## Math 423/823 Exercise Set 7 Solutions

25. Show that 'integration by parts' works with analytic functions: for any curve $\gamma(t)$, $a \leq t \leq b$, if $f, g, f'$ and $g'$ are all analytic along $\gamma$, then we have

$$\int_\gamma f(z)g'(z)\ dz = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))] - \int_\gamma f'(z)g(z)\ dz$$

[Hint: $F(z) = f(z)g(z)$ is the antiderivative of what (analytic) function?]
[Note: we will shortly be learning that the analyticity of $f'$ and $g'$ follow from that of $f$ and $g$, so the requirements on the derivatives are not, in the end, really neccesary...]

Since $f$ and $g$ are bboth analytic, their product $F(z) = f(z)g(z)$ is analytic, and, by the product rule, $F'(z) = f'(z)g(z) + f(z)g'(z)$ . Therefore,

$$\int_\gamma f'(z)g(z) + f(z)g'(z)\ dz = F(z)\Big|_{\gamma(a)}^{\gamma(b)} = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))]$$

But $\int_\gamma f'(z)g(z) + f(z)g'(z)\ dz = \int_\gamma f'(z)g(z)\ dz + \int_\gamma f(z)g'(z)\ dz$, so equating these two and rearranging terms, we have

$$\int_\gamma f(z)g'(z)\ dz = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))] - \int_\gamma f'(z)g(z)\ dz$$

as desired.

26. (Via the fundamental theorem of ('complex') calculus, )
Compute $\int_\gamma ze^{iz}\ dz$, where $\gamma$ is the (unit) circular arc running from $z = 1$ to $z = i$.
[Hint! Problem #25 will help...]
From the above, setting $f(z) = z$ and $g'(z) = e^{iz}$, so $f'(z) = 1$ and $g(z) = -ie^{iz}$, we have

$$\int_\gamma ze^{iz}\ dz = -ize^{iz}\Big|_1^i - \int_\gamma -ie^{iz}\ dz\ .$$

But $-e^{iz}$ is an antiderivative of $-ie^{iz}$, so

$$\int_\gamma ze^{iz}\ dz = (-ize^{iz}) - (-e^{iz})\Big|_1^i = (1 - iz)e^{iz}\Big|_1^i = (1 - i^2)e^{i^2} - (1 - i)e^i$$
$$= 2e^{-1} - (1 - i)(\cos(1) + i\sin(1) = [2e^{-1} - \cos(1) - \sin(1)] + i[\cos(1) - \sin(1)]\ .$$

27. [BC#4.49.7] Show that if $\gamma(t)$, $a \leq t \leq b$ is a simple closed curve traversed counterclockwise (so that the bounded region $R$ it encloses is always on the left), then

$$(**) = \frac{1}{2i}\int_C \overline{z}\ dz = \text{the area of the region } R.$$

[Hint: this is a "standard" consequence of Green's Theorem (from multivariate calculus), in disguise. Write $C(t) = x(t) + iy(t)$, and compute what the integral should be...note that the <u>real</u> part is an integral whose antiderivative we can write down!]

If we write this as an integral $dt$, writing $\gamma(t) = x(t) + iy(t)$, so $\gamma'(t) = x'(t) + iy'(t)$, we have

$(**) = \int_a^b (x(t) - iy(t))(x'(t) + iy'(t))\ dt = \int_a^b x(t)x'(t) - iy(t)x'(t) + ix(t)y'(t) - i^2 y(t)y'(t)\ dt = \int_a^b x(t)x'(t) + y(t)y'(t)\ dt + i\int_a^b x(t)y'(t) - y(t)x'(t)\ dt$

But $\int_a^b x(t)x'(t) + y(t)y'(t)\ dt = \frac{1}{2}([x(t)]^2 + [y(t)]^2\Big|_a^b$, which since $\gamma(a) = \gamma(b)$, is 0 (the two endpoints evaluate to the same (unknown) number).

On the other hand, $\int_a^b x(t)y'(t) - y(t)x'(t)\ dt = i\int_a^b (-y(t), x(t)) \cdot (x'(t), y'(t))\ dt$ is the line integral of the vector field $F(x, y) = (-y, x)$ around the closed curve $\gamma$. But by Green's Theorem, this is equal to the double integral

$$\iint_R \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\ dx\ dy = \iint_R (1) - (-1)\ dx\ dy = \iint_R 2\ dx\ dy = 2(\text{Area of R}).$$

Putting this all together, $(**) = \frac{1}{2i}[0 + i[2(\text{Area of R})]] = \text{Area of R}$, as desired.

28. Evaluate the following integrals:

(a): $\int_{\gamma_1} \frac{dz}{z^2+1}$, where $\gamma_1(t) = 1 + e^{2\pi ti}$, $0 \le t \le 1$

$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$ is analytic at every point of $\mathbb{C}$ except $z = i, -i$. But both $i$ and $-i$ lie outside of the simple closed curve $\gamma_1$ ; $\gamma_1$ describes the circle of radius 1 centered at $z = 1$, and both $i$ and $-i$ are $\sqrt{1^2+1^2} = \sqrt{2} > 1$ from $z = 1$. So $f$ is analytic on and inside of the curve $\gamma_1$, so Cauchy's Theorem tells us that $\int_{\gamma_1} \frac{dz}{z^2+1} = 0$ .

(b): $\int_{\gamma_2} \frac{dz}{z^2+1}$, where $\gamma_2(t) = i + e^{2\pi ti}$, $0 \le t \le 1$

The argument is similar to part (a), except with a different conclusion. Since $-i$ is a distance 2 from $i$, $-i$ lies outside of the simple closed curve $\gamma_2$ (which traces out the circle of radius 1 around $z = i$). So the function $g(z) = \frac{1}{z+i}$ is analytic on and inside of $\gamma_2$. SO by the Cauchy Integral Formula,

$$\int_{\gamma_2} \frac{dz}{z^2+1} = \int_{\gamma_2} \frac{\frac{1}{z+i}}{z-i}\ dz = \int_{\gamma_2} \frac{g(z)}{z-i}\ dz = 2\pi i g(i) = \frac{2\pi i}{i+i} = \frac{2\pi i}{2i} = \pi.$$

[Note: one of these requires the Cauchy integral formula (unless you have gotten very ambitious and are working them directly from the definition!).]