Math 856 Homework 4

Starred (*) problems to be handed in Thursday, October 19

(*) 19: If X, Y are smooth tangent vactor fields on M, and $f, g \in C^\infty(M)$, show that $[fX, gY] = (fg)[X, Y] + (fXg)Y - (gYf)X$.

20: Show that, in $\mathbb{R}^{2n}$, the vector field

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + \cdots + x^{2n} \frac{\partial}{\partial x^{2n-1}} - x^{2n-1} \frac{\partial}{\partial x^{2n}}$$

restricts to a nowhere-zero vector field tangent to the unit $(2n-1)$-sphere.

(*) 21: ["Bundle Section Extension Lemma"] Given a smooth vector bundle $p : E \to M$ over a smooth manifold M, a closed subset $A \subseteq M$, and a smooth section $s : A \to E$ defined over A (that is, for every $a \in A$ there is a neighborhood U_a of a in M and a smooth section $s_U : U \to E$ so that $s_u = s$ on $A \cap U$), show that there is a global smooth section $S : M \to E$ with $S|_A = s$. (Hint: partition of unity...)

22: [Lee, p. 101, problem 4-7] Let M, N be smooth manifolds, $f : M \to N$ a smooth map, and define $F : M \to M \times N$ by $F(x) = (x, f(x))$. Show that for every tangent vector field X on M there is a tangent vector field Y on $M \times N$ so that Y is F-related to X.

(*) 23: [Lee, p.101, problem 5-8] Let $p : E \to M$ be a smooth n-dimensional vector bundle and $X_1, \ldots, X_k$ be linearly independent smooth sections of E defined over an open subset $U \subseteq M$. Show that for every $a \in U$ there is a neighborhood V of a and smooth sections $Y_{k+1}, \ldots, Y_n$ defined over V so that $(X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_n)$ forms a local frame for E over $U \cap V$.

(Hint: if $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$, then why is it that if you wiggle the first k vectors a little bit, you still have a basis?)

24: Show that $M \times N$ is orientable $\Leftrightarrow$ both M and N are.