Math 314 Group Project: Fun with polynomials

Due date: Wednesday, December 4

This project explores methods for constructing bases for the vector space $\mathcal{P}_n$ of polynomials of degree up to n, and for verifying that collections of polynomials are bases for $\mathcal{P}_n$. Depending upon the set of polynommials you will use different characterizations of bases that we have developed in class. These bases arise in different contexts that you have met and/or will meet in your mathematics and other courses.

We know that $\mathcal{P}_n$ has the "standard" basis consisting of the polynomials (with one term, usually called *monomials*)

$$\{1, x, x^2, \ldots, x^n\}$$

because they span $\mathcal{P}_n$ - the definition, esssentially, of polynomials as the functions $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ express polynomials as linear combinations of monomials. The monomials are also linearly independent: If $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, then plugging in $x = 0$ gives $p(0) = a_n = 0$, so $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x = x[a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}] = 0$, so $a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1} = 0$, as well. Repeating this argument (plugging in $x = 0$) will repeatedly show another coefficient is equal to zero, resulting in $a_n = a_{n-1} = \cdots = a_1 = a_0 = 0$, establishing linear independence.

But what you may not realize is that you have already encountered other bases for these vector spaces $\mathcal{P}_n$. In particular, in second-semester calculus, the (integration) technique we call *partial fractions* made systematic use of other, alternative, bases for the vector space $\mathcal{P}_n$, although of course we did not use this terminology for it at the time! In this project, we will explore the construction and verification of some of these bases.

A refresher on partial fraction decompositions can be found at

http://www.math.unl.edu/∼mbrittenham2/classwk/314f13/project/partial.pdf

(which is based on the review notes I provided to one of my Calculus II classes). The basic point is that starting from a completely factored polynomial

$$q(x) = (x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_k)^{m_k}(x^2 + \alpha_1 x + \beta_1)^{n_1} \cdots (x^2 + \alpha_\ell x + \beta_\ell)^{n_\ell},$$

where the quadratic terms are irreducible, with degree$(q) = N = m_1 + \cdots m_k + 2n_1 + \cdots 2n_\ell$, partial fractions gives a prescription for writing any quotient $\dfrac{p(x)}{q(x)}$, with degree$(p) \le N - 1$ as a sum of the quotients

$$\frac{A}{(x - a_i)^m} \text{ for } 1 \le m \le m_i, \text{ and } \frac{Bx + C}{(x^2 + \alpha_j x + \beta_j \ell)^n} \text{ for } 1 \le n \le n_j.$$

What this really means, when we put things over a common denominator, is that $p(x)$ is a sum of constant multiples of the polynomials (the denominators divide the numerator)

$$\frac{q(x)}{(x - a_i)^m} \text{ for } 1 \le m \le m_i, \ \frac{q(x)}{(x^2 + \alpha_j x + \beta_j \ell)^n}, \text{ and } x\frac{q(x)}{(x^2 + \alpha_j x + \beta_j \ell)^n} \text{ for } 1 \le n \le n_j,$$

that is, these polynomials span $\mathcal{P}_{N-1}$. Noting that there are precisely

$$m_1 + \cdots m_k + 2n_1 + \cdots 2n_\ell = N = \dim(\mathcal{P}_{N-1})$$

polynomials in this list, this implies that these polynomials are a basis for $\mathcal{P}_{N-1}$. So partial fractions is really all about constructing a particular basis for $\mathcal{P}_{N-1}$ from a single degree N polynomial $q(x)$.

In the following exercises you will verify that for some choices of $q(x)$ these polynomials do indeed form a basis for $\mathcal{P}_{N-1}$, using the techniques that were developed for solving partial fractions problems, together with our knowledge that establishing linear independence can be as useful to verifying bases as showing that they span can.

We start with the technique usually called *plugging in*: Choosing specific values to evaluate an expression at tells us the value of one or more coefficients.

Problem 1. For the polynomial $q(x) = (x-1)(x-2)(x-3)(x-4)$, with degree 4, describe the basis for $\mathcal{P}_3$ that partial fractions asserts we have, and demonstrate that your collection of polynomials are *linearly independent*; that is, that

$a_1p_1(x) + a_2p_2(x) + a_3p_3(x) + a_4p_4(x) = 0$ implies that $a_1 = a_2 = a_3 = a_4 = 0$.

Explain why this implies that your collection of polynomials is therefore a basis for $\mathcal{P}_3$.

Problem 2. Explain how the same technique can be applied to the general situation where $q(x) = (x-a_1)(x-a_2)\cdots(x-a_k)$ with $a_1 < a_2 < \cdots < a_k$.

A second technique might be called *moving the origin*: by writing $x = (x-a)+a$, we can write powers of x as linear combinations of powers of $(x-a)$; e,g, by multiplying out $x^6 = [(x+1)-1]^6$.

Problem 3. For the polynomial $q(x) = (x-1)^4$, with degree 4, describe the basis for $\mathcal{P}_3$ that partial fractions asserts we have, and demonstrate that your collection of polynomials spans $\mathcal{P}_3$, by showing that elements of the 'standard' basis for $\mathcal{P}_3$ can be written as linear combinations of your partial fractions basis.

Problem 4. Explain how the same technique can be applied to the general situation where $q(x) = (x-a)^n$, for $a \in \mathbb{R}$ a constant and $n \geq 1$ an integer. Can you come up with an alternate approach which directly establishes that your polynomials are linearly independent (i.e., which does not first establish that they span)?

Next, we look at what happens when we have a combination of these two situations.

Problem 5. For the polynomial $q(x) = (x-1)^2(x-2)^2$, with degree 4, describe the basis for $\mathcal{P}_3$ that partial fractions asserts we have, and demonstrate that your collection of polynomials forms a basis for $\mathcal{P}_3$. You might try various combinations of the approaches explored in the previous problems (in some order), and/or other ideas, to see what works (best)!

Problem 6. Describe how your approach to problem #5 could be adapted to some more general situation. Try to describe the most general situation that you can manage to solve in a reasonably small amount of space (and time).

When irreducible quadratics appear in our factorization of $q(x)$, things get more 'interesting'. We no longer have a clear idea of what values to plug in to reveal information about coefficients in a linear combination, and there is no clear value to 'shift the origin' by to express our standard basis as linear combinations. But we can always apply the 'brute force' method, which is really just solving a system of linear equations! For example, by writing $a(x-1)+b(x-2)=0$ as $(a+b)x+(-a-2b)=0$, solving $a+b=0$ and $-a-2b=0$ for a and b establishes that $x-1$ and $x-2$ are linearly independent.

Problem 7. For the polynomial $q(x) = (x^2+2x+2)(x^2-x+2)$ [note that both factors are irreducible quadratics], with degree 4, describe the basis for $\mathcal{P}_3$ that partial fractions asserts we have, and show that your collection of polynomials forms a basis for $\mathcal{P}_3$. You can show either that they are linearly independent, or that they span (by showing that you can express another basis as linear combinations of them).

In practice, establishing that partial fractions 'works', even in the presence of irreducible quadratices, uses a standard trick of thinking of the polynomials as having *complex* coefficients, and then further factoring them into linear polynomials. Facts about complex numbers then are used to make certain that our solutions still have *real* coefficients (the phrase 'complex conjugate pairs' might mean something to you?). But exploring this further would take us too far afield, and so we will close our explorations here.

Guidelines for writing up your project.

The project is the solution to an open-ended multistep problem, formally presented. It will probably require several meetings for your group to find a solution to some of the problems and to present your solutions clearly and understandably. Everyone in the group should contribute to the project.

Your group should write up a short paper explaining the problem and the mathematics you used to solve it, and then discussing the significance of your solution. Your paper should be a grammatically correct, organized discussion of the problem, with an introduction and a conclusion. While you should answer the specific questions asked in the project, your report should not be a disconnected set of answers but a connected narrative with transitions. It should conform to proper English usage (yes, spelling counts!). You should show enough relevant calculations to justify your answers but not so much as to obscure the calculations' purpose. If you type your report (this is preferred but not required) it is fine to leave blank spaces and write the equations in. [There are certain advantages to typing: making (small or large) changes does not require the rewriting of the entire document!] Explain your results and conclusions, pointing out both strengths and weaknesses of your analysis. Assume that your reader is someone who took a linear algebra course a while ago and does not remember all of the details. Be sure to avoid plagiarism.

Preparing formal reports is an important job skill for mathematicians, scientists, and engineers. For example, the Columbia Investigation Board, in its report on the causes of the Columbia space shuttle accident, wrote:

"During its investigation, the board was surprised to receive [PowerPoint] slides from NASA officials in place of technical reports. The board views the endemic use of PowerPoint briefing slides instead of technical papers as an illustration of the problematic methods of technical communication at NASA."