If $x = \sqrt{n} + \lfloor \sqrt{n} \rfloor$, then $x = \langle \overline{a_0, \ldots, a_m} \rangle$ is <u>purely periodic</u>.

Pf. Set $x' = \lfloor \sqrt{n} \rfloor - \sqrt{n}$ = <u>conjugate</u> of x. Note that $-1 < x' < 0$

Set $x = \langle a_0, \ldots, a_k + x_k \rangle = \langle a_0, \ldots, a_k, \xi_{k+1} \rangle$ with $\xi_{k+1} = \frac{1}{x_k}$ [so: $a_k = \lfloor \xi_k \rfloor$, $x_k = \xi_k - a_k$]

<u>We know</u>: $x_k = \dfrac{\sqrt{n} - m_k}{q_k}$ for $1 \le m_k < \sqrt{n}$, $1 \le q_k \le n$

$$\text{so } \xi_k = \frac{1}{x_{k-1}} = \frac{q_{k-1}}{\sqrt{n} - m_{k-1}} = \frac{\sqrt{n} + m_{k-1}}{q_k}, \quad \xi_{k+1} = \frac{1}{\xi_k - a_k}$$

A direct (but ugly) check verifies that, for

$$\xi_k' = \text{conjugate of } \xi_k = \frac{-\sqrt{n} + m_{k-1}}{q_k}, \quad \boxed{\xi_{k+1}' = \frac{1}{\xi_k' - a_k}}$$

<u>But</u>! $x = \xi_0 = \langle \xi_0 \rangle$, so $\xi_0' = x'$ has $-1 < \xi_0' < 0$. <u>So</u> then by <u>induction</u>, $-1 < \xi_k' < 0 \implies \xi_k' - a_k < -1$

$$\implies -1 < \frac{1}{\xi_k' - a_k} = \xi_{k+1}' < 0, \text{ so } -1 < \xi_k' < 0 \text{ for } \underline{\text{all}}\ k.$$

But then $\left\lfloor \frac{-1}{\xi'_{k+1}} \right\rfloor = \lfloor a_k - \xi'_k \rfloor = a_k$, since

$a_k < a_k - \xi'_k < a_k + 1$.

But from HW, we know that $\sqrt{n}$ (and therefore $\sqrt{n} + \lfloor \sqrt{n} \rfloor$) has periodic continued fraction expansion, i.e., at some point $x_r = x_{r+m}$ for some $m > 0$.

But claim: the first time this happens is $r = 0$.

Because if $r > 0$ is first, then $x_r = x_{r+m} \Rightarrow \xi_{r+1} = \frac{1}{x_r} = \frac{1}{x_{r+m+1}} = \xi_{r+m+1}$

$$\Rightarrow \xi'_{r+1} = \frac{1}{\xi'_r - a_r} = \frac{1}{\xi'_{r+m} - a_{r+m}} = \xi_{r+m+1} \Rightarrow$$

$$\xi'_r - a_r = \xi'_{r+m} - a_{r+m} \Rightarrow a_r = \lfloor a_r - \xi'_r \rfloor = \lfloor a_{r+m} - \xi'_{r+m} \rfloor = a_{r+m}$$

$$\Rightarrow \xi_r = a_r + x_r = a_{r+m} + x_{r+m} = \xi_{r+m} \Rightarrow$$

$$\frac{1}{\xi_r} = x_{r-1} = x_{r+m-1} = x_{(r-1)+m} = \frac{1}{\xi_{r+m}}$$, contrad.

So for some m, $x_m = x_0 = \sqrt{n} - \lfloor \sqrt{n} \rfloor$, and

$$x = \langle a_0, \dots, a_{m-1}, a_m + x_0 \rangle = \langle \overline{a_0, \dots, a_m} \rangle$$