Math 417 Problem Set 7 Solutions

Starred (*) problems were due Friday, April 1.

(*) 51. Recall (!) that a subgroup $H \leq G$ is <u>characteristic</u> if $\varphi(H) = H$ for every $\varphi \in \mathrm{Aut}(G)$. Show that if K is a characteristic subgroup of H and H is a characteristic subgroup of G, then K is a characteristic subgroup of G.

We want to show that if $\varphi : G \to G$ is an automorphism of G, then $\varphi(K) = K$. What we know, since H is characteristic, is that $\varphi(H) = H$. But then if we define $\psi : H \to H$ by $\psi(h) = \varphi(h)$, then ψ is a homomorphism (since φ is) which is injective (since φ is!) and surjective, since $\psi(H) = \varphi(H) = H$. So ψ is an automorphism of H. Then since K is characteristic, we have $\psi(K) = K$ (thought of as a subgroup of H). But since $\psi = \varphi$ on elements of H, we then have $\varphi(K) = \psi(K) = K$. So K is carried to itself by any automorphism of G consequently, K is a characteristic subgroup of G.

(*) 54. Show that if G_1 and G_2 are groups and $H_1 \leq G_1$ and $H_2 \leq G_2$ are normal subgroups, then $H_1 \oplus H_2 \leq G_1 \oplus G_2$ is a normal subgroup and $(G_1 \oplus G_2)/(H_1 \oplus H_2) \cong (G_1/H_1) \oplus (G_2/H_2)$.

We know, since H_1 and H_2 are normal, that if $g_1 in G_1$, $h_1 \in H_1$, and if $g_2 \in G_2$, $h_2 \in H_2$, then $g_1h_1g_1^{-1} \in H_1$ and $g_2h_2g_2^{-1} \in H_2$. So given $(g_1, g_2) \in G_1 \oplus G_2$ and $(h_1, h_2) \in H_1 \oplus H_2$, we have

$(g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} = (g_1h_1, g_2h_2)(g_1^{-1}, g_2^{-1}) = (g_1h_1g_1^{-1}, g_2h_2g_2^{-1}) \in H_1 \oplus H_2$,

and so $H_1 \oplus H_2$ is a normal subgroup.

To show the needed isomorphism, we build a map $\varphi : G_1 \oplus G_2 \to (G_1/H_1) \oplus (G_2/H_2)$ by $\varphi(g_1, g_2) = (g_1H_1, g_2H_2)$. This is a homomomorphism, since

$\varphi((x_1, x_2)(y_1, y_2)) = \varphi(x_1y_1, x_2y_2) = (x_1y_1H_1, x_2y_2H_2) = ((x_1H_1)(y_1H_1), (x_2H_2)(y_2H_2)) = (x_1H_1, x_2H_2)(y_1H_1, y_2H_2) = \varphi(x_1, x_2)\varphi(y_1, y_2)$

It is also surjecive, since given (xH_1, yH_2) in the target group, we have $\varphi(x, y) = (xH_1, yH_2)$. Finally, we can find the kernel of φ, as

$\varphi(x, y) = (xH_1, yH_2) = e = (H_1, H_2) \Leftrightarrow x \in H_1$ and $y \in H_2 \Leftrightarrow (x, y) \in H_1 \oplus H_2$, so $\ker(\varphi) = H_1 \oplus H_2$.

Consequently, by the first isomorphism theorem, φ induces an <u>injective</u>, surjective, homomorphism $\overline{\varphi} : (G_1 \oplus G_2)/(H_1 \oplus H_2) \to (G_1/H_1) \oplus (G_2/H_2)$, that is, $\overline{\varphi}$ is an isomorphism.

(*) 57. (Gallian, p.222, # 42) Show that if $N, K \leq G$ are <u>normal</u> subgroups of G and $K \leq N$, then N/K is a normal subgroup of G/K, and $(G/K)/(N/K) \cong G/N$. [This is the "Third Isomorphism Theorem" of Emmy Noether. One approach: start by looking at the 'natural' map $G \to G/N$.]

$G/K = \{gK : g \in G\}$ is a group under multiplication of cosets, and $N/K = \{nK : n \in G\}$ is a subset of G/K. We start by showing it is a subgroup: $e \in N$ and so $eK = K \in N/K$ (the identity element of G/K). If $aK, bK \in N/K$, then $a, b \in N$ and so $(ab) \in N$ and $(aK)(bK) = (ab)K \in N/K$. Finally, if $a \in N$ then $a^{-1} \in N$ and so

$(aK)^{-1} = a^{-1}K \in N/K$. So N/K is closed under multiplication and inversion, and contains the identity, so N/K is a subgroup of G/K.

Even more, since N is normal in G, N/K is normal, since if $nK \in N/K$ and $gK \in G/K$, then $(gK)(nK)(gK)^{-1} = (gng^{-1})K \in N/K$, since $gng^{-1} \in N$.

Consequently, $(G/K)/(N/K)$ is a group. We have a 'natural' (surjective) homomorphism $\varphi_1 : G/K \to (G/K)/(N/K)$, with kernel N/K. But we <u>also</u> have a 'natural' (surjective) homomorphism $\varphi_2 : G \to G/K$, with kernel K. Composing these two homomorphisms, we get a (surjective!) homomorphism $\psi : G \to (G/K)/(N/K)$. The first isomorphism theorem then tells us that the induced homomorphism $\overline{\psi} : G/\ker(\psi) \to (G/K)/(N/K)$ will be an isomorphism; the only question is, what is $\ker(\psi)$?

To figure that out, start with $g \in G$ with $\psi(g) = e$ in $(G/K)/(N/K)$. This means $\varphi_1(\varphi_2(g)) = \varphi_1(gK) = (gK)(N/K) = e_{(G/K)/(N/K)}$. That is, $(gK)(N/K) = N/K$, so $gK \in N/K$, which means $g \in N$. So (!) $\ker(\psi) \subseteq N$. But conversely, if $g \in N$, then $gK \in N/K$, and so $\psi(g) = (gK)(N/K) = (N/K)$, so $g \in \ker(\psi)$, so $N \subseteq \ker(\psi)$. Together these give $\ker(\psi) = N$, and so

$\overline{\psi} : G/N \to (G/K)/(N/K)$ given by $\overline{\psi}(gN) = (gK)(N/K)$ is an isomorphism!

[N.B. The suggested approach will also work: The surjection $G \to G/N$ can be used to build a (surjective) homomorphism $G/K \to G/N$ given by $gK \mapsto gN$. Then we can show that the kernel of <u>this</u> homomorphism is N/K, yielding an isomorphism $(G/K)/(N/K) \to G/N$ (i.e., built in the opposite direction!). The diligent student can verify that this map is the inverse of the one built above....]

A selection of further solutions.

53. If G is an <u>abelian</u> group, let $K = \{a \in G \ : \ a^2 = 1\}$ and let $H = \{x^2 \ : x \in G\}$. Show that H and K are (normal) subgroups of G, and that $G/K \cong H$. [Hint: build a homomorphism $G \to H$ ...]

If $a, b \in K$ then $a^2 = 1$ and $b^2 = 1$, and so $(ab)^2 = abab = aabb = a^2b^2 = 1 \cdot 1 = 1$, so $ab \in K$. If $a \in K$ then $a^2 = 1$, so $1 = (a^2)^{-1} = a^{-2} = (a^{-1})^2$, so $a^{-1} \in K$. Toggether these two facts imply that K is a subgroup of G, and so (since G is abelian), K is a normal subgroup of G.

If $a, b \in H$ then $a = x^2$ and $b = y^2$ for some $x, y \in G$, and so $ab = x^2y^2 = xxyy = xyxy = (xy)^2$ with $xy \in G$, and so $ab \in H$. Also, $a^{-1} = (x^2)^{-1} = x^{-2} = (x^{-1})^2$ with $x^{-1} \in G$, and so $a^{-1} \in H$. So H is a subgroup of G, and so is a normal subgroup of G.

Finally, to show that $G/K \cong H$, we can look at the function $\varphi : G \to H$ given by $\varphi(g) = g^2$. Since G is abelian, this is in fact a homomorphism; $\varphi(gh) = (gh)^2 = ghgh = gghh = g^2h^2 = \varphi(g)\varphi(h)$. It is also surjective, since the definition of H state that everything in H is the square of something in G, i.e., is $\varphi(g)$ for some $g \in G$.

To establish the isomorphism $G/K \cong H$, it is (by the first isomorphism theorem) enough to show that $\ker(\varphi) = K$. But this follows essentially from the definition of K; K is the set og things whose square is the identity, i.e., it is the set og $g \in G$ so that $\varphi(g) = g^2 = 1$.

55. (Gallian, p.202, # 31) Let $G = (\mathbb{R}^*, \cdot, 1)$ and $H = (\mathbb{R}^+, \cdot, 1)$ be the groups of non-zero and positive real numbers, respectively, under multiplication. Show that $G \cong H \oplus \mathbb{Z}_2$ by directly building an isomorphism (and its inverse). [Hint: the absolute value function will likely play a role...]

The function $\varphi : G \to H$ given by $\varphi(x) = |x|$ is a homomorphism, since $\varphi(xy) = |xy| = |x| \cdot |y| = \varphi(x) varphi(y)$ (since we are using multiplication for both groups). More, it is a surjective homomorphism, since if $x \in H$ then $x > 0$, and then $\varphi(x) = |x| = x$ (where the x being fed to φ is thought of as living in G).

The kernel of φ is $\{x \neq 0 \ : \ |x| = 1\} = \{-1, 1\}$ (since 1 is the identity element of H), and so, in particular, two elements $x, y \in G$ have $\varphi(x) = \varphi(y) \Leftrightarrow xy^{-1} \in \{-1, 1\}$, i.e, $x = \pm y$. The idea is to use φ build our isomorphism, by using direct sum and another homomorphism to distinguish y from $-y$. This can be done using a homomorphism to $\mathbb{Z}_2$.

Let $\psi : G \to \mathbb{Z}_2$ (written additively!) be given by $\psi(x) = 0$ if $x > 0$ and $\psi(x) = 1$ if $x < 0$. This is a homomorphism: $\psi(xy) = 0 \Leftrightarrow xy > 0 \Leftrightarrow x, y > 0$ or $x, y < 0$. But then we have $\psi(x) + \psi(y) = 0 + 0 = 0$ in the first case, and $\psi(x) + \psi(y) = 1 + 1 = 0$ in the second, so $\psi(xy) = \psi(x) + \psi(y)$ when $\psi(xy) = 0$. The case $\psi(xy) = 1$ can be handled similarly (we then have x and y have opposite signs). [Alternatively, you can think of $\mathbb{Z} \cong \{-1, 1\}$ (written multiplicatively) via the homomorphism $x \to (-1)^x$, and then ψ can be defined as $\psi(x) = x/|x|$.]

Then we can put these two homomorphisms together to define $\omega : G \to H \oplus \mathbb{Z}_2$ by $\omega(g) = (\phi(g), \psi(g))$. This homomorphism is surjective, since positive numbers map onto $H \times \{0\}$ and negative number map onto $H \times \{1\}$, and it is also injective, since $\omega(g) = (1, 0)$ implies $|g| = 1$ <u>and</u> $g > 0$, so $g = 1$. Consequently, ω is a bijective homomorphism, and so it is an isomorphism.