Math 871 Problem Set 1 Solutions

1. [Munkres, p.14, #2 (part)] For each statement below, determine whether or not it is true. If true, show why; if not, give an example demonstrating this.

(*) (a) For any sets A, B, $A \setminus (A \setminus B) = B$.

Since $A \setminus$ (anything) is contained in A, this statement is false whenever B is not contained in A. For example, if $A = \{1, 2\}$ and $B = \{3\}$, then $A \setminus B = A$, and so $A \setminus (A \setminus B) = A \setminus A = \emptyset$, not B.

(*) (f) For any sets A, B, C, D, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

This is true; if $(a, b) \in (A \times B) \cap (C \times D)$, then $(a, b) \in A \times B$ and $(a, b) \in C \times D$. So $a \in A$, $b \in B$, $a \in C$, and $b \in D$, and so $a \in A \cap C$ and $b \in B \cap D$. Therefore $(a, b) \in (A \cap C) \times (B \cap D)$, and so $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. For the opposite inclusion, if $(a, b) \in (A \cap C) \times (B \cap D)$, then $a \in A \cap C$ and $b \in B \cap D$. So $a \in A$, $a \in C$, $b \in B$, $b \in D$, and so $(a, b) \in A \times B$ and $(a, b) \in B \times D$, and so $(a, b) \in (A \times B) \cap (C \times D)$, as desired. With both inclusions established, we know that the two sets are equal.

4. [Munkres, p.20, #2 (part)] If $f : A \to B$ is a function, $A_0, A_1 \subseteq A$, and $B_0, B_1 \subseteq B$, then

(*) (a) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

$$\begin{aligned} f^{-1}(B_0 \cap B_1) &= \{x \in A \ : \ f(x) \in B_0 \cap B_1\} \\ &= \{x \in A \ : \ f(x) \in B_0 \text{ and } f(x) \in B_1\} \\ &= \{x \in A \ : \ f(x) \in B_0\} \cap \{x \in A \ : \ f(x) \in B_1\} \\ &= f^{-1}(B_0) \cap f^{-1}(B_1) \ . \end{aligned}$$

Or: If $x \in f^{-1}(B_0 \cap B_1)$, then $f(x) \in B_0 \cap B_1$, so $f(x) \in B_0$ and $f(x) \in B_1$, so $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$, and so $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. This shows that $f^{-1}(B_0 \cap B_1) \subseteq f^{-1}(B_0) \cap f^{-1}(B_1)$.

For the reverse inclusion, if $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$, then $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$, so $f(x) \in B_0$ and $f(x) \in B_1$. This implies that $f(x) \in B_0 \cap B_1$ and so $x \in f^{-1}(B_0 \cap B_1)$. This gives $f^{-1}(B_0) \cap f^{-1}(B_1) \subseteq f^{-1}(B_0 \cap B_1)$. Taken together the two inclusions give $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

(*) (c) $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$, but equality does not always hold.

If $x \in f(A_0 \cap A_1)$, then $x = f(a)$ for some $a \in A_0 \cap A_1$, and so $a \in A_0$ and $a \in A_1$. Then $f(a) \in f(A_0)$ and $f(a) \in f(A_1)$, and so $x = f(a) \in f(A_0) \cap f(A_1)$. So everything that is in $f(A_0 \cap A_1)$ is also in $f(A_0) \cap f(A_1)$, giving containment.

In general, these sets are not equal; in the most extreme case, we may have $A_0 \cap A_1 = \emptyset$, and then $f(A_0 \cap A_1) = f(\emptyset) = \emptyset$, even though $f(A_0)$ and $f(A_1)$ may intersect. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 12$, then for $A_0 = [0, 1]$ and $A_1 = [2, 3]$, we have $f(A_0) = \{12\} = f(A_1)$, but $A_0 \cap A_1 = \emptyset$.

6. [Munkres, p.51, #5 (part)] For each of the following sets, determine whether or not it is countable:

(*) (c) $F = \{f : \mathbb{Z}_+ \to \mathbb{Z}_+ \ : \ \text{there is } N \in \mathbb{Z}_+ \text{ with} f(n) = 1 \text{ for all } n \geq N\}$, all eventually-1 functions.

F is the union of the sets $F_N = \{f : \mathbb{Z}_+ \to \mathbb{Z}_+ \;:\; f(n) = 1 \text{ for all } n \geq N\}$, for $N \in \mathbb{Z}_+$; that is, F is the union of countably many sets $\{F_N\}$. If we show that each of the sets F_N is countable, then as a countable union, F will be countable.

But each of the sets F_N is countable: this can be established either by building a surjective function from $\mathbb{N}$ to F_N or by building an injective function from F_N to $\mathbb{N}$. In the first case, it is quicker to build a function from an N-fold cartesian product of $\mathbb{N}$'s to F_N, as $g(n_1, \ldots, n_N) = f$, where $f(k) = n_k$ if $k \leq N$ and $f(k) = 1$ for $k > N$. But then we can use the fact that such a cartesian product is countable to build a surjective function $h : \mathbb{N} \to \mathbb{N} \times \cdots \times \mathbb{N}$; the composition $g \circ h : \mathbb{N} \to F_N$ is then surjective.

For an injective function $g : F_N \to \mathbb{N}$, we can steal from number theory (again): letting $p_1, \ldots, p_N$ be a set of distinct prime numbers greater than 1, we can define $g(f) = p_1^{f(1)} \cdots p_N^{f(N)}$. Since prime factorizations of integers are unique, if $f_1, f_2 \in F_N$ have $f_1 \neq f_2$, then $f_1(n) \neq f_2(n)$ for some n, and therefore $n \leq N$ (since $f_1(n) = 1 = f_2(n)$ for $n > N$). Therefore $g(f_1)$ and $g(f_2)$ are integers with different prime factorizations, and so $g(f_1) \neq g(f_2)$.

A selection of further solutions

1. [Munkres, p.14, #2 (part)] For each statement below, determine whether or not it is true. If true, show why; if not, give an example demonstrating this.

   (d) For any sets A, B, C, D, $\{A \subseteq C$ and $B \subseteq D\}$ implies that $A \times C \subseteq B \times D$.

We know that $A \subseteq C$ and $B \subseteq D$. Suppose that $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. But since $A \subseteq C$ this means that $x \in C$, and since $B \subseteq D$ we have $y \in D$. So $(x, y) \in C \times D$. Consequently, $(x, y) \in A \times B$ implies that $(x, y) \in C \times D$, so $A \times B \subseteq C \times D$.

(e) For any sets A, B, C, D, $A \times C \subseteq B \times D$ implies that $\{A \subseteq C$ and $B \subseteq D\}$.

This is not true! Although it is almost true.

Suppose that $A \times B \subseteq C \times D$, and suppose we pick $x \in A$ and $y \in B$. Then $(x, y) \in A \times B$, and so since $A \times B \subseteq C \times D$, we have $(x, y) \in C \times D$, so $x \in C$ and $y \in D$. This appears to show that $A \subseteq C$ and $B \subseteq D$, but it doesn't!

There is a subtle difference between "$\{x \in A$ and $y \in B\}$ implies $\{x \in C$ and $y \in D\}$" and "$\{x \in A$ implies $x \in C\}$ and $\{y \in B$ implies $y \in D\}$". The first is what we've shown, the second is what we want. The difference is that if, for example, A is empty ($A = \emptyset$), then the first statement is always true; you can't pick points in A and B, or to put it differently, $\emptyset \times B = \emptyset \subseteq C \times D$, no matter what B is. So, for example, if $A = \emptyset$, $C = B = \{0, 1\}$ and $D = \{0\}$, then $A \times B = \emptyset \subseteq C \times D$, but $B = \{0, 1\} \not\subseteq \{0\} = D$.

Put still differently, if $A \neq \emptyset$, we can pick an $x \in A$. Then for any $y \in B$ we have $(x, y) \in A \times B$, so $(x, y) \in C \times D$, so $y \in D$, and so $B \subseteq D$. Similarly, if $B \neq \emptyset$, then $A \times B \subseteq C \times D$ implies that $A \subseteq C$. But without knowing that A and B are nonempty, we cannot establish our desired conclusion.

3. [Munkres, p.20, #1] Show that if $f : A \to B$ is a function, then

   (a) If $A_0 \subseteq A$, then $A_0 \subseteq f^{-1}(f(A_0))$; the sets are equal, if f is injective.

If $a \in A_0$, then $f(a) \in f(A_0)$¡ and so a is in the set $\{x \in A : f(x) \in f(A_0)\} = f^{-1}(f(A_0)$. So $A_0 \subseteq f^{-1}(f(A_0)$.

5. [Munkres, p.44, #7] If A and B are finite sets, show that the set $B^A = \{f : A \to B\}$ of all functions from A to B is also finite.

The notation B^A almost gives away the idea; the set has $|B|^{|A|}$ elements. One way to see this is to build a bijective correspondence with an $|A|$-fold Cartesian product of copies of B; induction on $|A|$ together with the finitenes of B shows that this Cartesian product is finite, so B^A is finite.

We can build an injective/surjective map to/from $\{1, \ldots, |B|^{|A|}\}$ by, essentially, writing integers in base $b = |B|$. For example, given an injective map $\varphi : B \hookrightarrow \{1, \ldots, b\} \hookrightarrow \{0, \ldots, b-1\}$ (the second map is "subtract one") and a bijection $\theta : \{1, \ldots, |A|\} \to A$,, then the map

$$\Phi : B^A \to \{1, \ldots |B|^{|A|}\}$$

given by $\Phi(f) = \Sigma_{i=1}^{|A|} \varphi(f(\theta(i)))b^{i-1}$ is an injection; any two such sums (corresponding to functions f and g) representing the same number, since $0 \leq \varphi(f(\theta(i))) \leq b-1$ for each

i, must have $\varphi(f(\theta(i))) = \varphi(g(\theta(i)))$, so $f(\theta(i)) = g(\theta(i))$ for each i. Since θ is surjective, this means that $f = g$. A surjective map $\{1, \ldots, |B|^{|A|}\} \twoheadrightarrow B^A$ can similarly be built using the i-th 'digit' of the representation of a number in base b to determine the image of the i-th element of A.

6. [Munkres, p.51, #5 (part)] For each of the following sets, determine whether or not it is countable:

(a) $A = \{f : \{0,1\} \to \mathbb{Z}\}$, all functions from $0,1$ to $\mathbb{Z}$

Since a function is determined its values - its graph is all pairs $(a, f(a)$ - the function is completely determine by the set of pairs $\{(0, f(0)), (1, f(1))\}$, which in turn can be recovered fromt he ordered pair $(f(0), f(1))$. This means, really, that we can build a surjective map $\mathbb{Z} \times \mathbb{Z} \twoheadrightarrow A$ by send the pair (a, b) to the function f with $f(0) = 1$ and $f(1) = b$. But $\mathbb{Z} \times \mathbb{Z}$ is countable; the map $\mathbb{Z}_+ \times \{-1, 0, 1\} \to \mathbb{Z}$ give by $(n, s) \mapsto sn$ is onto. But $\mathbb{Z}_+$ and $\{-1, 0, 1\}$ are countable, so $\mathbb{Z}_+ \times \{-1, 0, 1\}$ is countable, so there is a surjection $\mathbb{Z}_+ \twoheadrightarrow \mathbb{Z}_+ \times \{-1, 0, 1\}$ which via composition gives a surjection $\mathbb{Z}_+ \twoheadrightarrow \mathbb{Z}$, so $\mathbb{Z}$ is countable. Then $\mathbb{Z} \times \mathbb{Z}$ is countable, meaning there is a surjection from $\mathbb{Z}_+$ to $\mathbb{Z} \times \mathbb{Z}$; composing with the surjection above gives a surjection from $\mathbb{Z}_+$ to A, so A is countable.

(e) $P = \{f : \mathbb{Z}_+ \to \mathbb{Z}_+ \ : \ n > m \text{ implies} f(n) > f(m)\}$, all increasing functions.

This set is <u>not</u> countable; we can show this, since we know, for example, that the set $2^{\mathbb{Z}_+} = \{f : \mathbb{Z}_+ \to \{0,1\}$ is not countable, by building an injective map $2^{\mathbb{Z}_+} \hookrightarrow P$. [If P were countable, composing this injection with an injection from P to $\mathbb{Z}_+$ would show that $2^{\mathbb{Z}_+}$ is countable, a contradiction.]

We can build the desired injection in many ways; perhaps the shortest is to define $\Phi(f) = g$ where $g(n) = 10^n + f(n)$. This is an injective map; if $\Phi(f_1) = \Phi(f_2)$, then $10^n + f_1(n) = 10^n + f_2(n)$ for all n, so $f_1(n) = f_2(n)$ for all n and $f_1 = f_2$. $\Phi(f) = g$ is an increasing function, since $m > n$ implies that $g(m) = 10^m + f(m) = 10^{m-n}10^n + f(m) > 10 \cdot 10^n + f(m) > 2 \cdot 10^n + f(m) = 10^n + 10^n + f(m) \geq 10 + 10^n + f(m) > 10^n + f(n) = g(n)$, where the inequality towards the end follows from $10 + f(m) > f(n)$, since $f(n) - f(m) \leq 1$.

[From the proof, it would seem that $g(n) = 2^n + f(n)$ would actually suffice...]

(f) The set F of all finite subsets of $\mathbb{N}$.

This <u>is</u> countable; it is the union, over all integers $n \geq 0$, of the set F_n of all subsets of $\mathbb{N}$ of size <u>at</u> <u>most</u> n. Treat size 0 differently, for $n \geq 1$ the function $f_n : \mathbb{N}^n \to F_n$ which sends $(k_1, \ldots k_n)$ to $\{k_1, \ldots, k_n\}$ is surjective, so each F_n is countable. So their (countable) union is countable, as well.