Math 325 Problem Set 5 Solutions

Starred (*) problems were due Friday, September 28.

(*) 25. (Belding and Mitchell, p.63, #1) Using the ϵ-δ formulation of limits,

(*) (b) show that $\lim_{x\to -1} 1-2x = 3$.

We want to control $|(1-2x)-3| = |-2x-2| = |(-2)(x+1)| = |-2|\cdot|x+1| = 2|x+1|$. We want this less than a (given) $\epsilon > 0$. We can control $|x-(-1)| = |x+1|$. Comparing this with what we want to control, we find that $|(1-2x)-3| = 2|x+1| < \epsilon$ so long as $|x-(-1)| = |x+1| < \epsilon/2$.

So if we set $\delta = \epsilon/2 > 0$, then $|x-(-1)| < \delta$ implies that $|(1-2x)-3| < \epsilon$. SO for every $\epsilon > 0$ we can find a $\delta = \epsilon/3$ so that $) < |x-(-1)| < \delta$ implies that $|(1-2x)-3| < \epsilon$, as desired.

(*) (e) show that $\lim_{x\to 3} \frac{1}{8-4x} = \frac{-1}{4}$.

We want to control
$|\frac{1}{8-4x} - \frac{-1}{4}| = |\frac{1}{8-4x} + \frac{1}{4}| = |\frac{4+(8-4x)}{(8-4x)(4)}| = |\frac{4(3-x)}{4\cdot 4\cdot(2-x)}| = |\frac{3-x}{4(2-x)}| = \frac{|3-x|}{|4|cdot|2-x|} = \frac{|x-3|}{4|x-2|}$.

Because we can make $|x-3|$ as small as we want, we can make this quantity small so long as $|x-2|$ does <u>not</u> get too small. This we can do, for example, by insisting that $|x-3| < 1/2$ [note that $|x-3| < 1$ won't quite do, because this will let x get close to 2 (!)], since then $|x-2| = |(x-3)+1| \geq |1| - |x-3| > 1-1/2 = 1/2$. So if $|x-3| < 1/2$. then $1/2 < |x-2|$, and so $\frac{1}{|x-2|} < \frac{1}{1/2}$. Then we have

$|\frac{1}{8-4x} - \frac{-1}{4}| = \frac{|x-3|}{4|x-2|} < \frac{|x-3|}{4(1/2)} = \frac{|x-3|}{2}$,

which we can make less than ϵ so long as $|x-3| < 2\epsilon$. So, for any $\epsilon > 0$, if we choose a $\delta < min\{1/2, 2\epsilon\}$, then $|x-3| < \delta$ impliees that $|x-3| < 1/2$ and so

$|\frac{1}{8-4x} - \frac{-1}{4}| = \frac{|x-3|}{4|x-2|} < \frac{|x-3|}{2} < \frac{2\epsilon}{2} = \epsilon$, so $|\frac{1}{8-4x} - \frac{-1}{4}| < \epsilon$, as desired.

(*) 28. (The 'Squeeze Play' Theorem) Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$, are functions with with $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$ with $0 < |x-a| < M$ for some $a \in \mathbb{R}$ and $M > 0$. Suppose further that

$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$. Show that $\lim_{x\to a} g(x) = L$.

[See Belding and Mitchell, p.64, #8 for an outline that you might follow.]

We want to show how to control $|g(x)-L|$, in particular, for an $\epsilon > 0$ we want to arrange that $|g(x)-L| < \epsilon$, which means that $-\epsilon < g(x)-L < \epsilon$. But since $f(x) \leq g(x) \leq h(x)$ for x with $|x-a| < M$, we have (*) $f(x)-L \leq g(x)-L \leq h(x)-L$.

So if we can manage to show that(**) $-\epsilon < f(x) - L$ <u>and</u> (***) $h(x) - L < \epsilon$, then we have $-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$, so $-\epsilon < g(x) - L < \epsilon$ and so $|g(x) - L| < \epsilon$, as desired.

But! we can make both of these happen, since $|f(x) - L| < \epsilon$ implies that $-\epsilon < f(x) - L$, and $|h(x) - L| < \epsilon$ implies that $h(x) - L < \epsilon$. And we know from our hypotheses that there are $\delta_1 > 0$ and $\delta_2 > 0$ so that $0 < |x - a| < \delta_1$ implies that $|f(x) - L| < \epsilon$, and $0 < |x - a| < \delta_2$ implies that $|h(x) - L| < \epsilon$. So if we choose a $\delta > 0$ smaller than M (so that (*) is true) and smaller than δ_1 (so that (**) is true) <u>and</u> smaller than δ_2 (so that (***) is true), <u>then</u> $0 < |x - a| < \delta$ implies that $-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$ and so $|g(x) - L| < \epsilon$.

So: given an $\epsilon > 0$ we can find δ_1 and δ_2 as above, and then setting $\delta = min\{M, \delta_1, \delta_2\} > 0$, we have $0 < |x - a| < \delta$ implies that $|g(x) - L| < \epsilon$, as desired.

(*) 29. (Belding and Mitchell, p.64, #9) If $f : \mathbb{R} \to \mathbb{R}$ is a function, $\lim_{x \to a} f(x) = L$, and for some $K, M \in \mathbb{R}$ with $M > 0$ we have $f(x) \leq K$ for all x with $0 < |x - a| < M$, show that $L \leq K$.

[What's the alternative?]

The alternative is that $L > K$. but then $L - K = \epsilon > 0$, and so since $\lim_{x \to a} f(x) = L$ we know that we can find a $\delta > 0$ so that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \epsilon = L - K$. But then $L - f(x) \leq |L - f(x)| = |f(x) - L| < L - K$, and so $L - f(x) < L - K$, so $f(x) > K$.

<u>But!</u> $f(x) \leq K$ for every x with $0 < |x - a| < M$. This is a problem, though, if we can find an x satisfying both $0 < |x - a| < \delta$ and $0 < |x - a| < M$. Which, of course, we can do: setting $x = a + (1/2)min\{\delta, M\}$, we have $|x - a| = (1/2)min\{\delta, M\}$, which is smaller than both δ and M. So for this value of x we have both $f(x) \leq K$ and $f(x) > K$. But this contradicts trichotomy.

So the only thing which we assumed, namely that $L > K$, is false. So $L \leq K$.

A selection of further solutions.

24. Show that if $0 \leq x < 1$ then for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ so that $x^n < \epsilon$.

[Hint: Suppose not! Then look at lower bounds for $A = \{x^n : n \in \mathbb{N}\}$.]

First, deal with $x = 0$. For any $\epsilon > 0$ we have $x^1 = x = 0 < \epsilon$, so $n = 1$ will work.

Now suppose that $0 < x < 1$ and we cannot find such an n, so that for every $n \in \mathbb{N}$ we have $\epsilon \leq x^n$. This means that $\epsilon > 0$ is a lower bound for the set $A = \{x^n : n \in \mathbb{N}\}$. Since the set A is non-empty ($x \in A$), completeness tells us that A has a greatest lower bound $\beta \in \mathbb{R}$. Since ϵ is a lower bound for A, we have $\epsilon \leq \beta$, and so, in particular, $\beta > 0$.

We then have that $\beta \leq x^n$ for every $n \in \mathbb{N}$, but no number larger than β will work. But! $0 < x < 1$ and $\beta > 0$ imply x^{-1} exists, and $x^{-1} > 1$ (since $x^{-1} \leq 1$ would imply that $1 = xx^{-1} < 1 \cdot x^{-1} = x^{-1} \leq 1$, so $1 < 1$, a contradiction). Then we have $x^{-1} \cdot \beta > 1 \cdot \beta = \beta$, so $x^{-1} \cdot \beta$ cannot be a lower bound for A, and so there is an $n \in \mathbb{N}$ with $x^n < x^{-1}\beta$. But then since $x > 0$ we then have $x^{n+1} = x \cdot x^n < xx^{-1}\beta = \beta$, so there is an $m = b + 1 \in \mathbb{N}$ so that $x^m < \beta$, and so β is <u>not</u> a lower bound for the set A.

This is a contradiction, so the only assumption we made must be false. That assumption was that we could not find an $n \in \mathbb{N}$ with $x^n < \epsilon$. So there <u>is</u> an n with $x^n < \epsilon$, as desired.