Computing $\langle a_0,\dots,a_n\rangle$ from $\langle a_0,\dots,a_{n-1}\rangle$:

Prop:

$$\langle a_0,\dots,a_i\rangle = \frac{h_i}{k_i} \quad \text{where}$$

$$h_{-2}=0,\ h_{-1}=1 \qquad h_i = a_i h_{i-1} + h_{i-2}$$

$$k_{-2} = \cancel{0}1,\ k_{-1}=0 \qquad k_i = a_i k_{i-1} + k_{i-2}$$

Pf: Induction $h_0 = a_0 \cdot 1 + 0 = a_0$ $\quad k_0 = a_0 \cdot 0 + 1 = 1$

$$\frac{h_0}{k_0} = a_0 = \langle a_0\rangle \ \checkmark$$

Suppose true for any ctd fraction of length i, $\langle a_0,\dots,a_{i-1}\rangle$. Then

$$\langle a_0,\dots,a_i\rangle = \langle a_0,\dots,a_{i-1},a_i\rangle = \langle \underbrace{a_0,\dots,a_{i-2}}, a_{i-1}+\frac{1}{a_i}\rangle$$

$$= \frac{\cancel{h_i}\,(a_{i-1}+\frac{1}{a_i})h_{i-2} + h_{i-3}}{(a_{i-1}+\frac{1}{a_i})k_{i-2} + k_{i-3}}$$

$$= \frac{\cancel{a_i}(\frac{1}{a_i}h_{i-2} + h_{i-1})}{\frac{1}{a}k_{i-2} + k_{i-1}} = \frac{h_{i-2} + a_i h_{i-1}}{k_{i-2} + a_i k_{i-1}} = \frac{h_i}{k_i}. \ ///$$

note: k_i's never "really" involve a_0 (which might be <0) so $k_i \geq 0$ all i and $k_{i} > k_{i-1}$ all $i \geq 0$.

Prop: $(h_i, k_i) = 1$ for all i. In fact

$$h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}, \text{ and}$$

$$h_i k_{i-2} - h_{i-2} k_i = (-1)^i a_i$$

Pf: By induction

~~$h_i k_{i-1} - h_{i-1} k_i$~~ ~~$k_2$~~ $h_{-1} \cdot k_{-2} - h_{-2} \cdot k_{-1} = 1 \cdot 1 - 0 \cdot 0 = 1 = (-1)^{-1-1}$

If $h_{i-1} k_{i-2} - h_{i-2} k_{i-1} = (-1)^{(i-1)-1} = (-1)^i$ then

~~$h_i k_{i-2}$~~ $$h_i k_{i-1} - h_{i-1} k_i = (a_i h_{i-1} + h_{i-2}) k_{i-1} - h_{i-1}(a_i k_{i-1} + k_{i-2})$$

$$= a_i h_{i-1} k_{i-1} - a_i h_{i-1} k_{i-1} - (h_{i-1} k_{i-2} - h_{i-2} k_{i-1}) = 0 - (-1)^i = (-1)^{i-1}$$

~~Similarly $h_0 k_{-2} - h_{-2} k_0 = a_0 \cdot 1 - 0 \cdot 1 = a_0 = (-1)^0 a_0$~~

Then ~~if $h_{i-1} k_{i-3} - h_{i-3} k_{i-1} = (-1)^{i-1} a_{i-1}$ then~~

$$h_i k_{i-2} - h_{i-2} k_i = (a_i h_{i-1} + h_{i-2}) k_{i-2} - h_{i-2}(a_i k_{i-1} + k_{i-2})$$

$$= a_i (h_{i-1} k_{i-2} - h_{i-2} k_{i-1}) + h_{i-2} k_{i-2} - h_{i-2} k_{i-2}$$

$$= a_i (-1)^{(i-1)-1} = (-1)^i a_i$$

Setting $r_i = \langle a_0, \dots, a_i \rangle$, we then have

$$r_i - r_{i-1} = \frac{h_i}{k_i} - \frac{h_{i-1}}{k_{i-1}} = \frac{h_i k_{i-1} - h_{i-1} k_i}{k_i k_{i-1}} = \frac{(-1)^{i-1}}{k_i k_{i-1}}$$

and

$$r_i - r_{i-2} = \dots = \frac{h_i k_{i-2} - h_{i-2} k_i}{k_i k_{i-2}} = \frac{(-1)^i a_i}{k_i k_{i-2}}$$

Fr, Mo. Rest

Note that the k_i's are all positive for $i \geq 0$.

$k_0 = 1 \quad k_1 = a_1 k_0 + k_{-1} = a_1 \qquad k_i = a_i k_{i-1} + k_{i-2} > k_{i-1}$

Now what do we have? for a given $x \in \mathbb{R}$

$$x = \langle a_0, \dots, a_{n-1}, a_n + x_n \rangle = \langle a_0, \dots, a_n, \tfrac{1}{x_n} \rangle$$

If we look at $r_n = \langle a_0, \dots, a_n \rangle$ then

Book writes $\xi_n = \frac{1}{x_n}$

$$r_0 < r_2 < r_4 < \dots < r_{2m} < \dots$$
$$r_1 > r_3 > \dots > r_{2m+1} > \dots$$

and $r_{2m} - r_{2m-1} = \frac{1}{k_{2m} k_{2m-1}} < \frac{1}{2m(2m-1)}$

And since

$$x = \langle a_0, \dots, a_{n-1}, a_n + x_n \rangle < \langle a_0, \dots, a_{n-1}, a_n \rangle$$

$$\frac{a_n(h_{n-1}) + h_{n-2}}{a_n(k_{n-1}) + k_{n-2}} \qquad \frac{(a_n + x_n)(h_{n-1}) + h_{n-2}}{(a_n + x_n) k_{n-1} + k_{n-2}}$$

$$x_n(k_{n-1} h_{n-2} - h_{n-1} k_{n-2})$$

So

$r_{2n} < x < r_{2n+1}$

So the convergents $\langle a_0, \dots, a_n \rangle = \frac{h_n}{k_n}$ <u>converge</u> to x!

Basic facts:

$\langle a_0, \dots, a_n \rangle \in \mathbb{Q}$ Pf. Induction

If $\langle a_0, \dots, a_n \rangle = \langle b_0, \dots, b_m \rangle$ with $a_n, b_m \geq 2$, then $n = m$ & $a_i = b_i$ all i.

<u>Needs</u>.

$\langle 0, a_1, \dots, a_n \rangle < 1$ (if $a_n \geq 2$ or $n \geq 2$) Pf $\frac{1}{\langle a_1, \dots, a_n \rangle} = \langle a_1, \dots, a_n \rangle$

with $a_1 \geq 1 \Rightarrow \langle a_1, \dots, a_n \rangle \geq a_1 > 1$.

If $x = \langle a_0, a_1, \dots$ with $i \geq 1$ all $i \geq 1$, then

$x \notin \mathbb{Q}$

<u>We know</u> $0 < |x - r_n| = |x - \frac{h_n}{k_n}| < \frac{1}{k_n k_{n+1}}$

$\Rightarrow 0 < |\cancel{x k_n} h_n x - k_n| < \frac{1}{k_{n+1}}$

<u>If</u> $x = \frac{a}{b}$ then $0 < |h_n \frac{a}{b} - k_n| < \frac{1}{k_{n+1}}$

$\Rightarrow 0 < |h_n a - k_n b| < \frac{|b|}{k_{n+1}}$

$\lightning$ $\hookrightarrow$ eventually small!

Prop: If $x = \langle a_0, a_1, \dots \rangle = \lim \langle a_0, \dots, a_n \rangle$

then $x = a_0 + \dfrac{1}{\langle a_1, a_2, \dots \rangle} = a_0 + \dfrac{1}{\lim \langle a_1, \dots, a_n \rangle}$

Pf: $a_0 + \dfrac{1}{\lim \langle a_1, \dots, a_n \rangle} = \lim\, a_0 + \dfrac{1}{\langle a_1, \dots, a_n \rangle}$

$= \lim \langle a_0, \dots, a_n \rangle$. ∎

So ~~since if $x = \langle a_0, a_1, \dots \rangle = \lim \langle a_0, \dots, a_n \rangle$~~

then ~~$\lfloor x \rfloor$ since $\lfloor \langle a_0, \dots, a_n \rangle \rfloor = a_0$ for all n~~

~~and $\langle a_0, \dots, a_n \rangle < \langle a_0, a_1 \rangle$ all n so~~

$\langle a_0 \rangle < x < \langle a_0, a_1 \rangle$ then

$a_0 = \lfloor \langle a_0 \rangle \rfloor \le \lfloor x \rfloor \le \lfloor \langle a_0, a_1 \rangle \rfloor = a_0$ so $\lfloor x \rfloor = a_0$.

Then same pf as for rationals gives

Prop: If $x = \langle a_0, a_1, \dots \rangle = \langle b_0, b_1, \dots \rangle$

then $a_i = b_i$ for all $i \ge 0$.

$x \notin \mathbb{Q}$. for any $a \in \mathbb{Z}$

If $1 \le b < k_n$ then $\left|x - \frac{a}{b}\right| \ge \left|x - \frac{h_n}{k_n}\right|$

In fact, if $1 \le b < k_{n+1}$ then $|bx - a| \ge |k_n x - h_n|$

$1 \le b < k_n \Rightarrow |bx-a| \ge |k_n x - h_n| \Rightarrow \left|x - \frac{a}{b}\right| = \frac{|bx-a|}{b} > \frac{|bx-a|}{k_n} \ge \frac{|k_n x - h_n|}{k_n} = \left|x - \frac{h_n}{k_n}\right|$

Suppose not! $1 \le b < k_{n+1}$ and $|bx - a| < |k_n x - h_n|$

Then set $\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \stackrel{\mathbb{R}}{=}$ solution to

$$\begin{pmatrix} h_n & h_{n+1} \\ k_n & k_{n+1}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix} a \\ b\end{pmatrix} \quad \text{ie.} \quad (\underline{\text{Math 314!}})$$

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \frac{1}{h_n k_{n+1} - h_{n+1} k_n}\begin{pmatrix} k_{n+1} & -h_{n+1} \\ -k_n & h_n\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = (-1)^{n+1}\begin{pmatrix} k_{n+1} a - h_{n+1} b \\ h_n b - k_n a\end{pmatrix}$$

both $\in \underline{\underline{\mathbb{Z}}}$.

Note $\alpha, \beta \ne 0$, since otherwise then either

$\frac{a}{b} = \frac{h_n}{k_n}$ or $\frac{h_{n+1}}{k_{n+1}} \Rightarrow$ $b = k_{n+1}$ or $\left|x - \frac{a}{b}\right| = \left|x - \frac{h_n}{k_n}\right| \not< \left|x - \frac{h_n}{k_n}\right|$

$k_{n+1} \mid h_{n+1} b \Rightarrow k_{n+1} \mid b$

$\Rightarrow k_{n+1} \le b$.

Note also that

$\alpha < 0 \Rightarrow$ ~~$h_n\alpha + h_{n+1}\beta = a$~~ $\Rightarrow$

$k_n\alpha + k_{n+1}\beta = b \Rightarrow k_{n+1}\beta = b - k_n\alpha > 0 \Rightarrow \beta > 0$

and $\alpha > 0 \Rightarrow k_{n+1}\beta = b - k_n\alpha < k_{n+1} - k_n\alpha < k_{n+1} \Rightarrow \beta < 1 \Rightarrow \beta < 0$.

So α & β have opposite signs.

OTOH, $xk_n - h_n$ and $xk_{n+1} - h_{n+1}$ <u>also</u> have opposite signs

GK

~~Then since~~ $\cancel{h_n\alpha + h_{n+1}\beta = a}$, $\cancel{k_n\alpha + k_{n+1}\beta = b}$

So $\alpha(xk_n - h_n)$ and $\beta(xk_{n+1} - h_{n+1})$ have the same sign.

Then
$$xa - b = x(h_n\alpha + h_{n+1}\beta) - (k_n\alpha + k_{n+1}\beta)$$
$$= (xh_n - k_n)\alpha + (xh_{n+1} - k_{n+1})\beta$$

So
$$|xa - b| = |(xh_n - k_n)\alpha| + |(xh_{n+1} - k_{n+1})\beta|$$
$$> |(xh_n - k_n)\alpha| \geq |xh_n - k_n|$$

$\divideontimes$