Math 856 Problem Set 3

Starred (*) problems to be handed in Friday, October 30

(*) 16. If X, Y are smooth tangent vector fields on M, and $f, g \in C^\infty(M)$, show that $[fX, gY] = (fg)[X,Y] + (fXg)Y - (gYf)X$. [Hint: evaluate on a third smooth function!]

17. [Lee, p. 101, problem 4-7] Let M, N be smooth manifolds, $f : M \to N$ a smooth map, and define $F : M \to M \times N$ by $F(x) = (x, f(x))$. Show that for every tangent vector field X on M there is a tangent vector field Y on $M \times N$ so that Y is F-related to X.

18. [Lee, p.101, problem 4-9] Suppose that the map $F : M \to N$ is a local diffeomorphism (that is, for every $a \in M$, there is a neighborhood $\mathcal{U}$ of a so that $F|_{\mathcal{U}} : \mathcal{U} \to F(\mathcal{U})$ is a diffeomorphism). Show that for every smooth vector field Y on N there is a unique smooth vector field X on M that is F-related to Y.

(*) 19. ["Bundle Section Extension Lemma"] Given a smooth vector bundle $p : E \to M$ over a smooth manifold M, a closed subset $A \subseteq M$, and a smooth section $s : A \to E$ defined over A (that is, for every $a \in A$ there is a neighborhood U_a of a in M and a smooth section $s_U : U \to E$ so that $s_u = s$ on $A \cap U$), show that there is a global smooth section $S : M \to E$ with $S|_A = s$. [Hint: partition of unity...]

20. [Lee, p.101, problem 5-8] Let $p : E \to M$ be a smooth n-dimensional vector bundle and $X_1, \ldots, X_k$ be linearly independent smooth sections of E defined over an open subset $U \subseteq M$. Show that for every $a \in U$ there is a neighborhood V of a and smooth sections $Y_{k+1}, \ldots, Y_n$ defined over V so that $(X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_n)$ forms a local frame for E over $U \cap V$.

(Hint: if $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$, then why is it that if you wiggle the first k vectors a little bit, you still have a basis?)

(*) 21. [Lee, p.346, Problem 13-1] If M is a smooth manifold that is the union of two open subsets U, V with $U \cap V$ connected, and if $TM|_U$ and $TM|_V$ are orientable bundles, show that M is orientable. Use this to show that S^n is orientable for every $n \geq 2$.

22. Show that $M \times N$ is orientable $\Leftrightarrow$ both M and N are.

23. The tangent space for a manifold M with boundary is defined in exactly the same way as for a manifold; the derivations at a point in ∂M are allowed to point "in all the directions" of $\mathbb{R}^n$.

We say that a tangent vector $X \in T_aM$ for $a \in \partial M$ "points inward" if in some set of local coordinates $h = (x^1, \ldots, x^n)$ we have $X = \displaystyle\sum_i v^i \frac{\partial}{\partial x^i}$ with $v^n > 0$. (Here h maps to the upper half-space, where $x^n \geq 0$.) Show that the notion of "pointing inward" is independent of coordinate chart.

24. [Lee, p.151, Problem 6-3] Show that the tangent bundle TM is trivial if and only if the cotangent bundle T^*M is also trivial.