Math 417 Problem Set 10 Solutions

Starred (*) problems were due Friday, April 22.

(*) 77. Show that if $H, K \subseteq G$ are subgroups of G, and HK is also a subgroup, then $|H| \cdot |K| = |HK| \cdot |H \cap K|$.

[Hint: show that if you pick coset representatives $A = \{a_1(H \cap K), \ldots, a_n(H \cap K)\}$ of the subgroup $H \cap K$ in H, then the map $A \times K \to HK$ given by $(a(H \cap K), k) \mapsto ak$ is a <u>bijection</u>.]

Let's do what the hint says. $H \cap K$ is a subgroup of G, and $H \cap K \subseteq H$, so we can treat $H \cap K$ as a subgroup of H, and so it has left cosets. If we call them $a_1(H \cap K), \ldots, a_n(H \cap K)$, then we can use them to build the function described in the hint: $(a_i(H \cap K), k) \mapsto a_i k$. We will show that this function is both injective and surjective.

For injective, since $A \times K$ is <u>not</u> a group (and we <u>don't</u> expect this function to be a homomorphism), we really need to show that (*) $a_{i_1} k_1 = a_{i_2} k_2$ implies $a_{i_1} = a_{i_2}$ and $k_1 = k_2$. But means that $x = a_{i_2}^{-1} a_{i_1} = k_2 k_1^{-1}$, and so x is in H (because the a_i's are) and in K (since the k_i's are), so $a_{i_2}^{-1} a_{i_1} \in H \cap K$, so $a_{i_1}(H \cap K) = a_{i_2}(H \cap K)$. So $a_{i_1} = a_{i_2}$ since the a_i come from distinct (and therefore disjoint) cosets. Then $x = k_2 k_1^{-1} = a_{i_2}^{-1} a_{i_1} = e_G$, so $k_1 = k_2$. So $(a_{i_1}, k_1) = (a_{i_2}, k_2)$, and so the function φ is injective.

For surjective, we start with $x \in HK$, so $x = hk$ with $h \in H$ and $k \in K$. Then $h(H \cap K)$ is a coset of $H \cap K$ in H and so $h(H \cap K) = a_i(H \cap K)$ for some i. But this means that $a_i^{-1} h \in H \cap K$, so $a_i^{-1} h = w$ for some $w \in H \cap K$, and so $h = a_i w$. Then $x = hk = (a_i w)k = a_i(wk)$ with $w \in H \cap K \subseteq K$ and $k \in K$¡ so $wk = k' \in K$. So $x = a_i k' = \varphi(a_i, k^prime)$, so w is in theimage of φ. So φ is surjective.

Consequently, φ is a bijection, so $|HK| = |A \times K| = |A| \cdot |K|$. But $|A| = [H : H \cap K] = |H|/|H \cap K|$ is the index of $H \cap K$ in H; rearrranging terms, we get $|HK| \cdot |H \cap K| = |H| \cdot |K|$, as desired.

(*) 80. (Gallian, p.422, # 26) Show that every group of order 175 is abelian.

This follows the pattern of other examples we have done. $|G| = 175 = 25 \cdot 7 = 5^2 \cdot 7$, and so Sylow theory tells us that G has a 5-Sylow subgroup H_5 (of order $25 = 5^2$) and a 7-Sylow subgroup H_7 (of order 7). We note that these Sylow subgroups, having prime-squared and prime orders, respectively, must be abelian. The number $|\mathcal{H}_5$ of 5-Sylow subgroups is 1 mod 5 and divides 7, and so must be 1; this means that H_5 is equal to its own conjugates, and so is normal. The number $|\mathcal{H}_7$ of 7-Sylow subgroups is 1 mod 7 and divides 25, and so must be one of 1,5, or 25; being 1 mod 7 means that it is also 1. This means that H_7 is also normal.

This means that we can form the quotient groups G/H_5 (which has order 7 and so is abelian) and G/H_7 (which has order 25 and so is also abelian). Then we can put together the quotient homomorphisms $G \to G/H_5$ and $G \to G/H_7$ to give a homomorphism $\varphi : G \to G/H_5 \oplus G/H_7$ fro G to an abelian group, given by $x \mapsto (xH_5, xH_7)$. The kernel of this homomorphism is $H_5 \cap H_7 = \{e_G\}$, since H_5 and H_7

have relatively prime orders, so φ is injective. This means that G is isomorphic to a subgroup of an abelian group, so G is abelian.

[Addendum: $G/H_5 \oplus G/H_7$ is a group of order 175, so φ is in fact an isomorphism. A problem on your second exam(!) will actually let you conclude that $G/H_5 \cong H_7$ and $G/H_7 \cong H_5$, and so G is isomorphic to $H_7 \oplus H_5$, which is either $\mathbb{Z}_7 \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_{175}$ or $\mathbb{Z}_7 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{35} \oplus \mathbb{Z}_5$.]

(*) 83. (Gallian, p.424, # 49) If G is a finite group and H is a <u>normal</u> p-Sylow subgroup of G, show that H is a <u>characteristic</u> subgroup of G (i.e., $\varphi(H) = H$ for every $\alpha \in \text{Aut}(G)$). On the other hand, if H is <u>not</u> normal, show that it is <u>not</u> characteristic.

If H is a p-Sylow subgroup of G, then every other p-Sylow subgroup in G (i.e., every other subgroup with the same order $|H| = p^k$) is a conjugate of H; since H is normal, all of the conjugates of H <u>are</u> H, and so H is only subgroup of G with order p^k. But then is $\varphi : G \to G$ is an automorphism of G, then $\varphi(H)$ is a subgroup of G, and since φ is a bijection, $|\varphi(H)| = |H|$ is a subgroup with the same ordre as H. By the above argument, we must have $\varphi(H) = H$. So every automorphism of G sends H to H; this means that H is a characteristic subgroup of G.

If, on the other hand, H is not normal, then there is a $g \in G$ so that $K = gHg^{-1}$ is (another subgroup) distinct from H. We wish to show that there is an automorphism of G that does not preserve H; but conjugation by g <u>is</u> an automorphism of G, $\varphi_g(x) = gxg^{-1}$. By our choice og g, $\varphi_g(H) \neq H$, and so there is an uatomorphism of G which does not preserve H, and so H is not a characteristic subgroup of G. [Our text states this last result (in an exercise?) as the contrapositive: a characteristic subgroup of a group G must be a normal subgroup.]

A selection of further solutions.

78. Show, using the Sylow Theorems, that a group of order 280 must have a <u>normal</u> Sylow subgroup.

$280 = 4 \cdot 70 = 2^3 \cdot 35 = 2^3 \cdot 5 \cdot 7$. So the group has a 2-Sylow subgroup (of order 8), a 5-Sylow subgroup (of order 5), and a 7-Sylow subgroup (of order 7). The number $|\mathcal{H}_5|$ of 5-Sylow subgroups must divide $[G : H_5] = 2^3 \cdot 7$, and so is one of 1, 2, 4, 7, 8, 14, 28, or 56; it must also be congruent to 1 mod 5, and so is either 1 or 56. Similarly, $|\mathcal{H}_7|$ divides $[G : H_7] = 2^3 \cdot 5$, so is 1, 2, 4, 5, 8, 10, 20, or 40, and is congruent to 1 mod 7, so is either 1 or 8. If either of them is 1 then the corresponding Sylow subgrop is normal (and we win); suppose instead that neither of them is 1. Then $|\mathcal{H}_5| = 56$ and $|\mathcal{H}_7| = 8$. This means that G has 56 distinct subgroups of order 5; since any two of them intersect only in e_G (any other common element will be a generator of both (cyclic) subgroups, making the two subgroups equal), this means that these subgroups account for $56(5-1) = 224$ distinct elements (of order 5). Similarly, G has 8 distinct subgroups of order 7, and so has $8(7-1) = 48$ distinct elements of order 7. But together these elements then account for $224 + 48 = 272$ elements of G, since the sets have no elements in common (you can't have order 5 <u>and</u> 7).

But none of these elements live in a 2-Sylow subgroup! And so the elements of 2-Sylow subgroups must all be found among the remaining $280 - 272 - 8$ elements. Since a 2-Sylow subgroup has 8 elements, these remaining elements must be the 2-Sylow subgroup. In particular, there can be only one 2-Sylow subgroup (since both would have to consist of the exact same 8 elements)! Therefore, every conjugate of the 2-Sylow subgroup H_2 is H_2, and so H_2 is normal.

81. (Gallian, p.423 # 32) Show that a group of order $375 = 3 \cdot 5^3$ contains a subgroup of order 15.

The group G has a 3-Sylow subgroup H_3 (of order 3) and a 5-Sylow subgroup H_5 (of order $5^3 = 125$). $|\mathcal{H}_5|$ divides 3, and is congruent to 1 mod 5, so $|\mathcal{H}_5| = 1$ and H_5 is normal. $|\mathcal{H}_3|$ divides 125, and so is one of 1,5,25, or 125. It is also congruent to 1 mod 3, and so is either 1 or 25.

If $|\mathcal{H}_3| = 1$, then H_3 is normal. We know that H_5 contains an element x of order 5, and then $xH_3x^{-1} = H_3$ means that $xH_3 = H_3x$, so setting $K = \langle x \rangle$, we have $H_3K = KH_3$ and so H_3K is a subgroup of G. By your problem #77 (!), $|H_3K| = (|H_3| \cdot |K|)/|H_3 \cap K|$, but $H_3 \cap K = \{e\}$ since the elements of the intersection must have order dividing both 3 and 5, and $|H_3| = 3$, $|K| = |x| = 5$, so $|H_3K| = 15$ and G has a subgroup of order 15.

On the other hand, if $|\mathcal{H}_3| = 25$, then G contains 25 distinct conjugates of H_3. The elements of H_5 act on these conjugates (by conjugation!), and since $|H_5| = 125$, there must be distinct elements $x \neq y$ in H_5 which take a fixed 3-Sylow subgroup H_3 to the same conjugate; $xH_3x^{-1} = yH_3y^{-1}$ (ince otherwise the 125 elements of H_5 would need to take H_3 to 125 distinct conjugates). This means that $(y^{-1}x)H_3(x^{-1}y) = (y^{-1}x)H_3(y^{-1}x)^{-1} = H_3$, so the element $y^{-1}x = w$, which lies in H_5 and is $\neq e_G$, satisfies $wH_3w^{-1} = H_3$, i.e, $wH_3 = H_3w$. This, as in the previous problem, means that, if we choose a power $z = w^k$ that has order 5 and set $K = \langle z \rangle$, we have $KH_3 = H_3K$ and $H_3 \cap K = \{e_G\}$, so H_3K is a subgroup and (since $|K| = 5$), $|H_3K| = 15$.

So in both cases we find a subgroup H_3K (with $K \leq H_5$) of G of order 15.

N.B.: Sylow theory will tell us that groups of order 15 are abelian, so our subgroup $H_3K \cong \mathbb{Z}_{15}$.