Danny Calegari, Circular groups and planar groups.

Conjecture: Let G be a countable group of homeos of $\mathbb{R}^2$.
$G < \mathrm{Homeo}^+(\mathbb{R}^2)$.
Then G is isomorphic to a subgroup of $\mathrm{Homeo}^+(S^1)$.

Possibly add some analytic conditions on planar action (but not OK to do so for action on S^1).

Why think so? Lots of natural action on the plane behave this way.

E.g. finite group: look at orbit of a pt. $\Rightarrow$ group acts on its convex cone $\Rightarrow$ on its bdry circle. Kernel of induced action on the ~~plane~~ circle is trivial (kernel acts on disk, permuting interior pts $\rightsquigarrow$ braid group action? $\rightsquigarrow$ torsion free!)

Defn G is left-orderable (LO) if $\exists$ total order on group elements $<$ so that
$$a<b \iff ga<gb \quad \forall g \in G.$$

G is circularly orderable (CO) if $\forall g \in G$ $\exists$ ordering $<_g$ on $G \setminus \{g\}$ so that
$$a <_g b \text{ and } b <_h a \iff a <_g h <_g b \text{ and }$$
$$b <_h g <_h a$$

a kind of cocycle condition..

For G countable, G is LO $\iff$ G ~~ac~~ $< \mathrm{Homeo}^+(\mathbb{R})$

G is CO $\iff$ $G < \mathrm{Homeo}^+(S^1)$

Idea: enumerate G, place them into $\mathbb{R}$ (or S^1) compatibly with the ordering, then show that action of G on itself (on left) extends to $\mathbb{R}$ (S^1).

Lack of LO or CO can be detected by looking at finite sets of group elements.

Fact: G is LO $\iff$ $\forall$ fin gen'd $H < G$ $\exists$ surjective homom $\phi_H : H \to G_H$

nontriv. LO group.

e.g. if G is locally indicable, then G is LO.
(every fin gen'd sg. $\twoheadrightarrow \mathbb{Z}$)

If $0 \to K \to G \to H \to 0$ and

K, H LO $\Rightarrow$ G LO.

K LO, H CO $\Rightarrow$ G is CO.

(algebraic analogue of "Denjoy'ing" a leaf of a codim-1 folia)

G LO $\Rightarrow$ $\exists$ central extension

$0 \to \mathbb{Z} \to \hat{G} \to G \to 0$ s.t. $\hat{G}$ is LO.

- extension is defined by the action on S^1.

('lift' action to univ cover of S^1)

Abstractly:

$$H^*(\mathrm{Homeo}^+(\mathbb{R}^2);\mathbb{Z}) \cong H^*(\mathrm{Homeo}^+(S^1);\mathbb{Z}) \cong H^*(\mathbb{CP}^\infty;\mathbb{Z}).$$

Instructive example: $x \in \mathbb{R}^2$, foliate $\mathbb{R}^2 \setminus x$ by circles

Group leaving foliation invariant G has = "bullseye group"

$$0 \to CO \to G \to LO \to 0$$

~~But this allows you to conclude~~ nothing about G..

Thm: (local circularity) G countable group of C^1 diffeos of $\mathbb{R}^2$ fixing a pt p. Then G is CO.

Pf $K \to G \to LG$ — linear part of G at p (diff'l map)

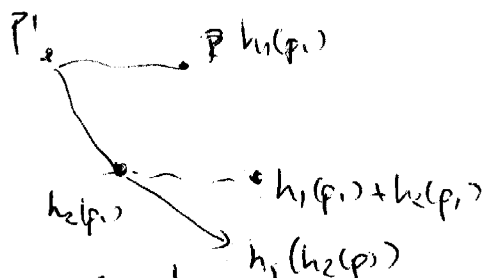
projectivize

$$\underset{LO}{D} \to LG \to \underset{CO}{P^+LG} \implies LG \text{ is CO.}$$

$\implies$ enough to show K is LO.

Pick $H_{f.g.} < K$. and $p_1, \dots, p_n \in \mathbb{R}^2$

$h_1, h_2 \in H$

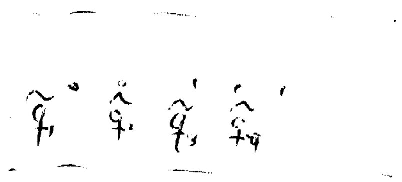


Rescale picture so that in limit these are equal $\implies$ H acts by translations on $\mathbb{R}^2$

$\Longrightarrow$ H is locally indicable $\Longrightarrow$ LO.

Lemma: If G fixes p & q (C^1 action on $\mathbb{R}^2$) then G is LO.

$\widetilde{\mathbb{R}^2\setminus p}$ =



G lifts to action of $\widetilde{\mathbb{R}^2\setminus p}$ which acts by translation on lifts of q.

Thm: If G acts on $\mathbb{R}^2$ by C^1 diffeos, and $\exists$ bounded invariant set X for G, then G is CO.

Pf: Take closure of X and fill in bounded complementary regions (still an invariant set)

$\longrightarrow$ WMA $\mathbb{R}^2\setminus X \cong \mathbb{R}^2\setminus K$, K closed totally disconnected

Get a combinatorial rep^n $\quad G \to MCG(\mathbb{R}^2\setminus K)$

$$= \widehat{MCG}(S^2\setminus K)$$

$$0 \longrightarrow \pi_1(S^2\setminus K) \longrightarrow \widehat{MCG}(S^2\setminus K) \longrightarrow MCG(S^2\setminus K) \longrightarrow 0$$

$=$ homeos of $\widetilde{S^2\setminus K}$ (univ cover) which are invariant under the deck group action.

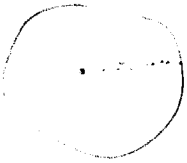
$\vdots$

Thm: K closed, ~~do~~ totally disconnected $\subset \mathring{\mathbb{D}}^2$ st.

$\bar{K}$ in D^2, does not contain $\partial \mathbb{D}^2$, then

$MCG(\mathbb{D}^2 \setminus K)$ is CO.

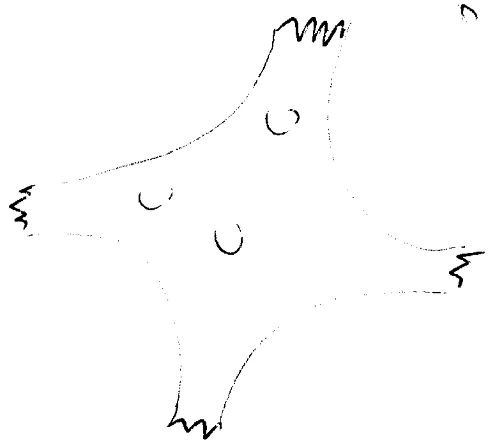
(generalizes Dehornoy: B_n is LO)

previously Fabel. $K =$  limiting on one pt in ∂D^2.

Natural partitions: G acting on $\mathbb{R}^2$, preserving some equiv reln $\sim$. Refine $\sim$ by $\sim'$ - equiv classes = path-components of equiv classes of $\sim$. G preserves $\sim'$.

Build $\mathcal{E} = \bigcup_{[p]} E_{[p]}$ = union of set of proper ends of $[p]$

this set is CO.

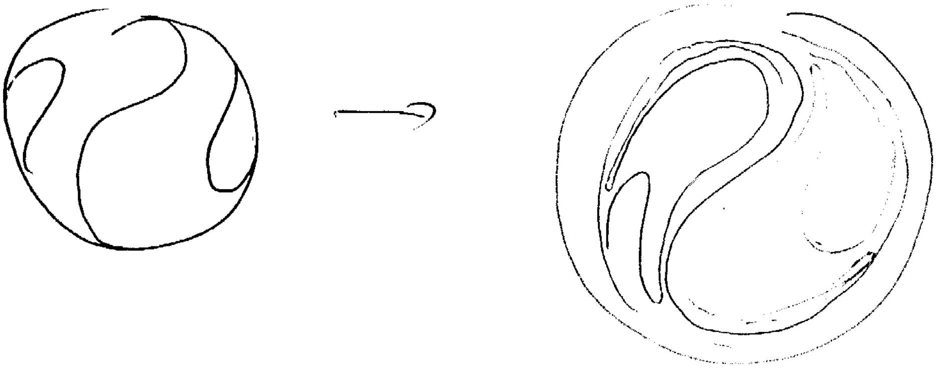


4 proper ends.

technical: if $[p]$ has no proper ends but $\mathbb{R}^2 \setminus [p]$ does, add "leftmost" complementary proper ends.

proper rays in $\mathbb{R}^2$, not in $[p]$...

Ex:



Has action by Klein 4-group (not C^0) which is proper on complementary components (but not on disk itself).

M^3, X = flow

X is product-covered if $(\widetilde{M}, \widetilde{X})$ is a product $(\mathbb{R}^3, \frac{\partial}{\partial z})$

Leaf space of $\widetilde{X} = \mathbb{R}^2$, $\pi_1(M)$ acts on $\mathbb{R}^2$.

(1) $l \sim m$ if l, m stay a bounded distance apart in the fibre.

(2) M hyperbolic, X quasigeodesic, $l \sim m$ if the endpts .....

(3) de Carvalho: zero-entropy equiv.

Thm (QG flow) M hyp 3-mfld, X quasi-geodesic flow, then $\pi_1(M)$ is CO.

Thm M hyperbolic, X product covered s.t. either

(1) X has a closed orbit

(2) X is decisive: (i.e. $\forall\ l, m$ or in forward time either $d(l,m) < C$ or $d(l,m) \to \infty$

then $\pi_1(M)$ is CO.

Thm (Calegari - Dunfield)

π_1(Weeks mfld) is not CO.

Cor: it has no such flow.