# Math 325 Problem Set 2 Solutions

Starred (*) problems were due Friday, September 7.

(*) 5. For the statements (a)-(c) below, state both the contrapositive and the 'proof by contradiction' form "it is not possible to have both..." versions of the given statement, and indicate (no explanation needed) which of the resulting collections of statements are <u>true</u>.

(a) If $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$ .

Contrapositive: If $a + b \notin \mathbb{Q}$, then it is not the case that $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$.

Equivalent: If $a + b \notin \mathbb{Q}$, then it <u>either</u> $a \notin \mathbb{Q}$ <u>or</u> $b \notin \mathbb{Q}$.

Contradiction: It is not possible to have $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ and <u>also</u> have $a + b \notin \mathbb{Q}$.

These statements are all true.

(b) If $a \notin \mathbb{Q}$, then $\dfrac{1}{a} \notin \mathbb{Q}$ .

Contrapositive: If $\dfrac{1}{a} \in \mathbb{Q}$, then $a \in \mathbb{Q}$.

Contradiction: It is not possible to have $\dfrac{1}{a} \in \mathbb{Q}$ and <u>also</u> have $a \notin \mathbb{Q}$.

All of these statements are true!

(c) If $a \notin \mathbb{Q}$ and $b \notin \mathbb{Q}$, then $ab \notin \mathbb{Q}$ .

Contrapositive: If $ab \in \mathbb{Q}$, then it is not the case that $a \notin \mathbb{Q}$ and $b \notin \mathbb{Q}$.

Contradiction: It is not possible to have $a \notin \mathbb{Q}$ and $b \notin \mathbb{Q}$ and <u>also</u> have $ab \notin \mathbb{Q}$ .

All of these statement are false!

(*) 6. Show, using the Rational Roots Theorem, that $\alpha = \sqrt{2 + \sqrt{3}}$ is not rational.

[Find a polynomial with integer coefficients that has $\alpha$ as a root!]

Setting $\alpha = \sqrt{2 + \sqrt{3}}$, then $\alpha^2 = 2 + \sqrt{3}$, so $\alpha^2 - 2 = \sqrt{3}$, so $(\alpha^2 - 2)^2 = 3$.

So $0 = (\alpha^2 - 2)^2 - 3 = (\alpha^2)^2 - 2(\alpha^2)(2) + 2^2 - 3 = \alpha^4 - 4\alpha^2 + 1$.

So $\alpha$ is a root of the polynomial $f(x) = x^4 - 4x^2 + 1$. Since $f$ has integer coefficients, the rational roots theorem tells us that if $r = p/q$ is a rational root for $f$, then $p$ divides the constant coeffficient, 1, and $q$ divides the leading coefficient, 1. So $p$ is $-1$ or 1, and so is $q$, so $r = p/q$ is $-1$ or 1. So if $\alpha$ is rational, either $\alpha = -1$ or $\alpha = 1$.

But! $f(-1) = (-1)^4 - 4(-1)^2 + 1 = 1 - 4 + 1 = -2 \neq 0$, and $f(a) = 1^4 - 4(1)^2 + 1 = 1 - 4 + 1 = -2 \neq 0$. So none of the roots of $f$ are rational (since the only possible values of a rational root are not roots!). So $\alpha$, which <u>is</u> a root of $f$, cannot be rational. So $\alpha$ is not rational.

(*) 8. By looking at the first few cases, find a (short) formula for the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\sum_{k=1}^{n}\frac{1}{k(k+1)}\ ;$$

then prove, using induction, that your formula is correct.

$\frac{1}{1\cdot 2}=\frac{1}{2}$, $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$, $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{2}{3}+\frac{1}{12}=\frac{8}{12}+\frac{1}{12}=\frac{3}{4}$,

and $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{3}{4}+\frac{1}{20}=\frac{15}{20}+\frac{1}{20}=\frac{4}{5}$.

This leads us to suspect that $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$ .

And we can prove this, by induction! If $n=1$, then $\displaystyle\sum_{k=1}^{1}\frac{1}{k(k+1)}=\frac{1}{1\cdot 2}=\frac{1}{2}=\frac{1}{1+1}$ which establishes the base case.

Then if we suppose that $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$, then

$$\sum_{k=1}^{n+1}\frac{1}{k(k+1)}=(\sum_{k=1}^{n}\frac{1}{k(k+1)})+\frac{1}{(n+1)(n+2)}=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$

$$=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2},$$

so $\displaystyle\sum_{k=1}^{n+1}\frac{1}{k(k+1)}=\frac{n+1}{n+2}=\frac{n+1}{(n+1)+1}$, establishing the inductive step.

So, by induction, for every $n\in\mathbb{N}$ we have $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$ .

**A selection of further solutions:**

9. Show, using induction, that for every $n\in\mathbb{N}$ we have

$$\sum_{k=1}^{n}k(k+1)(k+2)=\frac{1}{4}n^4+\frac{3}{2}n^3+\frac{11}{4}n^2+\frac{3}{2}n\ .$$

[Hint: going straight at it is fine; a ‘slicker’ way is to note that the quartic <u>is</u>

$\frac{1}{4}n(n+1)(n+2)(n+3)$ ...]

Checking $n=1$, we have $\displaystyle\sum_{k=1}^{1}k(k+1)(k+2)=1\cdot 2\cdot 3=6=\frac{24}{4}=\frac{1}{4}+\frac{6}{4}+\frac{11}{4}+\frac{6}{4}=$
$\frac{1}{4}1^4+\frac{3}{2}1^3+\frac{11}{4}1^2+\frac{3}{2}1$,
so the base case is true.

Then if we suppose that $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$, then

$$\sum_{k=1}^{n+1} k(k+1)(k+2) = (n+1)(n+2)(n+3) + \sum_{k=1}^{n} k(k+1)(k+2)$$
$$= (n^2+3n+2)(n+3) + \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$$
$$= n^3 + 3n^2 + 3n^2 + 9n + 2n + 6 + \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$$
$$= n^3 + 6n^2 + 11n + 6 + \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$$
$$= \frac{1}{4}n^4 + \frac{5}{2}n^3 + \frac{35}{4}n^2 + \frac{25}{2}n + 6 \ .$$

On the other hand,

$$\frac{1}{4}(n+1)^4 + \frac{3}{2}(n+1)^3 + \frac{11}{4}(n+1)^2 + \frac{3}{2}(n+1)$$
$$= \frac{1}{4}(n^4+4n^3+6n^2+4n+1) + \frac{3}{2}(n^3+3n^2+3n+1) + \frac{11}{4}(n^2+2n+1) + \frac{3}{2}(n+1)$$
$$= \frac{1}{4}n^4 + \frac{4}{4}n^3 + \frac{6}{4}n^2 + \frac{4}{4}n + \frac{1}{4} + \frac{3}{2}n^3 + \frac{9}{2}n^2 + \frac{9}{2}n + \frac{3}{2} + \frac{11}{4}n^2 + \frac{22}{4}n + \frac{11}{4} + \frac{3}{2}n + \frac{3}{2}$$
$$= \frac{1}{4}n^4 + \frac{4+6}{4}n^3 + \frac{6+18+11}{4}n^2 + \frac{4+18+22+6}{4}n + \frac{1+6+11+6}{4}$$
$$= \frac{1}{4}n^4 + \frac{10}{4}n^3 + \frac{35}{4}n^2 + \frac{50}{4}n + \frac{24}{4}$$
$$= \frac{1}{4}n^4 + \frac{5}{2}n^3 + \frac{35}{4}n^2 + \frac{25}{2}n + 6 \ .$$

So $\sum_{k=1}^{n+1} k(k+1)(k+2) = \frac{1}{4}(n+1)^4 + \frac{3}{2}(n+1)^3 + \frac{11}{4}(n+1)^2 + \frac{3}{2}(n+1)$, establishing the inductive step. So, by induction, we have $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$ for all $n \in \mathbb{N}$ .

An alternative approach establishes the inductive step by noting that

$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$ implies that

$$\sum_{k=1}^{n+1} k(k+1)(k+2) = \frac{1}{4}n(n+1)(n+2)(n+3) + (n+1)(n+2)(n+3)$$
$$= (1+\frac{1}{4}n)(n+1)(n+2)(n+3) = \frac{n+4}{4}(n+1)(n+2)(n+3) = \frac{1}{4}(n+1)(n+2)(n+3)(n+4)$$
$$= \frac{1}{4}(n+1)((n+1)+1)((n+1)+2)((n+1)+3) \ ,$$

which establsihes the inductive step.