# MATH 107

# SAMPLE FINAL

# FALL 2009

Math107
Calculus II

Fall 2009
University of Nebraska-Lincoln

**Name:** **Section/Instructor:**

________________________________________ ____________________

(print legibly)

# Final Exam

## Instructions

- Turn off all communication devices. If you do not do so, then you will not receive any credit for your exam.
- During the exam, you may use a calculator and one 3 inch by 5 inch note card. No other notes, text or reference may be used.
- The examination period is from 6:00pm to 8:00pm. If you wish to receive credit for your exam, then make sure that your exam is submitted to the proctor by 8:00pm.
- To receive full credit for a problem, you must provide a correct answer and a sufficient amount of work so that it can be determined how you arrived at your answer.
- Clearly indicate what your solutions are and any work that you do not want to be included in the grading process.
- There are 11 pages (including this one) in this exam with 10 problems. Before you begin, make sure that your exam has all 11 pages.
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| Problem | Score |
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| 1 | |
| 2 | |
| 3 | |
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| Problem | Score |
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| 6 | |
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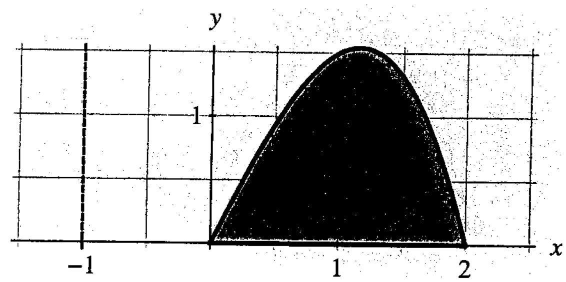
| 10 | |
| Total | |

1. Evaluate the following improper integrals. If an integral diverges, indicate so and provide calculations showing that it diverges.

   (a) (10 points) $\displaystyle\int_1^\infty \frac{1}{x^5}\, dx.$

   (b) (10 points) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1-x^4}}\, dx.$

2. The graph of the region bounded by $y = \frac{1}{2}x\left(4 - x^2\right)$ and $y = 0$, from $x = 0$ to $x = 2$, is given in the figure.




(a) (10 points) Set up, but DO NOT EVALUATE, an integral providing the volume of the solid generated by revolving the region in the figure about the $x$-axis.

(b) (10 points) Set up, but DO NOT EVALUATE, an integral providing the volume of the solid generated by revolving the region in the figure about the line $x = -1$.

3. (a) (5 points) Rewrite $\sum_{n=1}^{5} \frac{1}{n^{1.1}}$ without Sigma notation (DO NOT SIMPLIFY).

(b) (5 points) Is $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ a $p$-series, geometric series or alternating series? (Indicate all that apply.)

(c) (5 points) Does the $n^{\text{th}}$-term test tell you that the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges, diverges, or nothing at all?

(d) (5 points) Does $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converge or diverge? (JUSTIFY YOUR ANSWER.)

4. Determine if the following series converge absolutely, converge conditionally, or diverge. (JUSTIFY YOUR ANSWERS.)

(a) (10 points) $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n!}$

(b) (10 points) $\sum_{n=2}^{\infty}(-1)^n\frac{2}{\ln(n)}$

5. The following questions refer to the power series $\sum_{n=1}^{\infty} \frac{1}{n3^n}(x-2)^n$. (JUSTIFY YOUR ANSWERS.)

(a) (10 points) What is the radius of convergence for this power series?

(b) (10 points) What is the interval of convergence for this power series?

6. (a) (10 points) Find $P_3(x)$ (the Taylor polynomial of order 3) centered at 0 generated by the function $f(x) = x \sin x$.

(b) (10 points) With $F(x)$ the antiderivative of $f(x)$ satisfying $F(0) = 2$, find the Taylor polynomial of order 4 centered at 0 generated by $F(x)$.

7. An object's trajectory is given by the position vector

$$\vec{\mathbf{r}}(t) = \left(t^3 + 1\right)\vec{\mathbf{i}} + \left(t^2 - 5\right)\vec{\mathbf{j}}, \quad \text{for } 0 \le t \le 2,$$

with $t$ in seconds and distance in feet. The vector $\vec{\mathbf{i}}$ is the unit vector in the positive horizontal direction and $\vec{\mathbf{j}}$ is the unit vector in the positive vertical direction.

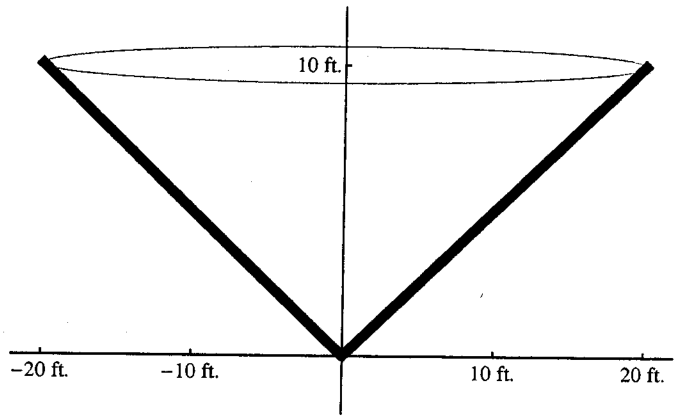
(a) (5 points) What is the vertical component of the velocity of the object when $t = 2$ seconds?

(b) (5 points) At $t = 2$, is the height of the object increasing, decreasing or neither?

(c) (5 points) What is the object's speed at $t = 2$ seconds?

(d) (5 points) What is the arclength of the object's path from $t = 0$ to $t = 2$ seconds?

8. (20 points) A fuel tank has the shape of an inverted right-circular cone. The height of the cone is 10 feet; the diameter of its top is 40 feet. The tank is full with fuel that has a weight density of 40 pounds/ft$^3$. The fuel is being pumped from the tank to just over its top. How much work is done to empty the tank?



9. The following problems refer to the vector $\vec{\mathbf{u}} = 4\vec{j} - 3\vec{k}$ and the points $P = (-2, 0, 1)$ and $Q = (2, 1, 3)$.

(a) (5 points) Compute $|\vec{\mathbf{u}}|$.

(b) (5 points) Provide a vector with 5 units of length that has the same direction as $\vec{\mathbf{u}}$.

(c) (5 points) Provide an equation for the sphere centered at $P$ and such that $Q$ is on the sphere.

(d) (5 points) Find the point $R$ satisfying $\overrightarrow{QR} = 3\vec{\mathbf{u}} + \overrightarrow{PQ}$.

10. This problem refers to the space curve described by

$$\vec{\mathbf{r}}(t) = 2\sin t\,\vec{\mathbf{i}} + 2\cos t\,\vec{\mathbf{j}} + 5t\,\vec{\mathbf{k}}.$$

(a) (10 points) Provide a vector equation for the line tangent to the space curve at the point with the position vector $\vec{\mathbf{r}}(4\pi)$.

(b) (10 points) What is the angle (in radians) between the line found in part (a) and the vector $\vec{\mathbf{u}} = -5\,\vec{\mathbf{i}} + \vec{\mathbf{j}} + 2\,\vec{\mathbf{k}}$?

Math107
Calculus II

Fall 2009
University of Nebraska-Lincoln

**Name:** SOLUTIONS

(print legibly)

**Section/Instructor:**

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1. Evaluate the following improper integrals. If an integral diverges, indicate so and provide calculations showing that it diverges.

(a) (10 points) $\int\limits_1^\infty \frac{1}{x^5}\,dx.$

**Solution:**

The integral is improper due to the infinite uppper limit of integration.

$$\int\limits_1^\infty \frac{1}{x^5}\,dx = \lim_{b\to\infty}\int\limits_1^b x^{-5}\,dx = \lim_{b\to\infty}\left[-\frac{1}{4}x^{-4}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{4b^4}\right)+\frac{1}{4}$$

$$\boxed{=\frac{1}{4}.}$$

(b) (10 points) $\int\limits_0^1 \frac{x^3}{\sqrt{1-x^4}}\,dx.$

**Solution:**

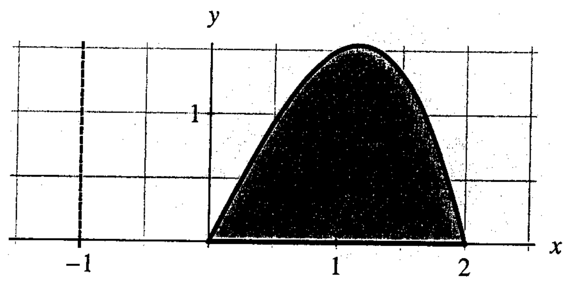
The integral is improper due to the discontinuity of the integrand at $x = 1$. We use the substitution $u = 1 - x^4$ and $du = -4x^3\,dx$.

$$\int\limits_0^1 \frac{x^3}{\sqrt{1-x^4}}\,dx = \lim_{b\to 1^-}\int\limits_0^b \frac{x^3}{\sqrt{1-x^4}}\,dx = \lim_{b\to 1^-}\left(-\frac{1}{4}\right)\int\limits_0^b \frac{-4x^3}{(1-x^4)^{\frac{1}{2}}}\,dx = -\lim_{b\to 1^-}\frac{1}{4}\int\limits_1^{1-b^4} u^{-\frac{1}{2}}\,du$$

$$= -\lim_{b\to 1^-}\frac{1}{4}\left[2u^{\frac{1}{2}}\right]_1^{1-b^4} = -\lim_{b\to 1^-}\frac{1}{4}\left(2(1-b^4)^{\frac{1}{2}}\right)+\frac{1}{4}\left(2(1)^{\frac{1}{2}}\right)$$

$$\boxed{=\frac{1}{2}.}$$

2. The graph of the region bounded by $y = \frac{1}{2}x\left(4 - x^2\right)$ and $y = 0$, from $x = 0$ to $x = 2$, is given in the figure.




(a) (10 points) Set up, but DO NOT EVALUATE, an integral providing the volume of the solid generated by revolving the region in the figure about the $x$-axis.

**Solution:**

We integrate with respect to $x$ and use the washer method. The inner radius of each washer in the solid is 0. The outer radius of each washer is $\frac{1}{2}x\left(4 - x^2\right)$. The limits of integration are from $x = 0$ to $x = 2$. The volume of the solid is

$$\boxed{V = \int_0^2 \pi \left[\left(\frac{1}{2}x\left(4 - x^2\right)\right)^2 - (0)^2\right] dx.}$$

(b) (10 points) Set up, but DO NOT EVALUATE, an integral providing the volume of the solid generated by revolving the region in the figure about the line $x = -1$.

**Solution:**

We integrate with respect to $x$ and use the cylindrical shell method. The radius of each shell in the solid is $x + 1$. The height of each shell is $\frac{1}{2}x\left(4 - x^2\right)$. The limits of integration are from $x = 0$ to $x = 2$. The volume of the solid is

$$\boxed{V = \int_0^2 2\pi(x + 1)\left(\frac{1}{2}x\left(4 - x^2\right)\right) dx.}$$

3. (a) (5 points) Rewrite $\sum_{n=1}^{5} \frac{1}{n^{1.1}}$ without Sigma notation (DO NOT SIMPLIFY).

**Solution:**

$$\sum_{n=1}^{5} \frac{1}{n^{1.1}} = \boxed{\frac{1}{1^{1.1}} + \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}}.}$$

(b) (5 points) Is $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ a $p$-series, geometric series or alternating series? (Indicate all that apply.)

**Solution:**

The series is a $\boxed{p\text{-series.}}$

(c) (5 points) Does the $n^{\text{th}}$-term test tell you that the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges, diverges, or nothing at all?

**Solution:**

Since

$$\lim_{n\to\infty} \frac{1}{n^{1.1}} = 0,$$

the $n^{\text{th}}$-term test tells us $\boxed{\text{nothing at all}}$. The series may or may not converge.

(d) (5 points) Does $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converge or diverge? (JUSTIFY YOUR ANSWER.)

**Solution:**

Since this is a $p$-series with $p = 1.1 > 1$, the series $\boxed{\text{converges.}}$ (It is also possible to use the integral test to determine this.)

4. Determine if the following series converge absolutely, converge conditionally, or diverge. (JUSTIFY YOUR ANSWERS.)

(a) (10 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n!}$

**Solution:**

First, we check for absolute convergence. We try the ratio test:

$$\lim_{n\to\infty} \frac{\frac{n+2}{(n+1)!}}{\frac{n+1}{n!}} = \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = 0.$$

Since $0 < 1$, the ratio test implies that the series $\boxed{\text{converges absolutely.}}$

(b) (10 points) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{2}{\ln(n)}$

**Solution:**

Again, we first check for absolute convergence. We try comparing $\displaystyle\sum_{n=2}^{\infty} \frac{2}{\ln(n)}$ to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$. We see that

$$\frac{2}{\ln(n)} \geq \frac{1}{n} \quad \text{for each integer } n \geq 2.$$

Also the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is a $p$-series with $p = 1$, so it diverges. By the (direct) comparison test, the series $\displaystyle\sum_{n=2}^{\infty} \frac{2}{\ln(n)}$ must also diverge. Hence the series $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{2}{\ln(n)}$ does not converge absolutely.

Now, we see if the series converges (conditionally). For this we try the alternating series test. We find that

- $\displaystyle\lim_{n\to\infty} \left|(-1)^n \frac{2}{\ln(n)}\right| = \lim_{n\to\infty} \frac{2}{\ln(n)} = 0$
- $\left|(-1)^{n+1} \frac{2}{\ln(n+1)}\right| = \frac{2}{\ln(n+1)} \leq \frac{2}{\ln(n)} = \left|(-1)^n \frac{2}{\ln(n)}\right|$ for each integer $n \geq 2$

By the alternating series test, the series $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{2}{\ln(n)}$ converges. Since it does not converge absolutelty, it $\boxed{\text{converges conditionally.}}$

5. The following questions refer to the power series $\sum_{n=1}^{\infty} \frac{1}{n3^n}(x-2)^n$. (JUSTIFY YOUR ANSWERS.)

(a) (10 points) What is the radius of convergence for this power series?

**Solution:**

We use the ratio test.

$$\lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)3^{n+1}}(x-2)^{n+1}\right|}{\left|\frac{1}{n3^n}(x-2)^n\right|} = \lim_{n\to\infty} \frac{n3^n}{(n+1)3^{n+1}}|x-2|$$
$$= \frac{1}{3}|x-2|.$$

By the ratio test, the series converges (absolutely) if $|x-2|<3$ and it diverges if $|x-2|>3$. Since $x=2$ is the center of the power series, the radius of convergence is $\boxed{3.}$

(b) (10 points) What is the interval of convergence for this power series?

**Solution:**

From the previous part, we already know that the series converges on the interval $(-1,5)$ and it diverges if $x<-1$ or $x>5$. It just remains to check what happens at $x=-1$ and at $x=5$.

If $x=-1$, then the power series reduces to $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which is an alternating harmonic series and converges (conditionally). It follows that $x=-1$ is in the interval of convergence.

If $x=5$, then the power series reduces to $\sum_{n=1}^{\infty}\frac{1}{n}$, which is the harmonic series and diverges. Hence $x=5$ is not in the interval of convergence.

The interval of convergence is $\boxed{[-1,5).}$

6. (a) (10 points) Find $P_3(x)$ (the Taylor polynomial of order 3) centered at 0 generated by the function $f(x) = x \sin x$.

**Solution:**

We compute

$$\begin{aligned} f^{(0)}(x) &= x \sin x & f^{(0)}(0) &= 0 \\ f^{(1)}(x) &= \sin x + x \cos x & f^{(1)}(0) &= 0 \\ f^{(2)}(x) &= 2 \cos x - x \sin x & f^{(2)}(0) &= 2 \\ f^{(3)}(x) &= -3 \sin x - x \cos x. & f^{(3)}(0) &= 0. \end{aligned}$$

The Taylor polynomial is

$$\boxed{P_3(x) = x^2.}$$

(b) (10 points) With $F(x)$ the antiderivative of $f(x)$ satisfying $F(0) = 2$, find the Taylor polynomial of order 4 centered at 0 generated by $F(x)$.

**Solution:**

One approach is to find the antiderivative $F(x)$ of $f(x)$ so that $F(0) = 2$ and then find the Taylor polynomial for $F(x)$. Instead, we can just use the antiderivative of the Taylor polynomial in the previous part:

$$\int P_3(x)\, dx = \int x^2\, dx = \frac{1}{3}x^3 + C.$$

We want the one antiderivative that equals 2 when $x = 0$, so we select $C = 2$. The required Taylor polynomial is

$$\boxed{2 + \frac{1}{3}x^3.}$$

7. An object's trajectory is given by the position vector

$$\vec{\mathbf{r}}(t) = \left(t^3 + 1\right) \vec{\mathbf{i}} + \left(t^2 - 5\right) \vec{\mathbf{j}}, \quad \text{for } 0 \le t \le 2,$$

with $t$ in seconds and distance in feet. The vector $\vec{\mathbf{i}}$ is the unit vector in the positive horizontal direction and $\vec{\mathbf{j}}$ is the unit vector in the positive vertical direction.

(a) (5 points) What is the vertical component of the velocity of the object when $t = 2$ seconds?

**Solution:**

We find that $\vec{\mathbf{r}}'(t) = \langle 3t^2, 2t \rangle$. Thus $\vec{\mathbf{r}}'(2) = \langle 12, 4 \rangle$. The vertical component of the velocity is $\boxed{4.}$

(b) (5 points) At $t = 2$, is the height of the object increasing, decreasing or neither?

**Solution:**

The vertical component of the velocity at $t = 2$ is 4, which is positive. It follows that the height of the object is $\boxed{\text{increasing}}$.

(c) (5 points) What is the object's speed at $t = 2$ seconds?

**Solution:**

$$|\vec{\mathbf{r}}'(2)| = |\langle 12, 4 \rangle| = \sqrt{12^2 + 4^2} = \boxed{\sqrt{160}.}$$

(d) (5 points) What is the arclength of the object's path from $t = 0$ to $t = 2$ seconds?

**Solution:**

Using the arc length formula,

$$L = \int_0^2 |\vec{\mathbf{r}}'(t)|\, dt = \int_0^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt = \int_0^2 \sqrt{9t^4 + 4t^2}\, dt$$

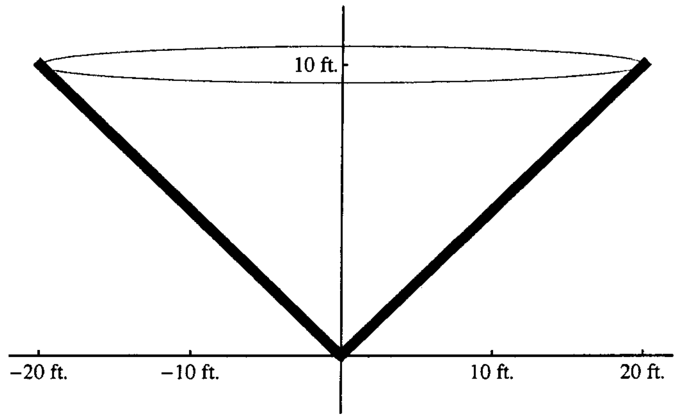
$$= \int_0^2 t\sqrt{9t^2 + 4}\, dt$$

Substituting $u = 9t^2 + 4$ and $du = 18t\, dt$, we have

$$L = \frac{1}{18}\int_4^{40} u^{\frac{1}{2}}\, du = \left[\frac{2}{27}u^{\frac{3}{2}}\right]_4^{40}$$

$$\boxed{= \frac{2}{27}\left(40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right).}$$

8. (20 points) A fuel tank has the shape of an inverted right-circular cone. The height of the cone is 10 feet; the diameter of its top is 40 feet. The tank is full with fuel that has a weight density of 40 pounds/ft$^3$. The fuel is being pumped from the tank to just over its top. How much work is done to empty the tank?




**Solution:**

The volume of a slice of fuel at height $y$ and with thickness $dy$ is

$$\pi(2y)^2\,dy = 4\pi y^2\,dy.$$

The weight of this slice is

$$40(4\pi y^2\,dy) = 160\pi y^2\,dy.$$

The slice needs to be lifted from a height of $y$ feet to a height of 10 feet to reach the top of the tank. The distance the slice is lifted is thus $10 - y$ feet. The work done on the slice of fuel to lift it to the top of the tank is

$$(10-y)(160\pi y^2\,dy) = 160\pi(10-y)y^2\,dy.$$

The total work done to empty the tank is

$$W = \int_0^{10} 160\pi(10-y)y^2\,dy = 160\pi\int_0^{10}\left(10y^2 - y^3\right)\,dy = 160\pi\left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_0^{10}$$

$$\boxed{= 160\pi\left(\frac{10000}{3} - \frac{10000}{4}\right) \text{ foot-pounds.}}$$

9. The following problems refer to the vector $\vec{u} = 4\vec{j} - 3\vec{k}$ and the points $P = (-2, 0, 1)$ and $Q = (2, 1, 3)$.

(a) (5 points) Compute $|\vec{u}|$.

**Solution:**

$$|\vec{u}| = \sqrt{4^2 + 3^2} = \boxed{5.}$$

(b) (5 points) Provide a vector with 5 units of length that has the same direction as $\vec{u}$.

**Solution:**

$\boxed{\vec{u}}$ itself is the required vector.

(c) (5 points) Provide an equation for the sphere centered at $P$ and such that $Q$ is on the sphere.

**Solution:**

If $Q$ needs to be on the sphere with center $P$, then the radius of the sphere is

$$|\overrightarrow{PQ}| = \sqrt{(2-(-2))^2 + (1-0)^2 + (3-1)^2} = \sqrt{21}.$$

The equation for the sphere is

$$\boxed{(x+2)^2 + y^2 + (z-1)^2 = 21.}$$

(d) (5 points) Find the point $R$ satisfying $\overrightarrow{QR} = 3\vec{u} + \overrightarrow{PQ}$.

**Solution:**

We have

$$3\vec{u} + \overrightarrow{PQ} = 3\langle 0, 4, -3\rangle + \langle 4, 1, 2\rangle = \langle 4, 13, -7\rangle.$$

It follows that

$$R = (x, y, z) = (2+4, 1+13, 3-7) = \boxed{(6, 14, -4).}$$

10. This problem refers to the space curve described by

$$\vec{\mathbf{r}}(t) = 2\sin t\,\vec{\mathbf{i}} + 2\cos t\,\vec{\mathbf{j}} + 5t\,\vec{\mathbf{k}}.$$

(a) (10 points) Provide a vector equation for the line tangent to the space curve at the point with the position vector $\vec{\mathbf{r}}(4\pi)$.

**Solution:**

The direction of the line is $\vec{\mathbf{r}}'(4\pi)$. Since

$$\vec{\mathbf{r}}'(t) = \langle 2\cos t, -2\sin t, 5\rangle,$$

the direction of the line is given by $\langle 2, 0, 5\rangle$.

The position vector for a point on the line is $\vec{\mathbf{r}}(4\pi) = \langle 0, 2, 20\pi\rangle$.

The equation for the line is

$$\langle 0, 2, 20\pi\rangle + t\langle 2, 0, 5\rangle = \boxed{\langle 2t, 2, 20\pi + 5t\rangle.}$$

(b) (10 points) What is the angle (in radians) between the line found in part (a) and the vector $\vec{\mathbf{u}} = -5\,\vec{\mathbf{i}} + \vec{\mathbf{j}} + 2\,\vec{\mathbf{k}}$?

**Solution:**

A vector giving the direction of the line from part (a) is $\langle 2, 0, 5\rangle$. The angle is

$$\theta = \cos^{-1}\left(\frac{\langle 2, 0, 5\rangle \cdot \langle -5, 1, 2\rangle}{|\langle 2, 0, 5\rangle||\langle -5, 1, 2\rangle|}\right) = \cos^{-1}\left(\frac{0}{\sqrt{29}\sqrt{30}}\right) = \boxed{\frac{\pi}{2}.}$$