Math 325 Problem Set 9

Starred (*) problems are due Friday, November 2.

(*) 53. (Belding and Mitchell, p.100, #11) Suppose that f is differentiable on $[a, b]$ with $f'(a) > 0$ and $f'(b) < 0$. Show that:

(a) Neither $f(a)$ nor $f(b)$ is a maximum value for f on $[a, b]$; that is, there is a $c \in (a, b)$ so that $f(a) < f(c)$ and $f(b) < f(c)$. [Hint: a previous problem will help...]

By Problem #47, since $f'(a) > 0$ there is a $\delta > 0$ so that $x \in (a, a + \delta)$ implies that $f(x) > f(a)$. In particular, $f(a + \delta/2) > f(a)$, so $f(a)$ is not a maximum value for f on $[a, b]$.

Also, setting $g(x) = -f(x)$, g is differentiable on $[a, b]$ and $g'(x) = -f'(x)$, so $g'(b) = -f'(b) > 0$, and so (again by Problem #47) there is a $\delta > 0$ so that $x \in (b - \delta, b)$ implies that $g(x) < g(b)$, and so $-f(x) < -f(b)$, so $f(x) > f(b)$. In particular, $f(b - \delta/2) > f(b)$, so $f(b)$ is not a maximum value for f on $[a, b]$.

Therefore, neither $f(a)$ nor $f(b)$ is a maximum value for f on $[a, b]$.

(b) Use this and Rolle's Theorem to show that there is a point $c \in (a, b)$ where $f'(c) = 0$.

By the Extreme Value Theorem, f achieves its maximum value on $[a, b]$, that is, there is a $c \in [a, b]$ so that $f(c) \geq f(x)$ for every $x \in [a, b]$. By part (a), c cannot equal a or b, so $c \in (a, b)$.

Applying Problem #47 again, it is not possible for $f'(c) > 0$, since then (as above!) there is a $\delta > 0$ so that $f(c+\delta/2) > f(c)$, contradicting the choice of c. But it is also not possible for $f'(c) < 0$, since then (as above) there is a $\delta > 0$ so that $f(c-\delta/2) > f(c)$, contradicting the choice of c. so $f'(c) = 0$.

Therefore, f is differentiable on $[a, b]$ and $f'(a) > 0$ and $f'(b) < 0$ implies that there is a $c \in (a, b)$ with $f'(c) = 0$.

(*) 55. Use Rolle's Theorem to show, by induction, that a polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

of degree n has <u>at</u> <u>most</u> n distinct roots (i.e., solutions to $p(x) = 0$).

[Hint: If p has degree n, then p' has degree $n - 1$ ...]

Working by induction, for $n = 1$ we have $p(x) = ax + b = 0$ only for $x = -b/a$; so there is one root if $a \neq 0$ and <u>no</u> root if $a = 0$. (But if $a = 0$ the polynomial actually has degree 0, not 1...)

Now suppose that we know that every polynomial of degree $n - 1$ has at most $n - 1$ distinct roots. Suppose that $p(x)$ is a polynomial of degree n; we want to show that it has at most n distinct roots. Well, suppose it <u>doesn't</u>! Suppose that $p(x)$ has

$n+1$ distinct roots, $x_1 < x_2 < \cdots < x_{n+1}$. Then for every $i = 1, \ldots n$ we have $p(x_i) = p(x_{i+1})$. But then p is continuous on $[x_i, x_{i+1}]$ (it is a polynomial), and p is differentiable on (x_i, x_{i+1}) (it is a polynomial!), so we can apply Rolle's Theorem. This tells us that there is a $c_i \in (x_i, x_{i+1})$ with $p'(c_i) = 0$.

But! since $x_i < c_i < x_{i+1}$, all of the c_i are distinct! This is becaus if $i \neq j$, then WOLOG $i < j$, so $i+1 \leq j$, so $x_{i+1} \leq x_j$, so $x_i < c_i < x_{i+1} \leq x_j < c_j$, so $c_i < c_j$. This means that $p'(x)$, which has degree $n-1$, has roots the n distinct numbers c_i, $i = 1, \ldots n$ (and possibly more!). But this contradicts our inductive hypothesis. So it is impossible for p to have $n+1$ distinct roots, so p has at most n distinct roots.

This completes our inductive step; so every polynomial of degree n has at most n distinct roots.

N.B.: You have probably seen this result before, proved in a different way: if p has no (real) roots, then we are done. But if c is a root of the degree-n polynomial $p(x)$ then $p(x) = (x-c)q(x)$ for some polynomial q having degree $n-1$. Then q (by an inductive argument) has at most $n-1$ roots, and $p(r) = (r-c)q(r) = 0$ only if either $r - c = 0$ (so $r = c$) or $q(r) = 0$ (so r is a root of q). So the roots of p are the roots of q (at most $n-1$ of them) plus c, so p has at most n roots.

(*) 57. Suppose that $f, g : [0,1] \to \mathbb{R}$ are both continuous on $[0,1]$, differentiable on $(0,1)$, $f(0) = g(0)$, and $f'(x) > g'(x)$ for every $x \in (0,1)$. Show that $f(x) > g(x)$ for all $x \in (0,1]$.

Suppose, by way of contradiction, that $a \in (0,1]$ and $f(a) \leq g(a)$. Then, setting $h(x) = f(x) - g(x)$, we have that h is continuous on the interval $[0,a]$, h is differentiable on $(0,a)$, $h(0) = 0$ and $h(a) = f(a) - g(a) \leq 0$. Therfore, by the Mean Value Theorem, there is a $c \in (0,a)$ so that $h'(c) = \dfrac{h(a) - h(0)}{a - 0} \leq 0$. But since $c \in (0,a) \subseteq (0,1)$, we have $c \in (0,1)$ and so $0 \geq h'(c) = f'(c) - g'(c) > 0$, and so $0 > 0$, a contradication. Therefore, every $a \in (0,1)$ has $f(a) > g(a)$, as desired.

A selection of further solutions.

54. (Belding and Mitchell, p.100, #12) Prove the **Intermediate Value Theorem for Derivatives**: If f is differentiable on $[a,b]$ and $f'(a) < k < f\prime(b)$, then there is a $c \in (a,b)$ with $f'(c) = k$.

[Hint: Consider the 'auxiliary' function $h(x) = kx - f(x)$ and apply the results of the preceding problem. Note that $f'(x)$ need not be continuous (although examples of this are tough to construct!), so we cannot 'just' apply IVT...!]

Following the hint, if we set $h(x) = kx - f(x)$, then h is differentiable on $[a,b]$ and $h'(x) = k - f'(x)$. Then $h'(a) = k - f'(a) > 0$, and $h'(b) = k - f'(b) < 0$, and so by Problem #53(b) there is a $c \in (a,b)$ so that $h'(c) = k - f'(c) = 0$. So $f'(c) = k$, as desired.