Math 971 Algebraic Topology

April 12, 2005

There is another piece of homological algebra that we will find useful ; the Five Lemma. It allows us to <u>compare</u> the information contained in two long exact sequences.

Five Lemma: If we have abelian groups and maps

$$\begin{array}{ccccccccc} A_n & \stackrel{f_n}{\to} & B_n & \stackrel{g_n}{\to} & C_n & \stackrel{h_n}{\to} & D_n & \stackrel{i_n}{\to} & E_n \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ A_{n-1} & \stackrel{f_{n-1}}{\to} & B_{n-1} & \stackrel{g_{n-1}}{\to} & C_{n-1} & \stackrel{h_{n-1}}{\to} & D_{n-1} & \stackrel{i_{n-1}}{\to} & E_{n-1} \end{array}$$

where the rows are exact, and the maps $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then γ is an isomorphism.

The proof is in some sense literally a matter of doing the only thing you can. To show injectivity, suppose $x \in C_n$ and $\gamma x = 0$, then $h_{n-1}\gamma x = \delta h_n x = 0$, so, since δ is injective, $h_n x = 0$. So by the exactness at C_n, $x = g_n y$ for some $y \in B_n$. Then $g_{n-1}\beta y = \gamma g_n y = \gamma x = 0$, so by exactness at B_{n-1}, $\beta y = f_{n-1} z$ for some $z \in A_{n-1}$. Then since α is surjective, $f_{n-1} z = \alpha w$ for some w. Then $0 = g_n f_n w$. But $\beta f_n w = f_{n-1}\alpha w$ $f_{n-1} z = \beta y$, so since β is injective, $y = f_n w$. So $0 = g_n f_n w = g_n y = x$. So $x = 0$.
For surjectivity, suppose $x \in C_{n-1}$. Then $h_{n-1} x \in D_{n-1}$, so since δ is surjective, $h_{n-1} x = \delta y$ for some $y \in D_n$. Then $\epsilon i_n y = i_{n-1}\delta y = i_{n-1} h_{n-1} x = 0$, so since ϵ is injective, $i_n y = 0$. So by exactness at D_n, $y = h_n z$ for some $z \in C_n$. Then $h_{n-1}\gamma z = \delta h_n z = \delta y = h_{n-1} x$, so $h_{n-1}(\gamma z - x) = 0$, so by exactness at C_{n-1}, $\gamma z - x = g_{n-1} w$ for some $w \in B_{n-1}$. Then since β is surjective, $w = \beta u$ for some $u \in B_n$. Then $\gamma g_n u = g_{n-1}\beta u = g_{n-1} w = \gamma z - x$, so $x = \gamma z - \gamma g_n u = \gamma(z - g_n u)$. So γ is onto.

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\mathrm{cl}_X(B) \subseteq \mathrm{int}_X(A)$, then for every k the inclusion-induced map $H_k(X \setminus B, A \setminus B) \to H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\mathrm{int}_X(A) \cup \mathrm{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \to H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$.] To prove the second statement, we know that the inclusion $C_n^{\{A,B\}}(X) \to C_n(X)$ induce isomorphisms on homology, as does $C_n(A) \to C_n(A)$, so, by the five lemma, the induced map $C_n^{\{A,B\}}(X)/C_n(A) \to C_n(X)/C_n(A) = C_n(X, A)$ induces isomorphisms on homology. But the inclusion $C_n(B) \to C_n^{\{A,B\}}(X)$ induces a map $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \to C_n^{\{A,B\}}(X)/C_n(A)$ which is an isomorphism of chain groups; a basis for $C_n^{\{A,B\}}(X)/C_n(A)$ consists of singular simplices which map into A or B, but don't map into A, i.e., of simplices mapping into B but not A, i.e., of simplices mapping into B but not $A \cap B$. But this is the <u>same</u> as the basis for $C_n(B, A \cap B)$!

With these tools, we can start making some <u>real</u> homology computations. First, we show that if $\emptyset \neq A \subseteq X$ is "nice enough", then $H_n(X, A) \cong \widetilde{H}_n(X/A)$. The definition of nice enough, like Seifert - van Kampen, is that A is closed and has an open neighborhood $\mathcal{U}$ that deformation retracts to A (think: A is the subcomplex of a CW-complex X). Then using $\mathcal{U}.X \setminus A$ as a cover of X, and $\mathcal{U}/A, (X \setminus A)/A$ as a cover of X/A, we have

$$\widetilde{H}_n(X/A) \stackrel{(1)}{\cong} H_n(X/A, A/A) \stackrel{(2)}{\cong} H_n(X/A, \mathcal{U}/A) \stackrel{(3)}{\cong} H_n(X/A \setminus A/A, \mathcal{U}/A \setminus A/A) \stackrel{(4)}{\cong} H_n(X \setminus A, \mathcal{U} \setminus A) \stackrel{(5)}{\cong} H_n(X, A)$$

Where (1),(2) follow from the LES for a pair, (3),(5) by excision, and (4) because the restriction of the quotient map $X \to X/A$ gives a homeomorphism of pairs.

Second, if X, Y are T_1, $x \in X$ and $y \in Y$ each have neighborhoods $\mathcal{U}, \mathcal{V}$ which deformation retract to each point, then the one-point union $Z = X \vee Y = (X \coprod Y)/(x = y)$ has $\widetilde{H}_n(Z) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$; this follows from a similar sequence of isomorphisms. Setting $z =$ the image of $\{x, y\}$ in Z, we have

$$\widetilde{H}_n(Z) \cong H_n(Z, z) \cong H_n(Z, \mathcal{U} \vee \mathcal{V}) \cong H_n(Z \setminus z, \mathcal{U} \vee \mathcal{V} \setminus z) \cong H_n([X \setminus x] \coprod [Y \setminus y], [\mathcal{U} \setminus x] \coprod [\mathcal{V} \setminus y]) \cong H_n(X \setminus x, \mathcal{U} \setminus x) \oplus H_n(Y \setminus y, \mathcal{V} \setminus y) \cong H_n(X, x) \oplus H_n(Y, y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

By induction, we then have $\widetilde{H}_n(\vee_{i=1}^k X_i) \cong \oplus_{i=1}^k \widetilde{H}_n(X_i)$

We have so far introduced two homologies; simplicial, H_*^Δ, whose computation "only" required some linear algebra, and singular, H_*, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For Δ-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub-Δ-complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X,A) \to H_{n-1}^\Delta(A) \to \cdots$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^{(k)},X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)},X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)}) \end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because, essentially, we can, in each case, identify these groups. $H_n(X^{(k)},X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$ are either 0 (for $n \neq k$) or $\oplus\mathbb{Z}$ (for $n = k$), one summand for each n-simplex in X. But the same is true for $H_n^\Delta(X^{(k)},X^{(k-1)})$; and for $n = k$ the generators are precisely the n-simplices of X. The inclusion-induced map takes generators to generators, so is an isomorphism. So by the Five Lemma, the middle rows are also isomorphisms, completing our inductive proof.

Returning to $H_n^\Delta(X) \xrightarrow{I_*} H_n(X)$, we wish now to show that this map is an isomorphism. Any $[z] \in H_n(X)$ is represented by a cycle $z = \sum a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$. But each $\sigma_i(\Delta^n)$ is a compact subset of X, and so meets only finitely-many cells of X. This is true for every singular simplex, and so there is a k for which all of the simplices map into $X^{(k)}$, and so we may treat $z \in C_n(X^{(k)}$. Thought of in this way, it is still a cycle, and so $[z] \in H_n(X^{(k)}) \cong H_n^\Delta(X^{(k)})$ so there is a $z' in C_n^\Delta(X^{(k)})$ and a $w \in C_{n+1}(X^{(k)})$ with $i_\# z' - z = \partial w$. But thinking of $z' in C_n^\Delta(X)$ and $w \in C_{n+1}(X)$, we have the same equality, so $[z'] \in H_n^\Delta(X)$ and $i_*[z'] = [z]$. So i_* is surjective. If $i_*([z]) = 0$, then the cycle $z = \sum a_i\sigma_i$ is a sum of characteristic maps of n-simplices of X, and so can be thought of as an element of $C_n^\Delta(X^{n)})$. Being 0 in $H_n(X)$, $z = \partial w$ for some $w \in C_{n+1}(X)$. But as before, $w \in C_n(X^{r)})$ for some r, and so thought of as an element of the image of the isomorphism $i_* : H_n^\Delta(X^{(r)}) \to H_n(X^{(r)})$, $i_*([z]) = 0$, so $[z] = 0$. So $z = \partial u$ for some $u \in C_{n+1}^\Delta(X^r) \subseteq C_{n+1}^\Delta(X)$. So $[z] = 0$ in $H_n^\Delta(X)$. Consequently, simplicial and singular homology groups are isomorphic.

Some topological results with homological proofs: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h, then $n = m$. This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n-1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n-1 = m-1$.

More generally, we can establish a result which is known as *invariance of domain*, a result which is useful in both topology and analysis.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

Note it is enough to proof this for our favorite open set, which in this context will be $\mathcal{V} = (-1,1)^n \subseteq \mathbb{R}^n$, since given any open $\mathcal{U}$ and $x \in \mathcal{U}$, we can find an injective linear map $h : (-1,1)^n \to \mathcal{U}$ taking 0 to x.

If we can show that $f \circ h$ has open image, then $f(x) \in f \circ h(\mathcal{V}) \subseteq f(\mathcal{U})$ shows that $f(x)$ has an open neighborhood in $f(\mathcal{U})$. Since x is arbitrary, $f(\mathcal{U})$ is open.

This in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.

Our next goal is to show that, when both make sense, simplicial and singular homology are isomorphic. In fact, the inclusion of the simplicial chain groups into the singular ones induces an isomorphism on homology.