Math 971 Algebraic Topology

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The main tool we will use turns a family of short exact sequences of chain maps between three chain complexes into a single *long exact homology sequence.* Given chain complexes $\mathcal{A} = (A_n, \partial)$, $\mathcal{B} = (B_n, \partial')$, and $\mathcal{C} = (C_n, \partial'')$ and short exact sequences of chain maps (i.e., $\partial' i_n = i_n \partial$, $\partial'' j_n = j_n \partial'$) $0 \to A_n \overset{i_n}{\to} B_n \overset{j_n}{\to} C_n \to 0$ there is a general result which provides us with a long exact sequence

$$\cdots \overset{\partial}{\to} H_n(\mathcal{A}) \overset{i_*}{\to} H_n(\mathcal{B}) \overset{j_*}{\to} H_n(\mathcal{C}) \overset{\partial}{\to} H_{n-1}(\mathcal{A}) \overset{i_*}{\to} \cdots$$

Most of the work is in defining the "boundary" map ∂. Given an element $[z] \in H_n(\mathcal{C})$, a representative $z \in C_n$ satisfies $\partial''(z) = 0$. But j_n is onto, so there is a $b \in B_n$ with $j_n(b) = z$, Then $i_{n-1}\partial'(b) = \partial'' j_n(b) = 0$, so $\partial'(b) \in \ker(j_{n-1} = \mathrm{im}(a_{n-1})$. So there is an $a \in A_{n-1}$ with $i_{n-1}(a) = \partial'(b)$. But then $i_{n-2}\partial(a) = \partial' i_{n-1}(a) = \partial'\partial'(b) = 0$, so, since i_{n-2} is injective, $\partial a = 0$, so $a \in Z_{n-1}(\mathcal{A})$, and so represents a homology class $[a] \in H_n(\mathcal{A})$. We define $\partial([z]) = [a]$.

To show that this is well-defined, we need to show that the class $[a]$ we end up with is independent of the choices made along the way. The choice of a was not really a choice; i_{n-1} is, by assumption, injective. For b, if $j_n(b) = z = j_n(b')$, then $j_n(b - b') = 0$, so $b - b' = i_n(w)$ for some $w \in A_n$. Then $\partial' b' = \partial' b - \partial' i_n(w) = \partial' b - i_{n-1}\partial(w)$, so choosing $a' = a - \partial(w)$ we have $i_{n-1}(a') = \partial'(b')$. But then $[a'] = [a - \partial w] = [a] - [delw] = [a]$. Finally, there is actually a choice of z ; if $[z] = [z']$, then $z' = z + \partial'' w$ for some $w \in C_{n+1}$; but then choosing b', w' with $j_n(b') = z'$, $j_{n+1}(w') = w$, we have $\partial'' w = \partial'' j_{n+1}(w') = j_n \partial'(w')$, so $z' = z + \partial'' w = j_n(b + \partial' w')$, so we may choose $b' = b + \partial' w'$ (since the result is independent of this choice!), then since $\partial' b' = \partial' b$ everything continues the same.

Now to exactness! We need to show three (types of) equalities, which means six containments. Three (image contained in kernel) are shown basically by showing that compositions of two consecutive homomorphisms are trivial. $j_n i_n = 0$ immediately implies $j_* i_* = 0$. From the definition of ∂, $i_*\partial[z] = [i_n(a)] = [\partial'(b)] = 0$, and $\partial j_*[z] = \partial[j_n(z)] = [a]$, where $i_{n-1}(a) = \partial'(z) = 0$, so $a = 0$ (since i_{n-1} is injective), so $[a] = 0$.

For the opposite containments, if $j_*[z] = [j_n(z)] = 0$, then $j_n(z) = \partial'' w$ for some w. Since j_{n+1} is onto, $w = j_{n+1}(b)$ for some b. Then $j_n(z - \partial' b) = \partial'' w - \partial'' j_{n+1} b = 0$, so $z = \partial' b = i_n(a)$ for some a, so $i_*[a] = [z - \partial' b] = [z]$. So $\ker j_* \subseteq \mathrm{im} i_*$. If $i_*[z] = 0$, then $i_n(z) = \partial' w$ for some $w \in B_{n+1}$. Setting $c = j_{n+1}(w)$, then $\partial'' c = j_n \partial' w - i_n i_n(Z) = 0$, so $[c] \in h_{n+1}(\mathcal{C})$, and computing $\partial[c]$ we find that we can choose w for the first step and z for the second step, so $\partial[c] = [z]$. So $\ker j_n \subseteq \mathrm{im}\partial$. Finally, if $\partial[z] = 0$, then $z = j_n(b)$ for some b, and $\partial' b = i_{n-1}(a)$ with $[a] = 0$, i.e., $a = \partial w$ for some w. So $\partial' b = i_{n-1}\partial w = \partial' i_n w$ But then $\partial'(b - i_n w) = 0$, and $j_n(b - i_n w) = z - 0 = z$, so $z \in \mathrm{im}(j_n)$, so $[z] \in \mathrm{im}(j_*)$. So $\ker\partial \subseteq \mathrm{im}(j_n)$. Which finishes the proof!

Now all we need are some new chain complexes. To start, we build the singular chain complex of a pair (X, A) , i.e., of a space X and a subspace $A \subseteq X$. Since as abelian groups we can think of $C_n(A)$ as a subgroup of $C_n(X)$ (under the injective homomorphism induced by the inclusion $i : A \to X$) we can set $C_n(X, A) = C_n(X)/C_n(A)$. Since the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ satisfies $\partial_n(C_n(A) \subseteq C_{n-1}(A)$ (the boundary of a map into A maps into A), we get an induced boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. These groups and maps $(C_n(X, A), \partial_n)$ form a chain complex, whose homology groups are the *singluar relative homology groups of the pair* (X, A) . To be a cycle in relative homology, you need to have a representative z with $\partial z \in C_{n-1}(A)$, i.e., you are a chain with boundary in A. To be a boundary, you need $z = \partial w + a$ for some $w \in C_{n+1}(X)$ and $a \in C_n(A)$, i.e., you *cobound* a chain in A ($\partial w = z - a$). Note that the relative homology of the pair $(X, \emptyset)$ is just the ordinary homology of X; we aren't modding out by anything.

The inclusion i_n and projection p_n maps give us short exact sequences $0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$, and since the boundary on chains in X restricts to the boundary on A and induces the boundary on (X, A), i_n and p_n are chain maps. So we get a long exact homology sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

There is also a long exact sequence of a triple (X, A, B), where by triple we mean $B \subseteq A \subseteq X$. From the short exact sequences $0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$ (i.e., $0 \to C_n(A)/C_n(B) \to C_n(X)/C_n(B) \to C_n(X)/C_n(A) \to 0$) we get the long exact sequence

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to H_{n-1}(X, B) \to \cdots$$

A map of pairs $f : (X, A) \to (Y, B)$ (meaning that $f(A) \subseteq B$) induces (by postcomposition) a map of relative homology $f_* : H_i(X, A) \to H_i(Y, B)$, just as with absolute homology. We also get a homotopy-invariance result: if $f, g : (X, A) \to (Y, B)$ are maps of pairs which are *homotopic as maps of pairs*, i.e., there is a map $(X \times I, A \times I) \to (Y, B)$ which is f on one end and g on the other, then $f_* = g_*$. The proof is identical to the one given before; the prism map P sends chains in A to chains in A, so induces a map $C_i(X \times I, A \times I) \to C_{i+1}(X, A)$ which does precisely what we want.

But the big result that allows us to get our homology machine really running is what is known as *excision.* To motivate it, let's try to imagine that we are trying to generalize Seifert - van Kampen. We start with $X = A \cup B$, and we want to try to express the homology of X in terms of that of A, B, and $A \cap B$. With our new-found tool of long exact homology sequences, we might try to first build a short exact sequence out of the chain complexes $C_*(A \cap B), C_*(A), C_*(B)$, and $C_*(X)$. If we take our cue from the proof of S-vK, we might think of chains in X as sums of chains in A and B, except that we mod out by chains in $A \cap B$. Putting this into action, we might try the sequence

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(X) \to 0$$

where $j_n : C_n(A) \oplus C_n(B) \to C_n(X)$ is defined as $j_n(a, b) = a + b$. In order to get exactness at the middle term (i.e., image = the kernel of this map, which is $\{(x, -x) : x \in C_n(A) \cap C_n(B)\}$), we set $i_n : C_n(A \cap B) \to C_n(A) \oplus C_n(B)$ to be $i_n(x) = (x, -x)$, since $C_n(A \cap B) = C_n(A) \cap C_n(B)$! i_n is then injective, and we certainly have that this sequence is exact at the middle term. But, in general, j_n is far from surjective! The image of j_n is the set of n-chains that can be expressed as sums of chains in A and B. Which of course not every chain in X can be; singular simplices in X need not map entirely into either A or B.

We can solve this by replacing $C_n(X)$ with the image of j_n, calling it, say, $C_n^{\{A,B\}}(X)$ ... [Note: these would form a chain complex.] Then we have a short exact sequence, and hence a long exact homology sequence. But it involves a "new" homology group $H_n^{A,B\}}(X)$. The point is that, like S-vK, under the right conditions, this new homology is the same as $H_n(X)$!

Starting from scratch, the idea is that, starting with an *open cover* $\{\mathcal{U}_\alpha\}$ of X, we build the *chain groups subordinate to the cover* $C_n^{CalU}(X) = \{\sum a_i \sigma_i^n : \sigma_i : n : \Delta^n \to X, \sigma_i^n(\Delta^n) \subseteq \mathcal{U}_\alpha \text{ for some } \alpha\} \subseteq C_n(X)$. Since the face of any simplex mapping into $\mathcal{U}_\alpha$ also maps into $\mathcal{U}_\alpha$, our ordinary boundary maps induce boundary maps on these groups, turning $(C_n^{\mathcal{U}}(X), \partial_n)$ into a chain complex. Our main result is that the inclusion i of these groups into $C_n(X)$ induces an isomophism on homology. And to show this, we once again use the notion of a chain homotopy.

Theorem: There is a chain map $b : C_n(X) \to C_n^{\mathcal{U}}(X)$ so that $i \circ b$ and $b \circ i$ are both chain homotopic to the identity. i consequently induces isomorphisms on homology.

And the key to building b (and the chain homotopies) is what is know as the *barycentric subdivision map.* The idea is really the same as for S-vK; we cut our singular simplices up into tiny enough pieces so that (via the Lebesgue number theorem) each piece maps into some $\mathcal{U}_\alpha$. Unlike S-vK, though, we want to do this in a more structured way, so that the cutting up process is "compatible" with our boundary maps. And the best way to describe this cutting up is through *barycentric coordinates.* Recall that an n-simplex is the set of convex linar combinations $\sum x_i v_i$ with $x_i \geq 0$ and $\sum x_i = 1$. The map which sends an n-simplex to the n-simplex Δ^n is literally the map $\sum x_i v_i \mapsto (x_0, \ldots, x_n)$. These are the barycentric coordinates of an n-simplex. Since, formally, all singular simplices are considered to have Δ^n for their domain, we can describe barycentric subdivision by describing how to cut up Δ^n. The idea is to build a process the is compatible with the boundary map, so that the subdivision, when restricted to a sub-simplex, is the subdivision of that sub-simplex. A 1-simplex $[v_0, v_1]$ is subdivided by adding the barycenter $w = (v + 0 + v_1)/2$ as a vertex, cutting $[v_0, v_1]$ into two 1-simplices ,$[v_0, w]$,$[w, v_1]$. A 2-simplex $[v_0, v_1, v_2]$ will, to be compatible withe boundary map, have its boundary cutinto 6 1-simplices; using the barycenter $(v_0 + v_1 + v_2)/3$ we can cone off each of these 1-simplices to subdivide $[v_0, v_1, v_2]$ into 6 2-simplices. Taking the cue that $2 = (1 + 1)!$, $6 = (2 + 1)!$ is probably no accident, we might expect that an n-simplex will be cut into $(n + 1)!$ n-simplices. Note that this is thew number of ways of ordering the vertices of our simplex. And following the "pattern" of our two test cases, where each new simplex was the convex span of vertices chosen as (vertex) , (barycenter of a 1-simplex having (vertex) as a vertex), (barycenter of a 2-simplex containing the previous 2 vertices), etc., we are led to the idea that the barycentric subdivision of an n-simplex $[v_0, \dots, v_n]$ is the $(n + 1)!$ n-simplices,

$$[v_{\alpha(0)}, (v_{\alpha(0)} + v_{\alpha(1)})/2, (v_{\alpha(0)} + v_{\alpha(1)} + v_{\alpha(2)})/3, \dots, (v_{\alpha(0)} + \cdots v_{\alpha(n)})/(n + 1)]$$

one for every permutation α of $\{0, \dots, n\}$. And since we want to take into account <u>orientations</u> as well, the natural thing to do is to define the barycentric subdivision of a singular n-simplex $\sigma : [v_0, \dots, v_n] \to X$ to be

$$S(\sigma) = \sum_\alpha (-1)^{\mathrm{sgn}(\alpha)} \sigma|_{[v_{\alpha(0)}, (v_{\alpha(0)}+v_{\alpha(1)})/2, (v_{\alpha(0)}+v_{\alpha(1)}+v_{\alpha(2)})/3, \dots, (v_{\alpha(0)}+\cdots v_{\alpha(n)})/(n+1)]}$$

where the sum is taken over all permutations of $\{0, \dots, n\}$. This (extending linearly over the chain group) is the subdivision operator, $S : C_n(X) \to C_n(X)$. A "routine" calculation establishes that $\partial S = S\partial$, i.e., it is a chain map (i.e., it behaves well on the boundary of our simplices). The point to this operator is that all of the subsimplices in the sum are a definite <u>factor</u> smaller than the original simplex. In fact, if the diameter of $[v_0, \dots, v_n]$ is d (the largest distance between points, which will, because it is the convex span of the vertices, be the largest distance between vertices), then every individual simplex in $S(\sigma)$ will have diameter at most $nd/(n+1)$ (the result of a little Euclidean geometry and induction). So by <u>repeatedly</u> applying the subdivision operator S to a singular simplex, we will obtain a singular chain $S^k(\sigma)$, which is "really" σ written as a sum of tiny simplices, whose singular simplices have image as small as we want. Or put more succinctly, if $\{\mathcal{U}_\alpha\}$ is an open cover of X and $\sigma : \Delta^n \to X$ is a singular n-simplex, then choosing a Lebesgue number ϵ for the open cover $\sigma^{-1}(\mathcal{U}_\alpha)$ of the compact metric space Δ^n, and choosing a k with $d(n/(n + 1))^k < \epsilon$, we find that $S^k(\sigma)$ is a sum of singular simplices each of which maps into one of the $\mathcal{U}_\alpha$, i.e., $S^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

This, in turn, allows us to define our chain map $b : C_n(X) \to C_n^{\mathcal{U}}(X)$; given a chain $\sum a_i\sigma_i$, we can, for each i find a k_i with $S^{k_i}(\sigma_i) \in C_n^{\mathcal{U}}(X)$; then define $b(\sum a_i\sigma_i) = \sum a_i S^{k_i}(\sigma_i)$. If we want to make sure this is well-defined, always choose the <u>smallest</u> $k_i \geq 0$ which works. Now that we have our putative homotopy-inverse, it suffices to show that $i \circ b$ and $b \circ i$ are chain homotopic to the identity. First note that $b \circ i$ <u>is</u> the identity; for every singular simplex in $C_n^{\mathcal{U}}(X)$, thinking of it as lying in $C_n(X)$ we find that the corresponding $k_i = 0$, and b does nothing to it. So all that we need to do is to come up with the chain homotopy $H : C_n(X) \to C_{n+1}(X)$ with $\partial_{n+1}H + H\partial_n = I - i \circ b$. Again, we really only need to define H for a single singular n-simplex σ (and extend linearly). And in the end, we only need to show how to define, for any k, an H_k so that $(\partial_{n+1}H_k + H_k\partial_n)(\sigma) = (I - i \circ S^k)(\sigma)$ (and then set $H = H_k$ on that simplex, for the appropriate k).

And to do <u>that</u>, we define a map $R : C_n(X) \to C_{n+1}(X \times I)$; when followed by the projection-induced map $p_\# : C_{n+1}(X \times I) \to C_{n+1}(X)$, we get a map $T : C_n(X) \to C_{n+1}(X)$ from which we will build H_k as $H_k = \sum TS^j$, where the sum is taken over $j = 0, \dots k - 1$. Once we define T (!) and show that $\partial T + T\partial = I - S$, we will have $\partial H_k + H_k\partial = \sum \partial TS^j + TS^j\partial = \sum(\partial T + T\partial)S^j = \sum(S^j - S^{j+1}) = I - S^k$ (since the last sum telescopes). And defining R, is, formally, just another particular sum. Setting up some notation, thinking of $\Delta^n \times I$, as before, as having vertices $\{v_0, \dots v_n\}$ on the 0-end and $\{w_0, \dots, w_n\}$ on the 1-end, $N = \{0, \dots, n\}$, $\Pi(Q) =$ the group of permuations of Q, and $\sigma' = \sigma \times I : \Delta^n \times I \to X \times I$), we have

$$R(\sigma) = \sum_{A \subseteq N} \sum_{\pi \in \Pi(N \setminus A)} \{(-1)^{|A|}(-1)^{\mathrm{sgn}(\pi)} \prod_{j \in N \setminus A} (-1)^j\}$$
$$\sigma'|_{[v_{i_0}, \dots, v_{i_j}, (w_{i_0}+\cdots w_{i_j})/(j+1), (w_{i_0}+\cdots w_{i_j}+w_{\pi(i_{j+1}})/(j+2), \dots, (w_{i_0}+\cdots w_{i_j}+w_{\pi(i_{j+1})}+\cdots w_{\pi(i_n)})/(n+1)]}$$

where we sum over all <u>non-empty</u> subsets of $\{0, \dots n\}$ (with the induced ordering on vertices from the ordering on $\{0, \dots, n\}$). Intuitively, this map "interpolates" between the simplex $[v_0, \dots v_n]$ and the barycentric subdivision on $w_0, \ldots, w_n$, by taking the (signed sums of the) convex spans of simplices on the bottom (0) and simplices on the top (1). Again, a "routine" calculation will establish that $\partial T + T\partial = I - S$, as desired. [At any rate, I verified it for n=1,2; the formula for the sign of each simplex was determined by working backwards from these examples.]

And the point to all of these calculations was that if $\{\mathcal{U}_\alpha\}$ is an open cover of X, then the inclusions $i_n : C_n^{\mathcal{U}}(X) \to C_n(X)$ induce isomorphisms on homology. This gives us two big theorems. The first is

Mayer-Vietoris Sequence: If $X = \mathcal{U} \cup \mathcal{V}$ is the union of two open sets, then the short exact sequences $0 \to C_n(\mathcal{U} \cap \mathcal{V}) \to C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \to C_n^{\{\mathcal{U},\mathcal{V}\}}(X) \to 0$, together with the isomorphism above, give the long exact sequence

$$\cdots \to H_n(\mathcal{U} \cap \mathcal{V}) \stackrel{(i_{\mathcal{U}*}, -i_{\mathcal{V}*})}{\to} H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \stackrel{j_{\mathcal{U}*}+j_{\mathcal{V}*}}{\to} H_n(X) \stackrel{\partial}{\to} H_{n-1}(\mathcal{U} \cap \mathcal{V}) \to \cdots$$

And just like Seifert - van Kampen, we can replace open sets by sets A, B having neigborhoods which deformation retract to them, and whose intersection deformation retracts to $A \cap B$. For example, subcomplexes $A, B \subseteq X$ of a CW-complex, with $A \cup B = X$ have homology satisfying a long exact sequence

$$\cdots \to H_n(A \cap B) \stackrel{(i_{A*}, -i_{B*})}{\to} H_n(A) \oplus H_n(B) \stackrel{j_{A*}+j_{B*}}{\to} H_n(X) \stackrel{\partial}{\to} H_{n-1}(A \cap B) \to \cdots$$

And this is also true for reduced homology; we just augment the chain complexes used above with the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$, where the first non-trivial map is $a \mapsto (a, -a)$ and the second is $(a, b) \mapsto a + b$.

And now we can do some meaningful calculations! An n-sphere S^n is the union $S^n_+ \cup S^n_-$ of its upper and lower hemispheres, each of which is contractible, and have intersection $S^n_+ \cap S^n_- = S^{n-1}_0$ the equatorial $(n-1)$-sphere. So Mayer-Vietoris gives us the exact sequence $\cdots \to \widetilde{H}_k(S^n_+) \oplus \widetilde{H}_k(S^n_-) \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to \widetilde{H}_{k-1}(S^n_+) \oplus \widetilde{H}_{k-1}(S^n_-) \to \cdots$ hhtp , i.e, $0 \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to 0$ i.e., $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for every k and n. So by induction,

$$\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z}, & if \text{k=n} \\ 0, & otherwise \end{cases}$$

And this, in turn, let's us prove a fairly sizable theorem:

Brouwer Fixed Point Theorem: For every n, every map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

Proof: If $f(x) \neq x$ for every x, then is with the $n = 2$ case that you may have seen before, we can construct a retraction $r : \mathbb{D}^n \to \partial\mathbb{D}^n = S^{n-1}$ by setting $r(x) =$ the (first) point past $f(x)$ where the ray from $f(x)$ to x meets $\partial\mathbb{D}^n$. This function is continuous, and is the identity on the boundary. So from our of your problem sets, the inclusion-induced map $i_* : H_{n-1}(S^n) \to H_{n-1}(\mathbb{D}^n)$ is injective. But this is impossible, since the first group is $\mathbb{Z}$ and the second is 0 .

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\mathrm{cl}_X(B) \subseteq \mathrm{int}_X(A)$, then for every k the inclusion-induced map $H_k(X \setminus B, A \setminus B) \to H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\mathrm{int}_X(A) \cup \mathrm{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \to H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$.] A proof of this second result follows from a relative version of the barycentric subdivision process;