Math 971 Algebraic Topology

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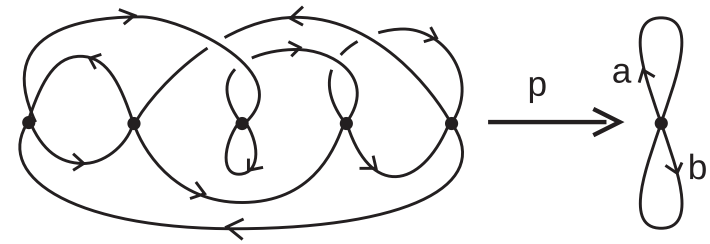
Covering spaces: The projective plane $\mathbb{R}P^2$ has $\pi_1 = \mathbb{Z}_2$. It is also the quotient of the simply-connected space S^2 by the antipodal map, which, together with the identity map, forms a group of homeomorphisms of S^2 which is isomorphic to $\mathbb{Z}_2$, and the quotient under the group action is $\mathbb{R}P^2$. The fact that $\mathbb{Z}_2$ has this dual role to play is no accident; codifying this relationship leads to covering spaces.

A map $p : E \to B$ is a *covering map* if for every $x \in B$, there is a neighborhood $\mathcal{U}$ of x (an *evenly covered neighborhood*) so that $p^{-1}(\mathcal{U})$ is a disjoint union $\mathcal{U}_\alpha$ of open sets in E, each mapped homeomorphically onto $\mathcal{U}$ by p . B is called the *base space* of the covering; E is called the *total space.* The quotient map from S^2 to $\mathbb{R}P^2$ is an example; (the image of) the complement of a great circle in S^2 will be evenly covered for any point it contains. The disjoint union of 42 copies of a space X, each mapping homeomorphically to X, is an example of a *trivial covering.* As a last example, we have the famous exponential map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi it} = (\cos(2\pi t), \sin(2\pi t))$. The image of any interval (a, b) of length less than 1 will have inverse image the disjoint union of the intervals $(a + n, b + n)$ for $n \in \mathbb{Z}$.

OK, maybe not the last. We can build many finite-sheeted (every point inverse is finite) coverings of a bouquet of two circles, say, by assembling n points over the vertex, and then, on either side, connecting the points by n (oriented) arcs, one each going in and out of each vertex. By choosing orientations on each 1-cell of the bouquet, we can build a covering map by sending the vertices above to the vertex, and the arcs to the one cells, homeomorphically, respecting the orientations. We can build infinite-sheeted coverings in much the same way.



Covering spaces of a (suitably nice) space X have a very close relationship to $\pi_1(X, x_0)$. The basis for this relationship is the

Homotopy Lifting Property: If $p : \widetilde{X} \to X$ is a covering map, $H : Y \times I \to X$ is a homotopy, $H(y, 0) = f(y)$, and $\widetilde{f} : Y \to \widetilde{X}$ is a *lift* of f (i.e., $p \circ \widetilde{f} = f$), then there is a unique lift $\widetilde{H}$ of H with $\widetilde{H}(y, 0) = \widetilde{f}(y)$.

The **proof** of this we will defer to next time, to give us sufficient time to ensure we finish it!

In particular, applying this in the case $Y = \{*\}$, where a homotopy $H : \{*\} \times I \to X$ is just a path $\gamma : I \to X$, we have the **Path Lifting Property**: "given a covering map $p : \widetilde{X} \to X$, a path $\gamma : I \to X$ with $\gamma(0) = x_0$, and a point $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{\gamma}$ lifting γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$." One of the immediate consequences of this is one of the cornerstones of covering space theory:

If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof: Suppose $\gamma : (I, \partial I) \to (\widetilde{X}, \widetilde{x}_0)$ is a loop $p_*([\gamma]) = 1$ in $\pi_1(X, x_0)$. So there is a homotopy $H : (I \times I, \partial I \times I) \to (X, x_0)$ between $p \circ \gamma$ and the constant path. By homotopy lifting, there is a homotopy $\widetilde{H}$ from γ to the lift of the constant map at x_0. The vertical sides $s \mapsto \widetilde{H}(0, s), \widetilde{H}(1, s)$ are also lifts of the constant map, beginning at $\widetilde{H}(0, 0), \widetilde{H}(1, 0) = \gamma(0) = \gamma(1) = \widetilde{x}_0$, so are the constant map at $\widetilde{x}_0$. Consequently, the lift at the bottom is the constant map at $\widetilde{x}_0$. So $\widetilde{H}$ represents a null-homotopy of γ, so $[\gamma] = 1$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. So $\pi_1(\widetilde{X}, \widetilde{x}_0) = \{1\}$.

Even more, the image $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))) \subseteq \pi_1(X, x_0)$ is precisely the elements whose representatives are loops at x_0, which when lifted to paths starting at $\widetilde{x}_0$), are loops. For if γ lifts to a loop $\widetilde{\gamma}$, then $p \circ \widetilde{\gamma} = \gamma$, so $p_*([\widetilde{\gamma}]) = [\gamma]$. Conversely, if $p_*([\widetilde{\gamma}]) = [\gamma]$, then γ and $p \circ \widetilde{\gamma}$ are homotopic rel endpoints, and so the homotopy lifts to a homotopy rel endpoints between the lift of γ at $\widetilde{x}_0$, and the lift of $p \circ \widetilde{\gamma}$ at $\widetilde{x}_0$ (which is $\widetilde{\gamma}$, since $\widetilde{\gamma}(0) = \widetilde{x}_0$ and lifts are unique). So the lift of γ is a loop, as desired.