Math 314 Matrix Theory

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In *reduced row echelon form (RREF)*, each of the resulting equations can be interpreted as reading either

(1) (bound variable) = (equation involving free variables)

(2) $0 = 0$ [which gives no new information]

(3) $0 = 1$ [so the system is inconsistent]

If the system is consistent, then *any* assignment of values to the free variables, leads, via the equations above, to a specific value of each of the bound variables, which gives a solution to the original system of equations.

Several things to note:

It is a fact, which we will establish later, that each augmented matrix has exactly one RREF; it doesn't matter what order we do our row reductions in, they will lead to the exact same place.

Our analysis above already allows us to understand solutions to systems in a general way; an inconsistent system has no solutions, a consistent system with no free variables has exactly one solution (the equations then read (bound variable) = (constant)), and a consistent system with one or more free variables has infinitely many distinct solutions (corresponding to different values of the free variables).

Linear systems as vector equations

A *vector* $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbf{R}^2$ is an arrow with tail at (x_0, y_0) and head at $(x_0 + a, y_0 + b)$. We generally think of vectors with the same description a, b , as the same, even if their tails are in different places. When we want uniformity, we put the tail at $(0, 0,)$, so the vector corresponds to the point (a, b) . More generally, we can think of vectors in $\mathbf{R}^n$ as having tails at $(0, \ldots, 0)$ and heads at $(a_1, \ldots a_n)$. *Vector addition* and *scalar multiplication* can be defined as they are in $\mathbf{R}^2$, working component by component:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \quad \text{and} \quad c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}.$$

A system of linear equations $\begin{matrix} a_{11}x_1 + \cdots + a1nx_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + amnx_n = b_m \end{matrix}$ can be re-written as $x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

The left side is a **linear combination** of vectors in $\mathbf{R}^m$. Essentially, a solving a system of equations amounts to finding the appropriate linear combination of *column vectors* from the coefficient latrix of the system, which equals the *target vector*. So a system of linear equations can be interpreted as a *single* vector equation in $\mathbf{R}^m$. This will be an important interpretation for us as we move forward!