**Math 325, Section 1**

**Exam 1 Solutions**

1. (20 pts.) What must be true of a (non-empty) set $S \subseteq \mathbb{R}$ if $\inf(S) = \sup(S)$ ?

$\inf(S)$ is a lower bound for $S$, so for every $x \in S$ we have $\inf(S) \leq x$ . $\sup(S)$ is an upper bound for $S$, so for every $x \in S$ we have $x \leq \sup(S)$.

But because $\inf(S) = \sup(S) = M$, this means that for every $x \in S$ we have $M \leq x \leq M$, so $x = M$ (since otherwise either $x < M$, a contradiction, or $x > M$, a contradiction!). So $x \in S$ implies that $x = M$, which means that $M$ is the only element of the set $S$; that is, $S = \{M\}$ ! So $S$ contains exactly one real number, namely $M$.

2. (25 pts.) Show, using the Rational Roots Theorem, that $\alpha = \sqrt{2+\sqrt{7}}$ is **not** a rational number.

There are (at least) two ways to show this. Via the Rat'l Roots Thm, we find a polynomial having $\alpha$ as a root:

$\alpha^2 = 2+\sqrt{7}$, so $\alpha^2 - 2 = \sqrt{7}$, so $(\alpha^2-2)^2 = 7$, so

$$(\alpha^2-2)^2 - 7 = \alpha^4 - 4\alpha^2 + 4 - 7 = \alpha^4 - 4\alpha^2 - 3 = 0.$$

So $\alpha$ is a root of the polynomial $p(x) = x^4 - 4x^2 - 3$. But the Rat'l Roots Thm. tells us that the only possible rational roots of this polynomial are $1, -1, 3$, and/or $-3$. But we can either plug all of these into $p$ and note that none of them are roots of $p$ (this is probably the preferred way?), or we can be a little sneakier. Note that $\alpha^2 = 2+\sqrt{7} > 2+\sqrt{4} = 2+2 = 4$, so $\alpha > 2$, but $\alpha^2 = 2+\sqrt{7} \leq 2+\sqrt{9} = 2+3 = 5 < 9$, so $\alpha < 3$. So $\alpha$ cannot be equal to any of these possible roots. In either case we then know that $\alpha$, which is a root of $p$, cannot be equal to any of the possible rational roots of $p$, so $\alpha$ cannot be rational!

Alternate proof: suppose $\alpha = p/q$ is rational. Then $\alpha^2 = p^2/q^2$ is also rational, so $\alpha^2 - 2 = (p^2-2q^2)/q^2$ is rational. But! by the work above, $\alpha^2 - 2 = \sqrt{7} = \beta$ is then rational. But $\beta$ is a root of $r(x) = x^2 - 7$, whose only possible rational roots, $1, -1, 7, -7$, aren't roots! So $\beta$ isn't rational. But if $\alpha$ is rational so is $\beta$ ! So $\alpha$ cannot be rational.

3. (30 pts.) We will define a sequence $(a_n)_{n=1}^{\infty}$ by setting $a_1 = 2$, and for $n \geq 1$ (inductively) setting

$$a_{n+1} = 3 + \sqrt{2a_n} \ .$$

Show that this sequence is both monotonically increasing and bounded from above (so the sequence converges).

$a_2 = 3+\sqrt{2\cdot 2} = 3+\sqrt{4} = 3+2 = 5 \geq 2 = a_1$, so $a_2 \geq a_1$, which gets us started on an induction. If we now suppose (as our inductive hypothesis) that $a_{n+1} \geq a_n$, then $2a_{n+1} \geq 2a_n$ (since $2a_{n+1} - 2a_n = 2(a_{n+1}-a_n)$ is the product of a positive number (2) and a non-negative one). But then $\sqrt{2a_{n+1}} \geq \sqrt{2a_n}$, from a result in class, and so $a_{n+2} = 3+\sqrt{2a_{n+1}} \geq 3+\sqrt{2a_n} = a_{n+1}$.

So $a_{n+1} \geq a_n$ implies that $a_{n+2} \geq a_{n+1}$, giving our inductive step. So $a_{n+1} \geq a_n$ for every $n \geq 1$, by induction.

To show that the sequence is bounded, we could just pick an impossibly large number and give it a try. Or we could use techniques like we have before to find out when $M = 3+\sqrt{2M}$, and use that. Or we could note that the thing which controls the size of $a_{n+1}$ is $\sqrt{2a_n}$, which for $a_n$ "large" is a lot smaller than $a_n$, for example, $a_n = 50$ gives $a_{n+1} = 3 + \sqrt{100} = 13$, which is a lot smaller than 50.

So let's pick $M = 50$, say, and show that $a_n \leq 50$ for every $n$, by induction! $a_1 = 2 \leq 50$ is true, so our base case works. Then if $a_n \leq 50$, then $2a - n \leq 100$¡ so $\sqrt{2a_n} \leq \sqrt{100} = 10$, so $a_{n+1} = 3 + \sqrt{2a_n} \leq 3 + 10 = 13 \leq 50$. This is our inductive step; $a_n \leq 50$ implies that $a_{n+1} \leq 50$. So $a_n \leq 50$ for all $n \geq 1$, by induction; so the sequence is bounded above.

Because it is a monotone increasing sequence which is bounded above, it then follows that the sequence converges.

[N.B.: We can, in fact, find the limit of the sequence; as with examples from class our limit properties allow us to conclude that the limit, $L$, satisfies $L = 3+\sqrt{2L}$, so $(L-3)^2 - 2L = L^2 - 8L + 9 = 0$. Using the quadratic formula, we conclude that

$$L = (8 \pm \sqrt{64 - 36})/2 = (8 \pm 2\sqrt{7})/2 = 4 \pm \sqrt{7}.$$

Since $L \geq a_2 = 5$ (since $a_n \geq a_2$ for every $n \geq 2$) and $4 - \sqrt{7} \leq 4 - \sqrt{4} = 4 - 2 = 2$, we conclude that $L = 4 + \sqrt{7}$.]

4. (25 pts.) Given sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, show that if the sequences

$$c_n = a_n + b_n \qquad \text{and} \qquad d_n = a_n - b_n$$

both converge, then the sequences $a_n$ and $b_n$ also both converge!

Since $c_n$ and $d_n$ both converge, we know that $c_n + d_n = (a_n + b_n) + (a_n - b_n) = 2a_n$ also converges. So $a_n = (1/2)(2a_n)$ also converges!

But then $a_n$ and $c_n = a_n + b_n$ converge, and so $c_n - a_n = (a_n + b_n) - a_n = b_n$ must converge, as well. So both $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$must be convergent sequences.

A somewhat different way to write the same thing is:

If $c_n = a_n + b_n \to L$ and $d_n = a_n - b_n \to M$, then $c_n + d_n = 2a_n \to L + M$, so $a_n = (1/2)(2a_n) \to (1/2)(L + M)$. In particular $a_n$ has a limit, so it converges! Then $b_n = (a_n + b_n) - a_n \to L - (1/2)(L + M) = (1/2)(L - M)$, so $b_n$ has a limit, so $b_n$ converges!

[There are several other, roughly equivalent, ways to see how to build $a_n$ and $b_n$ out of $c_n$ and $d_n$, leading to the same conclusions.]