Math 325 Problem Set 9 Solutions

Problems were due Friday, April 14.

32. [Zorn, p.182, # 6] Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous, and f is differentiable at $x = 0$, with $f(0) = f'(0) = 0$. Show that $h(x) = f(x)g(x)$ is also differentiable at $x = 0$ and $h'(0) = 0$.

    [Note that since we do <u>not</u> know that g is differentiable at $x = 0$, we <u>cannot</u> use the product rule....]

    What we need to show is that the difference quotient,
    $$\frac{h(x) - h(0)}{x - 0} = \frac{f(x)g(x) - f(0)g(0)}{x - 0} = \frac{f(x)g(x)}{x}$$
    must be close to 0 so long as $x - 0 = x$ is small enough. That is, given an $\epsilon > 0$ we need to produce a $\delta > 0$ so that $0 < |x - 0| = |x| < \delta$ implies that $|\frac{f(x)g(x)}{x}| < \epsilon$.

    But what we know is that $f'(0) = 0$, so we know that we can make $\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$ small, and that g is continuous (at $x = 0$), so we can make $|g(x) - g(0)|$ small. But this means that $g(x)$ cannot get <u>big</u>; in particular, as we have taken advantage of several times, there is a $\delta > 0$ so that $|x - 0| = |x| < \delta$ implies that $|g(x) - g(0)| < 11$, so $g(0) - 11 < g(x) < g(0) + 11$, so $|g(x)| <$max$\{|g(0) - 11|, |g(0) + 11|\} = N$. [This, formally, is because if $g(x) \geq 0$, then $|g(x)| = g(x) < g(a) + 11 = |g(a) + 11| \leq N$, while if $g(x) \leq 0$, then $|g(x)| = -g(x) < -(g(a) - 11) = |g(a) - 11| \leq N$; so, no matter which case we are in, we have $|g(x)| \leq N$.] So, so long as $|x - 0| < \delta$, we have (*) = $|\frac{f(x)g(x)}{x}| = |\frac{f(x)}{x}| \cdot |g(x)| < N|\frac{f(x)}{x}|$. If we ensure that <u>this</u> is less than ϵ, then we will have controlled (*).

    But this is something we can do! Given $\epsilon > 0$, we can find a $\delta' > 0$ so that $0 < |x - 0| = |x| < \delta'$ implies that $|\frac{f(x)}{x}| < \frac{\epsilon}{N}$. Then setting $\delta_0 =$min$\{\delta, \delta'\}$, we have $0 < |x| < \delta$ implies $|g(x)| < N$ <u>and</u> $|\frac{f(x)}{x}| < \frac{\epsilon}{N}$, so $|\frac{h(x) - h(0)}{x - 0} - 0| = |\frac{f(x)}{x}| \cdot |g(x)| < \frac{\epsilon}{N} \cdot N = \epsilon$, as desired.

33. As almost none of us learn, the angle sum formula for tangent is
    $$\tan(a + h) = \frac{\tan a + \tan h}{1 - \tan a \tan h}$$

    Use this to show directly from the ("limit as $h \to 0$" definition) that the derivative of $f(x) = \tan x$ is what you were told it is in calculus class.

[If you want something extra to do, derive this angle sum formula from the angle sum formulas for $\sin x$ and $\cos x$ (for fun!).]

We can go straight at this problem: starting from the angle sum formula, we can compute the difference quotient

$$\frac{\tan(a+h)-\tan a}{h} = \frac{1}{h}\left(\frac{\tan a+\tan h}{1-\tan a\tan h}-\tan a\right)$$

$$= \frac{1}{h}\cdot\frac{\tan a+\tan h-(\tan a)(1-\tan a\tan h)}{1-\tan a\tan h} = \frac{1}{h}\cdot\frac{\tan a+\tan h-\tan a+\tan^2 a\tan h}{1-\tan a\tan h}$$

$$= \frac{1}{h}\cdot\frac{\tan h+\tan^2 a\tan h}{1-\tan a\tan h} = \frac{\tan h}{h}\cdot\frac{1+\tan^2 a}{1-\tan a\tan h} = \frac{\sin h}{h}\cdot\frac{1}{\cos h}\cdot\frac{1+\tan^2 a}{1-\tan a\tan h}$$

But! As $h\to 0$, $\frac{\sin h}{h}\to 1$, $\cos h\to 1$, and $\tan h = \frac{\sin h}{\cos h}\to\frac{0}{1}=0$; the first is a computation from class (or use L'Hôpital!), and the second and third are because $\sin x$ and $\cos x$ are continuous at $x=0$. Putting these all together, we have, as $h\to 0$,

$$\frac{\tan(a+h)-\tan a}{h} = \frac{\sin h}{h}\cdot\frac{1}{\cos h}\cdot\frac{1+\tan^2 a}{1-\tan a\tan h}$$

$$\longrightarrow 1\cdot\frac{1}{1}\cdot\cdot\frac{1+\tan^2 a}{1-(\tan a)(0)} = 1+\tan^2 a = \sec^2 a$$

(since $1+\tan^2 a = \frac{\cos^2 a+\sin^2 a}{\cos^2 a} = \frac{1}{\cos^2 a} = \sec^2 a$).

So, from the difference quotient, $f(x)=\tan x$ has $f'(a)=\sec^2 a$, so $f'(x)=\sec^2 x$, just like your calculus instructor told you....

We can <u>prove</u> the angle sum formula for $\tan x$ by combining the angle sum formulas for $\sin x$ and for $\cos x$:

$\tan(a+h) = \frac{\sin(a+h)}{\cos(a+h)} = \frac{\sin a\cos h+\cos a\sin h}{\cos a\cos h-\sin a\sin h}$. Dividing top and bottom by $\cos a\cos h$ makes this

$\tan(a+h) = \frac{\frac{\sin a}{\cos a}\frac{\cos h}{\cos h}+\frac{\cos a}{\cos a}\frac{\sin h}{\cos h}}{\frac{\cos a}{\cos a}\frac{\cos h}{\cos h}-\frac{\sin a}{\cos a}\frac{\sin h}{\cos h}} = \frac{(\tan a)(1)+(1)(\tan h)}{(1)(1)-(\tan a)(\tan h)} = \frac{\tan a+\tan h}{1-\tan a\tan h}$, as desired.

34. [Zorn, p.193, # 2 (parts)]

(a) Use the product and chain rules to derive a general formula for the <u>second</u> derivative $(f\circ g)''(x)$; you should assume that $f''(x)$ and $g''(x)$ both exist.

$f''(x)$ exists means that $f'(x)$ exists (so f is differentiable) <u>and</u> $f'(x)$ is differentiable. Similarly, $g'(x)$ exists, so $g(x)$ is differentiable, and $g'(x)$ is differentiable. Then we can apply the product rule to $h(x)=f(x)g(x)$ and get $h'(x)=f'(x)g(x)+f(x)g'(x)$. But now each of the two products in this sum are differentiable, by the product rule, so their sum, $h'(x)$ is differentiable, and we find that

$$h''(x) = (f'(x)g(x))' + (f(x)g'(x))' = (f''(x)g(x) + f'(x)g'(x)) + (f'(x)g'(x) + f(x)g''(x))$$
$$= f''(x)g(x) + 2f'(x)g'(x)f(x)g''(x) \ .$$

(b) Find a 'hybrid product-chain rule' to express the derivative $(f \circ (gh))'(x)$; you should assume that f, g and h are all differentiable.

If we set $k(x) = g(x)h(x)$, then the product rule tells us that h is differentiable and $k'(x) = g'(x)h(x) + g(x)h'(x)$. We also have $(f \circ (gh))(x) = (f \circ k)(x)$ is differentiable, since f and k are, and the Chain Rule tells us that $(f \circ k)'(x) = f'(k(x)) \cdot k'(x) = f'(g(x)h(x)) \cdot (g'(x)h(x) + g(x)h'(x))$.

35. [Zorn, p.200, # 4] Use Rolle's Theorem to show, by induction, that a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ of degree n has <u>at</u> <u>most</u> n distinct roots (i.e., solutions to $p(x) = 0$).

[Hint: If p has degree n, then p' has degree $n - 1$ ...]

Working by induction, for $n = 1$ we have $p(x) = ax + b = 0$ only for $x = -b/a$; so there is one root if $a \neq 0$ and <u>no</u> root if $a = 0$. (But if $a = 0$ the polynomial actually has degree 0, not 1...)

Now suppose that we know that every polynomial of degree $n - 1$ has at most $n - 1$ distinct roots. Suppose that $p(x)$ is a polynomial of degree n; we want to show that it has at most n distinct roots. Well, suppose it <u>doesn't</u>! Suppose that $p(x)$ has $n + 1$ distinct roots, $x_1 < x_2 < \cdots < x_{n+1}$. Then for every $i = 1, \ldots n$ we have $p(x_i) = p(x_{i+1})$. But then p is continuous on $[x_i, x_{i+1}]$ (it is a polynomial), and p is differentiable on (x_i, x_{i+1}) (it is a polynomial!), so we can apply Rolle's Theorem. This tells us that there is a $c_i \in (x_i, x_{i+1})$ with $p'(c_i) = 0$.

But! since $x_i < c_i < x_{i+1}$, all of the c_i are distinct! This is becaus if $i \neq j$, then WOLOG $i < j$, so $i + 1 \leq j$, so $x_{i+1} \leq x_j$, <u>so</u> $x_i < c_i < x_{i+1} \leq x_j < c_j$, so $c_i < c_j$. This means that $p'(x)$, which has degree $n - 1$, has roots the n distinct numbers c_i, $i = 1, \ldots n$ (and possibly more!). But this contradicts our inductive hypothesis. So it is impossible for p to have $n + 1$ distinct roots, so p has at most n distinct roots.

This completes our inductive step; so every polynomial of degree n has at most n distinct roots.

N.B.: You have probably seen this result before, proved in a different way: if p has no (real) roots, then we are done. But if c is a root of the degree-n polynomial $p(x)$ then $p(x) = (x - c)q(x)$ for some polynomial q having degree $n - 1$. Then q (by an inductive argument) has at most $n - 1$ roots, and $p(r) = (r - c)q(r) = 0$ only if either $r - c = 0$ (so $r = c$) or $q(r) = 0$ (so r is a root of q). So the roots of p are the roots of q (at most $n - 1$ of them) plus c, so p has at most n roots.