Math 325 Problem Set 5 Solutions

Problems were due Friday, February 17.

16. [Zorn, p.81, #8 (part)] For the sequences below, determine the limit of the sequence and prove your assertion using the ϵ - N formulation of the limit. [You may use the fact that $|\sin x| \le 1$ for all $x \in \mathbb{R}$.]

(a) $a_n = \dfrac{2n}{3n-5}$ (b) $b_n = \dfrac{2n}{3n+\sin n+5}$

(a) Calculus tells us to expect that $a_n \to \frac{2}{3} = L$. To show this, we look at

$|a_n - L| = |\dfrac{2n}{3n-5} - \dfrac{2}{3}| = |\dfrac{(2n)(3)-(2)(3n-5)}{(3n-5)(3)}| = |\dfrac{10}{9n-15}| = \dfrac{10}{9n-15}$ (so long as $n \ge 2$). We want to know when this is less than $\epsilon > 0$. So we can solve:

$\frac{10}{9n-15} < \epsilon$, when $10 < \epsilon(9n-15)$, which occurs when $9n-15 > \frac{10}{\epsilon}$, or $n > \frac{1}{9}(\frac{10}{\epsilon}+15)$. So:

Given any $\epsilon > 0$, by the Archimedean principle, there is an $N \in \mathbb{N}$ so that $N > \frac{1}{9}(\frac{10}{\epsilon}+15)$. Then $n \ge N$ implies that $n \ge N > \frac{1}{9}(\frac{10}{\epsilon}+15)$, and so (running the argument above backwards) $|a_n - L| = |\dfrac{2n}{3n-5} - \dfrac{2}{3}| < \epsilon$.

So $n \ge N$ implies that $|a_n - L| < \epsilon$, showing that $a_n \to \frac{2}{3}$.

(b) Calculus tells us to expect, again, that $b_n = \dfrac{2n}{3n+\sin n+5} \to \dfrac{2]}{3}$. To show this, we look at

$|a_n - L| = |\dfrac{2n}{3n+\sin n+5} - \dfrac{2}{3}| = |\dfrac{(2n)(3)-(3n+\sin n+5)(2)}{(3n+\sin n+5)(3)}| = |\dfrac{-2\sin n - 10}{(3n+\sin n+5)(3)}| = \dfrac{|-(2\sin n+10)|}{|3n+\sin n+5|\cdot|3|} = \dfrac{2\sin n+10)}{(3n+\sin n+5)(3)}$,

since $2\sin n + 10 \ge -2+10 = 8 > 0$, and $3n+\sin n+5 \ge 3n-1+5 = 3n+4 \ge 4 > 0$. This is what we want to show is eventually small. But

$|a_n - L| = \dfrac{2\sin n+10}{(3n+\sin n+5)(3)} \le \dfrac{2\cdot 1+10}{(3n+\sin n+5)(3)} = \dfrac{12}{(3n+\sin n+5)(3)} = \dfrac{4}{3n+\sin n+5} \le \dfrac{4}{3n-1+5} = \dfrac{4}{3n+4} < \dfrac{4}{3n+4n} = \dfrac{4}{7n}$ and so for any $\epsilon > 0$, if we choose an N with $\frac{4}{7N} < \epsilon$, by making $7N > 4/\epsilon$, i.e., $N > 4/(7\epsilon)$, then $n \ge N$ implies that $|a_n - L| < \dfrac{4}{7n} \le \dfrac{4}{7N} < \epsilon$. So $b_n \to \frac{2}{3}$.

17. Show that if $a_n \to L$ as $n \to \infty$, then $|a_n| \to |L|$ as $n \to \infty$. Show, in fact, that for a notion of small, $\epsilon > 0$, the same $N \in \mathbb{N}$ that works to control a_n will work to control $|a_n|$.

[The results of section 1.7 will help with this.]

We what to show that for any $\epsilon > -$ there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $||a_n| - |L|| < \epsilon$. But the 'reverse' triangle inequality tells us that $|a_n - L| \geq |a_n| - |L|$, and (so) $|a_n - L| = |L - a_n| \geq |L| - |a_n|$. So $-|a_n - L| \leq |a_n| - |L| \leq |a_n - L|$. This implies that $||a_n| - |L|| \leq |a_n - L|$.

This means that if, given an $\epsilon > 0$, we (using $a_n \to L$) find an $N \in \mathbb{N}$ so that $n \geq N$ implies $|a_n - L| < \epsilon$, then for $n \geq N$ we also have $||a_n| - |L|| \leq |a_n - L| < \epsilon$. So $a_n| \to |L|$, and the same N's that work for bounding $|a_n - L|$ will also bound $||a_n| - |L||$.

18. [Zorn, p.89, #2] Show that if $b_n \neq 0$ for every $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} b_n = b$, and $b \neq 0$, then
$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b} \ .$$

[The textbook describes one possible approach; one could also do this without breaking it into so many steps....]

What we want to show is, for a given $\epsilon > 0$, that we can find an $N \in \mathbb{N}$ so that $n \geq N$ implies that $|\frac{1}{b_n} - \frac{1}{b}| < \epsilon$. But:

$$|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{b_n b}| = \frac{|b - b_n|}{|b_n| \cdot |b|} = \frac{|b_n - b|}{|b_n| \cdot |b|} \ .$$

We know from our hypothesis that we can make $|b_n - b|$ small; we can therefore control what we need so long as we can guarantee that (eventually) neither $|b_n|$ nor $|b|$ can also get small (since, being in the denominator, this keeps the quotient from getting large). $|b|$ gives us no trouble - it is a constant - so what we need to focus on is $|b_n|$. But since $b_n \to b$, we know that there is an N so that $n \geq N$ we have $|b_n - b| < \frac{|b|}{2}$ (any fraction of $|b|$ would really work), so (using the reverse triangle inequality) $|b| - |b_n| \leq |b - b_n| = |b_n - b| < |b|/2$, so (rearranging terms) $|b_n| > |b| - |b|/2 = |b|/2$. So:

Given an $\epsilon > 0$, if $n \geq N$ then $|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|b_n| \cdot |b|} < \frac{|b_n - b|}{(|b|/2) \cdot |b|} = |b_n - b| \cdot \frac{2}{|b|^2}$.

So if we choose an M so that if $n \geq N$ we have $|b_n - b| < \epsilon\frac{|b|^2}{2}$, then if $n \geq \max(N, M)$ we have $|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|b_n| \cdot |b|} < \frac{|b_n - b|}{(|b|/2) \cdot |b|} = |b_n - b| \cdot \frac{2}{|b|^2} < \epsilon\frac{|b|^2}{2} \cdot \frac{2}{|b|^2} = \epsilon$. So $1/b_n \to 1/b$ as $n \to \infty$.

19. Explain how our results about how convergent sequences combine (from section 2.2) enable us to determine the limit of the sequence

$$\frac{2n^2 + n \sin n - 9}{3n - 5n^2 + 2^{-n}} \ .$$

Starting from $a_n = \dfrac{2n^2 + n \sin n - 9}{3n - 5n^2 + 2^{-n}} = \dfrac{2 + (\sin n)/n - 9(1/n)^2}{3/n - 5 + 2^{-n}/n^2}$ (dividing top and bottom by n^2, where $n \neq 0$), then we can argue that $2 + (\sin n)/n - 9(1/n)^2 \to 2$ and $3/n - 5 + 2^{-n}/n^2 \to -5$, as $n \to \infty$, so since limits behave well with quotients, we have $a_n \to \frac{2}{-5} = -\frac{2}{5}$.

To show those two limits, we look first at $b_n = 2 + (\sin n)/n - 9(1/n)^2 = 2 + \frac{\sin n}{n} - 9(\frac{1}{n})^2$. since $1/n \to 0$ (from class!) and $-1 \leq \sin n \leq 1$, we have $-1/n \leq (\sin n)/n \leq 1/n$, with $-1/n \to (-1)(0) = 0$, so the Squeeze Theorem implies that $(\sin n)/n \to 0$. Also, $9(\frac{1}{n})^2 \to 9 \cdot 0 \cdot 0 = 0$, since limits behave well with products. So, since limits behave well with sums, $b_n = 2 + (\sin n)/n - 9(1/n)^2 \to 2 + 0 - 0 = 3$, as desired.

Finally, looking at $c_n = 3/n - 5 + 2^{-n}/n^2$, since $|2^{-n}| = (1/2)^n \leq 1$ (formally, we can establish this by induction!, using $(1/2)^{n+1} = (1/2) * 12)^n \leq (1)(1/2)^n = (1/2)^n$), then $-1/n^2 \leq 2^{-n}/n^2 \leq 1/n^2$ and $1/n^2 = (1/n)^2 \to 0 \cdot 0 = 0$ and so $-1/n^2 ra - 0 = 0$ (since limits behave well under multiplication), the Squeeze Theorem tells us that $2^{-n}/n^2 \to 0$. Also, $3/n = 3(1/n) \to 3(0) = 0$, and so $c_n = 3/n - 5 + 2^{-n}/n^2 \to 0 - 5 + 0 = -5$.

Putting these all together, this is what allows us to conclude that $c_n \to -2/5$.