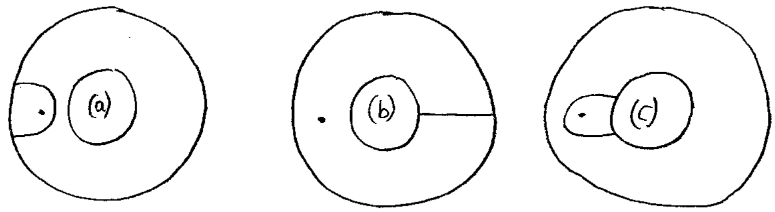
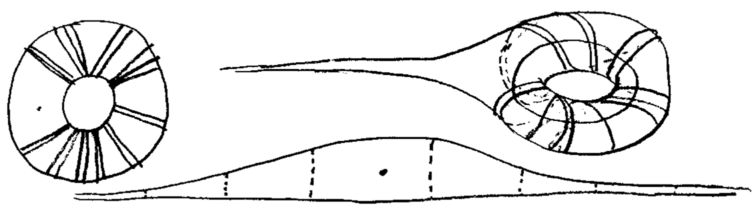
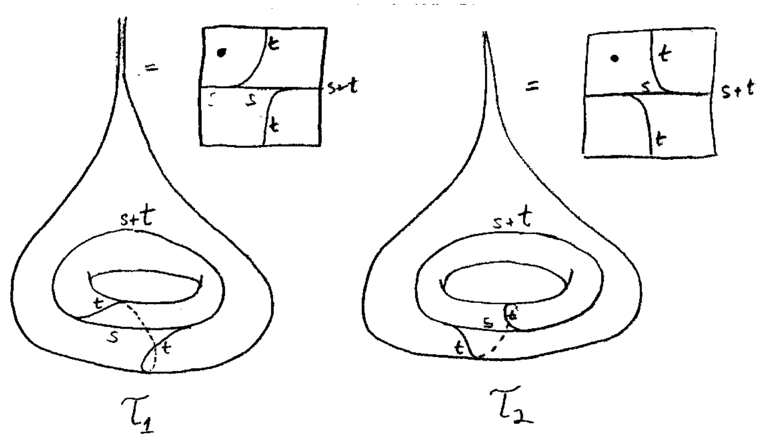


If the measure of the set of arcs of type (a) is m_a, etc., then (since the two boundary components match up) we have $2m_a + m_b = 2m_c + m_b$. But cases (a) and (c) 9.32
are incompatible with each other, so it must be that $m_a = m_c = 0$. Note that γ is orientable: it admits a continuous tangent vector field. By inspection we see a complementary region which is a punctured bigon.



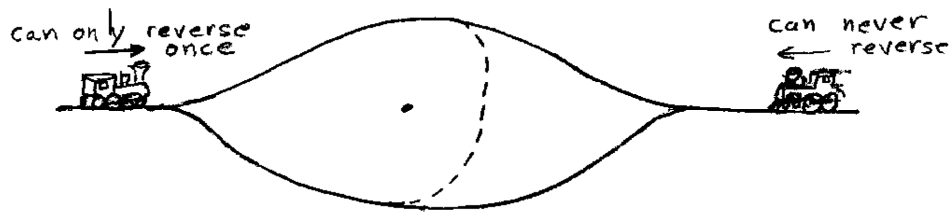
Since the area of a punctured bigon is 2π, which is the same as the area of $T - p$, this is the only complementary region.

It is now clear that a compactly supported measured lamination on $T - p$ with every leaf dense is essentially complete—there is nowhere to add new leaves under a small perturbation. If γ has a single closed leaf, then consider the families of measures on train tracks:

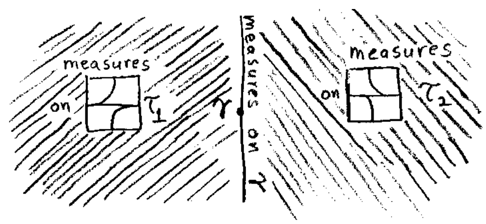


9.33

These train tracks cannot be enlarged to train tracks carrying measures. This can be deduced from the preceding argument, or seen as follows. At most one new branch could be added (by area considerations), and it would have to cut the punctured bigon into a punctured monogon and a triangle.



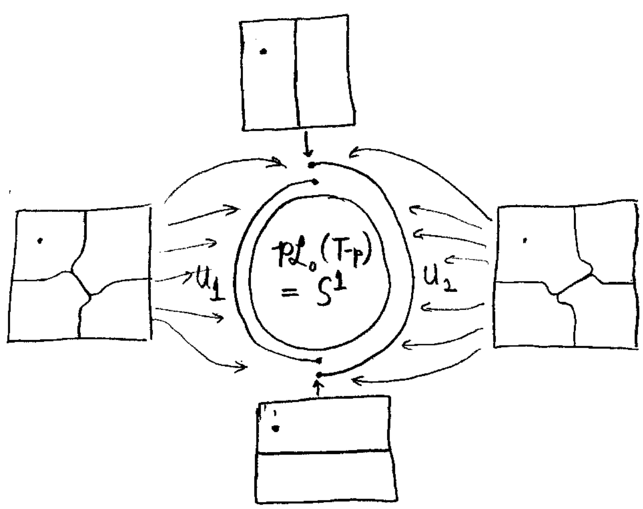
The train track is then orientable in the complement of the new branch, so a train can traverse this branch at most once. This is incompatible with the existence of a positive measure. Therefore $\mathcal{ML}_0(T-p)$ is two-dimensional, so τ_1 and τ_2 carry a neighborhood of γ.



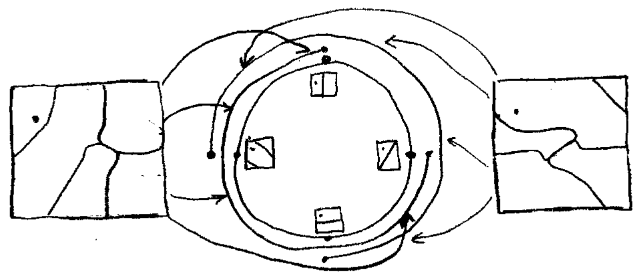
It follows that τ_γ is as shown. $\square$

PROPOSITION 9.5.3. *$\mathcal{PL}_0(T-p)$ is a circle.*

PROOF. The only closed one-manifold is S^1. That $\mathcal{PL}_0(T-p)$ is one-dimensional follows from the proof of 9.5.2. Perhaps it is instructive in any case to give a covering of $\mathcal{PL}_0(T-p)$ by train track neighborhoods: 9.34



or, to get open overlaps,

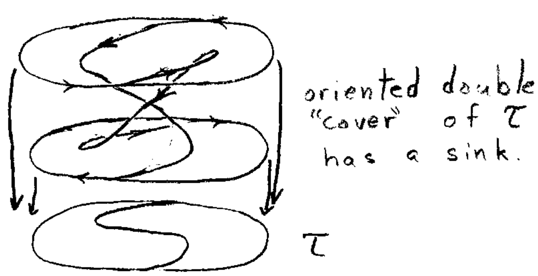


$\square$

PROPOSITION 9.5.4. *On any hyperbolic surface S which is not a punctured torus, an element $\gamma \in \overline{\mathcal{ML}}_0(S)$ is essentially complete if and only if $S - \gamma$ is a union of triangles and punctured monogons.*

PROOF. Let γ be an arbitrary lamination in $\mathcal{ML}_0(S)$, and let τ be any train track approximation close enough that the regions of $S - \tau$ correspond to those of $S - \gamma$. If some of these regions are not punctured monogons or triangles, we will add extra branches in a way compatible with a measure. 9.35

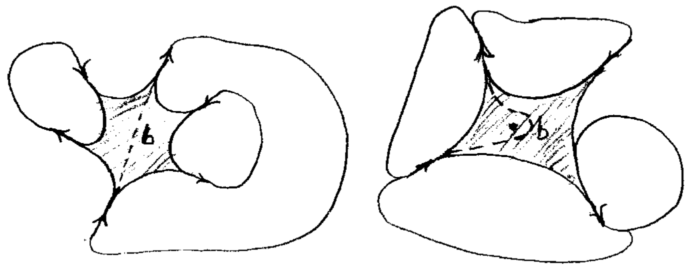
First consider the case that each region of $S - \gamma$ is either simply connected or a simple neighborhood of a cusp of S with fundamental group $\mathbb{Z}$. Then τ is connected. Because of the existence of an invariant measure, a train can get from any part of τ to any other. (The set of points accessible by a given oriented train is a "sink," which can only be a connected component.) If τ is not orientable, then every oriented train can get to any position with any orientation. (Otherwise, the oriented double "cover" of τ would have a non-trivial sink.)



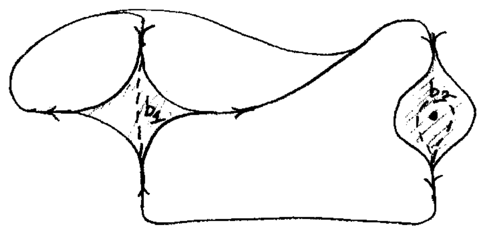
In this case, add an arbitrary branch b to τ, cutting a non-atomic region (of area $> \pi$). Clearly there is some cyclic train path through b, so $\tau \cup b$ admits a positive measure.

If τ is oriented, then each region of $S - \tau$ has an even number of cusps on its boundary. The area of S must be 4π or greater (since the only complete oriented surfaces of finite area having $\chi = -1$ are the thrice punctured sphere, for which $\mathcal{ML}_0$ is empty, and the punctured torus). If there is a polygon with more than four sides, it 9.36

can be subdivided using a branch which preserves orientation, hence admits a cyclic train path. The case of a punctured polygon with more than two sides is similar.

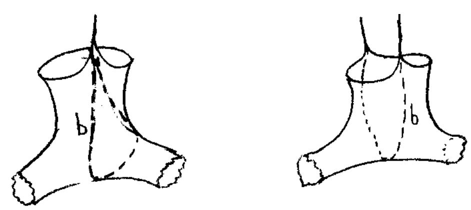


Otherwise, $S-\gamma$ has at least two components. Add one branch b_1 which reverses positively oriented trains, in one region, and another branch b_2 which reverses negatively oriented trains in another.

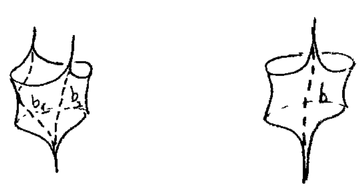


There is a cyclic train path through b_1 and b_2 in $\tau \cup b_1 \cup b_2$, hence an invariant measure.

Now consider the case when $S - \tau$ has more complexly connected regions. If 9.37
a boundary component of such a region R has one or more vertices, then a train pointing away from R can return to at least one vertex pinting toward R. If R is not an annulus, hook a new branch around a non-trivial homotopy class of arcs in R with ends on such a pair of vertices.

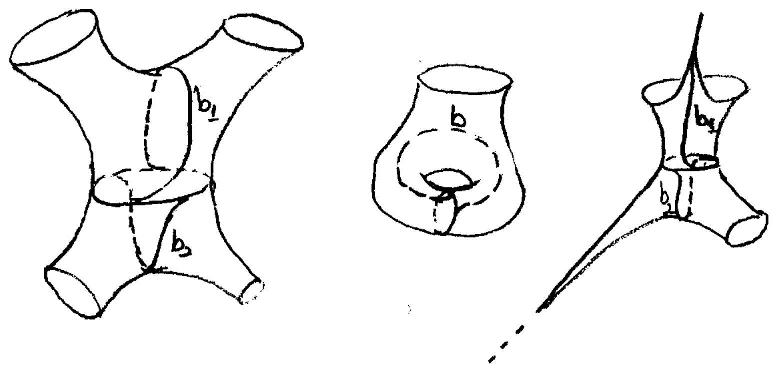


If R is an annulus and each boundary component has at least one vertex, then add one or two branches running across R which admit a cyclic train path.



If R is not topologically a thrice punctured disk or annulus, we can add an interior closed curve to R.

Any boundary component of R which is a geodesic α has another region R' (which may equal R) on the other side. In this case, we can add one or more branches in R and R' tangent to α in opposite directions on opposite sides, and hooking in ways 9.38
similar to those previously mentioned.



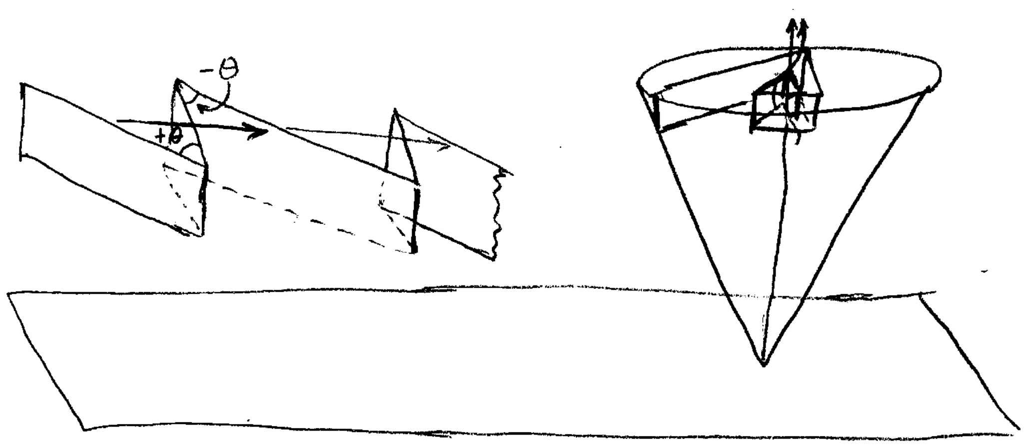
From the existence of these extensions of the original train track, it follows that an element $\gamma \in \mathcal{ML}_0$ is essentially complete if and only if $S - \gamma$ consists of triangles and punctured monogons. Furthermore, every $\gamma \in \overline{\mathcal{ML}_0}$ can be approximated by essentially complete elements $\gamma' \in \mathcal{ML}_0$. In fact, an open dense set has the property that the ϵ-train track approximation τ_ϵ has only triangles and punctured monogons as complementary regions, so generically *every* τ_ϵ has this property. The characterization of essential completeness then holds for $\overline{\mathcal{ML}_0}$ as well. ☐

Here is some useful geometric information about uncrumpled surfaces.

PROPOSITION 9.5.5. (i) *The sum of the dihedral angles along all edges of the wrinkling locus $w(S)$ tending toward a cusp of an uncrumpled surface S is 0. (The sum is taken in the group $S^1 = \mathbb{R} \bmod 2\pi$.)*

(ii) *The sum of the dihedral angles along all edges of $w(S)$ tending toward any side of a closed geodesic γ of $w(S)$ is $\pm\alpha$, where α is the angle of rotation of* 9.39
parallel translation around γ. (The sign depends on the sense of the spiralling of nearby geodesics toward γ.)

PROOF. Consider the upper half-space model, with either the cusp or the end of $\tilde{\gamma}$ toward which the geodesics in $w(S)$ are spiralling at ∞. About some level (in case (a)) or inside some code (in case (b)), S consists of vertical planes bent along vertical lines. The proposition merely says that the total angle of bending in some fundamental domain is the sum of the parts.

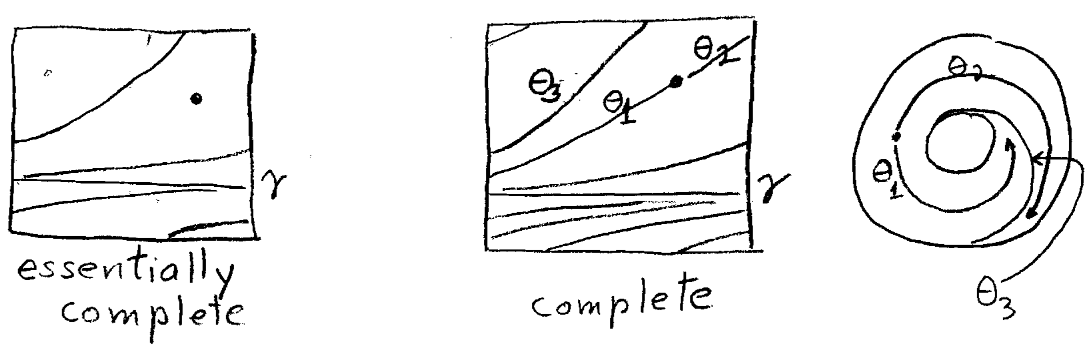


□

Corollary 9.5.6. *An uncrumpled surface realizing an essentially complete lamination in $\overline{\mathcal{ML}_0}$ in a given homotopy class is unique. Such an uncrumpled surface is totatlly geodesic near its cusps.*

Proof. If the surface S is not a punctured torus, then it has a unique completion obtained by adding a single geodesic tending toward each cusp. By 9.5.5, an uncrumpled surface cannot be bent along any of these added geodesics, so we obtain 9.40
9.5.6.

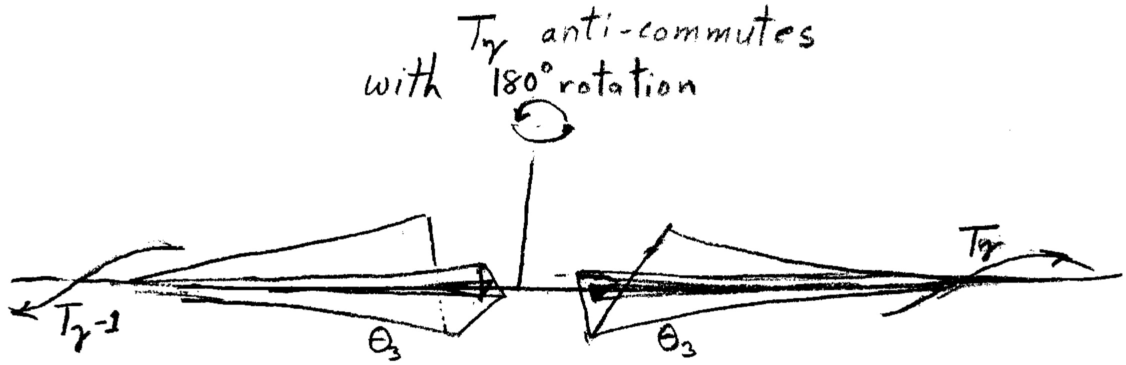
If S is the punctured torus $T - p$, then we consider first the case of a lamination γ which is an essential completion of a single closed geodesic. Complete γ by adding two closed geodesics going from the vertices of the punctured bigon to the puncture.



If the dihedral angles along the infinite geodesics are θ_1, θ_2 and θ_3, as shown, then by 9.5.5 we have

$$\theta_1 + \theta_2 = 0, \quad \theta_1 + \theta_3 = \alpha, \quad \theta_2 + \theta_3 = \alpha,$$

where α is some angle. (The signs are the same for the last two equations because any hyperbolic transformation anti-commutes with a 180° rotation around any perpendicular line.)



9.41

Thus $\theta_1 = \theta_2 = 0$, so an uncrumpled surface is totally geodesic in the punctured bigon. Since simple closed curves are dense in $\mathcal{ML}_0$, every element $g \in \mathcal{ML}_0$ realizable in a given homotopy class has a realization by an uncrumpled surface which is totally geodesic on a punctured bigon. If γ is essentially incomplete, this means its realizing surface is unique. □

PROPOSITION 9.5.7. *If γ is an essentially complete geodesic lamination, realized by an uncrumpled surface U, then any uncrumpled surface U' realizing a lamination γ' near γ is near U.*

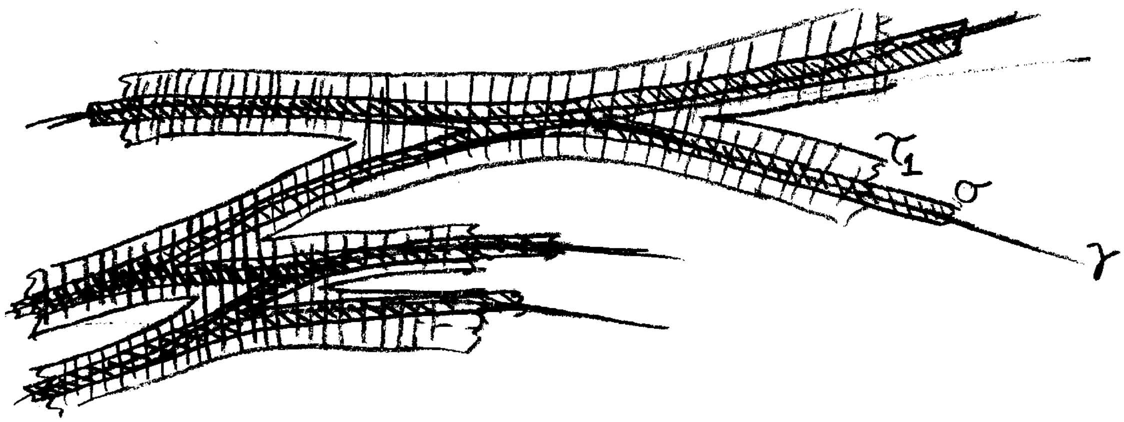
PROOF. You can see this from train track approximations. This also follows from the uniqueness of the realization of γ on an uncrumpled surface, since uncrumpled surfaces realizing laminations converging to γ must converge to a surface realizing γ. □

Consider now a typical path $\gamma_t \in \mathcal{ML}_0$. The path γ_t is likely to consist mostly of essentially complete laminations, so that a family of uncrumpled surfaces U_t realizing γ_t would be usually (with respect to t) continuous. At a countable set of values of t, γ_t is likely to be essentially incomplete, perhaps having a single complementary quadrilateral. Then the left and right hand limits U_{t-} and U_{t+} would probably exist, 9.42
and give uncrumpled surfaces realizing the two essential completions of γ_t. In fact, we will show that any path γ_t can be perturbed slightly to give a "generic" path in which the only essentially incomplete laminations are ones with precisely two distinct completions. In order to speak of generic paths, we need more than the topological structure of $\mathcal{ML}_0$.

PROPOSITION 9.5.8. *$\mathcal{ML}$ and $\mathcal{ML}_0$ have canonical PL (piecewise linear) structures.*

PROOF. We must check that changes of the natural coordinates coming from maximal train tracks (pp. 8.59-8.60) are piecewise linear. We will give the proof for $\mathcal{ML}_0$; the proof for $\mathcal{ML}$ is obtained by appropriate modifications.

Let γ be any measured geodesic lamination in $\mathcal{ML}_0(S)$. Let τ_1 and τ_2 be maximal compactly supported train tracks carrying γ, defining coordinate systems ϕ_1 and ϕ_2 from neighborhoods of γ to convex subsets of R^n (consisting of measures on τ_1 and τ_2). A close enough train track approximation σ of γ is carried by τ_1 and τ_2.



9.43

The set of measures on σ go linearly to measures on τ_1 and τ_2. If σ is a maximal compact train track supporting a measure, we are done—the change of coordinates $\phi_2 \circ \phi_2^{-1}$ is linear near γ. (In particular, note that if γ is essentially complete, change of coordinates is *always* linear at γ). Otherwise, we can find a finite set of enlargements of σ, $\sigma_1, \ldots, \sigma_k$, so that every element of a neighborhood of γ is closely approximated by one of the σ_i. Since every element of a neighborhood of γ is carried by τ_1 and τ_2, it follows that (if the approximations are good enough) each of the σ_i is carried by τ_1 and τ_2. Each σ_i defines a convex polyhedron which is mapped linearly by ϕ_1 and ϕ_2, so $\phi_2 \circ \phi_1^{-1}$ must be PL in a neighborhood of γ. □

Remark 9.5.9. It is immediate that change of coordinates involves only rational coefficients. In fact, with more care $\mathcal{ML}$ and $\mathcal{ML}_0$ can be given a piecewise integral linear structure. To do this, we can make use of the set $\mathcal{D}$ of integer-valued measures supported on finite collections of simple closed curves (in the case of $\mathcal{ML}_0$); $\mathcal{D}$ is analogous to the integral lattice in $\mathbb{R}^n$. $\mathrm{GL}_n\,\mathbb{Z}$ consists of linear transformations of $\mathbb{R}^n$ which preserve the integral lattice. The set V_τ of measures supported on a given train track τ is the subset of some linear subspace $V \subset \mathbb{R}^n$ which satisfies a finite number of linear inequalities $\mu(b_i) > 0$. Thus V_τ is the convex hull of a finite number of lines, each passing through an integral point. The integral points in U are closed under integral linear combinations (when such a combination is in U), so they determine an 9.44 integral linear structure which is preserved whenever U is mapped linearly to another coordinate system.

Note in particular that the natural transformations of $\mathcal{ML}_0$ are volume-preserving.

The structure on $\mathcal{PL}$ and $\mathcal{PL}_0$ is a piecewise integral *projective* structure. We will use the abbreviations PIL and PIP for piecewise integral linear and piecewise integral projective.

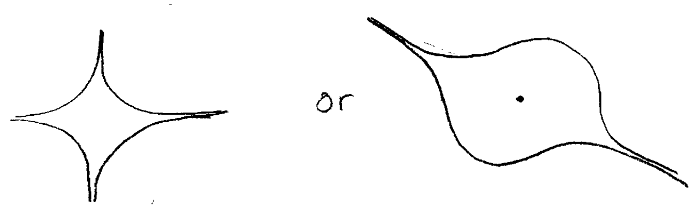
DEFINITION 9.5.10. The *rational depth* of an element $\gamma \in \mathcal{ML}_0$ is the dimension of the space of rational linear functions vanishing on γ, with respect to any natural local coordinate system. From 9.5.8 and 9.5.9, it is clear that the rational depth is independent of coordinates.

PROPOSITION 9.5.11. *If γ has rational depth 0, then γ is essentially complete.*

PROOF. For any $\gamma \in \mathcal{ML}_0$ which is not essentially complete we must construct a rational linear function vanishing on γ. Let τ be some train track approximation of γ which can be enlarged and still admit a positive measure. It is clear that the set of measures on τ spans a proper rational subspace in any natural coordinate system coming from a train track which carries τ. (Note that measures on τ consist of positive linear combinations of integral measures, and that every lamination carried by τ is approximable by one *not* carried by τ.) □ 9.45

PROPOSITION 9.5.12. *If $\gamma \in \mathcal{ML}_0$ has rational depth 1, then either γ is essentially complete or γ has precisely two essential completions. In this case either*

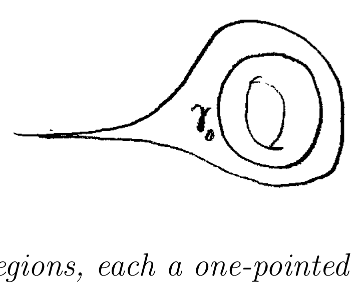
A. *γ has no closed leaves, and all complementary regions have area π or 2π. There is only one region with area 2π unless γ is oriented and* $\text{area}(S) = 4\pi$ *in which case there are two. Such a region is either a quadrilateral or a punctured bigon.*



or

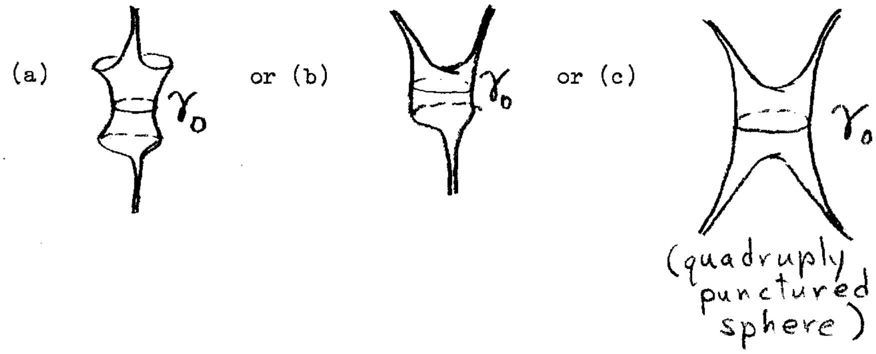
B. *γ has precisely one closed leaf γ_0. Each region touching γ_0 has area 2π. Either*

   1. *S is a punctured torus*

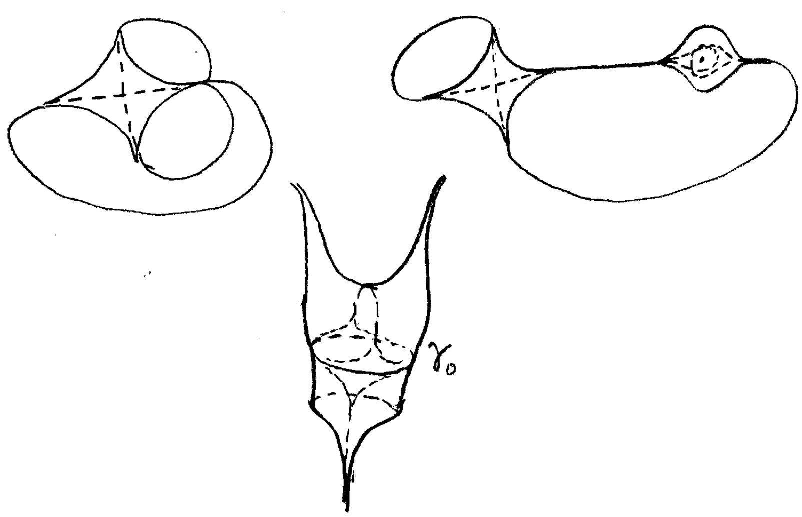


or

   2. *γ_0 touches two regions, each a one-pointed crown or a devils cap.*



Proof. Suppose γ has rational depth 1 and is not essentially complete. Let τ be a close train track approximation of γ. There is some finite set $\tau_1, \dots, \tau_k$ of essentially complete enlargements of τ which closely approximate every γ' in a neighborhood of γ. Let σ carry all the τ_i's and let V_σ be its coordinate system. The set of γ corresponding to measures carried by a given proper subtrack of a τ_i is a proper rational subspace of V_σ. Since γ is in a unique proper rational subspace, V_τ, the set of measures V_{τ_i} carried on any τ_i must consist of one side of V_τ. (If V_{τ_i} intersected both sides, by convexity γ would come from a measure positive on all branches of τ_i). Since this works for any degree of approximation of nearby laminations, γ has precisely two essential completions. A review of the proof of 9.5.4 gives the list of possibilities for $\gamma \in \mathcal{ML}_0$ with precisely two essential completions. The ambiguity in the essential completions comes from the manner of dividing a quadrilateral or other region, and the direction of spiralling around a geodesic.

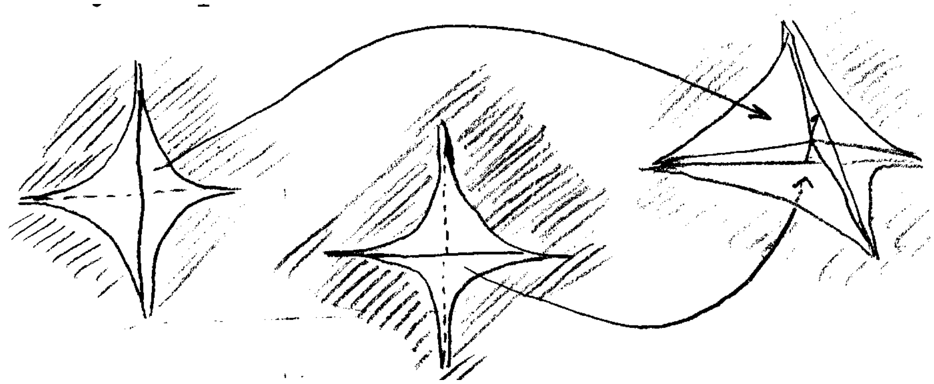


$\square$

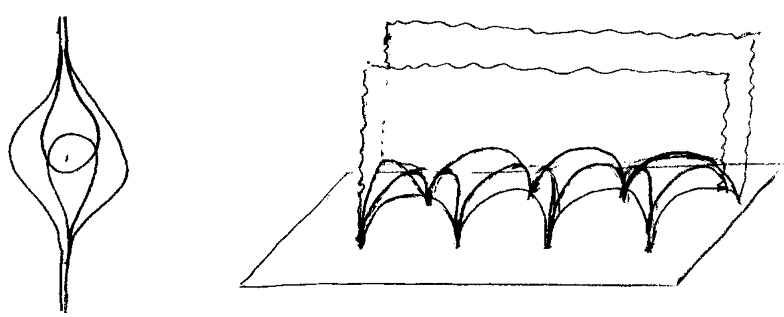
9.47

REMARK. There are good examples of $\gamma \in \mathcal{ML}_0$ which have large rational depth but are essentially complete. The construction will occur naturally in another context.

We return to the construction of continuous families of surfaces in a hyperbolic three-manifold. To each essentially incomplete $\gamma \in \mathcal{ML}_0$ of rational depth 1, we associate a one-parameter family of surfaces U_s, with U_0 and U_1 being the two uncrumpled surfaces realizing γ. U_s is constant where U_0 and U_1 agree, including the union of all triangles and punctured monogons in the complement of γ. The two images of any quadrilateral in $S - \gamma$ form an ideal tetrahedron. Draw the common perpendicular p to the two edges not in $U_0 \cap U_1$, triangulate the quadrilateral with 4 triangles by adding a vertex in the middle, and let this vertex run linearly along p, from U_0 to U_1. This extends to a homotopy of S straight on the triangles.

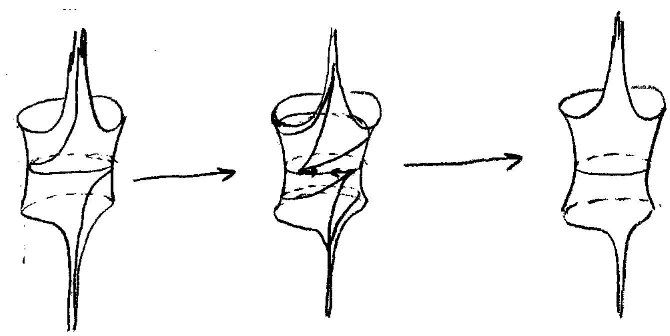


The two images of any punctured bigon in $S - \gamma$ form a solid torus, with the generating curve parabolic. The union of the two essential completions in this punctured bigon gives a triangulation except in a neighborhood of the puncture, with two new vertices at intersection points of added leaves. 9.48



Draw the common perpendiculars to edges of the realizations corresponding to these intersection points, and homotope U_0 to U_1 by moving the added vertices linearly along the common perpendiculars.

When γ has a closed leaf γ_0, the two essential completions of γ have added leaves spiralling around γ_0 in opposite directions. U_0 can be homotoped to U_1 through surfaces with added vertices on γ_0.



9.49

Note that all the surfaces U_s constructed above have the property that any point on U_s is in the convex hull of a small circle about it on U_s. In particular, it has curvature ≤ -1; curvature -1 everywhere except singular vertices, where negative curvature is concentrated.

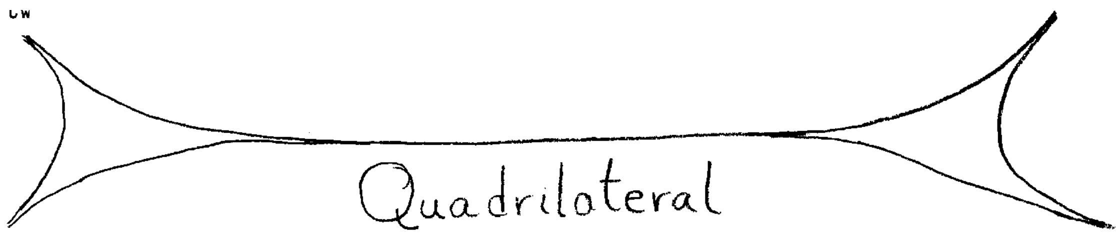
Theorem 9.5.13. *Given any complete hyperbolic three-manifold N with geometrically tame end E cut off by a hyperbolic surface $S_{[\epsilon,\infty)}$, there is a proper homotopy $F : S_{[\epsilon,\infty)} \times [0,\infty) \to N$ of S to ∞ in E.*

Proof. Let V_τ be the natural coordinate system for a neighborhood of $\epsilon(E)$ in $\mathcal{ML}_0(S)$, and choose a sequence $\gamma_i \in V_\tau$ limiting on $\epsilon(E)$. Perturb the γ_i slightly so that the path γ_t $[0 \leq t \leq \infty]$ which is linear on each segment $t \in [i,\, i+1]$ consists of elements of rational depth 0 or 1. Let U_t be the unique uncrumpled surface realizing γ_t when γ_t is essentially complete. When t is not essentially complete, the left and right hand limits U_{t+} and U_{t-} exist. It should now be clear that F exists, since one can cover the closed set $\{U_{t\pm}\}$ by a locally finite cover consisting of surfaces homotopic by small homotopies, and fill in larger gaps between U_{t+} and U_{t-} by the homotopies constructed above. Since all interpolated surfaces have curvature ≤ -1, and they all realize a γ_t, they must move out to ∞. An explicit homotopy can actually be defined, using a new parameter r which is obtained by "blowing up" all parameter values of t with rational depth 1 into small intervals. Explicitly, these parameter values can be enumerated in some order $\{t_j\}$, and an interval of length 2^{-j} inserted 9.50
in the r-parameter in place of t_j. Thus, a parameter value t corresponds to the point

or interval

$$r(t) = \left[t + \sum_{\{j | t_j < t\}} 2^{-}j,\ t + \sum_{\{j | t_j \le t\}} 2^{-j} \right].$$

Now insert homotopies as constructed above in *each* blown up interval. It is not so obvious that the family of surfaces is still continuous when an infinite family of homotopies is inserted. Usually, however, these homotopies move a very small distance—for instance, γ_t may have a quadrilateral in $S - \gamma_t$, but for all but a locally small number of t's, this quadrilateral looks like two asymptotic triangles to the naked eye, and the homotopy is imperceptible.



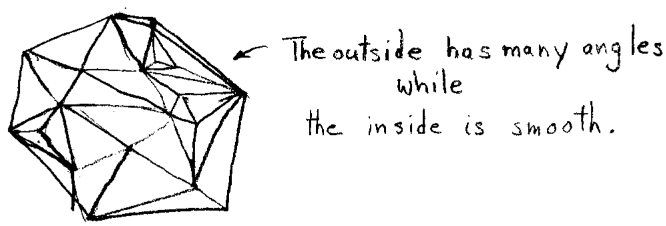
Formally, the proof of continuity is a straightforward generalization of the proof of 9.5.7. The remark which is needed is that if S is a surface of curvature ≤ -1 with a (pathwise) isometric map to a hyperbolic surface homotopic to a homeomorphism, then S is actually hyperbolic and the map is isometric—indeed, the area of S is not greater than the area of the hyperbolic surface. $\square$

Remarks. 1. There is actually a canonical line of hyperbolic structures on S joining those of U_{t+} and U_{t-}, but it is not so obvious how to map them into E nicely. 9.51

2. An alternative approach to the construction of F is to make use of a sequence of triangulations of S. Any two triangulations with the same number of vertices can be joined by a sequence of elementary moves, as shown.

Although such an approach involves more familiar methods, the author brutally chose to develop extra structure.

3. There should be a good analytic method of constructing F by using harmonic mappings of hyperbolic surfaces. Realizations of geodesic laminations of a surface are analogous to harmonic mappings coming from points at ∞ in Teichmüller space. The harmonic mappings corresponding to a family of hyperbolic structures on S moving along a Teichmüller geodesic to $\epsilon(E)$ ought to move nicely out to ∞ in E. A rigorous proof might involve good estimates of the energy of a map, analogous to §9.3.

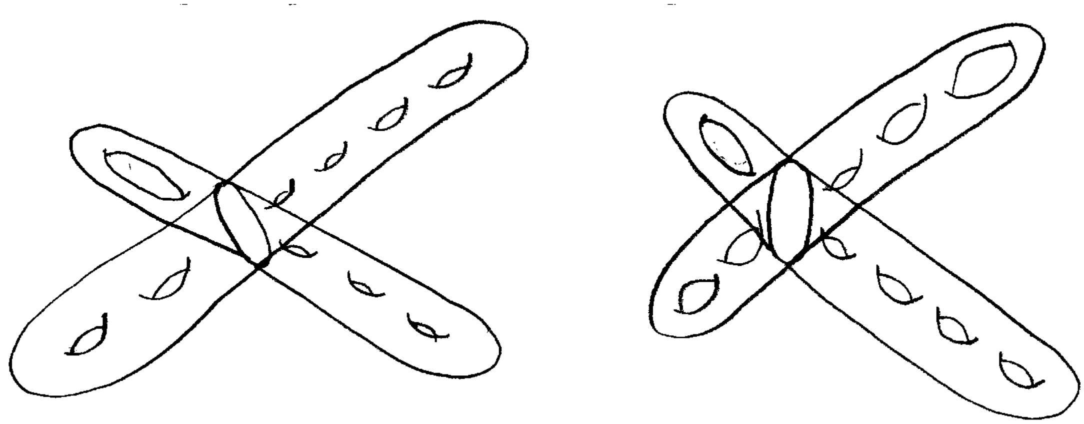


9.52

9.6. Strong convergence from algebraic convergence

We will take another step in our study of algebraic limits. Consider the space of discrete faithful representations ρ of a fixed torsion free group Γ in $\mathrm{PSL}_2(\mathbb{C})$. The set $\Pi_\rho \subset \Gamma$ of parabolics—i.e., elements $\gamma \in \Gamma$ such that $\rho(\gamma)$ is parabolic—is an important part of the picture; we shall assume that $\Pi_\rho = \Pi$ is constant. When a sequence ρ_i converges algebraically to a representation ρ where $\Pi = \Pi_{\rho_i}$ is constant by $\Pi_\rho \supset \Pi$ is strictly bigger, then elements $\gamma \in \Pi_\rho - \Pi$ are called *accidental parabolics.* The incidence of accidental parabolics can create many interesting phenomena, which we will study later.

One complication is that the quotient manifolds $N_{\rho_i\Gamma}$ need not be homeomorphic; and even when they are, the homotopy equivalence given by the isomorphism of fundamental groups need not be homotopic to a homeomorphism. For instance, consider three-manifolds obtained by gluing several surfaces with boundary, of varying genus, in a neighborhood of their boundary. If every component has negative Euler characteristic, the result can easily be given a complete hyperbolic structure. The homotopy type depends only on the identifications of the boundary components of the original surfaces, but the homeomorphism type depends on the order of arrangement around each image boundary curve.



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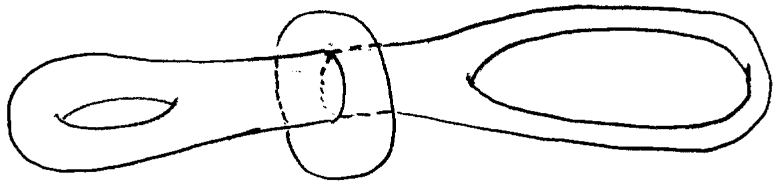
As another example, consider a thickened surface of genus 2 union a torus as shown.



It is also easy to give this a complete hyperbolic structure. The fundamental group has a presentation

$$\langle a_1, b_1, a_2, b_2, c : [a_1, b_1] = [a_2, b_2], \big[[a_1, b_1] = c, \big] = 1\rangle.$$

This group has an automorphism

$$a_1 \mapsto a_1,\ b_1 \mapsto b_1,\ c \mapsto c,\ a_2 \mapsto ca_2c^{-1},\ b_2 \mapsto cb_2c^{-1}$$

which wraps the surface of genus two around the torus. No non-trivial power of this automorphism is homotopic to a homeomorphism. From an algebraic standpoint there are infinitely many distinct candidates for the peripheral subgroups.

One more potential complication is that even when a given homotopy equivalence is homotopic to a homeomorphism, and even when the parabolic elements correspond, there might not be a homeomorphism which preserves cusps. This is easy to picture for a closed surface group Γ: when Π is the set of conjugates of powers of a collection of simple closed curves on the surface, there is not enough information in Π to say which curves must correspond to cusps on which side of S. Another example is when Γ is a free group, and Π corresponds to a collection of simple closed curves on the boundary of a handlebody with fundamental group Γ. The homotopy class of a 9.54
simple closed curve is a very weak invariant here.

Rather than entangle ourselves in cusps and handlebodies, we shall confine ourselves to the case of real interest, when the quotient spaces admit cusp-preserving homeomorphisms.

We shall consider, then, geometrically tame hyperbolic manifolds which have a common model, (N_0, P_0). N_0 should be a compact manifold with boundary, and P_0 (to be interpreted as the "parabolic locus") should be a disjoint union of regular neighborhoods of tori and annuli on ∂N_0, with fundamental groups injecting into $\pi_1 N_0$. Each component of $\partial N_0 - P_0$ should be incompressible.

THEOREM 9.6.1. *Let (N_0, P_0) be as above. Suppose that $\rho_i : \pi_1 N \to \mathrm{PSL}(2, \mathbb{C})$ is a sequence of discrete, faithful representations whose quotient manifolds N_i are geometrically tame and admit homeomorphisms* (*in the correct homotopy class*) *to N_0 taking horoball neighborhoods of cusps to P_0. If $\{\rho_i\}$ converges algebraically to a representation ρ, then the limit manifold N is geometrically tame, and admits a homeomorphism* (*in the correct homotopy class*) *to N_0 which takes horoball neighborhoods of cusps to P_0.*

We shall prove this first with an additional hypothesis:

9.6.1a. Suppose also that no non-trivial non-peripheral simple curve of a component of $\partial N_0 - P_0$ *is homotopic* (*in* N_0) *to* P_0. 9.55

REMARKS. The proof of 9.6.1 (without the added hypothesis) will be given in §9.8. There is no §9.8.

The main case is really when all N_i are geometrically finite. One of the main corollaries, from 8.12.4, is that $\rho(\pi_1 N_0)$ satisfies the property of Ahlfors: its limit set has measure 0 or measure 1.

PROOF OF 9.6.1a. It will suffice to prove that every sequence $\{\rho_i\}$ converging algebraically to ρ has a subsequence converging strongly to ρ. Thus, we will pass to subsequences whenever it is convenient.

Let $S_1, \ldots, S_k$ be the components of $\partial N_0 - P_0$, each equipped with a complete hyperbolic metric of finite area. (In other words, their boundary components are made into punctures.) For each i, let P_i denote a union of horoball neighborhoods of cusps of N_i, and let $E_{i,1}, \ldots, E_{i,k}$ denote the ends of $N_i - P_i$ corresponding to $S_1, \ldots, S_k$.

Some of the $E_{i,j}$ may be geometrically finite, others geometrically infinite. We can pass (for peace of mind) to a subsequence so that for each i, the $E_{i,j}$ are all geometrically finite or all geometrically infinite. We pass to a further subsequence so the sequences of bending or ending laminations $\{\beta_{i,j}\}_i$ or $\{\epsilon_{i,j}\}_i$ converge in $\mathcal{GL}_{(S_j)}$. Let χ_j be the limit.

If χ_j is realizable in N, then all nearby laminations have realizations for all representations near ρ, and the $E_{i,j}$ must have been geometrically finite. An uncrumpled surface U realizing χ_j is in the convex hull M of N and approximable by boundary 9.56
components of the convex hulls M_i Since the limit set cannot suddenly increase in the algebraic limit (p. 9.8), U must be a boundary component.

If χ_j is not realizable in N, then it must be the ending lamination for some geometrically infinite tame end E of the covering space of N corresponding to $\pi_1 S_j$, since we have hypothesized away the possibility that it represents a cusp. In view of 9.2.2 and 9.4.2, the image E_j of E in $N - P$ is a geometrically tame end of $N - P$, and $\pi_1 E = \pi_1 S_j$ has finite index in $\pi_1 E_j$.

In either case, we obtain embeddings in $N - P$ of oriented surfaces S'_j finitely covered by $S_{j[\epsilon,\infty)}$. We may assume (after an isotopy) that these embeddings are disjoint, and each surface cuts off (at least) one piece of $N - P$ which is homeomorphic to the product $S'_j \times [0, \infty)$. Since (N, P) is homotopy equivalent to (N_0, P_0), the image of the cycle $\sum \big[S_{j[\epsilon,\infty)}, \partial S_{j[\epsilon,\infty)}\big]$ in (N, P) bounds a chain C with compact support. Except in a special case to be treated later, the S'_j are pairwise non-homotopic and the fundamental group of each S'_j maps isomorphically to a unique side in $N - P$.

C has degree 0 "outside" each S'_j and degree some constant l elsewhere. Let N' be the region of $N-P$ where C has degree l. We see that N is geometrically tame, and homotopy equivalent to N'.

The Euler characteristic is a homotopy invariant, so $\chi(N) = \chi(N') = \chi(N_0)$. This imples $\chi(\partial N') = \chi(\partial N_0)$ (by the formula $\chi(\partial M^3) = 2\chi(M^3)$) so in fact the 9.57
finite sheeted covering $S_{j[\epsilon,\infty)} \to S'_j$ has only one sheet—it is a homeomorphism.

Let Q be the geometric limit of any subsequence of the N_i. N is a covering space of Q. Every boundary component of the convex hull M of N is the geometric limit of boundary components of the M_i; consequently, M covers the convex hull of Q. This covering can have only finitely many sheets, since $M-P$ is made of a compact part together with geometrically infinite tame ends. Any element $\alpha \in \pi_1 Q$ has some finite power $\alpha^k \in \pi_1 N$ $[k \geq 1]$. In any torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$, an element has at most one k-th root (by consideration of axes). If we write α as the limit of elements $\rho_i(g_i)$, $g_i \in \pi_1 N_0$, by this remark, g_i must be eventually constant so α is actually in the algebraic limit $\pi_1 N$. $Q = N$, and ρ_i converges strongly to ρ.

A cusp-preserving homeomorphism from N to some N_i, hence to N_0, can be constructed by using an approximate isometry of N' with a submanifold of $N_i - P_i$, for high enough i. The image of N' is homotopy equivalent to N_i, so the fundamental group of each boundary component of N' must map surjectively, as well as injectively, to the fundamental group of the neighboring component of $(N_i, P_i) - N'$. This implies that the map of N' into N_i extends to a homeomorphism from N to N_i.

There is a special case remaining. If any pair of the surfaces S'_i constructed in $N-P$ is homotopic, perform all such homotopies. Unless $N-P$ is homotopy equivalent to a product, the argument continues as before—there is no reason the cover of S'_i must be a connected component of $\partial N_0 - P_0$.

When $N-P$ is homotopy equivalent to the oriented surface S'_1 in it, then by a standard argument $N_0 - P_0$ must be homeomorphic to $S'_1 \times I$. This is the case essentially dealt with in 9.2. The difficulty is to control both ends of $N-P$—but the argument of 9.2 shows that the ending or bending laminations of the two ends of $N_i - P_i$ cannot converge to the same lamination, otherwise the limit of some intermediate surface would realize χ_i. This concludes the proof of 9.6.1a. $\square$

9.7. Realizations of geodesic laminations for surface groups with extra cusps, with a digression on stereographic coordinates

In order to analyze geometric convergence, and algebraic convergence in more general cases, we need to clarify our understanding of realizations of geodesic laminations for a discrete faithful representation ρ of a surface group $\pi_1(S)$ when certain non-peripheral elements of $\pi_1(S)$ are parabolic. Let $N = N_{\rho\pi_1 S}$ be the quotient three-manifold. Equip S with a complete hyperbolic structure with finite area. As in

§8.11, we may embed S in N, cutting it in two pieces the "top" N_+ and the "bottom" N_-. Let γ_+ and γ_- be the (possibly empty) cusp loci for N_+ and N_-, and denote by $S_{1+}, \dots, S_{j+}$ and $S_{1-}, \dots, S_{k-}$ the components of $S - \gamma_+$ and $S - \gamma_-$ (endowed with complete hyperbolic structures with finite area). Let $E_{1+}, \dots, E_{j+}$ and $E_{1-}, \dots, E_{k-}$ denote the ends of $N - P$, where P is the union of horoball neighborhoods of all cusps.

A compactly supported lamination on S_{i+} or S_{i-} defines a lamination on S. In particular, $\epsilon(E_{i\pm})$ may be thought of as a lamination on S for each geometrically infinite tame end of $E_{i\pm}$.

Proposition 9.7.1. *A lamination $\gamma \in \mathcal{GL}_0(S)$ is realizable in N if and only if γ contains no component of γ_+, no component of γ_-, and no $\epsilon(E_{i+})$ or $\epsilon(E_{i-})$.*

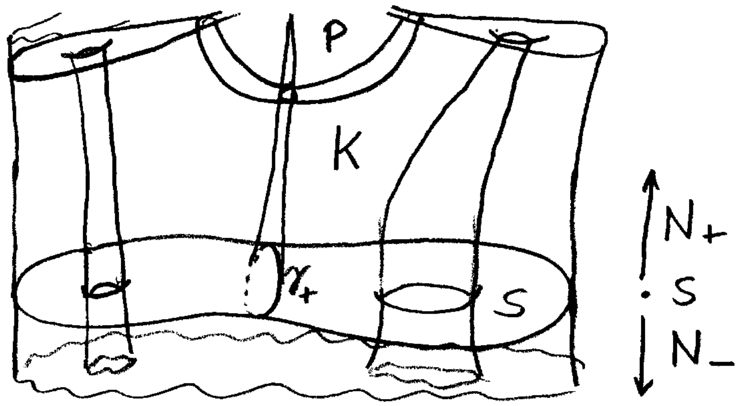
Proof. If γ contains any unrealizable lamination, it is unrealizable, so the necessity of the condition is immediate.

Let $\gamma \in \mathcal{ML}_0(S)$ be any unrealizable compactly supported measured lamination. If γ is not connected, at least one of its components is unrealizable, so we need only consider the case that γ is connected. If γ has zero intersection number with any components of γ_+ or γ_-, we may cut S along this component, obtaining a simpler surface S'. Unless γ is the component of γ_+ or γ_- in question, S' supports γ, so we pass to the covering space of N corresponding to $\pi_1 S'$. The new boundary components of S' are parabolic, so we have made an inductive reduction of this case.

We may now suppose that γ has positive intersection number with each component of γ_+ and γ_-. Let $\{\beta_i\}$ be a sequence of measures, supported on simple closed curves non-parabolic in N which converges to γ. Let $\{U_i\}$ be a sequence of uncrumpled surfaces realizing the β_i. If U_i penetrates far into a component of P 9.60
corresponding to an element α in γ_+ or γ_-, then it has a large ball mapped into P; by area considerations, this ball on U_i must have a short closed loop, which can only be in the homotopy class of α. Then the ratio

$$l_S(\beta_i)/i(\beta_i, \alpha) \geq l_{U_i}(\beta_i)/i(\beta_i, \alpha)$$

is large. Therefore (since $i(\gamma, \alpha)$ is positive and $l_S(\gamma)$ is finite) the U_i, away from their cusps, remain in a bounded neighborhood of $N - P$ in N. If γ_+ (say) is non-empty, one can now find a compact subset K of N so that any U_i intersecting N_+ must intersect K.



By the proof of 8.8.5, if infinitely many U_i intersected K, there would be a convergent subsequence, contradicting the non-realizability of γ. The only remaining possibility is that we have reached, by induction, the case that either N_+ or N_- has no extra cusps, and γ is an ending lamination. 9.61

A general lamination $\gamma \in \mathcal{GL}(S)$ is obtained from a possibly empty lamination which admits a compactly supported measure by the addition of finitely many non-compact leaves. (Let $\delta \subset \gamma$ be the maximal lamination supporting a positive transverse measure. If l is any leaf in $\gamma - \delta$, each end must come close to δ or go to ∞ in S, otherwise one could enlarge δ. By area considerations, such leaves are finite in number.) From §8.10, γ is realizable if and only if δ is. □

The picture of unrealizable laminations in $\mathcal{PL}_0(S)$ is the following. Let Δ_+ consist of all projective classes of transverse measures (allowing degenerate non-trivial cases) on $\chi_+ = \gamma_+ \cup U_i \epsilon(E_{i+})$. Δ_+ is convex in a coordinate system V_τ coming from any train track τ carrying χ_+.

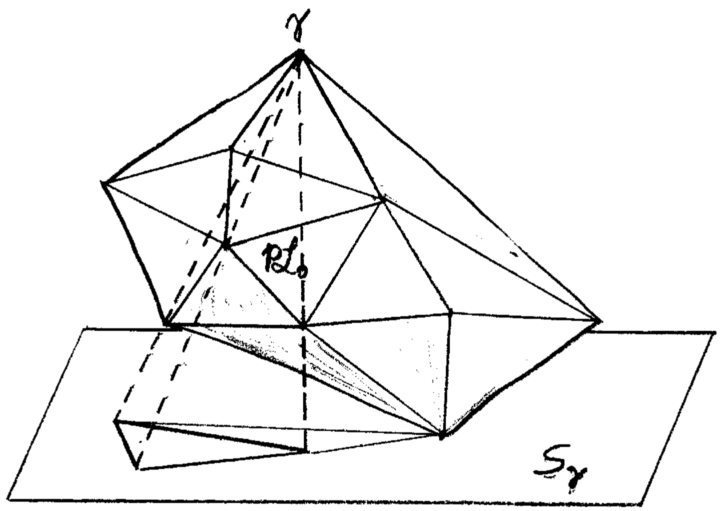
To see a larger, complete picture, we must find a larger natural coordinate system. This requires a little stretching of our train tracks and imaginations. In fact, it is possible to find coordinate systems which are quite large. For any $\gamma \in \mathcal{PL}_0$, let $\Delta_\gamma \subset \mathcal{PL}_0$ denote the set of projective classes of measures on γ.

Proposition 9.7.2. *Let γ be essentially complete. There is a sequence of train tracks τ_i, where τ_i is carried by τ_{i+1}, such that the union of natural coordinate systems $S_\gamma = U_i V_{\tau_i}$ contains all of $\mathcal{PL}_0 - \Delta_\gamma$.*

The proof will be given presently.

Since τ_i is carried by τ_{i+1}, the inclusion $V_{\tau_i} \subset V_{\tau+1}$ is a projective map (in $\mathcal{ML}_0$, 9.62
the inclusion is linear). Thus S_γ comes naturally equipped with a projective structure. We have not made this analysis, but the typical case is that $\gamma = \Delta_\gamma$. We think of S_γ as a stereographic coordinate system, based on projection from γ. (You may imagine

$\mathcal{PL}_0$ as a convex polyhedron in $\mathbb{R}^n$, so that changes of stereographic coordinates are piecewise projective, although this finite-dimensional picture cannot be strictly correct, since there is no fixed subdivision sufficient to make all coordinate changes.)



Corollary 9.7.3. *$\mathcal{PL}_0(S)$ is homeomorphic to a sphere.*

Proof that 9.7.2 implies 9.7.3. Let $\gamma \in \mathcal{PL}_0(S)$ be any essentially complete lamination. Let τ be any train track carrying γ. Then $\mathcal{PL}_0(S)$ is the union of two coordinate systems $V_\tau \cup S_\tau$, which are mapped to convex sets in Euclidean space. 9.63
If $\Delta_\gamma \neq \gamma$, nonetheless the complement of Δ_γ in V_τ is homeomorphic to $V_\tau - \gamma$, so $\mathcal{PL}_0(S)$ is homeomorphic to the one-point compactification of S_γ. □

Corollary 9.7.4. *When $\mathcal{PL}_0(S)$ has dimension greater than* 1*, it does not have a projective structure.* (In other words, the *pieces* in changes of coordinates have not been eliminated.)

Proof that 9.7.3 implies 9.7.4. The only projective structure on S^n, when $n > 1$, is the standard one, since S^n is simply connected. The binary relation of antipodality is natural in this structure. What would be the antipodal lamination for a simple closed curve α? It is easy to construct a diffeomorphism fixing α but moving any other given lamination. (If $i(\gamma, \alpha) \neq 0$, the Dehn twist around α will do.) □

Remark. When $\mathcal{PL}_0(S)$ is one-dimensional (that is, when S is the punctured torus or the quadruply punctured sphere), the PIP structure *does* come from a projective structure, equivalent to $\mathbb{R}P^1$. The natural transformations of $\mathcal{PL}_0(S)$ are necessarily integral—in $\mathrm{PSL}_2(\mathbb{Z})$.