Math 856 Homework 7

Starred (*) problems to be handed in Thursday, December 7

35: (Lee, p.287, Problem 11-8) Show that the dimension of the space of symmetric covariant k-tensors over an n-dimensional vector space V, $\Sigma^k(V)$, is $\binom{n+k-1}{k}$.

36: Show that a diffeomorphism $F : M \to N$ induces a bundle isomorphism $F^* : T^k(N) \to T^k(M)$.

37: (Lee, p.286, Problem 11-7) (a) Show that if S is a covariant k-tensor field and $X_1, \ldots, X_k$ are vector fields, then the assignment $(X_1, \ldots, X_k) \mapsto S(X_1, \ldots, X_k)$ defines a map $\mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$ which is multilinear over C^∞ functions, i.e., $S(X_1, \ldots, f_1X^i + f_2Y_i, \ldots, X_k) = f_1S(X_1, \ldots X^i, \ldots, X_k) + f_2S(X_1, \ldots Y_i, \ldots, X_k)$.

(b) Show the converse, i.e., a map $\mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$ is induced by a covariant k-tensor field $\Leftrightarrow$ the map is linear over C^∞ functions.

[N.B. This result provides the "standard" way to build covariant tensor fields without resorting to local coordinates to do it. Of course, the proof of this result requires using local coordinates...!]

38: (Lee, p.319, Problem 12-4) Show that two k-tuples $\{\omega_1, \ldots, \omega_k\}, \{\eta_1, \ldots, \eta_k\} \subseteq T^1(V)$ of linearly independent covectors have the same span $\Leftrightarrow \omega_1 \wedge \cdots \wedge \omega_k = c\eta_1 \wedge \cdots \wedge \eta_k$ for some $c \in \mathbb{R}$.

39: (a) Show that $\int_\gamma df = \int_{[0,1]} \gamma^*(df) = 0$ for every smooth closed curve $\gamma : [0, 1] \to M$ (i.e., $\gamma(0) = \gamma(1)$). [Hint: Stokes' Theorem!]

(b) Show, conversely, that if ω is a 1-form on M and $\int_\gamma \omega = \int_{[0,1]} \gamma^*\omega = 0$ for every smooth closed curve $\gamma : [0, 1] \to M$, then there is an $f \in C^\infty(M)$ with $\omega = df$.
[Hint: If $\omega = df$, then what is $\int_\gamma \omega$ for a **non**-closed curve γ ?]