Math 325 Problem Set 1 Solutions

Problems were due Friday, August 31.

1. Let $S = \{x \in \mathbb{R} \mid x^2 + x = 0\}$ and $T = \{x \in \mathbb{R} \mid x^2 + x < 5\}$.

(*) (a) Write S and T as (small) unions of points and/or intervals.

$x^2 + x = x(x + 1) = 0$ only when $x = 0$ or $x + 1 = 0$ (i.e., $x = -1$). So $S = \{-1, 0\}$.

$x^2+x = 5$ when $x^2+x-5 = 0$; using the Quadratic Formula, this is when $x = \dfrac{-1 \pm \sqrt{1 - 4 \cdot (-5)}}{2 \cdot 1} = \dfrac{-1 \pm \sqrt{21}}{2}$. So $x^2 + x - 5 = (x - \dfrac{-1 - \sqrt{21}}{2})(x - \dfrac{-1 + \sqrt{21}}{2})$. This quantity is less than 0 when one of the terms is negative and the other is positive. Since

$x - \dfrac{-1 - \sqrt{21}}{2} = x - \dfrac{1}{2} + \dfrac{\sqrt{21}}{2} > x - \dfrac{1}{2} - \dfrac{\sqrt{21}}{2} = x - \dfrac{-1 + \sqrt{21}}{2}$, this

then happens when $x - \dfrac{-1 - \sqrt{21}}{2} > 0$ and $x - \dfrac{-1 + \sqrt{21}}{2} < 0$, that is, when $\dfrac{-1 - \sqrt{21}}{2} < x$ and $x < \dfrac{-1 + \sqrt{21}}{2}$. So $x \in T$ precisely when $\dfrac{-1 - \sqrt{21}}{2} < x < \dfrac{-1 + \sqrt{21}}{2}$, so $T = (\dfrac{-1 - \sqrt{21}}{2}, \dfrac{-1 + \sqrt{21}}{2})$.

(*) (b) Decide whether each of the following statements is true, and (briefly) explain:

$$S \subseteq \mathbb{N} \ ; \ S \subseteq T \ ; \ T \cap \mathbb{Q} \neq \emptyset \ ; \ -2.8 \in \mathbb{Q} \setminus T \ .$$

FALSE: Since $-1 \in S$ and $-1 \notin \mathbb{N}$, we have $S \not\subseteq \mathbb{N}$.

TRUE: Since $0 < 5$, when $x^2 + x = 0$ we also have $x^2 + x < 5$, so everything in S is in T, so $S \subseteq T$.

TRUE: Since T is a (non-trivial) interval, it contains irrational numbers; for example, $\sqrt{21}/1,000,000$ is in T and is not rational. So $T \cap \mathbb{Q} \neq \emptyset$.

TRUE: $-2.8 = -14/5 \in \mathbb{Q}$. And $(-2.8)^2 + (-2.8) = (0.2 - 3)^2 + (0.2 - 3) = 0.04 - 1.2 + 9 + 0.2 - 3 = 0.04 - 1 + 6 = 5.04$, so $-2.8 \notin T$. So $-2.8 \in \mathbb{Q} \setminus T$.

3. Let L be the (linear) function $L(x) = ax + b$, where a and b are (real) constants and $a \neq 0$.

(*) (a) Explain why L is both one-to-one and onto.

If $L(x) = L(y)$, then $ax + b = ay + b$, so $a(x - y) = ax - ay = b - b = 0$. Then $x - y = \dfrac{1}{a} \cdot 0 = 0$ (since, because $a \neq 0$, it has a multiplicative inverse), so $x - y = 0$, so $x = y$. So we have found that $L(x) = L(y)$ implies $x = y$, so f is one-to-one.

On the other hand, for any $y \in \mathbb{R}$ we can solve $y = L(x)$, since $y = ax + b$ means $ax = y - b$ and so $x = \dfrac{y - b}{a}$. That is, $L(\dfrac{y - b}{a}) = a\dfrac{y - b}{a} + b = (y - b) + b = y$. So the image of L is all of $\mathbb{R}$, so L is onto.

(*) (b) Find a formula for the inverse function $M = L^{-1}$, and show that $L \circ M(x) = M \circ L(x) = x$ for every $x \in \mathbb{R}$.

We essentially did this above, when we solved $y = L(x)$; $x = \dfrac{y - b}{a}$ can be expressed as a function $M(x) = \dfrac{x - b}{a}$. Then:

$$L(M(x)) = L(\frac{x-b}{a} = a\frac{x-b}{a} + b = (x-b) + b = x, \text{ and}$$

$$M(L(x)) = \frac{L(x)-b}{a} = \frac{(ax+b)-b}{a} = \frac{ax}{a} = x.$$

4. Suppose the $f : A \to B$ and $g : B \to C$ are both functions.

(*) (b) Show that if $g \circ f$ is onto, then g is onto.

If $g \circ f$ is onto, then for any $y \in C$ we can find an $x \in A$ so that $(g \circ f)(x) = g(f(x)) = y$. But then if we set $z = f(x) \in B$, then $g(z) = y$. So for any $y \in C$ we can find a $z \in B$ so that $g(z) = y$. So g is onto.

A selection of further solutions:

2. Starting with a set S, we can construct a new set $P(S)$, the power set of S, consisting of all subsets of S. For example, $P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

(c) If we set $N_k = \{1, 2, \ldots, k\}$, explain why $P(N_{11})$ has twice as many elements as $P(N_{10})$.

Every element of $P(N_{11})$ is a subset of N_{11}. If U is such a subset, one of two things must be true: either $11 \in U$ or $11 \notin U$. If $11 \in U$, then we can write $U = \{11\} \cup V$, where V is all of the other elements of U. But then $U \subseteq N_{10}$, so $V \in P(N_{10})$. One the other hand, if $11 \notin U$, then $U \in P(N_{10})$. So elements of $P(N_{11})$ come in two flavors; either they are an element of $P(N_{10})$, or they are an element of $P(N_{10})$ together with the element 11. But there are as many elements of each kind as there are elements in $P(N_{10})$ (since one is precisely such an element and the other is built from it by adding 11). So P_{11} has the elements ob P_{10} plus the same number of elements again; so P_{11} has twice as many elements as P_{10}.