Investigating spline numerics

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Week's goals

- Fix splines CBC code
  - Done for the non-adaptive case
- Investigate whether the code now works
  - It doesn't
- Writing (continuation paper, extended conference paper)
  - Happening slowly

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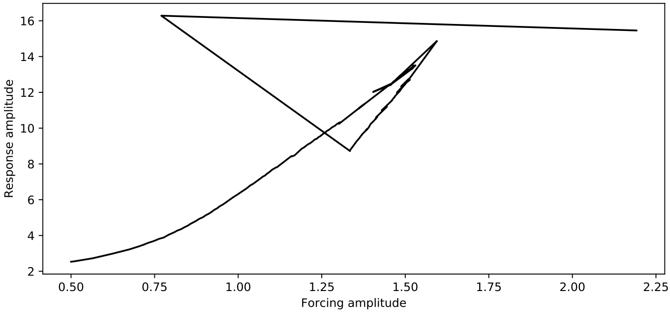
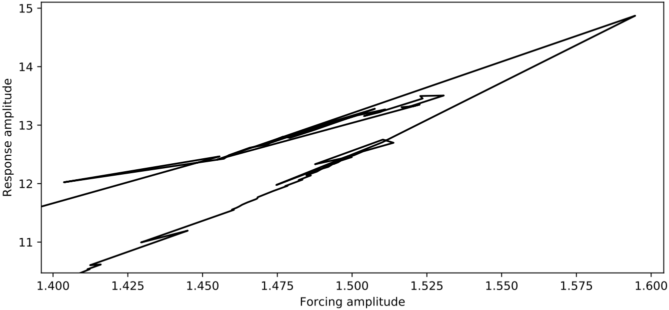
   - Haven't checked this; avoiding the adaptive knots method for now

Fixed plotting



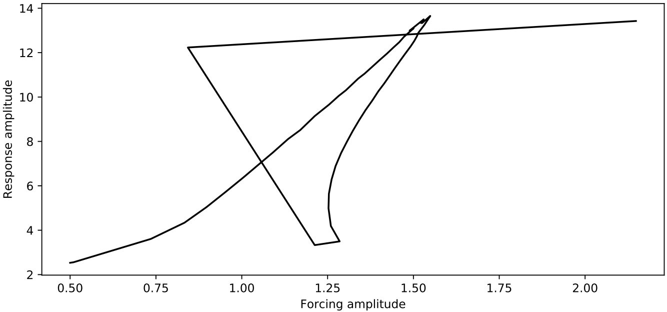
Stepsize 0.1; 3 interior knots; FDSS 0.2

Fixed plotting: zoomed in



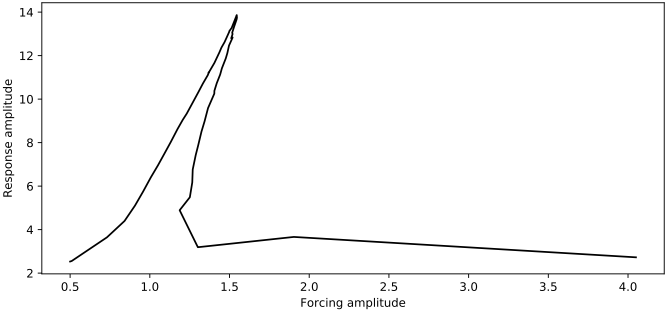
Bigger stepsize?

Fixed plotting, bigger steps



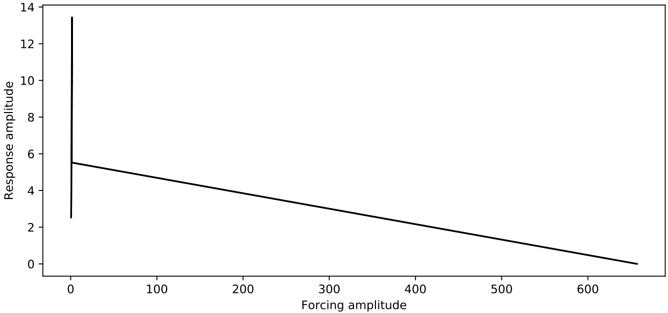
Stepsize 1; 3 interior knots; FDSS 0.2; better, but solution jumps; let’s change exterior knots

New exterior knots



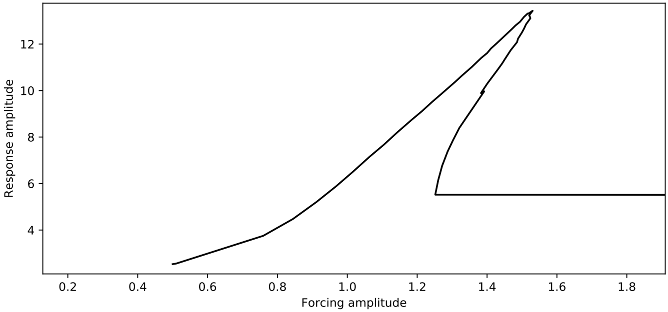
Stepsize 1; 3 interior knots; FDSS 0.5; fixed exterior knots; converged vectors are often not actually solutions

New convergence criteria



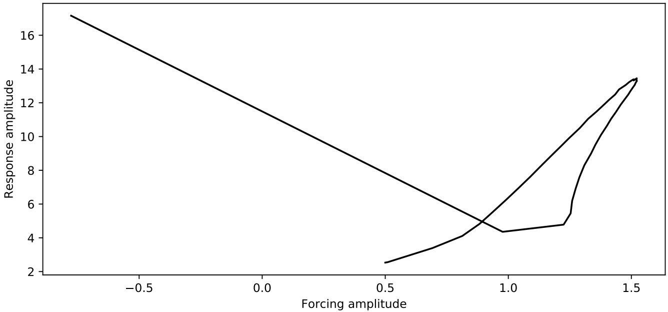
Same hyperparameters as before; convergence declared when continuation equation output has a norm below 5e-2

New convergence criteria



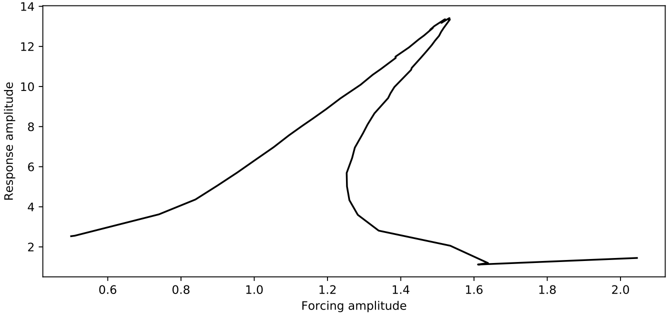
Zoom in to before the jump; more knots might make the results better

Old convergence criteria, but more knots



Stepsize 1; 8 interior knots; FDSS 0.5; solution still jumps!

Best I could get



'Simple' convergence criteria; stepsize 1; 3 interior knots; FDSS 0.2

Issues

- Hyperparameters are very hard to select
  - Lots of trial and error to get even remotely close to the real results
- Solution usually jumps near the second fold
  - Pseudo-arclength condition is met, so the equations are fine
  - Solution is often not actually a solution!
  - Either the solver is broken, or the full system is misbehaving

Non-solutions

- We're solving for $F(x_\omega) = 0$
  - Newton iterations: declare convergence when $\|x_\omega^i - x_\omega^{i-1}\| < tol$
  - Issue: converged x_ω typically does not solve $F(x_\omega) = 0$
  - Alternative: converge when $\|F(x_\omega)\| < tol$
  - Even for $tol \in \mathcal{O}(10^{-3})$, we never converge
  - Solution vector components jump around, rather than converging; unexpected for Newton solvers
- Either solver is problematic, or equations are
  - Using a Newton solver; simple code, tried and tested in the Fourier case
  - Finite differences are meaningful: $\mathcal{O}(0.1)$ perturbations to $\mathcal{O}(1)$ coefficients
  - If the solver and equations are correct, perhaps the equations are simply unsuitable?

Existence and uniqueness

Does a solution to $F(x_\omega) = 0$ actually exist?

- Continuous case:
  - A natural periodic orbit of the system exists
  - This natural periodic orbit necessarily gives noninvasive control
  - Noninvasive control means $F(x_\omega) = 0$, so solutions must exist
- Discretised case:
  - We can exactly represent the continuous problem as an infinite-dimensional Fourier problem
  - As the continuous solution exists, so too must the infinite-dimensional discretised problem
  - Due to how the Fourier errors decay, we can be sure that finite-dimensional Fourier discretisation produces a solvable continuation system
  - We don't get this guarantee with splines

Approximate solutions

Does a splines solution exist? When? Thought experiment:

- Run the system uncontrolled
- Discretise the output
- Use the discretised output as a control target

Imperfect discretisation: control target $\neq$ 'natural' oscillations

- Control becomes invasive
- Control target is not a solution to the continuation equations
  - Even though it was obtained from an exact solution, it is not actually a solution; discretisation error stops the natural system behaviour from being a solution

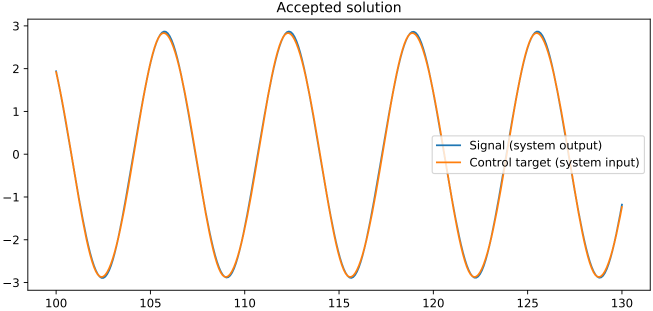
Discretisation error must be negligable for the standard CBC zero problem to become solvable

Key result

- If we have no discretisation error, solution exists to continuation equations
- If we have discretisation error, solution might not exist
- This explains why Fourier works, splines don't
  - No discretisation error for infinite Fourier
  - Can achieve negligable discretisation error for truncated Fourier
  - Harder to remove spline discretisation error

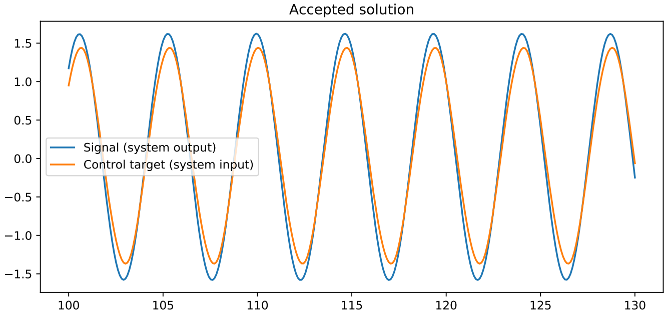
How accurate are splines?

Spline discretisation error



Splines is often very accurate

Spline discretisation error



But sometimes not

Minimization reformulation

- A solution is not guaranteed to exist when the spline fit isn't exact
- We can fix this with new, more general continuation equations
- Solve for least invasive control target, instead of noninvasive control
  - Solution will be noninvasive (same solution as for standard continuation equations) when discretisation is exact
  - Solution is still guaranteed to exist when discretisation is inexact
  - Solution is noise-robust

Minimization reformulation

- Let β be the discretisation
- Let $\mathrm{invasiveness}(\beta) = \int [\mathrm{signal}(t) - \mathrm{target}(\beta, t)]^2 \, \mathrm{d}t$
  - Valid for proportional control
  - Can be easily adapted for other control strategies
- Continuation equations:
  - $\frac{\partial \mathrm{invasiveness}}{\partial \beta_i} = 0$
  - predictor $\perp$ corrector
  - This can be solved using numerical integration and standard Newton iterations; no need for minimization: no experimental Hessians needed
  - Alternatively, solve using EGO minimizers; no experimental Jacobians needed

Next steps

- Write splines without SciPy
- Try minimizer approach; possibly slower; will guarantee finding an acceptable solution
- Try adaptive-knots BSplines
  - In general, optimization-based knot choice will minimize the discretisation error
  - Duffing is simple enough that adaptive knots shouldn't change the results much
- Talk to Krasi about approximation and existence of continuation solutions

Also, writing, annual review