ESSENTIAL LAMINATIONS IN SEIFERT-FIBERED SPACES

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by

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To my parents, who started it all....

Allen E Hatcher

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Over the past few decades, the importance of the incompressible surface in the study of 3-manifold topology has become apparent. In fact, nearly all of the important outstanding conjectures in the field have been proved, for 3-manifolds containing incompressible surfaces. Faced with such success, it becomes important to know just what 3-manifolds could contain an incompressible surface.

Historically, the first 3-manifolds (with infinite fundamental group) which were shown to contain no incompressible surfaces were a certain collection of Seifert-fibered spaces. Waldhausen, in the 1960's, showed that an incompressible surface in a Seifert-fibered space is isotopic to one which is either vertical or horizontal. This added structure severely restricts the existence of an incompressible surface, and led to the discovery of these 'small' Seifert-fibered spaces. (Later results have demonstrated that, in some sense, 'most' 3-manifolds do not contain incompressible surfaces.)

Now in recent years the essential lamination, a recently-defined hybrid of the incompressible surface and the codimension-one foliation, has begun to show similar power in tackling problems in 3-manifold topology. It has the added advantage of being far more widespread than the incompressible surface; in fact, in the same sense, 'most' 3-manifolds do contain essential laminations. In light of this, it becomes interesting to know if there are any 3-manifolds which contain no essential laminations, and only natural to look in the same place that Waldhausen found his examples.

In this thesis we carry out such a program. We show that an essential lamination in a Seifert-fibered space satisfies a structure theorem similar to the one given for surfaces by Waldhausen. Together with work of Eisenbud-Hirsch-Neumann on the existence of horizontal foliations, this structure theorem allows us to show that some of the 'small' Seifert-fibered spaces above cannot contain any essential laminations.

We also obtain, as a further application of the structure theorem, a result which states that any codimension-one foliation with no compact leaves in a 'small' Seifert-fibered space is isotopic to a horizontal foliation; this extends and completes (in some sense) a group of results on isotoping foliations in Seifert-fibered spaces, which began with Thurston's thesis.

Biographical Sketch

The author was born in 1961 in Milwaukee, Wisconsin, and was raised in Poughkeepsie, New York from the age of 2. He received a B.S. degree (with highest honors) from the State University of New York at Stony Brook in 1983, and an M.A. degree from Cornell University in 1986. In addition to mathematics, he enjoys photography, Tolkien, collecting (anything), walking, and backpacking.

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Introduction

This thesis is a study of essential laminations in Seifert-fibered spaces.

The notion of a essential lamination is the end result of an attempt to find a 'suitable' generalization of two more classically defined and studied objects, the (2-sided) incompressible surface and the codimension-1 foliation without Reeb components. Interest in such a generalization stems from the fact that for years these two 'parent' objects (and the incompressible surface in particular) have served as the 'workhorses' of the 3-dimensional (manifold) topologist. The idea has been that manifolds containing such surfaces and foliations enjoy nice topological properties; also, these objects serve as good 'backdrops' in which to control the behavior of topological constructions which one carries out in the 3-manifold.

This point of view has been demonstrated, with tremendous success, by the study of incompressible surfaces. Waldhausen (see [Wa 2]) has shown that a (closed, irreducible) 3-manifold M which contains a 2-sided incompressible surface has universal cover $\mathbf{R}^3$, and any map f from a (closed, irreducible, not simply-connected) 3-manifold N to M which induces an injection on fundamental groups is homotopic to a covering map. In particular, any irreducible N homotopy-equivalent to M is homeomorphic to M; i.e., the homeomorphism type of M is determined by its fundamental group alone.

On the other side of the spectrum, Palmeira [Pm] showed that a manifold which admits a codimension-1 foliation without Reeb components has universal cover $\mathbf{R}^3$. More recently, Gabai [Ga 1] has shown that 'taut' foliations can be used to determine interesting topological properties of 3-manifolds; in particular, he has proved the Property R conjecture [Ga 2] by constructing a 'suitably well-behaved' foliation in the complement of a knot in $\mathbf{S}^3$.

Into this tradition now comes the essential lamination. By its definition (see Chapter 1.A) a lamination is basically a codimension-1 foliation with 'gaps' in between its leaves. Thus in a sense it serves as a 'bridge' between (compact) embedded surfaces and foliations. The requirements which make a lamination 'essential' reflect both the incompressibility (i.e., π_1-injectivity) of a surface and the lack of Reeb components in a foliation.

The idea of an essential lamination is a very recent one (in some senses, the 'right' definition is only about 3 years old), but also a potentially powerful one. For example it has already been shown [G-O] that a 3-manifold which contains an essential lamination has universal cover $\mathbf{R}^3$. The ultimate goal is to show that essential laminations can be as useful to the study of 3-manifolds as incompressible surfaces have turned out to be. In particular, the goal of the theory is to show that Waldhausen's theorems remain true for 3-manifolds containing essential laminations.

The techniques used in studying essential laminations and the manifolds containing them are a hybrid of the techniques used with their parents. Because the leaves of a lamination $\mathcal{L}$ have 'gaps' in between them, one is able to use transversality (i.e., 'general position') arguments to make $\mathcal{L}$ meet surfaces and loops nicely. This allows for 'induction on dimension' arguments similar to those used with incompressible surfaces. Such techniques are available for foliations only in a very weak form (see, e.g, [Le]).

On the other hand, the leaves of $\mathcal{L}$ are usually non-compact, in distinction with an incompressible surface. These leaves then limit on other leaves, and themselves, and so foliation-theoretic techniques such as holonomy and Reeb stability come into play.

Through it all, our leaves are (usually non-compact) surfaces, and so our understanding of what surfaces look like and how they behave must be employed throughout. In particular, the fact that a null-homotopic embedded loop in a surface bounds an embedded <u>disk</u> in the surface (see [Ep]) is used almost constantly, usually without comment.

Chapter 1

Definitions, Notations, and Some Basic Constructions

Before discussing the main objects of interest to this thesis, essential laminations, and their interaction with the topology and geometry of a Seifert-fibered space, we must build up a somewhat sophisticated body of terminology about the objects and the spaces they will sit in. We also need to take a look a little at what these objects can do in some elementary settings, partly to help make us familiar with their abilities and limitations, but also to begin to develop a body of technique with which to carve out our proofs.

A. Laminations

Let M be a compact orientable 3-manifold, and F a compact surface.

A lamination $\mathcal{L} \subseteq$M (resp. $\lambda \subseteq$F) is a codimension-one foliation of a closed subset of M (resp, of F). Precisely, for every point $x \in$M, there is a neighborhood U of x and a homeomorphism $h : \mathrm{I}^2 \times \mathrm{I} \to \mathrm{U}$ so that $h^{-1}(\mathrm{U}\cap\mathcal{L}) = \mathrm{I}^2\times\mathrm{C}$, for some closed subset $\mathrm{C} \subseteq \mathrm{I}$, and for any two such coordinate charts, the transition function $h_1^{-1}(\mathrm{U} \cap \mathrm{V}) \to \mathrm{h}_2^{-1}(\mathrm{U} \cap \mathrm{V})$ respects the horizontal sheets of $\mathrm{I}^2 \times \mathrm{I}$. In addition we assume that $\mathcal{L} \cap \partial\mathrm{M} \equiv \partial\mathcal{L}$ is a 1-dimensional lamination in the compact surface ∂M, and that in a neighborhood of ∂M, $\mathcal{L}$ looks like a product $\partial\mathcal{L} \times \mathrm{I}$.

The set $\mathcal{L}$ has an associated topology, called the leaf topology: the sheets $h^{-1}(\mathrm{I}^2 \times \mathrm{x})$, taken over all coordinate maps h and all $x \in \mathrm{I}$, form a basis of open sets for the topology. A leaf of $\mathcal{L}$ is a path component L of $\mathcal{L}$ in this topology. It is a (usually non-compact) topological surface; the inclusion L$\hookrightarrow$M is a one-to-one immersion, which is generally assumed to be C^∞. The tangent planes to the leaves of $\mathcal{L}$ are assumed to vary continuously over $\mathcal{L}$.

Associated to $\mathcal{L}$ there is also a space M$|\mathcal{L}$, called M split on $\mathcal{L}$, defined as follows: for a path component V of M$\backslash\mathcal{L}$, define a metric on V by d(x,y)= infimum in M-metric of the length of an arc in V joining x and y (this is a metric because V is an open subset of M). Then let $\overline{V}$ be the metric completion of V. M$|\mathcal{L}$ is then the union over all V of these metric completions $\overline{V}$; it is a (often vastly disconnected, usually non-compact) 3 - manifold with boundary. (All of these definitions have their exact analogues for laminations λ in a surface F.)

A branched surface B (resp. train track τ) is a space locally modelled on the space shown in Figure 1a (resp. 1c). Given an embedding of a branched surface or train track (in a 3-manifold M or surface F, respectively), there is an associated fibered neighborhood N(B) (resp. N(λ)), which is locally modelled on the space shown in Figure 1b (resp. 1d). The fibers of such a neighborhood are the images of the vertical intervals in the model space. The fibered neighborhood has a horizontal boundary ∂_hN(B) and a vertical boundary ∂_vN(B); ∂_hN(B) runs transverse to the I-fibers, and ∂_vN(B) is a union of (subarcs of) the I-fibers.

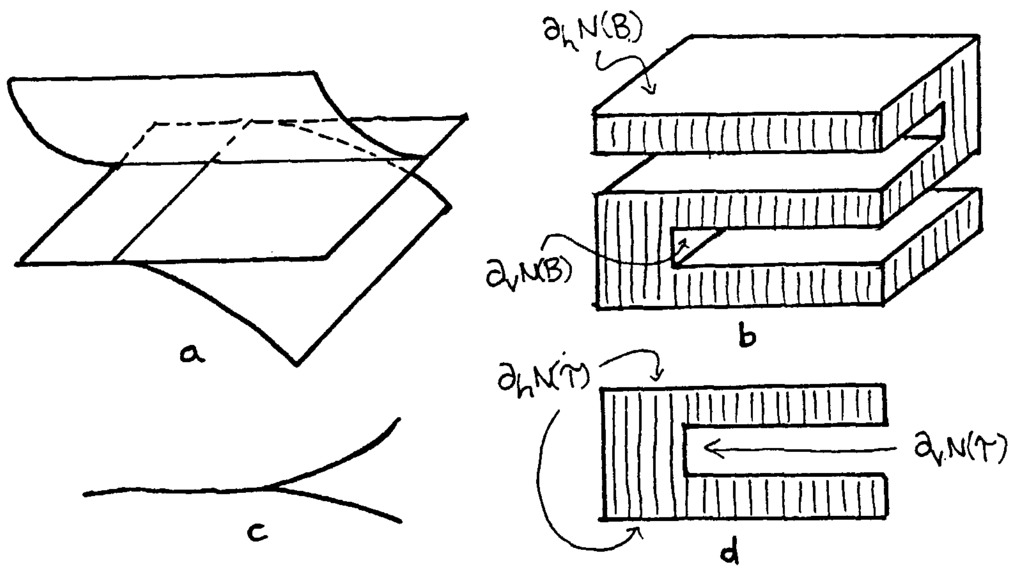

Figure 1: Branched surfaces and train tracks

A lamination $\mathcal{L} \subseteq M$ is carried by a branched surface B if $\mathcal{L}$ is contained in a fibered neighborhood of B, and all of the leaves of $\mathcal{L}$ are everywhere transverse to the fibers of N(B). If $\mathcal{L}$ meets every fiber of N(B) at least twice, then it can be embedded in a somewhat special way, so that it contains $\partial_h N(B)$; then splitting N(B) along $\mathcal{L}$, $N(B)|\mathcal{L}$ gives a collection of I-bundles over (compact and non-compact) surfaces, some with boundary; if we delete (the interiors of) those I-bundles over compact base, we get the fibered neighborhood of a new branched surface (which we still call B), carrying $\mathcal{L}$, for which $N(B)|\mathcal{L}$ contains no bundles over compact base. Such a branched surface B will be called a branched surface having no compact bundles w.r.t. $\mathcal{L}$.

Branched surfaces are useful for making general position and transversality arguments for laminations; it is therefore useful to assume that a lamination in question is carried by a branched surface. In this study, there will be an implicit assumption that any lamination considered is carried by a branched surface. In the case when $\mathcal{L}$ is a foliation (or otherwise cannot be carried by a branched surface), we can alter $\mathcal{L}$ slightly to create a new lamination by a process of splitting $\mathcal{L}$ open along some (finite) collection of its leaves. The idea of splitting is similar to replacing a compact surface in a 3-manifold by the boundary of a regular neighborhood. More precisely, we replace a leaf L of $\mathcal{L}$ by a closed (I-bundle) neighborhood of the leaf, creating the new lamination $\mathcal{L}_0=(\mathcal{L}\setminus L)\cup\partial N(L)$, with $M|\mathcal{L}_0=(M|\mathcal{L})\cup N(L)$. If we split our given lamination along some finite collection of leaves, we can then see, using coordinate charts, that the resulting lamination can be carried by a branched surface; choose a finite cover of $\mathcal{L}$ by charts, and do sufficient splittings so that every point of $\mathcal{L}$ is contained in the vertical interval of some chart which meets $M\setminus\mathcal{L}$ above and below the point.

B. Essential Laminations

Laminations, in such generality, are not terribly useful in the study of 3-manifold topology; any embedded compact surface, or any foliation, for example, is a lamination. Even the empty set $\emptyset$ is a lamination! To make them useful we must impose extra conditions on $\mathcal{L}$ to insure that the lamination succeeds in capturing the topology of the 3-manifold it is sitting in, so that it sits there, in the 3-manifold, in an 'essential' way. There are two standard ways to do this: one is by imposing conditions on the leaves of $\mathcal{L}$ and on $M|\mathcal{L}$, and the other is by requiring that $\mathcal{L}$ be carried by a branched surface B (called an <u>essential</u> branched surface) satisfying additional conditions. The two approaches are really the same (see [G-O]), and showing this basically amounts to turning one of the definitions into (a not terribly easy) theorem. Each approach has its own merits; for this study, the 'branched-surface-less' approach is better (although the other approach (and its terminology) is occasionally used in some of the more technical situations).

A lamination $\mathcal{L} \subseteq M^3$ is <u>essential</u> if:

(-1) $\mathcal{L} \neq \emptyset$,

(0) no leaf of $\mathcal{L}$ is a sphere or boundary-parallel disk,

(1) for every leaf L of $\mathcal{L}$, the inclusion $L \hookrightarrow M$ induces an injection on π_1 and relative π_1:

$$\pi_1(L) \hookrightarrow \pi_1(M) \qquad \pi_1(L, \partial L) \hookrightarrow \pi_1(M, \partial M)$$

(2) $M|\mathcal{L}$ is irreducible,

(3) $\mathcal{L}$ is <u>end-incompressible</u>, defined as follows:

Let $D_0 = D^2 \setminus x_0$, $x_0 \in \partial D^2$ (see Figure 2). Then D_0 can be thought of as a non-compact 2-manifold with boundary $\partial D_0 = \partial D^2 \setminus x_0$. Let $\mathbf{B_0}$ be a 3-ball with a point removed from its boundary; its boundary consists of 2 copies of D_0, which we will call $\partial_- \mathbf{B_0}$ and $\partial_+ \mathbf{B_0}$. A lamination $\mathcal{L}$ is <u>end-incompressible</u> if for every (proper) embedding $(D_0, \partial D_0) \hookrightarrow (M|\mathcal{L}, \partial(M|\mathcal{L}))$ there is a proper embedding $\partial \mathbf{B}_0 \to M|\mathcal{L}$ with $\partial_- \mathbf{B}_0 = D_0$ and $\partial_+ \mathbf{B}_0 \subseteq L$.

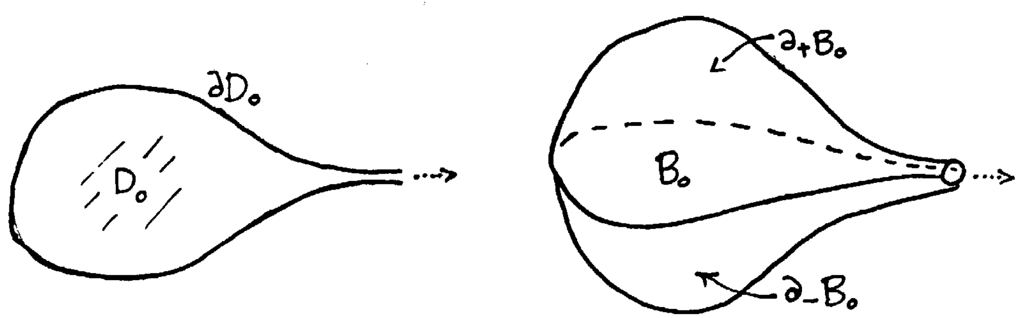

Figure 2: End-incompressibility

For example, a 2-sided incompressible, ∂-incompressible surface F (often called an essential surface) in an irreducible 3-manifold is an essential lamination; (1) is proved by way of the Loop Theorem [Pa], (2) is a straightforward exercise in 3-manifold topology, and (3) is vacuous (there don't exist any proper maps $D_0 \to M|F$, since $M|F$ is compact). Also a codimension-one foliation $\mathcal{F}$ of M without Reeb components (often called a taut foliation; see [No] for a definition) is an essential lamination ; (1) is proved in [No], and (2) and (3) are vacuous ($M|\mathcal{F}$ is empty). Other examples include the suspension of the stable lamination of a pseudo-Anosov surface diffeomorphism; see [G-K] for these and other examples.

A sublamination of $\mathcal{L}$ is a closed, saturated subset of $\mathcal{L}$ (and hence is also a lamination). It is not hard to see [G-O] that a sublamination of an essential lamination is essential. It is also worth noting now that if a lamination $\mathcal{L}$ is essential, then the lamination $\mathcal{L}_0$ obtained by splitting $\mathcal{L}$ along a leaf is still essential; the inclusion- and projection-induced homomorphisms

$$\pi_1(\partial N(L)) \hookrightarrow \pi_1(N(L)) \hookrightarrow \pi_1(L) \hookrightarrow \pi_1(M)$$

are all injective, so every leaf still π_1-injects, and $M|\mathcal{L}_0$ differs from $M|\mathcal{L}$ by the I-bundle N(L), which is irreducible and end-incompressible.

The notion of an essential lamination arose, in fact, out of a conscious attempt to find a suitable generalization of the incompressible surface and the taut foliation. These objects have served as useful 'backdrops' against which to understand the topology of the 3-manifold which contains them. The success of this generalization can be demonstrated by the following theorem:

Theorem [G-O]: If a 3-manifold M contains an essential lamination $\mathcal{L}$, then

(1) M is irreducible,

(2) $\pi_1(M)$ is infinite,

(3) the universal cover of M, $\widetilde{M}$, is homeomorphic to $\mathbf{R}^3$.

If M contains a 2-sided incompressible surface, (1) and (2) are basically by definition, and (3) was proved by Waldhausen in [Wa1]; if M admits a taut foliation, (1) was proved in [Ro], (2) in [No], and (3) in [Pa].

C. Seifert-fibered Spaces

A good reference for this section is [Ha 2].

A Seifert-fibered space is a compact 3-manifold M which admits a foliation by circles, called the Seifert-fibering of M, having the following structure. Each circle fiber γ has a saturated solid torus neighborhood whose induced Seifert-fibering is a model fibering; the foliation of the neighborhood is homeomorphic to the fibering of a solid torus obtained by taking a solid cylinder $D^2 \times I$ and gluing its ends together by a $2\pi p/q$-rotation about the center $\underline{0}$ of the disk. The foliation of the cylinder by vertical intervals gives the foliation of the solid torus by circles.

In a model fibering there is a core circle (from $\underline{0} \times I$), which generates the fundamental group of the solid torus; all other fibers represent $q \times$(generator). Fibers of M whose model neighborhoods have $q \neq 1$ are called exceptional fibers (or multiple fibers); they are isolated in M. All other fibers are called regular fibers of M.

The quotient space of M obtained by crushing each fiber to a point is a compact surface F (the quotient of a (model-fibered) neighborhood of a fiber is a disk, so every point of F has a disk neighborhood); it is called the base surface of the Seifert-fibering of M. Under the quotient map $\pi : M \to F$, the images of the multiple fibers form a discrete set in F, the set of multiple points of F; there are therefore only finitely many of them. All other points of F are called regular points.

There are two other ways to view a Seifert-fibered space, which take a more constructive, rather than existential, approach, and are therefore more useful to us in carrying out the constructions involved in proofs.

If we remove the interiors of small disks Δ_i around a collection of points p_i in F which contains all of the multiple points, giving F_0, then $\pi^{-1}(F_0) = M_0$ is a Seifert-fibered space with base F_0 and no multiple fibers. The projection $\pi{:}M_0 \to F_0$ therefore gives a circle-bundle (see [Ha 2]); and M can be recovered by gluing solid tori to the torus boundary-components of M_0 (these are solid tori $= \pi^{-1}(\Delta_i)$, not solid Klein bottles, because M is assumed orientable). This gives us a new way to think of a Seifert-fibered space: it is obtained from the total space of a circle-bundle over a compact surface by doing Dehn-filling on some (possibly empty) collection of boundary-tori of the bundle.

This approach also gives us a way to assign numerical invariants to a Seifert-fibered space, namely the Dehn-surgery coefficients for the Dehn-fillings. More precisely, if we choose a section $F_0 \hookrightarrow M_0$ of the circle bundle, the images of the boundary components give loops η_i in each of the boundary tori of M_0. These, together with a fiber ν_i of the bundle in each torus, yield a basis for the fundamental group of each torus. Dehn filling is uniquely determined by the isotopy class that the boundary of the meridian μ_i of the solid torus is glued to; this can be expressed in terms of our basis as $[\mu_i] = a_i[\eta_i] + b_i[\nu_i]$, where a_i and b_i are relatively prime.

There is some question of signs involved; one uses an orientation of M to make a consistent choice. An orientation of M makes the ratios a_i/b_i well-defined. These numbers form the major part of the Seifert invariants of M: we write

$$M = \Sigma(\pm g, b; a_1/b_1, \ldots, a_n/b_n)$$

where g=genus of F=genus of F_0 (+ if F is orientable, - if not), b=number of ∂-components of M=number of ∂-components of F, and the a_i, b_i are as above.

The above invariant is not uniquely determined by M; there is an ambiguity present in how many solid tori you drill out of M (there's nothing really wrong with removing the neighborhood of alot of regular fibers of M), and in the choice of section for the bundle M_0. These choices alter the Seifert invariant in the following ways (see [Ha 2]):

(0) re-ordering the points p_i changes the a_i/b_i by the corresponding permutation,

(0′) changing the orientation of M makes the a_i/b_i change sign,

(1) adding a regular point to the list p_i adds a $0/1$ to the Seifert invariant (by drilling a hole out of the section),

(2) you can delete a $0/1$ from the Seifert invariant (it corresponds to a regular point, and the section $F_0 \hookrightarrow M_0$ can be extended over the Dehn-filled solid torus,

(3) choosing a different section for the same bundle has the effect of adding integers to the Seifert-invariants,

$$a_i/b_i \longmapsto (a_i + k_i b_i)/b_i,\ k_i \in \mathbb{Z}$$

but (if M is closed) leaves the sum $\sum_{i=1}^{n} a_i/b_i$ unchanged.

As a consequence of (0)-(3), for closed M the sum $\sum_{i=1}^{n} a_i/b_i$ is in fact an invariant (up to sign) of the Seifert-fibering of M; it is called the Euler number, e(M), of M.

This ambiguity also allows us to define a normalized Seifert invariant for M:

$$M = \Sigma(\pm g, b; k, a_1/b_1, \dots, a_n/b_n)$$

where g,b are as above, k is an integer, and $0 < a_i/b_i < 1$, for all i. One just uses (3) to collect the integer parts of the Dehn surgery coefficients together into a single number. This invariant is well defined up to a permutation of the a_i/b_i, and a change of orientation of M; this change replaces k by n-k and a_i/b_i by $(b_i - a_i)/b_i$.

There is yet another way to view a Seifert-fibered space, one which, apart from the Seifert invariants just described, will in fact constitute our only working point of view in our proofs. We will think of a Seifert-fibered manifold M as a collection of solid tori glued together along their boundaries.

This view can be obtained from the previous one. Consider the base surface F of the Seifert-fibering; it is a compact surface, with or without boundary. It will be convenient to have two somewhat different kinds of representations of M as solid tori, depending on whether F (i.e., M) has non-empty boundary or not.

If $\partial F \neq \emptyset$, then it is well-known that F can be split open along a collection of disjoint properly-embedded arcs α_j to yield a disk: $\Delta = F \setminus \bigcup \mathrm{int}(N(\alpha_j)) = D^2$. By general position, we can assume these arcs miss the multiple points of M in F. If we then add more disjoint arcs α_k in Δ which each split off a disk containing a single multiple point, their union gives a disjoint collection of properly embedded arcs which splits F into a collection Δ_i of disks each containing *at most* one multiple point of M.

Then the $\pi^{-1}(\alpha_j) = A_j$ form a collection of annuli in M, which split M into a collection $\pi^{-1}(\Delta_i) = M_i$ of solid tori (they are Seifert-fibered spaces with base D^2 and at most one multiple fiber), which intersect along the annuli A_j in their boundaries.

If $\partial F = \emptyset$, then we don't have a boundary component to put the ends of arcs in. So we split M up in a different way. Choose a triangulation τ of F, in general position with respect to the collection of multiple points (i.e., so that every multiple point is in the interior of some 2-simplex), so that each 2-simplex contains at most one multiple point. Then, like the above, every 2-simplex Δ_i^2 has inverse image $\pi^{-1}(\Delta_i^2) = M_i$ a solid torus, and these solid tori meet along the inverse image of the 1-skeleton of τ, which meets each solid torus in its boundary.

The inverse images of the points of $\tau^{(0)}$ =the 0-skeleton of τ form a finite collection S of regular fibers of M in the boundary of the solid tori (they in fact constitute the points where three or more of the solid tori meet). These fibers will be of central importance to us (as we will eventually see), so we will give them a special name; we will call them the sentinel fibers of M (for reasons which will eventually become apparent).

D. Incompressible Surfaces in Seifert-fibered Spaces

Our primary interest in this thesis is to understand what an essential lamination in a Seifert-fibered space looks like. Since the essential lamination represents a generalization of the incompressible surface, it will serve us well, as a guide, to understand what an incompressible surface in a Seifert-fibered space looks like.

In [Wa 2], Waldhausen showed that a 2-sided incompressible surface T in a Seifert-fibered space $\pi : M \to F$ is isotopic to a surface which is either vertical or horizontal. A surface $T \subseteq M$ is called vertical if $T = \pi^{-1}(\pi(T))$, i.e., T is saturated: it is the full preimage of a collection circles and arcs in F, so any fiber of M which meets T is contained in T. T is therefore foliated by circles, so it consists of annuli, Möbius bands, tori, and Klein bottles. A surface $T \subseteq M$ is called horizontal (or transverse) if T is transverse to the circle fibers of M at every point. Therefore, $\pi|_T : T \to F$ is a branched cover of F, branched over the multiple points of F.

This theorem leads directly to a complete determination of when a Seifert-fibered space M contains an essential surface. A vertical essential surface T exists if and only if either the base F of M is not a sphere or disk, or, if F is a sphere or disk, M has at least 2 multiple fibers if $F = D^2$, or at least 4 multiple fibers if $F = S^2$. A vertical (connected) surface is essential unless, if $\partial T = \emptyset$, the projection circle of T bounds a disk in F containing at most 1 multiple point of F, or, if $\partial T \neq \emptyset$, the projection arc of T cuts off a disk from F which contains no multiple point. Every horizontal surface, on the other hand, is essential, and a horizontal surface exists provided either $\partial M \neq \emptyset$, or the Euler number, e(M), of M is =0.

One of the main goals of this thesis is to provide a classification scheme, similar to Waldhausen's, for essential laminations. In particular, we intend to show that the analogue of Waldhausen's theorem holds true for essential laminations in a Seifert-fibered space. In anticipation of this, we will therefore extend the definitions above to apply to laminations as well; a lamination $\mathcal{L} \subseteq M$ is <u>vertical</u> if every circle fiber of M which meets $\mathcal{L}$ is contained in $\mathcal{L}$; $\mathcal{L}$ is <u>horizontal</u> if it is transverse to every fiber of M at every point.

Notice that these are in fact somewhat local conditions; in our splittings of Seifert-fibered spaces into (vertical) solid tori, we will be able to tell, solid torus by solid torus, if a given lamination is vertical or horizontal or not.

In practice, we will (especially in the statements of theorems) abuse this notation somewhat, and say that a lamination is horizontal or vertical if it is <u>isotopic</u> to one which is. This should cause no confusion.

E. Some Basic Constructions

This section will describe some of the basic properties of laminations and essential laminations which will be utilized in this thesis. These properties show that laminations are not as 'out of control' as their generality might suggest. In fact they share many of the nice properties that compact surfaces in 3-manifolds do. One reason for this is the existence of an associated branched surface.

a. Transversality.

Let F$\subseteq$M be a compact surface properly embedded in a 3-manifold M and $\mathcal{L}\subseteq$M a lamination carried by a branched surface B. we say that $\mathcal{L}$ is <u>transverse</u> to F if for every point $x \in \mathcal{L}\cap$F, there is a coordinate neighborhood $h : \mathrm{I}^2 \times \mathrm{I} \rightarrow$U for $\mathcal{L}$ around x so that h^{-1}(F $\cap$ U)=r$\times$I $\times$ I, where $h^{-1}(x)$ =(r,s,t) (see Figure 3).

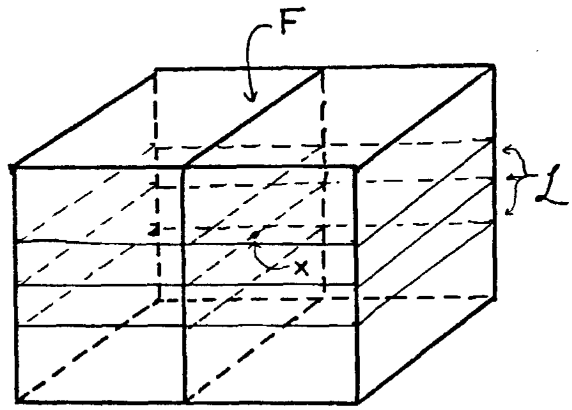

Figure 3: Transversality

Achieving transversality is most easily done by choosing a branched surface B carrying $\mathcal{L}$ (see [G-O]) and making B transverse to F; then any lamination (e.g. $\mathcal{L}$) carried by B will be transverse to F, too (in the sense that, for $x \in \mathcal{L}\cap$S, $T_x\mathcal{L}$ and T_xF <u>*span*</u> T_xM). The notion of making B transverse to F makes sense if you think of a branched surface as a compact 2-dimensional simplicial complex with a well-defined tangent plane at each point. Then standard general position arguments (inducting on dimension, triple points to branch curves to sectors of B) show that transversality can be achieved by arbitrarily small isotopies of B.

b. Monogons.

Until you work with essential laminations for awhile it may seem odd that, although the condition of end-incompressibility is meaningless for an essential surface or taut foliation, it was deemed necessary to add this condition to the definition of an essential lamination, in order to have an 'appropriate' generalization of these two more classical objects. It is, in fact, part of the definition more or less by 'accident', since from the original point of view, via branched surfaces, this property was achieved 'for free'. It is, however, an important part of the definition, so tools for detecting end-compressing disks and for realizing when they cannot exist will be useful.

In practice, when we build end-compressing disks, we in fact do it in a rather special way; we build what we will call a tail-compressing disk. A tail-compressing disk is a properly embedded disk $D_0 \subseteq M|\mathcal{L}$ with an arc $\alpha \subseteq D_0$ splitting D_0 into a disk Δ and a 'tail' R, with $R \subseteq N(B)$ (for some branched surface carrying $\mathcal{L}$); the tail is foliated by arcs parallel to α, each contained in an I-fiber of N(B). Because D_0 is properly embedded in $M|\mathcal{L}$, it follows that the lengths of these arcs are tending to 0.

Gabai and Oertel [G-O] have shown (utilizing an argument due to Hatcher) that the existence of an end-compressing disk implies the existence of a tail-compressing disk (if $\mathcal{L}$ is π_1-injective), and that a tail-compressing disk is an end-compressing disk for $\mathcal{L}$. Therefore, the notions of end-incompressibility and tail-incompressibility (i.e., the non-existence of any tail-compressing disks) are the same. The latter, however, is often easier to verify or disprove.

For example, this equivalence makes it easy to show that if $M_0 \subseteq M$ is a codimension-0 submanifold of M and if $\mathcal{L}$ is an essential lamination transverse to ∂M_0, then $\mathcal{L}_0 = \mathcal{L} \cap \partial M_0$ is end-incompressible in M_0; this is because any tail-compressing disk for $\mathcal{L}_0$ w.r.t. the branched surface $B_0 = B \cap M_0$ is a tail-compressing disk for $\mathcal{L}$ w.r.t. B.

Now for a lamination $\mathcal{L} \subseteq M$ carried by a branched surface B (having no compact bundles w.r.t. $\mathcal{L}$) and γ a loop transverse to $\mathcal{L}$ (i.e., to B), we can define a number

ϵ, called a monogon number for $\mathcal{L}$ w.r.t. γ, in terms of the branched surface B, as follows:

N(B) meets γ in a collection of vertical fibers, and $\mathcal{L} \cap \gamma$ is contained in these subarcs of γ; we let $\epsilon = 1/2$ of the smallest distance (along γ) from one of these subarcs to another. It then follows that any two points of $\mathcal{L} \cap \gamma$ which are within ϵ of one another are contained in the same vertical fiber of N(B).

The following lemma will be used many times in this thesis; it serves as one of the most common ways for us to show that certain situations and configurations which arise in the course of our proofs are in fact impossible. Its statement is highly technical and includes a great many hypotheses which at this time must remain mysterious; these hypotheses will with luck come to seem natural and self-evident to the reader as more and more occasions arise to apply the lemma in our proofs.

Monogon Lemma (Lemma 1.1): Suppose I_t is an isotopy of M and $\mathcal{L}$ a lamination. Suppose there is a disk D in M, transverse to $I_1(\mathcal{L})$, with $\partial D = \beta \cup \delta$, $\beta \subseteq$ a leaf of $I_1(\mathcal{L})$, δ contained in a loop γ transverse to $\mathcal{L}$, and suppose that I_t is constant on $I_1(\mathcal{L}) \cap \delta$ and the length of δ is $< \epsilon$, where ϵ is a monogon number for $\mathcal{L}$ w.r.t γ. Then $\mathcal{L}$ is not an essential lamination.

Proof: The idea is fairly straightforward; assume that $\mathcal{L}$ is essential, and use the information given to build an (tail-, hence) end-compressing disk.

$I_1(\mathcal{L}) \cap D = \lambda$ is a 1-dimensional lamination in the disk D; after an isotopy of D supported away from ∂D, we may assume λ does not contain any (trivial) loops. So λ consists of a collection of (∂-parallel) arcs in D with endpoints in δ. Let α be an outermost arc in λ (i.e., one furthest from β). α splits off a disk Δ from D which meets δ in a subarc δ_0; in particular, Δ does not meet $\mathcal{L}$ in its interior. Let y_0, y_1 be the endpoints of δ_0.

Now let Δ flow back along the isotopy I_t (i.e., consider the isotopy I_{1-t} of M, and Δ's behavior under it). Because $y_0, y_1 \in I_1(\mathcal{L}) \cap \delta$, this isotopy is constant on these two points, and so we may assume that it is constant on small neighborhoods of these points (by altering I_t in a trivial way). This gives an embedded disk $\Delta_0 = I_1^{-1}(\Delta)$ in M meeting $\mathcal{L}$ in an arc $I_1^{-1}(\alpha)$ of its boundary.

If we add onto this disk the track, under the isotopy, of the short arc δ_0, we obtain a singular disk $f{:}D^2 \to M$ representing a homotopy, rel endpoints, between an arc of $\mathcal{L}$ and the short arc $\delta_0 \subseteq \delta$, which is transverse to $\mathcal{L}$ near its boundary, and which has a subarc γ of its boundary mapping into $\mathcal{L}$. We can, therefore, by a small homotopy of f supported away from ∂D^2, make f transverse to $\mathcal{L}$. The inverse image of $\mathcal{L}$ is a lamination in a disk, which (by homotopy of f supported away from ∂D^2) can be made to consist of arcs. An outermost arc γ_0 (i.e., one furthest from γ) splits off disk D_0 missing γ; f restricted to this disk represents a homotopy, in $M|\mathcal{L}$, from the arc $f(\gamma_0) \subseteq \mathcal{L}$ to a subarc of $\delta_0 \subseteq \delta$, which therefore has length $< \epsilon$. But this situation violates the fact that $\mathcal{L}$ has monogon number ϵ; this homotopy, together with a tail in $N(B)|\mathcal{L}$, allows us to demonstrate that $\mathcal{L}$ is not essential.

To see this, look at the image of D_0 in $M|\mathcal{L}$. It is a compact set, so by a finite splitting of B, we can arrange that this image is contained in $M\backslash\mathrm{int}(N(B))=M_0$, and we can further assume (after perhaps a bit more splitting) that B is an essential branched surface. There is a first arc α of the tail which is contained in N(B) (because the tail is properly embedded in $M|\mathcal{L}$, so cannot be contained in M_0, because M_0 is compact); it therefore lies in $\partial_v N(B)$. D_0 together with the compact part of the tail that α splits off form a homotopy disk in $M_0$with boundary in ∂M_0, meeting $\partial_v N(B)$ in a single I-fiber. This boundary therefore has intersection number one with the core circle or arc of $\partial_v N(B)$, and therefore represents a non-trivial element of the first homology of $\partial N(B)$, and therefore is non-trivial in the fundamental group of ∂N(B). By the Loop Theorem , there is an embedded such loop α_0, non-trivial in $\pi_1\partial(N(B))$, bounding a disk in M_0. Furthermore (see [Hn] for this stronger statement), we may assume that either α_0 misses $\partial_v N(B)$ or it contains the arc α. But in the first case this gives a compressing disk for $\partial_h(N(B)$, and in the second gives a monogon for B, both contradicting the fact that B is an essential branched surface ∎

c. Making intersections taut, I: surfaces

In the course of our proofs, we will continually be isotoping essential laminations $\mathcal{L}$, making them transverse to surfaces T$\subseteq$M, and among other things looking at the intersections $\lambda = \mathcal{L} \cap T$ which occur; this intersection is a 1-dimensional lamination in T. All of the surfaces we will be interested in will either be compact (meridional) disks (in our solid tori), or vertical surfaces in our Seifert-fibered space; they are the inverse images of arcs or circles in the base space F, and hence are annuli or tori.

Any trivial loop γ in λ, i.e., one which bounds a disk in T, will then be null-homotopic in M; because $\mathcal{L}$ is essential, γ then bounds a disk in $\mathcal{L}$. There can therefore be no holonomy around γ in λ (see [No] for the notion of holonomy), because there can be no holonomy in $\mathcal{L}$ around the boundary of a disk in $\mathcal{L}$. It then follows from Reeb Stability [Re] that the collection of trivial loops τ in λ is an open and closed set in λ. Therefore $\lambda_0 = \lambda \setminus \tau$ is also a lamination in T, and by construction it contains no trivial loops.

As it turns out, such 1-dimensional laminations are not too complicated. First consider the case T= T^2 =torus. λ_0 cannot have any monogons (a monogon for λ_0 in T would be a monogon for $\mathcal{L}$ in M, since $\mathcal{L}$ is assumed transverse to T), and every compact (i.e., non-simply-connected) leaf is essential in T. So λ_0 is 'essential' in T. Now, there is a folk theorem (attributed to Thurston) which says that every essential lamination in a surface contains a measured sub-lamination, which, in the torus, consists of either parallel compact circles, or a lamination with 'irrational slope' consisting of non-compact leaves. This can also be established by using an Euler χ argument (see [Ca]) to show that λ_0 can be completed to a foliation of the torus T, and then use facts from dynamical systems about foliations of the torus (see, e.g., [H-H]) to show the existence of one of these sublaminations. This implies that there are only three kinds of behavior in T: either

(a) λ_0 contains no compact leaves; it then contains an irrational (measured) sublamination, and all other leaves are parallel to leaves of this lamination. In particular, such a lamination can be isotoped to be transverse to any foliation of T by compact loops (just make it transverse to one of them, split T open to an annulus, and apply the result below for laminations in a annulus), or

(b) λ_0 contains compact leaves. The collection of compact leaves then form a (closed) subset of λ_0, and all other leaves lie in the annular regions between two of the leaves, and are either of (1)'Reeb' type or (2) 'not' (see Figs. 4ab); leaves of the second type are sometimes called 'Kronecker' (for 'Kronecker flow').

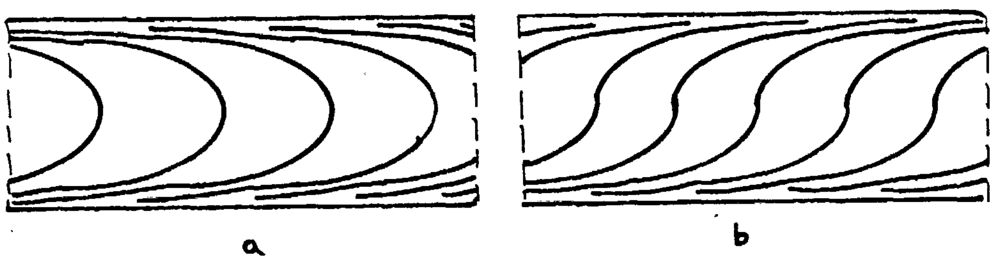

Figure 4: Laminations in a 2-torus

A lamination in a disk is even easier; there can be no non-compact leaves in $\lambda_0 = \mathcal{L} \cap D^2$. The collection of compact arcs is open and closed in D^2 (for the same reasons as above), so the collection of non-compact leaves would form a sublamination λ_1 of λ_0. If any leaf of λ_1 missed ∂D^2, then its closure would be a lamination in the interior of D^2; but any such lamination would admit a monogon (see [Ca] or the proposition in Chapter 2.B below), giving an end-compressing disk for $\mathcal{L}$. If a leaf ℓ of λ_1 did meet ∂D^2, then $\bar{\ell} \setminus \ell$ is a sublamination of λ_1 which does miss ∂D^2. So every leaf of λ is a compact arc; these (because they are carried by a train track) fall into a finite collection of parallel families in D^2.

In the case that λ_0 is in an annulus A, there will then be some (possibly empty) collection of ∂-parallel arcs in λ_0; this collection will be open and closed in λ_0, and so its complement will be a lamination λ_1 in the annulus. If this lamination contains a compact arc, it is essential; it splits A into a disk, so the above implies that <u>all</u> of the leaves of λ_1 are compact arcs, hence parallel in A.

If λ_1 is contained in the <u>interior</u> of an annulus, then it contains a (vertical) compact loop; just think of the annulus as sitting in a torus (in a π_1-injective way), then case (a) above cannot occur because the lamination is disjoint from the complementary annulus.

If λ_1 still meets ∂A, then a leaf doing so will be non-compact; arguments like the above (for the disk D^2) lead us to conclude that λ_1 then contains a sub-lamination which misses ∂A; it therefore also contains a (vertical) essential loop.

Summing up, after removing trivial loops and ∂-parallel arcs (where applicable) from a 1-dimensional lamination $\lambda \subseteq T$, we can completely determine what λ looks like, if T is a torus, disk, or annulus (the only surfaces which we will be concerned with in this thesis). In a torus T λ will either contain compact leaves with Reeb or Kronecker leaves in between, or can be made transverse to any foliation of T by compact circles. In a disk, λ will be empty; and in an annulus, λ will either consist of parallel essential arcs, or will contain a collection of vertical loops (with Reeb or Kronecker leaves in between).

d. Recognizing good laminations in a solid torus.

Let $\mathcal{L}$ be an essential lamination in the 3-manifold M, and let M_0 be a solid torus in M. By a small isotopy of $\mathcal{L}$ we can arrange that $\mathcal{L}$ is transverse to M_0 (this amounts to making $\mathcal{L}$ transverse to ∂M_0). Then $\mathcal{L} \cap M_0 = \mathcal{L}_0$ is a lamination in M_0. This lamination is almost certain not to have π_1-injective leaves. However, this lack of π_1-injectivity, basically, lives in the boundary $\partial\mathcal{L}_0 = \mathcal{L} \cap \partial M_0$, as the following lemma shows:

Lemma 1.2: Let $\mathcal{L}$, M_0. and $\mathcal{L}_0$ be as above, with $M \setminus \mathrm{int}(M_0)$ irreducible. If every embedded loop γ_0 in $\partial\mathcal{L}_0$ which is null-homotopic in M_0 bounds a disk in $\mathcal{L}_0$, then every leaf of $\mathcal{L}_0$ is π_1-injective in M_0.

Proof: This lemma is a little strange, since, as stated, it is in fact true, but also, as stated, it seems impossible to prove. The problem is very small, but to get around it we will need to first alter $\mathcal{L}$slightly; this has the effect of adding a (hidden) hypothesis to the lemma.

Look at the collection τ of loops in $\partial\mathcal{L}_0 = \mathcal{L}_0 \cap \partial M_0$ which are trivial in ∂M_0. These loops, by hypothesis are trivial in the leaves of $\mathcal{L}_0$ which contain them, and therefore bound (a collection $\mathcal{T}$ of) boundary-parallel disk leaves in $\mathcal{L}_0$. By a Reeb Stability argument, this collection $\mathcal{T}$ forms an open and closed set in $\mathcal{L}_0$, and so is a sublamination of $\mathcal{L}_0$, and so τ is a sublamination of $\partial\mathcal{L}_0$. τ therefore consists of a finite number of parallel families of trivial loops in ∂M_0, bounding parallel families of ∂-parallel disks in M_0. We can then by an isotopy of $\mathcal{L}$ (choosing an outermost family of disks (meaning an innermost family of loops) and working in) remove these families of disks from $\mathcal{L}_0$. Since $\mathcal{T}$ is closed in $\mathcal{L}_0$, nothing else is changed, so after the isotopy $\mathcal{L}_0$ has been altered to $\mathcal{L}_0 \setminus \mathcal{T}$, i.e., $\partial\mathcal{L}_0$ no longer contains any loops trivial in ∂M_0. We will prove the lemma for this altered lamination; note that this does succeed in proving it for the original lamination, too, since they only differ by a bunch of disk leaves, which clearly π_1-inject.

Let γ be a (singular) loop in a leaf L_0 of $\mathcal{L}_0$, which is null-homotopic in M_0. Then γ is also null-homotopic in M. Since L_0 is contained in a leaf L of $\mathcal{L}$, and this leaf is π_1-injective in M, it follows then that γ is null-homotopic in L. Let $H : D^2 \to L$, be a null-homotopy, and make it transverse to ∂M_0. Then $\Gamma = H^{-1}(\partial M_0)$ is a (finite) collection of circles in a disk D^2. Consider a circle γ_0 of Γ innermost in D^2, and consider the leaf l_0 of the lamination $\lambda_0 = \mathcal{L}_0 \cap \partial M_0$ which H maps it into. This leaf is homeomorphic to either S^1 or $\mathbf{R}$. If it is homeomorphic to $\mathbf{R}$, then γ_0 is null-homotopic in l_0, and so by redefining H on the disk Δ_0 of D^2 cut off by γ_0 so that it maps into l_0 (and then homotoping H off of l_0 slightly), we get a new null-homotopy for γ_0 with fewer circles of intersection in Γ. If l_0 is a circle, then one of two things will be true. In the most usual case γ_0 again maps into l_0 null-homotopically, in which case we proceed as before, finishing the proof by induction. When γ_0 is essential in l_0, we must use a different argument which avoids the induction.

Because γ_0 is innermost, it bounds a disk Δ_0 in D^2 which misses Γ, so the image of Δ_0 under H misses ∂M_0, and hence maps into M_0 or $M_1 = M \setminus \mathrm{int}(M_0)$. So some non-trivial power of l_0 is null-homotopic in M_0 or M_1.

If γ_0 is null-homotopic in M_1, this means that the torus ∂M_1 is <u>compressible</u> in M_1. Because M_1 is irreducible by hypothesis, it follows (see [Ha 2]) that M_1 is in fact a solid torus. This implies that our original 3-manifold M is a union of two solid tori glued along their boundary, and hence is a lens space. But this is impossible, since a lens space cannot contain an essential lamination (it does not have universal cover $\mathbf{R}^3$).

If γ_0 is null-homotopic in M_0, then because $\pi_1(M_0)$ is torsion-free (it's $\mathbf{Z}$), l_0 is also null-homotopic in M_0, and therefore bounds a disk leaf of $\mathcal{L}_0$, by hypothesis.

By our additional hypothesis, this disk is not ∂-parallel in M_0, so it must be essential in M_0; in particular, $\mathcal{L}_0$ contains a meridian disk leaf. Now consider the collection μ of meridian loops of $\partial\mathcal{L}_0$. By hypothesis, these loops bound a collection $\mathcal{M}$ of meridian disk leaves of $\mathcal{L}_0$. Again, Reeb Stability implies that this collection $\mathcal{M}$ is closed in $\mathcal{L}_0$, so μ is closed in $\partial\mathcal{L}_0$. But then the leaves of $\partial\mathcal{L}_0$ not in μ live in the annular regions between loops of μ; they cannot be compact (they would then be trivial or meridional), but they cannot be non-compact, because they would have to limit on μ, giving non-trivial holonomy around a loop which bounds a disk. Therefore, $\mathcal{M} = \mathcal{L}_0$, so every leaf of $\mathcal{L}_0$ is a meridional disk, which obviously π_1-injects. ∎

e. Making intersections taut, II: solid tori.

Because a Seifert-fibered space can be thought of as a union of solid tori, which meet along their boundaries, it will also be useful to have a general procedure to isotope an essential lamination $\mathcal{L}$ so that it meets a vertical solid torus M_0 in a Seifert-fibered M in a lamination, $\mathcal{L} \cap M_0 = \mathcal{L}_0$, which has π_1-injective leaves. We will show later that such a lamination $\mathcal{L}_0$ in fact has a rather simple structure; this structure theorem will then be exploited to give our structure theorems for essential laminations in Seifert-fibered spaces.

Now there is in fact a very easy way to do this: just think of a solid torus M_0 as a regular neighborhood of its core circle γ_0, make γ_0 transverse to a branched surface carrying $\mathcal{L}$, and then $\mathcal{L} \cap M_0$ will be a collection of meridian disks in M_0, which certainly has π_1-injective leaves.

Unfortunately, this is a far too destructive process for our uses (it loses alot of the information that we will be gathering in the proofs of our theorems), as well as being far too unimaginative to be useful. Instead we will construct an isotopy which is much more 'conservative' (and which, incidentally, allows much more interesting laminations $\mathcal{L} \cap M_0$ to be created).

We have seen already that in order to make a lamination meet a (nice) solid torus M_0 in a π_1-injective lamination $\mathcal{L}_0=\mathcal{L}\cap M_0$, we need only arrange that any loop of $\partial\mathcal{L}_0$ which is null homotopic in M_0 bounds a disk in $\mathcal{L}_0$. What we will now do is to describe an isotopy process which, given an essential lamination, will arrange exactly that.

First we deal with trivial loops of $\lambda_0 = \partial\mathcal{L}_0$. If $\partial\mathcal{L}_0$ contains loops which are trivial in ∂M_0, then because $\mathcal{L}$ is essential (in particular, there is no non-trivial holonomy around loops in $\mathcal{L}$ which are null-homotopic in M), the collection C of such loops in ∂M_0 is open and closed in $\partial\mathcal{L}_0$, and (by transversality) consists of a finite number of families of parallel loops in $\partial\mathcal{L}_0$.

Now take an *outermost* loop γ of an *innermost* family of trivial loops. γ bounds a disk D in ∂M_0, and a disk D_0 in the leaf of $\mathcal{L}$ containing it, and they are isotopic, rel γ (because M is irreducible). An (ambient) isotopy of $\mathcal{L}$ taking D_0 to D and a bit beyond has the effect of removing the family of loops containing γ from λ_0 (and possibly more). To be more exact, such an isotopy must be done in stages, since it is not immediate that $D_0 \cap D=\gamma$; it could consist of (a finite number of) loops in D_0 (one then argues from innermost out). Then by induction on the number of parallel families in all the λ_i, we can assume that $\mathcal{L}\cap F$ contains no trivial loops.

Now if $\partial\mathcal{L}_0$ still contains loops which are null-homotopic in M_0, then these loops must be meridional, i.e., bound disks in M_0 but not in ∂M_0. What we first must establish is that at least one of these loops in $\partial\mathcal{L}_0$ bounds a disk leaf of $\mathcal{L}_0$.

Choose a meridional loop γ of $\partial\mathcal{L}_0$. Because $\mathcal{L}$ is essential, this (embedded) loop bounds a disk D in $\mathcal{L}$. Consider the intersection $D \cap \partial M_0 \subseteq \partial\mathcal{L}_0$; this intersection consists of (a finite number of) closed loops. Choose an innermost such loop γ_0 in D, bounding a disk Δ in D (possibly $\gamma_0 = \gamma$).

Claim: Δ is contained in M_0.

If not, then $\Delta \subseteq M\backslash\mathrm{int}(M_0)$ (because γ_0 is innermost). But then, as before, if γ_0 is essential in ∂M_0, then Δ represents a compressing disk for $\partial(M\backslash\mathrm{int}(M_0))$. Therefore, $M\backslash\mathrm{int}(M_0)$ is a solid torus, making M a lens space, a contradiction (it couldn't contain $\mathcal{L}$). So γ_0 must be trivial in ∂M_0. But this contradicts the fact that we have already isotoped $\mathcal{L}$ so that $\partial\mathcal{L}_0$ contains no loops trivial in ∂M_0, thus establishing the claim.

Therefore there is a disk Δ in $\mathcal{L}_0 \subseteq M_0$ with boundary a loop $\gamma_0 \subseteq \partial\mathcal{L}_0 \subseteq \partial M_0$. Consider now the collection $\mathcal{M}$ of meridian disk leaves of $\mathcal{L}_0$. Reeb Stability implies that this collection is open and closed in M_0, as before. Moreover, because $\mathcal{M} \neq \emptyset$, the lamination $\mathcal{L}_1 = \mathcal{L}_0\backslash\mathcal{M}$ must have $\partial\mathcal{L}_1$ consisting of meridional loops; it cannot contain any trivial loops, by construction, and any non-compact leaf of $\partial\mathcal{L}_1$ would have to limit on a meridional loop, implying non-trivial holonomy around a loop null-homotopic in a leaf of $\mathcal{L}$, a contradiction.

Every leaf of $\mathcal{L}_1$ has more than one boundary component; if a leaf had only one and were compact, then it would contain a non-separating loop contained in the ball $M_0\backslash\Delta$, implying the leaf of $\mathcal{L}$ containing it was not π_1-injective in M. If the leaf were non-compact, then the limit set of an end (see [Ni] for a definition) would be a lamination which did not meet ∂M_0; it would then be contained in the interior of a ball, implying the existence of an essential lamination in a sphere, which is impossible.

This implies that, although $M_0|\mathcal{M}$ is a possibly infinite collection of balls, only finitely many of them can contain any leaves of $\mathcal{L}_1$. To see this, look at a loop γ having intersection number 1 with each loop of the meridional lamination $\partial\mathcal{L}_0$. If there are an infinite number of regions containing leaves of $\mathcal{L}_1$, then there are an infinite number of (distinct) arcs of $\gamma|\mathcal{M}$ meeting these leaves. Such a collection of arcs must have their lengths tending to 0. If we look at the top endpoints (in some orientation of γ) of these arcs, we have an infinite sequence of (distinct) points in $\partial\mathcal{M}$, which (because $\mathcal{M}$ is closed) must limit on some point of $\gamma \cap \mathcal{M}$. This point therefore lies in a meridional disk leaf D of $\mathcal{L}_0$; therefore by Reeb Stability, all nearby leaves are also meridian disks. But the top endpoints of the arcs are limiting on this leaf, and the lengths of the arcs are tending to zero (so the bottoms endpoints are limiting on D, too), implying that these non-disk leaves pass arbitrarily close to D, a contradiction.

Now look at a component N of $M_0|\mathcal{M}$, and the leaves of $\mathcal{L}_1$ contained in it. N is a ball with two leaves of $\mathcal{M}$ in its boundary.

Every loop of $\partial\mathcal{L}_1 \cap N = \lambda$ bounds a disk D in the leaf of $\mathcal{L}$ containing it; thinking of $\mathcal{L} \subseteq N(B)$, the set of these disks which are parallel to D in N(B) have boundaries forming an open and closed set in λ. Consequently, they fall into <u>finitely-many</u> parallel families (in N(B)). For (choosing an arc β running from the top to the bottom of the ball) every point of $\lambda \cap \beta$ has an open neighborhood in β whose points are in loops bounding parallel disks in N(B); because $\lambda \cap \beta$ is compact in β, there is a finite subcover, giving the finite number of families.

Therefore, the loops of $\partial\mathcal{L}_0 \backslash \partial\mathcal{M}$ fall into a finite number of such parallel families.

It is possible to see a finite sequence of <u>surgeries</u> of $\mathcal{L}$ in M_0 which makes every loop in ∂M_0 bound a disk in M_0 (see Figure 5). These surgeries represent our 'template'; what we wish to do now is use this surgery picture to find an isotopy of $\mathcal{L}$ which will do the same thing.

We have a finite number $\lambda_1, \ldots, \lambda_n$ of families of loops in $\partial\mathcal{L}_0 \backslash \partial\mathcal{M}$ which bound a collection $\mathcal{D}_i$ of disks in $\mathcal{L}$ parallel in N(B). Think of doing these surgeries family by family. Choose a collection $\mathcal{D}_i$; note that every disk in the collection meets λ_i only in its boundary (a disk cannot be parallel in N(B) to a proper subdisk of itself - it would imply that the disk met an I-fiber of N(B) infinitely often).

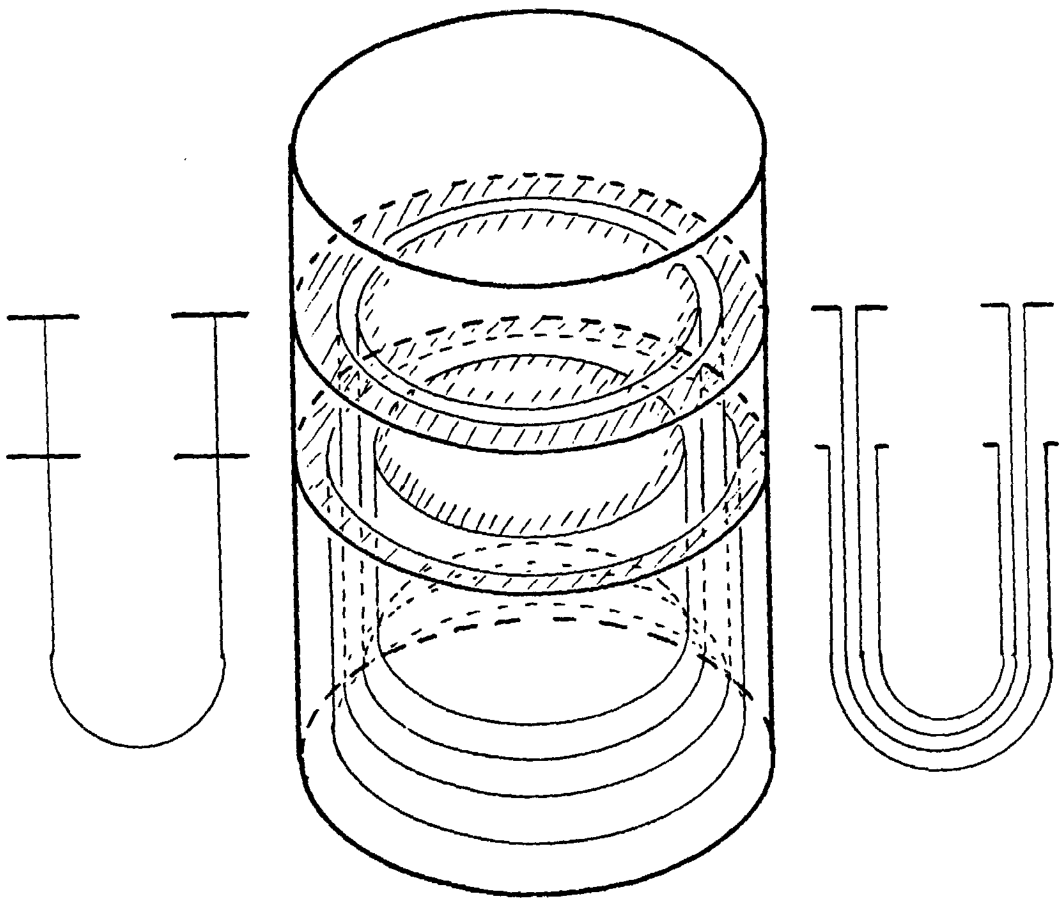

Figure 5: Surgery in the solid torus

Therefore the disk in $\mathcal{D}_i$ together with one of the disks from the surgery form an embedded sphere in M (all of which are parallel to one another); because M is irreducible, they bound (nested) balls (see Figure 6). This ball, together with the ball that the two 'outermost' surgery disks bound, forms a ball which can be used to describe an isotopy taking the disks in $\mathcal{D}_i$ to the (other) collection of disks in M_0, making the collection of loops λ_i bound disks in M_0. This isotopy may have removed leaves of $\mathcal{M}$, as well as loops from some of the λ_i, but since it can be thought of as a replacement (surgering, and then throwing away the spheres created), it <u>adds nothing</u> to any intersection $\mathcal{L}$ has with any object outside of the

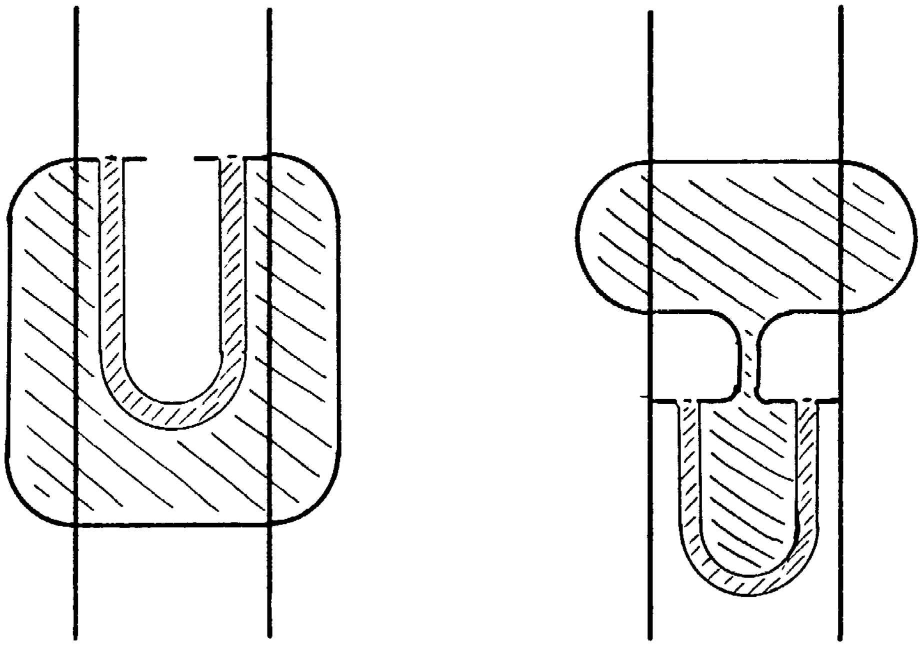

Figure 6: Surgeries to isotopies

interior of M_0; in particular, it adds no new intersections with ∂M_0, and moves none of the disks which it didn't erase. By a finite application of this process, then, we can arrange that every loop in $\mathcal{L} \cap \partial\mathcal{L}$ bounds a disk leaf in M_0, completing our isotopy.

Chapter 2

The Main Results and Their Consequences

Throughout this chapter, M will denote an orientable Seifert-fibered space.

A. The Main Theorems

Theorem 2.1: Let $\mathcal{L}$ be an essential lamination in M, with $\partial M \neq \emptyset$.

(A) If, for some ∂-component T of M, $\mathcal{L} \cap T \subseteq \partial\mathcal{L}$ is either empty or contains a vertical sublamination $\ell \subseteq T$, Then $\mathcal{L}$ contains a (non-empty) vertical sublamination.

(B) If, for some component T of ∂M, $\mathcal{L} \cap T$ consists of a horizontal lamination in T, and if $\mathcal{L}$ contains no (non-empty) vertical sublaminations, then $\mathcal{L}$ is a horizontal lamination.

This theorem will be proved in Chapter 4.

Corollary 2.2: $\mathcal{L}$ and M as above. If $\partial\mathcal{L}$ consists of collections of parallel compact circles in ∂M, then either $\mathcal{L}$ contains a vertical sublamination or $\mathcal{L}$ is a horizontal lamination.

Proof: One or the other of the conditions (A), (B) above must then be satisfied.

Corollary 2.3: $\mathcal{L}$ and M as above. If $\mathcal{L} \cap \partial M = \emptyset$, then $\mathcal{L}$ contains a vertical sublamination.

Theorem 2.4: If $\mathcal{L}$ is an essential lamination in M, with $\partial M = \emptyset$, then either $\mathcal{L}$ contains a horizontal sublamination, or (after possibly splitting $\mathcal{L}$ along a single one-sided leaf) $\mathcal{L}$ contains a vertical sublamination.

This theorem will be proved in Chapter 5.

Corollary 2.5: An essential lamination $\mathcal{L} \subseteq M$, $\partial M = \emptyset$ containing no non-orientable leaves contains a vertical or horizontal sublamination.

Corollary 2.6: If a Seifert-fibered 3-manifold M with ∂M= $\emptyset$ contains an essential lamination, then it contains a horizontal or vertical one.

This corollary therefore tells us that if we wish to show that a Seifert-fibered space contains no essential laminations, it suffices to show that it contains no horizontal or vertical ones. We will now describe those spaces which cannot contain vertical or horizontal essential laminations.

B. Vertical Essential Laminations

If $\mathcal{L}$ is a vertical essential lamination in $\pi : M \to F$, then possibly after splitting $\mathcal{L}$ along a finite collection of leaves, we may assume that $\mathcal{L}$ misses the multiple fibers of M ($\mathcal{L}$ may have contained some of them, if the multiplicity of the fiber is 2). Note that this split lamination is still vertical and essential in M.

Recall that (under the assumption that ∂M= $\emptyset$) M contains a vertical essential surface if and only if F$\neq$S^2, or, if F=S^2, M has at least 4 multiple fibers. Since we are interested in the non-existence of vertical laminations, we will therefore now make the assumption that M has base S^2, and 3 multiple fibers (if M has $\leq$2 multiple fibers, then it is a lens space, which doesn't have universal cover R^3, and so can't contain an essential lamination).

Let F_0 =F with small (open) neighborhoods of the multiple points of removed; F_0 is then a pair of pants = $S^2\backslash 3D^2$. $\mathcal{L}$ is contained in $\pi^{-1}(F_0) = M_0$, and $\lambda = \pi(\mathcal{L})$ is a 1-dimensional lamination in F_0.

There is a (partial) section s:$F_0 \hookrightarrow M_0$ of the Seifert-fibering, with $\mathcal{L}$ transverse to s(F_0). We can therefore conclude that $\lambda \subseteq F_0$ contains no trivial loops (the disk in F_0 it bounds would lift under s to a compressing disk for $\mathcal{L}$ in M, because $\mathcal{L}$ is vertical) or monogons (it would lift under s to an end-compressing disk for $\mathcal{L}$, because $\mathcal{L}$ is transverse to s(F_0)). It is therefore an essential lamination in F_0. Therefore, the next lemma becomes relevant.

Lemma 2.7: An essential lamination λ in the interior of a pair of pants contains a compact (∂-parallel) loop.

Proof: Because λ is incompressible, it is carried by an incompressible train track τ, i.e., one that has no 'oudenogons' or 'monogons' (see Figure 7). The proof then runs by utilizing a calculation of the Euler characteristic of a surface S from the complementary regions S$\backslash\tau$ of τ (see [Ca]) together with a kind of induction.

(1) To begin with we show: any train track in the interior of a disk D^2 bounds a 'bad region' (= oudenogon or monogon). For each component s of $D^2\backslash\tau$ define $\chi_b(s)=\chi(s)-1/2$(number of cusps of s), where a cusp in s is a branch point of τ which 'meets' s.

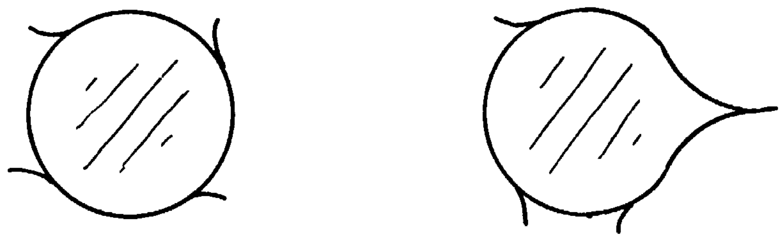

Figure 7: Oudenogons and Monogons

Then from [Ca] we have $\chi(D^2)=1=\Sigma\ \chi_b(s)$. This implies that for some s, $\chi_b(s)>0$, but the only such are oudenogons or monogons.

(2) Every incompressible train track in the interior of an annulus A has a ∂-parallel region (i.e., a region of A$|\tau$ which is an annulus one of whose ∂-components is a ∂-loop of A). To see this, suppose not, and consider the train track τ in the disks formed by filling in each of the ∂-components of the annulus. Each has a bad region, which because it does not live in A, must contain the filled-in disk. This region cannot be an oudenogon, because the would give a ∂-parallel region for τ in A, so it must be a monogon. This gives either a twice-punctured monogon in

$A \backslash \tau$ (which has $\chi_b = -3/2$) or two once-punctured monogons (which each have $\chi_b = -1/2$). Since $\chi(A)$=0, this means that some other region of $A \backslash \tau$ is bad, a contradiction.

(3) Every essential lamination λ in a pair of pants P contains a ∂-parallel loop. For suppose not. λ is carried by an incompressible train track τ; consider the effect on τ of filling in a ∂-component with a disk. If τ is still incompressible, then τ must have a ∂-parallel region. This implies that λ contains a ∂-parallel loop in A; no matter where you remove a disk, this loop will still be ∂-parallel in P.

So we must assume that τ is never incompressible; this means that in each annulus A_i it has a bad region. This bad region must contain the filled-in disk, otherwise τ is not incompressible in P. If the region is an oudenogon, then τ has a ∂-parallel region in P, implying λ contains a ∂-parallel loop.

Therefore every such region is a monogon; because such occurs every time a single ∂-component of P is filled in, it implies that there are three once-punctured monogons (total $\chi_b = -3/2$) for τ in P. Since $\chi(P)$=-1, this still implies that τ has a bad region in P, a contradiction, proving the lemma. ∎

Corollary 2.8: There are no vertical essential laminations in a Seifert-fibered space M with base S^2 and 3 multiple fibers.

Proof: By the above discussion, if such a lamination did exists, we could find a vertical essential lamination $\mathcal{L}$ in $M_0 = \pi^{-1}(F_0)$. $\pi(\mathcal{L})$ then contains a ∂-parallel loop in F_0; therefore, $\mathcal{L}$ contains a ∂-parallel torus T in M_0. But then thought of in M, T is <u>compressible</u> (because we get M from M_0 by gluing solid tori to its boundary components), a contradiction.

Corollary 2.9: An essential lamination $\mathcal{L}$ in a Seifert-fibered space M with base S^2 and 3 multiple fibers contains a horizontal sublamination.

Proposition 2.10: If an essential lamination $\mathcal{L}$ with no compact leaves in a Seifert-fibered space M contains a horizontal sublamination $\mathcal{L}_0$, then $\mathcal{L}$ is a horizontal lamination.

Proof: Since $\mathcal{L}_0$ is horizontal, M$|\mathcal{L}_0$ is a collection of I-bundles foliated by subarcs of the circle fibers of M. Let N be a component of M$|\mathcal{L}_0$, an I-bundle over some non-compact surface E, $\pi : N \to E$, and consider $\mathcal{L}_1 = \mathcal{L} \cap N \subseteq N$. Every leaf L of $\mathcal{L}_1$ is π_1-injective in N (since the composition $\pi_1(L) \to \pi_1(N) \to \pi_1(M)$ is injective).

Now let $\{C_i\}$ be an <u>exhaustion</u> of E by compact, connected subsurfaces, i.e., $\cup C_i = E$, and let $E_i = E \setminus \mathrm{int}(C_i)$. Because the angle that the tangent space of a circle fiber of M makes with the tangent space at x to a leaf of $\mathcal{L}$, for $x \in \mathcal{L}$, is a continuous function of x, and because $\mathcal{L}$ is carried by a branched surface (so as $x \in E$ tends to ∞, the length of the fiber containing $x \to 0$), for some i, every leaf of $\mathcal{L}_1$ is horizontal over E_i. So to show $\mathcal{L}$ can be made horizontal, it suffices to show that $\mathcal{L} \cap \pi^{-1}(C_i)$ can be isotoped to be horizontal in $N_i = \pi^{-1}(C_i)$, rel $\pi^{-1}\partial(C_i) = A$. Note that N_i is a compact handlebody.

We proceed by induction on the genus of N_i (see Figure 8). If genus=0, then C_i is a disk, and $N_i = C_i \times I$, with $C_i \times \partial I \subseteq \mathcal{L}_0$, and $\mathcal{L}_1$ meeting $\partial C_i \times I$ horizontally. Therefore $\mathcal{L}_1 \cap N_i$ is a collection of taut disks, which can be pulled horizontal.

If genus> 0, then choose an essential arc α in C_i and look at the disk $\Delta = \pi^{-1}\alpha$. $\partial\Delta$ can be separated into four arcs, two contained in $\mathcal{L}_0$ and two transverse to $\mathcal{L}_1$. By the usual methods we can remove any trivial loops of intersection $\mathcal{L}_1 \cap \Delta$; then $\mathcal{L}_1$ meets Δ in compact arcs. None of these arcs can have both endpoints in the same arc of $\partial\Delta$; the disk it cuts off together with a (vertical) half-infinite rectangle going off to infinity in N would give an end-compressing disk for $\mathcal{L}$.

So all of the arcs run from one side of Δ to the other; in particular, these arc can be pulled taut w.r.t. the I-fibering of Δ from N. If we then split open N_i along Δ, we get an I-bundle of smaller genus, with $\mathcal{L}_1$ meeting the ∂I-bundle horizontally,

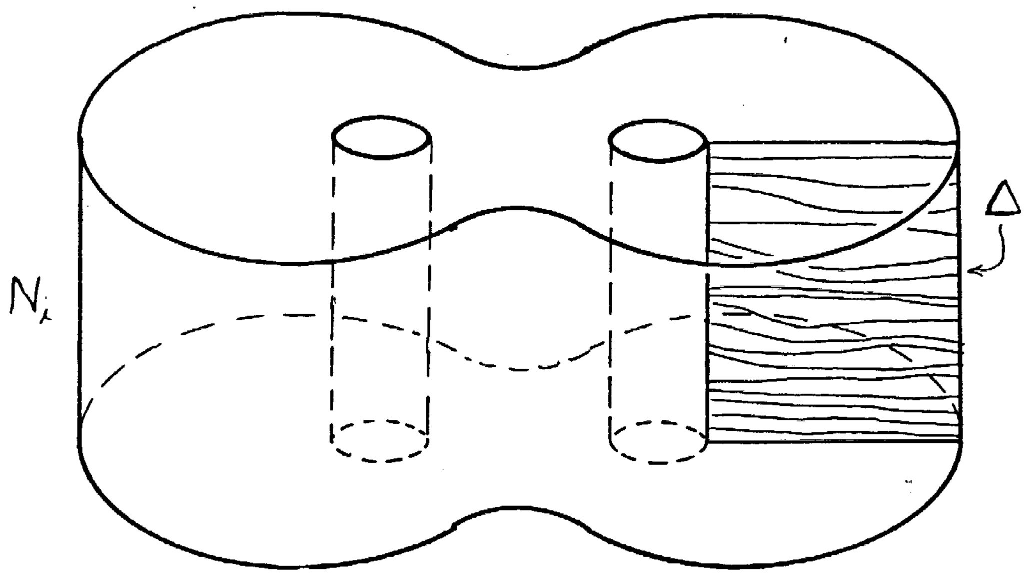

Figure 8: Horizontal laminations

and with horizontal complement in N. By induction, therefore, we can isotope $\mathcal{L}_1$ (rel $\mathcal{L}_0$) to be horizontal in N. Doing this simultaneously for all of the components of $M|\mathcal{L}_0$, we see that we can isotope $\mathcal{L}$ to be horizontal in M. ∎

Note: It is not hard to build counterexamples to this proposition, if we allow $\mathcal{L}$ to have compact leaves. For example, in the 3-torus M= $S^1 \times S^1 \times S^1$, we can build the essential lamination lamination in Figure 9, containing 3 (horizontal) torus leaves, with 'Reeb' annular leaves in between. The lamination is not transversely oriented, but any horizontal lamination would be (inheriting a transverse orientation from a coherent orientation of the circle-fibers of M), so it cannot be horizontal. More complicated examples can also be constructed.

Corollary 2.11: Every taut foliation with no compact leaves $\mathcal{F}$ in a Seifert-fibered space M with base S^2 and 3 multiple fibers is ($C^{(0)}$-) isotopic to a transverse foliation.

Proof: We can split $\mathcal{F}$ along a finite collection of leaves to give an essential lamination $\mathcal{L}$ carried by a branched surface. By the corollary above, $\mathcal{L}$ contains

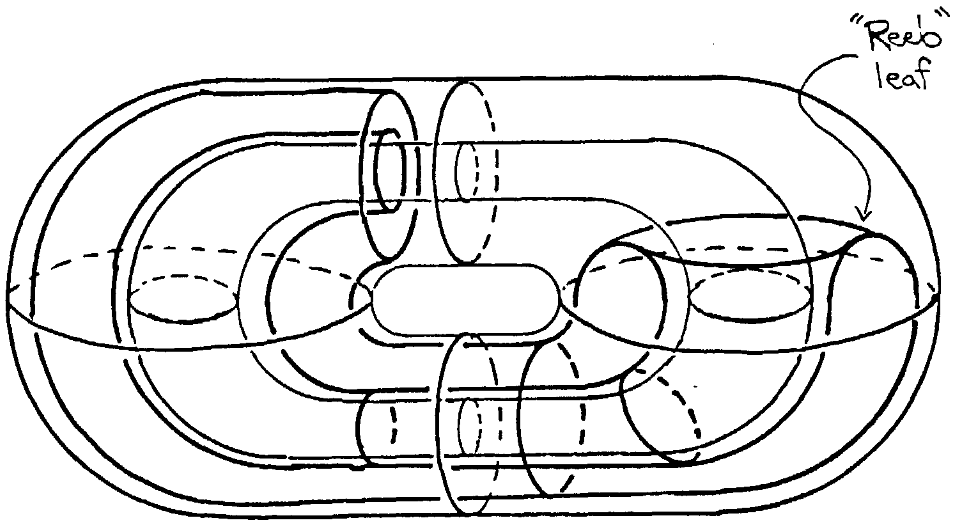

Figure 9: A lamination in the 3-torus

a horizontal sublamination. By the proposition (since $\mathcal{L}$ has no compact leaves) $\mathcal{L}$ itself is a horizontal lamination. The I-bundles M$|\mathcal{L}$ then are fibered by arcs in the circle fibers; crushing each fiber to a point retrieves $\mathcal{F}$ in M, and it is now transverse to the fibers of M. ∎

This result can be thought of as an extension and completion (in the C^0-case) of results of Thurston [Th] and Levitt [Le], Eisenbud-Hirsch-Neumann [E-H-N], and Matsumoto [Ma]. Taken together these papers show that a C^2-foliation with no compact leaves, in any (closed) Seifert-fibered space other than the ones in the corollary, can be C^2-isotoped to a transverse one. The corollary says that a C^2-foliation in M with base S^2 and 3 multiple fibers can be C^0-isotoped to a transverse one; it leaves open the question of whether such a foliation can be C^2-isotoped (the argument above cannot be adapted; at the very beginning, the splitting of the foliation to obtain a branched surface destroys the transverse C^2 structure).

It is worth noting that an extension in the other direction is not possible; there exist C^0-foliations of Seifert-fibered spaces, with no compact leaves, which contain vertical sublaminations. For example, consider the following train track τ in a surface F of genus 2 (see Figure 10). Solving the branch equations (see [Ha 1]) gives a 4-parameter family of solutions, generating a 4-parameter family of incompressible measured laminations λ carried with full support by τ. Then $\mathcal{L} = \lambda \times S^1 \subseteq F \times S^1 = M$ is a vertical essential lamination; for most choices of weights, $M|\mathcal{L}$ is a (non-compact) 6-gon crossed with S^1; $\mathcal{L}$ can then be extended to a foliation with no compact leaves, as in the figure.

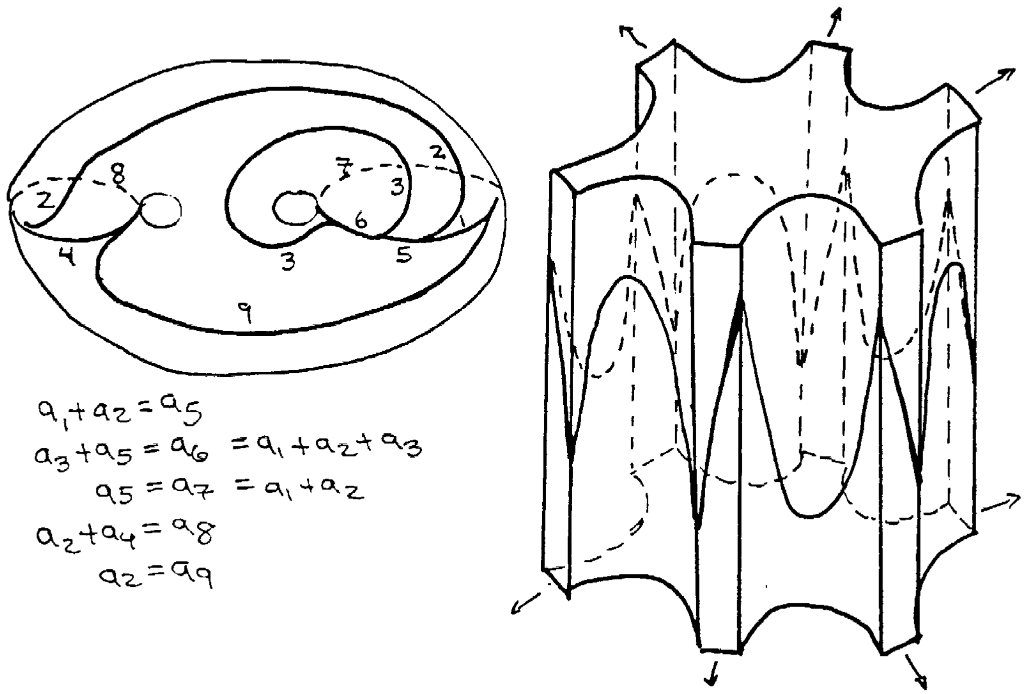

Figure 10: Foliations containing vertical sublaminations

C. Horizontal Essential Laminations

We have seen that the only Seifert-fibered spaces which might not contain any essential laminations are those with base S^2 and 3 multiple fibers. We have also seen that any such space M which contains an essential lamination contains a horizontal one. We now turn our attention to the existence of horizontal laminations in these spaces.

It is easy to see (from the discussion above) that any horizontal lamination $\mathcal{L}$ can be completed to a transverse foliation of M; $M|\mathcal{L}$ is a collection of I-bundles, and these bundles can be foliated by surfaces transverse to the I-fibers, completing $\mathcal{L}$ to a foliation of M. Because the I-fibers are contained in the circle fibers of M, this foliation is therefore everywhere transverse to the circle fibers of M.

[E-H-N] and [J-N] contain results about the existence of such foliations, and give criteria based on the normal Seifert invariants of M for determining their existence. More precisely, suppose M is a Seifert-fibered space with normal Seifert invariant M= $\Sigma(0,0;k,a_1/b_1,a_2/b_2,a_3/b_3)$ and suppose either

(a) k$\neq$-1, -2, or

(b) k=-1, and (possibly after a permutation of the a_i/b_i) $a_i/b_i \geq a'_i/b'_i > 0$, where

$$a'_1/b'_1 = 1-(a'_2/b'_2 + a'_3/(b'_2(b'_3-1)))$$

or

(c) k=-2; then after replacing M= $\Sigma(0,0;-2,a_1/b_1,a_2/b_2,a_3/b_3)$ with M= $\Sigma(0,0;-1,(b_1-a_1)/b_1,(b_2-a_2)/b_2,(b_3-a_3)/b_3)$ (by reversing the orientation of M), apply the criterion (b).

Then M does not admit a transverse foliation.

In particular, M contains no essential laminations. Since it is well known that Seifert-fibered spaces M as above with $1/b_1+1/b_2+1/b_3<1$ have universal cover R^3 (see [Or]), we have the following corollary.

Corollary 2.12: There exist Seifert-fibered spaces M with $\widetilde{M} = R^3$ which contain no essential laminations.

Chapter 3

π_1-injective, end-incompressible laminations in a solid torus

We have seen that a Seifert-fibered space can be thought of as a collection of solid tori glued together along their boundary. So it should not be hard to imagine (this is hindsight for me) that we could understand essential laminations in Seifert-fibered spaces if we understood 'essential' laminations in solid tori well enough. We will see, however, that requiring <u>all</u> of the properties of an essential lamination severely restricts the field in a solid torus; the only one which will survive ∂-injectivity is a collection of meridian disks. Therefore, while working in a solid torus, we will weaken our definition of essential lamination to require only

(1) No spheres or ∂-parallel disks

(2) π_1-injective leaves, and

(3) end-incompressibility.

It is somewhat remarkable that we are able to give a structure theorem for laminations of this type, and that it is of such a simple form; and it seems even more remarkable (to me) that this is precisely the kind of result that we need to provide a classification for essential laminations in Seifert-fibered spaces.

Theorem 3.1: Given an 'essential' lamination $\mathcal{L}_0$ in a solid torus M_0, either it is a collection of meridian disks, or there is a (model) Seifert-fibering of M_0 so that $\mathcal{L}$ contains a <u>vertical</u> sublamination $\mathcal{L}_1$ (whose leaves are then annuli, and possibly one Möbius band); all leaves of $\mathcal{L}_0\backslash\mathcal{L}_1$ are non-compact, simply-connected, and <u>horizontal</u>.

The proof contains two essential ingredients; first one needs that the ∂-lamination $\partial\mathcal{L}_0$ contains compact loops (which determine the regular fiber of the Seifert-fibering), and then that every such compact loop is in the boundary of a compact leaf of $\mathcal{L}_0$. The union of these leaves is the vertical sublamination $\mathcal{L}_1$. First, though, we need a small catalogue of basic facts, so that we can more easily recognize when these two things are happening.

A. Some basic facts about laminations in a solid torus

Fact 1: An 'essential' lamination $\mathcal{L}_0$ in $D^2 \times S^1$ must meet the boundary torus; $\partial\mathcal{L}_0 \neq \emptyset$.

This is true more generally; an essential lamination cannot live in the interior of a handlebody. To see this, take a meridian disk D (or, in the general case, a compressing disk in one of the handles), and make $\mathcal{L}_0$ transverse to it. If $\mathcal{L}_0 \cap D = \lambda$ contains any compact loops, then by the same argument as in Chapter 1.E.b, we can isotope $\mathcal{L}_0$ to remove them. So we can assume λ contains no compact loops. λ is carried by the train track $\tau = B_0 \cap D$ (where $\mathcal{L}_0$ is carried by B_0), and contains only non-compact leaves; the Euler characteristic calculation carried out in Chapter 2.B implies that, if $\lambda \neq \emptyset$, λ will contain a monogon, so $\mathcal{L}_0$ does, which is essential since $\mathcal{L}_0$ is transverse to D. So $\lambda = \emptyset$, so $\mathcal{L}_0$ misses D, implying that $\mathcal{L}_0$ is contained in a ball, **B** (or, inductively, is contained in the interior of a handlebody of lower genus). It is π_1-injective there (same argument as before), and contains no spheres ($\mathcal{L}$ didn't), and so all of its leaves are planes. Capping this ball off with a ball, we get a lamination in S^3, which is essential (because monogons can be pushed off the capping ball), a contradiction.

Fact 2: Every leaf L of $\mathcal{L}_0$ meets $T = \partial M_0$.

Otherwise, the closure $\overline{L}$ of L would give a π_1-injective lamination missing T (because $\mathcal{L}_0$ meets T transversely; basically, if $L_1 \subseteq \overline{L_2}$ and L_1 meets T, then L_2 does, too (by looking at coordinate charts for $\mathcal{L}_0$)). Applying the argument above to this sublamination gives the same conclusion, unless $\overline{L} \cap D$ contains a monogon;

but then Euler-χ arguments will find a monogon for $\mathcal{L}_0 \cap D$ inside that one, which is essential because $\mathcal{L}_0$ is transverse to D.

Fact 3: If a leaf L of $\mathcal{L}_0$ has more than one compact ∂-component, then it is an annulus.

This is standard; the two loops are parallel, otherwise one of them is trivial (making L a boundary-parallel disk). Draw an arc α in the leaf joining the two components; then $\gamma_1 * \alpha * \overline{\gamma_2} * \overline{\alpha}$ is (almost) an embedded loop in L null-homotopic in $D^2 \times S^1$, hence bounds a disk in L. It follows that L is a disk with two arcs in its boundary identified, i.e. an annulus.

Fact 4: An annulus A with ∂A vertical (in a model fibering of a solid torus) is vertical.

This is also standard; from the previous argument it is easy to see that A is ∂-parallel, and so isotopic to an (of necessity vertical) annulus in the boundary of the solid torus. Pushing it back into the solid torus slightly, we also then see that A is isotopic to a (properly embedded) vertical annulus.

Fact 5: A non-orientable surface L with $\pi_1(L)=\mathbf{Z}$ and a compact ∂-component γ is a Mõbius band.

Proof: Let $p: L_0 \to L$ be the orientable double cover of L. γ is orientation-preserving in L, so $p^{-1}(\gamma) = \gamma_1 \cup \gamma_2$, disjoint simple loops mapping homeomorphically down to γ under p. Being simple loops, they do not represent a proper power in $\pi_1(L_0) = \mathbf{Z}$ [Ca]. So both represent the generator (up to reorienting the curves), hence are freely-homotopic. By [Ep], they are then isotopic, and cobound an annulus A in L_0. Since γ_1 and γ_2 are ∂-components, this implies that L_0 itself is an annulus, hence compact.

So $p(L_0)=L$ is compact; by the classification of surfaces, it is therefore a Möbius band.

Fact 6: A Möbius band L with ∂L vertical (in a model fibering of a solid torus) is vertical.

This follows from a result of [Ru], which says that one-sided incompressible surfaces in a solid torus with a single boundary curve are determined up to isotopy by the slope of that curve (π_1-injective surfaces are incompressible). With this result in hand it remains then only to show that ∂L represents 2x the generator of π_1(solid torus), because a vertical π_1-injective Möbius band with that boundary slope can easily be constructed.

But this in turn follows readily from some π_1 considerations; let M=solid torus, and consider $M_0=M\setminus\text{int}(N(L))$. It π_1-injects into M (since L is π_1-injective), is irreducible (since M is) and has boundary $=(\partial M\setminus\partial N(L))\cup(\partial N(L)\cap\text{int}(M)= A_1\cup A_2$ =annulus∪annulus=torus. So M_0 is a solid torus, and $M=M_0\cup_{A_2}N(L)$.

Claim: The core of A_2 represents a generator of $\pi_1(M_0)$. For if the map $\pi_1(A_2)\to\pi_1(M_0)$ sends 1 to n, then by Van Kampen's theorem $\pi_1(M)$ is equal to $\pi_1(M_0) *_{\pi_1(A_2)} \pi_1(N(L))$, and since the core of A_2 represents 2 in $\pi_1(N(L))$, this implies that $\pi_1(M)= \mathbf{Z} =(a,b : a^2 = b^n) = G$. But the subgroup generated by a^2 is normal (a^2 commutes with both a and b), and G modulo this subgroup is $(a,b : a^2 = 1, b^n = 1) = \mathbf{Z}_2 * \mathbf{Z}_n$. But every quotient group of $\pi_1(M)=\mathbf{Z}$ is cyclic, implying n=1. (Alternatively, this could be seen by recognizing that G is the fundamental group of a (2,n)-torus knot, which has incompressible boundary (so G contains $\mathbf{Z}\oplus\mathbf{Z}$, unless n=-1 or 1.)

In particular, M_0 deformation retracts to A_2, so M deformation retracts to the regular neighborhood N(L) of L. Since ∂L represents 2×generator in $\pi_1(N(L))$ (it's parallel to the core of A_2), it therefore represents 2×generator in $\pi_1(M)$.

Note also that there cannot be two disjoint such Möbius bands in a solid torus M, because any other L′ would be contained in the solid torus complement M_0 of the other. The boundary of L′ is parallel to ∂L in M, but ∂L now generates the fundamental group of M_0 (this is easy to see when L is vertical; draw a picture!), and so $\partial L'$ cannot in fact bound a Möbius band in M_0 (the generator can't be divisible by 2).

B. Proof of the theorem

Lemma 3.2: Any π_1-injective, end-incompressible lamination $\mathcal{L}_0$ in a solid torus M_0 contains a compact ∂-leaf.

Proof: Suppose not; we know from Fact 1 above that $\partial\mathcal{L}_0$ is non-empty. From the catalogue of ∂-laminations in Chapter 1.E.b, $\partial\mathcal{L}_0$ contains an irrational lamination, and so can be isotoped so that it is everywhere transverse to the meridional foliation.

Pick a meridian disk D in M_0. By an isotopy of $\mathcal{L}_0$ (supported away from $\partial\mathcal{L}_0$) we can make $\mathcal{L}_0$ transverse to D. By the usual argument, $\mathcal{L}_0 \cap D = \lambda_0$ consists of circles and arcs, and by an isotopy of $\mathcal{L}_0$ we can remove the circles of intersection, using the π_1-injectivity of $\mathcal{L}_0$. Pick an outermost arc α of this intersection. It cuts D into two disks, one of which D_0 meets $\mathcal{L}_0$ only in an arc of its boundary. The other arc of ∂D_0 lies in $\partial M_i | \partial\mathcal{L}_0$, and and splits the component containing it into two half-infinite rectangles. Pick one rectangle R, then it is easy to see that $D_0 \cup R$ is an end-compressing disk for $\mathcal{L}_0$, because $\mathcal{L}_0$ is transverse to ∂M_i, contradicting the end-incompressibility of $\mathcal{L}_0$. ∎

Proposition 3.3: Every compact ∂-loop $\gamma_0 \subseteq \partial\mathcal{L}_0$ is contained in a compact leaf L_0 of $\mathcal{L}_0$.

Proof: If γ is meridional (null-homotopic in M_0), then by π_1-injectivity it is null-homotopic in L, hence bounds a disk in L. Since γ is a boundary component, it follows that L is itself a disk, hence compact. γ cannot be trivial in ∂M_0; it would then bound a (compact) ∂-parallel disk in M_0. If L is non-orientable, then, because L has fundamental group **Z**= the integers (since it injects into $\pi_1(D^2 \times S^1)=\mathbf{Z}$) and has a compact ∂-component, it follows that L is a Möbius band, hence also compact. So if we assume by way of contradiction that L is not compact, we may assume that L is orientable and that γ is essential and not meridional in ∂M_0, hence is non-trivial in $\pi_1(M_0)$. Now by an isotopy of $\mathcal{L}_0$ we can make $\partial\mathcal{L}_0$ transverse to the meridional foliation of $T=\partial M_0$, except for 'Reeb' leaves which are tangent at one point. Notice also that we can assume that $\partial\mathcal{L}_0$ is semi-transverse to the circle fibering of $T=\partial M_0$ parallel to the vertical loop γ (i.e. transverse in the weak sense - it contains a (possibly empty) sublamination λ_0 consisting of vertical loops, and every leaf of $\lambda \setminus \lambda_0$ is everywhere transverse to the circle fibers; see Figure 11).

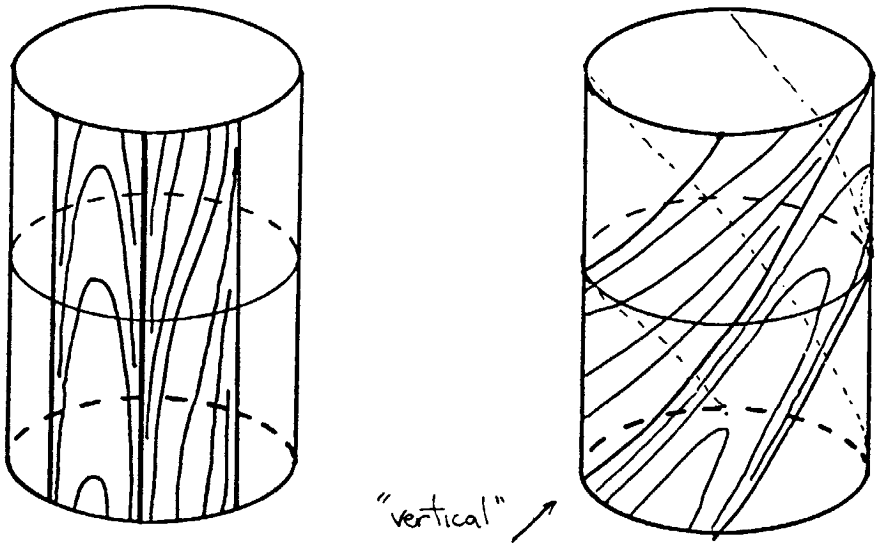

Figure 11: Structure of the ∂-lamination

Now choose a meridian disk D whose boundary doesn't meet a Reeb leaf of $\partial\mathcal{L}_0$ tangentially (i.e. for which ∂D is transverse to $\partial\mathcal{L}_0$). By an isotopy of $\mathcal{L}_0$ supported away from T, we may make $\mathcal{L}_0$ transverse to D; as before, $\mathcal{L}_0 \cap$D consists of circles and compact arcs. We can, as usual, remove the circles of intersection by an isotopy of $\mathcal{L}_0$. γ meets D in n points $x_1, ..., x_n$ (where n=multiple of the generator of $\pi_1(M_0)=\mathbf{Z}$ that γ represents). Consider the arcs $\alpha_1, ..., \alpha_n$ of D$\cap\mathcal{L}_0$ emanating from these points. These arcs are distinct; if α joins γ to itself, then $\alpha \cup$ one of the arcs in γ cut off by the two points would represent a one-sided loop in L, since γ is transverse to the meridional foliation.

Let $\alpha = \alpha_i$ be an outermost arc in this collection, cutting off an (outermost) disk Δ in D. The other endpoint y of α is in a boundary component of L different from γ (and hence non-compact, by Fact 3); call it ℓ. Notice that Δ meets γ only at $x_i = x$ (otherwise α wouldn't be outermost), so the arc $\beta = \Delta \cap \partial$D is contained in a component of ∂D split on γ.

Now γ splits T=∂M$_0$ into an annulus, **A**, and we can in fact think of $\partial\mathcal{L}_0$ as living in this annulus. The monotonicity of $\partial\mathcal{L}_0$ w.r.t. the meridional foliation means that, in this annulus, if we orient a non-compact leaf of $\partial\mathcal{L}_0 = \lambda$, it meets our meridional arc β in the annulus <u>monotonically</u> as we traverse the leaf in either direction (see 1.E.b); in particular, if we follow the leaf ℓ in one or the other direction, it will return after travelling around ℓ n-times vertically (or, in a Reeb leaf, a net 0-times vertically) to a point y' in β, and the arc in β between y' and y contains no points of ℓ (Figure 11).

We will 'orient' the arc β so that we can talk of $x \in \gamma$ as the 'leftmost' point of β, and $y \in \ell$ as the 'rightmost' point. Now the observation above can be extended to see that the points of $\ell \cap \beta$ will occur as consecutive discrete points $y_0 = y, y_1 = y', y_2, y_3, ..$ with y_{i+1} lying to the left of y_i (there are an infinite number of such points of intersection, since ℓ can't push its way past x in **A**). Now look at the arcs ρ_i of λ with endpoint y_i, and consider the other endpoints z_i of these arcs. These points live in β, because the ρ_i can't cross α. They also must lie in non-compact leaves of $\partial\mathcal{L}_0$, because γ doesn't meet β, and because otherwise L would have more than one compact boundary component, making it compact by Fact 3. Now, one of two things can happen: either z_i lies to the right of y_i for every i, or some z_i lies to the left of y_i (and we say that ρ_i turned left).

Suppose the second possibility occured, and z_i represents the first i in which the arc emanating out of y_j turned left. Set $z_i = w$, and consider the leaf ℓ_1 of $\partial\mathcal{L}_0$ containing it. We can again travel in one direction or the other along ℓ_1 and find a sequence of points w_i lying to the left of w in β, and again ask which of the two above conditions occur. Continuing in this way, we will either find an infinite number of arcs σ_i emanating from the consecutive points v_i of some ℓ_0 turning right, or an infinite sequence of arcs η_i emanating from points $y_{i_1}, z_{i_2}, w_{i_3}, v_{i_4}, \ldots$ turning left, and with other endpoints $z, w, v, u, \ldots$ (see Figure 12).

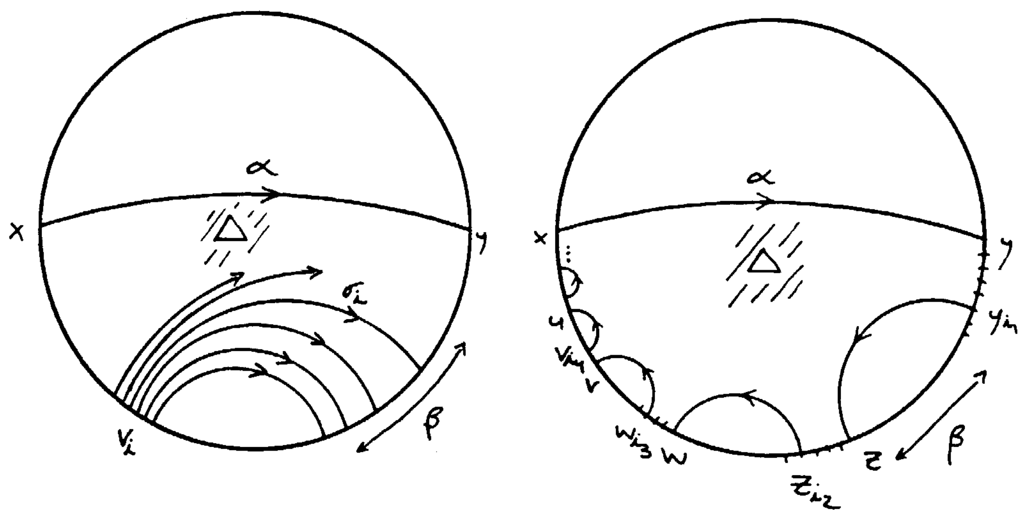

Figure 12: Tracking the arcs

We show first that this second situation leads to a contradiction. First, for simplicity, rename the right endpoints of the η_i u_i, and the left endpoints v_i. Now, if we look from right to left in β, these points occur, in order, as $u_1, v_1, u_2, v_2, u_3, v_3, \ldots$. This sequence is monotone in β, and so has a limit, z. But the sequences u_i and v_i are also monotone, and so have a limit, and it is clear that they in fact share the common limit z. But because the arcs η_i are living in the lamination λ, this situation is impossible: $z \in \lambda$, because it is a limit point of point in λ, and λ is closed. So z lies in an arc α_0 of λ. The points u_i limit on z, so the arcs η_i must be limiting on α_0. So the other endpoints v_i of the η_i must be limiting on the other endpoint of α_0. But those points are in fact limiting on z, so the other endpoint of α_0 must in fact be z. But this is an absurd situation (because λ is transverse to ∂D).

So the second situation cannot occur, so, after a finite number of attempts, we will find our σ_i, v_i, and ℓ_0. We show now that this situation also leads us to a contradiction, thus proving the proposition.

First, we must understand what occurred before we managed to turn up these arcs (see Figure 13). There were a finite number of times (call them $t_1, t_2, \ldots, t_N$)

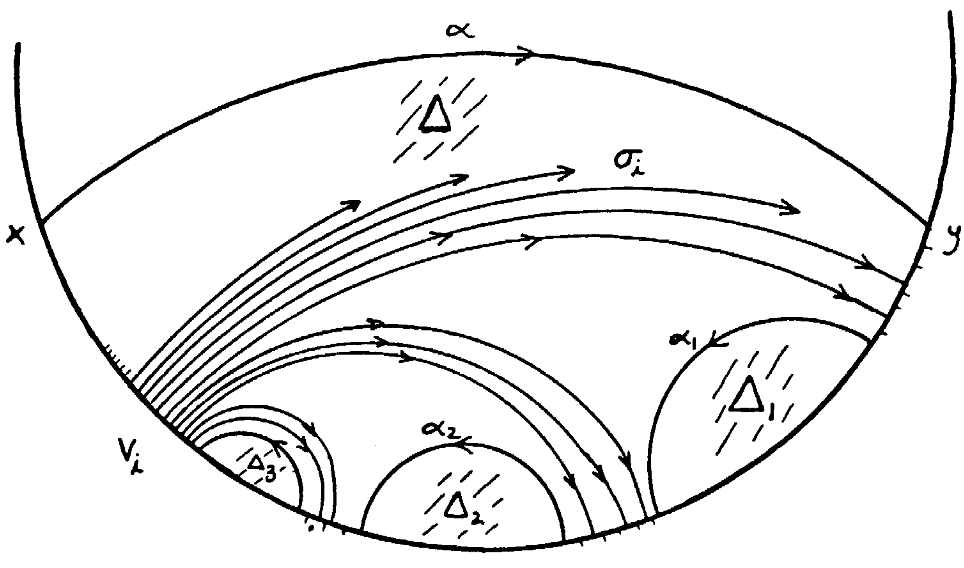

Figure 13: Finding the arc of limits

in which we began to find arcs turning right, but at some point $i_1, i_2, ..., i_N$, an arc α_1, etc., appeared which turned left, and whose other endpoint initiated the next search. The endpoint of α_N gave us the search which succeeded.
Now, the question to ask is, where do the other endpoints of the σ_i in fact lie? One answer is that the other endpoint of σ_i lies between v_i and v_{i-1} (see Figure 14a,b). But in this case, it is easy to use the disk cut off by σ_i together with a tail in $\partial\mathrm{M}_0$ (in one direction or the other) to create an end-compressing disk for $\mathcal{L}_0$. Therefore, we may assume that this situation never occurs.

Look now at the arcs α_i. Each α_i cuts off a disk Δ_i from D, in which the other endpoints of the σ_i cannot lie. Further, these disks all lie to the right of v_1. The arcs of β, lying to the right of v_1, and lying outside of the Δ_i, can be expressed as a finite union of arcs lying between consecutive points of leaves of $\partial\mathcal{L}_0$. More specifically, each connected component of $\beta\backslash$(the above) represents one of the episodes t_j, and each one of those components can be cut up into i_j intervals between the endpoints of the arcs which did turn right. The endpoints of these intervals represent consecutive points in some leaf of $\partial\mathcal{L}_0$.

Now we have an infinite number of arcs σ_i whose right endpoints are falling (monotonically to the right) into a finite number of intervals. It follows therefore that an infinite number of them are falling into one interval, δ. Now, no ∂-leaf can meet this arc more than twice (and twice only if the endpoints join the 'turnaround' arc in a Reeb leaf); this follows from the 'partial' monotonicity of λ w.r.t. the meridional foliation of T. Therefore, the endpoints of an infinite number of the σ_i all lie on distinct boundary leaves of L; L thus has an infinite number of boundary components.

In particular, there are two such leaves of ∂L passing between the two points a and b at the ends of this arc, where a and b are consecutive points on some leaf ℓ'

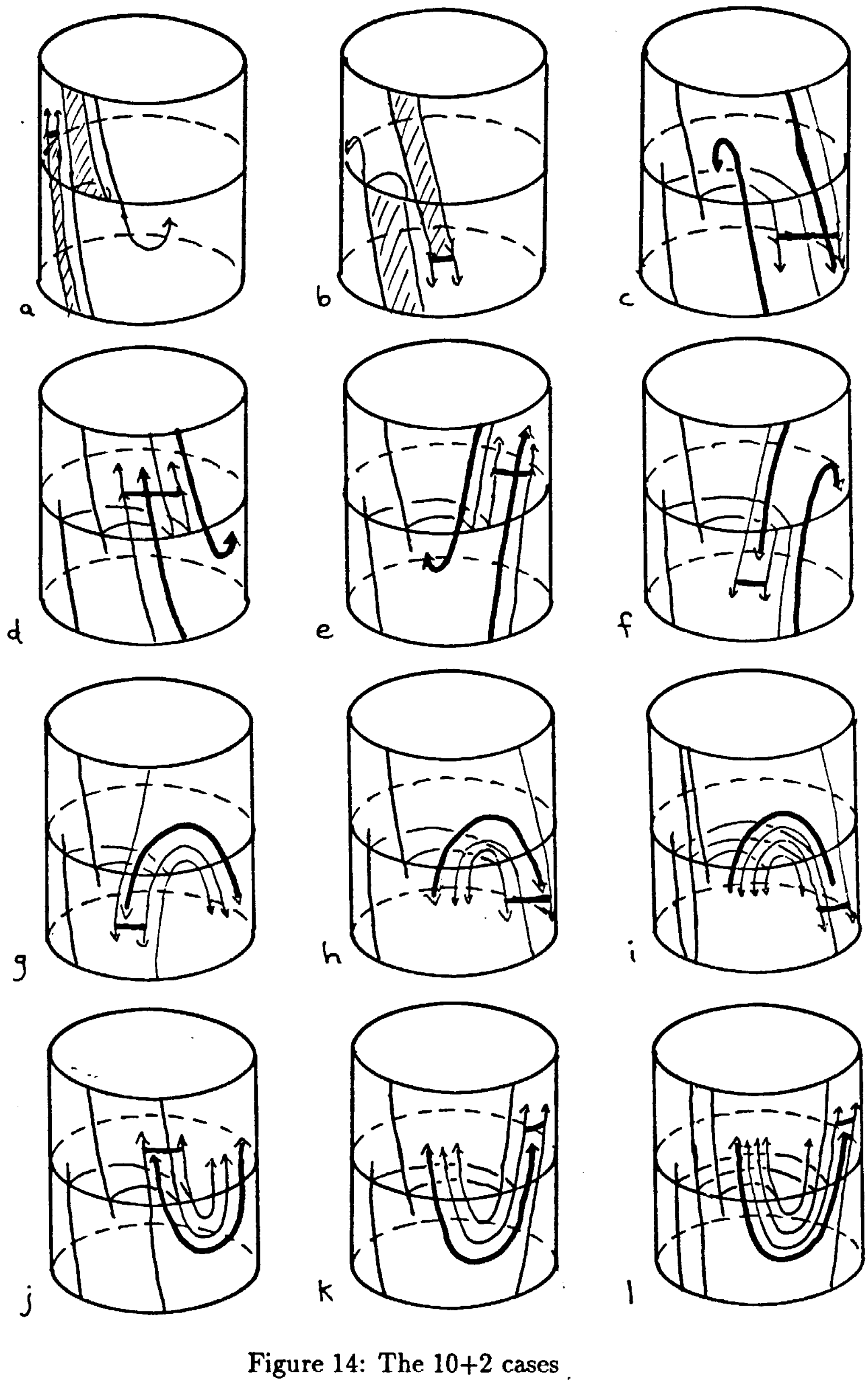

Figure 14: The 10+2 cases

of $\lambda \cap L$. The two components ℓ_1 and ℓ_2 that we have identified have the same limit set as ℓ' (this follows from Chapter 1.E.b); so we are in the situation of Figure 14. The arcs α_i, α_{i+1}, an arc in ℓ_0, an arc in (say) ℓ_1, and an arc in β together give a loop in $D^2 \times S^1$, which bounds a disk, D_0, together with a 'tail' bounded by arcs in ℓ_1 and ℓ_2 which follow ℓ' to their common limit loop (see below). This situation will look like one of those in Figure 14c-14l; the differences in the figures represent the different (possible) 'Reeb' behaviors of the bounding leaf ℓ', and the different disks D_0 that we therefore find. For brevity we will deal only with the case 14c (the remaining cases are all essentially the same).

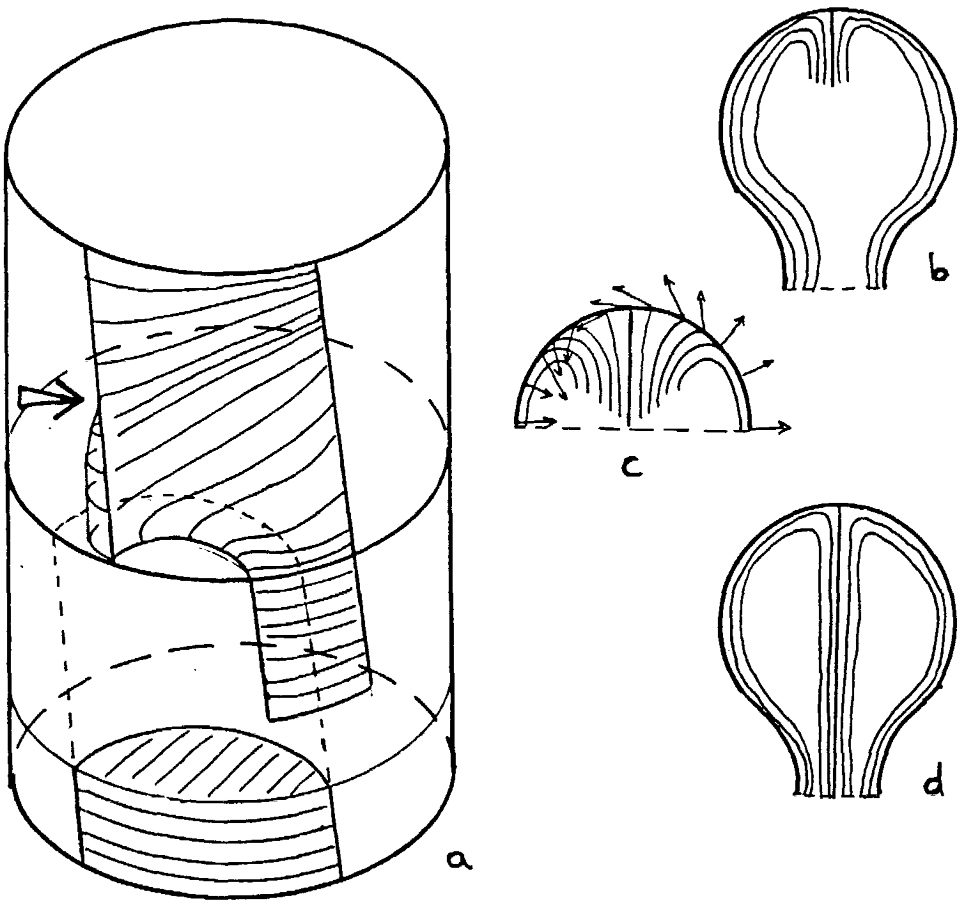

Figure 15: Finding a 'transversely-oriented' monogon

This disk can be chosen as in Figure 15a; it then turns out that ∂D_0 is transverse to $\mathcal{L}_0$, <u>except</u> <u>at</u> <u>one</u> <u>point</u>, where locally the intersection looks like Figure 15b. But because D_0 is an <u>embedded</u> disk, we can see that by shrinking D_0 slightly we can make it transverse to $\mathcal{L}_0$ near the boundary (there is a tangent direction in the plane of the disk which is normal to $\mathcal{L}_0$ at each point of the boundary; see Figure 15c), so by an isotopy of the disk supported away from the boundary, we can make this slightly smaller disk transverse to $\mathcal{L}_0$. Then as before, if we look at $\mathcal{L}_0 \cap D_0$, we can remove circles of intersection by an isotopy of $\mathcal{L}_0$; we can also then see that there are no non-compact arcs of intersection (using the Euler characteristic argument employed before).

So $\mathcal{L}_0 \cap$(the smaller D_0) consists of compact arcs, so the intersection with the bigger D_0 also consists of compact arcs, except that the arc in ∂D_0 also has a compact tail growing out of its middle. If we now choose an outermost arc in D_0, cutting off the outermost disk Δ, and add to Δ the infinite rectangle in the tail that abuts it, we have built a monogon for $\mathcal{L}_0$, which, because its tail is contained in the ∂-torus T, is essential (see Figure 15d). This provides the necessary contradiction, so L must in fact be compact. ∎

To complete the theorem, consider our 'essential' lamination $\mathcal{L}_0$. By the lemma, $\partial\mathcal{L}_0$ contains a compact loop γ. Choose the Seifert-fibering of the solid torus M_0 whose regular fiber in ∂M_0 is isotopic to γ. Since every compact loop of $\partial\mathcal{L}_0$ is parallel to γ, we can, after an isotopy of of $\mathcal{L}_0$ supported near ∂M_0, assume that every compact loop of $\partial\mathcal{L}_0$ is a fiber of M. Now by the proposition every leaf of $\mathcal{L}_0$ which contains a compact ∂-loop is compact. They have vertical boundaries, and so by the facts above, each can be isotoped to be vertical in M. They can in fact be so isotoped simultaneously; the leaves fall into a finite collection of parallel families, and each can be isotoped in turn, from the innermost out; think of isotoping the innermost leaf of the family to the boundary and then back in slightly; this is

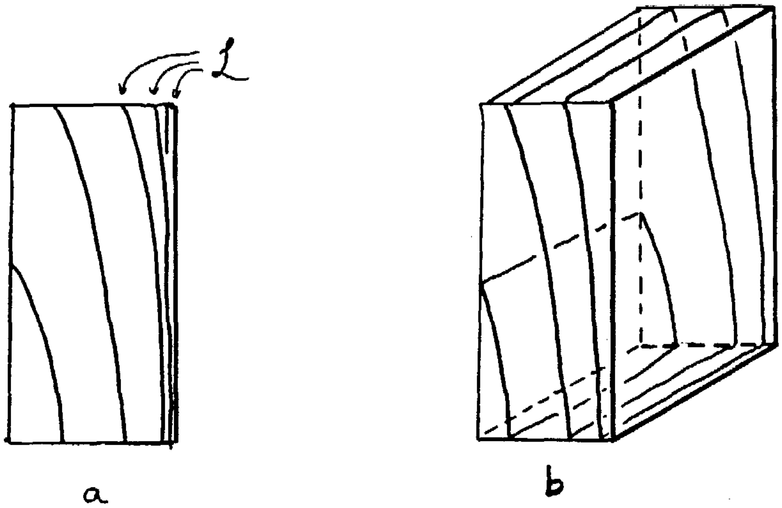

Figure 16: Making the other leaves horizontal

an ambient isotopy which makes the entire family vertical. Subsequent isotopies will be supported away from the ones which you have already straightened, so you always make progress. This gives us the vertical sublamination $\mathcal{L}_1$ of the theorem.

Now consider the leaves of $\mathcal{L}_0$ which are not in $\mathcal{L}_1$. These leaves all have non-compact boundary (which we assume runs transverse to the foliation of ∂M_0 by fiber circles), and so limit on leaves of $\mathcal{L}_1$. From holonomy considerations, this limiting takes place in a very simple way; see Figure 16a.

Thus in each component M_1 of $M_0|\mathcal{L}_1$, it is possible to arrange the leaves of $\mathcal{L}_0$, by an isotopy supported away from ∂M_0, to meet a saturated neighborhood of the boundary of the component as in Figure 16b. It is easy then to see that $\mathcal{M}_0 = \mathcal{L}_0 \cap (M_1 \setminus \mathrm{int}(N(\mathcal{L}_1))$ is π_1-injective in $M_2 = M_1 \setminus \mathrm{int}(N(\mathcal{L}_1))$ (all we've done is to remove half-infinite rectangular 'tails' from the leaves of $\mathcal{L}_0$, and the solid torus M_2 π_1-injects into M_0), and end-incompressible (a monogon for $\mathcal{M}_0$ is a monogon for $\mathcal{L}_0$, since $\mathcal{L}_0$ is transverse to ∂M_1). Also, its ∂-lamination is transverse to the circle fibering of ∂M_1 induced from M_0, so it has no trivial leaves. Consequently, by the proof above, it either consists of meridian disks, or it contains an annulus or Mõbius band leaf L. If the latter occurs, then L has boundary transverse to the vertical fibering of ∂M_2 induced from M_0, and so meets every fiber of ∂M_2. In particular, since $\partial M_2 \cap \partial M_0 \neq \emptyset$, ∂L meets ∂M_0.

Now, there is an arc α in L which together with an arc δ in ∂M_2 bounds a disk D in M_2 (if L is an annulus this is because it is ∂-parallel; if L is a Mõbius band, look at the boundary of a regular neighborhood of L; it is a ∂-parallel annulus, which supports such a disk, and then project back). By an isotopy of D (leaving α in L and δ in ∂M_2) we can arrange that δ is contained in an annulus A of $\partial M_2 \cap \partial M_0$, and so we can make it lie in a circle fiber of this annulus. We may also assume (by techniques of Chapter 1.E.b above) that D is transverse to $\mathcal{L}_0$, meeting it in a collection of compact arcs.

Now consider in what leaves of $\partial\mathcal{L}_0 \subseteq \partial M_0$ the endpoints of δ are lying in. None of the circle loops of $\partial\mathcal{L}_0$ meet A, so these points are contained in (distinct; these leaves run transverse to the circle fibering of ∂M_0) non-compact leaves of $\partial\mathcal{L}_0$. Therefore (see Figure 17) δ together with a pair of half-infinite arcs in $\partial\mathcal{L}_0$ cut off a half-infinite rectangle in ∂M_0; this together with the disk D form a 'monogon' for $\mathcal{L}_0$; embedded in the 'monogon' is a monogon for $M_0|\mathcal{L}_0$, which forms an end-compressing disk for $\mathcal{L}_0$, because its 'tail' is in ∂M_0, which is transverse to $\mathcal{L}_0$.

This gives us a contradiction, so $\mathcal{M}_0$ consists of meridian disks with boundary transverse to the circle fibering; an isotopy rel boundary makes this a collection of <u>horizontal</u> disks. Doing this simultaneously for all of the components of $M_0 \backslash \mathcal{L}_1$ gives an isotopy of $\mathcal{L}_0$ which makes every leaf of $\mathcal{L}_0 \backslash \mathcal{L}_1$ horizontal, in our chosen Seifert-fibering of M_0.

Since the lamination in the saturated neighborhood is also clearly horizontal, this implies that the leaves of $\mathcal{L}_0 \backslash \mathcal{L}_1$ can be isotoped, rel $\mathcal{L}_1$, to be horizontal in M_0. By gluing back, we have then arranged that

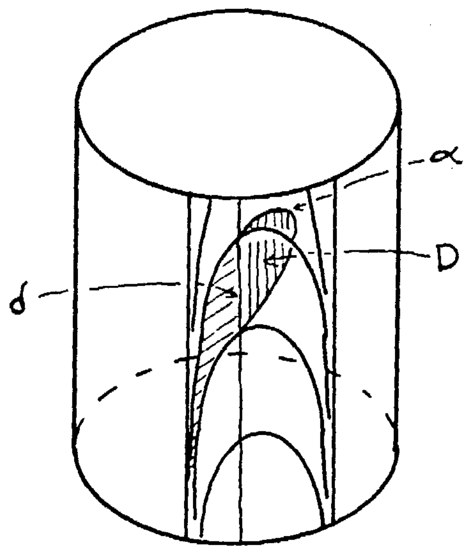

Figure 17: Finding meridian disks

(*) the leaves in the complement of the vertical sublamination of $\mathcal{L}_0$ found above can be isotoped (rel the vertical sublamination) so that they are <u>horizontal</u>. Since these leaves are just disks with half-infinite rectangles glued to them, they are also simply-connected.

This completes our proof.

Chapter 4

The Case of Non-empty Boundary

In this chapter we will prove the first of our structure theorems, describing essential laminations in a Seifert-fibered space which has non-empty boundary.

Theorem 2.1: Let $\mathcal{L}$ be an essential lamination in M = Seifert-fibered, with $\partial M \neq \emptyset$.

(A) If, for some component T of ∂M, $\mathcal{L} \cap T \subseteq \partial\mathcal{L}$ is either empty or contains a vertical sublamination $\ell \subseteq T$, Then $\mathcal{L}$ contains a (non-empty) vertical sublamination.

(B) If, for some component T of ∂M, $\mathcal{L} \cap T$ consists of a horizontal lamination in T, and if $\mathcal{L}$ contains no (non-empty) vertical sublaminations, then $\mathcal{L}$ is a horizontal lamination.

The idea of the proof is to split M up into a collection of solid tori M_i (this splitting has been described in Chapter 1.C), with boundary $\partial M_i = T_i$, and then isotope $\mathcal{L}$ so that it meets each solid torus in a π_1-injective lamination $\mathcal{L}_i \subseteq M_i$ with no ∂-parallel disk leaves. It therefore is an 'essential' lamination, and so our structure theorem in Chapter 2 tells us what each looks like. An analysis of the structures that are possible under our assumptions (A),(B), completes the proof.

We have already described what essential laminations in $D^2 \times S^1$ look like; the only example from Chapter 2 which survives ∂-injectivity is a collection of meridional disks. If we make the additional assumption in the theorem above that $M \neq D^2 \times S^1$, then we can also assume that ∂M is incompressible (see [Ha 2]).

Now recall some of our notation from Chapter 1. Given a Seifert-fibered manifold M with base surface F, $\partial F \neq \emptyset$, choosing a collection α_j of arcs in F we can split M open along the annuli A_j to obtain a collection of solid tori M_i whose union is M. For the purposes of our proof we will assume in addition that all of the annuli A_j meet (our special ∂-component) T, i.e., all of the arcs meet the ∂-component of F which corresponds to T under projection. We will also assume that, in case (A), $\mathcal{L}$ has been isotoped so that the vertical loops of $\partial\mathcal{L}$ miss the ∂A_j, and the other leaves of $\partial\mathcal{L}$ are everywhere transverse to the circle fibers of T (and in particular are transverse to ∂A_j); in case (B), we will assume that $\mathcal{L}$ has been isotoped so that $\mathcal{L}\cap T$ is everywhere transverse to the circle fibers of T, and $\partial\mathcal{L}$ is transverse to ∂A_j.

Make $\mathcal{L}$ transverse to the surface $A = \cup A_j$, and look at $\lambda = \mathcal{L}\cap A$, a (1-dimensional) lamination in a collection of annuli. By the methods in Chapter 1.E.d, we can by isotopy of $\mathcal{L}$ remove any trivial loops from λ. If there are any ∂-parallel arcs in λ, they can be removed by a similar means: choose an outermost family of arcs, and consider the innermost arc α of the family. It splits off a disk Δ_0 from A; by the ∂-injectivity of the leaf L of $\mathcal{L}$ containing it, this arc is homotopic rel endpoints to ∂L. Because α is embedded in L, this implies that it is also ∂-parallel in L, and so splits off a disk Δ_1 from L. Look at $\Delta_0 \cap \Delta_1$, a (finite) collection of arcs in Δ_1 (see Figure 18). Choose an outermost such arc β splitting off a subdisk D_1 in Δ_1 missing Δ_0; in Δ_0, β splits off a subdisk D_0 (which doesn't necessarily miss Δ_1, but it does miss D_1).

Together these disks form an embedded disk D in M (by identifying them along β) with $\partial D \subseteq \partial M$; because ∂M is incompressible, this means that D is ∂-parallel; and so the disks D_0 and D_1 are isotopic rel β. By extending this isotopy to an ambient isotopy, we an isotope $\mathcal{L}$ so that $D_1 \subseteq \mathcal{L}$ is contained in A; by isotoping a little further we have arranged that Δ_0 and Δ_1 meet in (at least) one less arc (while possibly removing some <u>other</u> subfamilies of arcs from $\mathcal{L}\cap A$; the point is, it isn't <u>adding</u> any). After finitely many such isotopies, we have removed a family of

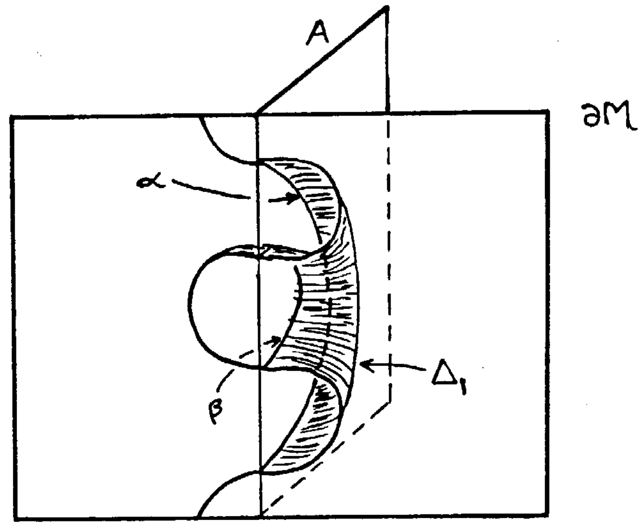

Figure 18: Removing trivial arcs

trivial arcs from λ; so after finitely many isotopies we may assume that λ contains no trivial arcs.

So now $\lambda = \mathcal{L} \cap A$ consists of a lamination containing no trivial loops or arcs; it therefore either consists entirely of essential compact arcs, or contains compact essential loops in its interior.

The following notation will be convenient: we will think of the ∂M_i as alternately consisting of (the images under the splitting of) the annuli A_j in M, and call these <u>cut</u>-<u>annuli</u> A_{ij}, and of (the images of) the boundary tori of M, and call these <u>boundary</u>-<u>annuli</u>, or ∂-annuli A'_{ij}. The boundaries of these annuli in M_i consist of vertical fibers in the model fiberings of the M_i induced from M. Recall that we say that a 1-dimensional lamination λ in a (circle-fibered) torus or annulus is <u>semi-transverse</u> if it contains a (possibly empty) sublamination λ_0 consisting of vertical loops, and every leaf of $\lambda \setminus \lambda_0$ is everywhere transverse to the circle fibering.

Now we have a particular ∂-component T of M in which $\partial\mathcal{L}$ is well-behaved; also, we arranged that every A_i met T, which in our new language means that every cut-annulus in ∂M_i meets a ∂-annulus <u>coming</u> <u>from</u> <u>T</u> on one side or the other. This will allow us to show that the laminations $\mathcal{L}_i = \mathcal{L} \cap M_i$ are in fact of a very nice form.

First, $\mathcal{L}_i \cap A'_{ij}$, a (1-dimensional) lamination in $A'_{ij} \subseteq \partial M$, contains no trivial arcs. For if it did, then let α be an outermost such arc (see Figure 19). Note that A'_{ij} cannot be contained in T, because $\partial\mathcal{L}$ was made semi-transverse to the circle fibers in T. But then the cut annulus A_{ij} containing $\partial\alpha$ then meets T on its <u>other</u> side, and there are compact essential arcs β_1, β_2 in A_{ij} containing the endpoints of α. For otherwise these endpoints are contained in a pair of half-infinite arcs in A_{ij}, and the disk of A'_{ij} that α cuts off together with the half-infinite rectangle of A_{ij} that the arcs cut off would form an end-compressing disk for $\mathcal{L}$ in M. Then the arc $\beta_1 \cup \alpha \cup \beta_2 = \alpha'$ lies in some leaf L of $\mathcal{L}$, and is homotopic in M (in fact, in M_i) to an arc of ∂A_{ij} which lies in a circle fiber of T. The ∂-injectivity of $\mathcal{L}$ then requires that α' be homotopic rel endpoints to an arc α'' in $\partial L \cap T$. But this is impossible, because the endpoints of α' lie in leaves of $\partial\mathcal{L} \cap T$ which are horizontal in T, and so α'' together with the arc in ∂A_{ij} would form an <u>essential</u> loop in T. But the arc in ∂A_{ij} is homotopic (rel endpoints) to α', which is homotopic (rel endpoints) to α'', so the loop is <u>trivial</u> in M, proving that T is compressible, a contradiction.

Therefore, since $\partial\mathcal{L}_i \cap A'_{ij}$ contains no trivial loops (this would imply trivial loops for $\partial\mathcal{L}$ in ∂M, giving ∂-parallel disks in $\mathcal{L}$), $\lambda_i = \partial\mathcal{L}_i$ meets the A'_{ij} in a semi-transverse lamination, and hence either consists of essential arcs or contains essential (vertical) loops. Notice that this means that if $\mathcal{L}$ meets <u>one</u> boundary component in a semi-transverse lamination, then it meets <u>all</u> of the boundary components in semi-transverse laminations (two semi-transverse laminations in two annuli, glued along a boundary component of each, gives a semi-transverse lamination).

This also means that $\partial\mathcal{L}_i$ contains no trivial loops (a trivial loop would have to create a trivial arc in one of the cut- or boundary-annuli).

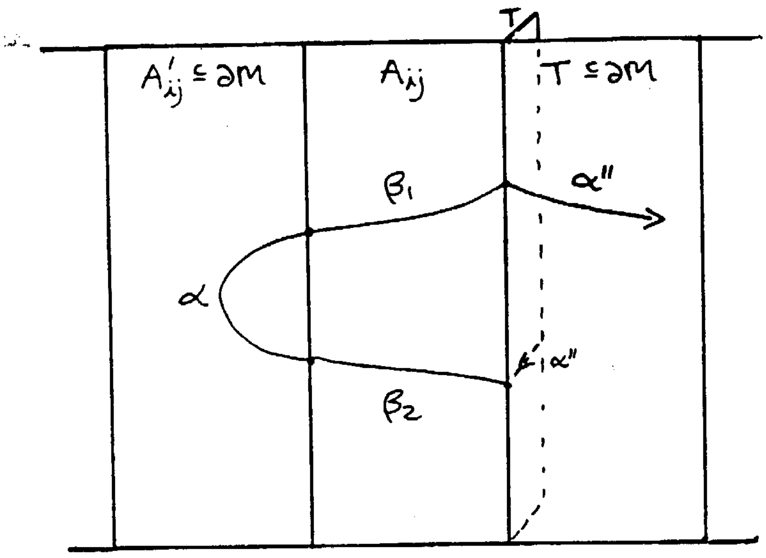

Figure 19: Finding a ∂-compressing disk

Lemma 4.1: Each $\mathcal{L}_i$ is π_1-injective in M_i.

Proof: If γ is a loop in $\mathcal{L}_i$ which is null-homotopic in M_i, then thought of in M it is a null-homotopic loop which <u>misses</u> A. Without loss of generality, we may assume that $\gamma \subseteq \mathrm{int}(M_i)$. By the π_1-injectivity of $\mathcal{L}$, γ is null-homotopic in the leaf L of $\mathcal{L}$ containing it. We will show how to replace this null-homotopy $H{:}D^2 \to L$ with one in L which <u>misses</u> A. We may homotope H (rel ∂D) to make the map transverse to A. $H^{-1}(A)$ then consists of (a finite number of) loops in $\mathrm{int}(D^2)$ (since $H(\gamma) \cap A = \emptyset$) which bound disks in D. Choose an innermost disk Δ, $\partial\Delta = \delta$. $H|_\delta$ is null-homotopic in L and contained in A; since $\mathcal{L} \cap A$ contains no trivial circles, it is also in fact null-homotopic in $L \cap A$ (δ null-homotopic in L and contained in A means it is null homotopic in A, since the components of A are π_1-injective in M_i (they are vertical); null-homotopic in A and contained in L means it either goes around a trivial disk, which isn't there, or it is contained in a contractible or π_1-injective set in $L \cap A$, hence null-homotopic in that component). Redefining H on Δ to be in $L \cap A$, and pushing off A, we have reduced the number of loops in H^{-1}(A). By induction, we can find a null-homotopy H with H(D^2) $\subseteq$L\A, i.e., with image in $\mathcal{L}_i$. ∎

Therefore, each $\mathcal{L}_i$ is an 'essential' lamination in the solid torus M_i which contains it. By the theorem of Chapter 3, it therefore consists either of meridian disks, or (in some possibly different circle fibering) contains a vertical sublamination.

Lemma 4.2: $\mathcal{L}_i'$ either consists of meridian disks or contains a vertical sublamination in the Seifert-fibering induced from M.

Proof: Suppose not; suppose $\mathcal{L}_i$ contains a vertical sublamination in a circle fibering distinct from the one induced from M. Then, in particular, the boundary loop of a compact leaf L of $\mathcal{L}_i$ would meet a boundary-annulus A'_{ij} of ∂M_i corresponding to T (see Figure 20). The leaf L is ∂-compressible in M_i; $\pi_1(L,\partial L)\neq 0$, while $\pi_1(M_i,\partial M_i)=0$. In particular, there is an arc in L which is homotopic rel endpoints to a (singular) arc in $\partial A'_{ij}$; just choose any arc in L essential in relative π_1, and drag its endpoints around in ∂L until they lie in the same component of $\partial A'_{ij}$. This arc is homotopic to an arc in ∂M_i, and one can kill any meridional component of this arc using the fact that the meridian is homotopic to a point in M_i.

If the endpoints of this arc γ are distinct, then the homotopy of this arc in L to an arc in $\partial A'_{ij}\subseteq$T gives a ∂-compressing disk for $\mathcal{L}$ in M, a contradiction. If the endpoints of γ are identical, then it follows that L is a Möbius band; a slightly different argument must be used in this case.

γ is now a closed loop, homotopic rel a basepoint $\gamma(0) = \gamma(1) \subseteq\partial A'_{ij}$ to a closed loop β contained in $\partial A'_{ij}$, a circle fiber of M in T. Further, β is non-trivial in this fiber; L is orientation-reversing around γ, so it can't be null-homotopic, so β is essential in M_i. γ is also still an arc in L homotopic rel endpoints to an arc in T$\subseteq\partial$M; so (thinking now in M) by ∂-injectivity of $\mathcal{L}$ it is homotopic rel

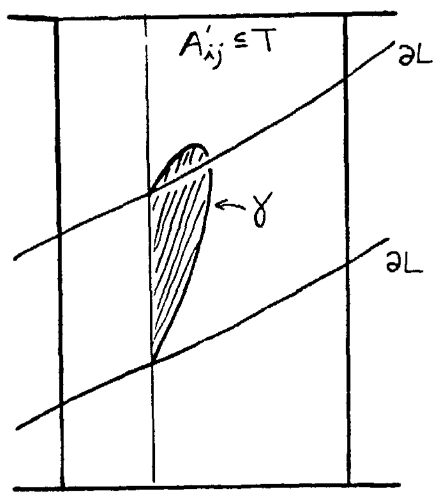

Figure 20: Ruling out other annuli

endpoints (in the leaf of $\mathcal{L}$ containing L) to a loop in T. This loop is homotopic rel basepoint to the loop in the fiber $\partial A'_{ij}$; because T is incompressible, they are therefore homotopic in T. $\mathcal{L} \cap T$ therefore contains a leaf containing an essential loop homotopic in T to a non-trivial multiple of a circle fiber of M; this implies that the leaf is a circle fiber of M. But this leaf (contains the basepoint of γ and therefore) meets A, a contradiction. ∎

With this lemma we can finish the proof of the theorem. If every $\mathcal{L}_i$ consists of meridian disks (whose boundaries have already been made transverse to the circle fiberings of ∂M_i), then we can, by an isotopy of $\mathcal{L}_i$ supported away from ∂M_i, make $\mathcal{L}_i$ a collection of horizontal disks. $\mathcal{L} = \cup \mathcal{L}_i$ is then a lamination which meets each of the solid tori M_i in a horizontal lamination, and is therefore a horizontal lamination in M.

If one of the $\mathcal{L}_i$ contains a vertical sublamination, then consider all of the vertical sublaminations in all of the $\mathcal{L}_i$. They each meet ∂M_i in the (entire) collection of vertical loops of $\partial \mathcal{L}_i$, and so they glue together across the A_j to give a lamination which is a vertical sublamination $\mathcal{L}_1$ of $\mathcal{L}$. All of the other leaves of the $\mathcal{L}_i$ (meet ∂M_i transverse to the circle fibering and so) can be isotoped to be horizontal in the M_i; they then glue together across the A_j to give a horizontal lamination $\mathcal{L}\backslash\mathcal{L}_1$ asymptotic to $\mathcal{L}_1$.

Chapter 5

The Case of Empty Boundary

This chapter is devoted to proving the following result:

Theorem 2.4: Let M be a closed orientable Seifert-fibered 3-manifold, and $\mathcal{L}$ an essential lamination in M. Then either $\mathcal{L}$ contains a horizontal sublamination, or (possibly after splitting $\mathcal{L}$ along a (single) one-sided leaf) $\mathcal{L}$ contains a vertical sublamination.

Let us recall some of our notation from Chapter 1. Let M be an orientable closed Seifert-fibered space. The base F of the fibering is a closed surface, with some collection of distinguished points $p_1, ..., p_n$ which are the images of the multiple fibers in M. If we triangulate F so that each p_i is in the interior of a 2-simplex, and no 2-simplex contains more than one of the p_i, then we have that

π^{-1}(0-simplex)= regular fiber of the fibering of M

π^{-1}(1-simplex)= vertical annulus = A_j

π^{-1}(2-simplex)= vertical (saturated) solid torus in M (with model fibering induced from the fibering of M).

In this way, we think of M as a union of (embedded) solid tori M_i, i=1,...,r which meet one another in the annuli A_j in their boundaries. We set $S=\pi^{-1}(F^{(0)})$, the collection of *sentinel fibers* of the decomposition of M into solid tori.

A. The isotopy process

The strategy of the proof is to set up an isotopy process, i.e., a sequence of isotopies I_j which will, one by one, isotope $\mathcal{L}$ to meet the i^{th} solid torus ($j \equiv i$(mod r)) only in horizontal disks, while at the same time controlling the intersection of $I_j(\mathcal{L})$ with the sentinel fibers S. What we will see is that if at any stage of the process we are unable to continue the isotopy process, we can use this information to find a vertical sublamination of $\mathcal{L}$ (after possibly splitting one of the leaves of $\mathcal{L}$). Otherwise, we are able to continue the isotopy process indefinitely, and then we will be able to see that (larger and larger pieces of) $\mathcal{L}$ begin to limit on (larger and larger pieces of) some lamination $\mathcal{L}_0$, which, by its construction, is horizontal; as it turns out, $\mathcal{L}_0$ is in fact a sublamination of $\mathcal{L}$.

Now recall that in Chapter 1 we showed how to isotope a lamination so that it meets a (vertical) solid torus M_i in a lamination $\mathcal{L}_i$ with π_1-injective leaves. Consider now how this isotopy affects $\mathcal{L} \cap S$, the intersection of $\mathcal{L}$ with the sentinel fibers S. This isotopy was described in terms of doing surgery on $\mathcal{L}$ in the boundary of the solid torus, and then throwing away any 2-spheres which are created. In terms of the sentinel fibers, this means that $\mathcal{L} \cap S$ (after surgery) is contained in $\mathcal{L} \cap S$ (from before the surgery). This is what we mean by controlling the isotopies. We will call an isotopy which has this control conservative.

Now after this (preliminary) isotopy, we have arranged that $\mathcal{L} \cap M_i = \mathcal{L}_i$ is π_1-injective in M_i. It is also end-incompressible, and contains no spheres or ∂-parallel disks (by construction), so it is 'essential'. By the Theorem it then either consists of meridional disks, or contains a vertical sublamination w.r.t. some Seifert-fibering of M_i (not necessarily the one that it inherits from M).

Let us consider first the case that $\mathcal{L}_i$ consists of meridional disks. We wish to show that, by an isotopy of $\mathcal{L}$ which controls the intersection of $\mathcal{L}$ with S, we can make $\mathcal{L}$ meet M_i in a collection of taut disks (meaning each disk meets each annulus of ∂M_i |S in essential arcs). To do this, consider $\lambda = \partial \mathcal{L}_i \subseteq \partial M_i$, and its intersection with each annulus complement A_j of S in ∂M_i (see Figure 21). This intersection consists of a finite number of parallel families of essential and trivial arcs in A_j.

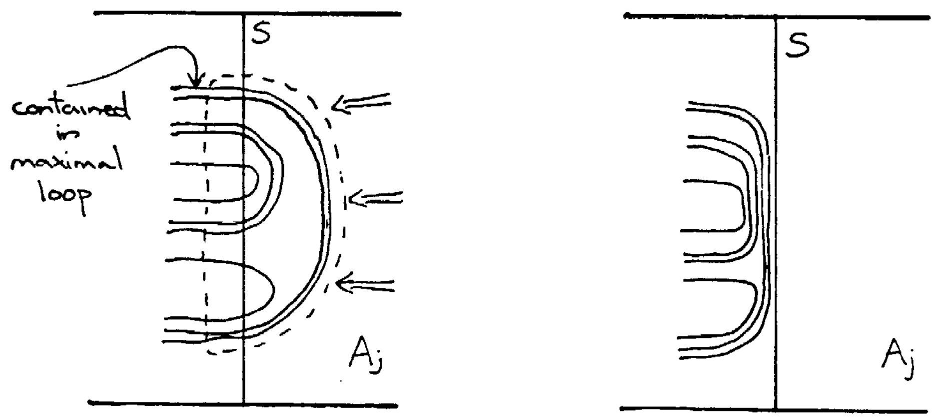

Figure 21: Pulling meridional disks taut

Note that because λ is (assumed to be) carried by a train track $\tau = B \cap \partial M_i$, there is an upper bound on the number of times a loop in λ can meet S (the loops fall into a finite number of loops parallel in τ; each loop in a family meets S the same number of times).

Any collection of trivial arcs in an A_j can be removed by an isotopy of $\mathcal{L}$ supported in a neighborhood of the disk which the innermost arc of the family splits off from A_j. This reduces the number of times the loops of λ containing these arcs meets S. By an inductive use of this process (make sure you push trivial arcs contained in every maximal loop before continuing - this is possible because we need not worry about choosing outermost families), we are then done: eventually every loop of λ must be taut. Note that this isotopy never adds points to $\mathcal{L} \cap S$, only removes them.

If, on the other hand, $\partial \mathcal{L}_i$ contains non-meridional loops, then $\mathcal{L}_i$ contains an annulus or Möbius band leaf. Look at the collection C of compact leaves of $\mathcal{L}_i$; it is a (closed) sublamination of $\mathcal{L}_i$. $C \cap \partial M_i$ consists of a collection of parallel loops in ∂M_i; by a process similar to that just described, we can make these loops meet S tautly.

There are now two cases to consider:

(a) $\partial C \subseteq \partial M_i$ runs parallel to S (i.e., $C \cap S = \emptyset$), or C contains a Möbius band leaf. Then (see Fact 6 of Chapter 3 for the Möbius band case) we can isotope C (in so doing isotoping $\mathcal{L}$) so that C contains a circle fiber of M. Therefore, possibly after splitting $\mathcal{L}$ along the leaf containing the fiber, we may assume that $\mathcal{L}$ misses a circle fiber γ of M, and therefore misses a small (fibered) neighborhood of γ, and so we can consider $\mathcal{L} \subseteq M \setminus \mathrm{int}(N(\gamma)) = M_0$. Now, thought of in M_0, $\mathcal{L}$ is still essential: π_1-injectivity of leaves follows from the injectivity of the composition $\pi_1(L) \rightarrow \pi_1(M_0) \rightarrow \pi_1(M)$, ∂-injectivity is vacuous ($\mathcal{L}$ misses ∂M_0), irreducibility of $M_0|\mathcal{L}$ follows because γ is essential in $M|\mathcal{L}$, and end-incompressibility follows easily (using tail-incompressibility).

Therefore by Corollary 2.3, $\mathcal{L}$ contains a vertical sublamination in M_0, and hence contains a vertical sublamination in M, proving the theorem.

(b) $\partial C \subseteq \partial M_i$ meets S, and C does not contain a Möbius band leaf. Then every leaf of C is a ∂-parallel annulus, and the loops of ∂C meet S non-trivially and tautly. The leaves of C again fall into a finite number of parallel families in M_i. Choose an innermost leaf L of an outermost family in C, and choose a ∂-compressing disk Δ for L, $\partial \Delta = \alpha \cup \delta$, with $\alpha \subseteq L$ and δ contained in a loop of S. By the usual methods we can assume that $\mathcal{L}$ meets Δ transversely in a collection of arcs.

Then by doing a ∂-surgery on $\mathcal{L}$ using (a disk slightly larger than) Δ, we can split the annulus leaves in the same family as L into a collection of trivial disks (see Figure 22), which we can then isotope away using our previous methods. Note that this creates no new families of annuli or Möbius bands; the effect of surgery on leaves near L is to cut off half-infinite rectangular tails from simply-connected

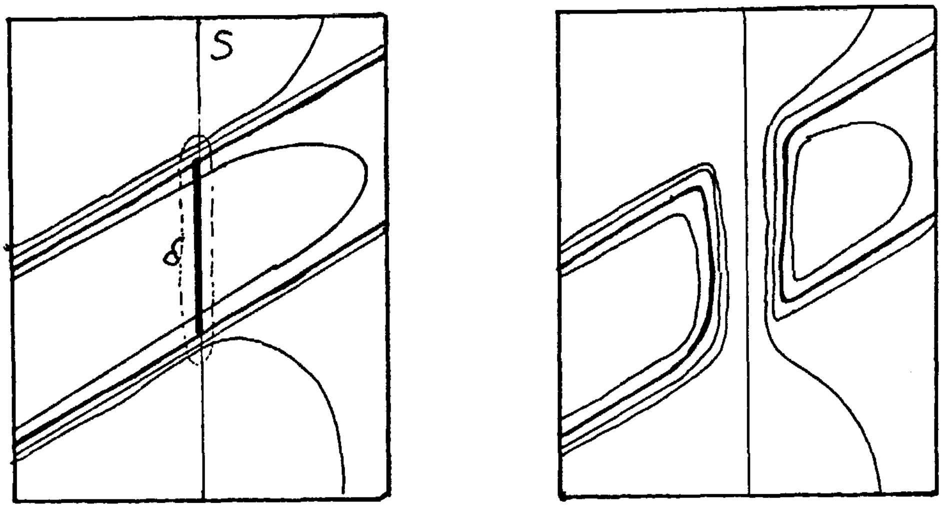

Figure 22: Killing annuli

leaves (each parallel family is open and closed in $\mathcal{L}_i$), and cut them into trivial disks. So simply-connected leaves remain simply-connected. It also adds no new points of intersection to S.

Therefore, after a finite number of such surgeries, we can kill off all of the annulus leaves of C; $\mathcal{L} \cap M_i$ then must consist of meridional disks (because it is still π_1-injective and end-incompessible), which we treat as before.

The construction above forms the core of our isotopy process. Starting with $\mathcal{L}$, either it contains a vertical sublamination or there is a conservative isotopy I_1 so that $I_1(\mathcal{L})$ meets M_1 in a collection of taut disks. We now continue cyclically through our list of solid tori $M_1, \ldots, M_r$, so that at stage j, we are adding to the previous isotopies, trying to make $I_j(\mathcal{L})$ meet M_i in taut disks, where $j \equiv i (\mathrm{mod}\ r)$. By the above construction, either this isotopy can be built, or $\mathcal{L}$ contains a vertical sublamination.

If we therefore assume that $\mathcal{L}$ does not contain a vertical sublamination, then are able to construct an infinite sequence of isotopies I_j with the property that $I_j(\mathcal{L})$ meets M_i in a collection of taut disks. If at any stage $I_j(\mathcal{L})$ meets <u>all</u> of the solid tori $M_1, \ldots, M_r$ in taut disks, then as in Chapter 4 these disks can be 'straightened' out, completing the isotopy of $\mathcal{L}$ to a horizontal lamination. Thus $\mathcal{L}$ is itself a horizontal lamination.

Because each of the above two situations justify the theorem, we (can and) will assume from now on that neither of them hold; i.e. $\mathcal{L}$ does not contain a vertical sublamination (up to splitting a leaf), and is not itself isotopic to a horizontal lamination. In other words, not only can we build the isotopies that we want to, but we in fact need to. In what follows now, we will think of these isotopies as defining an infinite isotopy process; we find ourselves forever pushing $\mathcal{L}$ around, and are 'not quite' able to make it all horizontal.

We will need a little more notation to continue. We have defined I_j as the composition of the first j isotopies of $\mathcal{L}$, making $\mathcal{L}$ meet the solid tori M_i cyclically in taut disks. We will let $I_{(j)}$ represent any stage of the isotopy between I_{j-1} and I_j. We will also let $I_{j,k}$ denote the composition $I_k \circ I_j^{-1}$ (i.e., the composition of the isotopies built between the j^{th} and the k^{th} stages), so that $I_{j,k} \circ I_j = I_k$.

B. Finding stable arcs

Now we have an isotopy process, and we assume that it continues indefinitely. This means that at no stage does it succeed in pulling $\mathcal{L}$ horizontal, but for all j, the isotopy I_j succeeds in making $\mathcal{L}$ meet M_i in taut disks, where $j \equiv i (\text{mod } r)$. Now for each j, the points $I_j(\mathcal{L}) \cap S$ form a (closed) collection of points in S. By the construction of the isotopy I_j, these points were never moved by any of the isotopies that went into the construction of I_j, i.e., they are stable under these isotopies. In particular, for $j \leq k$, $I_j(\mathcal{L}) \cap S \supseteq I_k(\mathcal{L}) \cap S$, i.e., these sets are nested. They are also non-empty; if $I_j(\mathcal{L}) \cap S = \emptyset$, then by the argument above, $\mathcal{L}$ misses a fiber of M (i.e., any of those in S), and so contains a vertical sublamination.

So we have a nested sequence of closed, non-empty subsets of the compact set S, their intersection $\cap(I_j(\mathcal{L}) \cap S) = P_0$ is therefore non-empty. P_0 in fact meets every component of S; for otherwise, the argument above applies to the component it missed. By construction, P_0 consists of all of the points of $\mathcal{L} \cap S$ which are never moved by any of the isotopies in our isotopy process, i.e., they represent the stable points of our isotopy process. What we will now show is that, as we watch the isotopies progress, these points become 'islands of stability' for the process; a stable (horizontal) lamination starts to grow out of them.

Now, consider a 1-simplex $e_i \in B^{(1)}$ and the annulus $A_i=\pi^{-1}(e_i)$, $\partial A_i \subseteq S$. Pick points x, x' of P_0, one in each component of ∂A_i. What we wish to look at now are the arcs of $I_j(\mathcal{L}) \cap A_i$ containing x, x' (call them, respectively, α, α'), and how they change under further isotopies. Because for each arc one of its endpoints is anchored down (x, x' are stable), the only way these arcs can change is by 'boundary compressions' (see Figure 23). Our intent is to show that for some $k \geq j$, each of these arcs $I_{j,k}(\alpha)$, $I_{j,k}(\alpha')$ has both of its endpoints in P_0. This arc would therefore be stable, i.e., $I_{j,k}(\alpha)$ (say) would be fixed under all further isotopies.

We proceed as follows. Given α, $\alpha' \subseteq A_i$, there exists an arc ω_j (for 'winding number') joining x to x' and not meeting α, α' except at their endpoints. This is because A_i split on α, α' has 2 or 3 (if α, α' are both trivial arcs in A_i) components, at least one of which contains both x and x'.

Lemma 5.1: If at some further stage $I_{j'}$ of the isotopy process, one of the arcs emanating from x, x' has non-zero winding number wrt. ω_j (meaning it is not isotopic rel endpoints to an arc meeting ω_j only at its endpoints), then at some stage of the isotopy process between I_j and $I_{j'}$, one of the arcs emanating from x or x' was trivial, i.e., ∂-parallel in A_i.

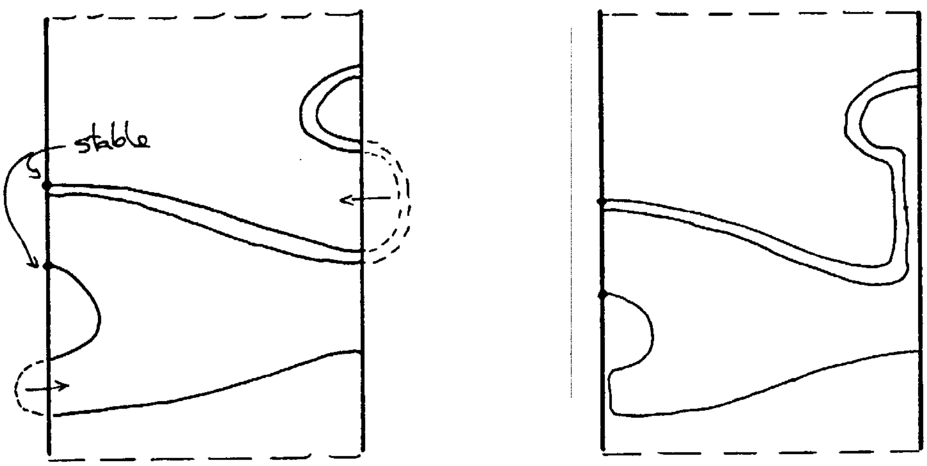

Figure 23: Boundary-compressions

Proof: Since α, α', have zero winding number wrt. ω_j, and change only by ∂-compressions, there is a first ∂-compression after which one of the arcs has non-zero winding number. We claim that, at the time of this compression, one of the arcs is trivial.

For suppose not; note that since the stable ends of the arcs α, α' are on opposite sides of A_i, the ∂-compression leaves one of the arcs, say α', fixed. Since this is the first ∂-compression where the winding number changes, we have that the winding number of α' is zero. Now if α' is not trivial, then its other endpoint is on the same side as x (see Figure 24). Since α is not trivial, its other endpoint is on the x'-side of A_i, so the ∂-compression is taking place on that side. But now just note that because after the compression the arc emanating from x cannot meet α' (because after the compression, $\mathcal{L}$ still meets A_i in a lamination, which can't have leaves intersecting), which hasn't been moved, only one of two things can have occurred: either **(1)** the new arc α_{new} is a trivial arc, in which case it is isotopic rel endpoints to an arc in ∂A_i, with x as an endpoint, so has zero winding number, or (@) α_{new} is an essential arc (which lies in $A_i|\alpha'$, which is a disk), and so is isotopic rel x to

α, by an boundary-preserving isotopy which does not meet x'; and therefore α_{new} also has winding number zero w.r.t. ω_j, since it must then have the same winding number that α has. Both of these situations, however, violate our hypothesis, giving the necessary contradiction. ∎

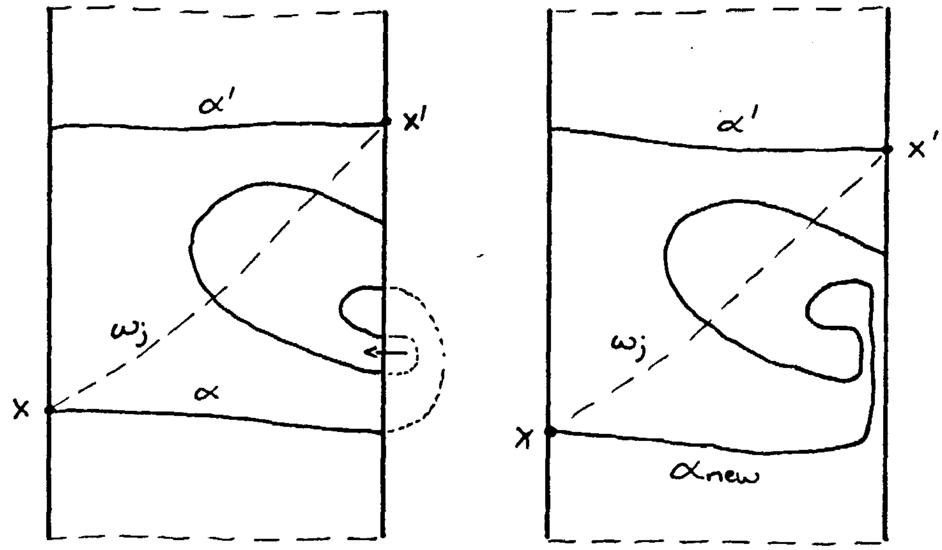

Figure 24: Winding numbers

In other words, if one of the arcs moves alot, then one of the arcs had to be trivial (at some time). Note that it is possible to achieve non-zero winding number w.r.t. ω_j, provided one allows trivial arcs to occur; see Figure 25.

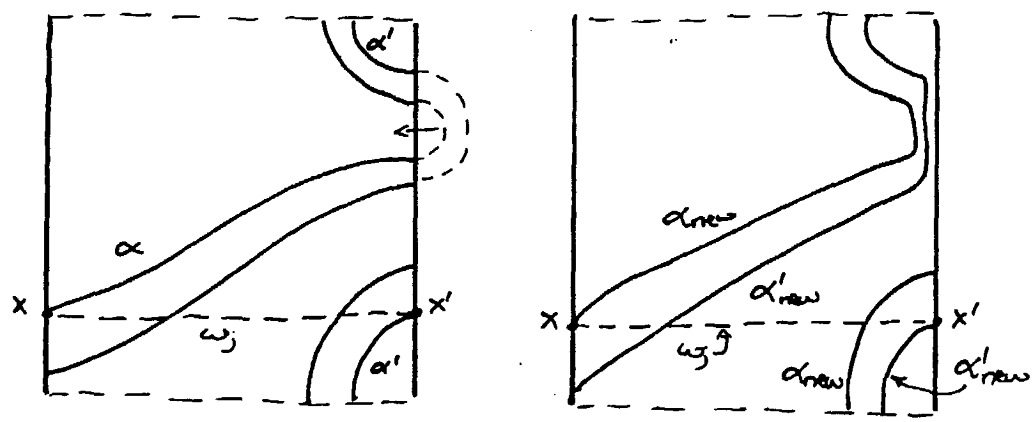

Figure 25: Achieving non-zero winding number

It then follows, by an inductive use of the lemma (since the arcs emanating from x, x' in $\mathrm{I}_{\mathrm{nr+i}}(\mathcal{L}) \cap \mathrm{A_i}$ are non-trivial (they are contained in the boundary of taut disks in $\mathrm{M_i}$)), that one of two things will happen:

(1) one of the points x, x', is the endpoint of a trivial arc in $\mathrm{I}_{(\mathbf{k})}(\mathcal{L}) \cap \mathrm{A_i}$ infinitely often (i.e., for arbitrarily large values of k),

or

(2) eventually, neither point is contained in a trivial arc, and there exists j, and ω_j so that for k$\geq$j, the arcs of $\mathrm{I}_{(\mathrm{k})}(\mathcal{L}) \cap \mathrm{A_i}$ emanating for x, x', never have non-zero winding number wrt. ω_j.

(Recall that $\mathrm{I}_{(\mathrm{k})}$ stands for <u>any</u> boundary compression in the isotopies used to get $\mathrm{I_k}$ from $\mathrm{I_{k-1}}$.)

What we now show is that the first of these possibilities must necessarily lead to a contradiction, while the second leads to the eventual stability of the arcs emanating from x, x' (in order to avoid a contradiction similar to the one encountered in the first case).

First case: x (say) is contained in a trivial arc α_k of $I_{(k)}(\mathcal{L}) \cap \mathrm{A_i}$ for arbitrarily large values of k.

What we will do now is watch the proliferation of the intersections of these trivial arcs with the <u>neighbor</u> <u>loop</u> γ, a loop in $\mathrm{A_i}$ lying parallel to the component of $\partial\mathrm{A_i}$ containing x. Recall that our isotopies are conservative, so that the only points of the intersection of $\mathcal{L}$ with the sentinel fibers which move are those which disappear. Now the effect of a ∂-compression on the arc α_k is to cut off a short arc near its non-stable end, and splice it to another arc by an arc running in the annulus between γ and the loop of $\partial\mathrm{A_i}$ it runs next to. Such compressions do not remove points of intersection of α_k with γ. (Note that we can also assume that this is true whenever any arc moves by some isotopy, but one of its endpoints doesn't.) We can therefore further assume that the points of $\alpha_k \cap \gamma$ are fixed under all further isotopies, i.e., $\alpha_k \cap \gamma \subseteq \alpha_{k'} \cap \gamma$ whenever $k' \geq k$. Further, since the arc containing x periodically becomes essential (every time $\mathcal{L} \cap M_i$ is pulled taut), it follows that this inclusion is usually proper, i.e., these trivial arcs continue to pick up more and more points of intersection with the neighbor loop as k gets larger and larger. It is the fact that these points must be piling up on one another in the neighbor loop that is going to give us our contradiction.

First we need some notation. Let ω (for winding number) be an essential arc in A_i whose endpoints in ∂A_i are not in $\mathcal{L}$ (in fact, since $\mathcal{L} \cap \partial A_i$ is closed, we may assume ϵ -neighborhoods (in ∂A_i) of the endpoints do not meet $\mathcal{L}$, for sufficiently small ϵ). Orient ω with tail z on the component S of ∂A_i containing x. z and x separate S into two arcs, called the left side and the right side of ω. Orient the α_k with tail at x, and orient the neighbor loop γ. Using these orientations, we can assign local orientations to the points of $\alpha_k \cap \gamma$, and winding numbers to the arcs of α_k between x and a point of $\alpha_k \cap \gamma$. Note that because the isotopies are constant near the points of $\alpha_k \cap \gamma$, and $(\alpha_k \cap \gamma) \subseteq (\alpha_{k+1} \cap \gamma)$, it follows that the local orientation assigned to a point is the same as the one assigned when thought of as living in every further arc α_k. Also, the winding numbers associated to a subarc of α_k is actually a function of its endpoint $t \in \alpha_k \cap \gamma$, because the arcs in α_k and in α_{k+1} between x and t are identical.

Call the other endpoint of α_k (i.e. the one which isn't x) x_k, and the intersection point of α_k with γ, which is adjacent to x_k along α_k, y_k (see Figure 26).

Then the winding number of the arc β_i of α_i between x and y_i is always either -1, 0, or 1. This is because β_i differs from α_i only in the short arc between x_i and y_i (which doesn't meet ω), and α_i has one of the above mentioned winding numbers

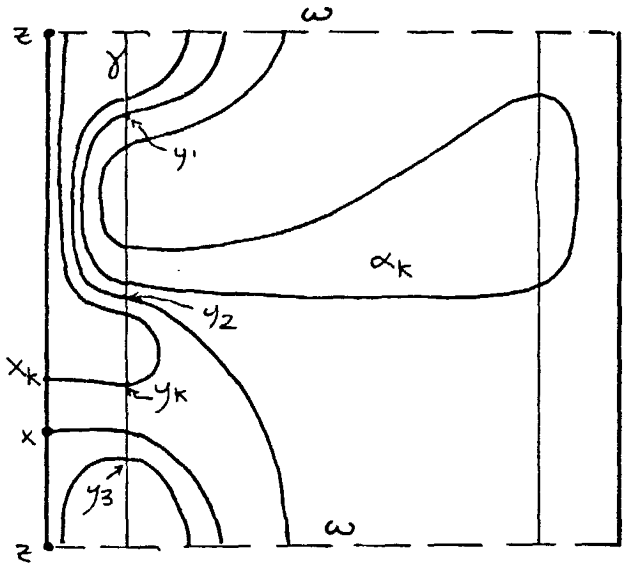

Figure 26: Stabilization: first case

because, being trivial, it is homotopic (in fact isotopic) rel endpoints to a subarc of S$\in \partial$A$_i$, which meets the winding arc ω at most once. Therefore, the winding number of the points y_i in α_k is either -1, 0, or 1, i.e. the (running) winding number of the arc β_i in α_k between x and y_i can take on only 3 values. So at least one of the values will be attained an arbitrarily large number of times.

Now lift the α_i to the universal cover π:R$\times I \rightarrow$ A$_i$ of A$_i$, sending x to (0,0)=$\tilde{x}$, and let $\tilde{y}_i$ be the resulting lifts of the points y_i. Let $\tilde{\gamma}$=$\pi^{-1}(\gamma)$ = the 'neighbor line', so $y_i \in \tilde{\gamma}$. Because we could calculate the winding number of α_i w.r.t. ω by lifting α_i to $\tilde{\alpha}_i$ and count the winding number w.r.t. <u>all</u> of the lifts of ω in $\mathbf{R}\times$I, and this amounts basically to calculating the integer part of the first coordinate of $\tilde{y}_i \in \mathbf{R}\times$I, it follows that the points $\tilde{y}_i$ must lie in a compact piece [-2,2]$\times$I of $\mathbf{R}\times$I.

So these points $\tilde{y}_i$ must be piling up on one another. In particular, for any $\epsilon > 0$, there exist points $\tilde{y}_i$, $\tilde{y}_j$, $j < i$, which are within ϵ of one another along $\tilde{\gamma}$. $\tilde{\beta}_i \setminus \tilde{\beta}_j$ is the arc of $\tilde{a}_i$ between $\tilde{y}_j$ and $\tilde{y}_i$, which together with the arc of $\tilde{\gamma}$ between these two points, forms a (null-homotopic; $\mathbf{R}\times$I is contractible) loop in $\tilde{A}_i$. This loop projects down in A_i to a loop consisting of the arc $\beta=\beta_i\backslash\beta_j$ in α_i, together with an arc of length $< \epsilon$ in γ, and this loop is null-homotopic.

Now, consider this short arc δ between y_i and y_j in γ. If $\beta \cap \delta \subseteq \partial\delta$, then $\beta \cup \delta$ is an embedded null-homotopic loop in A_i, hence bounds a disk D in A_i with $\partial D = \beta \cup \delta$, where $\beta \subseteq \mathcal{L}$, and δ is a short arc (of length $< \epsilon$) transverse to $\mathcal{L}$. Thus the Monogon Lemma applies and we can find an end-compressing disk for $\mathcal{L}$, a contradiction.

If β meets δ in the interior of δ, then since $\beta \subseteq \alpha_i$, it follows that α_i meets δ in interior points. Now α_i cuts off a disk Δ in A_i; think of it as being colored green. Δ meets γ in subarcs of $\gamma\backslash\alpha_i$; think of these as being colored green as well. Because α_i separates A_i, it follows that $\gamma\backslash\alpha_i$ consists of an even number of arcs, which (travelling along γ) are alternately colored green and left uncolored (locally, α_i is colored green on only one side).

Since α_i meets $\delta \subseteq \gamma$ in interior points, it follows that $\delta\backslash\alpha_i \subseteq \gamma\backslash\, \alpha_i$ contains a colored subarc, δ_0. δ_0 is contained in Δ, properly embedded, and so it splits Δ into two disks, one of which, Δ_0, does not contain the arc $\eta \subseteq \partial\Delta$. Therefore, $\partial\Delta_0 = \delta_0 \cup \alpha_0$, with $\alpha_0 \subseteq \alpha_i \subseteq \mathcal{L}$, and $\delta_0 \subseteq \gamma$, transverse to $\mathcal{L}$, with length $<$ the length of $\delta < \epsilon$. Therefore the Monogon Lemma applies, giving, again, an end-compressing disk for $\mathcal{L}$, a contradiction.

Therefore, this first situation is impossible.

Second case: α_i and α_i' are always essential (for $i > i_0$), and there is some essential arc $\omega \subseteq A_i$ joining x and x' so that α_i and α_i' always have winding number zero w.r.t. ω.

We wish now to show that eventually α_i (say) becomes stable, i.e., for some i, $\alpha_k=\alpha_i$, for all $k \geq i$. This amounts to saying that $x_k = x_i$, for all $k \geq i$, i.e., $x_i \in P_0$.

So assume the contrary; assume that $x_{k_i} \neq x_{k_{i-1}}$, for $k_i > k_{i-1}$, infinitely often (to save the reader's eyesight, we will conveniently forget that this expression has a 'k' in it, and write x_i instead). We will then obtain a contradiction, in a manner similar to the first case (with some slight technical additions).

We get an arbitrarily large collection of distinct points $y_i \in \gamma$, i= $1, 2, \ldots$, in the α_i which are near neighbors to the endpoints x_i of the α_i. Now, as before, we can lift the α_i, α_i' to $\mathbf{R} \times \mathrm{I} = \tilde{\mathrm{A}}_i$, with $\tilde{x}$ =(a,0), $\tilde{x}'$ =(b,1) fixed. Because the winding number of the lifts of α_i can be counted across the lifts of ω, it follows that the endpoints $\tilde{x}_i$ of the lifts of the α_i based at $\tilde{x}$ all lie in the interval [b-1,b+1]×1 and so the points y_i are contained in a compact piece ([b-1,b+1]×I)$\cap\tilde{\gamma}\subseteq\tilde{\gamma}$ of the neighbor line on the $\tilde{x}_i'$-side of $\mathbf{R}\times$I. So as before we have an arbitrarily large number of $\tilde{y}_i$ accumulating in a fixed compact piece of $\tilde{\gamma}$, so eventually we can find (adjacent) points of (some) $\tilde{\alpha}_i \cap \tilde{\gamma}$ which are within ϵ of one another. the arc of $\tilde{\alpha}_i$ joining these two points, together with the arc of $\tilde{\gamma}$ joining them, form an (embedded) loop in $\mathbf{R}\times$I, which descends to a (singular) null-homotopic loop in $\mathrm{A_i}$.

Finding an <u>embedded</u> loop in $\mathrm{A_i}$ with this property, however, now takes a bit more work. We must learn something about the <u>normal</u> <u>orientations</u> of the arcs α_i, $\tilde{\alpha}_i$, resp., when they meet γ, $\tilde{\gamma}$.

Lemma 5.2: If we orient α_i, α_i' so that x, x' are at their tails, and look at the normal orientations that this induces on the set T= $(\alpha_i \cap \gamma) \cup (\alpha_i' \cap \gamma)$ of (transverse) intersection points with γ, then seen from γ they occur with <u>opposite</u> <u>sign</u>.

Proof: This is basically because α_i and α_i' together separate $\mathrm{A_i}$ (although each separately doesn't) into two disks $\mathrm{D_1}$, $\mathrm{D_2}$ (see Figure 27), with the orientations of α_i, α_i', giving orientations two two arcs in each boundary, as shown. Any arc δ of γ between two adjacent points of T must lie in either $\mathrm{D_1}$ or $\mathrm{D_2}$ ($\mathrm{D_1}$, say). If the

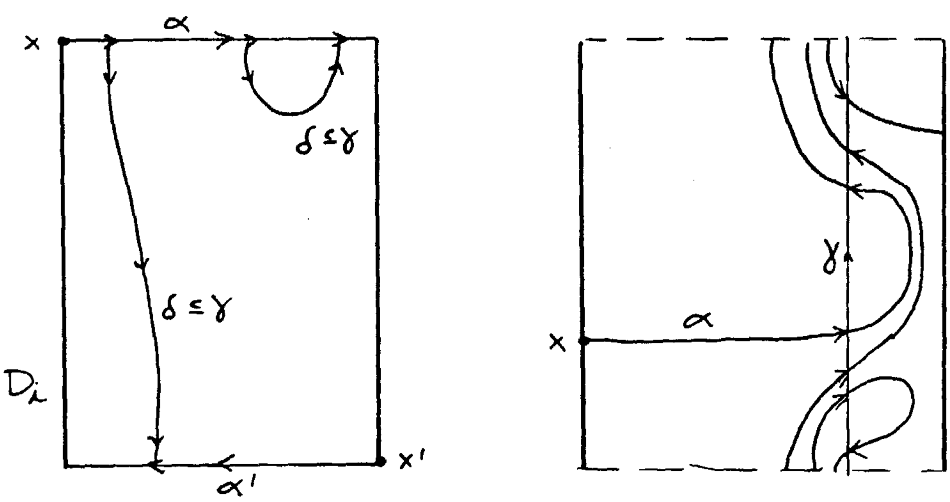

Figure 27: Normal orientations

endpoints of δ both lie on the same end of ∂D_1 then measured along δ the normal orientations of its endpoints are opposite; if they lie on opposite ends of ∂D_1, then, because we chose the orientations of α_i and α_i' to complement one another as they do, measured along δ the normal orientations of its endpoints are again opposite. ∎

Note that this lemma would not be true if we dealt with only one arc (α_i, say) at a time; this is because by itself α_i, say, does not separate A_i (see Figure 27). Note also that if we <u>lift</u> α_i to $\tilde{\alpha}_i$ in $\mathbf{R}\times I$, with the lifted orientation, and look at the normal orientations with which $\tilde{\alpha}_i$ meets the neighbor line $\tilde{\gamma}$, as you travel along $\tilde{\gamma}$ these also alternate; this is because $\tilde{\alpha}_i$ now <u>does</u> separate $\mathbf{R}\times I$, so the situation is just as in the first case of the lemma above.

Now, we have already found adjacent points of (some) $\tilde{\alpha}_i \cap \tilde{\gamma}$ which are within ϵ of one another along $\tilde{\gamma}$. By the note above, these two points inherit opposite normal orientations in $\tilde{\gamma}$ from $\tilde{\alpha}_i$. Together with the arc of $\tilde{\gamma}$ between them, the arc of $\tilde{\alpha}_i$ joining them forms an embedded null-homotopic loop in $\tilde{A}_i$, which descends to a null-homotopic loop in A_i, consisting of an arc β of α_i between points y_{i_0} and y_{i_1}

of $\alpha_i \cap \gamma$, together with the short arc δ of γ between them. If $\beta \cap \delta = \partial\delta$, then, as before, $\beta \cup \delta$ is an embedded null-homotopic loop; the disk it bounds satisfies the hypotheses of the Monogon Lemma, and so $\mathcal{L}$ has an end-compressing disk, a contradiction.

If $\beta \cap \delta \neq \partial\delta$, then in particular $\alpha_i \cup \alpha_i'$ meets δ in interior points. Now these points of intersection inherit normal orientations from α_i and α_i', which when seen along δ occur with opposite sign. The endpoints of δ also have opposite sign (their lifts did in $\tilde{\gamma}$, and they remain the same when projected); it then follows that there are an <u>even</u> number of points in $C=(\alpha_i \cup \alpha_i') \cap \delta$. Since the endpoints of δ both belong to α_i, it then also follows that some pair of points of C, adjacent along δ, both belong to α_i or α_i' (say α_i), joined by a subarc δ_0 of δ. Now α_i and α_i' together separate A_i into two disks D_1 and D_2, and since δ_0 doesn't meet α_i or α_i' except at its endpoints, δ_0 is contained in one of these disks, say D_1. δ_0 separates this disk into two sub-disks; because both of the endpoints of δ_0 are in α_i, one of these disks Δ_0 does not meet ∂A_i (see Figure 28), so its boundary $\partial\Delta_0 = \delta_0 \cup \beta_0$, where β_0 is a subarc of α_i. This disk Δ_0 also satisfies the hypothesis of the Monogon Lemma, and so $\mathcal{L}$ again has an end-compressing disk, a contradiction.

So all other possibilities lead us to the existence of an end-compressing disk for $\mathcal{L}$; we must therefore conclude that, eventually, the arcs of α_i, α_i', for some i, emanating from the points $x,x' \in P_0$ are stable: the other endpoints are also in P_0.

C. Proof of the theorem

We are now in a position to complete the proof of the theorem.

Given a point $x \in P_0$ in the stable set of our isotopy process, and an annulus A_i containing it, in the boundary of a solid torus M_i, we have shown that for some j, the arc α_j of $I_j(\mathcal{L}) \cap A_i$ which contains x is <u>stable</u>; all further isotopies of $\mathcal{L}$ are fixed on α_j. This is equivalent to saying that its other endpoint is also in P_0; since

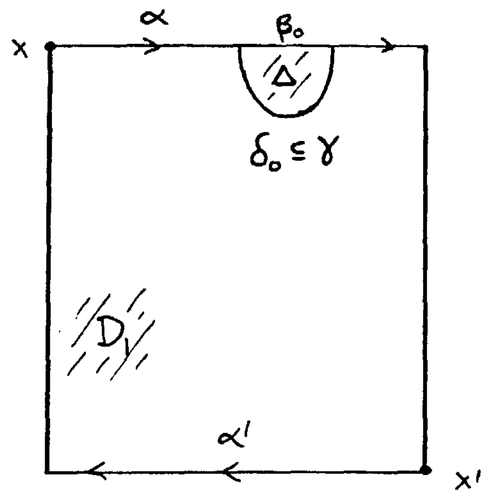

Figure 28: Stabilization: second case

such an arc would only be changed by ∂-compressions, and both its endpoints are stable, this means that the arc cannot be move by further isotopies. Conversely, if an arc is never moved by any of the isotopies, then its endpoints are also never moved, so they survive to the stable set of the isotopy process.

We wish now to show that for some (possibly larger) j, the point x is in fact contained in a stable disk of $I_j(\mathcal{L}) \cap M_i$. A stable disk D is one which every further isotopy leaves fixed; this amounts to saying that $D \cap S \subseteq P_0$.

Consider $x \in \partial M_i$. It is contained in two of the annuli $A_i \subseteq \partial M_i$ to the left and right of it; by the above, for some j the arcs of $I_j(\mathcal{L}) \cap A_i$ have their other endpoints x_{-1}, x_1 in P_0. Work now with these points and the annuli A_i in ∂M_i containing them; the above gives further points x_{-2}, x_2 in P_0in stable arcs containing x_{-1}, x_1. Continue this process. This gives a collection of points $x_{-r}, \ldots, x_{-1}, x, x_1, \ldots, x_r$ in P_0 strung together by (stable) arcs of some $I_j(\mathcal{L}) \cap \partial M_i$.

But after every n isotopies, $I_{kn+i}(\mathcal{L}) \cap M_i$ consists of a collection of taut disks in M_i. Therefore this collection of points is every so often contained in the boundary of a taut disk. But a taut disk always meets $S \cap \partial M_i$ in a fixed number pq of points, where p=the number of loops in S$\cap\partial$M$_i$, and q=the multiple of a generator of π_1(M$_i$) that a loop of S$\cap\partial$M$_i$ represents in M$_i$. So our list of points above cannot grow indefinitely; i.e, eventually some $x_{-r_1} = x_{r_2}$, and the arcs stringing the collection together closes up into a stable (taut) loop in ∂M$_i$; after at most n isotopies, it is contained in a taut disk, so it must in fact be the (entire) boundary of this taut disk. A taut disk with stable boundary can never move (all of our isotopies are ∂-compressions of one sort or another), so this is in fact a stable disk, which contains x.

Since the point x $\in$P$_0$ is contained in only finitely many of the solid tori M$_i$, by applying the above finitely many times, we can conclude that there is a j so that x is contained in a stable disk of I$_j$($\mathcal{L}$)$\cap$M$_i$ for each M$_i$ containing x.

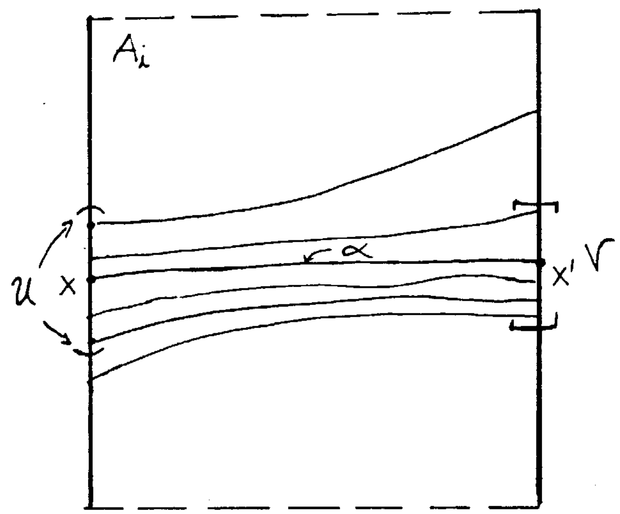

Figure 29: Stabilizing neighborhoods, I

Lemma 5.3: Given x $\in$P$_0$, there is a neighborhood $\mathcal{U}$ of x in S and a j so that for any $x' \in \mathcal{U}$$\cap$P$_0$ and A$_i$ $\subseteq\partial$M$_i$ containing x', x' is contained in a stable arc of I$_j$($\mathcal{L}$)$\cap$A$_i$.

Proof: Fix an annulus A_i containing x. By the above, there is a j so that x is contained in stable arc α of $I_j(\mathcal{L})\cap A_i$, with other endpoint x'. Let $\mathcal{U}$ be a (closed) ϵ-neighborhood of x in the loop of S containing x, intersected with P_0, and consider the (taut) arcs of some $I_{kn+i}(\mathcal{L})\cap A_i$, with $kn + i \geq j$, emanating from these points (then set $j = kn + i$). P_0 is closed, so $P_0\cap\mathcal{U}$ is closed in $\mathcal{U}$; there is therefore a highest and lowest point of P_0 in $\mathcal{U}$. By choosing a larger j, if necessary, we may additionally assume that the arcs of $I_j(\mathcal{L})\cap A_i$ emanating from these points are also stable (see Figure 29). This collection $I_j(\mathcal{L})\cap A_i$ of arcs is a 1-dimensional lamination in A_i, which are all parallel to one another.

Now, suppose an arc β of $I_j(\mathcal{L})\cap A_i$ emanating from a point in $P_0 \cap \mathcal{U}$ moves under a further isotopy (see Figure 30). Consider the <u>first</u> time such a move occurs.

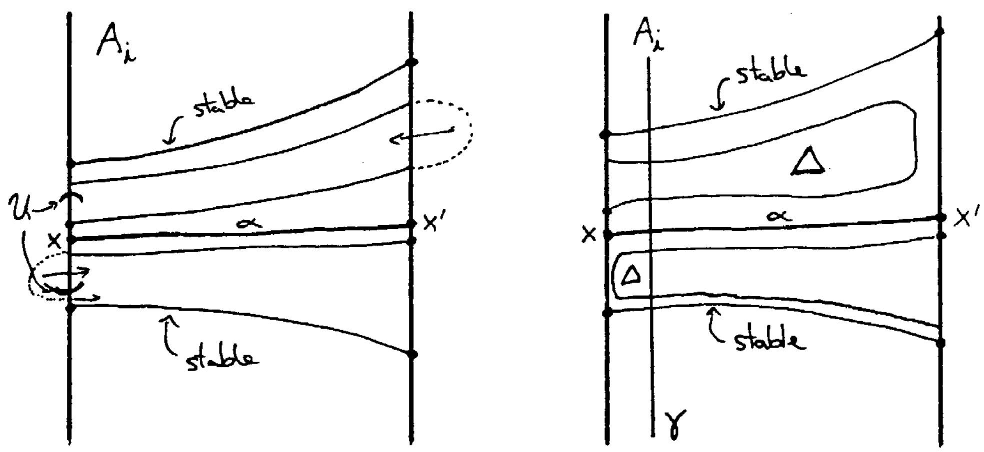

Figure 30: Stabilizing neighborhoods, II

Because the endpoint of arc on the x-side is stable, the change occurs as a ∂-compression on the x'-side of A_i.

If the resulting arc is trivial, then because the points at either end of $\mathcal{U}$ are in stable (essential) arcs, the disk that it cuts off of A_i therefore meets ∂A_i in an arc of $\mathcal{U}\backslash x$ (because x is contained in a stable arc, too), which therefore has length less than ϵ.

If the resulting arc is still essential, then the ∂-compression joined β to a trivial arc on the x'-side of A_i. But such a trivial arc (since all of the arcs between the highest and lowest (essential) arcs out of $\mathcal{U}$ were essential at stage j) had to be created by some x-side ∂-compression at some stage after k; this trivial arc (immediately after the compression) had to meet the neighbor loop γ on the x-side; because it is trivial, arguments as above imply and 'innermost' disk in the disk it cuts off which meets γ only in its boundary (in an arc of length less than ϵ.

In each case we therefore have a situation which gives a disk satisfying the hypotheses of the Monogon Lemma, giving an end-compressing disk for $\mathcal{L}$, a contradiction.

Repeating this argument for each of the annuli containing x, taking the maximum of the j's generated and the intersection of the $\mathcal{U}$'s generated, completes the proof. ∎

Now we have that for each x in P_0 there exists a pair $(\mathcal{U}_x, j_x)$ guaranteed by the lemma. The collection of $\mathcal{U}_x$'s form an open cover of P_0, which, because it is compact (P_0 is closed in S, which is compact), has a finite subcover, $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$. Set $j=\max\{j_1, \ldots, j_n\}$, then it follows that every arc of $I_j(\mathcal{L})\cap A_i$ emanating from any point of P_0, for any A_i, is stable; it has both of its endpoints in P_0.

Now choose a point $x \in P_0 \cap M_i$, for any given M_i. For some r, $0 \leq r < n$, x is contained in a taut disk D of $I_{j+r}(\mathcal{L})\cap M_i$. But by dragging ourselves around ∂D starting from x, we see inductively (using the above) that every point of $\partial D \cap S$ is in fact contained in P_0, i.e., the boundary of this disk is stable, and therefore the disk containing x is stable. It therefore follows that for every $x \in P_0$, and every M_i containing x, x is contained in a stable, taut, disk of $I_{j+n}(\mathcal{L})\cap M_i$. Because $P_0 \cap M_i$ is a closed set in ∂M_i, it follows that the collection of disks of $I_{j+n}(\mathcal{L}) \cap M_i$ containing points of P_0 is a (closed) sublamination of $I_{j+n}(\mathcal{L}) \cap M_i$; the union of these disks over all of the M_i then forms a sublamination $\mathcal{L}_0$ of $I_{j+n}(\mathcal{L})$ (they meet correctly along the ∂M_i, in the (stable) arcs emanating from P_0), which meets each M_i in a collection of tauts disks. By a small further isotopy of $I_{j+n}(\mathcal{L})$ (first supported in a neighborhood of the ∂M_i to make the boundaries of the taut disks transverse to the circle fibering of ∂M_i, then supported away from ∂M_i to make the entire disks transverse) we can make $\mathcal{L}_0$ into a lamination meeting each solid torus in a collection of <u>transverse</u> disks, i.e., $\mathcal{L}_0$ is a horizontal lamination.

Therefore, $\mathcal{L}$ contains a sublamination $I_{j+n}^{-1}(\mathcal{L}_0)$ which is isotopic to a horizontal lamination.

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