Math 325 Exam 1 practice problems

1. Show that if $x, y \geq 0$, then the *arithmetic mean* $m = \dfrac{x+y}{2}$ and the *geometric mean* $\mu = \sqrt{xy}$ of x and y always satisfies $m \geq \mu$.

   [Hint: show that $2(m-\mu)$ is a square!]

Following the hint, $2(m-\mu) = (x+y) - 2\sqrt{xy} = (\sqrt{x})^2 - 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2 = (\sqrt{x} - \sqrt{y})^2$; $\sqrt{x}$ and $\sqrt{y}$ both make sense since $x, y \geq 0$. So since $2(m-\mu) = a^2 \geq 0$ for some $a \in \mathbb{R}$, we have $2(m-\mu) \geq 0$, so $m - \mu \geq 0$, so $m \geq \mu$.

   Show by an example that this inequality can be strict (that is, $m > \mu$).

Most any pair of numbers will do! Setting $x = 1$ and $y = 9$, $m = (1+9)/2 = 5$ and $\mu = \sqrt{1 \cdot 9} = \sqrt{9} = 3$, so $m > \mu$.

2. Show that $\alpha = \sqrt{2 + \sqrt{7}}$ is **not** a rational number.

We show that α is the root of a polynomial p with integer coefficients, and then show that p has no rational roots. We have $\alpha^2 = 2 + \sqrt{7}$, so $\alpha^2 - 2 = \sqrt{7}$ and so $(\alpha^2 - 2)^2 = 7$. This means that $\alpha^4 - 5\alpha^2 + 4 = 7$, so $\alpha^4 - 4\alpha^2 - 3 = 0$. So α is a root of the polynomial $p(x) = x^4 - 4x^2 - 3$.

But by the Rational Roots Theorem, the only possible rational roots of $p(x)$ are $1, -1, 3$, or -3, since these are the numbers p/q with p dividing -3 and q dividing 1. But: $p(1) = p(-1) = 1 - 4 - 3 = -6 \neq 0$, and $p(3) = p(-3) = 3^4 - 4 \cdot 3^2 - 3 = 81 - 106 - 3 = 81 - 109 = -28 \neq 0$. So no potential rational roots are roots, so p has no rational roots.

So α, which is a root of p, cannot be rational.

3. Use induction to show that for every $n \geq 1$,

$$a_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} = f(n)$$

   (Hint: write out what $f(n+1)$ is; it will help.)

To establish the result "For $n \in \mathbb{N}$ we have $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$ by induction, we show:

(True for $n = 1$) $\displaystyle\sum_{k=1}^{1} \frac{1}{k(k+2)} \frac{1}{1(1+2)} = \frac{1}{3} = \frac{8}{24} = \frac{(1)(8)}{4(2)(3)} = \frac{1(3 \cdot 1 + 5)}{4(1+1)(1+2)}$, so the result is true for $n = 1$.

(n implies $n+1$) If we suppose that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} = f(n)$, then

$\displaystyle\sum_{k=1}^{n+1} \frac{1}{k(k+2)} = \frac{1}{(n+1)(n+3)} + \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{(n+1)(n+3)} + \frac{n(3n+5)}{4(n+1)(n+2)}$. But!

$$\frac{1}{(n+1)(n+3)} + \frac{n(3n+5)}{4(n+1)(n+2)} = \frac{4(n+2)}{4(n+1)(n+2)(n+3)} + \frac{n(n+3)(3n+5)}{4(n+1)(n+2)(n+3)}$$

$$= \frac{4(n+2)+n(n+3)(3n+5)}{4(n+1)(n+2)(n+3)} = \frac{4n+8+n(3n^2+14n+15)}{4(n+1)(n+2)(n+3)} = \frac{4n+8+3n^3+14n^2+15n)}{4(n+1)(n+2)(n+3)}$$
$$= \frac{3n^3+14n^2+19n+8)}{4(n+1)(n+2)(n+3)} = \frac{(n+1)(3n^2+11n+8)}{4(n+1)(n+2)(n+3)} = \frac{(3n+8)(n+1))}{4(n+2)(n+3)}$$
$$= \frac{(n+1))(3(n+1)+5)}{4((n+1)+1)((n+1)+2)} = f(n+1).$$

So both the base ($n = 1$) and inductive steps are true, so $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$ is true by induction.

4. (a) Use induction to show that for all $n \geq 1$, $n(n+1)$ is divisible by 2.

We want to show that for all $n \in \mathbb{N}$ we have $n(n+1) = 2K$ for some $K \in \mathbb{N}$. Arguing by induction, for $n = 1$ we have $1(1+1) = 2 = 2 \cdot 1$ so $K = 1$ works, and for the inductive step we have that if $n(n+1) = 2K$, then $(n+1)((n+1)+1) = (n+2)(n+2) = n^2+3n+2 = (n^2+n)+(2n+2) = n(n+1)+2(n+1) = 2K+2(n+1) = 2(K+n+1)$, so $(n+1)(n+2)$ is also a multiple of 2.

Having established the base ($n = 1$) and inductive steps, for every $n \in \mathbb{N}$ we have $n(n+1) = 2K$ for some $K \in \mathbb{N}$.

(b): Use induction to show that for every n$\geq$1, $n^3 + 5n$ is divisible by 6.

Arguing in the same way, for $n = 1$ we have $n^3 + 5n = 1 + 5 = 6 = 6 \cdot 1$ is a multiple of 6. for the inductive step, if we assume that $n^3 + 5n = 6K$ for some $K \in \mathbb{N}$, then

$(n+1)^3 + 5(n+1) = (n^3 + 3n^2 + 3n + 1) + (5n + 5) = n^3 + 3n^2 + 8n + 6 = (n^3 + 5n) + (3n^2 + 3n) + (6) = (n^3 + 5n) + 3(n^2 + n) + 6 = 6K + 3(2L) + 6 = 6(K + L + 1)$,

since by part (a) $n^2 + n$ is always even. So $n^3 + 5n$ = a multiple of 6 implies that $(n+1)^3 + 5(n+1)$ = a(nother) multiple of 6. This establishes the inductive step, and therefore, $n^3 + 5n$ is a multiple of 6 for every $n \in \mathbb{N}$.

5. Use the rational roots theorrem to show that $\alpha = \sqrt{2} - \sqrt{5}$ is not a rational number.

We show that α is the root of a polynomial p with integer coefficients, and then show that p has no rational roots. We have $\alpha^2 = (\sqrt{2} - \sqrt{5})^2 = 2 - 2\sqrt{2}\sqrt{5} + 5 = 7 - 2\sqrt{10}$, so $\alpha^2 - 7 = -2\sqrt{10}$ and so $(\alpha^2-7)^2 = \alpha^4 - 14\alpha^2 + 49 = 40$. This means that $\alpha^4 - 14\alpha^2 + 9 = 0$, and so α is a root of the polynomial $p(x) = x^4 - 14x^2 + 9$.

But by the Rational Roots Theorem, the only rational numbers that could be a root of $p(x) = x^4 - 14x^2 + 9$ are $1, -1, 3, -3, 9$ or -9. But $p(1) = p(-1) = 1 - 14 + 9 = -4 \neq 0$, $p(3) = p(-3) = 9 \cdot 9 - 14 \cdot 9 + 9 = -4 \cdot 9 = -35 \neq 0$, and $p(9) = p(-9) = 9 \cdot 729 - 14 \cdot 9 \cdot 9 + 9 = 9(729 - 126 + 1) \neq 0$. So none of these numbers are roots of p, so p has no rational roots. So α, which is a root fo p, cnnot be rational.

6. Suppose that S and T are both non-empty subsets of the real line, and both are bounded from above. Show that if $\text{l}ub(S) \leq \text{l}ub(T)$, then $\text{l}ub(S \cup T) = \text{l}ub(T)$.

Set $\alpha = \text{l}ub(S)$ and $\beta = \text{l}ub(T)$. Then $x \leq \alpha$ for every $x \in S$, and so since $\alpha \leq \beta$, we have $x \leq \beta$ for all $x \in S$. Since we also have $x \leq \beta$ for every $a \in T$, we then have that $x \leq \beta$ for every $x \in S \cup T$, so β is an upper bound for $S \cup T$.

To show that β is the <u>least</u> upper bound, suppose that $\gamma < \beta$. Then since β is the least upper bould of T, there is an $x \in T$ so that $\gamma < x$. But then $x \in S \cup T$, so there is an $x \in S \cup T$ with $\gamma < x$. So no number smaller than β is an upper bound for $S \cup T$, and so $\beta = \mathrm{l}ub(S \cup T)$.

7. Show, directly from the ϵ-δ definition, that $f(x) = x^2 - 3x - 5$ is continuous at $x = a$ for every $a \in \mathbb{R}$.

We want to show that for every $a \in \mathbb{R}$, then for every $\epsilon > 0$ there is a $\delta > 0$ so that $|x - a| < \delta$ implies that $|f(x) - f(a)| = |(x^2 - 3x - 5) - (a^2 - 3a - 5)| < \epsilon$.

But $|f(x)-f(a)| = |(x^2-3x-5)-(a^2-3a-5)| = |(x^2-a^2)-3(x-a)| = |(x-a)(x+a-3)| = |x - a| \cdot |x + a - 3| = |x - a| \cdot |(x - a) + (2a - 3)| \leq |x - a| \cdot (|x - a| + |2a - 3|)$.

So if we insist, first, that $|x - a| < 13$, then $|x - a| + |2a - 3| < 13 + |2a - 3| = R$, where R is a constant (bigger then 13). Then so longer as $|x - a| < 13$ we have $|f(x) - f(a)| \leq |x - a| \cdot (|x - a| + |2a - 3|) = R \cdot |x - a|$, and this is less than ϵ so long as $|x - a| < \epsilon/R$.

So for a given $\epsilon > 0$, if we set $\delta = min\{13, \epsilon/R\}$, then $|x - a| < \delta$ implies that $|x - a| + |2a - 3| < R$, and so $|f(x) - f(a)| < R\delta \leq R(\epsilon/r) = \epsilon$. So f is continuous at every $a \in \mathbb{R}$.

8. Prove, directly from the definition of a limit, that

$$\lim_{x\to 1} (x^2 - 3x + 1) = -1$$

Well, this is practically a special case of the previous problem, but....

We want to show that for any $\epsilon > 0$ there is a $\delta > 0$ so that $|x - 1| < \delta$ implies that $|(x^2 - 3x + 1) - (-1)| = |x^2 - 3x + 2| = |(x - 1)(x - 2)| = |x - 1| \cdot |x - 2| < \epsilon$. But $|x - 2| = |(x - 1) - 1| \leq |x - 1| + | - 1| = |x - 1| + 1 < 4$ so long as $|x - 1| < 3$, and then $|(x^2 - 3x + 1) - (-1)| \leq |x - 1|(|x - 1| + 1) < 4|x - 1| < \epsilon$ so long as, in addition, $|x - 1| < \epsilon/4$. So if we set $\delta = min\{3, \epsilon/4\}$, then $|x - 1| < \delta$ implies that $|x - 1| + 1 < 4$, and so $(x^2 - 3x + 1) - (-1)| < 4|x - 1| \leq 4(\epsilon/4) = \epsilon$, as desired.

9. Find the following limits (you need not give ϵ-δ level explanations):

(a): $\displaystyle\lim_{x\to 1} \frac{x^3 - 3x^2 + x + 1}{x^2 - 3x + 2}$

$x^3 - 3x^2 + x + 1 = (x - 1)(x^2 - 2x - 1)$ and $x^2 - 3x + 2 = (x - 1)(x - 2)$, so $f(x) = \dfrac{x^3 - 3x^2 + x + 1}{x^2 - 3x + 2} = \dfrac{x^2 - 2x - 1}{x - 2}$, so long as $x \neq 1$. Since $x^2 - 2x - 1 \to 1 - 2 - 1 = -2$ and $x - 2 \to 1 - 2 = -1$ as $x \to 1$ (since polynomials are continuous), we know that $\dfrac{x^2 - 2x - 1}{x - 2} \to \dfrac{-2}{-1} = 2$ as $x \to 1$ (since the limit of a quotient is the quotient of the limits). And since $f(x)$ and $\dfrac{x^2 - 2x - 1}{x - 2}$ agree everywhere except at $x = 1$, and the limit as we approach 1 ignores what happens at 1, we have $f(x) \to 2$ as $x \to 1$.

(b): $\displaystyle\lim_{x\to -1} \frac{x^2 + 4x + 3}{x^3 - x^2 + x - 1}$

This one is quicker: $x^2 + 4x + 3 \to (-1)^2 + 4(-1) + 3 = 1 - 4 + 3 = 0$ as $x \to -1$, by continuity, and $x^3 - x^2 + x - 1 \to (-1)^3 - (-1)^2 + (-1) = 1 = -1 - 1 + 1 - 1 = -2$ as

$x \to -1$, again by continuity. Therefore, $f(x) = \dfrac{x^2+4x+3}{x^3-x^2+x-1} \to \dfrac{0}{-2} = 0$ as $x \to -1$, since the limit of a quotient is the quotient of the limits.

(c): $\displaystyle\lim_{x\to 2} \frac{(x+1)^2-9}{x^4-3x^2-3x+2}$

Since $(x+1)^2-9 = x^2+2x-8 = (x-2)(x+4)$ and $x^4-3x^2-3x+2 = (x-2)(x^3+2x^2+x-1)$, we have $f(x) = \dfrac{(x+1)^2-9}{x^4-3x^2-3x+2} = \dfrac{(x-2)(x+4)}{(x-2)(x^3+2x^2+x-1)} = \dfrac{x+4}{x^3+2x^2+x-1}$ so long as $x \neq 2$. Then since $\dfrac{x+4}{x^3+2x^2+x-1} \to \dfrac{2+4}{2^3+2\cdot 2^2+2-1} = \dfrac{6}{17}$ as $x \to 2$ (since the limit of a quotient is the quotient of the limits, and the numerator and denomenator are both continuous functions), and $f(x)$ and this quotient agree everywhere but $x = 2$, we have $f(x) \to \dfrac{6}{17}$ as $x \to 2$.