Math 417 Problem Set 2 Solutions REDUX

(*) 12. Find the inverse of the element $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 2 \\ 0 & 5 & 1 \end{pmatrix}$ in $GL_3(\mathbb{Z}_7)$.

Apparently *I* found the inverse to a different matrix? Here is the right one!

We can find the inverse either by using a formula for the entries of the inverse of the 3×3 matrix (which involves the inverse of the determinant of A, computed mod 7), or by solving the (implied) system of linear equations, in the equation $A \cdot A^{-1} = I$ (again, solved mod 7), or we can use the shorthand for esssentially solving this system of equations, via the super-augmented matrix and row reduction. (Below we take the approach of <u>adding</u> a multiple of one row to another to make an entry equal to 0 mod 7, rather than subtracting to make it 0; many different routes work.)

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 2 & 2 & 0 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 7 & 2 & 14 & 4 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 4 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 16 & 4 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

[in the first reduction we multiplied the middle row by $2^{-1} = 4$]

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 7 & 1 & 4 & 8 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 17 & 4 & 4 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & 4 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 \end{array}\right)$$

and so $A^{-1} = \begin{pmatrix} 3 & 4 & 4 \\ 2 & 4 & 0 \\ 4 & 1 & 1 \end{pmatrix}$. And we can check this by direct computation!

$$\begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 2 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 4 \\ 2 & 4 & 0 \\ 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 7 & 7 \\ 21 & 22 & 14 \\ 14 & 21 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(again, the equalities hold modulo 7).