MATH 107

SAMPLE FINAL

SPRING 2009

Math 107 — Final Exam — Spring 2009 — May 5, 2009

Name: ____

Lecturer Name (circle one): Bill Rogge Mark Walker Roger Wiegand

Lecture Class Time (circle one):

8:30–9:20 MWF 10:30–11:20 MWF 12:30–1:20 MWF 9:30–10:45 TTh 6:30–8:40 MW

TA Name: ____

General Instructions:

1. Please write you name on each page. There should be 12 problems on 8 pages (including this one). Please check that you have all the pages.
2. Please show all of your work and provide explanation as needed. An answer with no indication of how it was arrived at will not receive much credit, even if the answer is correct.
3. No books or notes are allowed.
4. Please turn off all communication devices
5. Do not spend too much time on any one problem — note the point value of the problem when deciding how much time to spend on it.
6. Calculators without computer algebra systems are allowed. For example, the TI-83, TI-84 and TI-86 *are allowed* but the TI-89, TI-92, TI-Nspire, HP-40, HP-41, Casio ALGEBRA FX 2.0, Casio ClassPad 300, and Casio ClassPad 330 *are not allowed.*
7. Best of luck!

Problem	Points Possible	Your Score
1	20	
2	12	
3	8	
4	27	
5	18	
6	10	
7	20	
8	15	
9	20	
10	18	
11	12	
12	20	
Total	200	

(a) (*5 points*) $3\vec{u} - 4\vec{v}$

(b) (*5 points*) the unit vector in the direction of $\vec{v}$

(c) (*5 points*) $\vec{u} \cdot \vec{v}$

(d) (*5 points*) $\mathbf{proj}_{\vec{v}}\,\vec{u}$ (the projection of $\vec{u}$ onto $\vec{v}$)

2. (*12 points*) Does $\lim_{n\to\infty} \dfrac{\ln(5n^2)}{\ln(n^3)}$ exist? If so, what is the limit? If not, why not?

3. (*8 points*) Give a parametric description of the *line segment* starting at $P = (3, 5, 6)$ and ending at $Q = (8, 2, 6)$. Be sure to include inequalities giving the range of the parameter.

4. Determine whether the following series converge or diverge. Give reasons.

   (a) (*9 points*) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

   (b) (*9 points*) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^4 - 6}}{n^2 + 1}$

   (c) (*9 points*) $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^{-\frac{3}{2}}}{n}$

5. Consider the power series $\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$.

(a) (*10 points*) Find the radius of convergence. Show details.

(b) (*8 points*) Find the interval of convergence, that is, the set of values of x for which the series converges. (In particular, you need to determine whether or not the series converges at the endpoints of the interval.) Explain what you are doing.

6. (*10 points*) Evaluate $\int_0^1 \left\langle \frac{1}{1+t}, e^{5t} \right\rangle dt$. (Note: In case you prefer $\vec{i}, \vec{j}$ notation, this integral is the same as $\int_0^1 \left(\frac{1}{1+t}\vec{i} + e^{5t}\vec{j} \right) dt$.)

7. Starting with the Taylor series centered at $x = 0$ for $\frac{1}{1-x}$, namely, $\sum_{n=0}^{\infty} x^n$, find the following. (For each, either write your answer using sigma notation or include enough terms so that the pattern is evident.)

(a) (*10 points*) The Taylor series centered at $x = 0$ for $\frac{1}{(1+2x)}$.

(b) (*10 points*) The Taylor series centered at $x = 0$ for $\frac{1}{(1+2x)^2}$. *Hint:* Evaluate $\frac{d}{dx}\left(\frac{1}{1+2x}\right)$.

8. (*15 points*) Find $\displaystyle\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx.$

9. Consider the vector-valued function $\vec{r}(t) = \langle 3t^2, 3-t, t^3-t \rangle$.

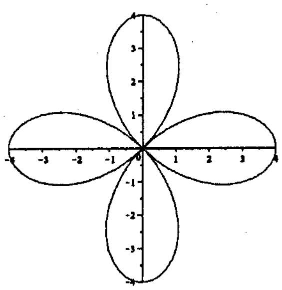
   (a) (*5 points*) Find $\vec{v}(t)$ (i.e., velocity).

   (b) (*5 points*) Find $\vec{a}(t)$ (i.e., acceleration).

   (c) (*5 points*) What is the value of $\|\vec{r}\,'(1)\|$ and what does it represent?

   (d) (*5 points*) Evaluate $\vec{r}\,'(1) \cdot \vec{r}\,''(1)$.

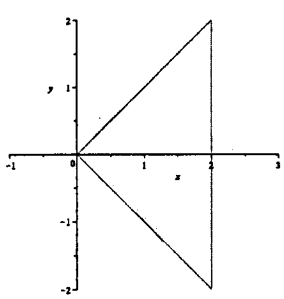
10. Use the graph of $r = 4\cos(2\theta)$ given here to answer the questions that follow.



(a) (*12 points*) Find the area of one of the petals.

(b) (*6 points*) Set up *but do not bother to evaluate* a definite integral that gives the arclength of one petal.

11. (*12 points*) Consider the triangular region R bound by $y = x$, $y = -x$ and $x = 2$. (See picture.) Find the volume of the solid obtained by rotating R about the y-axis. Show work and provide explanations making it clear what you are doing.



12. For each of the following improper integrals, if it converges, find its exact value. If it diverges, prove that it does.

(a) *(10 points)* $\displaystyle\int_1^\infty 3xe^{-2x}\,dx$

(b) *(10 points)* $\displaystyle\int_1^\infty \frac{2+\sin^3(x)}{x}\,dx$

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Name: SOLUTIONS

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Math 107 — Final Exam — Spring 2009 — May 5, 2009 — Solutions

1. Given $\vec{u} = \langle 2, -4, 4 \rangle$ and $\vec{v} = \langle -2, 6, -3 \rangle$, find

   (a) (*5 points*) $5\vec{u} - 4\vec{v}$

   Solution: $5\vec{u} - 4\vec{v} = \langle 10, -20, 20 \rangle - \langle -8, 24, -12 \rangle = \langle 18, -44, 32 \rangle$

   (b) (*5 points*) the unit vector in the direction of $\vec{v}$

   Solution: $\|\vec{v}\| = \sqrt{4 + 36 + 9} = 7$. So the answer is $\frac{\vec{v}}{7} = \langle -\frac{2}{7}, \frac{6}{7}, -\frac{3}{7} \rangle$.

   (c) (*5 points*) $\vec{u} \cdot \vec{v}$

   Solution: $\vec{u} \cdot \vec{v} = -4 - 24 - 12 = -40$

   (d) (*5 points*) $\mathbf{proj}_{\vec{v}}\, \vec{u}$ (the projection of $\vec{u}$ onto $\vec{v}$)

   Solution: $\mathbf{proj}_{\vec{v}}\, \vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{-40}{49}\vec{v} = \left\langle \frac{80}{49}, \frac{-240}{49}, \frac{120}{49} \right\rangle$

2. (*12 points*) Does $\displaystyle\lim_{n\to\infty} \frac{\ln(5n^2)}{\ln(n^3)}$ exist? If so, what is the limit? If not, why not?

   Solution: Method I: Apply l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(5x^2)}{\ln(x^3)} = \lim_{x\to\infty} \frac{\frac{10x}{5x^2}}{\frac{3x^2}{x^3}} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}.$$

   Method II: Simplify using rules of logarithms:

$$\lim_{n\to\infty} \frac{\ln(5n^2)}{\ln(n^3)} = \lim_{n\to\infty} \frac{\ln(5) + 2\ln(n)}{3\ln(n)} = \lim_{n\to\infty} \frac{\ln(5)}{3\ln(n)} + \frac{2}{3} = \frac{2}{3}.$$

3. (*8 points*) Give a parametric description of the *line segment* starting at $P = (3, 5, 6)$ and ending at $Q = (8, 2, 6)$. Be sure to include inequalities giving the range of the parameter.

   Solution: Let $\vec{v} = \vec{PQ} = \langle 5, -3, 0 \rangle$. Then $\vec{r}(t) = \vec{OP} + t\vec{PQ} = \langle 3 + 5t, 5 - 3t, 6 \rangle$ gives the line passing though P and parallel to $\vec{PQ}$. This line is at P when $t = 0$ and is at Q when $t = 1$. So the answer is

$$\vec{r}(t) = \langle 3 + 5t, 5 - 3t, 6 \rangle \qquad 0 \le t \le 1$$

   or, equivalently,

$$\begin{aligned} x &= 3 + 5t \\ y &= 5 - 3t \qquad\qquad 0 \le t \le 1. \\ z &= 6 \end{aligned}$$

4. Determine whether the following series converge or diverge. Give reasons.

(a) (*9 points*) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

Solution: All the terms are positive and thus we may apply the n-th Root Test:

$$\sqrt[n]{\frac{n^{10}}{10^n}} = \frac{(\sqrt[n]{n^{10}})}{\sqrt[n]{10^n}} = \frac{(\sqrt[n]{n})^{10}}{10} \to \frac{1}{10} \qquad \text{as } n \to \infty.$$

Here, we have used that $\sqrt[n]{n} \to 1$ as $n \to \infty$. Since this limit is < 1, the n-th Root Test tells us the series CONVERGES.

Or, since all terms are positive, one could also use the Ratio Test:

$$\frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}} = \frac{\left(\frac{n+1}{n}\right)^{10}}{10} \to \frac{1}{10} \qquad \text{as } n \to \infty.$$

Since this limit is < 1, the Ratio Test tells us the series CONVERGES.

(b) (*9 points*) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^4-6}}{n^2+1}$

Solution: *Note*: The limits of the summation should have been $n = 2$ to ∞ (since the expression is undefined when $n = 1$).

We examine the limit of the terms:

$$\lim_{n\to\infty} (-1)^n \frac{\sqrt{n^4-6}}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{n^2\sqrt{1-\frac{6}{n^4}}}{n^2(1+\frac{1}{n^2})} = \lim_{n\to\infty} (-1)^n \frac{\sqrt{1-\frac{6}{n^4}}}{1+\frac{1}{n^2}}.$$

The even terms tend to 1 and the odd terms tend to -1, and so the limit does not exit. Since this limit is not 0, the n-th term test tells us the series DIVERGES.

Or, one could take absolute values of the terms, and then prove the limit of these terms is 1 by using the above argument or a similar one. Since the terms don't approach 0 in absolute values, the series diverges.

Alternatively, one could examine this limit using several applications of l'Hôpital's Rule, but that would be considerably harder.

(c) (*9 points*) $\sum_{n=2}^{\infty} \frac{(\ln n)^{-\frac{3}{2}}}{n}$

Solution: Since all the terms are positive, we may use the Integral Test, which says that this series converges if and only if the improper integral $\int_2^\infty \frac{(\ln x)^{-\frac{3}{2}}}{x}\,dx$ converges. By using $u = \ln(x)$, we get

$$\int_2^\infty \frac{(\ln x)^{-\frac{3}{2}}}{x}\,dx = \int_{\ln(2)}^\infty u^{-\frac{3}{2}}\,du = \lim_{b\to\infty}\left[-2u^{-\frac{1}{2}}\right]_{\ln(2)}^{b} = \lim_{b\to\infty}(-2\frac{1}{\sqrt{b}} + 2\frac{1}{\sqrt{\ln(2)}}) = 2\frac{1}{\sqrt{\ln(2)}}.$$

In particular, the integral converges, and so the series also CONVERGES.

5. Consider the power series $\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$.

(a) (*10 points*) Find the radius of convergence. Show details.

Solution: We combine the Absolute Convergence Test and Root Test:

$$\sqrt[n]{\frac{n^2|x|^n}{2^n}} = \frac{\sqrt[n]{n^2}|x|}{2} = \frac{\sqrt[n]{n}^2|x|}{2} \to \frac{|x|}{2} \qquad \text{as } n \to \infty.$$

Alternatively, one can combine the Absolute Convergence Test and Ratio Test:

$$\frac{\frac{(n+1)^2|x|^{n+1}}{2^{n+1}}}{\frac{n^2|x|^n}{2^n}} = \frac{(\frac{n+1}{n})^2|x|}{2} \to \frac{|x|}{2} \qquad \text{as } n \to \infty.$$

Thus, the series converges if $\frac{|x|}{2} < 1$ and diverges if $\frac{|x|}{2} > 1$; equivalently, it converges for $|x| < 2$ and diverges for $|x| > 2$. This shows the radius of convergence is 2.

(b) (*8 points*) Find the interval of convergence, that is, the set of values of x for which the series converges. (In particular, you need to determine whether or not the series converges at the endpoints of the interval.) Explain what you are doing.

Solution: We know from (a) it converges for $-2 < x < 2$ and diverges for $x < -2$ or $x > 2$. When $x = 2$, we get $\sum_{n=1}^{\infty} n^2$, which clearly diverges (by the n-th term test). When $x = -2$, we get $\sum_{n=1}^{\infty}(-1)^n n^2$, which also diverges (by the n-th term test). Thus the series converges for $-2 < x < 2$ and diverges for all other values of x. In other words, the interval of convergence is $(-2, 2)$.

6. (*10 points*) Evaluate $\int_0^1 \left\langle \frac{1}{1+t}, e^{5t} \right\rangle dt$. (Note: In case you prefer $\vec{i}, \vec{j}$ notation, this integral is the same as $\int_0^1 \left(\frac{1}{1+t}\vec{i} + e^{5t}\vec{j} \right) dt$.)

Solution: We have

$$\int_0^1 \frac{1}{1+t}\, dt = [\ln(1+t)]_0^1 = \ln(2)$$

and

$$\int_0^1 e^{5t}\, dt = \left[\frac{1}{5}e^{5t}\right]_0^1 = \frac{e^5 - 1}{5}.$$

Thus

$$\int_0^1 \left\langle \frac{1}{1+t}, e^{5t} \right\rangle dt = \left\langle \ln(2), \frac{e^5-1}{5} \right\rangle.$$

7. Starting with the Taylor series centered at $x = 0$ for $\frac{1}{1-x}$, namely, $\sum_{n=0}^{\infty} x^n$, find the following. (For each, either write your answer using sigma notation or include enough terms so that the pattern is evident.)

(a) (*10 points*) The Taylor series centered at $x = 0$ for $\frac{1}{(1+2x)}$.

Solution: Replacing x by $-2x$ in the given Taylor series gives $\sum_{n=0}^{\infty}(-2x)^n$, or equivalently, $\sum_{n=0}^{\infty}(-2)^n x^n$, or equivalently $\sum_{n=0}^{\infty}(-1)^n 2^n x^n$, and this is the Taylor series of $\frac{1}{1+2x}$. (Each of these three forms is correct.)

(b) (*10 points*) The Taylor series centered at $x = 0$ for $\frac{1}{(1+2x)^2}$. *Hint*: Evaluate $\frac{d}{dx}\left(\frac{1}{1+2x}\right)$.

Solution: By differentiating both sides of

$$\frac{1}{1+2x} = \sum_{n=0}^{\infty}(-2)^n x^n$$

we get

$$\frac{-2}{(1+2x)^2} = \sum_{n=1}^{\infty}(-2)^n n x^{n-1}.$$

(The summation can now be taken to start at $n = 1$, but if it starts at $n = 0$ that is still correct.) Divide by -2 to get

$$\frac{1}{(1+2x)^2} = \sum_{n=1}^{\infty}(-2)^{n-1} n x^{n-1} = 1 - 2 \cdot 2x + 4 \cdot 3x^2 - 8 \cdot 4x^3 + 16 \cdot 5x^4 - 32 \cdot 6x^5 + \cdots.$$

8. (*15 points*) Find $\displaystyle\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx.$

Solution: We perform the trigonometric substitution $x = 2\cos(t)$, $dx = -2\sin(t)\,dt$:

$$\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx = \int \frac{-2\sin(t)}{\sqrt{1-\cos^2(t)}}\,dt = \int \frac{-2\sin(t)}{\sin(t)}\,dt = \int -2\,dt = -2t + C.$$

Since $t = \arccos\left(\frac{x}{2}\right)$, the answer is

$$-2\arccos\left(\frac{x}{2}\right) + C.$$

Or, if you use the trigonometric substitution $x = 2\sin(t)$ instead, you would be led by a similar calculation to the equivalent answer of

$$2\arcsin\left(\frac{x}{2}\right) + C.$$

Yet another acceptable method is to use $u = \frac{x}{2}$, $du = \frac{dx}{2}$ to get

$$\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx = 2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arcsin(u) + C = 2\arcsin\left(\frac{x}{2}\right) + C,$$

using the standard formula for the derivative of $\arcsin(u)$. Similarly, the following would be acceptable:

$$\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx = \int \frac{1}{\frac{1}{2}\sqrt{4-x^2}}\,dx = 2\int \frac{1}{\sqrt{4-x^2}}\,dx = 2\int \frac{1}{\sqrt{4-x^2}}\,dx = 2\arcsin\left(\frac{x}{2}\right) + C.$$

9. Consider the vector-valued function $\vec{r}(t) = \langle 3t^2, 3-t, t^3-t \rangle$.

(a) (*5 points*) Find $\vec{v}(t)$ (i.e., velocity).

Solution: $\vec{v}(t) = \vec{r}\,'(t) = \langle 6t, -1, 3t^2-1 \rangle$.

(b) (*5 points*) Find $\vec{a}(t)$ (i.e., acceleration).

Solution: $\vec{a}(t) = \vec{r}\,''(t) = \langle 6, 0, 6t \rangle$.

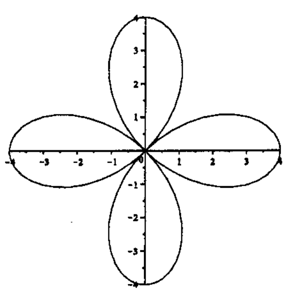
(c) (*5 points*) What is the value of $\|\vec{r}\,'(1)\|$ and what does it represent?

Solution: $\|\vec{r}\,'(1)\| = \| \langle 6, -1, 2 \rangle \| = \sqrt{36+1+4} = \sqrt{41}$. This number gives the speed (of the parametrization or the particle) at $t=1$.

(d) (*5 points*) Evaluate $\vec{r}\,'(1) \cdot \vec{r}\,''(1)$.

Solution: $\vec{r}\,'(1) \cdot \vec{r}\,''(1) = \langle 6, -1, 2 \rangle \cdot \langle 6, 0, 6 \rangle = 48$.

10. Use the graph of $r = 4\cos(2\theta)$ given here to answer the questions that follow.



(a) (*12 points*) Find the area of one of the petals.

Solution: The bottom petal is traced out when $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$. Using the standard formula for the area within a polar curve $r = f(\theta)$, we get

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{r^2}{2}\, d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 8\cos^2(2\theta)\, d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 + 4\cos(4\theta))\, d\theta,$$

using the double-angle formula $\cos(4\theta) = 2\cos^2(2\theta) - 1$. Thus the area is

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 + 4\cos(4\theta))\, d\theta = [4\theta + \sin(4\theta)]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 2\pi.$$

Of course, one could find the area of any of the other petals instead. All the work (and the final answer) would be the same, except the limits of integration would be different.

Alternatively, one could find the area of half of one of the petals and double the answer. For example, doing this for the top half of the right petal gives

$$2\int_0^{\frac{\pi}{4}} \frac{r^2}{2} d\theta = 2\int_0^{\frac{\pi}{4}} 8\cos^2(2\theta)\, d\theta = 2\int_0^{\frac{\pi}{4}} (4 + 4\cos(4\theta))\, d\theta = 2\,[4\theta + \sin(4\theta)]_0^{\frac{\pi}{4}} = 2\pi.$$

(b) (*6 points*) Set up *but do not bother to evaluate* a definite integral that gives the arclength of one petal.

Solution: Recall that the arclength of a polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is

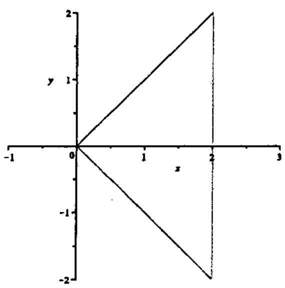
$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Since $\frac{dr}{d\theta} = -8\sin(2\theta)$ in our example, the answer is

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{16\cos^2(2\theta) + 64\sin^2(2\theta)}\, d\theta.$$

(Again, this is one of four correct answers; the others differ only in the choice of limits of integration.)

11. (*12 points*) Consider the triangular region R bound by $y = x$, $y = -x$ and $x = 2$. (See picture.) Find the volume of the solid obtained by rotating R about the y-axis. Show work and provide explanations making it clear what you are doing.



Solution: Method I: We use the "washer method". By symmetry, we can compute the volume of the top half and double this to get the answer. For $0 \le y \le 2$, the cross section at height y is a washer with inner radius y and outer radius of 2. The area of this cross section is thus $\pi 2^2 - \pi y^2 = \pi(4 - y^2)$, and the volume is then

$$\int_0^2 \pi(4 - y^2)\, dy = \pi\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{16}{3}\pi.$$

We double this to get the answer: $\dfrac{32}{3}\pi$.

Method II: We use the method of "cylindrical shells". For any x with $0 \le x \le 2$, the vertical line segment at x traces out, upon rotation about the y-axis, a cylinder of radius $r = x$ and height $h = x - (-x) = 2x$. The surface area of this cylinder is $2\pi r h = 2\pi x(2x) = 4\pi x^2$. Thus the total volume is

$$\int_0^2 4\pi x^2\, dx = \left[\frac{4}{3}\pi x^3\right]_0^2 = \frac{32}{3}\pi.$$

(a) (10 points) $\int_1$

Solution: The integral CONVERGES and we will find its exact value.
We start by finding an anti-derivative using integration by parts with $u = 3x$ and $dv = e^{-2x}\,dx$ (so that $du = 3\,dx$ and $v = -\frac{1}{2}e^{-2x}$):

$$\int 3xe^{-2x}\,dx = -\frac{3}{2}xe^{-2x} - \int (-\frac{3}{2}e^{-2x})\,dx = -\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + C. = -\frac{3x}{2e^{2x}} - \frac{3}{4e^{2x}} + C.$$

Thus

$$\int_1^\infty 3xe^{-2x}\,dx = \lim_{b\to\infty}\left[-\frac{3x}{2e^{2x}} - \frac{3}{4e^{2x}}\right]_1^b = \frac{3}{2e^2} + \frac{3}{4e^2}.$$

Here, we have used that $\lim_{x\to\infty}\frac{1}{e^{2x}} = 0$ (which is evident) and that $\lim_{x\to\infty}\frac{x}{e^{2x}} = 0$. The latter holds by l'Hôpital's Rule.

(b) *(10 points)* $\displaystyle\int_1^\infty \frac{2+\sin^3(x)}{x}\,dx$

Solution: We will show this integral diverges by using the Direct Comparison Test. Observe that $\frac{2+\sin^3(x)}{x} \geq \frac{1}{x}$ since $\sin(x)$ (and hence $\sin^3(x)$) is ≥ -1 for all x. Moreover, both functions are positive. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges (by the p-test, or from the calculation $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}[\ln(x)]_1^b = \infty$), it follows from the Direct Comparison Test that $\int_1^\infty \frac{2+\sin^3(x)}{x}\,dx$ DIVERGES too.