Math 423/823 Exercise Set 1 Solutions

Due Thursday, Jan. 27

1. [BC#1.2.6(b)] For complex numbers $z_1 = a_1 + b_1 i$, etc., verify the distributive law:
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$
We can write out both sides, using our rules for addition and muliplication:
$$\begin{aligned} z_1(z_2+z_3) &= (a_1+b_1 i)((a_2+b_2 i)+(a_3+b_3 i)) \\ &= (a_1+b_1 i)((a_2+a_3)+(b_2+b_3)i) \\ &= (a_1(a_2+a_3)-b_1(b_2+b_3))+(a_1(b_2+b_3)+(a_2+a_3)b_1)i \\ &= (a_1a_2+a_1a_3-b_1b_2-b_1b_3)+(a_1b_2+a_1b_3+a_2b_1+a_3b_1)i \end{aligned}$$
$$\begin{aligned} z_1z_2+z_1z_3 &= (a_1+b_1 i)(a_2+b_2 i)+(a_1+b_1 i)(a_3+b_3 i) \\ &= ((a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i)+((a_1a_3-b_1b_3)+(a_1b_3+a_3b_1)i) \\ &= ((a_1a_2-b_1b_2)+(a_1a_3-b_1b_3))+((a_1b_2+a_2b_1)+(a_1b_3+a_3b_1))i \\ &= (a_1a_2+a_1a_3-b_1b_2-b_1b_3)+(a_1b_2+a_1b_3+a_2b_1+a_3b_1)i \end{aligned}$$
But these last two expressions are identical! So $z_1(z_2+z_3) = z_1z_2 + z_1z_3$, as desired. [N.B.: apparently I didn't follow my own advice....]

2. [BC#1.3.1] Reduce each of the quantities to a real number:
$$\text{(a) } \frac{1+2i}{3-4i} + \frac{2-i}{5i} \qquad\qquad \text{(c) } (1-i)^4$$
$\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = \dfrac{(1+2i)(3+4i)}{3^2+4^2} + \dfrac{(2-i)(-5i)}{0^2+5^2}$
$= \dfrac{(3-8)+(4+6)i}{25} + \dfrac{(0-5)+(-10+0)i}{25}$
$= \dfrac{(-5-5)+(10-10)i}{25} = \dfrac{-10}{25} = \dfrac{-2}{5}$
$(1-i)^4 = [(1-i)^2]^2 = [(1-i)(1-i)]^2$
$= [((1)(1)-(-1)(-1))+((1)(-1)+(1)(-1))i]^2$
$= [-2i]^2 = (-2)^2 i^2 = (4)(-1) = -4$

3. [BC#1.5.11] Use mathematical induction to show that for all natural numbers n, and complex numbers $z_1, \ldots, z_n$,
$$\overline{z_1 + \cdots + z_n} = \overline{z_1} + \cdots + \overline{z_n} \qquad \text{and} \qquad \overline{z_1 \cdots z_n} = \overline{z_1} \cdots \overline{z_n}$$
From class, or direct computation, we know that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$. This is the base case of our induction. [Technically, $n = 1$ really is, and $\overline{z_1} = \overline{z_1}$ is even more immediately true.] For our inductive hypothesis, we suppose that
$$\overline{z_1 + \cdots + z_n} = \overline{z_1} + \cdots + \overline{z_n} \qquad \text{and} \qquad \overline{z_1 \cdots z_n} = \overline{z_1} \cdots \overline{z_n}$$
and show that the result is also true with n replaced by $n+1$:

$\overline{z_1 + \cdots + z_{n+1}}$
$= \overline{(z_1 + \cdots + z_n) + z_{n+1}} = \overline{(z_1 + \cdots + z_n)} + \overline{z_{n+1}} = (\overline{z_1} + \cdots + \overline{z_n}) + \overline{z_{n+1}}$
$= \overline{z_1} + \cdots + \overline{z_{n+1}}$

where the third equality is the base case of our induction, and the fourth equality is our inductive hypothesis. So by induction, the result holds for all n. Similarly,

$\overline{z_1 \cdots z_{n+1}}$
$= \overline{(z_1 \cdots z_n) \cdot z_{n+1}} = \overline{(z_1 \cdots z_n)} \cdot \overline{z_{n+1}} = (\overline{z_1} \cdots \overline{z_n}) \cdot \overline{z_{n+1}}$
$= \overline{z_1} \cdots \overline{z_{n+1}}$

where, again, the third equality is the base case of our induction, and the fourth equality is our inductive hypothesis. So by induction, the result also holds for all n.

4. Show that if $p(x) = a_n x^n + \cdots + a_0$ is a polynomial with real coefficients, and $z = a+bi$ is a complex root of p [i.e., $p(z) = a_n z^n + \cdots + a_0 = 0$], then $\overline{z}$ is <u>also</u> a root of p.

   The main point is that the coefficients, being real, are equal to their own complex conjujates; $a_i = a_i + 0i$, so $\overline{a_i} = a_i - 0i = a_i$. Then if z is a root of f, since, by problem #3, $\overline{a_i z^i} = \overline{a_i}(\overline{z})^i = a_i(\overline{z})^i$, we have

   $$\begin{aligned}\overline{p(z)} &= \overline{a_n z^n + \cdots + a_1 z + a_0} \\ &= \overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} \\ &= \overline{a_n}(\overline{z})^n + \cdots + \overline{a_1 z} + \overline{a_0} \\ &= a_n(\overline{z})^n + \cdots + a_1\overline{z} + a_0 \\ &= p(\overline{z})\end{aligned}$$

   But since z is a root of p, $p(z) = 0$, and so $p(\overline{z}) = \overline{p(z)} = \overline{0} = 0$, so $\overline{z}$ is a root of p, as well.

   [N.B.: Note that this line of work only works for polynomials with real coeeficients! For example, the roots of $p(z) = z^2 - i$ (find them!) are not complex conjugates of one another....]