Math 971 Algebraic Topology

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We have so far introduced two homologies; simplicial, H_*^Δ, whose computation "only" required some linear algebra, and singular, H_*, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For Δ-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub-Δ-complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X,A) \to H_{n-1}^\Delta(A) \to \cdots$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^{(k)},X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)},X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)}) \end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because, essentially, we can, in each case, identify these groups. $H_n(X^{(k)},X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$ are either 0 (for $n \neq k$) or $\oplus\mathbb{Z}$ (for $n = k$), one summand for each n-simplex in X. But the same is true for $H_n^\Delta(X^{(k)},X^{(k-1)})$; and for $n = k$ the generators are precisely the n-simplices of X. The inclusion-induced map takes generators to generators, so is an isomorphism. So by the Five Lemma, the middle rows are also isomorphisms, completing our inductive proof.

Returning to $H_n^\Delta(X) \xrightarrow{I_*} H_n(X)$, we wish now to show that this map is an isomorphism. Any $[z] \in H_n(X)$ is represented by a cycle $z = \sum a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$. But each $\sigma_i(\Delta^n)$ is a compact subset of X, and so meets only finitely-many cells of X. This is true for every singular simplex, and so there is a k for which all of the simplices map into $X^{(k)}$, and so we may treat $z \in C_n(X^{(k)}$. Thought of in this way, it is still a cycle, and so $[z] \in H_n(X^{(k)}) \cong H_n^\Delta(X^{(k)})$ so there is a $z' in C_n^\Delta(X^{(k)})$ and a $w \in C_{n+1}(X^{(k)})$ with $i_\# z' - z = \partial w$. But thinking of $z' in C_n^\Delta(X)$ and $w \in C_{n+1}(X)$, we have the same equality, so $[z'] \in H_n^\Delta(X)$ and $i_*[z'] = [z]$. So i_* is surjective. If $i_*([z]) = 0$, then the cycle $z = \sum a_i\sigma_i$ is a sum of characteristic maps of n-simplices of X, and so can be thought of as an element of $C_n^\Delta(X^{n)})$. Being 0 in $H_n(X)$, $z = \partial w$ for some $w \in C_{n+1}(X)$. But as before, $w \in C_n(X^{r)})$ for some r, and so thought of as an element of the image of the isomorphism $i_* : H_n^\Delta(X^{(r)}) \to H_n(X^{(r)})$, $i_*([z]) = 0$, so $[z] = 0$. So $z = \partial u$ for some $u \in C_{n+1}^\Delta(X^{r)}) \subseteq C_{n+1}^\Delta(X)$. So $[z] = 0$ in $H_n^\Delta(X)$. Consequently, simplicial and singular homology groups are isomorphic.

Some topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A,B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists.

Another: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h, then $n = m$. This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n-1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$.

More generally, we can establish a result which is known as *invariance of domain*, a result which is useful in both topology and analysis.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

Note it is enough to proof this for our favorite open set, which in this context will be $\mathcal{V} = (-1,1)^n \subseteq \mathbb{R}^n$, since given any open $\mathcal{U}$ and $x \in \mathcal{U}$, we can find an injective linear map $h : (-1,1)^n \to \mathcal{U}$ taking 0 to x. If we can show that $f \circ h$ has open image, then $f(x) \in f \circ h(\mathcal{V}) \subseteq f(\mathcal{U})$ shows that $f(x)$ has an open neighborhood in $f(\mathcal{U})$. Since x is arbitrary, $f(\mathcal{U})$ is open.

This in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.

Our next goal is to show that, when both make sense, simplicial and singular homology are isomorphic. In fact, the inclusion of the simplicial chain groups into the singular ones induces an isomorphism on homology.