Math 417 Problem Set 1 Solutions

(*) 1. We have seen that rotation $R(\theta)$ by angle θ and reflection $S(\theta)$ in the line making angle θ are given in matrix terms as multiplication by

$$A(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \text{ and } B(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Show that $S(\theta)\circ S(\psi)$ is a rotation, and $S(\theta)\circ R(\psi)$ and $R(\psi)\circ S(\theta)$ are both reflections, and determine which angle they rotate or reflect by.

We can recognize each of these compositions as a rotation or reflection by multiplying the two matrices together and using angle sum/difference formulas for $\sin x$ and $\cos x$. In particular,

$$B(\theta)B(\psi) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}\begin{pmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta)\cos(2\psi) + \sin(2\theta)\sin(2\psi) & \cos(2\theta)\sin(2\psi) - \sin(2\theta)\cos(2\psi) \\ \sin(2\theta)\cos(2\psi) - \cos(2\theta)\sin(2\psi) & \sin(2\theta)\sin(2\psi) + \cos(2\theta)\cos(2\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta - 2\psi) & \sin(2\psi - 2\theta) \\ \sin(2\theta - 2\psi) & \cos(2\theta - 2\psi) \end{pmatrix} = \begin{pmatrix} \cos(2\theta - 2\psi) & -\sin(2\theta - 2\psi) \\ \sin(2\theta - 2\psi) & \cos(2\theta - 2\psi) \end{pmatrix}$$

$= A(2\theta - 2\psi)$, so $S(\theta)\circ S(\psi)$ is a rotation by angle $2\theta - 2\psi$.

Similarly,

$$B(\theta)A(\psi) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}\begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta)\cos(\psi) + \sin(2\theta)\sin(\psi) & -\cos(2\theta)\sin(\psi) + \sin(2\theta)\cos(\psi) \\ \sin(2\theta)\cos(\psi) - \cos(2\theta)\sin(\psi) & -\sin(2\theta)\sin(\psi) - \cos(2\theta)\cos(\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta - \psi) & \sin(2\theta - \psi) \\ \sin(2\theta - \psi) & -\cos(2\theta - \psi) \end{pmatrix}$$

$= B(\theta - \psi/2)$, so $S(\theta)\circ R(\psi)$ is a reflection in the line with angle $\theta - \frac{1}{2}\psi$, and

$$A(\psi)B(\theta) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix}\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi)\cos(2\theta) - \sin(\psi)\sin(2\theta) & \cos(\psi)\sin(2\theta) + \sin(\psi)\cos(2\theta) \\ \sin(\psi)\cos(2\theta) + \cos(\psi)\sin(2\theta) & \sin(\psi)\sin(2\theta) - \cos(\psi)\cos(2\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi + 2\theta) & \sin(\psi + 2\theta) \\ \sin(\psi + 2\theta) & -\cos(\psi + 2\theta) \end{pmatrix}$$

$= B(\theta + \psi/2)$, so $R(\psi)\circ S(\theta)$ is a reflection in the line with angle $\theta + \frac{1}{2}\psi$.

(*) 4. Show that the set $G = \{1, 5, 9, 13\}$ forms a group, with group multiplication being multiplication modulo 16. (One approach: build the 'Cayley' table! This helps to see why some needed properties hold.)

We need to show that G has an identity, inverses, and its multiplication is associative. However, multiplication modulo 16 is associative!, so we know that multiplication <u>of these</u> <u>numbers</u> modulo 16 is associative. If we build the Calyey table, by multiplying all pairs of numbers together and reducing modulo 16, we get:

$*$	1	5	9	13
1	1	5	9	13
5	5	25	45	65
9	9	45	81	117
13	13	65	117	169

i.e.,

$*$	1	5	9	13
1	1	5	9	13
5	5	9	13	1
9	9	13	1	5
13	13	1	5	9

since $117 = 5+112 = 5+16\cdot 7$. From this table we see that 1 is an identity element for the group, and since a 1 appears in every row and column, every element has a right and a left inverse; in particular, $1^{-1} = 1$, $5^{-1} = 13$, $9^{-1} = 9$, and $13^{-1} = 5$. So G has an associative multiplication, an identity, and inverse, so G is a group. The important point, really, is that products of things in G are <u>in</u> G! (The set G is "closed" under multiplication.)

(*) 5. (Gallian, p.55, #18) Which elements $x \in D_4 =$ the group of symmetries of a regular 4-gon (i.e., square) satisfy $x^2 = e$? Which satisfy $x = y^2$ for some $y \in D_4$?

[Problem #1 can help you decide what an element y^2 can look like...]

Problem #1 tells us that for a reflection $S(\theta)$, $S(\theta)\circ S(\theta)$ is a rotation by angle $2\theta - 2\theta = 0$, i.e., is the identity. On the other hand, for a rotation $R(\theta)$, $R(\theta) \circ R(\theta) = R(2\theta)$ is a rotation by angle 2θ (this is what our understanding of rotations tells us; it can also be verified computationally like in Problem #1). So every reflection x in D_4 satisfies $x^2 = e$, and the rotations in D_4 that satisfy $x^2 = e$ must be by an angle θ so that 2θ is a multiple of 2π. So the four reflections $S(0)$, $S(\pi/4)$, $S(\pi/2)$, and $S(3\pi/4)$, and the rotations $R(0) = e$ and $R(\pi)$, are the elements x with $x^2 = e$.

When looking for squares (i.e., $y \in D_4$ with $y = x^2$ for some $x \in D_4$), we now know that all elements x^2 are the identity if x is a reflection, or a rotation, by an angle twice the angle of rotation of x, if x is a rotation. So e is a square, and any rotation having the rotation by half its angle also in D_4, is a square. The only angle which works (other than 0) is π, so e and $R(\pi)$ are the perfect squares in D_4.

A completely different, but perfectly valid, approach to this problem is to list all eight elements e, $R(\pi/2)$, $R(\pi)$, $R(3\pi/2)$, $S(0)$, $S(\pi/4)$, $S(\pi/2)$, and $S(3\pi/4)$, and square them! The ones that square to e answer the first part, and the elements you <u>get</u> as squares answers the second part!

A selection of further solutions

2. (Gallian, p.38, #14) If we build a rhombus R (a quadrilateral with all four sides having equal length) by gluing two equilateral triangles together along a pair of sides, describe the symmetries of R in terms of rotations and reflections.

Call the vertices, in order, A, B, C, D. The centroid of R is where the two diagonals AC and BD meet, and two (opposite) vertices A, C will be farther from the centroid than the other two. So any symmetry must either fix both A and C, or swap them. Since it must permute the vertices, it must <u>also</u> either fix both B and D or swap them. Every possible combination of these options is possible; the identity fixes both pairs, rotation by π swaps both pairs, reflection in AC swaps B and D but fixes the other two, and reflection in BD swaps A and C but fixes the other two. Since the symmetry is determined by where it sends the vertices (linearity tells us where every other point goes), there are precisely four symmetries.

3. Describe the symmetries of a cylinder (of height h).

   If we assume we are talking about a right circular cylinder, then the centroid is the center of the circle halfway up the cylinder. A symmetry of the cylinder will fix that point, and so it will either preserve the top and bottom of the cylinder, or swap them. If it preserves the top and bottom then it sends the top circle to itself, and so fixes the center of that circle, too. So it will fix the line running through the center of the cylinder, and be either a rotation or reflection on the top of the cylinder. This gives either a rotation (by any angle) around the line passing through the centers of the top and bottom of the cylinder, or a reflection in a (vertical) plane containing that line.

   If, on the other hand, the symmetry S swaps the top and bottom, then composing with the reflection R in the horizontal plane that contains the centroid of the cylinder (which also swaps ends) will build a symmetry RS that preserves the ends (and so is a rotation or reflection as above). But since $RR = I$ (repeating the reflection takes us back to the identity, then $S = (RR)S = R(RS)$, so S is the composition with a rotation/reflection preservng ends and R. It takes some effort, but the composition of a reflection preserving the ends with R can be seen to be be a rotation around a horizontal line through the centroid; looking at the horizontal circle with center equal to the centroid, you can convince yourself that two antipodal points are fixed by the composition, and they determine the line. On the other (third?) hand, the composition of a rotation around the vertical line with R will be, well, that composition! This might be best described by matrices, by putting the centroid of the cylinder at the origin. Then the rotation will rotate the first two coordinates, and fix the third, after which R will fix the first two coordinates and send z to $-z$. (Note: these two operations commute!) A reflection followed by R can be handled the same way....

7. (Gallian, p.57, #48) Show that the collection of all 3×3 matrices

$$H = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

   with $a, b, c \in \mathbb{R}$ forms a group under matrix multiplication.

   [This group is known as the *Heisenberg group*, and arises in the study of one-dimensional quantum systems. You may find your row reduction prowess useful in finding inverses!]

   On some level, this is just a lesson in matrix multiplication and inversion? Matrix multiplication is associative, and so what we really need to check is that the product of two matrices in H is again in H. But

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r+a & s+at+b \\ 0 & 1 & t+c \\ 0 & 0 & 1 \end{pmatrix},$$

   so matrix multiplication takes matrices in H to another matrix in H. Our friend the identity matrix is in H ($a = b = c = 0$), and gives us an identity element in H. The last thing to verify is the elements of H have inverses, and for this we can memorize a formula for the inverse of a 3×3 matrix, or directly compute by row reduction, or rather, back-solving (it is already in row echelon form), like so:

$$\begin{pmatrix} 1 & a & b & | & 1 & 0 & 0 \\ 0 & 1 & c & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & a & b & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & a & 0 & | & 1 & 0 & -b \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & | & 1 & -a & ac-b \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$

So $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$, which is again in H. So elements of H have inverses in H.

Having an identity, inverses and associativity implies that H is a group.