Math 423/823 Exercise Set 3 Solutions

9. [BC#2.12.3] If $z = x + yi$ and $f(z) = (x^2 - y^2 - 2y) + (2x - 2xy)i$, use the formulas

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}$$

to write $f(z)$ in terms of z (and $\overline{z}$) and simplify the result.

$$f(z) = \left(\left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2 - 2\left(\frac{z-\overline{z}}{2i}\right)\right) + \left(2\left(\frac{z+\overline{z}}{2}\right) - 2\left(\frac{z+\overline{z}}{2}\right)\left(\frac{z-\overline{z}}{2i}\right)\right)i$$
$$= \left(\left(\frac{z+\overline{z}}{2}\right)^2 - (-i)^2\left(\frac{z-\overline{z}}{2}\right)^2 + 2i\left(\frac{z-\overline{z}}{2}\right)\right) + \left(2\left(\frac{z+\overline{z}}{2}\right) + 2i\left(\frac{z+\overline{z}}{2}\right)\left(\frac{z-\overline{z}}{2}\right)\right)i$$
$$= \left(\frac{z+\overline{z}}{2}\right)^2 + \left(\frac{z-\overline{z}}{2}\right)^2 + i(z-\overline{z})) + (z+\overline{z})i - (z+\overline{z})\left(\frac{z-\overline{z}}{2}\right)$$
$$= \frac{1}{4}(z^2 + 2z\overline{z} + \overline{z}^2 + z^2 - 2z\overline{z} + \overline{z}^2 - 2z^2 + 2\overline{z}^2) + 2zi$$
$$= \frac{1}{4}(4\overline{z}^2) + 2zi = \overline{z}^2 + 2zi$$

10. [BC#2.14.3] Sketch the regions onto which the sector

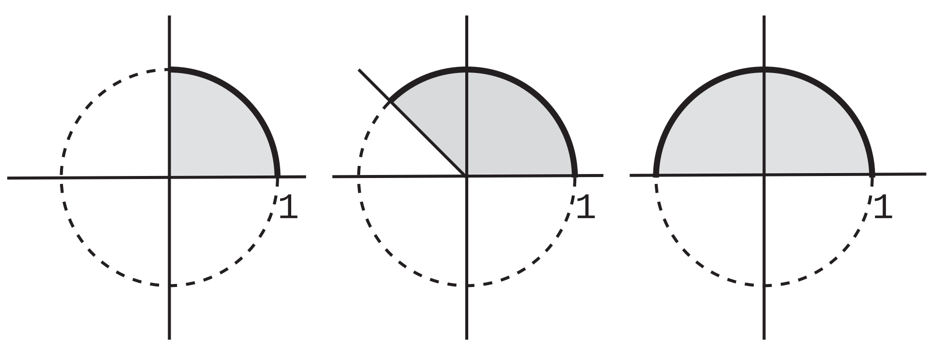
$$A = \{z = re^{i\theta} : 0 \le r \le 1, 0 \le \theta \le \pi/4\}$$

is mapped by the functions

(a) $w = z^2$ (b) $w = z^3$ (c) $w = z^4$

Writing the maps in exponential notation, we have
(a) $w = (re^{i\theta})^2 = r^2e^{i2\theta}$ (b) $w = (re^{i\theta})^3 = r^3e^{i3\theta}$ (c) $w = (re^{i\theta})^4 = r^4e^{i4\theta}$

So the complex numbers with $0 \le r \le 1$ will, in all cases, be carried to complex numbers with modulus between 0 and 1 (inclusive), and the complex numbers with arguments between 0 and $\pi/4$ (inclusive) will be carried to the complex numbers with arguments between 0 and (a) $\pi/2$, (b) $3\pi/4$, and (c) π (inclusive), respectively. So our region A will be mapped to the points in the unit disk whose arguments lie in the ranges given above. These regions are sketched below.



11. Show that the reciprocal function, $f(z) = 1/z$, maps the disk $D = \{z : |z-1| < 2 \text{ and } z \neq 0\}$ onto the region that lies <u>outside</u> of the circle $\{w : |w + 1/3| = 2/3\}$.

$$|1/z + 1/3| > 2/3 \Leftrightarrow |\frac{1}{x+yi} + \frac{1}{3}|^2 > \frac{4}{9}$$
$$\Leftrightarrow |\frac{x+yi}{x^2+y^2} + \frac{1}{3}|^2 > \frac{4}{9}$$
$$\Leftrightarrow |(\frac{x}{x^2+y^2} + \frac{1}{3}) + i\frac{y}{x^2+y^2}|^2 > \frac{4}{9} \qquad \Leftrightarrow (\frac{x}{x^2+y^2} + \frac{1}{3})^2 + (\frac{y}{x^2+y^2})^2 > \frac{4}{9}$$
$$\Leftrightarrow \frac{x^2}{(x^2+y^2)^2} + \frac{2}{3}\frac{x}{x^2+y^2} + \frac{1}{9} + \frac{y^2}{(x^2+y^2)^2} > \frac{4}{9}$$
$$\Leftrightarrow \frac{x^2+y^2}{(x^2+y^2)^2} + \frac{2}{3}\frac{x}{x^2+y^2} + \frac{1}{9} > \frac{4}{9} \qquad \Leftrightarrow \frac{1}{x^2+y^2} + \frac{2}{3}\frac{x}{x^2+y^2} + \frac{1}{9} > \frac{4}{9}$$
$$\Leftrightarrow 9 + 6x + (x^2+y^2) > 4(x^2+y^2) \text{ [muliplying both sides by } 9(x^2+y^2)]$$
$$\Leftrightarrow 3(x^2+y^2) - 6x < 9 \qquad \Leftrightarrow x^2+y^2-2x < 3 \qquad \Leftrightarrow x^2 - 2x + 1 + y^2 < 4$$
$$\Leftrightarrow (x-1)^2 + y^2 < 4 \qquad \Leftrightarrow |(x+yi) - 1|^2 < 4 \qquad \Leftrightarrow |(x+yi) - 1| < 2$$
$$\Leftrightarrow |z-1| < 2$$

Then, reading bottom to top, every point in $D = \{z : |z-1| < 2 \text{ and } z \neq 0\}$ lands in $\{w : |w+1/3| > 2/3\}$ (notes that $z \neq 0$ is used, because at one point we divide by x^2+y^2), and reading top to bottom every point in $\{w : |w+1/3| > 2/3\}$ <u>is</u> $1/z$ for some point in D. So D is mapped (into and) onto $\{w : |w+1/3| > 2/3\}$ under f.

Alternatively, $|1/z + 1/3| > 2/3 \Leftrightarrow |\frac{3+z}{3z}| > \frac{2}{3} \Leftrightarrow |3+z| > 2|z|$ (this uses $|z| \neq 0$)
$\Leftrightarrow |3+z|^2 > 4|z|^2 \Leftrightarrow (x+3)^2 + y^2 > 4(x^2+y^2)$
and the argument can be finished as above.

12. Find $\displaystyle\lim_{z \to 1+i} \frac{z^2+z-1-3i}{z^2-2z+2}$.

Plugging in $z = 1+i$ yields $0/0$, which implies that $z-(1+i)$ evenly divides both top and bottom. Since $z^2+z-1-3i = (z-r)(z-(1+i))$, we must have $-1-3i = r(1+i)$, so $r(1+i)(1-i) = 2r = (-1-3i)(1-i) = -1-3i+i-3 = -4-2i$, and so $r = -2-i$. And since $1+i$ is a root of z^2-2z+2, the other root is its conjugate, $1-i$. So:
$$\lim_{z\to 1+i} \frac{z^2+z-1+-3i}{z^2-2z+2} = \lim_{z\to 1+i} \frac{(z-(1+i))(z-(-2-i))}{(z-(1+i))(z-(1-i))} = \lim_{z\to 1+i} \frac{z-(-2-i)}{z-(1-i)} =$$
$$\frac{(1+i)-(-2-i)}{((1+i)-(1-i)} = \frac{3+2i}{2i} = \frac{-3i+2}{2} = \frac{2-3i}{2} = 1 - \frac{3}{2}i$$