

Math 971 Algebraic Topology

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Homology on “small” chains = singular homology: The point to all of these calculations was that if $\{\mathcal{U}_\alpha\}$ is an open cover of X , then the inclusions $i_n : C_n^{\mathcal{U}}(X) \rightarrow C_n(X)$ induce isomorphisms on homology. This gives us two big theorems. The first is

Mayer-Vietoris Sequence: If $X = \mathcal{U} \cup \mathcal{V}$ is the union of two open sets, then the short exact sequences $0 \rightarrow C_n(\mathcal{U} \cap \mathcal{V}) \rightarrow C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \rightarrow C_n^{\{\mathcal{U}, \mathcal{V}\}}(X) \rightarrow 0$, together with the isomorphism above, give the long exact sequence

$$\cdots \rightarrow H_n(\mathcal{U} \cap \mathcal{V}) \xrightarrow{(i_{\mathcal{U}*}, -i_{\mathcal{V}}^*)} H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \xrightarrow{j_{\mathcal{U}*} + j_{\mathcal{V}}^*} H_n(X) \xrightarrow{\partial} H_{n-1}(\mathcal{U} \cap \mathcal{V}) \rightarrow \cdots$$

And just like Seifert - van Kampen, we can replace open sets by sets A, B having neighborhoods which deformation retract to them, and whose intersection deformation retracts to $A \cap B$. For example, subcomplexes $A, B \subseteq X$ of a CW-complex, with $A \cup B = X$ have homology satisfying a long exact sequence

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{(i_{A*}, -i_{B*})} H_n(A) \oplus H_n(B) \xrightarrow{j_{A*} + j_{B*}} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \cdots$$

And this is also true for reduced homology; we just augment the chain complexes used above with the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$, where the first non-trivial map is $a \mapsto (a, -a)$ and the second is $(a, b) \mapsto a + b$.

And now we can do some meaningful calculations! An n -sphere S^n is the union $S_+^n \cup S_-^n$ of its upper and lower hemispheres, each of which is contractible, and have intersection $S_+^n \cap S_-^n = S_0^{n-1}$ the equatorial $(n-1)$ -sphere. So Mayer-Vietoris gives us the exact sequence

$$\cdots \rightarrow \tilde{H}_k(S_+^n) \oplus \tilde{H}_k(S_-^n) \rightarrow \tilde{H}_k(S^n) \rightarrow \tilde{H}_{k-1}(S_0^{n-1}) \rightarrow \tilde{H}_{k-1}(S_+^n) \oplus \tilde{H}_{k-1}(S_-^n) \rightarrow \cdots, \text{ i.e.,}$$

$$0 \rightarrow \tilde{H}_k(S^n) \rightarrow \tilde{H}_{k-1}(S_0^{n-1}) \rightarrow 0 \quad \text{i.e., } \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S_0^{n-1}) \text{ for every } k \text{ and } n. \text{ So by induction,}$$

$$\tilde{H}_k(S^n) \cong \tilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z}, & \text{if } k=n \\ 0, & \text{otherwise} \end{cases}$$

There is another piece of homological algebra that we will find useful ; the Five Lemma. It allows us to compare the information contained in two long exact sequences.

Five Lemma: If we have abelian groups and maps

$$\begin{array}{ccccccccc} A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & D_n & \xrightarrow{i_n} & E_n \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \xrightarrow{h_{n-1}} & D_{n-1} & \xrightarrow{i_{n-1}} & E_{n-1} \end{array}$$

where the rows are exact, and the maps $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then γ is an isomorphism.

The proof is in some sense literally a matter of doing the only thing you can. To show injectivity, suppose $x \in C_n$ and $\gamma x = 0$, then $h_{n-1}\gamma x = \delta h_n x = 0$, so, since δ is injective, $h_n x = 0$. So by the exactness at C_n , $x = g_n y$ for some $y \in B_n$. Then $g_{n-1}\beta y = \gamma g_n y = \gamma x = 0$, so by exactness at B_{n-1} , $\beta y = f_{n-1}z$ for some $z \in A_{n-1}$. Then since α is surjective, $f_{n-1}z = \alpha w$ for some w . Then $0 = g_n f_n w$. But $\beta f_n w = f_{n-1} \alpha w = f_{n-1} z = \beta y$, so since β is injective, $y = f_n w$. So $0 = g_n f_n w = g_n y = x$. So $x = 0$. For surjectivity, suppose $x \in C_{n-1}$. Then $h_{n-1}x \in D_{n-1}$, so since δ is surjective, $h_{n-1}x = \delta y$ for some $y \in D_n$. Then $\epsilon i_n y = i_{n-1} \delta y = i_{n-1} h_{n-1} x = 0$, so since ϵ is injective, $i_n y = 0$. So by exactness at D_n , $y = h_n z$ for some $z \in C_n$. Then $h_{n-1} \gamma z = \delta h_n z = \delta y = h_{n-1} x$, so $h_{n-1}(\gamma z - x) = 0$, so by exactness at C_{n-1} , $\gamma z - x = g_{n-1} w$ for some $w \in B_{n-1}$. Then since β is surjective, $w = \beta u$ for some $u \in B_n$. Then $\gamma g_n u = g_{n-1} \beta u = g_{n-1} w = \gamma z - x$, so $x = \gamma z - \gamma g_n u = \gamma(z - g_n u)$. So γ is onto.

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\text{cl}_X(B) \subseteq \text{int}_X(A)$, then for every k the inclusion-induced map $H_k(X \setminus B, A \setminus B) \rightarrow H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\text{int}_X(A) \cup \text{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \rightarrow H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$.] To prove the second statement, we know that the inclusion $C_n^{\{A, B\}}(X) \rightarrow C_n(X)$ induce isomorphisms on homology, as does $C_n(A) \rightarrow C_n(X)$, so, by the five lemma, the induced map $C_n^{\{A, B\}}(X)/C_n(A) \rightarrow C_n(X)/C_n(A) = C_n(X, A)$ induces isomorphisms on homology. But the inclusion $C_n(B) \rightarrow C_n^{\{A, B\}}(X)$ induces a map $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \rightarrow C_n^{\{A, B\}}(X)/C_n(A)$ which is an isomorphism of chain groups; a basis for $C_n^{\{A, B\}}(X)/C_n(A)$ consists of singular simplices which map into A or B , but don't map into A , i.e., of simplices mapping into B but not A , i.e., of simplices mapping into B but not $A \cap B$. But this is the same as the basis for $C_n(B, A \cap B)$!

With these tools, we can start making some real homology computations. First, we show that if $\emptyset \neq A \subseteq X$ is "nice enough", then $H_n(X, A) \cong \tilde{H}_n(X/A)$. The definition of nice enough, like Seifert - van Kampen, is that A is closed and has an open neighborhood \mathcal{U} that deformation retracts to A (think: A is the subcomplex of a CW-complex X). Then using $\mathcal{U} \cdot X \setminus A$ as a cover of X , and $\mathcal{U}/A, (X \setminus A)/A$ as a cover of X/A , we have

$$\tilde{H}_n(X/A) \stackrel{(1)}{\cong} H_n(X/A, A/A) \stackrel{(2)}{\cong} H_n(X/A, \mathcal{U}/A) \stackrel{(3)}{\cong} H_n(X/A \setminus A/A, \mathcal{U}/A \setminus A/A) \stackrel{(4)}{\cong} H_n(X \setminus A, \mathcal{U} \setminus A) \stackrel{(5)}{\cong} H_n(X, A)$$

Where (1),(2) follow from the LES for a pair, (3),(5) by excision, and (4) because the restriction of the quotient map $X \rightarrow X/A$ gives a homeomorphism of pairs.

Second, if X, Y are T_1 , $x \in X$ and $y \in Y$ each have neighborhoods \mathcal{U}, \mathcal{V} which deformation retract to each point, then the one-point union $Z = X \vee Y = (X \amalg Y)/(x = y)$ has $\tilde{H}_n(Z) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$; this follows from a similar sequence of isomorphisms. Setting $z =$ the image of $\{x, y\}$ in Z , we have

$$\tilde{H}_n(Z) \cong H_n(Z, z) \cong H_n(Z, \mathcal{U} \vee \mathcal{V}) \cong H_n(Z \setminus z, \mathcal{U} \vee \mathcal{V} \setminus z) \cong H_n([X \setminus x] \amalg [Y \setminus y], [\mathcal{U} \setminus x] \amalg [\mathcal{V} \setminus y]) \cong H_n(X \setminus x, \mathcal{U} \setminus x) \oplus H_n(Y \setminus y, \mathcal{V} \setminus y) \cong H_n(X, x) \oplus H_n(Y, y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$

By induction, we then have $\tilde{H}_n(\bigvee_{i=1}^k X_i) \cong \bigoplus_{i=1}^k \tilde{H}_n(X_i)$

We have so far introduced two homologies; simplicial, H_*^Δ , whose computation "only" required some linear algebra, and singular, H_* , which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For Δ -complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X . In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub- Δ -complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$$\cdots \rightarrow H_n^\Delta(A) \rightarrow H_n^\Delta(X) \rightarrow H_n^\Delta(X, A) \rightarrow H_{n-1}^\Delta(A) \rightarrow \cdots$$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k -skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^{(k)}, X^{(k-1)}) & \rightarrow & H_n^\Delta(X^{(k-1)}) & \rightarrow & H_n^\Delta(X^{(k)}) & \rightarrow & H_n^\Delta(X^{(k)}, X^{(k-1)}) & \rightarrow & H_{n-1}^\Delta(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)}, X^{(k-1)}) & \rightarrow & H_n(X^{(k-1)}) & \rightarrow & H_n(X^{(k)}) & \rightarrow & H_n(X^{(k)}, X^{(k-1)}) & \rightarrow & H_{n-1}(X^{(k-1)}) \end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because we can, in each case, identify these groups. $H_n(X^{(k)}, X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \tilde{H}_n(\bigvee S^k)$

Some topological results with homological proofs: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h , then $n = m$. This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong (\mathbb{R}^m, \mathbb{R}^m \setminus 0)$, which gives

$$\begin{aligned} \tilde{H}_i(S^{n-1}) &\cong H_{i+1}(\mathbb{D}^n, \partial \mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0) \\ &\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial \mathbb{D}^m) \cong \tilde{H}_i(S^{m-1}) \end{aligned}$$

Setting $i = n - 1$ gives the result, since $\tilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$.

More generally, we can establish a result which is known as *invariance of domain*, a result which is useful in both topology and analysis.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \rightarrow \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

Note it is enough to prove this for our favorite open set, which in this context will be $\mathcal{V} = (-1, 1)^n \subseteq \mathbb{R}^n$, since given any open \mathcal{U} and $x \in \mathcal{U}$, we can find an injective linear map $h : (-1, 1)^n \rightarrow \mathcal{U}$ taking 0 to x . If we can show that $f \circ h$ has open image, then $f(x) \in f \circ h(\mathcal{V}) \subseteq f(\mathcal{U})$ shows that $f(x)$ has an open neighborhood in $f(\mathcal{U})$. Since x is arbitrary, $f(\mathcal{U})$ is open.

This in turn implies the “other” invariance of domain; if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in \mathbb{R}^n), a contradiction.

Our next goal is to show that, when both make sense, simplicial and singular homology are isomorphic. In fact, the inclusion of the simplicial chain groups into the singular ones induces an isomorphism on homology.