**Math 325, Section 1**

**Exam 2 Solutions**

**1.** Show that every *subsequence* $(a_{n_k})_{k=1}^{\infty}$ of a *monotonic* sequence $(a_n)_{n=1}^{\infty}$ is also monotonic.

Suppose first that $a_n$ is monotone increasing, so $n \geq m$ implies that $a_n \geq a_m$. [If you take the approach that increasing means that $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$, then this statement can be established by induction on $n$: for $n = m$ we have $a_n = a_m \geq a_m$, and if $a_n \geq a_m$ then $a_{n+1} \geq a_n \geq a_m$, so $a_{n+1} \geq a_m$, giving the inductive step.]

Now suppose that $a_{n_k}$ is a subsequence of $a_n$. Then $n_{k+1} > n_k$ for every $k$, and so since $a_n$ is increasing we have that $a_{n_{k+1}} \geq a_{n_k}$. Then by the same induction argument as above we again have that $r \geq s$ implies that $a_{n_r} \geq a_{n_s}$, so $a_{n_k}$ is monotone increasing. This establishes our result.

A symmetric argument, reversing all of the inequalities involving the sequence $a_n$, establishes the analogous result for monotone decreasing sequences.

**2.** Show, by example, that it is possible for a function $f : D \to \mathbb{R}$ to be continuous, for a number $a$ to be an accumulation point of $D$, but the limit $\lim\limits_{x \to a} f(x)$ <u>does</u> <u>not</u> <u>exist</u>.

We wish the limit not to exist; but if $a \in D$ then continuity <u>at</u> <u>$a$</u> would require that $\lim\limits_{x \to a} f(x) = f(a)$, and so in particular the limit must exist! So our example must rely on the number $a$ not being in the domain $D$ of our function $f$.

From here we can construct many examples; forcing the limit to not exist can be accomplished by making $f$ 'blow up' as $x$ approaches $a$, or oscillate wildly, or approach one value from one side and another value from the other. So, for example,

$f(x) = 1/x$, with domain $D = (0, \infty)$, has 0 as an accumulation point of $D$ but the limit as $x$ approach 0 does not exist, because if $1/x \to L$ as $x \to 0$, then $x = 1/(1/x) \to 1/L$ so by uniqueness of limits, $1/L = 0$, so $1 = L \cdot 0 = 0$, which is absurd. Note that $f$ is continuous on $D$, since it is the reciprocal of $x$, which is continuous and non-zero on $D$.

$g(x) = \sin(1/x)$, with domain $(0, \infty)$, has 0 as an accumulation point of $D$ but the limit as $x$ approach 0 does not exist, because as $x \to 0$, $1/x$ grows arbitrarily large, so $\sin(1/x)$ takes the values 1 and $-1$ repeatedly as $x \to 0$. Put more bluntly, $\sin(1/(1/(n+1/2)\pi)) = \sin(x_n) = 1$ and $\sin(1/(1/(n+3/2)\pi)) = \sin(y_n) = -1$, with $x_n \to 0$ and $y_n \to 0$, which violates the uniquness of limits (since $1 \neq -1$), unless $g(x)$ has no limit as $x \to 0$. Note that $g$ is continuous on $D$, since it is the composition of $\sin(x)$ and the function $f$ above.

$h(x) = x/|x|$, with domain $D = \mathbb{R} \setminus \{0\}$ is continuous, since it is $-1$ for $x < 0$ and 1 for $x > 0$, so for any point $c$ in $D$ there is a $\delta > 0$ so that $h$ is constant (hence continuous) on $(c-\delta, c+\delta)$. But the limit of $h$ as $x$ approaches 0 does not exist, since there are sequences $x_n = -1/n$ and $y_n = 1/n$ so that $h(x_n) = -1 \to -1$ and $h(y_n) = 1 \to 1$, so for the limit to exist we would require $1 = -1$, which is (still) absurd.

**3.** Show that if $f : [0,2] \to \mathbb{R}$ is *continuous* and $f(0) = f(2)$, then there is a(t least one) $c \in [0,1]$ satisfying $f(c) = f(c+1)$.

[Hint: construct a second function that you can apply the intermediate value theorem to, to get the conclusion that we want!]

The function $f_1 = f : [0,1] \to \mathbb{R}$ (i.e., with smaller domain) is continuous, as is $f_2 = f : [1,2] \to \mathbb{R}$. Also, the function $g(x) = x+1$, $g : [0,1] \to [1,2]$ is continuous (it is a polynomial!). So the function $h : [0,1] \to \mathbb{R}$ given by

$$h(x) = f(x) - f(x+1) = f_1(x) - f_2(g(x))$$

is continuous (as the difference of two continuous functions, one of them continuous as the composition of two continuous functions).

But then $h(0) = f(0) - f(1) = \alpha$ and

$$h(1) = f(1) - f(2) = f(1) - f(0) = -[f(0) - f(1)] = -\alpha.$$

So one of three things is true: $\alpha > 0$ and so $-\alpha = f(1) \leq 0 \leq f(0) = \alpha$, or $\alpha < 0$ and so $\alpha = f(0) \leq 0 \leq f(1) = -\alpha$, or $\alpha = 0$ and so $\alpha = f(0) \leq 0 \leq f(1) = -\alpha$. In every case, 0 lies between $h(0)$ and $h(1)$, and so by the Intermediate Value Theorem, there is a $c \in [0,1]$ so that $h(c) = f(c) - f(c+1) = 0$, i.e., $f(c) = f(c+1)$. This establishes our result.

**4.** Show that if $A, B, C \subseteq \mathbb{R}$ and the functions $f : A \to B$ and $g : B \to C$ are both *uniformly continuous*, then the composition $g \circ f : A \to C$ [defined by $(g \circ f)(x) = g(f(x))$] is <u>also</u> uniformly continuous.

Since $f$ is uniformly continuous, for every $\eta > 0$ there is a $\delta > 0$ so that, if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \eta$.

Since $g$ is uniformly continuous, for every $\epsilon > 0$ there is an $\eta > 0$ so that, if $z, w \in B$ and $|z - w| < \eta$, then $|g(z) - g(w)| < \epsilon$.

But now suppose that $\epsilon > 0$ is given; then pick $\eta > 0$ as in the second statement, and <u>then</u> pick a $\delta > 0$ as in the first statement. Then if $x, y \in A$ and $|x - y| < \delta$, then we have $|f(x) - f(y)| < \eta$. But then $f(x), f(y) \in B$ and $|f(x) - f(y)| < \eta$, and so we have $|g(f(x)) - g(f(y))| < \epsilon$ .

So we have that for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x, y \in A$ and $|x - y| < \delta$, then $|(g \circ f)(x) - (g \circ f)(y)| = |g(f(x)) - g(f(y))| < \epsilon$, and so $g \circ f$ is uniformly continuous.

**5.** Show that if $f, g : [a,b] \to \mathbb{R}$ are a pair of bounded functions, and $U(h)$ denotes the upper Riemann integral of $h$ over the interval $[a,b]$, then

$$U(f+g) \leq U(f) + U(g) \ .$$

We wish to show that

$$\begin{aligned} U(f+g) &= \inf\{U(f+g,P) \ : \ P \text{ a partition of } [a,b]\} \\ &\leq \ \inf\{U(f,P) \ : \ P \text{ a partition of } [a,b]\} + \inf\{U(g,P) \ : \ P \text{ a partition of } [a,b]\} \end{aligned}$$

We can do this by showing that $U(f+g) \leq U(f,P) + U(g,Q)$ for any <u>pair</u> of partitions $P, Q$ of $[a,b]$, since then $U(f+g) - U(g,Q) \leq U(f,P)$ for every partition $P$, so $U(f+g) - U(g,Q)$

is a lower bound for the $U(f,P)$, so $U(f+g)-U(g,Q)\le U(f)$, since $U(f)$ is the greatest such lower bound. But then $U(f+g)-U(f)\le U(g,Q)$ for every partition $Q$ of $[a,b]$, so $U(f+g)-U(f)$ is a lower bound for the $U(g,Q)$, so $U(f+g)-U(f)\le U(g)$, yielding $U(f+g)\le U(f)+U(g)$, as desired.

But we know that more points in a partition leads to smaller upper Riemann sums; if $P\subseteq R$ then $U(f,R)\le U(f,P)$. So if we set $R=P\cup Q$, then $P\subseteq R$ and $Q\subseteq R$ and so $U(f,R)\le U(f,P)$ and $U(g,R)\le U(g,P)$, so $U(f,R)+U(g,R)\le U(f,P)+U(g,Q)$.

So it is enough to show that $U(f+g)\le U(f,R)+U(g,R)$, since then

$$U(f+g)\le U(f,R)+U(g,R)\le U(f,P)+U(g,Q)$$

gives our needed result. That is, we can assume that the partitions used for $f$ and $g$ are the <u>same</u>. Our intended result then follows if we show that $U(f+g,R)\le U(f,R)+U(g,R)$, since then

$U(f+g)=\inf\{U(f+g,P)\ :\ P \text{ a partition of } [a,b]\}\le U(f+g,R)\le U(f,R)+U(g,R)$, so $U(f+g)\le U(f,R)+U(g,R)$, so $U(f+g)\le U(f,P)+U(g,Q)$, as desired.

But $U(f+g,R)\le U(f,R)+U(g,R)$ follows from the fact that

$U(f+g,R)=\sum_i \sup\{(f+g)(x)\ :\ x\in[x_i,x_{i+1}]\}(x_{i+1}-x_i)$

and, for each $i$, we have

$$\sup\{(f+g)(x)\ :\ x\in[x_i,x_{i+1}]\}$$
$$\le \sup\{f(x)\ :\ x\in[x_i,x_{i+1}]\}+\sup\{g(x)\ :\ x\in[x_i,x_{i+1}]\}=F+G\ ;$$

this is, effectively, from an old problem set, although we can see this directly, since $f(x)\le F$ and $g(x)\le G$ for every $x\in[x_i,x_{i+1}]$ (since they are suprema), so $(f+g)(x)=f(x)+g(x)\le F+G$ for every $x\in[x_i,x_{i+1}]$, making $F+G$ an upper bound, so it is $\ge$ the supremum.

Putting this all together, we find that since

$$\sup\{(f+g)(x)\ :\ x\in[x_i,x_{i+1}]\}$$
$$\le \sup\{f(x)\ :\ x\in[x_i,x_{i+1}]\}+\sup\{g(x)\ :\ x\in[x_i,x_{i+1}]\},$$

for every $i$, we have

$U(f+g,R)=\sum_i \sup\{(f+g)(x)\ :\ x\in[x_i,x_{i+1}]\}(x_{i+1}-x_i)$
$\le \sum_i[\sup\{f(x)\ :\ x\in[x_i,x_{i+1}]\}+\sup\{g(x)\ :\ x\in[x_i,x_{i+1}]\}](x_{i+1}-x_i)$
$=\sum_i[\sup\{f(x)\ :\ x\in[x_i,x_{i+1}]\}](x_{i+1}-x_i)+\sum_i[\sup\{g(x)\ :\ x\in[x_i,x_{i+1}]\}](x_{i+1}-x_i)$
$=U(f,R)+U(g,R)$,

<u>So</u> $U(f+g,R)\le U(f,R)+U(g,R)$ for every partition $R$, so $U(f+g)\le U(f,R)+U(g,R)$ for every partition $R$, so $U(f+g)\le U(f,P)+U(g,Q)$ for every <u>pair</u> of partitions $P,Q$ of $[a,b]$, so $U(f+g)\le U(f)+U(g)$, as desired.