Math 856 Problem Set 4

Starred (*) problems to be handed in Friday, November 13

(*) 25. Let $\pi : E \to B$ be an orientable vector bundle, and $f : X \to B$ a smooth map. Show that the pullback $\pi' : f^*E \to X$ is an orientable bundle.

26. If ξ is a vector bundle, show that the Whitney sun of ξ with itself, $\xi \oplus \xi$, is orientable.

27. Let $X_1, \ldots , X_k$ be linearly independent vector fields on a manifold M with Riemannian metric $\langle \cdot, \cdot \rangle$. Show that the Gram-Schmidt orthonormalization procedure can be applied to the vector fields at every point simultaneously, to give a collection of orthonormal vector fields $Y_1, \ldots , Y_k$.

28. [Lee, p.289, problem 11-17] We defined the metric induced by a Riemannian metric $\langle \cdot, \cdot \rangle_a$ to be the infimum of lengths of piecewise smooth curves from p to q. Show that this cannot be a <u>minimum</u>: For $M = \mathbb{R}^2 \setminus (0, 0)$ with the usual Riemannian metric from $\mathbb{R}^2$, show that no curve in M from $(-1, 0)$ to $(1, 0)$ has length equal to the distance between the two points. [Also known as: I lied in class....]

29. (a) Show that an immersion from one n-manifold (with<u>out</u> boundary...) to another is an open map.

(b) Show that if M and N are n-manifolds, M is compact, N is connected, and $F : M \to N$ is an immersion, then F is onto.

(*) 30. [Lee, p.171, problem 7-2] Show that under the quotient map $p : S^2 \to \mathbb{R}P^2$ given by $p(x) = \{x, -x\}$ that the map $f : S^2 \to \mathbb{R}^4$ given by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$ descends to a smooth <u>embedding</u> $\overline{f} : \mathbb{R}P^2 \to \mathbb{R}^4$.

(*) 31. [Lee, p.172, problem 7-9] Show that if $\pi : M \to N$ is a submersion and Y is a vector field on N, then there is a vector field X (called a *lift* of X) on M that is π-related to Y. (Hint: partition of unity!)

32. If $S \subseteq M$ is a **closed**, embedded submanifold, $U \supseteq S$ is an open neighborhood of S, and $f : S \to \mathbb{R}$ is a smooth function, show that there is a smooth function $F : M \to \mathbb{R}$ with $F|_S = f$ and $\mathrm{supp}(F) \subseteq U$.