Math 423/823 Exercise Set 6 Solutions

21. [BC#4.38.4] The integral $\int_0^\pi e^{(1+i)x}\ dx$ is, technically, equal to

$$\int_0^\pi e^x \cos x\ dx + i\int_0^\pi e^x \sin x\ dx$$

Evaluate these two integrals $\int_0^\pi e^x \cos x\ dx$ and $\int_0^\pi e^x \sin x\ dx$ by applying the Fundamental Theorem of Calculus (p.119, bottom) directly to the top integral and equating the real and imaginary parts.

We can find an antiderivative of $f(x) = e^{(1+i)x}$ via the chain rule: $\frac{d}{dx}(e^{cx}) = ce^{cx}$, so $F(x) = \frac{1}{1+i}e^{(1+i)x} = \frac{1-i}{2}e^{(1+i)x}$ has derivative $f(x)$.

So $\int_0^\pi e^{(1+i)x}\ dx = \frac{1-i}{2}e^{(1+i)x}\Big|_0^\pi$

$$= \frac{1-i}{2}(e^{(1+i)\pi} - e^{(1+i)0}) = \frac{1-i}{2}(e^\pi e^{i\pi} - 1) = -(e^\pi + 1)/2 + i(e^\pi + 1)/2.$$

Equating the real and imaginary parts, we then have

$\int_0^\pi e^x \cos x\ dx = -(e^\pi + 1)/2$ and $\int_0^\pi e^x \sin x\ dx = (e^\pi + 1)/2.$

22. Find a parametriation of the curve which follows the circle of radius 2 counterclockwise from $z = 2$ to $z = 2i$, followed by the line segment that runs from $z = 2i$ to $z = -1$.

[Note: there are literally an infinite number of ways to answer this question (correctly!); take pity on your poor instructor when choosing your parametrization....]

We can parametrize the (one-fourth of the) circle as $\alpha(t) = 2e^{it}$ for $0 \le t \le \pi/2$, and we can parametrize the line segment as $\beta(t) = (1-t)(2i) + t(-1)$ for $0 \le t \le 1$. If we shift the time interval for α to $-\pi/2 \le t \le 0$ we can stitch the two parametrizations together:

$$\gamma(t) = \begin{cases} 2e^{i(t+\pi/2)} & \text{if } -\pi/2 \le t \le 0 \\ (1-t)(2i) - t & \text{if } 0 \le t \le 1 \end{cases}$$

is one possible parametrization.

23. [BC#3.42.1(part)] Find the integrals $\int_C \frac{z+2}{z}\ dz$, where

(a): C is the semicircle $z = 2e^{i\theta}$, $0 \le \theta \le \pi$

We now have much better tools to compute these contour integrals, but in the spirit in which the problems were intended:

$C(\theta) = 2e^{i\theta}$, so $C'(\theta) = 2ie^{i\theta}$, and

$$\int_C \frac{z+2}{z}\, dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\, 2ie^{i\theta}\, d\theta = \int_0^{\pi} 2ie^{i\theta} + 2i\, d\theta = [2e^{i\theta} + 2i\theta]\Big|_0^{\pi}$$
$$= [2e^{i\pi} + 2i\pi] - [2e^{i0} + 2i(0)] = -2 + 2\pi i - 2 = 2\pi i - 4.$$

(c): C is the circle $z = 2e^{i\theta}$, $0 \le \theta \le 2\pi$

Stealing much of the work from the first part,

$$\int_C \frac{z+2}{z}\, dz = \int_0^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\, 2ie^{i\theta}\, d\theta = \int_0^{2\pi} 2ie^{i\theta} + 2i\, d\theta = [2e^{i\theta} + 2i\theta]\Big|_0^{2\pi}$$
$$= [2e^{2i\pi} + 4i\pi] - [2e^{i0} + 2i(0)] = 2 + 4\pi i - 2 = 4\pi i.$$

24. [BC#4.42.8] Find the integral $\int_C z^n(\overline{z})^m\, dz$, where C is the unit circle $|z| = 1$ traversed in a counterclockwise direction.

[Note: you will fnd it helpful to know that $\int_0^{2\pi} e^{ik\theta}\, d\theta$ is 0 if $k \neq 0$, and 2π if $k = 0$. You need not prove this.]

Extra credit: why does it not matter <u>where</u> we choose to start our parametrization of the circle (i.e., at what point along the circle)?

Parametrizing the circle as $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, we have $\gamma'(t) = ie^{it}$ and

$$\int_C z^n(\overline{z})^m\, dz = \int_0^{2\pi} (e^{it})^n(\overline{e^{it}})^m ie^{it}\, dt = i\int_0^{2\pi} (e^{it})^{n+1}(e^{-it})^m\, dt$$
$$= i\int_0^{2\pi} (e^{it})^{n-m+1}\, dt = i\int_0^{2\pi} e^{i(n-m+1)t}\, dt,$$

which, according to our note, is 0 if $n - m + 1 \neq 0$, and is $2\pi i$ if $n - m + 1 = 0$. More precisely,

$\int_C z^n(\overline{z})^m\, dz$ equals 0, if $m \neq n + 1$, and equals $2\pi i$, if $m = n + 1$.

It doesn't matter where we start the parametrization, so long as we go counterclockwise, since if we chose a different starting point, then using <u>both</u> starting points, we could cut C into two pieces, and treat it as <u>two</u> paths taken one after the other. Then we would write C as C_1 followed by C_2. Using the other starting point, it would be C_2 followed by C_1, so we would be comparing

$$\int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz \qquad \text{to} \qquad \int_{C_2} f(z)\, dz + \int_{C_1} f(z)\, dz,$$

which are the same!