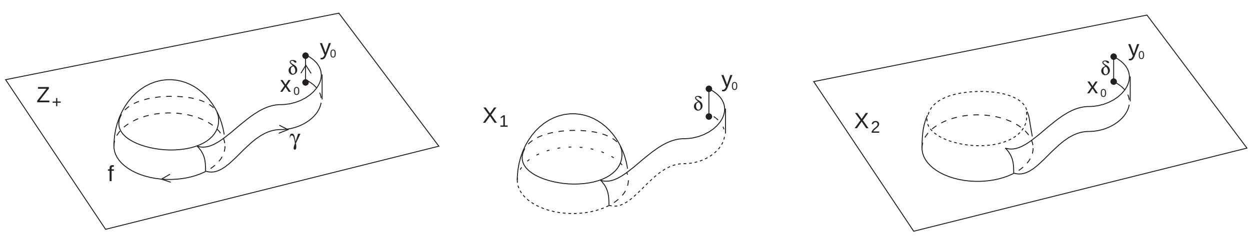
Math 971 Algebraic Topology

February 3, 2005

Gluing on a 2-disk: If X is a topological space and $f : \partial\mathbb{D}^2 \to X$ is continuous, then we can construct the quotient space $Z = (X \coprod \mathbb{D}^2)/\{x \sim f(x) : x \in \partial\mathbb{D}^2\}$, the result of gluing $\mathbb{D}^2$ to X along f. We can use Seifert - van Kampen to compute π_1 of the resulting space, although if we wish to be careful with basepoints x_0 (e.g., the image of f might not contain x_0, and/or we may wish to glue several disks on, in remote parts of X), we should also include a rectangle R, the mapping cylinder of a path γ running from $f(1,0)$ to x_0, glued to $\mathbb{D}^2$ along the arc from $(1/2, 0)$ to $(1,0)$ (see figure). This space Z_+ deformation retracts to Z, but it is technically simpler to do our calculations with the basepoint y_0 lying above x_0. If we write $D_1 = \{x \in \mathbb{D}^2 : ||x|| < 1\} \cup (R \setminus X)$ and $D_2 = \{x \in \mathbb{D}^2 : ||x|| > 1/3\} \cup R$, then we can write $Z_+ = D_+ \cup (X \cup D_2) = X_1 \cup X_2$. But since $X_1 \simeq *$, $X_2 \simeq X$ (it is essentially the mapping cylinder of the maps f and γ) and $X_1 \cap X_2 = \{x \in \mathbb{D}^2 : 1/3 < ||x|| < 1\} \cap (R \setminus X) \sim S^1$, we find that

$$\pi_1(Z, y_0) \cong \pi_1(X_2, y_0) *_{\mathbb{Z}} \{1\} = \pi_1(X_2)/ < \mathbb{Z} >^N \cong \pi_1(X_2)/ < [\overline{\delta} * \overline{\gamma} * f * \gamma * \delta] >^N$$

If we then use δ as a path for a change of basepoint isomorphism, and then a homotopy equivalence from X_2 to X (fixing x_0), we have, in terms of group presentations, if $\pi_1(X, x_0) =< \Sigma | R >$, then $\pi_1(Z) =< \Sigma | R \cup \{[\overline{\gamma} * f * \gamma]\} >$. So the effect of gluing on a 2-disk on the fundamental group is to add a new relator, namely the word represented by the attaching map (adjusting for basepoint). All of this applies equally well to attaching several 2-disks; each adds a new relator.



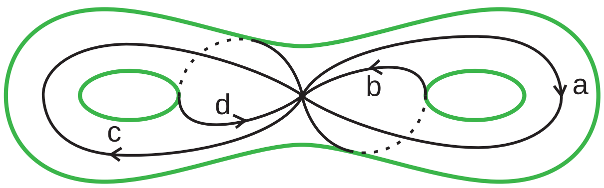
The inherent complications above derived from needing open sets can be legislated away, by introducing additional hypotheses:

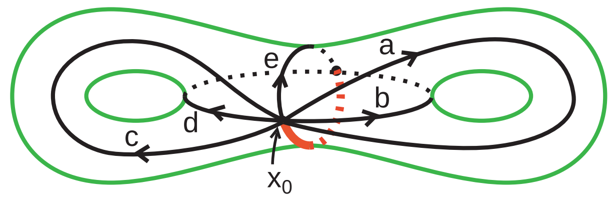
Theorem: If $X = X_1 \cup X_2$ is a union of closed sets X_1, X_2, with $A = X_1 \cap X_2$ path-connected, and if X_1, X_2 have open neighborhood $\mathcal{U}_1, \mathcal{U}_2$ so that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2$ deformation retract onto X_1, X_2, A respectively, then $\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(A)} \pi_1(X_2)$ as before.

The hypotheses are satisfied, for example, if $X_1.X_2$ are subcomplexes of the cell complex X.

This in turn opens up huge possibilities for the computation of $\pi_1(X)$. For example, for cell complexes, we can inductively compute π_1 by starting with the 1-skeleton, with free fundamental group, and attaching the 2-cells one by one, which each add a relator to the presentation of $\pi_1(X)$. [**Exercise:** (Hatcher, p.53, # 6) Attaching n-cells, for $n \geq 3$, has no effect on π_1.] For example, the 2-sphere S^2 can be thought of as a 2-disk with a 2-disk attached, along a circle, and so has $\pi_1(S^2) \cong \{1\}_{\mathbb{Z}}\{1\} = \{1\}$. We can also compute the fundamental group of any compact surface:

The *real projective plane* $\mathbb{R}P^2$ is the quotient of the 2-sphere S^2 by the antipodal map $x \mapsto -x$; it can also be thought of as the upper hemisphere, with identification only along the boundary. This in turn can be interpreted as a 2-disk glued to a circle, whose boundary wraps around the circle twice. So $\pi_1(\mathbb{R}P^2) \cong < a | a^2 > \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. A surface F of genus 2 can be given a cell structure with 1 0-cell, 4 1-cells, and 1 2-cell, as in the figure, as in the first of the figures below. The fundamental group of the 1-skeleton is therefore free of rank 4, and $\pi_1(F)$ has a presentation with 4 generators and 1 relator. Reading the attaching map from the figure, the presentation is $< a, b, c, d | [a, b][c, d] >$.





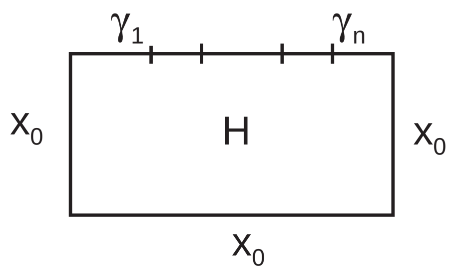
Giving it a different cell structure, as in the second figure, with 2 0-cells, 6 1-cels, and 2 2-cells, after choosing a maximal tree, we can read off the two relators from the 2-cells to arrive at a different presentation $\pi_1(F) =< a, b, c, d, e|aba^{-1}eb^{-1}, cde^{-1}c^{-1}d^{-1} >$. A posteriori, these two presentations describe isomorphic groups.

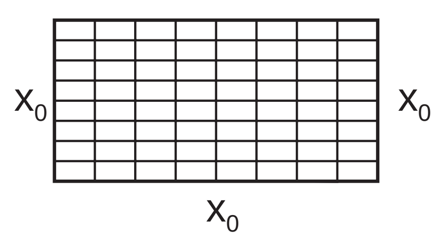
Using the same technology, we can also see that, in general, any group is the fundamental group of some 2-complex X; starting with a presentation $G =< \Sigma|R >$, build X by starting with a bouquet of $|\Sigma|$ circles, and attach $|R|$ 2-disks along loops which represent each of the generators of R. (This works just as well for infinite sets Σ and/or R; essentially the same proofs as above apply.)

Understanding that darn kernel.

We now turn our attention to proving Seifert - van Kampen; understanding the kernel of the map $\phi : \pi_1(X_1) * \pi_1(X_2) \to \pi_1(X)$, under the hypotheses that X_1, X_2 are open, $A = X_1 \cap X_2$ is path-connected, and the basepoint $x_0 \in A$. So we start with a product $g = g_1 \cdots g_n$ of loops alternately in X_1 and X_2, which when thought of in X is null-homotopic. We wish to show that g can be expressed as a product of conjugates of elements of the form $i_{1*}(a)(i_{2*}(a))^{-1}$ (and their inverses). The basic idea is that a "big" homotopy can be viewed as a large number of "little" homotopies, which we essentially deal with one at a time, and we find out how little "little" is by using the same Lebesgue number agument that we used before.

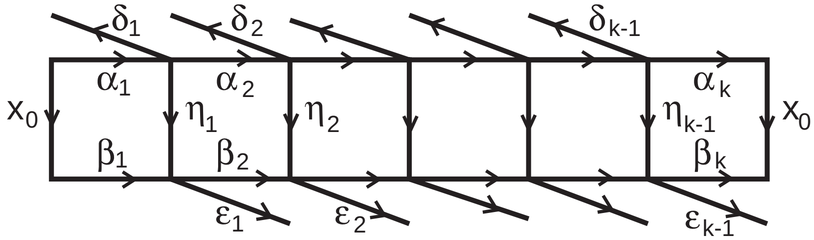
Specifically, if H is the homotopy, rel basepoint, from $\gamma_1 * \cdots * \gamma_n$, where γ_i is a based loop representing g_i, and the constant loop, then, as before, $\{H^{-1}(X_1), H^{-1}(X_2)\}$ is an open cover of $I \times I$, and so has a Lebesgue number ϵ. If we cut $I \times I$ into subsquares, with length $1/N$ on a side, where $1/N < \epsilon$, then each subsquare maps into either X_1 or X_2. The idea is to think of this as a collection of horizontal strips, each cut into squares. Arguing by induction, starting from the bottom (where our conclusion will be obvious), we will argue that if the bottom of the strip can be expressed as an element of the group
$N =< i_{1*}(\gamma)(i_{2*}(\gamma))^{-1} : \gamma \in \pi_1(A) >^N \subseteq \pi_1(X_1) * \pi_1(X_2)$
(i.e., as a product of conjugates of such loops), then so can the top of the strip.





And to do <u>this</u>, we work as before. We have a strip of squares, each mapping into either X_1 or X_2. If adjacent squares map into the same subpace, amalgamate them into a single larger rectangle. Continuing in this way, we can break the strip into subrectangles which alternately map into X_1 or X_2. This means that the vertical arcs in between map into $X_1 \cap X_2 = A$, and represent paths η_i in A. Their endpoints also map into A, and so can be joined by paths (δ_i on the top, ϵ_i on the bottom) in A to the basepoint. The top of the strip is homotopic, rel basepoint, to
$(\alpha_1 * \delta_1) * (\overline{\delta_1} * \alpha_2 * \delta_2) * \cdots * (\overline{\delta_{k-1}} * \alpha_k)$
each grouping mapping into either X_1 or X_2. The rectangles demonstrate that each grouping is homotopic, rel basepoint, to the product of loops

$(\overline{\delta_i} * \eta_i * \epsilon_i) * (\overline{\epsilon_i} * \beta_i * \epsilon_{i+1}) * (\overline{\epsilon_{i+1}} * \overline{\eta_{i+1}} * \delta_{i+1}) = a_i b_i a_{i+1}^{-1}$
where this is thought of as a product in either $\pi(X_1)$ or $\pi_1(X_2)$. The point is that when strung together, this appears to give $(b_1 a_2^{-1})(a_2 b_2 a_3^{-1}) \cdots (a_k b_k)$, with lots of cancellation, but in reality, the terms $a_i^{-1} a_i$ represent elements of N, since the two "cancelling" factors are thought of as living in the different groups $\pi_1(X_1), \pi_1(X_2)$. The remaining terms, if we delete these "cancelling" pairs, is $b_1 \cdots b_k = \beta_1 * \epsilon_1 * \cdots * \overline{\epsilon_i} * \beta_i * \epsilon_{i+1} * \cdots * \overline{\epsilon_k} * \beta_k$, which is homotopic rel endpoints to $\beta_1 * \cdots * \beta_k$, which, by induction, can be represented as a product which lies in N.



So, we can obtain the element represented by the top of the strip by inserting elements of N into the bottom, which is a word having a representation as an element of N. The final problem to overcome is that the insertions represented by the vertical arcs might not be occuring where we want them to be! But this doesn't matter; inserting a word w in the middle of another uv (to get uwv) is the same as multiplying uv by a conjugate of w; $uwv = (uv)(v^{-1}wv)$, so since the bottom of the strip is in N, and we obtain the top of the strip by inserting elements of N into the bottom, the top is represented by a product of conjugates of elements of N, so (since N is normal) is in N. And a <u>final</u> final point; the subrectangles may not have cut the bottom of the strip up into the same pieces that the inductive hypothesis used to express the bottom as an element of N. It didn't even cut it into loops; we added paths at the break points to make that happen. The inductive hypothesis would have, in fact, added its own extra paths, at possibly different points! But if we add <u>both</u> sets of paths, and cut the loop up into even more pieces, then we end up with a loop, which we have expressed as a product in $\pi_1(X_1) * \pi_1(X_2)$ in two (possibly different) ways, since the two points of view will have interpreted pieces as living in different subspaces. But when this happens, it must be because the subloop really lives in $X_1 \cap X_2 = A$. Moving from one to the other amounts to repeatedly changing ownership between the two sets, which in $\pi_1(X_1) * \pi_1(X_2)$ means <u>inserting</u> an element of N into the product (that is literally what elements of N do). But as before, these insertions can be collected at one end as products of conjugates. So if one of the elements is in N, the other one is, too.

Which completes the proof!

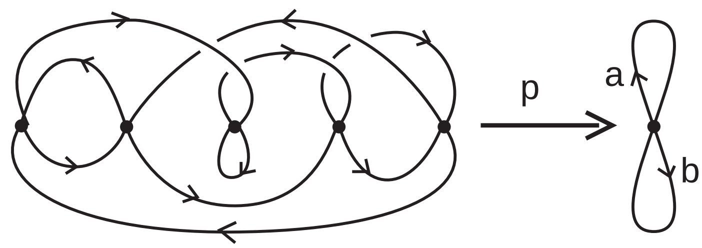
Postscript: why should we care? The role of the fundamental group in distinguishing spaces has already been touched upon; if two (path-connected) spaces have non-isomorphic fundamental groups, then the spaces are not homeomorphic, and even not homotopy equivalent. It is one of the most basic, and in many cases the best such invariant we have in our arsenal (hence the name "fundamental"). As we have seen with the circle, it captures the notion of how many times a loop "winds around" in a space. And the idea of using paths to understand a space is very basic; we explore a space by mapping familiar objects into it. (This is a theme we keep returning to in this course.) The concepts we have introduced play a role in analysis, for instance with the notion of a path integral; the invariance of the integral under homotopies rel endpoints is an important property, related to Green's Theorem and (locally) conservative vector fields. And the <u>space</u> of all paths in X plays an important (theoretical, although pprobably not practical) role in what we will do next.

Covering spaces: We can motivate our next topic by looking more closely at one of our examples above. The projective plane $\mathbb{R}P^2$ has $\pi_1 = \mathbb{Z}_2$. It is also the quotient of the simply-connected space S^2 by the antipodal map, which, together with the identity map, forms a group of homeomorphisms of S^2

which is isomorphic to $\mathbb{Z}_2$. The fact that $\mathbb{Z}_2$ has this dual role to play in describing $\mathbb{R}P^2$ is no accident; codifying this relationship requires the notion of a covering space.

The quotient map $q : S^2 \to \mathbb{R}P^2$ is an example of a *covering map*. A map $p : E \to B$ is called a covering map if for every point $x \in B$, there is a neighborhood $\mathcal{U}$ of x (an *evenly covered neighborhood*) so that $p^{-1}(\mathcal{U})$ is a disjoint union $\mathcal{U}_\alpha$ of open sets in E, each mapped homeomorphically onto $\mathcal{U}$ by (the restriction of) p . B is called the *base space* of the covering; E is called the *total space*. The quotient map q is an example; (the image of) the complement of a great circle in S^2 will be an evenly covered neighborhood of any point it contains. The disjoint union of 43 copies of a space, each mapping homeomorphically to a single copy, is an example of a *trivial covering*. As a last example, we have the famous exponential map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$. The image of any interval (a, b) of length less than 1 will have inverse image the disjoint union of the intervals $(a + n, b + n)$ for $n \in \mathbb{Z}$.

OK, maybe not the last. We can build many finite-sheeted (every point inverse is finite) coverings of a bouquet of two circles, say, by assembling n points over the vertex, and then, on either side, connecting the points by n (oriented) arcs, one each going in and out of each vertex. By choosing orientations on each 1-cell of the bouquet, we can build a covering map by sending the vertices above to the vertex, and the arcs to the one cells, homeomorphically, respecting the orientations. We can build infinite-sheeted coverings in much the same way.



Covering spaces of a (suitably nice) space X have a very close relationship to $\pi_1(X, x_0)$. The basis for this relationship is the

Homotopy Lifting Property: If $p : \widetilde{X} \to X$ is a covering map, $H : Y \times I \to X$ is a homotopy, $H(y, 0) = f(y)$, and $\widetilde{f} : Y \to \widetilde{X}$ is a *lift* of f (i.e., $p \circ \widetilde{f} = f$), then there is a unique lift $\widetilde{H}$ of H with $\widetilde{H}(y, 0) = \widetilde{f}(y)$.

The **proof** of this property follows a pattern that we will become very familiar with: we lift maps a little bit at a time. For every $x \in X$ there is an open set $\mathcal{U}_x$ evenly covered by p . For each fixed $y \in Y$, since I is compact and the sets $H^{-1}(\mathcal{U}_x)$ form an open cover of $Y \times I$, then since I is compact, the Tube Lemma provides an open neighborhood $\mathcal{V}$ of y in Y and finitely many $p^{-1}\mathcal{U}_x$ whose union covers $\mathcal{V} \times I$.

To define $\widetilde{H}(y, t)$, we (using a Lebesgue number argument) cut the interval $\{y\} \times I$ into finitely many pieces, the ith mapping into $\mathcal{U}_{x_i}$ under H. $\widetilde{f}(y)$ is in one of the evenly covered sets $\mathcal{U}_{x_1\alpha_1}$, and the restricted map $p^{-1} : \mathcal{U}_{x_1} \to \mathcal{U}_{x_1\alpha_1}$ following H restricted to the first interval lifts H along the first interval to a map we will call $\widetilde{H}$. We then have lifted H at the end of the first interval = the beginning of the second, and we continue as before. In this way we can define $\widetilde{H}$ for all (y, t) . To show that this is independent of the choices we have made along the way, we imagine two ways of cutting up the interval $\{y\} \times I$ using evenly covered neighborhoods $\mathcal{U}_{x_i}$ and $\mathcal{V}_{w_j}$, and take intersections of both sets of intervals to get a common refinement of both sets, covered by the intersections $\mathcal{U}_{x_i} \cap \mathcal{V}_{w_j}$, and imagine building $\widetilde{H}$ using the refinement. At the start, at $\widetilde{f}(y)$, we are in $\mathcal{U}_{x_1\alpha_1} \cap \mathcal{V}_{w_1\beta_1}$. Because at the start of the lift $(y, 0)$ we lift to the same point, and p^{-1} restricted to this intersection agrees with p^{-1} restricted to each of the two pieces, we get the same lift acroos the first refined subinterval. This process repeats itself across all of the subintervals, showing that the lift is independent of the choices made. This also shows that the lift is unique; once we have decided what $\widetilde{H}(y, 0)$, the rest of the values of the $\widetilde{H}$ are determined by the requirement of being a lift. also, once we know the map is well-defined, we can see that it is continuous, since for any y, we can make the same choices across the entire open set V given by the Tube Lemma, and find that $\widetilde{H}$, restricted to $\mathcal{V} \times (a_i - \delta, b_i + \delta)$ (for a small delta; we could wiggle the endpoints in the construction without changing the resulting function, by its well-definedness) is H estricted to this set followed by p^{-1} restriced in domain and range, so this composition is continuous. So $\widetilde{H}$ is locally continuous, hence continuous.

In particular, applying this in the case $Y = \{*\}$, where a homotopy $H : \{*\} \times I \to X$ is generally thought of as a path $\gamma : I \to X$, we have the **Path Lifting Property**: "given a covering map $p : \widetilde{X} \to X$, a path $\gamma : I \to X$ with $\gamma(0) = x_0$, and a point $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{\gamma}$ lifting γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$." One of the immediate consequences of this is one of the cornerstones of covering space theory:

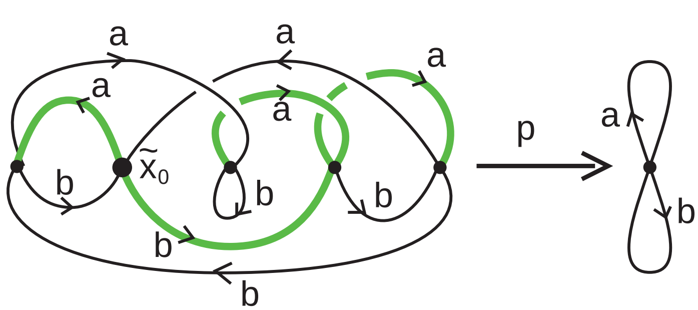
If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof: Suppose $\gamma : (I, \partial I) \to (\widetilde{X}, \widetilde{x}_0)$ is a loop $p_*([\gamma]) = 1$ in $\pi_1(X, x_0)$. So there is a homotopy $H : (I \times I, \partial I \times I) \to (X, x_0)$ between $p \circ \gamma$ and the constant path. By homotopy lifting, there is a homotopy $\widetilde{H}$ from γ to the lift of the constant map at x_0. The vertical sides $s \mapsto \widetilde{H}(0, s), \widetilde{H}(1, s)$ are also lifts of the constant map, beginning from $\widetilde{H}(0, 0), \widetilde{H}(1, 0) = \gamma(0) = \gamma(1) = \widetilde{x}_0$, so are the constant map at $\widetilde{x}_0$. Consequently, the lift at the bottom is the constant map at $\widetilde{x}_0$. So $\widetilde{H}$ represents a null-homotopy of γ, so $[\gamma] = 1$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. So $\pi_1(\widetilde{X}, \widetilde{x}_0) = \{1\}$.

Even more, the image $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$ is precisely the elements whose representatives are loops at x_0, which when lifted to paths starting at $\widetilde{x}_0$), are loops. For if γ lifts to a loop $\widetilde{\gamma}$, then $p \circ \widetilde{\gamma} = \gamma$, so $p_*([\widetilde{\gamma}]) = [\gamma]$. Conversely, if $p_*([\widetilde{\gamma}]) = [\gamma]$, then γ and $p \circ \widetilde{\gamma}$ are homotopic rel endpoints, and so the homotopy lifts to a homotopy rel endpoints between the lift of γ at $\widetilde{x}_0$, and the lift of $p \circ \widetilde{\gamma}$ at $\widetilde{x}_0$ (which is $\widetilde{\gamma}$, since $\widetilde{\gamma}(0) = \widetilde{x}_0$ and lifts are unique). So the lift of γ is a loop, as desired.

So, for example, if we build a 5-sheeted cover of the bouquet of 2 circles, as above, (after choosin maximal tree upstairs) we can read off the images of the generators of the fundamental group of the total space; we have labelled each ede by the ereator it traces out downstairs, and for each ede outside of the maximal tree chosen, we read from basepoint out the tree to one end, across the edge, and then back to the basepoint in the tree. In our example, this gives:

$$< ab, aaab^{-1}, baba^{-1}, baa, ba^{-1}bab^{-1}, bba^{-1}b^{-1} | >$$



This is (from its construction) a copy of the free group on 6 letters, in the free group $F(a, b)$. In a similar way, by explicitly building a covering space, we find that the fundamental group of a closed surface of genus 3 is a subgroup of the fundamental group of the closed surface of genus 2.