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Solutions

Math 325 Exam 1

Show all work! How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (25 pts.) Show that $\sqrt{3}+\sqrt{5}$ is not a rational number.

If $\alpha=\sqrt{3}+\sqrt{5}$, then $\alpha^2=(\sqrt{3}+\sqrt{5})^2=3+2\sqrt{3}\sqrt{5}+5$
$=8+2\sqrt{15}$

So $(\alpha^2-8)=2\sqrt{15}$, so $(\alpha^2-8)^2=(2\sqrt{15})^2=4\cdot15=60$

So $(\alpha^2-8)^2-60=0=\alpha^4-16\alpha^2+64-60$
$=\alpha^4-16\alpha^2+4$

So α is a root of $f(x)=x^4-16x^2+4$.

The only possible rational roots of f are $\pm1, \pm2$, or ±4, by the rational roots theorem.

But! $f(\pm1)=1-16+4=5-16=-11\neq0$
$f(\pm2)=16-16\cdot4+4=20-64=-44\neq0$
$f(\pm4)=4^4-16\cdot4^2+4=256-256+4=4\neq0$

So no root of f is rational. Since α is a root, $\alpha=\sqrt{3}+\sqrt{5}$ is not rational.

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2. (25 pts.) Suppose that S and T are both non-empty subsets of the real line, and both are bounded from above. Show that if $sup(S) \leq sup(T)$, then $sup(S \cup T) = sup(T)$.

Since T is bdd above (by sup(T)!), $x \leq \sup(T)$ for every $x \in T$. Since S is bdd above (by sup(S)), $y \leq \sup(S)$ for every $y \in S$, so since $\sup(S) \leq \sup(T)$, $y \leq \sup(T)$ for every $y \in S$.

So $z \leq \sup(T)$ for every $z \in S \cup T$, so $\sup(T)$ is an upper bd for $S \cup T$.

But! if $N < \sup(T)$ ~~$\varepsilon > 0$~~, then ~~$\sup(T) - \varepsilon$~~ N is not an upper bd for T, so there is an $x \in T \subseteq S \cup T$ with ~~$\sup(T) - \varepsilon$~~ $N < x$. So there is an $x \in S \cup T$ with ~~$\sup(T) - \varepsilon$~~ $N < x$. So ~~$\sup(T)$~~ N is not an upper bd for $S \cup T$. So $\sup(T)$ is the least upper bd for $S \cup T$; that is,

$\sup(S \cup T) = \sup(T)$. ///

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3. (25 pts.) Show, using the ϵ - N definition of convergence, that

$$\lim_{n\to\infty} \frac{7n-4}{3n+11} = \frac{7}{3}.$$

we wish to show that for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$

so that $\left|\frac{7n-4}{3n+11} - \frac{7}{3}\right| < \varepsilon$ so long as $n \geq N$.

But $\left|\frac{7n-4}{3n+11} - \frac{7}{3}\right| = \left|\frac{(7n-4)(3) - 7(3n+11)}{(3n+11)(3)}\right|$

$= \left|\frac{21n - 12 - 21n - 77}{3(3n+11)}\right| = \left|\frac{-89}{3(3n+11)}\right| = \frac{89}{3(3n+11)}$ (since $n \geq 1$)

$< \frac{89}{3(3n)} = \frac{89}{9n} < \frac{90}{9n} = \frac{10}{n}$

since $3n < 3n+11$

So if we pick $N \in \mathbb{N}$ with $\frac{10}{N} < \varepsilon$

(i.e. $N > \frac{10}{\epsilon}$) then $n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N}$, so

$\left|\frac{7n-4}{3n+11} - \frac{7}{3}\right| < \frac{10}{n} \leq \frac{10}{N} < \varepsilon$.

So given $\varepsilon > 0$ we can find $N \in \mathbb{N}$ so that $n \geq N$

implies $\left|\frac{7n-4}{3n+11} - \frac{7}{3}\right| < \varepsilon$. So $\frac{7n-4}{3n+11} \to \frac{7}{3}$ as $n \to \infty$. //

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4. (25 pts.) We define a sequence inductively, as $a_1 = 3$ and, for $n \geq 1$, $a_{n+1} = 1 + \frac{1}{10}a_n^2$. Show that the sequence $(a_n)_{n=1}^{\infty}$ is monotonically *decreasing*, and bounded from below (so it has a limit).

We show ~~that~~ this by induction.

We want $a_{n+1} \leq a_n$ for all $n \geq 1$.

$n=1$: $a_2 = 1 + \frac{1}{10}a_1^2 = 1 + \frac{1}{10}3^2 = 1 + \frac{9}{10} = 1.9 < 3 = a_1$,

so $a_2 \leq a_1$.

If we have $a_{n+1} \leq a_n$, then

since $a_{n+1} \geq 0$ (see below!)

$a_{n+1}^2 = a_{n+1} \cdot a_{n+1} \leq a_{n+1} \cdot a_n \leq a_n \cdot a_n = a_n^2$

since $a_n \geq 0$

So $\frac{1}{10}a_{n+1}^2 \leq \frac{1}{10}a_n^2$, so

$a_{n+2} = 1 + \frac{1}{10}a_{n+1}^2 \leq 1 + \frac{1}{10}a_n^2 = a_{n+1}$

So $a_{n+2} \leq a_{n+1}$. So $a_{n+1} \leq a_n$, by induction.

(This is the induction step.)

We also have $a_{n+1} = 1 + \frac{1}{10}a_n^2 \geq 1$ (since $a_n^2 \geq 0$)

for every $n \geq 1$ (and $a_1 \geq 1$ (!))

So $a_n \geq 1$ for every $n \geq 1$, so a_n is bdd below.

So $(a_n)_{n=1}^{\infty}$ is decreasing & bdd below, so it converges!

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