Math 325 Problem Set 2 Solutions

Problems were due Friday, January 27.

5. [Zorn, p.36, #3] For the statements (a)-(d) below, state both the converse and the contrapositive of the given statement, and indicate (no explanation needed) which of statement, converse, and contrapositive are <u>true</u>.

(a) If $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$. Converse: If $a + b \in \mathbb{Q}$, then $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$.

Contrapositive: If $a + b \notin \mathbb{Q}$, then it is not the case that $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$.

Equivalent: If $a + b \notin \mathbb{Q}$, then it <u>either</u> $a \notin \mathbb{Q}$ <u>or</u> $b \notin \mathbb{Q}$.

The statement and contrapositive are true; the converse is false.

(b) If $a \notin \mathbb{Q}$, then $\frac{1}{a} \notin \mathbb{Q}$. Converse: If $\frac{1}{a} \notin \mathbb{Q}$, then $a \notin \mathbb{Q}$.

Contrapositive: If $\frac{1}{a} \in \mathbb{Q}$, then $a \in \mathbb{Q}$. All of these statements are true!

(c) If $a \notin \mathbb{Q}$ and $b \notin \mathbb{Q}$, then $ab \notin \mathbb{Q}$. Converse: If $ab \notin \mathbb{Q}$, then $a \notin \mathbb{Q}$ and $b \notin \mathbb{Q}$.

Contrapositive: If $ab \in \mathbb{Q}$, then it is not the case that $a \notin \mathbb{Q}$ and $b \notin \mathbb{Q}$.

Equivalent: If $ab \in \mathbb{Q}$, then <u>either</u> $a \in \mathbb{Q}$ <u>or</u> $b \in \mathbb{Q}$. All of these statement are false!

(d) If $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

OK, how to handle this one isn't really clear to <u>me</u>...

Converse: If $\lim_{n\to\infty} a_n = 0$, then $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges.

Alternative: If $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, <u>then</u>: if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Contrapositive: If $\lim_{n\to\infty} a_n \neq 0$, then it is not the case that $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges.

Alternative: If $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, <u>then</u>: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge.

The statement and contrapositive are true; the converse is false.

6. Show, using the Rational Roots Theorem, that $\alpha = \sqrt{2+\sqrt{3}}$ is not rational.

[Find a polynomial with integer coefficients that has α as a root!]

Setting $\alpha = \sqrt{2+\sqrt{3}}$, then $\alpha^2 = 2 + \sqrt{3}$, so $\alpha^2 - 2 = \sqrt{3}$, so $(\alpha^2 - 2)^2 = 3$.

So $0 = (\alpha^2 - 2)^2 - 3 = (alpha^2)^2 - 2(\alpha^2)(2) + 2^2 - 3 = \alpha^4 - 4\alpha^2 + 1$.

So α is a root of the polynomial $f(x) = x^4 - 4x^2 + 1$. Since f has integer coefficients, the rational roots theorem tells us that if $r = p/q$ is a rational root for f, then p divides the constant coeffficient, 1, and q divides the leading coefficient, 1. So p is -1 or 1, and so is q, so $r = p/q$ is -1 or 1. So if α is rational, either $\alpha = -1$ or $\alpha = 1$.

But! $f(-1) = (-1)^4 - 4(-1)^2 + 1 = 1 - 4 + 1 = -2 \neq 0$, and $f(a) = 1^4 - 4(1)^2 + 1 = 1 - 4 + 1 = -2 \neq 0$. So none of the roots of f are rational (since the only possible values of a rational root are not roots!). So α, which is a root of f, cannot be rational. So α is not rational.

7. [Zorn, p.45, #6] By looking at the first few cases, find a (short) formula for the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} \ ;$$

then prove, using induction, that your formula is correct.

$\frac{1}{1\cdot 2} = \frac{1}{2}$, $\quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $\quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{8}{12} + \frac{1}{12} = \frac{3}{4}$, and $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{3}{4} + \frac{1}{20} = \frac{15}{20} + \frac{1}{20} = \frac{4}{5}$.

This leads us to suspect that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

And we can prove this, by induction! If $n = 1$, then $\displaystyle\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1}$ which establishes the base case.

Then if we suppose that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$, then

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = (\sum_{k=1}^{n} \frac{1}{k(k+1)}) + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2},$$

so $\displaystyle\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{n+2} = \frac{n+1}{(n+1)+1}$, establishing the inductive step.

So, by induction, for every $n \in \mathbb{N}$ we have $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

8. Show, by induction, that if $x > 0$ then $(1+x)^n \geq nx + 1$ for every $n \in \mathbb{N}$.

For the base case, if $n = 1$ we have $(1+x)^1 = 1+x = 1\cdot x+1 \geq 1\cdot x+1$, so $(1+x)^1 \geq 1\cdot x+1$.

Then if we suppose that $(1+x)^n \geq nx+1$, then since $x > 0$ we have $x+1 \geq 1$ (so multiplying by $1+x$ will not change the direction of an inequality), so $(1+x)^{n+1} = (1+x)^n(1+x) \geq (nx+1)(1+x) = nx+1+nx^2+x = (n+1)x+1+nx^2$. But since $x > 0$ we have $x^2 > 0$ and $n \in \mathbb{N}$ means $n \geq 1$, so $nx^2 > 0$, so $(1+x)^{n+1} \geq (n+1)x+1+nx^2 > (n+1)x+1+0 > (n+1)x+1$, so $(1+x)^{n+1} \geq (n+1)x+1$.

So we have found that if $(1+x)^n \geq nx+1$, then $(1+x)^{n+1} \geq (n+1)x+1$. This gives our inductive step.

Therefore, by induction, we find tht if $x > 0$, then for every $n \in \mathbb{N}$ we have $(1+x)^n \geq nx+1$.