Math 325 Problem Set 10 Solutions

Problems were due Friday, April 21.

36. ['another' L'Hôpital's Rule] Show that if $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$, and

$$\frac{f'(x)}{g'(x)} \to L \text{ as } x \to \infty, \text{ then } \frac{f(x)}{g(x)} \to L \text{ as } x \to \infty\ .$$

[Hint: Look at Proposition 3.6(ii), as a way to convert this into an 'ordinary' L'Hôpital's Rule problem...]

From Proposition 3.6, we have that $\lim_{x\to\infty}(f(x) = \lim_{x\to 0^+} f(1/x)$. So we can try to compute $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ as $\lim_{x\to 0^+}\frac{f(1/x)}{g(1/x)}$, instead. That is, if we write $F(x) = f(1/x)$ and $G(x) = g(1/x)$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{F(x)}{G(x)}$.

But! since $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$, we know, by the same proposition that $F(x) \to 0$ and $G(x) \to 0$ as $x \to 0^+$. So we can try to apply L'Hôpitals's Rule! Since $F'(x) = (f(1/x))' = f'(1/x)\cdot(-1/x^2)$, we have $f'(1/x) = -x^2F'(x)$. Similarly, we find that $g'(1/x) = -x^2G'(x)$.

But now since $\frac{f'(x)}{g'(x)} \to L$ as $x \to \infty$, the proposition again says that $\frac{f'(1/x)}{g'(1/x)} \to L$ as $x \to 0^+$, that is, $\frac{f'(1/x)}{g'(1/x)} = \frac{-x^2F'(x)}{-x^2G'(x)} = \frac{F'(x)}{G'(x)} \to L$ as $x \to 0^+$. So! the ordinary L'Hôpital's Rules tells us that $\lim_{x\to 0^+}\frac{F(x)}{G(x)} = L$, and so $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$, as desired.

37. [Zorn, p.226, # 9] Show that if f is integrable on $[a,b]$, and you can show (from the definition!) that $\int_a^b f(x)\,dx = L$ and $\int_a^b f(x)\,dx = M$, then $L = M$. [I.e., 'the value of an integral is unique'.]

[Suppose not! Show that there is a partition P that gets you into trouble...]

$\int_a^b f(x)\,dx = L$ and $\int_a^b f(x)\,dx = M$, anf $L \neq M$. The $L - M \neq 0$, and so $|L - M| = \epsilon > 0$. But then by integrability we can find a $\delta > 0$ so that for any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ (and any sample points: we'll pick $c_i = x_i$) we have $||P|| < \delta$ implies

$|R(f,P,\{x_i\}) - L| < \epsilon/2$ and $|R(f,P,\{x_i\}) - L| < \epsilon/2$. But then $\epsilon = |L - M| = |(L - R(f,P,\{x_i\})) + (R(f,P,\{x_i\}) - M)| \le |L - R(f,P,\{x_i\})| + |R(f,P,\{x_i\}) - M| = |R(f,P,\{x_i\}) = L| + |R(f,P,\{x_i\}) - M| < \epsilon/2 + \epsilon/2 = \epsilon$, so $\epsilon < \epsilon$, wich is absurd. Therefore, it is not possible to have $L \neq M$, so $L = M$.

[Formally, in the above argument we need to know that there is a partition P with $||P|| < \delta$ (if there weren't one, we couldn't insert it into the inequalities above!). But there is one: picking an $n \in \mathbb{N}$ with $1/n < \delta$, the partition $P = \{a + i\frac{b-a}{n}\ :\ i = 0,\ldots,n\}$ has $||P|| = 1/n < \delta$.]

38. [Zorn, p.236, # 1]

(a): Show that if h is integrable on the interval $[a,b]$ and $h(x) \ge 0$ for every $x \in [a,b]$, then for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ and set of 'samples' $S = \{c_1,\ldots,c_n\}$ with $x_{i-1} \le c_i \le x_i$ for each i, we have $R(h,P,S) \ge 0$. Explain why we can then conclude that $\int_a^b h(x)\,dx \ge 0$.

If $h(x) \geq 0$ for every $x \in [a,b]$, then $h(c_i) \geq 0$ for every sample point c_i, since $a \leq x_{i-1} \leq c_i \leq x_i \leq b$, so $c_i \in [a,b]$. Then since $x_i - x_{i-1} > 0$ for every i (since $x_{i-1} < x_i$), we have $f(c_i)(x_i - x_{i-1}) \geq 0$ for every i. So $R(f,P,S) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ is a sum of non-negative numbers, which is (by induction!) nonnegative. So $R(f,P,S) \geq 0$ for every P and S.

This, in turn, implies that $\int_a^b f(x)dx \geq 0$, since if $\int_a^b f(x)dx = L < 0$, then for every P and S, we have $|R(f,P,s) - L| = |R(f,P,s) + (-L)| = R(f,P,s) + (-L) \geq -L = \epsilon > 0$ (since $R(f,P,S)$ and $-L$ are both ≥ 0), and so for that choice of $\epsilon > 0$ there is <u>no</u> $\delta > 0$ so that $||P|| < delta$ implies that $|R(f,P,S) - L| < \epsilon$. So $\int_a^b f(x)dx$ <u>cannot</u> equal L.

(b): Use part (a) and the properties of integrals (Theorem 5.5) to show that if f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for every $x \in [a,b]$, then $\int_a^b f(x)\ dx\ \geq \int_a^b g(x)\ dx$.

We know from work in class (sort of...) that if f and g are integrable on $[a,b]$, then $f - g = f + (-1)g$ is also integrable on $[a,b]$, and $\int_a^b (f-g)(x)\ dx = \int_a^b f(x)\ dx + (-1)\int_a^b g(x)\ dx$. But since $f(x) \geq g(x)$ on $[a,b]$, we have $(f-g)(x) \geq 0$ on $[a,b]$, and so by part (a) we have $\int_a^b (f-g)(x)\ dx \geq 0$. So $\int_a^b (f-g)(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \geq 0$, and so $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$, as desired.

39. [Zorn, p.236, # 2] Suppose that f is integrable on $[a,b]$, and $m \leq f(x) \leq M$ for every $x \in [a,b]$. Show that $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$.

We can solve this one of two ways. If we define functions $g(x) = m$ and $h(x) = M$ for all $x \in [a.b]$, then bboth g and h are intgrable on $[a,b]$. (This is because <u>every</u> Riemann sum for g is equal to $m(b-a)$ and every Riemann sum for h is $M(b-a)$. This in turn means that $\int_a^b g(x)\ dx = m(b-a)$ and $\int_a^b h(x)\ dx = M(b-a)$.) Then the previous problem, part (b), allows us to conclude that since $f(x) \geq g(x)$ on $[a,b]$ we have $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx = m(b-a)$, and since $h(x) \geq f(x)$ on $[a,b]$ we have $M(b-a) = \int_a^b h(x)\ dx \geq \int_a^b f(x)\ dx$. So $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$.

The other, more back-to-basics, approach is to look at the Riemann sums of f. Since $m \leq f(x) \leq M$ on $[a,b]$¡ for any partition P of $[a,b]$ and sample points $S = \{c_i\}$, we have $m \leq f(c_i) \leq M$, so

$$m(b-a) = \sum_{i=1}^{n} m \cdot (x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(c_i) \cdot (x_i - x_{i-1}) = R(f,P,S) \leq \sum_{i=1}^{n} M \cdot (x_i - x_{i-1}) = M(b-a).$$

So every Riemann sum is trapped between $m(b-a)$ and $M(b-a)$. But since f is integrable, we can choose any sequence of partitions P_n with $||P_n|| \to 0$ and $n \to \infty$ and compute the integral as $\lim_{n\to\infty} R(f,P_n,S_n)$. But since $m(b-a) \leq R(f,P_n\S_n) \leq M(b-a)$, the Squeeze Play Theorem (!) tells us that $m(b-a) \leq \lim_{n\to\infty} R(f,P_n\S_n) = \int_a^b f(x)\ dx \leq M(b-a)$.