CARDINALITY AND THE AXIOM OF CHOICE

Fall, 1981

Definition. Two sets, A and B, have the same cardinality iff there exists a 1 - 1, onto function $f : A \to B$.
The cardinality of a set A is denoted $|A|$.
Let N be the set of positive integers, Z the set of all integers, Q the set of rational numbers, R the set of real numbers.

Theorem Car. 1. $|N| = |Z|$.

Theorem Car. 2. Every subset of N is either finite or has the same cardinality as N.

Definition. A set which is finite or has the same cardinality as N is countable or has countable cardinality.

Theorem Car. 3. Q is countable.

Theorem Car. 4. The countable union of countable sets is countable.

Definition. For any set A, 2^A denotes the set of all subsets of A. (The empty set, denoted $\emptyset$, is a subset of any set.) 2^A is called the power set of A.

Theorem Car. 5. For any set A, there is a 1 - 1, function f from A into 2^A.

Theorem Car. 6. For a set A, let P be the set of all functions from A to the two point set $\{0,1\}$. Then $|P| = |2^A|$.

Theorem Car. 7.(Cantor). There is no one-to-one function from a set A to 2^A.

Definition. A set is infinite iff it contains a subset ~~with the same cardinality~~ as N.

Theorem Car. 8. A set is infinite if and only if there is a one-to-one function from the set into a proper subset of itself.

Theorem Car 9. (Schroeder-Bernstein) If A and B are sets so that there exist one-to-one functions f from A into B and g from B into A, then $|A| = |B|$.

Below are listed Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle. These three statements are equivalent and are used freely in most standard mathematics. We will use them freely in this course.

Definitions 1. A set X is partially ordered by the relation $\leq$ iff, for any elements x, y, and z in X,

(i) if $x \leq y$ and $y \leq z$, then $x \leq z$, and

(ii) if $x \leq y$ and $y \leq x$, then $x = y$.

2. Let X be a set partially ordered by $\leq$. Then an element m in X is a maximal element iff for any x in X, $m \leq x$ implies that $m = x$.

3. A set is totally ordered iff it is partially ordered and every two elements are comparable.

4. A set is <u>well-ordered</u> iff it is totally ordered and every non-empty subset has a least element.

<u>Theorem Car. 10</u>. R with the usual ordering is totally ordered, but not well-ordered. N is well-ordered.

<u>Example</u>. For any set A , the set 2^A is partially ordered by set inclusion. The set A is a maximal element, and, in fact, the only maximal element in this ordering.

<u>Theorem Car. 11</u>. Any subset of a well-ordered set is well-ordered by the same ordering restricted to the subset.

<u>Zorn's Lemma</u>. Let X be a partially ordered set in which each totally ordered subset has an upper bound. Then X has a maximal element.

<u>Axiom of Choice</u>. Let $\{A_\alpha\}_{\alpha\in\lambda}$ be a collection of non-empty sets. Then there is a function $f: \lambda \longrightarrow \bigcup_{\alpha\in\lambda} A_\alpha$ so that for each α in λ , $f(\alpha)$ is an element of A_α .

<u>Well-ordering Principle</u>. Every set can be well-ordered.

<u>Definition</u>. The ordinal mumbers with which everyone is familiar are the non-negative integers: 0,1,2,3, ...
We can continue to count beyond the finite ordinals by the following method. Let the set of non-negative integers be

given a name, namely ω_0. The next ordinal will be defined to be the set of its predecessors (which have already been defined). In this manner the ordinals are defined, each as a set, namely its predecessors. Below are written the first ordinals:

$$0, 1, 2, \ldots \; \omega_0, \omega_0+1, \omega_0+2, \ldots \; 2\omega_0, 2\omega_0+1, \ldots \ldots \; \omega_1, \omega_1+1, \ldots \ldots \omega_2, \ldots$$

The ordinal ω_1 is the first ordinal whose cardinality is greater than the cardinality of ω_0. Likewise, ω_2 is by definition the first ordinal whose cardinality is greater than ω_1, etc. $\omega_0, \omega_1, \omega_2 \ldots$ are called cardinal numbers and the bars are ommitted when referring to them as cardinalities, even though technically they should be there.

<u>Theorem Car. 12</u>. If A is an infinite set, then the countable union of sets of $|A|$ has $|A|$.

<u>Definition</u>. Let X be a set totally ordered by $\leq$ and let $x \in X$. Then $I(x) = \{y \in X \mid y \lneq x$ is called an <u>initial segment</u>.

<u>Theorem Car. 13</u>. Let X and Y be well-ordered sets. Then precisely one of the following is true:

(i) There is a function f from X to Y which is one-to-one, onto , and order preserving.

(ii) There is a y in Y and a function f from X to $I(y)$ which is one-to-one, onto, and order preserving.

(iii) There is an x in X and a function f from Y to $I(x)$ which is one-to-one, onto and order preserving.

<u>Theorem Car. 14</u>. Let A and B be sets. Then either there is a one-to-one function from A to B or from B to A.

<u>Definition</u>. Cardinalities are ordered. We write $|A| \leq |B|$ iff there is a one-to-one function from A to B.

<u>Theorem Car. 15</u>. Cardinalities are well-ordered by $\leq$ above.

General Topology

Fall 1981

Definitions. 1. Suppose X is a set. Then $\mathcal{T}$ is a topology for X if and only if $\mathcal{T}$ is a collection of subsets of X such that

i) $\emptyset \in \mathcal{T}$,

ii) $X \in \mathcal{T}$,

iii) if $A \in \mathcal{T}$ and $B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,

iv) if $\{A_\alpha\}_{\alpha\in\lambda}$ is any collection of sets each of which is in $\mathcal{T}$, then $\bigcup_{\alpha\in\lambda} A_\alpha \in \mathcal{T}$.

2. A topological space is an ordered pair $(X,\mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology for X.

3. If $(X,\mathcal{T})$ is a topological space, then U is an open set in $(X,\mathcal{T})$ if and only if $U \in \mathcal{T}$.

Several examples of topological spaces are listed below.

Example 1. For a set X, let 2^X be the set of all subsets of X. Then 2^X is called the discrete topology on X. The space $(X,2^X)$ is called a discrete topological space.

Example 2. For a set X, $\{\emptyset,X\}$ is called the indiscrete topology for X. So $(X,\{\emptyset,X\})$ is an indiscrete topological space.

Example 3. For any set X, the finite complement topology for X is described as follows: a subset U of X is open if and only if $U=\emptyset$ or $X-U$ is finite.

Example 4. Let R^n be the set of all n-tuples of real numbers. We will define the distance $d(x,y)$ between points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ by the equation $d(x,y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}}$. A topology T for R^n is defined as follows: a subset U of R^n belongs to T if and only if for each point p of U there is a positive number ε so that $\{x \mid d(p,x) < \varepsilon\}$ is a subset of U. This topology T is called the usual topology for R^n.

Definitions. Let $(X,\mathcal{T})$ be a topological space, A be a subset of X, and p be a point in X. Then:

1. p is a limit point of A if and only if for each open set U containing P, $(U-\{p\}) \cap A \neq \emptyset$. Notice that p may or may not belong to A.

2. If $p \in A$ but p is not a limit point of A, then p is an isolated point of A.

3. The closure of A (denoted $\overline{A}$ or $Cl(A)$) is A together with all limit points of A.

4. The set A is closed iff A contains all its limit points, i.e. $\overline{A} = A$.

Theorem 1. For any topological space $(X,\mathcal{T})$ and subset A of X, $\overline{A}$ is closed.

Theorem 2. Let X be a topological space, i.e., $(X,\mathcal{T})$ is really the topological space but the topology is not named. Then a subset A of X is closed if and only if $X - A$ is open.

Theorem 3. The union of finitely many closed sets in a topological space is closed.

<u>Theorem</u> 4. Let $\{A_\alpha\}_{\alpha\in\lambda}$ be a collection of closed subsets of a topological space X. Then $\bigcap_{\alpha\in\lambda} A_\alpha$ is closed.

<u>Theorem</u> 5. Suppose A is a subset of X, a topological space. Then $\overline{A}$ = the intersection of all closed sets containing A.

<u>Theorem</u> 6. Let $A,B \subset X^{\text{top.sp.}}$. Then

a) if $A \subset B$, $\overline{A} \subset \overline{B}$ and

b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

<u>Definition</u>. Let $\mathcal{T}$ be a topology on a set X and let $\mathcal{B}$ be a subset of $\mathcal{T}$. Then $\mathcal{B}$ is a <u>basis</u> for the topology $\mathcal{T}$ if and only if every element of $\mathcal{T}$ is the union of elements in $\mathcal{B}$.

<u>Theorem</u> 7. Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if $\mathcal{B} \subset \mathcal{T}$ and for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$.

<u>Theorem</u> 8. Let $\mathcal{B} = \{(a,b) \subset \mathbf{R}^1 | a$ and b are rational numbers$\}$. Then $\mathcal{B}$ is a basis for the usual topology on $\mathbf{R}^1$.

Suppose you are given a set X and a collection $\mathcal{B}$ of subsets of X. Under what circumstances is $\mathcal{B}$ a basis for a topology on X? This question is answered in the following theorem.

Theorem 9. Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basic for a topology for X if and only if the following conditions hold.

i) $\emptyset \in \mathcal{B}$

ii) for each point p in X there is a set U in $\mathcal{B}$ with $p \in U$, and

iii) if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ so that $p \in W \subset (U \cap V)$.

Theorem 9 allows one to describe topological spaces by first specifying a set X and then a collection $\mathcal{B}$ of subsets of X which satisfy the conditions of Theorem 9. The topology $\mathcal{T}$ whose basis is $\mathcal{B}$ is thereby described.

Definitions. Suppose X is a set. A function d from $X \times X$ into R^1_+, the non-negative reals, is a metric for X if and only if the following conditions are satisfied.

(i) $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$, and

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

If d is a metric for X, then $d(x,y)$ is called the distance from x to y.

Suppose X is a set, d is a metric for X, $p \in X$, and $\varepsilon \in R^1_+$. Then the open ε ball about p is defined by $B_\varepsilon(p) = \{x \in X \mid d(x,p) < \varepsilon\}$. The d-metric topology for X is the topology whose basis is all the $B_\varepsilon(p)$'s. (Check that the collection of all open ε balls is a basis.)

Now suppose that $(X,\mathscr{T})$ is a topological space. Then $(X,\mathscr{T})$ is a <u>metric space</u> (or <u>metrizable</u>) iff there is a metric d on X for which $\mathscr{T}$ is the d-metric topology. If X is a metric space, then the statement that d <u>is a metric for</u> X means that the d-metric topology is the topology for X.

Notice that the same metric space may have many different metrics. As an exercise find several metrics for $\mathbf{R}^n$.

<u>Theorem</u> 10. If X is a metric space, then there is a metric d for X so that for each $x,y \in X$, $d(x,y) < 1$.

<u>Example</u> 5. Let X be a set totally ordered by $<$. Let $\mathscr{B}$ be the collection of all subsets of X of one of the following three forms: $\{x \in X | x < a$ for some $a \in X\}$, $\{x \in X | a < x$ for some $a \in X\}$, or $\{x \in X | a < x < b$ for some $a,b \in X\}$. Then $\mathscr{B}$ is a basis for a topology $\mathscr{T}$ on X. The topology $\mathscr{T}$ is called the <u>order topology</u> for X.

<u>Example</u> 6. The usual topology on $\mathbf{R}^1$ is the order topology given by the usual order.

<u>Example</u> 7. For each ordinal α, the predecessors of α with the order topology form a space called α.

<u>Definition</u>. Let $(X,\mathscr{T})$ be a topological space and let $\mathscr{S}$ be a collection of subsets of X. Then $\mathscr{S}$ is a <u>sub-basis</u> of $\mathscr{T}$ if and only if the collection $\mathscr{B}$ of all finite intersections of sets in $\mathscr{S}$ is a basis for $\mathscr{T}$.

<u>Theorem</u> 11. Let $(X,\mathscr{T})$ be a topological space and let $\mathscr{S}$ be a collection of subsets of X. Then $\mathscr{S}$ is a sub-basis for $\mathscr{T}$ if and only if each element of $\mathscr{S}$ is in $\mathscr{T}$ and for each set U in $\mathscr{T}$ and point p in U there is a finite collection $\{V_i\}_{i=1}^n$ of elements of $\mathscr{S}$ so that $p \in \bigcap_{i=1}^n V_i$ and $\bigcap_{i=1}^n V_i \subset U$.

Theorem 12. Let $\mathcal{S}$ be the collection of all subsets of R^1 of one of the following two forms: $\{x|x < a$ for some $a \in R^1\}$. Then $\mathcal{S}$ is a sub-basis for R^1 with the usual topology.

Once again we seek to answer the question of when a given collection $\mathcal{S}$ of subsets of a set X is a sub-basis for a topology on X.

Theorem 13. Let $\mathcal{S}$ be a collection of subsets of a set X. Then $\mathcal{S}$ is a sub-basis for a topology on X if and only if every point of X is in some element of $\mathcal{S}$ and there are sets $\{U_i\}_{i=1}^n$ in $\mathcal{S}$ so that $\bigcap_{i=1}^n U_i = \emptyset$.

Theorem 13 can be used to describe topologies by presenting a sub-basis for them.

Example 8. Let 2^X be the set of all functions from the set X into the two point set $\{0,1\}$. Let $\mathcal{S}$ be the collection of all subsets of 2^X of the form $U(x,\varepsilon) = \{f \in 2^X | f(x) = \varepsilon\}$. Let $\mathcal{T}$ be the topology on 2^X with sub-basis $\mathcal{S}$. (This topology is really the product topology, but we will not give a general definition of product topology until later.)

Theorem 14. Suppose $(X,\mathcal{T})$ is a topological space, $Y \subset X$, and $\mathcal{T}_Y = \{U | \text{for some } V \text{ in } \mathcal{T},\ U = V \cap Y\}$. Then $\mathcal{T}_Y$ is a topology for Y.

Theorem 14 allows us to define a topology on a subset Y of X when $(X,\mathcal{T})$ is a topological space. The topology $\mathcal{T}_Y$ of Y of Theorem 14 is called the relative topology or subspace topology. The topological space $(Y,\mathcal{S})$ is a subspace of $(X,\mathcal{T})$ if and only if Y is a subset of X and $\mathcal{S}$ is the relative topology on Y.

Theorem 15. If X is a metric space and $Y \subset X$, then Y is a metric space.

Separation Properties

Definitions. Let $(X,\mathcal{T})$ be a topological space:

1) X is T_1 iff every point in X is a closed set.

2) X is Hausdorff (or T_2) iff for each pair of points x,y in X, there are disjoint open sets U and V in $\mathcal{T}$ so that $x \in U$ and $y \in V$.

3) X is regular iff for each $x \in X$ and closed set A in X with $x \notin A$, there are open sets U,V so that $x \in U$, $A \subset V$ and $U \cap V = \emptyset$.

4) X is normal iff for each pair of disjoint closed sets A and B in X, there are open sets U,V so that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Theorem 16. A topological space X is regular if and only if for each point p in X and open set U containing p there is an open set V so that $p \in V$ and $\overline{V} \subset U$.

Theorem 17. A topological space X is normal if and only if for each closed set A in X and open set U containing A there is an open set V so that $A \subset V$, and $\overline{V} \subset U$.

Theorem 18. A topological space X is normal if and only if for each pair of disjoint closed sets A and B, there are disjoint open sets U and V so that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 19. A metric space is normal.

Definition. Let P be a property of a topological space (such as T_1, Hausdorff, etc.). A topological space X is hereditarily P iff for each subspace Y of X, Y has property P.

Theorem 20. A Hausdorff space is hereditarily Hausdorff.

Theorem 21. A regular space is hereditarily regular.

Theorem 22. Let A be a closed subset of a normal space X. Then X is normal when given the relative topology.

Normality Lemma 23. Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\omega_0}$ and $\{V_i\}_{i\in\omega_0}$ be two collections of open sets such that

(i) $A \subset \bigcup_{i\in\omega_0} U_i$,

(ii) $B \subset \bigcup_{i\in\omega_0} V_i$,

(iii) for each i in ω_0, $\overline{U}_i \cap B = \emptyset$ and $\overline{V}_i \cap A = \emptyset$.

Then there are open sets U and V so that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Definitions. 1) A subset B of a topological space X is a G_δ iff B is the intersection of countably many open sets.

2) A subset B of a topological space X is an F_σ iff B is the union of countably many closed sets.

?Theorem 24. An F_σ subset of a normal space is normal.

Countability Properties

Definitions. 1) Let A be a subset of a topological space X. Then A is dense in X iff $\overline{A} = X$.

2) A space X is separable iff X has a countable dense subset.

3) A space X is 2nd countable iff X has a countable basis.

4) Let p be a point in a space X. A collection of open sets $\{U_\alpha\}_{\alpha\in\lambda}$ in X is a <u>neighborhood basis for</u> p iff for each $\alpha \in \lambda$, $p \in U_\alpha$, and for open set U in X with p in U, there is an α in λ so that $U_\alpha \subset U$.

5) A space X is <u>1st countable</u> iff for each point x in X, x has a neighborhood basis consisting of a countable number of sets.

6) A space X has the <u>Souslin property</u> iff X does <u>not</u> contain uncountably many disjoint open sets.

<u>Theorem</u> 25. A 2nd countable space is separable.

<u>Theorem</u> 26. A 2nd countable space is 1st countable.

<u>Theorem</u> 27. A 2nd countable space is hereditarily 2nd countable.

<u>Theorem</u> 28. A separable space has the Souslin property.

<u>Theorem</u> 29. If X is a separable, Hausdorff space, then $|X| \leq |2^{2^\omega}|$.

<u>Theorem</u> 30. For any X, 2^X has the Souslin property.

<u>Theorem</u> 31. The space 2^{R^1} is separable.

<u>Definition</u>. Let $P = \{p_i\}_{i\in\omega_0}$ be a sequence of points in a space X. Then the sequence P <u>converges to</u> a point x iff for every open set U containing x there is an integer M so that for each $m > M$, $p_m \in U$.

<u>Theorem</u> 32. Suppose x is a limit point of the set A in a 1st countable space X. Then there is a sequence of points in A which converges to x.

<u>Theorem</u> 33. Every uncountable set in a 2nd countable space has a limit point.

Covering Properties

Definition. 1) Let A be a subset of X and let $\mathscr{B} = \{B_\alpha\}_{\alpha\epsilon\lambda}$ be a collection of subsets of X. Then $\mathscr{B}$ is a cover of A iff $A \subset \bigcup_{\alpha\epsilon\lambda} B_\alpha$. Also, $\mathscr{B}$ is an open cover iff each B_α is open.

2) A space X is compact iff every open cover $\mathscr{B}$ of X has a finite subcover $\mathscr{C}$. That is, $\mathscr{C}$ is an open cover of X each of whose elements is a set in $\mathscr{B}$.

3) A space X is countably compact iff every countable open cover of X has a finite subcover.

4) A space X is Lindelöf iff every open cover of X has a countable subcover.

5) A collection $\mathscr{B} = \{B_\alpha\}_{\alpha\epsilon\lambda}$ of subsets of a space X is locally finite iff for each point p in X there is an open set U containing p so that U intersects only finitely many elements of $\mathscr{B}$.

Example. Let $\mathscr{B} = \{[n,n+1] \subset R^1 | n$ is an integer$\}$. Then $\mathscr{B}$ is a locally finite collection in R^1 (usual).

Definition. 6) Let $\mathscr{B} = \{B_\alpha\}_{\alpha\epsilon\lambda}$ be a cover of X. Then $\mathscr{C} = \{C_\beta\}_{\beta\epsilon\mu}$ is a refinement of $\mathscr{B}$ iff (i) $\mathscr{C}$ is a cover of X and (ii) for each $\beta \epsilon \mu$ there is an $\alpha \epsilon \lambda$ such that $C_\beta \subset B_\alpha$. The collection $\mathscr{C}$ is an open refinement iff each C_β is an open set.

7) A space X is paracompact iff every open cover of X has a locally finite open refinement and X is Hausdorff.

Theorem 34. Every countably compact and Lindelöf space is compact.

Theorem 35. Every compact, Hausdorff space is paracompact.

Theorem 36. Let A be a closed subspace of a compact space (respectively, countably compact, Lindelöf, paracompact). Then A is compact (resp., countably compact, Lindelöf, paracompact).

Theorem 37. The closed subspace [0,1] in the $\mathbb{R}^1$ (usual) topology is compact.

Theorem 39. Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Theorem 40. If X is a Lindelöf space, then every uncountable subset of X has a limit point.

Theorem 41. Let X be a T_1 space. Then X is countably compact if and only if every infinite subset of X has a limit point.

Theorem 42. ω_1 is countably compact but not compact.

Theorem 43. Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets has a finite subcover.

Theorem 44. (The Alexander Sub-basis Theorem) Let $\mathcal{S}$ be a sub-basis for a space X. Then X is compact if and only if every sub-basic open cover has a finite subcover. (A sub-basic open cover is a cover of X each element of which is a set in the sub-basis.)

Theorem 45. A compact, Hausdorff space is normal. Show regular then normal

Theorem 46. A regular, Lindelöf space is normal.

<u>Theorem</u> 47. A regular, T_1, Lindelöf space is paracompact.

<u>Theorem</u> 48. Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a locally finite collection of subsets of a space X. Let C be a subset of λ. Then $\mathrm{Cl}(\bigcup_{\alpha\in C} B_\alpha) = \bigcup_{\alpha\in C} \overline{B}_\alpha$.

<u>Theorem</u> 49. A paracompact space is normal.

<u>Theorem</u> 50. A metric space is paracompact.

<u>Theorem</u> 51. In a metric space X, the following are equivalent:

(a) X is separable,

(b) X is 2nd countable,

✓ (c) X has the Souslin property,

(d) X is Lindelöf,

(e) every uncountable set in X has a limit point.

Continuity and homeomorphisms

Definition. Let X and Y be topological spaces. A function $f : X \to Y$ is a continuous function or map if and only if for every open set U in Y, $f^{-1}(U)$ is open in X.

Theorem 52. Let $f : X \to Y$ be a function. Then the following are equivalent:

(a) f is continuous,

(b) for every closed set K in Y, $f^{-1}(K)$ is closed in X,

$Y - K$ open
$f^{-1}(Y-K)$ open.
claim $X - f^{-1}(Y-K) = \cancel{X} = f^{-1}(K)$
$k \notin f^{-1}(Y-K)$
$\Rightarrow f(k) \notin Y-K \Rightarrow f(k) \in K \Rightarrow k \in f^{-1}(K)$

(c) if p is a limit point of A in X, then f(p) belongs to Cl(f(A)).

$k \in f^{-1}(K) \Rightarrow f(k) \in K$
$\Rightarrow f(k) \notin f^{-1}(Y-K)$
$\Rightarrow k \in X - f^{-1}(Y-K)$

Theorem 53. Let X be a compact (resp. Lindelöf, countably compact) space and let $f : X \to Y$ be a continuous function that is onto. Then Y is compact (resp. Lindelöf, countably compact).

Theorem 54. Let X be a separable space and let $f : X \to Y$ be a continuous, onto map. Then Y is separable.

Theorem 55. Let A and B be disjoint closed sets in a normal space X. Then there exist open sets U_r for each diadic rational r (that is, r can be written as a quotient of integers with denominator a power of 2) so that $A \subset U_0$, $B \subset (X - U_1)$, and for $r < s$, $Cl(U_r) \subset U_s$.

<u>Theorem</u> 56 (Urysohn's Lemma). A space X is normal if and only if for each pair of disjoint open sets A and B in X, there exists a continuous function $f : X \to [0,1]$ so that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

<u>Theorem</u> 57 (The Tietze Extension Theorem). A space X is normal if and only if every continuous function f from a closed set A in X into $[0,1]$ can be extended to a continuous function $F : X \to [0,1]$. (F extends f means for each point x in A, $F(x) = f(x)$.)

<u>Theorem</u> 58 (The Tietze Extension Theorem). A space X is normal if and only if every continuous function f from a closed set A in X into $(0,1)$ can be extended to a continuous function $F : X \to (0,1)$.

<u>Theorem</u> 59. If X and Y are metric spaces with metrics d_X and d_Y respectively, then a function $f\colon X \to Y$ is continuous if and only if for each point x in X and $\varepsilon > 0$, there is a $\delta > 0$ so that for each $y \in X$ with $d_X(x,y) > \delta$, $d_Y(f(x), f(y)) < \varepsilon$.

<u>Definition</u>. A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is <u>uniformly continous</u> if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ so that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

<u>Theorem</u> 60. Let $f\colon X \to Y$ be a map from a compact metric space to a metric space Y. Then f is uniformly continuous for any choice of metrics for X and Y.

<u>Theorem</u> 61. Let $f_i\colon (X, d_X) \to (Y, d_y)$ $(i \in \omega)$ be a sequence of maps so that for each $i \in \omega$, and point x in X, $d_y(f_i(x), f_{i+1}(x)) < 1/2^i$. Then $\lim_{i \to \infty} f_i$ exists and is continuous.

<u>Definition</u>. A map $f\colon X \to Y$ is <u>closed</u> (resp. <u>open</u>) if and only if for every closed (resp. open) set A in X, $f(A)$ is closed (resp. open) in Y.

<u>Theorem</u> 62. Let X be compact and Y Hausdorff. Then any map $f\colon X \to Y$ is a closed map.

A closed $\subseteq X$
$\Rightarrow A$ compact
$\Rightarrow f(A)$ compact $\subseteq Y$ Hausdorff $\Rightarrow f(A)$ closed

<u>Definition</u>. A map $f: X \to Y$ is a <u>homeomorphism</u> if and only if f is continuous, 1 - 1 and onto and $f^{-1}: Y \to X$ is also continuous.

<u>Theorem</u> 63. For a map $f: X \to Y$, the following are equivalent:

(a) f is a homeomorphism. a) ⇒ b) closed

$f(A) =$

(b) f is 1 - 1, onto and closed. $(f^{-1})^{-1}(A)$, closed

(c) f is 1 - 1, onto and open. $= f(A)$ closed

<u>Definition</u>. Spaces X and Y are <u>homeomorphic</u> if and only if there is a homeomorphism $f: X \to Y$ which is onto.

<u>Theorem</u> 64. For points $a < b$ in E^1, the interval (a, b) is homeomorphic to E^1.

<u>Theorem</u> 65. Suppose $f: X \to Y$ is a 1 - 1 and onto continuous map, X is compact and Y is Hausdorff. Then f is a homeomorphism.

see 63 (b)

Products

Let $\{X_\alpha\}_{\alpha\varepsilon\lambda}$ be a collection of spaces. The <u>product</u> $\prod_{\alpha\varepsilon\lambda} X_\alpha$, or <u>Cartesian product</u>, is a generalization of the familiar n-tuples. Define $\prod_{\alpha\varepsilon\lambda} X_\alpha$ to be $\{f: \lambda \to \bigcup_{\alpha\varepsilon\lambda} X_\alpha \mid f(\alpha) \varepsilon X_\alpha\}$. So a point in $\prod_{\alpha\varepsilon\lambda} X_\alpha$ can be thought of as a function from the indexing set into $\bigcup_{\alpha\varepsilon\lambda} X_\alpha$. So if $f \varepsilon \prod_{\alpha\varepsilon\lambda} X_\alpha$, $f(\alpha)$ is the α^{th} <u>coordinate</u> of f. We could write f as $\{f_\alpha\}_{\alpha\varepsilon\lambda}$ where $f(\alpha) = f_\alpha$.

For each β in λ, define the projection function $\Pi_\beta: \prod_{\alpha\varepsilon\lambda} X_\alpha \to X_\beta$ by $\Pi_\beta(f) = f(\beta)$. A subbasis for the <u>product topology</u> on $\prod_{\alpha\varepsilon\lambda} X_\alpha$ is the collection of all sets of the form $\Pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β. Why is it appropriate to refer to this topology as the finite gate topology?

<u>Theorem</u> 66. The space 2^X described before is really the product, $\prod_{x\varepsilon X} \{0,1\}_x$.

<u>Theorem</u> 67. The function $\Pi_\beta: \prod_{\alpha\varepsilon\lambda} X_\alpha \to X_\beta$ is a continuous, open, onto map.

<u>Theorem</u> 68. The function $\Pi_\beta: \prod_{\alpha\varepsilon\lambda} X_\alpha \to X_\alpha$ need not be closed.

<u>Theorem</u> 69. A function $g: Y \to \prod_{\alpha\varepsilon\lambda} X_\alpha$ is continuous if and only if $\Pi_\beta o g$ is continuous for each β in λ.

✓<u>Theorem</u> 70. Let $\{Xi\}_{i\varepsilon\omega}$ be a collection of metric spaces. Then $\prod_{i\varepsilon\omega} X_i$; is a metric space.

<u>Theorem</u> 71. The space $\mathbb{R}^n$ is homeomorphic to $\prod_{i=1}^{n} \mathbb{R}_i^1$ where $\mathbb{R}_i^1 = \mathbb{R}^1$.

<u>Theorem</u> 72. Let $\{X_\beta\}_{\beta\varepsilon\mu}$ be a collection of Hausdorff (resp. regular) spaces. Then $\prod_{\beta\varepsilon\mu} X_\beta$ is Hausdorff (resp. regular).

✓<u>Theorem</u> 73. Let $\{X_\beta\}_{\beta\varepsilon\mu}$ be a collection of separable spaces where $|\mu| \leq 2^{\omega_0}$, then $\prod_{\beta\varepsilon\mu} X_\beta$ is separable.

✓<u>Theorem</u> 74. Let $\{X_\beta\}_{\beta\varepsilon\mu}$ be a collection of separable spaces. Then $\prod_{\beta\varepsilon\mu} X_\beta$ has the Souslin property.

<u>Theorem</u> 75. Let $\{X_\beta\}_{\beta\varepsilon\mu}$ be a collection of compact spaces. Then $\prod_{\beta\varepsilon\mu} X_\beta$ is compact. Use Alexander Subbase Thm

Connectedness

Definitions.

1. Subsets A, B of X are separated if and only if $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

2. A space X is connected if and only if X is not the union of two non empty separated sets. The notation $X = A|B$ means $X = A \cup B$ and A and B are separated sets.

Theorem 76. A space X is connected if and only if there is not a continuous function $f: X \to \mathbb{R}^1$ so that $f(x) = \{0,1\}$.

Theorem 77. The space $\mathbb{R}^1$ is connected.

Theorem 78. Let A,B be separated subsets of a space X. If C is a connected subset of $A \cup B$, then $C \subset A$, or $C \subset B$.

Theorem 79. Let C be a connected subset of X. If D is a subset of X so that $C \subset D \subset \bar{C}$, then D is connected.

Example. Let

$$X = \{(x,y) \in \mathbb{R}^2 | x = 0, y \in [-1,1]\} \cup \{(x,y) \in \mathbb{R}^2 | x \in (0,1], y = \sin \tfrac{1}{x}\}.$$

This example is the closure of the $\sin 1/x$ curve.

Theorem 80. The closure of the $\sin 1/x$ curve is connected.

Theorem 81. Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of connected subsets of X and E be another connected subset of X so that for each α in λ,

$E \cap C_\alpha \neq \emptyset$. Then $E \cup (\bigcup_{\alpha\varepsilon\lambda} C_\alpha)$ is connected.

<u>Theorem</u> 82. Let $f: X \overset{\text{onto}}{\twoheadrightarrow} Y$ be a continuous function. If X is connected, then Y is connected.

<u>Theorem</u> 83. For spaces X and Y, $X \times Y$ is connected if and only if each of X and Y is connected.

<u>Theorem</u> 84. For spaces $\{X_\alpha\}_{\alpha\varepsilon\lambda}$, $\prod_{\alpha\varepsilon\lambda} X_\alpha$ is connected if and only if for each α in λ, X_α is connected.

<u>Theorem</u> 85. Let A be a countable subset of $\mathbb{R}^n (n \geq 2)$. Then $\mathbb{R}^n - A$ is connected.

<u>Theorem</u> 86. Let X be a countable, regular, T_1 space. Then X is not connected.

<u>Theorem</u> 87. Let X be a connected space, C a connected subset of X, and $X - C = A|B$. Then $A \cup C$ and $B \cup C$ are each connected.

<u>Definition</u>. Let X be a space and $p \in X$. The <u>component of p in X</u> is the union of all connected subsets of X which contain p.

<u>Theorem</u> 88. Each component of X is connected and closed.

Theorem 90. Let A and B be closed subsets of a compact, Hausdorff space X so that no component intersects both A and B. Then $X = H|K$ where $A \subset H$ and $B \subset K$.

Example. This example will demonstrate the necessity of the "compactness" hypothesis of Theorem 90. Let X be the subset of $\mathbb{R}^2$ equal to $([0,1] \times \bigcup_{i\in\omega}\{1/i\}) \cup \{(0,0), (1,0)\}$. Show that the conclusion to Theorem 90 fails when $A = \{(0,0)\}$ and $B = \{(1,0)\}$.

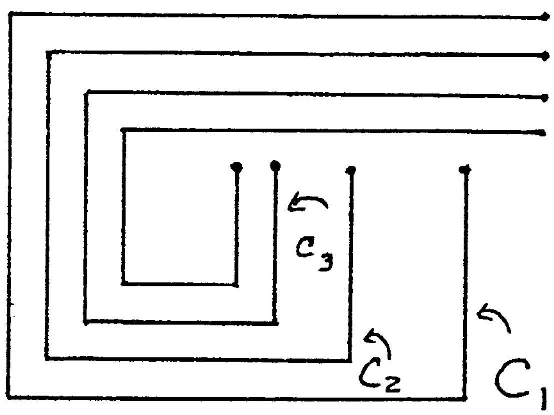
Definition. A continuum is a connected, compact, Hausdorff space.

Theorem 91. Let U be a proper, open subset of a continuum X. Then each component of $\bar{U}$ contains a point of Bd U. (Note: Bd $U = \bar{U} - U$.)

Theorem 92. ("To the boundary" theorem). Let U be a proper, open subset of a continuum X. Then each component of U has a limit point on Bd U.

Theorem 93. No continuum X is the union of a countable number (>1) of disjoint closed subsets.

Example. This example shows the necessity of the compactness hypothesis on X



The example X pictured above is a subset of the plane which is the union of a countable number of arcs as shown. Show that X is connected.

<u>Theorem</u> 94. Let $\{C_i\}_{i\varepsilon\omega}$ be a collection of continua so that for each i, $C_{i+1} \subset C_i$. Then $\bigcap_{i\varepsilon\omega} C_i$ is a continuum .

<u>Theorem</u> 95. Let $\{C_\alpha\}_{\alpha\varepsilon\lambda}$ be a collection of continua indexed by a well-ordered set λ so that if $\alpha < \beta$, then $C_\beta \subset C_\alpha$. Then $\bigcap_{\alpha\varepsilon\lambda} C_\alpha$ is a continuum.

<u>Definition</u>. Let X be a connected set. A point p in X is a <u>non-separating point</u> iff $X - \{p\}$ is connected. Otherwise p is a <u>separating</u> point.

<u>Theorem</u> 96. Let X be a continuum, p be a point of X, and $X - \{p\} = H|K$. Then $H \cup \{p\}$ is a continuum and if $q \neq p$ is a non-separating point of $H \cup \{p\}$, then q is a non-separating point of X.

<u>Theorem</u> 97. Let X be a metric continuum. Then X has at least two non-separating points.

<u>Theorem</u> 98. Let X be a continuum. Then X has at least two non-separating points.

Theorem 99. Let X be a metric continuum with exactly two non-separating points. Then X is homeomorphic to $[0,1]$.

Definition. A space X is *locally connected at the point p* of X if and only if for each open set U containing p, there is a connected open set V so that $p \in V \subset U$. A space X is *locally connected* if and only if it is locally connected at each point.

Theorem 100. The following are equivalent:

(i) X is locally connected.

(ii) X has a basis of connected open sets.

(iii) For each ρ in X and open set U containing ρ, the component of ρ in U is open.

(iv) For each ρ in X and open set U containing ρ, there is a connected set C so that $\rho \in \operatorname{Int} C \subset C \subset U$.

(v) For each ρ in X and open set U containing ρ, there is an open set V containing ρ and $V \subset$ (the component of ρ in U).

Theorem 101. Let X be a locally connected space and $f: X \to Y$ be an onto, closed or open map. Then Y is locally connected.

Definition. A *Peano Continuum* is a locally connected metric continuum.

Theorem 102. A Hausdorff space X is a Peano Continuum if and only if X is the image of $[0,1]$ under a continuous function.

Definitions. A space X is *arc-wise connected* iff for each pair of points $\rho, q \in X$ there is an embedding $h: [0,1] \to X$ so that $h(0) = \rho$ and $h(1) = q$.

A space X is *locally arc-wise connected at* ρ iff for each open set U containing ρ there is an open set V containing ρ so that for each pair of points $x,y \in V$, there is an arc in U which contains x and y. (Note: "an arc" means the homeomorphic image of $[0,1]$).

A space is *locally arc-wise connected* iff it is locally arc-wise connected at each point.

Theorem 103. An arc-wise connected space is connected.

Theorem 104. A locally arc-wise connected space is locally connected.

Theorem 105. A Peano Continuum is arc-wise connected and locally arc-wise connected.

31C. *Theorem* 106. An open, connected subset of a Peano continuum is arc-wise connected.

Metric Spaces

<u>Definitions</u>. Suppose X is a set. A function d from $X \times X$ into R^1_+, the non-negative reals, is a <u>metric</u> for X if and only if the following conditions are satisfied.

(i) $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$, and

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

If d is a metric for X, then $d(x,y)$ is called the distance from x to y.

Suppose X is a set, d is a metric for X, $p \in X$, and $\varepsilon \in R^{\cdot}_+$. Then the <u>open</u> ε <u>ball about</u> p is defined by $B_\varepsilon(p) = \{x \in X \mid d(x,p) < \varepsilon\}$. The <u>d-metric topology</u> for X is the topology whose basis is all the $B_\varepsilon(p)$'s. (Check that the collection of all open ε balls is a basis.)

Now suppose that $(X,\mathcal{T})$ is a <u>metric space</u> (or <u>metrizable</u>) iff there is a metric d on X for which $\mathcal{T}$ is the d-metric topology. If X is a metric space, then the statement that d <u>is a metric for</u> X means that the d-metric topology is the topology for X.

Notice that the same metric space may have many different metrics. As an exercise find several different metrics for R^n.

<u>Example</u>. For any set X, define a metric d on X by $d(x,y) = 1$ if $x \neq y$, $d(x,x) = 0$. What is the d-metric topology on X?

<u>Theorem</u> M.1. If X is a metric space and $Y \subset X$, then Y is a metric space.

Theorem M.2. If X is a metric space, then there is a metric d for X so that for each $x, y \in X$, $d(x,y) < 1$.

Theorem M.3. Let X be a metric space. Then X is perfectly normal.

Definition. A collection $\{B_\alpha\}_{\alpha\in\lambda}$ is a discrete collection of subsets of X if and only if for each point p in X there is an open set U containing p so that U intersects at most one B_α.

Definition. A space X is collectionwise normal if and only if for each discrete collection of closed sets $\{H_\alpha\}_{\alpha\in\lambda}$ in X, there is a collection of disjoint open sets $\{U_\alpha\}_{\alpha\in\lambda}$ so that for each α in λ, $H_\alpha \subset U_\alpha$.

Theorem M.4. Every metric space is collectionwise normal.

Theorem M.5. If X is metrizable and Y is metrizable, then $X \times Y$ is metrizable.

Theorem M.6. If $\{X_i\}_{i\in\omega_0}$ is a collection of metric spaces, then $\prod_{i\in\omega_0} X_i$ is metrizable.

Theorem M.7. Let d_1 be a metric for X and d_2 be a metric for Y. A function $f\colon X \to Y$ is continuous if and only if for each $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ so that $d_1(x,y) < \delta$ implies that $d_2(f(x),f(y)) < \varepsilon$.

Theorem M.8. In a metric space X, the following are equivalent:

(a) X is 2nd countable,

(b) X has the Souslin property,

(c) X is Lindelöf,

(d) X is separable,

(e) every uncountable set in X has a limit point.

<u>Theorem</u> M.9. If a metric space is countably compact, it is compact.

<u>Theorem</u> M.10. Let C be a compact subset of a metric space X and $\{U_\alpha\}_{\alpha\epsilon\lambda}$ be a collection of open sets in X so that $C \subset \bigcup_{\alpha\epsilon\lambda} U_\alpha$. Then there is an $\varepsilon > 0$ so that for every set S with diameter less than ε in X where $S \cap C \neq \emptyset$, there is a U_α so that $S \subset U_\alpha$.

<u>Definition</u>. Let S be a subset of a metric space X. Then the <u>diameter of</u> S equals $\sup\{d(x,y)\,|\,x,y \in S\}$.

<u>Definition</u>. In the situation described in Theorem M.10, any number ε satisfying the conclusion is called a <u>Lebesgue number</u>.

<u>Definition</u>. Let X be a metric space with metric d. A sequence $\{x_i\}_{i\epsilon\omega_0}$ of points in X is a <u>Cauchy sequence</u> if and only if for each $\varepsilon > 0$, there is an integer M so that for all $m,n > M$, $d(x_m, x_n) < \varepsilon$.

<u>Definition</u>. Let d be a metric for X. Then d is a <u>complete metric</u> for X if and only if every d-Cauchy sequence in X converges.

<u>Definition</u>. A space X is <u>complete</u> or is a <u>complete metric space</u> iff there is a complete metric for X.

<u>Theorem</u> M.11. The space $\mathbb{R}^n$ is complete.

<u>Theorem</u> M.12. There is a metric for $\mathbb{R}^1$ which is not complete.

<u>Theorem</u> M.13. A closed subset of a complete space is complete.

<u>Theorem</u> M.14. An open set U of a metric space X can be embedded as a closed subset of $X \times \mathbb{R}^1$.

<u>Theorem</u> M.15. If X and Y are complete metric spaces, then $X \times Y$ is complete.

<u>Theorem</u> M.16. If $\{X_i\}_{i\in\omega_0}$ is a collection of complete spaces, then $\prod_{i\in\omega_0} X_i$ is complete.

<u>Theorem</u> M.17. An open set U of a complete space X is complete.

<u>Theorem</u> M.18. Let X be a complete metric space and $Y \subset X$. Then Y is complete if and only if Y is a G_δ subset of X.

<u>Theorem</u> M.19. Let X be a compact metric space. Then every metric for X is a complete metric for X.

<u>Theorem</u> M.20. Let X be a metric space. If X is not compact, there is a metric for X which is not complete.

<u>Definition</u>. Let Y be a space with metric d. A sequence of continuous maps $f_i: X \to Y$ <u>converges uniformly</u> iff for every $\varepsilon > 0$, there is an integer M so that for every $x \in X$ and $m,n > M$, $d(f_m(x), f_n(x)) < \varepsilon$.

<u>Theorem</u> M.21. Let Y be a metric space with a complete metric d. If a sequence of continuous maps $f_i: X \to Y$ converges uniformly, then $\lim_{i\to\infty} f_i = f$ exists and is continuous.

<u>Theorem</u> M.22. Let X be a complete metric space and $\{U_i\}_{i\in\omega_0}$ be a collection of dense open sets. Then $\bigcap_{i\in\omega_0} U_i$ is a dense set.

<u>Definition</u>. A subset Y of a space X is <u>nowhere dense</u> if and only if $\mathrm{Int}(\overline{Y}) = \emptyset$.

<u>Theorem</u> M.23. Let X be a complete metric space. Then X is not the union of countably many nowhere dense sets.

<u>Note</u> M.24. For a space X the following are equivalent:

(a) no open subset of X is the union of countably many nowhere dense sets, and

(b) if $\{U_i\}_{i\in\omega_0}$ is a collection of dense open sets in X, then $\bigcap_{i\in\omega_0} U_i$ is dense.

<u>Definition</u>. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of a space X. Then $\mathcal{W} = \bigcup_{i\in\omega_0}\mathcal{W}_i$ is a σ-<u>discrete</u> <u>open</u> (resp. <u>closed</u>) <u>refinement</u> of $\{U_\alpha\}_{\alpha\in\lambda}$ iff $\mathcal{W}$ is an open (resp. closed) refinement and for each i, $\mathcal{W}_i$ is a discrete collection of open (resp. closed) sets.

<u>Theorem</u>. M.25. Let X be a regular, T_1 space in which every open cover has a σ-discrete open refinement. Then X is paracompact.

<u>Theorem</u> M.26. Let X be a collectionwise normal, T_1 space in which every open cover has a σ-discrete closed refinement. Then X is paracompact.

<u>Lemma</u> M.27. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of X where λ is a well-ordered set, for each $\alpha\in\lambda$, $U_\alpha = \bigcup_{i\in\omega_0} F_{\alpha,i}$ where each $F_{\alpha,i}$ is a closed set, and for each $\alpha\in\lambda$ and $i\in\omega_0$, $C\ell(\bigcup_{\beta<\alpha} F_{\beta,i}) \subset \bigcup_{\beta<\alpha} U_\beta$.

Then $\{U_\alpha\}_{\alpha\in\lambda}$ has a σ-discrete closed refinement.

<u>Theorem</u> M.28. Every metric space is paracompact.

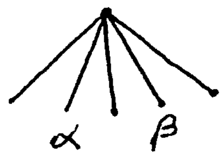
<u>Definition</u>. A collection $\mathcal{B} = \bigcup_{i\in\omega_0}\mathcal{B}_i$ is σ-<u>locally finite</u> (resp. σ-<u>discrete</u>) if and only if each $\mathcal{B}_i$ is locally finite (resp. discrete). Note: In general

a collection is σ-something if and only if it can be broken into a countable number of pieces each of which is something.

Theorem M.29. If X is a metric space, then X has a σ-locally finite basis.

Theorem M.30. If X is a metric space, then X has a σ-discrete basis.

Example M.1.



For a set λ consider the set of all ordered pairs (α, t) where $\alpha \in \lambda$ and $0 < t \leq 1$. Then one additional point 0 is added. Think of the cone pictured above. A metric d is put on this space as follows: $d((\alpha,t),(\beta,s)) = s + t$ if $\alpha \neq \beta$, $d((\alpha,t),(\alpha,s)) = |t-s|$, and $d((\alpha,t),0) = t$. This space is called a hedgehog.

Theorem M.31. Every metric space can be embedded in a countable product of hedgehogs.

Theorem M.32. Every separable metric space can be embedded in a countable product of intervals.

Urysohn's metrization
Theorem M.33. A 2nd countable, regular, T_1 space is metrizable.

Theorem M.34. Let X be a regular, T_1 space with a σ-discrete basis. Then X is metrizable.

Theorem M.25. Let X be a regular T_1 space with a σ-locally finite basis. Then X is a metrizable space.

<u>Metrization</u> <u>Theorem</u>. For a regular, T_1 space the following are equivalent:

(a) X is metrizable,

(b) X has a σ-discrete basis,

(c) X has a σ-locally finite basis,

(d) X can be embedded in a countable product of hedgehogs.

<u>Theorem</u> M.36. Let X be a metric space and $f: X \twoheadrightarrow Y$ be a closed, onto map so that for each $y \in Y$, $f^{-1}(y)$ is compact. Then Y is metrizable.

<u>Theorem</u> M.37. Let X be a compact metric space, Y be a Hausdorff space, and $f: X \twoheadrightarrow Y$ be an onto map. Then Y is a compact metric space.

<u>Theorem</u> M.38. Let X be a compact metric space. Then there is a continuous function from the Cantor set onto X.

The Cantor Set

Definition. Let $A_0 = [0,1]$, $A_1 = [0,1/3] \cup [2/3,1]$, $A_2 = [0,1/9] \cup [2/9,3/9]$ $[6/9,7/9] \cup [8/9,1]$, ... Then $\bigcap_{i\varepsilon\omega_0} A_i$ is the Standard Cantor Set. A space homeomorphic to the Standard Cantor Set is called a Cantor Set.

Theorem C.S.1. Let C be a Cantor Set. Then $|C| = |\mathbb{R}| = 2^{\omega_0}$.

Theorem C.S.2. The Standard Cantor Set has meausre 0. And for any $\lambda \varepsilon [0,1)$ there is a Cantor Set in $[0,1]$ with measure λ. ?

Theorem C.S.3. $\pi_{i\varepsilon\omega_0} \{0,1\}$ is a Cantor Set. ✓

Theorem C.S.4. Let C be a Cantor Set in $\mathbb{R}^1$, A a countable subset of $\mathbb{R}^1$ and $\varepsilon > 0$. Then $\exists$ a rigid translation h of $\mathbb{R}^1$ of distance less than ε so that $h(c) \cap A = \emptyset$.

Theorem C.S.5. Let X be a compact, metric space and let $\{A_i\}_{i\varepsilon\omega_0}$ be a collection of closed sets in X such that

(i) $A_{i+1} \subset A_i$,

(ii) $A_i = \cup\{A_{ik}\}_{k=1}^{n_i}$, where $\{A_{ik}\}_{k=1}^{n_i}$ is a collection of disjoint closed sets.

(iii) $A_0 \neq \emptyset$ and each A_{ik} contains at least two $A_{(i+1)k}$'s,

(iv) no A_{ik} has an isolated point,

(v) diam $A_{ik} < 1/2^i$; for each $i\varepsilon\omega_0$ and $k = 1,\ldots,n_i$. Then $\bigcap_{i\varepsilon\omega_0} A_i$ is a Cantor Set.

✓ Theorem C.S.6. There is a Cantor Set in $\mathbb{R}^2 - \{(0,0)\}$ such that every ray from the origin intersects it.

Theorem C.S.7. There is a Cantor Set C in $\mathbb{R}^2$ so that the graph of every continuous function from $[0,1] \to [0,1]$ intersects C.

Theorem C.S.8. A space X is a Cantor Set if and only if X is non-$\emptyset$ compact, metric, 0-dimensional and has no isolated points.

Theorem C.S.9. Let X be 2nd countable. Then $\exists$ a countable set $B \subset X$ so that every open set in X containing a point in $X - B$ contains an uncountable number of points in $X - B$.

Theorem C.S.10. Let X be a compact, metric space. Then $|X| \leq \omega_0$ or $|X| = 2^{\omega_0}$. Also if $|X| > \omega_0$, then X contains a Cantor Set.

Theorem C.S.11. Let X be a countable, compact, metric space. Then X has an isolated point.

Theorem C.S.12. There is a map from the Cantor Set onto $[0,1]$.

Theorem C.S.13. Let X be a compact, metric space. Then there is a map from the Cantor Set onto X.

Definition. Let C be the Standard Cantor Set. Then a point in C of the form $k/3^n$ where $n = 0,1,\ldots$ and $k = 0,1,\ldots,3^n$ is called an accessible point. All other points are non-accessible or inaccessible.

Theorem C.S.14. If C is the Standard Cantor Set, then there is a homeomorphism $h: C \twoheadrightarrow C$ which takes all the accessible points to non-accessible points.

Definition. Let C be a Cantor Set in some space X. Suppose $C = \bigcap_{i \in \omega_0} A_i$, where $A_{i+1} \subset \operatorname{Int} A_i$, each A_i has a finite number of components, and each component of each A_i is a cell (respectively, ______). Then C is definable by cells (respectively, by ______).

Definition. Let C be a Cantor Set in $\mathbb{R}^n$. Then C is tame iff $\exists$ a homeomorphism $H: \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $H(C)$ lies on a straight line. Otherwise, C is wild.

Theorem C.S.15. Every Cantor Set in $\mathbb{R}^n$ is definable by PL n-manifolds with boundary.

Theorem C.S.16. Let C be a Cantor Set in $\mathbb{R}^n$. Then C is tame iff C is definable by n-cells.

Theorem C.S.17. Let C be a Cantor Set in $\mathbb{R}^2$. Then C is tame.

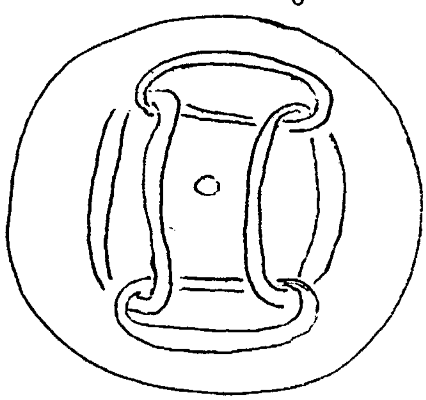
Theorem C.S.18. Let C be a Cantor Set in $\mathbb{R}^n \times \{0\}$. Then C is tame in $\mathbb{R}^{n+1}$.

Theorem C.S.19. Let C be a Cantor Set in $\mathbb{R}^n$. Then $\exists$ an embedding h of the arc $[0,1]$ into $\mathbb{R}^n$ so that $C \subset h([0,1])$.

Theorem C.S.20. Let C be a Cantor Set in $\mathbb{R}^n$. Then $\exists$ an embedding h of the n-cell B into $\mathbb{R}^n$ such that $C \subset h(\mathrm{Bd}\, B)$.

Theorem C.S.21. Let C be a Cantor Set in $\mathbb{R}^n$ and let X be a non-degenerate continuum in $\mathbb{R}^n$. Then $\exists$ a re-embedding h of X into $\mathbb{R}^n$ so that $C \subset h(X)$.

Theorem C.S.22. Let T_0 be a standard solid torus and T_1 be the union of the four solid tori in T_0 (see picture). Let T_{i+1} be obtained from T_i as T_i is obtained from T_0. Then $\bigcap_{i \in \omega_0} T_i$ is a wild Cantor Set in $\mathbb{R}^3$ called Antoine's Necklace.



Theorem C.S.23. There is a wild Cantor Set C in $\mathbb{R}^3$ with $\pi_1(\mathbb{R}^3 - C) = 1$.

Theorem C.S.24. There exist wild arcs, n-cells and (n-1)-spheres in $\mathbb{R}^n$ for $n \geq 3$.

Theorem C.S.25. Let C be a Cantor Set in $\mathbb{R}^n$ $(n \geq 2)$, $x, y \in \mathbb{R}^n - C$, $\varepsilon > 0$ and $\overline{xy}$ the straight line segment joining x and y. Then $\exists$ a homeomorphism $h: \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that

(i) $h(x) = x$ and $h(y) = y$

(ii) h equals the identity outside an ε-nbhd of $\overline{xy}$ and h moves points less than a distance of ε.

(iii) $h(\overline{xy}) \cap C = \emptyset$.

Classification of 2-manifolds

Definition. An n-manifold is a separable metric space M^n so that for each $p \in M^n$, there is an open set U containing p so that U is homeomorphic to $\mathbb{R}^n$.

Theorem 1. Let v_0, v_1 be two points in $\mathbb{R}^n$. Then $\sigma^1 = \{\mu v_0 + (1-\mu)v_1 \mid 0 \leq \mu \leq 1\}$ is the straight line segment between v_0 and v_1.

Definition. A set σ^1 as above is called a 1-simplex or edge with vertices v_0 and v_1.

Theorem 2. Let v_0, v_1, v_2 be non-colinear points in $\mathbb{R}^n$. Then $\sigma^2 = \{\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_0 + \lambda_1 + \lambda_2 = 1 \text{ and } 0 \leq \lambda_i \leq 1 \text{ for each } i = 0,1,2\}$ is a triangle with edges v_0v_1, v_1v_2, v_0v_2 and vertices $v_0, v_1,$ and v_2.

Definitions. 1. A set σ^2 as above is a 2-simplex with vertices $v_0, v_1,$ and v_2 and edges $v_0v_1, v_1v_2,$ and v_0v_2.

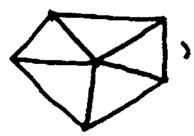
2. A triangulated compact 2-manifold is a space homeomorphic to a subset M^2 of $\mathbb{R}^n$ so that $M^2 = \bigcup_{i=1}^{k} \sigma_i$ so that:

(a) each σ_i is a 2-simplex,

(b) for $i \neq j$, $\sigma_i \cap \sigma_j$ is either $\emptyset$, an edge of each σ_i and σ_j, or a vertex of each,

(c) each edge of any σ_i is an edge of exactly two σ_i's, and

(d) for each vertex v of a σ_i, the union of all σ_i's containing v is homeomorphic to a polygonal disk, where v goes to the center and each simplex containing v goes linearly to one of the sectors.

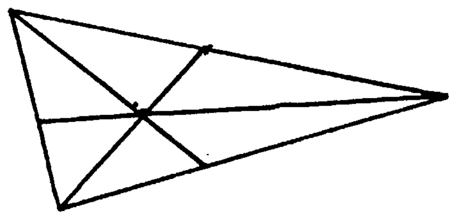


The set of 2-simplexes $\{\sigma_i\}_{i=1}^{k}$ above is called a <u>triangulation</u> of the 2-manifold.

<u>Theorem</u> 3. A triangulated, compact 2-manifold is a 2-manifold.

<u>Definitions</u>. 1. Let σ^2 be a 2-simplex with vertices v_0, v_1 and v_2. Then $p = 1/3v_0 + 1/3v_1 + 1/3v_2$ is the <u>barycenter</u> of σ^2.

2. Let $T = \{\sigma_i\}_{i=1}^{k}$ be a triangulation for a triangulated, compact 2-manifold M^2. The <u>first derived subdivision</u> of T, denoted T', is a collection of 2-simplexes obtained from T by breaking each σ_i in T into six pieces as shown:



where the new vertices are the barycenter of σ_i and the centers of each edge. The 2nd <u>derived subdivision</u>, denoted T", is (T')'.

<u>Theorem</u> 4. The first derived subdivision of a triangulation of a 2-manifold is also a triangulation of the 2-manifold.

<u>Definitions</u>. 1. Let M^2 be a 2-manifold with triangulation $T = \{\sigma_i\}_{i=1}^{k}$. Let A be the union of any subset of the elements of T or their edges or their vertices. The <u>regular neighborhood</u> of A, denoted N(A), equals $\cup \{\sigma_j'' | \sigma_j'' \in T'' \text{ and } \sigma_j'' \cap A \neq \emptyset\}$.

2. The <u>1-skeleton</u> of a triangulation T equals $\cup \{\sigma_j | \sigma_j$ is an edge of a 2-simplex in T$\}$ and is denoted $T^{(1)}$.

3. The <u>dual 1-skeleton</u> of a triangulation T equals $\cup \{\sigma_j | \sigma_j$ is an edge of a 2-simplex in T' and neither vertex of σ_j is a vertex of a 2-simplex of T$\}$.

Exercise. The boundary of a tetrahedron is naturally triangulated with four 2-simplexes. On the boundary of a tetrahedron locate the first and second derived subdivisions, the 1-skeleton, and its regular neighborhood, and the dual 1-skeleton for the natural triangulation.

Definitions 1. A graph G is the union of 1-simplexes $\{\sigma_i\}_{i=1}^k$ in R^n so that for $i \neq j$, $\sigma_i \cap \sigma_j$ is empty or an endpoint of each of σ_i and σ_j. The σ_i's are the edges of G.

2. A tree is a connected graph with no circuits.

3. Given a connected graph G with edges $\{\sigma_i\}_{i=1}^k$, a subgraph T of G is a maximal tree if and only if T is a tree and for any edge e of G not in T, $T \cup e$ has a circuit.

Theorem 5. Let G be a connected graph. Then G contains a maximal tree and every maximal tree for G contains every vertex of G.

Theorem 6. Let A_0 and A_1 be two subsets of a Hausdorff space X and let h_0 and h_1 be homeomorphisms of A_0 and A_1, respectively to D^2 $(= [0,1] \times [0,1])$. Suppose $A_0 \cap A_1$ is homeomorphic to an arc of the form $h_0^{-1}(\alpha) = h_1^{-1}(\beta)$ where α and β are arcs on $\mathrm{Bd}\, D^2$. Then $A_0 \cup A_1$ is homeomorphic to D^2

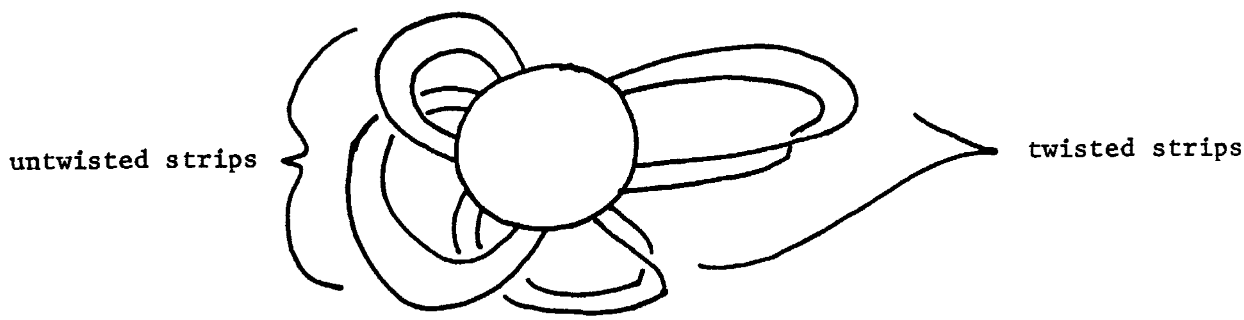
Theorem 7. Let M^2 be a compact, triangulated 2-manifold with triangulation T. Let S be a tree whose edges are 1-simplexes in the 1-skeleton of T. Then $N(S)$, the regular neighborhood of S, is homeomorphic to D^2.

Theorem 8. Let M^2 be a compact, triangulated 2-manifold with triangulation T. Let S be a tree whose edges are edges in the dual 1-skeleton of T. Then $\cup \{\sigma_j' \mid \sigma_j' \in T'' \text{ and } \sigma_j' \cap S \neq \emptyset\}$ is homeomorphic to D^2.

Theorem 9. Let M^2 be a connected, compact, triangulated 2-manifold with triangulation T. Let S be a maximal tree in the 1-skeleton of T. Let S' be the subgraph of the dual 1-skeleton of T whose edges do not intersect S. Then S' is connected.

Theorem 10. Let M^2 be a connected, compact, triangulated 2-manifold. Then $M^2 = D_0 \cup D_1 \cup (\bigcup_{i=1}^{k} H_i)$ where D_0, D_1, and each H_i is homeomorphic to D^2, $\operatorname{Int} D_0 \cap D_1 = \emptyset$, the H_i's are disjoint, $\bigcup_{i=1}^{k} \operatorname{Int} H_i \cap (D_0 \cup D_1) = \emptyset$, and for each i, $H_i \cap D_1 = 2$ disjoint arcs each arc on the boundary of each of H_i and D_1.

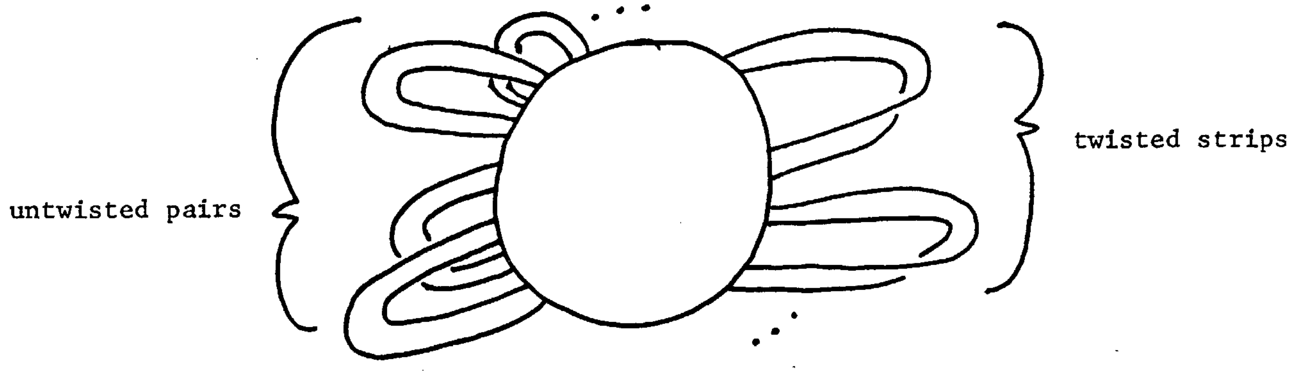
Theorem 11. Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 so that $M^2 - (\operatorname{Int} D_0)$ is homeomorphic to the following subset of R^3: a disk D_1 with a finite number of disjoint strips attached to boundary of D_1 where each strip has no twist or 1/2 twist. (See Example below.)



Figure

Note that the boundary of the disk with strips is one simple closed curve. (Why?)

<u>Theorem</u> 12. Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 so that $M^2 - \mathrm{Int}\, D_0$ is homeomorphic to a disk D_1 with strips attached as follows: first come a finite number of strips with 1/2 twist each whose attaching arcs are consecutive along $\mathrm{Bd}\, D_1$, next come a finite number of pairs of untwisted strips, each pair with attaching arcs entwined as pictured with the four arcs from each pair consecutive along $\mathrm{Bd}\, D_1$.



<u>Theorem</u> 13. Let M^2 be a connected compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 so that $M^2 - \mathrm{Int}\, D_0$ is homeomorphic to one of the following:

(a) a disk D_1,

(b) a disk D_1 with k 1/2 twisted strips with consecutive attaching arcs, or

(c) a disk D_1 with k pairs of untwisted strips, each pair in entwining position with the four attaching arcs from each pair consecutive.