Math 856 Problem Set 1

Starred (*) problems to be handed in Friday, September 18

(*) 1. You've probably heard that a connected, locally path-connected space X is path connected; the set of points reachable from $x \in X$ by a path is open (and its complement is also open). So a connected manifold is path connected.

Show, further, that a connected manifold M is *arcwise connected*, that is, for every pair of points $x, y \in M$ there is a *one-to-one* path $\gamma : [0,1] \to M$ with $\gamma(0) = x, \gamma(1) = y$.

(There is a theorem, due to Hahn and Mazurkiewicz (circa 1914), which says that a Hausdorff space is path connected iff it is arcwise connected (which is kind of funny, since the term "Hausdorff" wasn't really introduced until the 1920s?); but for manifolds you can give a much more elementary proof...)

2. Show that every topological n-manifold has a countable basis consisting of open sets homeomorphic to $\mathbb{R}^n$. [Hint: start with any old countable basis....]

3. (Lee, p. 28, problem 1-4) If $0 \le k \le \min\{m, n\}$, show that the set $R_k \subseteq M(m \times n, \mathbb{R})$ of m-by-n matrices with rank $\ge k$ is an open subset of $M(m \times n, \mathbb{R}) \cong \mathbb{R}^{mn}$ (and therefore admits a smooth structure). (*Hint:* look at Lee's linear algebra appendix...)

Note: This implies that the space $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices is a smooth manifold, of dimension n^2.

(*) 4. We will call two C^∞ atlases $\mathcal{A}$ and $\mathcal{B}$ for a manifold M *equivalent* if their union $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$ is also a C^∞ atlas for M. Show that equivalence is an equivalence relation!

5. We say that two charts $\phi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$, $U, V \subseteq M^n$ are $\underline{C^\infty\text{-related}}$ if $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ and $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ are both C^∞. Show that the relation " is C^∞-related to " is **not** an equivalence relation. (Hint: $M^n = \mathbb{R}$ will suffice for an example...)

6. Show that $\mathbb{R}$ has uncountably many distinct smooth structures. [(Perhaps) show first that it is enough to find uncountably many charts, with intersecting domains and ranges, no two of which are C^∞-related to one another.]

(*) 7. If M and N are smooth manifolds, show that $M \times N$ and $N \times M$, with the (two) product smooth structures $\{(U_\alpha \times V_\beta, h_\alpha \times k_\beta)\}$ are diffeomorphic. [I.e., exhibit (and verify) a diffeomorphism!]

8: Show that a function $f : M^n \to N^m$ is $C^\infty \Leftrightarrow g \circ f : M^n \to \mathbb{R}$ is C^∞ for *every* C^∞ function $g : N^m \to \mathbb{R}$.