Math 971 Algebraic Topology

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The fun comes when you try to compute them. $C_n(X) = \{\sum a_i\sigma_i : a_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ is continuous$\}$ is typically a huge group, since there will be immense numbers of maps $\Delta^n \to X$. About the only space for which this is not true is the one-point space $*$; then there are, for each n, exactly one (distinct) map $\sigma_n : \Delta^n \to *$; the constant map. Therefore each face of Δ^n gives the same restriction map σ^{n-1}, and so the boundary maps can be dirctly computed (the depend on the parity of the number $n+1$ of faces an n-simplex has). We find that $\partial_{2n} = Id$ and $\partial_{2n-1} = 0$. so in computing homology groups, we either have kernel everything ($\partial_i = 0$) and image everything ($\partial_{i+1} = Id$) or kernel nothing ($\partial_i = Id$) and image nothing ($\partial_{i+1} = 0$), so in both cases $H_i(*) = 0$. Except for $i = 0$; then $\partial_0 = 0$ (by definition) and $\partial_1 = 0$, so $H_0(*) = \mathbb{Z}$. But other than this example (and, well, OK, spaces with the discrete topology; it's the same calculation as above for every point!), computing singular homology from the definition is quite a chore! so we need to build some labor-saving devices, namely, some theorems to help us break the problem of computing these groups into smaller, more managable pieces.

First set of managable pieces: if we decompose X into its path components, $X = \bigcup X_\alpha$, then $H_i(X) \cong \bigoplus H_i(X_\alpha)$ for every i. This is because every singular simplex, since Δ^i is path-connected, maps into some X_α . So $C_i(X) \cong \bigoplus C_i(X_\alpha)$. Since the boundary of a simplex mapping into X_α consists of simplices in X_α, the boundary maps respect the decomposistions of the chain groups, so $B_i(X) \cong \bigoplus B_i(X_\alpha)$ and $Z_i(X) \cong \bigoplus Z_i(X_\alpha)$, and so the quotients are $H_i(X) \cong \bigoplus H_i(X_\alpha)$.

So, if we wish to, we can focus on computing homology groups for path-connected spaces X. For such a space, $H_0(X) \cong \mathbb{Z}$, generated by any map of a 0-simplex (= a point) into X. This is because any pair of 0-simplices are homologous; given any two points $x, y \in X$, there is a path $\gamma : I \to X$ from x to y, This path can be interpreted as a singular 1-simplex, and $\partial\gamma = y - x$. So $H_0(X)$ is generated by a single point $[x]$. No multiple of this point is null-homologous, because for any 1-chain $\sum n_i\sigma_i$, the sum of the coefficients of its boundary is 0 (since this is true for each singular 1-simplex), and any 0-chain $\sum n_i[x_i]$ is homologous to $(\sum n_i)[x]$ by the above argument.

Perhaps the most important property of the fundamental group is that a continuouos map between spaces induces a homomorphism between groups. (This implied, for instance, that homeomorphic spaces have isomorphic π_1). The same is true for homology groups, for essentially the same reason. Given a map $f : X \to Y$, there is an induced map $f_\# : C_n(X) \to C_n(Y)$ defined by postcomposition; for a singular simplex σ, $f_\#(\sigma) = f \circ \sigma$, and we extend the map linearly. Since $f \circ (g|_A) = (f \circ g)|_A$ (postcomposition commutes with restriction of the domain), $f_\#$ commutes with ∂ : $f_\#(\partial\sigma) = \partial(f_\#(\sigma))$. A homomorphism between chain complexes (i.e., a sequence of such maps, one for each chain group) which commutes with the boundaries maps in this way, is called a *chain map*. A chain map, such as $f_\#$, therefore, takes cycles to cycles, and boundaries to boundaries, and so $f_\# : Z_i(X) \to Z_i(Y)$ (which is linear, hence a homomorphism) induces a homomorphism $f_* : H_i(X) \to H_i(Y)$ by $f_*[z] = [f_\#(z)]$. Since it is defined by composition with singular simplices, it is immediate that, for a map $g : Y \to Z$, $(g \circ f)_* = g_* \circ f_*$. And since the identity map $I : X \to X$ satisfies $I_\# = Id$, so $I_* = Id$, homeomorphic spaces have isomorphic homology groups.

Another important property of π_1 is that homotopic maps give the same induced map (after correcting for basepoints). This is also true for homology; if $f \sim g : X \to Y$, then $f_* = g_*$. The proof, however, is not quite as straightforward as for homotopy. And it requires some new technology; the chain homotopy. A chain homotopy H between the chain complexes $f_\#, g_\# : C_*(X) \to C_*(Y)$ is a sequence of homomorphisms $H_i : C_i(X) \to C_{i+1}(Y)$ satisfying $H_{i-1}\partial_i + \partial_{i+1}H_i = f_\# - g_\# : C_i(X) \to C_i(Y)$. The existence of H implies that $f_* = g_*$, since for an i-cycle z (with $\partial_i(z) = 0$) we have

$f_*[z] - g_*[z] = [f_\#(z) - g_\#(z)] = [H_{i-1}\partial_i(z) + \partial_{i+1}H_i(z)] = [H_{i-1}(0) + \partial_{i+1}(w)] = [\partial_{i+1}(w)] = 0.$

And the existence of a homotopy between f and g implies the existence of a chain homotopy between $f_\#$ and $g_\#$. This is because the homotopy gives a map $H : X \times I \to Y$, which induces a map $H_\# : C_{i+1}(X \times I) \to C_{i+1}(Y)$. Then we pull, from our back pocket, a *prism map* $P : C_i(X) \to C_{i+1}(X \times I)$; the composition $H_\# \circ P$ will be our chain homotopy. The prism map takes a (singular) i-simplex σ and sends it to a sum of singular $(i+1)$-simplices in $X \times I$. and the way we define it is to take the i-simplex Δ^i, and taking it to $\Delta^i \times I$ (i.e., a *prism*), and thinking of this as a sum of $(i+1)$-simplices. Using the map $\sigma' = \sigma \times Id : \Delta^i \times I \to X \times I$ restricted to each of these $(i+1)$-simplices yields the prism map. Now, there are many ways of decomposing a prism into simplices, but we need to be careful to choose one which restricts well to each of the faces of Δ^i, in order to get the chain homotopy property we require. In the end, what this requires is that the decomposition, when restricted to any face of Δ^i (which we think of as a copy of Δ^{i-1}), is the same as the decomposition we would have applied to a prism over an $(i-1)$-simplex. After some exploration, we are led to the following formulation.

If we write $\Delta^n \times \{0\} = [v_0, \dots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \dots, w_n]$, then we can decompose $\Delta^n \times I$ as the (n+1)-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$. We then define $P(\sigma) = \sum (-1)^i \sigma'|_{[v_0,\dots,v_i,w_i,\dots,w_n]}$. A routine calculation verifies that $(\partial P + P\partial)(\sigma) = \sigma'|_{[w_0,\dots,w_n]} - \sigma'|_{[v_0,\dots v_n]}$; Composing with $H_\#$ yields our result.

Consequently, for example, homotopy equivalent spaces have isomporphic (reduced) homology groups; homotopy equivalences induce isomorphisms. So all contractible spaces have trivial reduced homology in all dimensions, since they are all homotopy to a point. If we think of a cell complex as a collection of disks glued together, this lends some hope that we can compute their homology groups, since we can compute the homology of the building blocks. Our next goal is to make turn this idea into action; but we need another tool, to frame our answer in the best way possible.

Exact sequences: Most of the fundamental properties of homology groups are described in terms of exact sequences. A sequence of homomorphisms $\cdots \overset{f_{n+1}}{\to} A_n \overset{f_n}{\to} A_{n-1} \overset{f_{n-1}}{\to} a_{n-2} \to \cdots$ of abelian groups is called *exact* if $\mathrm{im}(f_n) = \ker(f_{n-1})$ for every n. In most cases, we get the most mileage out of an exact sequence when some of the groups are trivial; $0 \to A \overset{f}{\to} B$ is exact $\Leftrightarrow$ f is injective, and $A \overset{f}{\to} B \to 0$ is exact $\Leftrightarrow$ f is surjective. An exact sequence $0 \to A \to B \to C \to 0$ is called a *short exact sequence.*

The main tool we will use turns a family of short exact sequences of chain maps between three chain complexes into a single *long exact homology sequence.* Given chain complexes $\mathcal{A} = (A_n, \partial)$, $\mathcal{B} = (B_n, \partial')$, and $\mathcal{C} = (C_n, \partial'')$ and short exact sequences of chain maps (i.e., $\partial' i_n = i_n \partial$, $\partial'' j_n = j_n \partial'$)
$0 \to A_n \overset{i_n}{\to} B_n \overset{j_n}{\to} C_n \to 0$ there is a general result which provides us with a long exact sequence

$$\cdots \overset{\partial}{\to} H_n(\mathcal{A}) \overset{i_*}{\to} H_n(\mathcal{B}) \overset{j_*}{\to} H_n(\mathcal{C}) \overset{\partial}{\to} H_{n-1}(\mathcal{A}) \overset{i_*}{\to} \cdots$$

Most of the work is in defining the "boundary" map ∂. Given an element $[z] \in H_n(\mathcal{C})$, a representative $z \in C_n$ satisfies $\partial''(z) = 0$. But j_n is onto, so there is a $b \in B_n$ with $j_n(b) = z$, Then $i_{n-1}\partial'(b) = \partial'' j_n(b) = 0$, so $\partial'(b) \in \ker(j_{n-1} = \mathrm{im}(a_{n-1})$. So there is an $a \in A_{n-1}$ with $i_{n-1}(a) = \partial'(b)$. But then $i_{n-2}\partial(a) = \partial' i_{n-1}(a) = \partial'\partial'(b) = 0$, so, since i_{n-2} is injective, $\partial a = 0$, so $a \in Z_{n-1}(\mathcal{A})$, and so represents a homology class $[a] \in H_n(\mathcal{A})$. We define $\partial([z]) = [a]$.
To show that this is well-defined, we need to show that the class $[a]$ we end up with is independent of the choices made along the way. The choice of a was not really a choice; i_{n-1} is, by assumption, injective. For b, if $j_n(b) = z = j_n(b')$, then $j_n(b - b') = 0$, so $b - b' = i_n(w)$ for some $w \in A_n$. Then $\partial' b' = \partial' b - \partial' i_n(w) = \partial' b - i_{n-1}\partial(w)$, so choosing $a' = a - \partial(w)$ we have $i_{n-1}(a') = \partial'(b')$. But then $[a'] = [a - \partial w] = [a] - [delw] = [a]$. Finally, there is actually a choice of z ; if $[z] = [z']$, then $z' = z + \partial'' w$ for some $w \in C_{n+1}$; but then choosing b', w' with $j_n(b') = z'$, $j_{n+1}(w') = w$, we have
$\partial'' w = \partial'' j_{n+1}(w') = j_n \partial'(w')$, so
$z' = z + \partial'' w = j_n(b + \partial' w')$, so we may choose $b' = b + \partial' w'$ (since the result is independent of this choice!), then since $\partial' b' = \partial' b$ everything continues the same.

Now to exactness! We need to show three (types of) equalities, which means six containments. Three (image contained in kernel) are shown basically by showing that compositions of two consecutive homomorphisms are trivial. $j_n i_n = 0$ immediately implies $j_* i_* = 0$. From the definition of ∂, $i_*\partial[z] = [i_n(a)] = [\partial'(b)] = 0$, and $\partial j_*[z] = \partial[j_n(z)] = [a]$, where $i_{n-1}(a) = \partial'(z) = 0$, so $a = 0$ (since i_{n-1} is injective), so $[a] = 0$.

For the opposite containments, if $j_*[z] = [j_n(z)] = 0$, then $j_n(z) = \partial'' w$ for some w. Since j_{n+1} is onto, $w = j_{n+1}(b)$ for some b. Then $j_n(z - \partial' b) = \partial'' w - \partial'' j_{n+1} b = 0$, so $z = \partial' b = i_n(a)$ for some a, so

$i_*[a] = [z - \partial' b] = [z]$. So $\ker j_* \subseteq \text{im} i_*$. If $i_*[z] = 0$, then $i_n(z) = \partial' w$ for some $w \in B_{n+1}$. Setting $c = j_{n+1}(w)$, then $\partial'' c = j_n \partial' w - i_n i_n(Z) = 0$, so $[c] \in h_{n+1}(\mathcal{C})$, and computing $\partial[c]$ we find that we can choose w for the first step and z for the second step, so $\partial[c] = [z]$. So $\ker j_n \subseteq \text{im}\partial$. Finally, if $\partial[z] = 0$, then $z = j_n(b)$ for some b, and $\partial' b = i_{n-1}(a)$ with $[a] = 0$, i.e., $a = \partial w$ for some w. So $\partial' b = i_{n-1}\partial w = \partial' i_n w$ But then $\partial'(b - i_n w) = 0$, and $j_n(b - i_n w) = z - 0 = z$, so $z \in \text{im}(j_n)$, so $[z] \in \text{im}(j_*)$. So $\ker \partial \subseteq \text{im}(j_n)$. Which finishes the proof!

Now all we need are some new chain complexes. To start, we build the singular chain complex of a pair (X, A) , i.e., of a space X and a subspace $A \subseteq X$. Since as abelian groups we can think of $C_n(A)$ as a subgroup of $C_n(X)$ (under the injective homomorphism induced by the inclusion $i : A \to X$) we can set $C_n(X, A) = C_n(X)/C_n(A)$. Since the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ satisfies $\partial_n(C_n(A) \subseteq C_{n-1}(A)$ (the boundary of a map into A maps into A), we get an induced boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. These groups and maps $(C_n(X, A), \partial_n)$ form a chain complex, whose homology groups are the *singluar relative homology groups of the pair* (X, A) . To be a cycle in relative homology, you need to have a representative z with $\partial z \in C_{n-1}(A)$, i.e., you are a chain with boundary in A. To be a boundary, you need $z = \partial w + a$ for some $w \in C_{n+1}(X)$ and $a \in C_n(A)$, i.e., you *cobound* a chain in A ($\partial w = z - a$).

The inclusion i_n and projection p_n maps give us short exact sequences $0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$, and since the boundary on chains in X restricts to the boundary on A and induces the boundary on (X, A), i_n and p_n are chain maps. So we get a long exact homology sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

There is also a long exact sequence of a triple (X, A, B) , where by triple we mean $B \subseteq A \subseteq X$. From the short exact sequences $0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$ (i.e., $0 \to C_n(A)/C_n(B) \to C_n(X)/C_n(B) \to C_n(X)/C_n(A) \to 0$) we get the long exact sequence

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to H_{n-1}(X, B) \to \cdots$$

A small techincal aside: te fact that $H_0(*) = \mathbb{Z}$ is annoying to some, and opften required treating 0-dimensional homology as a special case. But since the boundary of a singular 1-simplex is always $v - w$, we find that the image of ∂_1 is always contained in the subgroup of $C_0(X)$ consisting of chains whose coefficients sum to 0. This means that we can, for free, *augment* the singular chain complex by a map $\cdots \to C_1(X) \overset{\partial_1}{\to}$
From these humble beginnings we can do some meaningful calculations!