Math 423/823 Exercise Set 4 Solutions

13. [BC#2.18.11] Show that if $T(z) = \dfrac{az+b}{cz+d}$ (where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ then

(a) if $c = 0$ then $\lim\limits_{z\to\infty} T(z) = \infty$.

In this case $ad - bc = ad \neq 0$, so $a, d \neq 0$, and so $a/d = \alpha \neq 0$. Setting $\beta = b/d$, we then have $T(z) = \alpha z + \beta$ with $\alpha \neq 0$, so $|T(z)| = |\alpha z + \beta| \geq |\alpha z| - |\beta| = |\alpha||z| - |\beta|$, which, since $|\alpha| > 0$, will grow large when $|z|$ grows large. So $T(z) \to \infty$ as $z \to \infty$.

(b) if $c \neq 0$ then $\lim\limits_{z\to\infty} T(z) = \dfrac{a}{c}$ and $\lim\limits_{z\to -d/c} T(z) = \infty$.

In this case we can look at $T\Big(\dfrac{1}{z}\Big) = \dfrac{a\frac{1}{z} + b}{c\frac{1}{z} + d} = \dfrac{a + bz}{c + dz}$ and investigate what happens as $z \to 0$. Since $c \neq 0$ and the numerator and denominator are continuous (at $z = 0$), we get $\lim\limits_{z\to\infty} T(z) = \lim\limits_{z\to 0} T\Big(\dfrac{1}{z}\Big) = \dfrac{a + 0b}{c + 0d} = \dfrac{a}{c}$.

For $\lim\limits_{z\to -d/c} T(z)$, we look at $\dfrac{1}{T(z)} = \dfrac{cz+d}{az+b}$. Then since $a(-\dfrac{d}{c}) + b = \dfrac{bc - ad}{c}$ is (finite and) non-zero, both the numerator and denominator are continuous at $-\dfrac{d}{c}$, and the denominator is non-zero there, we have $\lim\limits_{z\to -d/c} \dfrac{1}{T(z)} = \dfrac{c(-\frac{d}{c}) + d}{a(-\frac{d}{c}) + b} = \dfrac{-dc + cd}{-ad + cb} = 0$, so $\lim\limits_{z\to -d/c} T(z) = \infty$.

14. [BC#2.20.9] Let f be the function $f(z) = \begin{cases} (\overline{z})^2/z & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$.

Show that f is not differentiable at 0, even though the limit of the difference quotient exists (and both agree) when you let $\Delta z \to 0$ along the vertical and horizontal axes; show that if you approach 0 along the line $h = k$ (where $\Delta z = h + ik$) you find a different limit.

$f(z) = f(x+yi) = \dfrac{(x-yi)^2}{x+yi} = \dfrac{(x-yi)^3}{(x+yi)(x-yi)} = \dfrac{x^3 - 3x^2yi - 3xy^2 + y^3i}{x^2+y^2} = \dfrac{x^3 - 3xy^2}{x^2+y^2} + \dfrac{-3x^2y + y^3}{x^2+y^2}i$.

So as we approach 0 along the x-axis, $z = x + 0i$ and

$\dfrac{f(z) - f(0)}{z - 0} = \dfrac{x^3 - 3x0^2}{x(x^2+0^2)} + \dfrac{-3x^2 0 + 0^3)}{x(x^2 + y0^2)}i = \dfrac{x^3}{x^3} + \dfrac{-3x^2 0 + 0^3)}{x^3}i = 1$, with limit 1.

But as we approach 0 along the y-axis, $z = 0 + yi$ and

$\dfrac{f(z) - f(0)}{z - 0} = \dfrac{0^3 - 3\cdot 0y^2}{(yi)(0^2+y^2)} + \dfrac{-3\cdot 0^2 y + y^3)}{(yi)(0^2+y^2)}i = 0 + \dfrac{y^3}{y^3 i}i = 1$, which <u>also</u> has limit 1.

But as we approach 0 along the line $y = x$, $z = x + xi$ and
$\dfrac{f(z) - f(0)}{z - 0} = \dfrac{x^3 - 3xx^2}{(x + xi)(x^2 + x^2)} + \dfrac{-3x^2x + x^3)}{(x + xi)(x^2 + x^2)} i = \dfrac{-2x^3}{(x + xi)2x^2} + \dfrac{-2x^3}{(x + xi)2x^2} i =$
$\dfrac{-x - xi}{x + xi} = -\dfrac{x + xi}{x + xi} = -1$, which has limit -1.

Therefore, the difference quotient in fact has no limit, and so f is not differentiable at $z = 0$.

15. [BC#2.23.6] Revisit problem #14 from the viewpoint of the Cauchy-Riemann equations. That is, write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ (noting that we define $u(0, 0) = v(0, 0) = 0$). Show that u_x, u_y, v_x, and v_y all exist at $(0, 0)$ *and* that they satisfy the Cauchy-Riemann equations at $(0, 0)$.

    From the work above we see that $u(x, y) = \dfrac{x^3 - 3xy^2}{x^2 + y^2}$ and $v(x, y) = \dfrac{-3x^2y + y^3)}{x^2 + y^2}$ (filled in to have value 0 at $z = 0$. Most of the work for this problem can be lifted out of the computations above: at $z = 0$ we have

    $$u_x = \lim_{x\to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x\to 0} \frac{x^3}{xx^2} = \lim_{x\to 0} 1 = 1$$
    $$u_y = \lim_{y\to 0} \frac{u(0, y) - u(0, 0)}{y - 0} = \lim_{y\to 0} \frac{0}{yy^2} = \lim_{y\to 0} 0 = 0$$
    $$v_x = \lim_{x\to 0} \frac{v(x, 0) - u(0, 0)}{x - 0} = \lim_{x\to 0} \frac{0}{xx^2} = \lim_{x\to 0} 0 = 0$$
    $$v_y = \lim_{y\to 0} \frac{v(0, y) - u(0, 0)}{y - 0} = \lim_{y\to 0} \frac{y^3}{yy^2} == \lim_{y\to 0} 1 = 1$$

    So at $z = 0$, we have $u_x = 1 = v_y$ and $u_y = 0 = -v_x$. So the Cauchy-Riemann equations are satisfied at $z - 0$, even though $f(z)$ is not differentiable at $z - 0$!

16. Let $f(z) = z^3 + 1$ and $a = \dfrac{1 + \sqrt{3}i}{2}$, $b = \dfrac{1 - \sqrt{3}i}{2}$. Show that there is *no* value of w on the line segment $\{\dfrac{1 + t\sqrt{3}i}{2} \ : \ -1 \le t \le 1\}$ where $f'(w) = \dfrac{f(b) - f(a)}{b - a}$.

    Note that $a = \dfrac{1 + \sqrt{3}i}{2} = \cos(\pi/3) + i\sin(\pi/3)$ and $b = \dfrac{1 - \sqrt{3}i}{2} = \cos(-\pi/3) + i\sin(-\pi/3)$, so $a^3 = \cos(\pi) + i\sin(\pi) = -1$ and $b^3 = \cos(-\pi) + i\sin(-\pi) = -1$

    So we find that $f(a) = f\big(\dfrac{1 + \sqrt{3}i}{2}\big) = (-1) + 1 = 0 = (-1) + 1 = f\big(\dfrac{1 - \sqrt{3}i}{2}\big) = f(b)$ and so $\dfrac{f(b) - f(a)}{b - a} = 0$.

    But $f'(z) = (z^3 + 1)' = 3z^2 = 0$ only for $z = 0$. And since $z = 0$ does not lie on the line $\gamma(t) = \{\dfrac{1 + t\sqrt{3}i}{2}$ through a and b (all such points have real part 1/2), there is no w on the line segment between a and b so that $f'(w) = 0 = \dfrac{f(b) - f(a)}{b - a}$.