FINAL EXAM

Math 107, Spring Semester 2010

Name (Print): Solutions

Student ID Number: 00000000

TA Name: Zach

Please Circle your professor's name:

M. Brittenham M. Rammaha B. Rogge J. DeVries

Please Circle your class time:

8:30 M,W,F 9:30 T,R 10:30 M,W,F 12:30 M,W,F 6:30 M,W

INSTRUCTIONS:

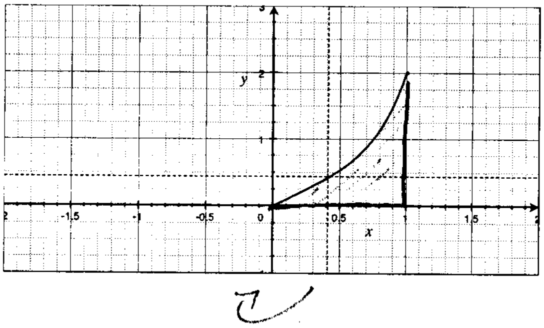
- *There are 9 pages of questions and this cover sheet.*
- *SHOW ALL YOUR WORK. Partial credit will be given only if your work is relevant and correct.*
- *This examination is closed book. Calculators that perform symbolic manipulations such as the TI-89, TI-92 or their equivalence, are* **not permitted**. *Other calculators may be used. Turn off and put away all* **cell phones.**

Question	Points	Score
1	12	
2	12	
3	24	
4	18	
5	12	
6	16	
7	12	
8	8	
9	24	
10	16	
11	26	
12	20	
Total	200	

1. [12 Points] Let **R** be the bounded region in the first quadrant enclosed by the graphs of $y = x^5 + x$, $y = 0$, and $x = 1$. Let **S** be the solid obtained by revolving **R** about the **y-axis**. Find **(but don't evaluate)** an integral whose value gives the volume of **S**.

Volume by shells:

$$\text{Volume} = 2\pi \int_0^1 (x-0)(x^5+x)\,dx$$



2. [12 Points] For what values of x does the following series converge? For these values of x, what is the sum of the series?

$$\sum_{k=1}^{\infty}(-1)^{k+1}(\frac{1}{2}\,x)^k = (\frac{1}{2}\,x) - (\frac{1}{2}\,x)^2 + (\frac{1}{2}\,x)^3 - \cdots \qquad (1)$$

$$a_k = (-1)^{k+1}(\tfrac{1}{2}x)^k \qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}(\frac{1}{2}x)^{n+1}}{(-1)^{n+1}(\frac{1}{2}x)^n}\right| = \left|\tfrac{1}{2}x\right|$$

So $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{2}|x| < 1$ for $|x|<2$

Check $|x|=2$: $\quad x=2 \quad \sum (-1)^{k+1}(1)^k = \sum (-1)^{k+1}$ diverges

$\quad x=-2 \quad \sum (-1)^{k+1}(-1)^k = \sum (-1)$ diverges.

Converges for $|x|<2$

$$\sum_{k=1}^{\infty} (-1)^{k+1}(\tfrac{1}{2}x)^k = (\tfrac{1}{2}x)\sum_{k=0}^{\infty}(-1)^{k+2}(\tfrac{1}{2}x)^k = \tfrac{1}{2}x\sum_{k=0}^{\infty}(-\tfrac{1}{2}x)^k$$

$$= \frac{1}{2}x\,\frac{1}{1-(-\frac{1}{2}x)}$$

3. [24 Points] Evaluate the following integrals:

a) [12 Points] $\int 2x \arctan x \, dx$

$u = \arctan x \quad dv = 2x\,dx$

$du = \frac{1}{1+x^2}dx \quad v = x^2$

$= x^2 \arctan x - \int \frac{x^2}{1+x^2} dx = x^2 \arctan x - \int 1 - \frac{1}{1+x^2} dx$

$= x^2 \arctan x - (x - \arctan x) + C$

b) [12 Points] $\int \frac{2x-13}{(2x+1)(x-3)} dx = \int \frac{A}{2x+1} + \frac{B}{x-3} dx$

$= \int \frac{A(x-3)+B(2x+1)}{(2x+1)(x-3)} dx \qquad A(x-3)+B(2x+1) = 2x-13$

$x=3: \quad B(7) = -7 \quad B = -1$

$x = -\frac{1}{2} \quad A(-\frac{7}{2}) = -14 \quad A = \frac{-14}{-7}2 = 4$

$= \int \frac{4}{2x+1} - \frac{1}{x-3} dx = 4\frac{1}{2}\ln(2x+1) - \ln(x-3) + C$

4. [18 Points] Determine whether the following improper integrals are convergent or divergent. If the integral is convergent, find its exact value.

a) [9 Points] $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

$u = \ln x \quad du = \frac{1}{x}dx$

$x = e \to u = \ln e = 1$

$x \to \infty \to u \to \infty$

$= \displaystyle\int_1^\infty \frac{du}{u^2} = \lim_{N\to\infty} \left.\frac{-1}{u}\right|_1^N = \lim_{N\to\infty} -\frac{1}{N} - (-1) = \lim_{N\to\infty} 1 - \frac{1}{N} = 1$

converges

b) [9 Points] $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1-x^4}}\,dx$ integrand blows up at $x=1$.

$= \displaystyle\lim_{a\to 1^-} \int_0^a \frac{x^3}{\sqrt{1-x^4}}\,dx$

$\displaystyle\int \frac{x^3}{\sqrt{1-x^4}}\,dx \quad u = 1-x^4 \quad du = -4\cancel{x}x^3\,dx$

$= \displaystyle\int \frac{-\frac{1}{4}du}{\sqrt{u}} = -\frac{1}{4}\int u^{-1/2}du = -\frac{1}{4}2u^{1/2}+C$

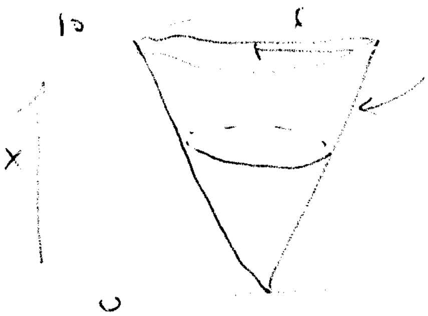
$= -2\sqrt{1-x^4} \cancel{x} + C$

$= \displaystyle\lim_{a\to 1^-} \left.-2\sqrt{1-x^4}\right|_0^a$

$= \displaystyle\lim_{a\to 1^-} -2\left(\sqrt{1-a^4} - 1\right) = -2(0-1) = 2$

converges

5. [12 Points] A tank has the shape of a right circular cone with its vertex on the ground. The height of the tank is 10 feet; the radius of its top is 6 feet. Assume that the tank is filled with water weighing 62.4 lb/ft^3. Write down **but do not evaluate** an integral whose value is the work required to pump all of the water over the top of the tank.



radius $= \frac{6}{10}x$ height lifted $= 10 - x$

$$\text{Work} = \int_0^{10} \underbrace{(62.4)}_{\text{density}} \underbrace{\left(\pi\left(\tfrac{6}{10}x\right)^2\right)}_{\text{area of \cancel{shell} slice}} \underbrace{(10-x)}_{\text{distance travelled}}\, dx$$

6. [16 Points] Determine whether the following series converge absolutely, converge conditionally or diverge. You must show all details to receive credit.

a) [6 Points] $\displaystyle\sum_{k=1}^{\infty} \frac{k+4}{3k+1} = \sum c_n$

$c_n = \dfrac{k+4}{3k+1} = \dfrac{1+4/k}{3+1/k} \to \dfrac{1+0}{3+0} = \dfrac{1}{3}$ as $k \to \infty$

$\frac{1}{3} \neq 0$ so diverges by n^{th} (k^{th}!) term test.

b) [10 Points] $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{\sqrt{k}-1}{4k^2+k-1} = \sum a_k$

$|a_k| = \dfrac{\sqrt{k}-1}{4k^2+k-1}$ limit compare with $\dfrac{\sqrt{k}}{4k^2} = \frac{1}{4}k^{-3/2} = b_k$

$\dfrac{|a_k|}{b_k} = \dfrac{1 - 1/\sqrt{k}}{1 + \frac{1}{4k} - \frac{1}{4k^2}} \to \dfrac{1-0}{1+0+0} = 1 \neq 0$ and $\sum b_k$ conv (p-series, $p = 3/2 > 1$) so $\sum |a_k|$ converges by limit comparison so $\sum a_k$ converges absolutely.

7. [12 Points] A particle is traveling in space from the point $P = (1, 4, 0)$ to the point $Q = (3, 2, 1)$ on a line segment with speed $0.5\, cm/min$, where the xyz-coordinate system is measured in centimeters.

a) [6 Points] Find the velocity vector of the particle.

direction $= \vec{v} = \vec{PQ} = (2, 2, -1)$ $\quad \|\vec{v}\| = (2^2+2^2+(-1)^2)^{1/2} = 9^{1/2} = 3$

want speed $= 1/2$, so use $\vec{w} = \frac{1}{6}\vec{v} = (\frac{1}{3}, \frac{1}{3}, -\frac{1}{6})$

velocity $= \vec{w} = (\frac{1}{3}, \frac{1}{3}, -\frac{1}{6})$

b) [6 Points] Find the correct position function of the particle, and find and total time needed for the trip.

position $= P + t\vec{w} = (1, 4, 0) + t(\frac{1}{3}, \frac{1}{3}, -\frac{1}{6})$

total time $= 6$ minutes

8. [8 Points] By using a suitable comparison Theorem, determine whether the following improper integral is convergent or divergent:

$$(*) = \int_1^\infty \frac{x}{4x^2 + x + 1}\, dx = \int_1^\infty f(x)\, dx$$

compare with $\frac{1}{4x} = g(x) = \frac{x}{4x^2}$

$$\frac{f(x)}{g(x)} = \frac{1}{1 + \frac{1}{4x} + \frac{1}{4x^2}} \to \frac{1}{1+0+0} = 1 \neq 0 \quad \text{so since}$$

$$\int_1^\infty \frac{1}{4x}\, dx = \lim_{N\to\infty} \left(\frac{1}{4} \ln x \Big|_1^N \right) = \lim_{N\to\infty} \frac{1}{4} \ln N = \infty,$$

(*) diverges by limit comparison

9. [24 Points] Find the radius of convergence and the largest interval on which the following power series converges absolutely. At each endpoint of the interval, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3k-1}(x-2)^k = \sum a_n (x-2)^k \qquad (2)$$

$a_k = \frac{(-1)^k}{3k-1}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{3(n+1)-1} \cdot \frac{3n-1}{(-1)^n}\right|$

$$= \frac{3n-1}{3n+2} = \frac{3-1/n}{3+2/n} \to \frac{3-0}{3+0} = 1 = L$$

so radius of convergence is $\frac{1}{L} = R = 1$

check $|x-2| = 1 \to x-2 = 1, -1$

$x-2=1$ $(x=3)$ $\quad \sum \frac{(-1)^k}{3k-1}(1)^k = \sum \frac{(-1)^k}{3k-1}$ $\quad \frac{1}{3k-1} > 0, \to 0$, decreasing

so converges by alt series test

$x-2=-1$ $(x=1)$ $\quad \sum \frac{(-1)^k(-1)^k}{3k-1} = \sum \frac{1}{3k-1}$

diverges by comparison with $\frac{1}{3k} < \frac{1}{3k-1}$

$\sum \frac{1}{3k} = \frac{1}{3}\sum\frac{1}{k}$ diverges (p-series, $p=1$)

at $x=3$ $\quad \sum |a_n(x-2)^k| = \sum \frac{1}{3k-1}$ diverges

So interval of convergence is $(1,3]$ with conditional convergence at $x=3$.

10. [16 Points] The Taylor series of $\cos(x^2)$ about $x = 0$ is:

$$\cos(x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\, x^{4k}, \quad x \in \mathbb{R}. \tag{3}$$

a) [6 Points] Find a series whose sum is $\int_0^1 \cos(x^2)dx$.

term-by-term integration: $\int \cos(x^2)dx = \int \sum \frac{(-1)^k}{(2k)!} x^{4k}\, dx$

$= \sum \frac{(-1)^k}{(4k+1)(2k)!} x^{4k+1}$

So $\int_0^1 \cos(x^2)dx$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(2k)!}(1)^{4k+1} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(2k)!}(0)^{4k+1}$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(2k)!}$

b) [10 Points] Use the series you found in part a) to approximate the integral $\int_0^1 \cos(x^2)dx$ with an error that does not exceed 0.01.

Approximate $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(2k)!}$ to within .01

alternating series! $\sum (-1)^k a_k$ $a_k = \frac{1}{(4k+1)(2k)!}$ is >0

decreasing $\left(4(k+1)+1 > 4k+1 \right.$ and
and $\left. (2(k+1))! > (2k)! \right)$

$\frac{1}{(4k+1)(2k)!} < \frac{1}{4k+1} \to 0$ as $k \to \infty$ so by alt series remainder thm,

$\left| \sum_{k=0}^{\infty} (-1)^k a_k - \sum_{k=0}^{N-1} (-1)^k a_k \right| < a_N = \frac{1}{(4N+1)(2N)!}$

so want $\frac{1}{(4N+1)(2N)!} < \frac{1}{100}$ ie $(4k+1)!(2N)! > 100$ check:

$N=2$ $9(4!) = 9 \cdot 24 = 216 > 100$ so $N=2$ works

So $\sum_{k=0}^{1} \frac{1}{(4k+1)(2k)!} = \frac{1}{1 \cdot 1} + \frac{1}{5 \cdot 2} = 1.1$ is within .01.

11. [26 Points] The graphs of the following polar curves are as shown:

$$C_1: \quad r = 2(1-\cos\theta); \quad 0 \le \theta \le 2\pi, \qquad C_2: \quad r = 2\cos\theta; \quad 0 \le \theta \le \pi.$$

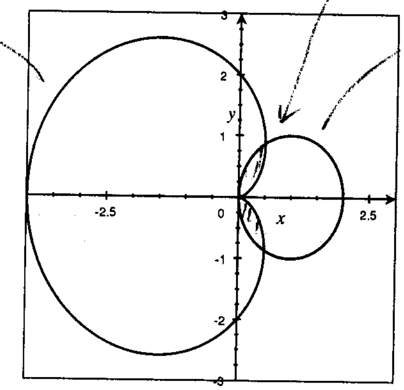
$2(1-\cos\theta) = 2\cos\theta$

$2 - 2\cos\theta \qquad 2 = 4\cos\theta$

$\cos\theta = 1/2 \to \theta = \frac{\pi}{3}$

$2(1-\cos\theta) = 0 \to \cos\theta = 1 \to \theta = 0$

$2\cos\theta = 0 \to \cos\theta = 0 \to \theta = \frac{\pi}{2}$



a) [10 Points] Find, but don't evaluate, an integral whose value is the area of the region **in the first quadrant** that lies inside C_1 **and** inside C_2.

symmetry → work in 1^{st} quadrant, multiply by 2

$$\text{Area} = 2\left(\int_0^{\frac{\pi}{3}} \frac{1}{2}(2(1-\cos\theta))^2\,d\theta + \int_{\frac{\pi}{3}}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta\right)$$

b) [8 Points] Find the slope of C_2 at the point that corresponds to $\theta = \frac{\pi}{6}$.

$y = 2\cos\theta\sin\theta \qquad x = 2\cos\theta\cos\theta = 2\cos^2\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos\theta\cos\theta + (-\sin\theta)\sin\theta)}{2(2\cos\theta(-\sin\theta))} = \frac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta}$$

At $\theta = \pi/6$

$$\frac{dy}{dx} = \frac{(1/2)^2 - (\sqrt{3}/2)^2}{2(1/2)(\sqrt{3}/2)} = \frac{\frac{1}{4} - \frac{3}{4}}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} \cdot \frac{-2}{4} = -\frac{1}{\sqrt{3}}$$

c) [8 Points] Find, but don't evaluate, an integral whose value is the arc length of C_1.

$0 \le \theta \le 2\pi$

$$\text{Arclength} = \int_0^{2\pi} \left((2(1-\cos\theta))^2 + ((2(1-\cos\theta))')^2\right)^{1/2} d\theta$$

$$= \int_0^{2\pi} \left[(2(1-\cos\theta))^2 + (2\sin\theta)^2\right]^{1/2} d\theta$$

12. [20 Points] Consider the vectors $\overrightarrow{v} = \langle 2, -1, -2\rangle = 2\overrightarrow{i} - \overrightarrow{j} - 2\overrightarrow{k}$ and $\overrightarrow{w} = \langle 2, 2, -1\rangle = 2\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$.

a) [5 Points] Find the vector $2\overrightarrow{v} + \overrightarrow{w}$.

$$2\vec{v} + \vec{w} = 2\langle 2,-1,-2\rangle + \langle 2,2,-1\rangle$$
$$= \langle 4,-2,-4\rangle + \langle 2,2,-1\rangle = \boxed{\langle 6,0,-5\rangle}$$

b) [5 Points] Find the cosine of the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$.

$$\cos\theta = \frac{V\cdot W}{\|V\|\,\|W\|} = \frac{2\cdot 2+(-1)\cdot 2+(-2)\cdot(-1)}{(2^2+1^2+2^2)^{1/2}(2^2+2^2+1^2)^{1/2}} = \frac{4-2+2}{9^{1/2}\,9^{1/2}} = \boxed{\frac{4}{9}}$$

from above!

c) [5 Points] Find $\text{proj}_{\overrightarrow{w}}\overrightarrow{v}$, i.e., the vector projection of $\overrightarrow{v}$ onto $\overrightarrow{w}$.

$$\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{4}{9}\langle 2,2,-1\rangle = \left\langle \frac{8}{9}, \frac{8}{9}, \frac{-4}{9}\right\rangle$$

d) [5 Points] Does there exist a **unit** vector $\overrightarrow{u}$ that is parallel to the vector $\overrightarrow{w}$ and orthogonal to $\overrightarrow{v}$? If "yes", find it; and if "no" explain why not.

Is there a vector $c\vec{w} = \vec{u}$ with $\|\vec{u}\|=1$ and $\vec{u}\cdot\vec{v}=0$?

$(c\vec{w})\cdot\vec{v} = c(\vec{w}\cdot\vec{v}) = c\cdot 4 = 4c = 0$ only if $c=0$

so $\|\vec{u}\| = |0|\,\|\vec{w}\| = 0$, so no.