Math 107 Project: Balancing on the point of a pin

Due date: Thursday, December 5

This project explores the mathematics behind the *center of mass* (or *center of gravity*) of an object.

In many physical situations, an object behaves as if all of its mass were concentrated at a single point, called the center of mass (COM) of the object. For example, an object allowed to rotate freely will rotate around a line through its center of mass; an object thrown through the air, in the absence of air resistance, will have its center of mass trace out the perfect parabolic arc that physics predicts. See, for example,

http://www.schooltube.com/video/ef4699826e6448bf9703/Elmo-Center-of-Mass

for experiments carried out with an Elmo doll! In this project we will focus on center of mass computations for an object modeled as a thin plate of uniform density shaped like a region R in the plane; under these hypotheses, the COM is usually called the *centroid* of the region R. For a region R having a line of symmetry, the centroid will always lie along the line, a fact which can greatly simplify calculations of centroids. Knowlege of the centroid of an region, in turn, can greatly simplify other calculations; the Thoerem of Pappus states that when a region R of the plane is rotated in space around a line not meeting R, the volume of the resulting solid of revolution is equal to the area of R times the distance traveled by the centroid R under the rotation. Our goal is to verify some of these observations and carry out a variety of computations.

Some basic material on centers of mass can be found in section 8.4 of our text, pages 415-423, which makes a good starting point for your studies. To summarize, a collection of masses m_i distributed at the points x_i along the real line will "balance" at the point $\overline{x}$ where

$$\overline{x}\sum m_i = \sum x_i m_i, \text{ or, in a different form, } \sum m_i(\overline{x} - x_i) = 0$$

This is essentially the principle of the lever; a small mass far from the balance point can balance a larger mass close to the balance point but on the other side. $\overline{x}$ is the center of mass of the collection of masses. More generally, a finite collection of masses m_i distributed at points (x_i, y_i) in the plane has center of mass $(\overline{x}, \overline{y})$, where

$$\sum m_i(\overline{x} - x_i) = 0 \text{ and } \sum m_i(\overline{y} - y_i) = 0$$

$\overline{x} - x_i$ represents the "signed" distance from the point (x_i, y_i) to the line $x = \overline{x}$; the condition $\sum m_i(\overline{x} - x_i) = 0$ ensures that the masses, if placed on a massless plate supported along the line $x = \overline{x}$, will balance. The other condition ensures that the masses balance when supported along the horizontal line $y = \overline{y}$.

Even more is true: for $a, b \in \mathbb{R}$, if we set $c = a\overline{x} + b\overline{y}$, then the distance from the mass m_i sitting at the point (x_i, y_i) to the line $ax + by = c$ is equal to

$$\frac{|a(\overline{x} - x_i) + b(\overline{y} - y_i)|}{\sqrt{a^2 + b^2}}\ ,$$

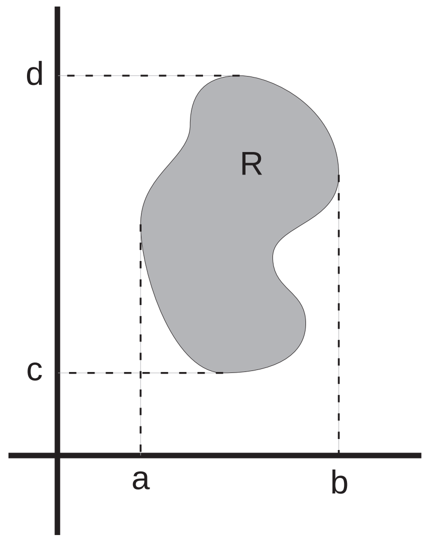
and so, if we define the 'signed' distance from (x_i, y_i) to the line $ax + by = c$ to be

$$d_{a,b}(x_i, y_i) = \frac{a(\overline{x} - x_i) + b(\overline{y} - y_i)}{\sqrt{a^2 + b^2}} \ ,$$

then $\sum m_i d_{a,b}(x_i, y_i) = 0$. So the collection of masses will also balance when supported along any line through the center of mass.

From this it follows that a collection of masses will "balance" on the point of a pin placed at the center of mass, since they will not tip in any direction.

Now assume we have a thin plate occupying a region R as shown. Also, assume the density of the plate is a constant $\rho \ kg/m^2$.



In order to find the centroid of the plate, we start by finding $\overline{x}$. We partition the interval $[a, b]$ via the regular partition $\{a = x_0, x_1, x_2, \cdots, x_n = b\}$, with $\Delta x = \frac{b-a}{n}$. This process results in dividing the plate into thin vertical strips which can be approximated as a rectangle of a small width Δx. Let $L(z_k)$ be the total length of the line segments of intersection of the vertical line $x = z_k$ with R, where $z_k \in [x_{k-1}, x_k]$ is any point of your choice. Now, we think of each vertical strip of the plate as a discrete mass in the plane whose coordinate is (z_k, w_k), for some $w_k \in \mathbb{R}$, which is irrelevant in the following calculations. Let us note that the mass of the kth vertical strip is given by: $m_k =$ (density)(area) $\approx \rho L(z_k)\Delta x$. So, by thinking of the whole plate as a discrete system of n masses $m_k \approx \rho L(z_k)\Delta x$ each located at a point (z_k, w_k) in the plane, we find

$$\overline{x} = \frac{M_y}{M} \approx \frac{\sum_{k=1}^{n} \rho z_k L(z_k)\Delta x}{\sum_{k=1}^{n} \rho L(z_k)\Delta x} = \frac{\sum_{k=1}^{n} z_k L(z_k)\Delta x}{\sum_{k=1}^{n} L(z_k)\Delta x}.$$

By letting $n \to \infty$, we obtain the formula $\quad \overline{x} = \dfrac{\int_a^b xL(x)dx}{\int_a^b L(x)dx}.$

A similar collection of steps will obtain a formula for the y-coordinate of the centroid:

$$\overline{y} = \frac{\int_c^d yS(y)dy}{\int_c^d S(y)dy},$$

where $S(w_k)$ is the total length of the line segments of intersection of the horizontal line $y = w_k$ with R.

Problem # 1: Explain why the area $A(R)$ of the region R is equal to

$$A(R) = \int_a^b L(x)\ dx = \int_c^d L(y)\ dy\ .$$

Hence we have:

$$\overline{x} = \frac{1}{A(R)} \int_a^b xL(x)dx, \quad \overline{y} = \frac{1}{A(R)} \int_c^d yS(y)dy\ .$$

Use this to explain why, if the region R has a vertical line of reflection symmetry $x = A$, then $\overline{x} = A$, and if R has a horizontal line of reflection symmetry $y = B$, then $\overline{y} = B$. [Hint: a line of symmetry tells us something about the function $L(x)$ or $L(y)$. You might, to start, assume that $A = 0$ and $B = 0$.]

By computing $L(x)$ and $L(y)$ for specific examples, together with symmetry considerations, we can compute the centroids of a wide variety of regions in the plane:

Problem # 2: Compute the centroids of

(a): the disk $D = \{(x, y)\ :\ (x + 2)^2 + y^2 \leq 1\}$;
(b): the triangle with vertices $(1, 0)$, $(5, 0)$, and $(4, 4)$;
(c): the triangle with vertices $(1, 0)$, $(4, 0)$, and $(5, 4)$;
(d): the region lying between the parabolas $y = 2x - x^2$ and $y = 2x^2 - 4x$

[You may use familiar formulas for area to streamline your computations, as well as any symmetries that you can identify.]

In your answers for 3(b) and 3(c), if you compare the coordinates of the centroids with the coordinates of the vertices of the triangles, you may begin to suspect that there is a general formula for the centroid of a triangle, in terms of the coordinates of the vertices. Your final task is to find that formula, by carrying out the integral computations for a triangle with unknown values for its vertices.

Problem # 3: Find a general formula for the centroid of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3).

To do this, we may assume that the x-coordinates of the points are ordered

$$x_1 \leq x_2 \leq x_3$$

(otherwise we would just rename them), but you should treat the two cases, where (x_2, y_2) lies above or below the side joining the other two vertices, separately. You will, as with 3(b) and 3(c), need to find formulas for the lines joining the vertices, in order to carry out your integral computations. You will also probably find it helpful, while in the middle of your calculations, to introduce some shorthand, like $\Delta_1 = x_2 - x_1$ or $m_3 = \dfrac{y_3 - y_1}{x_3 - x_1}$, to keep the readability of what you are doing to an acceptably high level!

You should, for this problem, simplify your answers as much as possible, hopefully to the point where your answers for both cases look identical!

Guidelines for writing up your project.

The intent of projects is to expose you to mathematics as you might meet it in the real world, i.e., working as a team. Your group must understand the problem; translate it into mathematics; learn, read about, or develop mathematical methods to find the answer; show that the answer is correct; translate the mathematical answer back into the original problem and, finally, explain the significance of the translated answer. Projects are easier than real world problems, in that we make sure that the problem can be solved using the methods of this course. You may need to learn some new information to do the project.

The project is the solution to an open-ended multistep problem, formally presented. It will probably require several meetings for your group to find a solution to the problem and to present that solution clearly and understandably. Everyone in the group should contribute to the project.

Your group should write up a short paper explaining the problem and the mathematics you used to solve it, and then discussing the significance of your solution. Your paper should be a grammatically correct, organized discussion of the problem, with an introduction and a conclusion. While you should answer the specific questions asked in the project, your report should not be a disconnected set of answers but a connected narrative with transitions. It should conform to proper English usage (yes, spelling counts!) and should include appropriate diagrams and/or graphs, clearly labeled. You should show enough relevant calculations to justify your answers but not so much as to obscure the calculations' purpose. If you type your report (this is preferred but not required) it is fine to leave blank spaces and write the equations in. [There are certain advantages to typing: making (small or large) changes does not require the rewriting of the entire document!] Explain your results and conclusions, pointing out both strengths and weaknesses of your analysis. Assume that your reader is someone who took a calculus class course a while ago and does not remember all of the details. Be sure to avoid plagiarism.

Preparing formal reports is an important job skill for mathematicians, scientists, and engineers. For example, the Columbia Investigation Board, in its report on the causes of the Columbia space shuttle accident, wrote:

"During its investigation, the board was surprised to receive [PowerPoint] slides from NASA officials in place of technical reports. The board views the endemic use of PowerPoint briefing slides instead of technical papers as an illustration of the problematic methods of technical communication at NASA."

For **extra credit**, you can build models of some of the regions described in Problem # 2, out of some relatively stiff material, and demonstrate that they balance on the head of a pin placed at the centroid that you computed! [This can also serve as an independent check that your computed solution is correct!]