Math 325 Problem Set 3 Solutions

8. [Lay, p.127, # 12.8] If S and T are subsets of $\mathbb{R}$ with $S \subseteq T$ and T is bounded both above and below, then show that S is also bounded, and
$$\inf(T) \le \inf(S) \le \sup(S) \le \sup(T) \ .$$

Because T is bounded, both $\inf(T)$ and $\sup(T)$ make sense, and $\inf(T) \le x \le \sup(T)$ for every x in T. But since $S \subseteq T$, every y in S also lies in T, so $\inf(T) \le y \le \sup(T)$ for every $y \in S$. So S is bounded both above and below.

Even more, that pair of inequalities states that $\inf(T)$ is a lower bound for S, and $\sup(T)$ is an upper bound for S. This implies that $\inf(T) \le \inf(S)$, since $\inf(S)$ is the largest of the lower bounds for S, and $\sup(S) \le \sup(T)$, since $\sup(S)$ is the smallest of the upper bounds for S (so it cannot be larger than $\sup(T)$). Finally, so long as S is non-empty (which the problem really should have asserted), picking an $x \in S$ we have $\inf(S) \le x \le \sup(S)$ (since the numbers on either end <u>are</u> lower and upper bounds), giving the final inequality we need.

9. [Lay, p.104, # 10.8] Show that for every $n \ge 1$ we have
$$\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{n}{2n+1}$$
[One way: Factor $4k^2 - 1$!]

We establish this by induction. For $n = 1$ we have
$\sum_{k=1}^{1} \frac{1}{4k^2 - 1} = \frac{1}{4 \cdot 1^2 - 1} = \frac{1}{3} = \frac{1}{2 \cdot 1 + 1}$, establishing the initial step.

Now if we suppose that $\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{n}{2n+1}$, then

$$\sum_{k=1}^{n+1} \frac{1}{4k^2 - 1} = \{\sum_{k=1}^{n} \frac{1}{4k^2 - 1}\} + \frac{1}{4(n+1)^2 - 1} = \frac{n}{2n+1} + \frac{1}{4(n+1)^2 - 1}$$
$$= \frac{n}{2n+1} + \frac{1}{4n^2 + 8n + 3} = \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)}{(2n+1)(2n+3)} + \frac{1}{(2n+1)(2n+3)}$$
$$= \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}$$

so $\sum_{k=1}^{n+1} \frac{1}{4k^2 - 1} = \frac{n+1}{2(n+1)+1}$, establishing our induction step.

So, by induction, we have $\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{n}{2n+1}$ for every $n \ge 1$.

10. [Lay, p.105, # 10.22] Use induction to establish *Bernoulli's Inequality*: If $x \in \mathbb{R}$ and $x+1>0$, then for every $n \in \mathbb{N}$ we have $(x+1)^n \geq 1+nx$.

For our base case $n=1$ we have $(x+1)^1 = x+1 = 1+x = 1+1\cdot x \geq 1+1\cdot x$, as required. Now if we suppose that (*) $(x+1)^n \geq 1+nx$, then $(x+1)^{n+1} = (x+1)^n(x+1)$. But because $(x+1)>0$ we can multiply both sides of (*) by $x+1$ while maintaining the inequality, so
$(x+1)^{n+1} = (x+1)^n(x+1) \geq (1+nx)(x+1) = x+1+nx^2+nx = 1+(n+1)x+nx^2$. But since $x \in \mathbb{R}$ we know from our previous work that $x^2 \geq 0$, and since $n \in \mathbb{N}$ we have $n \geq 1 > 0$ so $n>0$, and so $nx^2 \geq 0$. So
$(x+1)^{n+1} \geq 1+(n+1)x+nx^2 \geq 1+(n+1)x+0 = 1+(n+1)x$, so $(x+1)^{n+1} \geq 1+(n+1)x$, establishing our induction step. So by induction, $(x+1)^n \geq 1+nx$ for every $n \in \mathbb{N}$, as desired.

11. [Lay, p.106, # 10.26] Show that for every $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$ so that $n \leq k^2 \leq 2n$.

This <u>can</u> be established by induction (in a slightly roundabout way): for the base case $n=1$ we have $1 = 1^2 \leq 1^2 = 1 \leq 2 = 2(1)$, so for $n=1$ we find the $k=1$ works. Now suppose that $n \geq 1$ and $n \leq k^2 \leq 2n$ for some $k \in \mathbb{N}$.
If $k^2 \geq n+1$ as well, then $n+1 \leq k^2 \leq 2n < 2n+2 = 2(n+1)$, so $n+1 \leq k^2 \leq 2(n+1)$, so the same k works for $n+1$ as well, establishing the induction step. The only other alternative is that $k^2 < n+1$, and so we have that $n \leq k^2 < n+1$ and $k^2 \in \mathbb{N}$, and so we must have $k^2 = n$. But then $(k+1)^2 = k^2+2k+1 \geq k^2+1 = n+1$ (since $k \in \mathbb{N}$, so $2k \geq 0$), so $n+1 \leq (k+1)^2$. But since $(k-1)^2 \geq 0$, we also have
$(k+1)^2 \leq (k+1)^2+(k-1)^2 = k^2+2k+1+k^2-2k+1 = 2k^2+2 = 2(k^2+1) = 2(n+1)$. So $n+1 \leq (k+1)^2 \leq 2(n+1)$, as desired. This establishes the induction step, and so for every $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ with $n \leq k^2 \leq 2n$, as desired.

An alternative approach proceeds by showing that for every $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ so that $\sqrt{n} \leq k \leq \sqrt{2n}$. This implies our sought-after result, since
$n = (\sqrt{n})^2 \leq k\sqrt{n} \leq k \cdot k = k^2 \leq k\sqrt{2n} \leq (\sqrt{2n})^2 = 2n$, so $n \leq k^2 \leq 2n$.
And to establish $\sqrt{n} \leq k \leq \sqrt{2n}$, we need only establish that $\sqrt{2n}-\sqrt{n}$ is at least 1; then a result from class implies that there is an integer lying between $\sqrt{n}$ and $\sqrt{2n}$.
[Technically, this is false for small values of n (!) We deal with those directly...]
But if $\sqrt{2n}-\sqrt{n} < 1$ then $\sqrt{2n} < 1+\sqrt{n}$, so

$2n = (\sqrt{2n})^2 < (1+\sqrt{n})(\sqrt{2n}) < (1+\sqrt{n})^2 = 1+2\sqrt{n}+n$,
so $n < 1+2\sqrt{n}$, so $0 \leq n-1 < 2\sqrt{n}$, so
$n^2-2n+1 = (n-1)^2 < 2\sqrt{n}(n-1) < (2\sqrt{n})^2 = 4n$,
so $n^2 < 6n-1 < 6n$, so $n < 6$.

So we can establish that $\sqrt{2n}-\sqrt{n} \geq 1$ for $n \geq 6$, allowing us to find our integer k for these values. Since we can deal with the remaining cases $n \leq 5$ by an alternate argument (find values of k directly: $1 \leq 1^2 \leq 2$, $2 \leq 2^2 \leq 4$, $3 \leq 2^2 \leq 6$,$4 \leq 2^2 \leq 8$,$5 \leq 3^3 \leq 10$), we have found, for all $n \in \mathbb{N}$, a $k \in \mathbb{N}$ with $n \leq k^2 \leq 2n$, as desired.