Math 325 Problem Set 4 Solutions

Problems were due Friday, February 10.

13. [Zorn, p.64, #11] Our text says that a set A is *bounded away from* 0 if there is an $\epsilon > 0$ so that for every $x \in A$ we have $|x| > \epsilon$. Show that A is bounded away from 0 if and only if the set $B = \{\frac{1}{x} \mid x \in A\}$ is bounded.

    [N.B. "P if and only if Q" means P implies Q and Q implies P.]

    If A is bounded away from 0, then we have an $\epsilon > 0$ so that $x \in A$ implies $|x| > \epsilon$. but then $|x| > 0$, so $1/|x| > 0$ and $1/\epsilon > 0$, so $1/(\epsilon|x|) > 0$. Then multiplication by $1/(\epsilon|x|)$ will not change the direction of an inequality, so

    $$1/|x| = \epsilon/(\epsilon|x|) < |x|/(\epsilon|x|) = 1/\epsilon, \text{ for every } x \in A.$$

    So $-1/|x| > -1/\epsilon$, as well, But then since $-|x| \leq x \leq |x|$ for any $x \in \mathbb{R}$ (x equals one of them...), we have $-1/|x| \leq 1/x \leq 1/|x|$ (again, $1/x$ equals one of them), so $-1/\epsilon < -1/|x| \leq 1/x \leq 1/|x| < 1/\epsilon$, and so $-1/\epsilon < 1/x < 1/\epsilon$, for every $x \in A$. So B is bounded below (by $-1/\epsilon$) and bounded above (by $1/\epsilon$), so B is bounded.

    For the other direction, if we suppose that $B = \{\frac{1}{x} \mid x \in A\}$ is bounded, then there are N and M so that $M \leq 1/x \leq N$ for every $x \in A$. This statement alone requires that $x \neq 0$, since $1/0$ doesn't make sense and the statement assumes that $1/x$ always does make sense. This direction is a little trickier, since we can't 'just' invert our newly-found inequalities (and get a reversed inequality), because, for example, $a < 0 < b$ implies $1/a < 0 < 1/b$ (and the inequality was not reversed). But we can instead sort of borrow from a previous homework problem...

    $M \leq 1/x$ does mean that $-1/x \leq -M$, so since we have $1/x \leq N$, we have $-1/x \leq \max(-M, N)$ and $1/x \leq \max(-M, N)$. But since $1/|x|$ must equal one of these two values ($1/x$ or $-1/x$), and both are $\leq \max(-M, N)$, we can conclude that $0 \leq 1/|x| \leq \max(-M, N) = K$ for every $x \in A$. But now we can invert things! Since $1/|x| > 0$ we have $|x| > 0$, and $0 < 1/|x| \leq K$, so $K > 0$. Then $1/|x| \leq K$ means that $1/K = (1/|x|)(|x|/K) \leq K(|x|/K) = |x|$. So $|x| geq 1/K > 1/(2K) = L > 0$ for every $x \in A$, so there is an $L > 0$ so that $|x| > L$ for every $x \in A$. So A is bounded away from 0.

14. If we set $A = \{x \in \mathbb{R} \mid x^3 < 2\}$, show that A is bounded above, so has a supremum $\alpha = \sup(A)$. Then show (in a manner similar to our classroom demonstrations) that $\alpha^3 < 2$ is not possible. (If you are feeling like doing even more, show that $\alpha^3 > 2$ is also impossible! From that, we can conclude that $\alpha^3 = 2$.)

    We showed in class that $f(x) = x^3$ is an increasing function. So if we find a single $a \in \mathbb{R}$ so that $a^3 > 2$, then $x \geq a$ will imply that $x^3 \geq a^3 > 2$, so $x \notin A$. This means that $x \in A$ implies that $x < a$, so A will be bounded above by a. But such an a is readily available; $2^3 = 8 > 2$, so 2 is an upper bound for A.

    We therefore have a least upper bound $\alpha = \sup(A)$. To show that $\alpha^3 < 2$ is impossible, suppse that $\alpha^3 < 2$! (We will get ourselves into trouble.) Then $2 - \alpha^3 = \epsilon > 0$. What we show is that (as in class) α could not be an upper bound for A, by finding a $\delta > 0$ so that $(\alpha+\delta)^3 < 2$, so $\alpha+\delta \in A$ and $\alpha < \alpha+\delta$, a contradiction.

To determine δ, we note that $(\alpha+\delta)^3 = \alpha^2 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3$. Since we intend to have $\delta > 0$ and we know, from above, that $\alpha \leq 2$, then

$(\alpha+\delta)^3 = \alpha^3 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3 \leq \alpha^3 + 3\cdot 2^2\delta + 3cdot2\delta^2 + \delta^3 = \alpha^3 + 12\delta + 6\delta^2 + \delta^3$.

So if we make sure that $12\delta + 6\delta^2 + \delta^3 < \epsilon$, then $(\alpha+\delta)^3 < \alpha^3 + \epsilon = \alpha^3 + (2-\alpha^3) = 2$, as desired.

There are many ways to arrange this. Perhaps the least tortuous way is to insist, first, that $0 < \delta \leq 1$. Then $12\delta + 6\delta^2 + \delta^3 \leq 12\delta + 6\delta\cdot 1 + \delta\cdot 1^2 = 19\delta$. So to ensure that $12\delta + 6\delta^2 + \delta^3 < \epsilon$ we can <u>also</u> insist that $\delta < \epsilon/19$. <u>So</u> if we set $\delta = \min(1, \epsilon/20)$, then

$(\alpha+\delta)^3 = \alpha^3 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3 \leq \alpha^3 + 12\delta + 6\delta^2 + \delta^3 \leq \alpha^3 + 12\delta + 6\delta + \delta = \alpha^3 + 19\delta \leq \alpha^3 + 19\epsilon/20 < \alpha^3 + \epsilon = \alpha^3 + (2-\alpha^3) = 2$.

So $\alpha+\delta > \alpha$ and $(\alpha+\delta)^3 < 2$, so $\alpha+\delta \in A$, contradicting the choice of $\alpha = \sup(A)$. So $\alpha^3 < 2$ is impossible.

For the extra part: Showing $\alpha^3 > 2$ is impossible proceeds similarly. Setting $\epsilon = \alpha^3 - 2 > 0$, we find a $\delta > 0$ so that $(\alpha-\delta)^3 > 2$, so (by our reasoning at the start of the problem) $\alpha - \delta < \alpha$ is an upper bound for A, so α <u>cannot</u> be the least upper bound for A.

Finding an appropriate δ follows the same line as our argument above. $(\alpha-\delta)^3 = \alpha^3 - 3\alpha^2\delta + 3\alpha\delta^2 - \delta^3 > \alpha^3 - 3\alpha^2\delta - \delta^3$ (since $\alpha > 0$; 0 is not an upper bound for A). But if we insist that $0 < \delta \leq 1$, then $\alpha^3 - 3\alpha^2\delta - \delta^3 \geq \alpha^3 - 3\alpha^2\delta - \delta\cdot 1^2 = \alpha^3 - (3\alpha^2+1)\delta$, and we can make $(3\alpha^2+1)\delta < \epsilon$ by choosing $0 < \delta < \epsilon/(3\alpha^2+1)$. For this δ, we find that $(\alpha-\delta)^3 > 2$, a contradiction.

So both $\alpha^3 < 2$ and $\alpha^3 > 2$ are impossible; this means that $\alpha^3 = 2$.

15. For subsets $A, B \subseteq \mathbb{R}$, we define their 'sum' $A + B = \{a + b \ : \ a \in A, b \in B\}$.

Show that if A and B are both non-empty and bounded from above, then so is $A+B$, and

$$\sup(A+B) = \sup(A) + \sup(B) \ .$$

[Hint: show that $\sup(A) + \sup(B)$ is an upper bound! Then worry about whether there might be a smaller one...]

If $x \in A$ then $x \leq \sup(A)$, and if $y \in B$ then $y \leq \sup(B)$. So if $z \in A+B$ then $z = x+y$ for some $x \in A$ and $y \in B$, so $z = x+y \leq \sup(A)+\sup(B)$. So $\sup(A)+\sup(B)$ is an upper bound for $A+B$. On the other hand, if $N < \sup(A)+\sup(B)$, then $(\sup(A)+\sup(B)) - N = \epsilon > 0$. What we can do, then, is 'split' this exccess betweeen A and B to identify 'good' elements of the set. Specifically, if we set $N_1 = \sup(A) - \epsilon/2 < \sup(A)$, then N_1 is <u>not</u> an upper bound for A, so there is an $x \in A$ with $x > N_1$. Also, setting $N_2 = \sup(B) - \epsilon/2 < \sup(B)$, then N_2 is not an upper bound for B, so there is a $y \in B$ with $y > N_2$.

Then $z = x+y \in A+B$, and $z = x+y > N_1+y > N_1+N_2 = (\sup(A)-\epsilon/2)+(\sup(B)-\epsilon/2) = (\sup(A)+\sup(B)) - \epsilon = N$. So for any $N < \sup(A)+\sup(B)$ we can find a $z \in A+B$ with $z > N$. So $\sup(A)+\sup(B)$ is the least upper bound for $A+B$.