Math 314 Matrix Theory

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The three steps of *row reduction*, swap rows, rescale a row by multiplication, and add a multiple of a row to another, are enough to change a system of equations to a new, "simpler", system, **without changing the set of solutions**. In practice, we will carry out the row reduction in two stages:

Moving **left-to-right, top-to-bottom**, creating as many 0 coefficients as we can, we can reach *row echelon form (REF)*:

(1) Reading top to bottom, the first non-zero entry in each row (the *lead coefficient*) occurs further and further to the right in the rows,

(2) Every entry directly below a lead coefficient is equal to 0 [This is exactly what MoSS would do], and, as a consequence,

(3) Rows with all coefficients 0 appear at the bottom.

REF allows us to read off the solutions to the original system by *back-substitution*: We can solve for each lead term in each row, and substitute them, working from bottom to top, into each equation above it. *Or*, we can do this at the level of the augmented matrix, working **right-to-left, bottom-to-top**, to reach *reduced row echelon form (RREF)*:

(4) Every lead coefficient is equal to 1 (just divide the row through by what we have)

(5) Every entry directly above a leading 1 is equal to 0 (by adding multiples of the row with leading 1 to the rows above).

Row reduction operations taking an augmented matrix to one in RREF do not change the set of solutions; but those solutions can be transparently read off from the RREF.

An inconsistent system is one in which no assignement of values to the variables can satisfy all of the equations in the system. In RREF, this can be detected by a row which reads

$$(\ 0\ 0 \cdots 0) \mid 1\ )$$

This equation translates to "$0 = 1$", which certainly no assignment of values can satisfy!

In all other cases, the system is consistent; and we can assemble solutions from the RREF. The entry where a leading 1 occurs is called a *pivot*. The columns of the coefficient matrix correspond to the variables in our system of equation ; a column containing a pivot gives a *bound variable*, and all other columns give *free variables*. If we follow the lead of MoSS, and write the equations corresponding to the rows of our RREF as (it turns out)

$$\text{(bound variable)} = \text{(equation involving free variables)}$$

by subtracting the free variables over to the other side, then *any* assignment of values to the free variables forces, through these equations, assignment of values to the bound variables which makes the equations corresponding to the rows of the RREF true, and hence giving a solution to the (original) system of equations. For example, the augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 7 & | & 11 \\ 1 & 2 & 4 & 2 & 15 & | & 23 \\ 2 & 4 & 1 & 2 & 12 & | & 19 \end{pmatrix} \quad \text{row reduces to} \quad \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & | & 5 \\ 0 & 0 & 1 & 0 & 2 & | & 3 \\ 0 & 0 & 0 & 1 & 2 & | & 3 \end{pmatrix}$$

From which we can read off the solutions
$x_1 = 5 - 2x_2 - 3x_5$, x_2 is free, $x_3 = 3 - 2x_5$, $x_4 = 3 - 2x_5$, and x_5 is free.