Math 417 Problem Set 3 Solutions

Starred (*) problems were due Friday, February 12.

(*) 18. (Gallian, p.70, #32) Show that if G is a group and $H, K \subseteq G$ are subgroups of G, then their intersection $H \cap K$ is also a subgroup of G. Does this extend to the intersection of <u>any</u> number of subgroups of G ?

Since we don't know that G is finite, we need to estabish the three properties (a) $e \in H \cap K$, (b) if $g, h \in H \cap K$ then $gh \in H \cap K$, and (c) if $g \in H \cap K$ then $g^{-1} \in h \cap K$. [Although it is true that our one-step condition will work...]

Each of these properties follows the same pattern. For (a), since H and K are subgroups of G, we know that $e \in H$ and $e \in K$, and so $e \in H \cap K$ since it lies in both. For (b), if $g, h \in H \cap K$, then $g, h \in H$ and so $gh \in H$ since H is a subgroup, and $g, h \in K$ so $gh \in K$ since K is a subgroup. Therefore, gh is in both H and K, and so $gh \in H \cap K$. Finally, if $g \in H \cap K$, then $g \in H$ (so $g^{-1} \in H$) and $g \in K$ (so $g^{-1} \in K$), and so g^{-1} is in both H and K. Therefore $g^{-1} \in H \cap K$.

Having established all three properties, we have shown that $H \cap K$ is a subgroup of G. This argument <u>does</u> go through for any number of subgroups of G; e is in every one of them, so it lies in the intersection; and gh and g^{-1} will lie in all of the subgroups and so they will lie in the intersection.

(*) 22. (Gallian, p.72, #46) Suppose that G is a group and $g \in G$ has $|g| = 5$. Show that the centralizer of g, $C(g) = C_G(g) = \{x \in G \ : \ xg = gx\}$, is equal to the centralizer of g^3, $C_G(g^3)$.

[Hint: show that anything that commutes with g must commute with g^3, <u>and</u> <u>vice versa</u>! What, if anything, is special about the numbers 5 and 3 in this problem?]

What we need to show is that both $C_G(g) \subseteq C_G(g^3)$ and $C_G(g^3) \subseteq C_G(g)$. That is, if an element $x \in G$ satisfies $xg = gx$ then we also have $xg^3 = g^3x$, and, conversely, if $xg^3 = g^3x$ then we must also have $xg = gx$.

For the first, if $x \in C_G(g)$, then $xg = gx$, and so

$xg^3 = x(ggg) = (xg)gg = (gx)gg = g(xg)g = g(gx)g = gg(xg) = gg(gx) = (ggg)x = g^3x$,

so $x \in C_G(g^3)$. Conversely, if $y \in C_G(g^3)$, then $yg^3 = g^3y$, and so

$yg = yg(e) = yg(g^5) = yg^6 = y(g^3g^3) = (yg^3)g^3 = g^3y)g^3 = g^3(yg^3) = g^3(g^3y) = (g^3g^3)y = g^6y = (g^5)gy = (e)gy = gy$,

so $y \in C_G(g)$. Here we have used that we were told that $|g| = 5$, so $g^5 = e$. So we have found that $C_G(g)$ and $C_G(g^3)$ contain the same elements, when $|g| = 5$, and so $C_g(g) = C_G(g^3)$.

(*) 25. (Gallian, p.74, #68) Let $G = GL_2(\mathbb{R})$ = the 2×2 invertible matrices, under matrix multiplication, and let $H = \{A \in GL_2(\mathbb{R}) \ : \ \det(A) = 2^k \text{ for some } k \in \mathbb{Z}\}$. Show that H is a subgroup of G.

We again wish to show that H contains the identity (matrix), and is closed under both matrix multiplication and matrix inversion. This we can do, following the model of the previous problem above. Here we will instead apply our one-step approach: H is a subgroup of G so long as whenever $A, B \in H$ we have $AB^{-1} \in H$ But since we have $A \in H$ and $B \in H$, we know that $\det(A) = 2^k$ for some integer k, and $\det(B) = 2^\ell$ for some integer ℓ. But then

$$\det(AB^{-1}) = \det(A)(\det(B^{-1})) = \det(A)(\det(B))^{-1} = 2^k \cdot (2^\ell)^{-1} = 2^k \cdot 2^{-\ell} = 2^{k-\ell},$$

for the integer $k - \ell$, using the properties of determinants which we learn in linear algebra, and so $AB^{-1} \in H$. So H is a subgroup of G.

A selection of further solutions.

19. (Gallian, p.70, #16) If G is a group, and $H \subseteq G$ is a subset of G so that, whenever $a, b \in H$ we have $a^{-1}b^{-1} \in H$, is this enough to guarantee that H is a subgroup of G? If yes, explain why! If not, give an example which shows that it doesn't work.

[Hint: if $a \in H$, start listing other elements that you can guarantee are in H ...]

Taking our cue from the hint, if we know that $a \in H$, then $a, a \in H$ and so $a^{-1}a^{-1} = a^{-2} \in H$. Then we know that $a, a^{-2} \in H$, and so $a^{-1}(a^{-2})^{-1} = a^{-1}a^2 = a \in H$, which tells us nothing new.... But $a^{-2}, a^{-2} \in H$, so $(a^{-2})^{-1}(a^{-2})^{-1} = a^2a^2 = a^4 \in H$. Then $a, a^4 \in H$, and so $a^{-1}(a^4)^{-1} = a^{-1}a^{-4} = a^{-5} \in H$.

We can keep this up for awhile; so far we have showwn that if $a \in H$, then $a^{-5}, a^{-2}, a, a^4 \in H$. Even with this list we might get suspicious; the exponents differ by (multiples of) 3. This pattern will continue; the only powers of a we will discover in H are of the form $1 + 3k$ for some $k \in \mathbb{Z}$. This suggests that

$$H = \{a^{3k+1} : k \in \mathbb{Z}\}$$

is in fact a set satisfying the condition we have imposed. And we can show this: $(a^{3k+1})^{-1}(a^{3\ell+1})^{-1} = a^{-3k-3\ell-2} = a^{3(-k-\ell-1)+1} = a^{3m+1}$ for $m = -k - \ell - 1$. But! If $|a| = \infty$, then $e \notin H$, since the only power of a which is equal to e is a^0, and a^0 is not in H. [Actually, all of the conditions for a group fails for this example!] We can get the same result if we choose our favorite group G and element $a \in G$ having $|a| = 3n$ for some k (like, for example, $\mathbb{Z}_{3n}$?), since then the set H we have built has exactly n elements, none of which are e.

[It is in fact true that the property that $a, b \in H$ implies $a^{-1}b^{-1} \in H$ together with $e \in H$ does imply that H is a subgroup.]

21. (Gallian, p.57, #38) Show that if G is a group and $a, b \in G$, then there is an $x \in G$ so that $axb = bxa$. Show, therefore, that if G has the property that whenever $axb = cxd$ we must have $ab = cd$ ('middle cancellation'), then G must be abelian.

One 'hint' is that we don't know much of anything about G, and so the only elements we know how to built can be built from a and b by taking powers, inverses, and products. To make axb to be bxa, we need to make axb start with a b ! So we need to get the a on the left out of the way. We can do that by multiplying by a^{-1}, but since the only thing we can do is multiply on the right of a, we can try $x = a^{-1} \cdot (something) = a^{-1}y$. Then:

$axb = a(a^{-1}y)b = (aa^{-1})yb = yb$, which we want equal to $bxa = ba^{-1}ya$.

But iff we just get the y out of the way, the a^{-1} and a in '$ba^{-1}ya$' can cancel! So set $x = a^{-1}$, and then

$axb = a(a^{-1})b = (aa^{-1})b = eb = b = be = b(a^{-1}a) = b(a^{-1})a = bxa$, as desired.

This means, if we have the property that $axb = cxd$ implies $ab = cd$, that for any $a, b \in G$, since $a(a^{-1})b = b(a^{-1})a$, tha we have $ab = ba$. Since this hold for any pair of elements of G, we have shown that G is abelian.

26. If G is an abelian group and $n \in \mathbb{Z}$, show that $H_n = \{g \in G \ : \ g = x^n \text{ for some } x \in G\}$ is a subgroup of G. Give an example where this fails if G is not abelian.

We show the three needed properties:

Since for $e \in G$, $e = e^n$ (by induction!), we have $e \in H_n$.

If $g, h \in H_n$, then $g = x^n$ and $h = y^n$ for some elements $x, y \in G$. Then, since G is abelian, $gh = x^n y^n = x \cdots x \cdot y \cdots y = xy \cdots xy = (xy)^n$ (by induction!), so $gh \in H_n$.

If $g \in H_n$, then $g = x^n$ for some $x \in g$, and then $g^{-1} = (x^n)^{-1} = x^{-n} = (x^{-1})^n = y^n$ for $y = x^{-1} \in G$, and so $g^{-1} \in G$.

Having established the three needed properties, we have shown that H_n is a subgroup.

This does not work, however, when G is not abelian. An example can be found using our work on problem #5. In D_4, the elements that are squares are all reflections, the identity, and the rotation by π. But these elements together do not form a subgroup!, since the reflections in lines with angle 0 and $\pi/4$ have composition the rotation by angle $\pi/2$, which is not a square. So this collection of elements is not closed under the group multiplication, and so do not form a subgroup.