Math 325 Problem Set 6 Solutions

20. Show *directly* (i.e., without quoting "Cauchy implies convergent" and "convergent implies Cauchy") that if a_n and b_n are Cauchy sequences, then so are the sequences $c_n = a_n + b_n$ and $d_n = a_n b_n$. [Hint: for the second, you will need to use Cauchy implies bounded?]

If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are both Cauchy sequences, then we know that they are both bounded. [There is an N so that $n \geq N$ implies $|a_n - a_N| < 1$, so $a_N - 1 < a_n < a_N + 1$, so $-|a_N| - 1 < a_n < |a_N| + 1$¡ so $|a_n| < |a_N| + 1$; then setting $M = \max\{|a_1|, \dots, |a_N|, |a_N| + 1\}$, we have $|a_n| \leq M$ for all n.]. Then, borrowing from our proofs about convergent sequences, given $\epsilon > 0$ we can choose N_1 and N_2 so that $n, m \geq N_1$ implies that $|a_n - a_m| < \epsilon/2$ and $n, m \geq N_2$ implies that $|b_n - b_m| < \epsilon/2$. Then setting $N = \max\{N_1, N_2\}$ we have $n, m \geq N$ implies

$|(a_n + b_n) - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \epsilon/2 + \epsilon/2 = \epsilon$,

so $a_n + b_n$ is also Cauchy. Also, since

$|a_n b_n - a_m b_m| = |a_n(b_n - b_m) + b_m(a_n - a_m)| \leq |a_n(b_n - b_m)| + |b_m(a_n - a_m)| = |a_n| \cdot |b_n - b_m| + |b_m| \cdot |a_n - a_m|$

then we choose M_1 and M_2 so that $|a_n| \leq M_1$ and $|b_n| \leq M_2$ for all n, and choose N_1 and N_2 so that $n, m \geq N_1$ implies that $|a_n - a_m| < \epsilon/(2M_1)$ and $n, m \geq N_2$ implies that $|b_n - b_m| < \epsilon/(2M_2)$. Then setting $N = \max\{N_1, N_2\}$ we have $n, m \geq N$ implies

$|a_n b_n - a_m b_m| \leq |a_n| \cdot |b_n - b_m| + |b_m| \cdot |a_n - a_m| < |a_n|\epsilon/(2M_1) + |b_m|\epsilon/(2M_2) < M_1\epsilon/(2M_1) + M_2\epsilon/(2M_2) = \epsilon/2 + \epsilon/2 = \epsilon$,

so $a_n b_n$ is Cauchy as well. So the sum and product of Cauchy sequences are Cauchy.

21. [Lay, p.181, problem # 18.15] A sequence a_n is called *contractive* if for some constant $0 < k < 1$ we have $|s_{n+2} - s_{n+1}| < k|s_{n+1} - s_n|$ for all $n \in \mathbb{N}$. Show that every contractive sequence is Cauchy (and therefore converges).

[Hint: By induction, $|s_{n+2} - s_{n+1}| < k^n|s_2 - s_1|$, and $\displaystyle\sum_{r=m+1}^{n} k^r$ is something we know the exact value of...]

Since $|a_{n+2} - a_{n+1}| < k|a_{n+1} - a_n|$ for every n, we can show by induction that $|a_{n+2} - a_{n+1}| < k^n|a_2 - a_1|$ for every $n \geq 1$. The base case $n = 1$ is $|a_3 - a_2| < k^n|a_2 - a_1|$, which is true by our hypothesis. If we suppose that $|a_{n+2} - a_{n+1}| < k^n|a_2 - a_1|$, then

$|a_{(n+1)+2} - a_{(n+1)+1}| < k|a_{n+2} - a_{n+1}| < k(k^n|a_2 - a_1|) = k^{n+1}|a_2 - a_1|$

which establishes the inductive step, so the result is true by induction.

Then, again by induction, for any $m < n$ we have $|a_n - a_m| < \displaystyle\sum_{i=m-1}^{n-2} k^i|a_2 - a_1|$, since

$|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_m)| \leq |a_n - a_{n-1}| + |a_{n-1} - a_m| < k^{n-2}|a_2 - a_1| + \displaystyle\sum_{i=m-1}^{n-3} k^i|a_2 - a_1| = \sum_{i=m-1}^{n-2} k^i|a_2 - a_1|$

where the middle inequality follows from our inductive hypothesis. But! $\sum_{i=m-1}^{n-2} k^i = k^{m-1} \sum_{m=0}^{n-m-1} k^i = k^{m-1} \frac{1-k^{n-m}}{1-k}$ (by induction!), so we have

$|a_n - a_m| < k^{m-1} \frac{1-k^{n-m}}{1-k} |a_2 - a_1| < \frac{k^{m-1}}{1-k} |a_2 - a_1|$

But! Since $0 < k < 1$, we know that $k^r \to 0$ as $r \to \infty$, and so for every $\epsilon > 0$ we can choose an N so that $m \geq N$ implies that $k^m < \frac{\epsilon(1-k)}{|a_2 - a_1|}$, so then $n, m \geq N$ implies that (WOLOG $N \leq m < n$ and)

$|a_n - a_m| < \frac{k^{m-1}}{1-k} |a_2 - a_1| < \frac{k^m}{1-k} |a_2 - a_1| < \frac{|a_2-a_1|}{1-k} \frac{\epsilon(1-k)}{|a_2-a_1|} = \epsilon.$

So the sequence a_n is Cauchy, and therefore convergent!

22. [Lay, p.189, problem # 19.8, sort of] If b_n is a subsequence of the sequence a_n (so $b_n = a_{g(n)}$ for some strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$) <u>and</u> a_n is a subsequence of the sequence b_n, show by example that we <u>need</u> <u>not</u> have $a_n = b_n$ for every n.

If b_n is a subsequence of a_n and a_n is a subsequence of b_n, it need not be the case that $a_n = b_n$ for every n. We can build examples in many ways; here is one. Let $a_n = (-1)^n$, so $a_n = 1$ for n even and $a_n = -1$ for n odd. Now let b_n be the sequence which alternates <u>two</u> 1s followed by <u>two</u> -1s, alternating forever. Writing an explicit formula for this sequence is a challenge: the floor function can do it, as $b_n = (-1)^{\lfloor (n-1)/2 \rfloor}$. a_n is a subsequence of b_n, in fact $a_n = b_{2n}$ (the second occurance of each pair of 1s and -1s). But b_n is also a subsequence of a_n, although again explicitly expressing it as one is a challenge! The idea is that we skip every <u>third</u> term in the sequence a_n, starting with the second term, so that we get

1, (not -1), 1, -1, (not 1), -1, 1, (not -1), 1, -1, etc.

One way to write it is as $b_{2n+1} = a_{3n+1}$ and $b_{2n} = a_{3n}$, so $b_n = a_{\lfloor 3n/2 \rfloor}$.

Actually, after having discussed this with some of you, it seesm that we need not get so inventive. If $a_n = (-1)^n$, then $b_n = a_{n+1} = (-1)^{n+1}$ is a subsequence of a_n, and then $c_b = b_{n+1} = (-1)^{n+2} = (-1)^n = a_n$ is a subseuence of b_n ! But $b_n = -a_n \neq a_n$ for <u>every</u> n !

Here is a question to think about; can we do this with a <u>convergent</u> sequence? The answer is no; if we could do something like the above, then they, being subsequences of one another, they must converge to the same limit L, and cannot not be constant. But then, being subsequences of one another, for any $\epsilon > 0$ the <u>last</u> n, m with $|a_n - L| \geq \epsilon$ and $|b_m - L| \geq \epsilon$ (which for small ϵ must occur later and later) must, since $n_k \geq k$, occur <u>earlier</u> in b_n than in a_n (thinking of b_n as a subsequence of a_n) <u>and</u> <u>earlier</u> in a_n than in b_n (thinking of a_n as a subsequence of b_n). So they must occur with the <u>same</u> (ever larger) index n each time, meaning that when we write $a_k = b_{n_k}$ we must have $n_k = k$ for arbitrarily large values of k (and therefore for every <u>smaller</u> index, since once $n_k > k$

the gap between the indices will always be at least as big ever after). So the terms of the sequences must be equal to one another all the way up. So for convergent sequences being a subequence of one another implies that the two sequences are identical!

23. [Lay, p.189, problem # 19.13] If a_n and b_n are both bounded sequences, show that $a_n + b_n$ is also bounded. Then show that

$$\limsup_{n\to\infty}(a_n + b_n) \ \le\ \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \ .$$

<u>Then</u> find examples of sequences for which this inequality is *strict* (i.e., equality does not hold). [Hint: For the middle part, one of your previous problems will help... For the last part, at least one of your sequences (both of them?) must <u>not</u> be convergent....]

If $A_n = \sup\{a_k : k \ge n\}$, $B_n = \sup\{b_k : k \ge n\}$, and $C_n = \sup\{a_k + b_k : k \ge n\}$, then

$\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} C_n$, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} A_n$, and $\limsup_{n\to\infty} b_n = \lim_{n\to\infty} B_n$.

But $a_k \le A_n$ and $b_k \le B_n$ for every $k \ge n$ so $a_k + b_k \le A_n + B_n$ for every $k \ge n$, so $A_n + B_n$ is an upper bound for the set that defines C_n (as its supremum), so we have $C_n \le A_n + B_n$ for every n, implying that $= \lim_{n\to\infty} C_n \le \lim_{n\to\infty}(A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n$, which is precisely what the statement we are asked to prove says.

As an example where this inequality is strict, we can take $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$; then $A_n = \sup\{-1, 1\} = 1$ and $B_n = \sup\{-1, 1\} = 1$ for every n, so $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$. But $a_n + b_n = (-1)^n + (-1)^{n+1} = (-1)^n - (-1)^n = 0$ for every n, so $\limsup_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n = 0 < 1 + 1 = 2$, so the inequality established above is strict, for this pair of sequences. Many other examples can be constructed in a similar fashion; the basic idea is that where one sequence peaks need not match where the other one does, so the peak of their sum can be less than the sum of their peaks. [For example, $\sin x$ and $\cos 1$ repeatedly peak at 1, but their sum, $\sin x + \cos x$ has a maximum of $\sqrt{2}$ (as can be seen by using angle sum and difference formulas; it is $\sqrt{2}\sin(x + \pi/4)$.]