Math 325 Problem Set 3 Solutions

Problems were due Friday, February 3.

9. [Zorn, p.58, #4] Show, by induction, that the (ordinary) triangle inequality extends to show that for any $n \geq 2$ we have

$$|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k| \ .$$

Arguing by induction, our base case is $n = 2$, where $x_1 + x_2| \leq |x_1| + |x_2|$ is true, because this is the triangle inequality that we established in class.

If we then assume that for some $n \geq 2$ we have $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ when $a_1, \ldots, a_n \in \mathbb{R}$, then if we have $a_1, \ldots, a_{n+1} \in \mathbb{R}$, then $|\sum_{k=1}^{n+1} a_k| = |(\sum_{k=1}^{n} a_k) + a_{n+1}|$. but $\sum_{k=1}^{n} a_k$ is a real number, so the ordinary triangle inequality tells us that $|(\sum_{k=1}^{n} a_k) + a_{n+1}| \leq |(\sum_{k=1}^{n} a_k)| + |a_{n+1}|$. Then our inductive hypothesis tells us that $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$, and so, putting everything together,

$$|\sum_{k=1}^{n+1} a_k| = |(\sum_{k=1}^{n} a_k) + a_{n+1}| \leq |(\sum_{k=1}^{n} a_k)| + |a_{n+1}| \leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k| \ .$$

So $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ implies that $|\sum_{k=1}^{n+1} a_k| \leq \sum_{k=1}^{n+1} |a_k|$. This gives us our inductive step, and so, by induction, we have $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ for any $n \geq 2$.

10. [Zorn, p.58, #6] Show that the <u>maximum</u> of two numbers $x, y \in \mathbb{R}$ can be computed by

$$\max(x, y) = \frac{x + y + |x - y|}{2} \ .$$

[That is, if $x \leq y$ then $\dfrac{x + y + |x - y|}{2} = y$, while if $y \leq x$ then it equals x.]

Find a similar formula which gives the minimum of x and y.

Given $x, y \in \mathbb{R}$, by trichotomy we know that either $x < y$, $x = y$, or $x > y$. We look at each case separately.

If $x < y$, then $\max(x, y) = y$. But then $x - y < 0$, so $|x - y| = -(x - y) = y - x$, and so $\dfrac{x + y + |x - y|}{2} = \dfrac{x + y + (y - x)}{2} = \dfrac{2y}{2} = y$, so the two quantities agree.

If $x = y$, then $\max(x, y) = x = y$. But then $x - y = 0$, so $|x - y| = 0$, and so $\dfrac{x + y + |x - y|}{2} = \dfrac{x + y + 0}{2} = \dfrac{2x}{2} = x = y$, and so the two quantities again agree.

Finally, if $x > y$, then $\max(x, y) = x$. But then $x - y > 0$, so $|x - y| = x - y$, and so $\dfrac{x + y + |x - y|}{2} = \dfrac{x + y + (x - y)}{2} = \dfrac{2x}{2} = x$, so the two quantities again agree.

So, for any choice of x and y, we find that $\max(x, y) = \dfrac{x + y + |x - y|}{2}$, which is what we wished to show.

A formula fo $\min(x, y)$ can be found similarly; we want the exact opposite result (for $x < y$ versus $x > y$), so we want the exact opposite (i.e., negative) result to occur with $|x - y|$. We can do this by subtracting $|x - y|$ intead of adding it; so

$$\min(x, y) = \frac{x + y - |x - y|}{2}$$

which we can verify by the same "case analysis".

11. [Zorn, p.64, #7] (a) Show that if $B \subseteq \mathbb{R}$ is bounded, and $A \subseteq B$, then A is bounded.

Since B is bounded, it has both an upper and a lower bound, so there are $N, M \in \mathbb{R}$ with $x \leq N$ for every $x \in B$, and $M \leq x$ for every $x \in B$. But sincce $A \subseteq B$, if $x \in A$ then $x \in B$ (and so $x \leq N$ and $M \leq x$. So $x \leq N$ for every $x \in A$, so N is an uppoer bound for A. Also, $M \leq x$ for evvey $x \in A$, so M is a lower bound for A. So, A has both an upper and a lower bound, so A is bounded!

(b) If $S \subseteq \mathbb{R}$, then we define the set $|S|$ as $|S| = \{|s| \ : \ s \in S\}$. Show that if S is bounded, then $|S|$ is bounded.

As above, since S is bounded, there are $N, M \in \mathbb{R}$ so that $x \leq N$ and $M \leq x$ for every $x \in S$. But now if we pick $y \in |S|$, then $y = |x|$ for some $x \in S$. So $y = |x| \geq 0$ for every $y \in |S|$, so 0 is a lower bound for $|S|$.

But we also know that either $y = |x| = x$ (if $x \geq 0$) or $y = |x| = -x$ (if $x \leq 0$). If $y = x$, then $y \leq N$ since $x \in S$ so $x \leq N$. But if $y = -x$, then $y \leq -M$, since $x \in S$ so $M \leq x$, so $x \geq M$, so $-x \leq -M$ (since negating both sides of an inequality reverses the inequality). So for any $y \in |S|$ we have either $y \leq N$ or $y \leq -M$. So if we set $K = \max(N, -M)$, then $N \leq K$ and $-M \leq K$, and so no matter which one of $y \leq N$ or $y \leq -M$ is true, we can conclude that $y \leq N \leq K$ or $y \leq -M \leq K$, so $y \leq K$ in both cases. So for every $y \in |S|$ we have $y \leq K$¡ so K is an upper bound for $|S|$.

So since $|S|$ has both an upper (K) and a lower (0) bound, $|S|$ is bounded.

12. [Zorn, p.64, #8] For each of the following sets, either show that it is bounded (and find bounds), or explain why it isn't. [You can appeal to results from calculus in your answers.]

(a) $A = \left\{ \sum_{k=1}^{n} \frac{1}{k} \ : \ n \in \mathbb{N} \right\}$

Every term in each sum is greater then 0, so each sum is greater than 0, so 0 is a lower bound for the set. But since $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent series, calculus tells us that the partial sums tend to ∞ as $n \to \infty$, and so for any $N \in \mathbb{R}$ there is a partial sum $\sum_{k=1}^{n} \frac{1}{k} > N$ (you may recall a certain 'useless' fact from class about when this first exceeds 100...), and so no number can be an upper bound for the set, so A is not bounded from above. So A is not bounded.

(b) $B = \left\{ \sum_{k=1}^{n} \frac{1}{2^k} \ : \ n \in \mathbb{N} \right\}$

Again, each term in a sum is positive, so each sum is posisitve, so 0 is a lower bound. In this case, though, the related infinite series is $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} \sum_{k=0}^{i} nfty(\frac{1}{2})^k$ is a geometric series, which converges (by calculus) to $\frac{1}{2}1/(1-\frac{1}{2}) = 1$.

Since the terms being added are positive, each element of B is larger than the previous one, so the infinite sum is larger than them all. So the limit, 1, is an upper bound for all of the elements. [An alternative proof: use induction to show that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} < 1$, for every n .] So since B has both an upper (1, or anything larger than that!) bound and a lower 0, or anything further to the left than that!) bound, B is bounded.

(c) $C = \left\{ \frac{\ln n}{n} \ : \ n \in \mathbb{N} \right\}$

$\ln 1 = 0$, otherwise all of the terms are positive, since $\ln x$ is an increasing function. So every element of C is greater than or equal to 0, so 0 is a lower bound. All of the elements of C are ≤ 1, since $\ln x \leq x$ for all $x \geq 1$. The quickest way to show this is to note that $f(x) = \ln x - x$ has derivative $\frac{1}{x} - 1 \leq 0$ when $x \geq 1$, so f is a decreasing function for $x \geq 1$, and $f(1) = -1 \leq 0$. So $f(x) \leq 0$ for all $x \geq 1$, so $\ln x \leq x$. This means that $\frac{\ln x}{x} \leq \frac{x}{x} = 1$ for $x \geq 1$. This means that 1 is an upper bound for C, so C is bounded.

Alternate argument: If we look at the function $g(x) = \frac{\ln x}{x}$, then g has derivative $g'(x) = \frac{\frac{1}{x} \cdot x - (\ln x) \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} = 0$ when $\ln x = 1$, i.e., $x = e$. g is increasing from 0 to e (since $g'(x) > 0$) and decreasing for $x > e$ (since $g'(x) < 0$), so g has its maximum at $x = e$, so $g(x) \leq \frac{1}{e}$ for all $x > 0$. In particular, $\frac{\ln n}{n} \leq \frac{1}{e}$ for all $n \in \mathbb{N}$, so, in fact, $\frac{1}{e} = e^{-1}$ is an upper bound for the set C.

(d) $D = \left\{ \frac{2^n}{n^2} \ : \ n \in \mathbb{N} \right\}$

Yet again, every eleent is positive, so 0 is a lower bound. But as we learn in calculus, the numerator, an exponential, blows up faster than the denomenator, a (mere) power, as n grows large, so these terms approach ∞ as n grows large, so D has no upper bound. So D is not bounded.