Math 971 Algebraic Topology

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The main tool we will use turns a family of short exact sequences of chain maps between three chain complexes into a single *long exact homology sequence.* Given chain complexes $\mathcal{A} = (A_n, \partial)$, $\mathcal{B} = (B_n, \partial')$, and $\mathcal{C} = (C_n, \partial'')$ and short exact sequences of chain maps (i.e., $\partial' i_n = i_n \partial$, $\partial'' j_n = j_n \partial'$)
$0 \to A_n \overset{i_n}{\to} B_n \overset{j_n}{\to} C_n \to 0$ there is a general result which provides us with a long exact sequence

$$\cdots \overset{\partial}{\to} H_n(\mathcal{A}) \overset{i_*}{\to} H_n(\mathcal{B}) \overset{j_*}{\to} H_n(\mathcal{C}) \overset{\partial}{\to} H_{n-1}(\mathcal{A}) \overset{i_*}{\to} \cdots$$

Most of the work is in defining the "boundary" map ∂. Given an element $[z] \in H_n(\mathcal{C})$, a representative $z \in C_n$ satisfies $\partial''(z) = 0$. But j_n is onto, so there is a $b \in B_n$ with $j_n(b) = z$, Then $i_{n-1}\partial'(b) = \partial'' j_n(b) = 0$, so $\partial'(b) \in \ker(j_{n-1} = \mathrm{im}(a_{n-1})$. So there is an $a \in A_{n-1}$ with $i_{n-1}(a) = \partial'(b)$. But then $i_{n-2}\partial(a) = \partial' i_{n-1}(a) = \partial'\partial'(b) = 0$, so, since i_{n-2} is injective, $\partial a = 0$, so $a \in Z_{n-1}(\mathcal{A})$, and so represents a homology class $[a] \in H_n(\mathcal{A})$. We define $\partial([z]) = [a]$.
To show that this is well-defined, we need to show that the class $[a]$ we end up with is independent of the choices made along the way. The choice of a was not really a choice; i_{n-1} is, by assumption, injective. For b, if $j_n(b) = z = j_n(b')$, then $j_n(b - b') = 0$, so $b - b' = i_n(w)$ for some $w \in A_n$. Then $\partial' b' = \partial' b - \partial' i_n(w) = \partial' b - i_{n-1}\partial(w)$, so choosing $a' = a - \partial(w)$ we have $i_{n-1}(a') = \partial'(b')$. But then $[a'] = [a - \partial w] = [a] - [delw] = [a]$. Finally, there is actually a choice of z ; if $[z] = [z']$, then $z' = z + \partial'' w$ for some $w \in C_{n+1}$; but then choosing b', w' with $j_n(b') = z'$, $j_{n+1}(w') = w$, we have
$\partial'' w = \partial'' j_{n+1}(w') = j_n \partial'(w')$, so
$z' = z + \partial'' w = j_n(b + \partial' w')$, so we may choose $b' = b + \partial' w'$ (since the result is independent of this choice!), then since $\partial' b' = \partial' b$ everything continues the same.

Now to exactness! We need to show three (types of) equalities, which means six containments. Three (image contained in kernel) are shown basically by showing that compositions of two consecutive homomorphisms are trivial. $j_n i_n = 0$ immediately implies $j_* i_* = 0$. From the definition of ∂, $i_*\partial[z] = [i_n(a)] = [\partial'(b)] = 0$, and $\partial j_*[z] = \partial[j_n(z)] = [a]$, where $i_{n-1}(a) = \partial'(z) = 0$, so $a = 0$ (since i_{n-1} is injective), so $[a] = 0$.

For the opposite containments, if $j_*[z] = [j_n(z)] = 0$, then $j_n(z) = \partial'' w$ for some w. Since j_{n+1} is onto, $w = j_{n+1}(b)$ for some b. Then $j_n(z - \partial' b) = \partial'' w - \partial'' j_{n+1} b = 0$, so $z = \partial' b = i_n(a)$ for some a, so $i_*[a] = [z - \partial' b] = [z]$. So $\ker j_* \subseteq \mathrm{im} i_*$. If $i_*[z] = 0$, then $i_n(z) = \partial' w$ for some $w \in B_{n+1}$. Setting $c = j_{n+1}(w)$, then $\partial'' c = j_n \partial' w - i_n i_n(Z) = 0$, so $[c] \in h_{n+1}(\mathcal{C})$, and computing $\partial[c]$ we find that we can choose w for the first step and z for the second step, so $\partial[c] = [z]$. So $\ker j_n \subseteq \mathrm{im}\partial$. Finally, if $\partial[z] = 0$, then $z = j_n(b)$ for some b, and $\partial' b = i_{n-1}(a)$ with $[a] = 0$, i.e., $a = \partial w$ for some w. So $\partial' b = i_{n-1}\partial w = \partial' i_n w$ But then $\partial'(b - i_n w) = 0$, and $j_n(b - i_n w) = z - 0 = z$, so $z \in \mathrm{im}(j_n)$, so $[z] \in \mathrm{im}(j_*)$. So $\ker \partial \subseteq \mathrm{im}(j_n)$. Which finishes the proof!

Now all we need are some new chain complexes. To start, we build the singular chain complex of a pair (X, A) , i.e., of a space X and a subspace $A \subseteq X$. Since as abelian groups we can think of $C_n(A)$ as a subgroup of $C_n(X)$ (under the injective homomorphism induced by the inclusion $i : A \to X$) we can set $C_n(X, A) = C_n(X)/C_n(A)$. Since the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ satisfies $\partial_n(C_n(A) \subseteq C_{n-1}(A)$ (the boundary of a map into A maps into A), we get an induced boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. These groups and maps $(C_n(X, A), \partial_n)$ form a chain complex, whose homology groups are the *singluar relative homology groups of the pair* (X, A) . To be a cycle in relative homology, you need to have a representative z with $\partial z \in C_{n-1}(A)$, i.e., you are a chain with boundary in A. To be a boundary, you need $z = \partial w + a$ for some $w \in C_{n+1}(X)$ and $a \in C_n(A)$, i.e., you *cobound* a chain in A ($\partial w = z - a$). Note that the relative homology of the pair $(X, \emptyset)$ is just the ordinary homology of X; we aren't modding out by anything.

There is a reduced relative homology as well, since we can augment with the same map (1-simplices always have 2 ends!), but in this case it has (essentially) no effect; $\widetilde{H}_i(X, A) \cong H_i(X, A)$ for all i <u>unless</u> $A = \emptyset$, in which case we lose the $\mathbb{Z}$ in dimension 0 that we expect to.

The inclusion i_n and projection p_n maps give us short exact sequences $0 \to C_n(A) \to C_n(X) \to$

$C_n(X,A) \to 0$, and since the boundary on chains in X restricts to the boundary on A and induces the boundary on (X,A), i_n and p_n are chain maps. So we get a long exact homology sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

There is also a long exact sequence of a triple (X,A,B) , where by triple we mean $B \subseteq A \subseteq X$. From the short exact sequences $0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$ (i.e., $0 \to C_n(A)/C_n(B) \to C_n(X)/C_n(B) \to C_n(X)/C_n(A) \to 0$) we get the long exact sequence

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to H_{n-1}(X,B) \to \cdots$$

From these humble beginnings we can do some meaningful calculations! First note that if X is contractible then $\widetilde{H}_i(X) = 0$ for every i.