Math 971 Algebraic Topology

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We have so far introduced two homologies; simplicial, H_*^Δ, whose computation "only" required some linear algebra, and singular, H_*, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For Δ-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub-Δ-complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X,A) \to H_{n-1}^\Delta(A) \to \cdots$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^{(k)},X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)},X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)}) \end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because, essentially, we can, in each case, identify these groups. $H_n(X^{(k)},X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$ are either 0 (for $n \neq k$) or $\oplus\mathbb{Z}$ (for $n = k$), one summand for each n-simplex in X. But the same is true for $H_n^\Delta(X^{(k)},X^{(k-1)})$; and for $n = k$ the generators are precisely the n-simplices of X. The inclusion-induced map takes generators to generators, so is an isomorphism. So by the Five Lemma, the middle rows are also isomorphisms, completing our inductive proof.

Returning to $H_n^\Delta(X) \xrightarrow{I_*} H_n(X)$, we wish now to show that this map is an isomorphism. Any $[z] \in H_n(X)$ is represented by a cycle $z = \sum a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$. But each $\sigma_i(\Delta^n)$ is a compact subset of X, and so meets only finitely-many cells of X. This is true for every singular simplex, and so there is a k for which all of the simplices map into $X^{(k)}$, and so we may treat $z \in C_n(X^{(k)}$. Thought of in this way, it is still a cycle, and so $[z] \in H_n(X^{(k)}) \cong H_n^\Delta(X^{(k)})$ so there is a $z' in C_n^\Delta(X^{(k)})$ and a $w \in C_{n+1}(X^{(k)})$ with $i_\# z' - z = \partial w$. But thinking of $z' in C_n^\Delta(X)$ and $w \in C_{n+1}(X)$, we have the same equality, so $[z'] \in H_n^\Delta(X)$ and $i_*[z'] = [z]$. So i_* is surjective. If $i_*([z]) = 0$, then the cycle $z = \sum a_i\sigma_i$ is a sum of characteristic maps of n-simplices of X, and so can be thought of as an element of $C_n^\Delta(X^{n)})$. Being 0 in $H_n(X)$, $z = \partial w$ for some $w \in C_{n+1}(X)$. But as before, $w \in C_n(X^{r)})$ for some r, and so thought of as an element of the image of the isomorphism $i_* : H_n^\Delta(X^{(r)}) \to H_n(X^{(r)})$, $i_*([z]) = 0$, so $[z] = 0$. So $z = \partial u$ for some $u \in C_{n+1}^\Delta(X^{r)}) \subseteq C_{n+1}^\Delta(X)$. So $[z] = 0$ in $H_n^\Delta(X)$. Consequently, simplicial and singular homology groups are isomorphic.

One consequence of this fact is that we can prove the topological invariance of the **Euler characteristic** of a space X. If X is a Δ-complex made up of a finite number of simplices, then we can count the number m_i of i-simplices in the Δ-complex structure of X. The Euler characteristic of X is then defined to be the alternating sum $\chi(X) = \displaystyle\sum_{i=0}^{\infty}(-1)^i m_i$. Now, as a topological space, X can be given many different Δ-complex structures, and $\chi(X)$ is a priori a number which depends on the structure, not just on X. But once we note that m_i = the rank of the (simplicial) chain group $C_i^\Delta(X)$ (there is one generator for each i-simplex), we find that $\chi(X) = \sum_{i=0}^N(-1)^i \text{ rank}(C_i(X))$, and then the following result from homological algebra establishes the topological invariance of this number:

Proposition: If $\cdots 0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$ is a chain complex, with every chain group having finite rank, then

$\sum_{i=0}^n(-1)^i \text{ rank}(C_i) = \sum_{i=0}^n(-1)^i \text{ rank}(H_i(\mathcal{C})$.

The proof follows from cleverly applying the fact that since $H_i(\mathcal{C})=\ker\partial_i/\mathrm{im}\partial_{i+1}$, $z_i = \mathrm{rank}(\ker\partial_i) = \mathrm{rank}(H_i(\mathcal{C})) + \mathrm{rank}(\mathrm{im}\partial_{i+1}) = h_i + b_{i+1}$, so $h_i = z_i - b_{i+1}$, together with the fact that since (by Noether) $\mathrm{im}(\partial_i) \cong C_i/\ker(\partial_i)$, so $c_i = \mathrm{rank}(C_i) = z_i + b_i$. We therefore have
$\sum_{i=0}^n (-1)^i \ \mathrm{rank}(H_i(\mathcal{C}) = \sum_{i=0}^n (-1)^i h_i = \sum_{i=0}^n (-1)^i (z_i - b_{i+1}) = \sum_{i=0}^n (-1)^i z_i - \sum_{i=0}^n (-1)^i b_{i+1} = \sum_{i=0}^n (-1)^i z_i + \sum_{i=0}^n (-1)^i b_i = \sum_{i=0}^n (-1)^i (z_i + b_i) = \sum_{i=0}^n (-1)^i \ \mathrm{rank}(C_i)$ as desired.
Consequently, $\chi(X) = \sum_{i=0}^N (-1)^i \ \mathrm{rank}(C_i^\Delta(X)) = \sum_{i=0}^N (-1)^i \ \mathrm{rank}(H_i^\Delta(X)) = \sum_{i=0}^N (-1)^i \ \mathrm{rank}(H_i(X))$, which is an invariant of X, since the singular homology groups are!

The fact that this number has two different interpretations leads to some non-trivial results. First, it tells us that the Euler charactistic calculation is independent of how we express a space X as a Δ-complex. χ is also actually invariant under homotopy equivalence, since the homology groups are; so homotopy equivalent spaces have the same Euler *chi*. Consequently, all contractible spaces, for example, must have Euler characteristic $= 1$.

Next, by the lifting criterion, if $p: \widetilde{X} \to X$ is a k-fold covering space of a Δ-complex X, then $\widetilde{X}$ can be given a Δ-complex structure with k times as many i-simplices as X, for every i (lift the characteristic maps of the simplices of X....). So $\chi(\widetilde{X}) = k \cdot \chi(X)$. This give a necessary condition for one space to be a covering of another; it's Euler χ must be a multiple of the other. For example, from our homology calculations, it follows that for a closed orientable surface F_g of genus g, $\chi(F_g) = 2 - 2g$. So a k-fold covering of F_g will have Euler χ equal to $k(2-2g) = 2k - 2kg = 2 - 2(kg - k + 1)$, and so is a surface of genus $kg - k + 1$. [The converse, that a surface with this genus k-fold covers F_g, can be established by building the coverings directly.] Consequently, F_5 is a 2-fold covering of F_3, so there is a subgroup of index 2 of $\pi_1(F_3)$ isomorphic to $\pi_1(F_5)$, but F_6 is not a finite-sheeted cover of F_3, because $-4 \not| -10$. [It is also not an inifinite-sheeted covering, because their total spaces are non-compact...] Consequently, $\pi_1(F_6)$ is not isomorphic to a subgroup of $\pi_1(F_3)$.

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A, B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex K whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to K, with boundary a (for the exact same reasons as above) closed orientable surface.

Another: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h, then $n = m$. This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n-1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$.

More generally, we can establish a result which is known as *invariance of domain*, which is useful in both topology and analysis.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f: \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

We will defer this proof for awhile (perhaps permanently?).
Note it is enough to proof this for our favorite open set, which in this context will be $\mathcal{V} = (-1,1)^n \subseteq \mathbb{R}^n$, since given any open $\mathcal{U}$ and $x \in \mathcal{U}$, we can find an injective linear map $h : (-1,1)^n \to \mathcal{U}$ taking 0 to x. If we can show that $f \circ h$ has open image, then $f(x) \in f \circ h(\mathcal{V}) \subseteq f(\mathcal{U})$ shows that $f(x)$ has an open neighborhood in $f(\mathcal{U})$. Since x is arbitrary, $f(\mathcal{U})$ is open.

This in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.