, \ldots, n\}$
- Solve for β_i satisfying the above

We're free to choose basis func's B_i; lots of collocation literature, lots of basis func choices

Collocation vs the standard method

Standard method: basis func. coeff's are the object of interest; input and output functions are a means to an end

- We seek the basis function coefficients that remain unchanged when passed through the IO map
- We don't really care what the input and output functions are, because if their coefficients are the same, so are the signals
- This ceases to solve the undiscretised problem when we have any discretisation error

Collocation: coefficients are a means to an end; input and output functions are the object of interest

- We seek coefficients s.t. chosen points remain unchanged by the IO map
- Coefficients are just the parameters we adjust to find this fixed-point; we don't care about their values
- The discretisation can be inexact and we can still find a solution!

Why doubt the standard method?

- The standard method is valid when the discretisation and un-discretisation operators are each other's inverse
  - Can transform between functions, discretisations, and back, with zero error
  - We assume the solution to a discretised map is representative of a solution to the undiscretised map
  - We can prove *[quite easily]* that this assumption breaks when there's discretisation error
- Not been an issue so far, as Fourier discretisation can be made exact to working precision on the comparatively simple signals used so far
- Bonus: collocation is thoroughly tried-and-tested; accepted method because it's very accurate and computationally efficient

Challenges of collocation

- Noise-robustness
    - We're requiring the signals to be exactly equal at the collocation points; this removes all the noise-filtering abilities of the basis functions
    - Surrogates would fix this!
    - Alternatively: collocate statistically – model noise and find the statistically optimal coefficients

- Differentiation
    - Perhaps there's alternative methods to finite-differences?
    - Differentiation matrices? Spectral methods? Generalised secant methods?

Interesting aside: take the infinite limit of number of collocation points; we then get the 'minimally invasive' control reformulation I've discussed previously

Next steps

- Take a break from *[but don't abandon!]* standard-continuation, BSpline discretisation using the 'standard' CBC discretisation method

- Try CBC BSpline discretisation using collocation
- Compare collocation basis functions

- Then… numerical methods
  - Efficient collocation-system solvers, Jacobian estimation

Target result: demonstrate efficient CBC discretisation using collocation methods