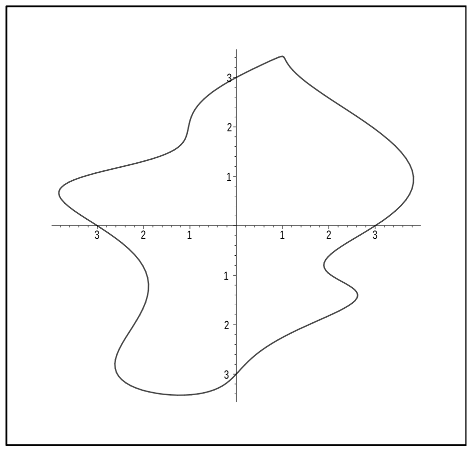
Math 423/823 Final Exam Practice Problems

1. For $z = x + yi$, does 1^z always equal 1 ?

2. Find the value of $\displaystyle\int_C f(z)\, dz$, where $f(z) = f(x + iy) = x^2 - iy^2$ and $C(t) = e^{it}$ for $0 \le t \le \pi$.

3. Find the integral of the function $f(z) = \dfrac{z}{z^3 + 1}$ around the simple closed curve $C(t) = [3 + \sin(5t)]\cos t + i[3 + \sin(2t)]\sin t$, $0 \le t \le 2\pi$. [See figure below.]



4. If $w = f(z)$ is analytic **and non-constant** on and inside of the simple closed curve C and, for some constant K, $|f(z)| = K$ for every point on C, show that there is a point z_0 inside of C where $f(z_0) = 0$.

   [Hint: Suppose not! Then show that we can apply the Maximum Principle to both $f(z)$ and $g(z) = \dfrac{1}{f(z)}$ and get ourselves into trouble!]

5. Show that if $|z| = 1$, then for any complex number b we have $\left|\dfrac{z + b}{\bar{b}z + 1}\right| = 1$.

6. Find the values of $z = \sqrt{1 + \sqrt{i}}$.

7. Show that if f is an entire function and $f(x + 2\pi) = f(x)$ for every <u>real</u> value of x, then $f(z + 2\pi) = f(z)$ for every <u>complex</u> value z. [Hint: what can you say about $g(z) = f(z + 2\pi) - f(z)$?]

8. Use residues to compute $\displaystyle\int_0^\infty \frac{dx}{x^6 + 1}$.

9. Use residues to compute $\displaystyle\int_0^\infty \frac{x^2 dx}{x^4 + 1}$.

10. Find the integral of $f(z) = \dfrac{z}{1 + \overline{z}}$ over the line segment $\gamma(t) = t$, $0 \le t \le 1$.

11. Determine, for the branch of the analytic function $f(z) = z^{1/2}$ with domain all z except for $\{x + 0i : x \le 0$ and with $f(1) = 1$, whether or not $f(z_1 z_2) = f(z_1)f(z_2)$

hold for every z_1, z_2 in the domain of f. Is there a different choice of branch cut which would change the answer?

12. Write the function $f(z) = \dfrac{z}{z^2 - 4z + 3}$ as a Laurent series which converges for $1 < |z| < 3$, and as (another!) Laurent series which converges for $3 < |z| < \infty$.

13. Find the residue at $z = 1$ for the functions $f(z) = \dfrac{z}{z^2 - 1}$ and $g(z) = \dfrac{\sin(2\pi z)}{(z-1)^2}$

14. Let C be any simple closed curve in the plane, oriented counterclockwise, and for z not on C, define

$$f(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\, dz \ .$$

Show that for every z inside of C, $f(z) = 6\pi i z$, while for every z outside of C, $f(z) = 0$.

15. Show that if

$$f(z) = f(x + yi) = u(x, y) + iv(x, y) \text{ and } g(z) = g(x + yi) = p(x, y) + iq(x, y)$$

both satisfy the Cauchy-Riemann equations at $z = 0$, then $h(z) = f(z)g(z)$ also satisfies the CR-equations at $z = 0$.

[There is nothing at all special about 0; it was chosen for notational convenience.]

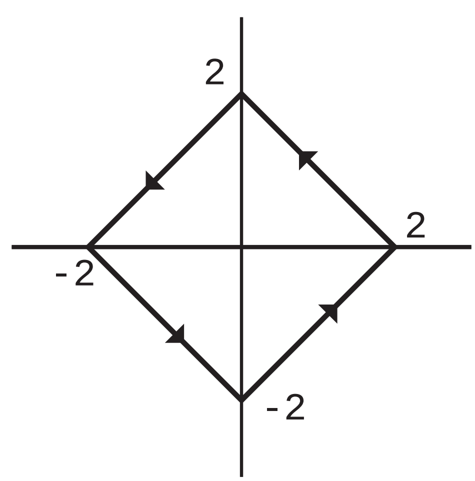
16. Show that setting $z = e^{it}$, we can rewrite $\dfrac{\cos 5t}{\cos t}$ as $z^4 - z^2 + 1 - z^{-2} + z^{-4}$.

Use this to find the value of $\displaystyle\int_0^{2\pi} \frac{\cos 5t}{\cos t}\, dt$ by converting to an integral over the unit circle $C(t) = e^{it}$, $0 \le t \le 2\pi$.

17. Find the Laurent series expansion of the function $f(z) = \dfrac{z^3}{(z-1)^2}$ centered at $z = 0$, valid for $1 < |z| < \infty$.

18. Find the value of $\displaystyle\int_C \frac{dz}{(z^2 + 1)(2z + 5)}$,

where C is the boundary of the 'diamond' $S = \{(x + iy : |x| + |y| \le 2\}$, traversed counterclockwise (see figure below).



Some potentially useful formulas

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\arcsin(z) = -i\log(iz + \sqrt{1-z^2})$$

$$\arctan z = \frac{i}{2}\log\left(\frac{i-z}{i+z}\right)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ , for } |z| < 1$$

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\Big(\frac{1}{1-z}\Big)$$

$$\frac{d}{dz}\Big(\log(1-z)\Big) = \frac{-1}{1-z}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^n$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^n$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^n$$

$$\frac{1}{z^2+1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \text{ , for } |z| < 1$$