Math 971 Algebraic Topology

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The isomorphism between simplicial and singular homology provides very quick proofs of several results about singular homology, which would other would require some effort:

If the Δ-complex X has no simplices in dimension greater than n, then $H_i(X) = 0$ for all $i > n$.

This is because the simplicial chain groups $C_i^\Delta(X)$ are 0, so $H_i^\Delta(X) = 0$.

If for each n, the Δ-complex X has finitely many n-simplices, then $H_n(X)$ is finitely generated for every n.

This is because the simplicial chain groups $C_n^\Delta(X)$ are all finitely generated, so $H_n^\Delta(X)$, being a quotient of a subgroup, is also finitely generated. [We are using here that the number of generators of a subgroup H of an *abelian* group G is no larger than that for G; this is not true for groups in general!]

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A,B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex K whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to K, with boundary a (for the exact same reasons as above) closed orientable surface.

Another: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h, then $n = m$. This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n-1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n-1 = m-1$.

More generally, we can establish a result which is known as *invariance of domain*, which is useful in both topology and analysis.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

We will defer this proof for awhile (perhaps permanently?).
Note it is enough to proof this for our favorite open set, which in this context will be $\mathcal{V} = (-1,1)^n \subseteq \mathbb{R}^n$, since given any open $\mathcal{U}$ and $x \in \mathcal{U}$, we can find an injective linear map $h : (-1,1)^n \to \mathcal{U}$ taking 0 to x. If we can show that $f \circ h$ has open image, then $f(x) \in f \circ h(\mathcal{V}) \subseteq f(\mathcal{U})$ shows that $f(x)$ has an open neighborhood in $f(\mathcal{U})$. Since x is arbitrary, $f(\mathcal{U})$ is open.

This in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.