Math 325 Problem Set 7 Solutions

Problems were due Friday, March 10.

24. Show that for every $a \in \mathbb{R}$ we have $\lim_{x\to a} x = a$. (Your argument should be quite short!) Then, using induction and our limit theorems, show that for every $n \in \mathbb{N}$ we have $\lim_{x\to a} x^n = a^n$.

From the definition, we need to shwo that, given an $\epsilon > 0$, there is a $\delta > 0$ so that $0 < |x - a| < \delta$ implies that $|x - a| < \epsilon$. but $\delta = \epsilon > 0$ works!, since $0 < |x - a| < \epsilon$ implis that $|x - a| < \epsilon$. So $\lim_{x\to a} x = a$.

This is also the base case ($n = 1$) for an argument by induction that $\lim_{x\to a} x^n = a^n$ for $n \in \mathbb{N}$. If we then assume that $\lim_{x\to a} x^n = a^n$, then since $\lim_{x\to a} x = a$ by our base case and because the limit of a product is the product of their limits, we have $\lim_{x\to a} x^{n+1} = \lim_{x\to a} x^n \cdot x = \lim_{x\to a} x^n \cdot \lim_{x\to a} x = a^n \cdot a = a^{n+1}$.

So $\lim_{x\to a} x = a$, and $\lim_{x\to a} x^n = a^n$ implies that $\lim_{x\to a} x^{n+1} = a^{n+1}$. So, by induction, we have $\lim_{x\to a} x^n = a^n$ for every $n \in \mathbb{N}$.

25. (a) Show that the (not very well-known? but it follows from angle sum formulas) trig identity

$$\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \qquad \text{for every } A, B \in \mathbb{R},$$

and the fact (which we will show in class!) that $|\sin(C)| \le |C|$ for every $C \in \mathbb{R}$, together imply that for every $x, y \in \mathbb{R}$ we have $|\sin(x) - \sin(y)| \le |x - y|$.

From the identity above,

$$|\sin(x) - \sin(y)| = |2\sin\left(\tfrac{x-y}{2}\right)\cos\left(\tfrac{x+y}{2}\right)| = 2|\sin\left(\tfrac{x-y}{2}\right)| \cdot |\cos\left(\tfrac{x+y}{2}\right)| \ .$$

But $|\cos\left(\frac{x+y}{2}\right)| \le 1$, since the cosine of <u>anything</u> is ≤ 1, so

$$|\sin(x) - \sin(y)| \le 2|\sin\left(\tfrac{x-y}{2}\right)|.$$

But setting $C = \dfrac{x-y}{2}$ and using our result from class, as have

$$|\sin\left(\frac{x-y}{2}\right)| \le |\frac{x-y}{2}|, \text{ and so}$$

$$|\sin(x) - \sin(y)| \le 2|\sin\left(\frac{x-y}{2}\right)| \le 2|\frac{x-y}{2}| = |x-y|, \text{ as desired.}$$

(b) Using (a), show that the function $f(x) = \sin x$ is continuous at a for every $a \in \mathbb{R}$.

We wish to show that for a given $a \in \mathbb{R}$ and $\epsilon > 0$ we can find a $\delta > 0$ so that $|x-a| < \delta$ implies that $|\sin(x) - \sin(a)| < \epsilon$. But since $|\sin(x) - \sin(a)| \le |x-a|$, bounding $|x-a|$

automatically bounds $|\sin(x) - \sin(a)|$. That is, is we set $\delta = \epsilon > 0$, then $|x - a| < \delta$ implies that $|\sin(x) - \sin(a)| \leq |x - a| < \delta = \epsilon$, so $|\sin(x) - \sin(a)| < \epsilon$. So $f(x) = \sin x$ is continuous at $x = a$ for every $a \in \mathbb{R}$.

26. Show that if $f : (a, b) \to \mathbb{R}$ is a continuous function, then the function $g : (a, b) \to \mathbb{R}$ given by $g(x) = |f(x)|$ is also continuous. (You should argue directly from ϵ's and δ's.)

There is a definite pattern to these problems... We start by knowing that for any $c \in (a, b)$ and given an $\epsilon > 0$, we can always find a $\delta > 0$ so that $|x - a| < \delta$ and $x \in (a, b)$ implies that $|f(x) - f(a)| < \epsilon$. What we wish to show is that for $c \in (a, b)$ and $\epsilon > 0$, there is a $\delta' > 0$ so that $|x - a| < \delta'$ and $x \in (a, b)$ implies that $|g(x) - g(a)| < \epsilon$.

But! $|g(x) - g(a)| = \Big||f(x)| - |f(a)|\Big|$, and, by the 'reverse' triangle inequality,

$\Big||A| - |B|\Big| \leq |A - B|$ for any $A, B \in \mathbb{R}$.

[The proof is fairly short: $|A| = |B+(A-B)| \leq |B|+|A-B|$ implies $|A|-|B| \leq |A-B|$, which is half of what we need; $|B| = |A + (B - A)| \leq |A| + |B - A| = |A| + |A - B|$ provides the other half.] So $|g(x) - g(c)| = \Big||f(x)| - |f(c)|\Big| \leq |f(x) - f(c)|$ for any $x, c \in (a, b)$. $|f(x) - f(c)| < \epsilon$ automatically implies that $\Big|g(x) - g(c)\Big| < \epsilon$.

So, given $\epsilon > 0$, choose $\delta > 0$ so that $|x - c| < \delta$ and $x \in (a, b)$ implies that $|f(x) - f(c)| < \epsilon$. Then $|x - c| < \delta$ implies that $|g(x) - g(c)| \leq |f(x) - f(c)| < \epsilon$, as well. So for every $\epsilon > 0$ we can find $\delta > 0$ so that $|x - c| < \delta$ and $x \in (a, b)$ implies that $|g(x) - g(c)| < \epsilon$. So g is continuous at $x = c$ for every $c \in (a, b)$.

27. Using the previous problem (and a problem from a previous problem set!), show that if $f, g : (a, b) \to \mathbb{R}$ are continuous functions, then the function $M : (a, b) \to \mathbb{R}$ given by $M(x) = \max\{f(x), g(x)\}$ is also continuous.

From a previous problem, we know that

$$M(x) = \max\{f(x), g(x)\} = \frac{(f(x) + g(x) + |f(x) - g(x)|}{2}$$
$$= \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|) \ .$$

But since f and g are continuous, we know that $f(x)-g(x)$ is continuous [the difference of two continuous functions is continuous], and so by the previous problem, $|f(x)-g(x)|$ is continuous. Then $f(x) + g(x) + |f(x) - g(x)|$ is continuous [the sum of continuous functions is continuous], and so $M(x) = \max\{f(x), g(x)\} = \frac{1}{2}(f(x)+g(x)+|f(x)-g(x)|)$ is continuous [a constant multiple of a continuous function is continuous], as desired.