Math 971 Algebraic Topology

Homework # 1 Solutions

(p.19, # 14): Given $v, e, f > 0$ with $v - e + f = 2$, build a cell structure on S^2 with v 0-cells, e 1-cells, and f 2-cells.

Here is one way to more or less systematically do it. Starting from the smallest case, $(v, e, f) = (1, 0, 1)$ [we always need at least one top- and bottom-dimensional cell, each] as a 2-cell with its boundary quotiented out to a point, we can proceed to the cases $(1, n, n+1)$ by adding a bouquet of circles off of our 0-cell. Each new loop cuts out a new 2-cell from our original one, so the edges and faces each increase by 1 each time. Then we can choose one of the 1-cells, and continually cut it into pieces, each time creating one more vertex and edge, to build the cases $(1+m, n+m, n+1)$. This covers all cases, except for $f = 1, v > 1$ (since increasing v, above, required at least one e, which you don't get unless $f > 1$); this we can handle, for example, by starting from $(2, 1, 1)$ as an arc in the sphere, and continually subdividing the arc. Formally, we should probably describe the gluing maps for the 2-cells, but these should be evident from the pictures. See the pictures on the accompanying page.

(p.38, # 10): Since $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$, elements represented by loops $a(t) = (\gamma(t), y_0)$ and $b(s) = (x_0, \delta(s))$, with $\gamma : I \to X$, $\delta : I \to Y$, commute. Construct an explicit homotopy.

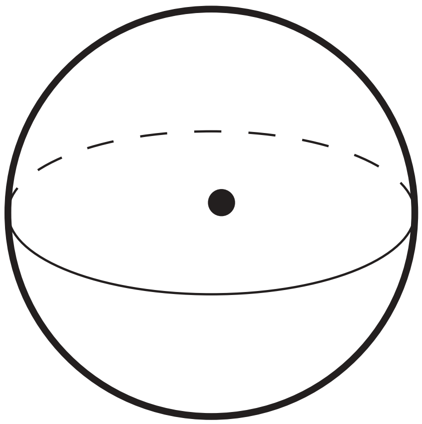
We wish to build a based homotopy $H : I \times I \to X \times Y$ between $a * b$ and $b * a$ (see the pictures on the accompanying page). The basic idea is really to write down the only map we can, that has a vague chance of looking like a homotopy! Define $K : I \times I \to X \times Y$ by $K(t, s) = (\gamma(t), \delta(s))$. This is continuous, because

$$(t, s) \longmapsto t \longmapsto \gamma(t) \text{ and } (t, s) \longmapsto s \longmapsto \delta(s)$$

both are. Since $\gamma(0) = \gamma(1) = x_0$ and $\delta(0) = \delta(1) = y_0$, this homotopy, on its boundary, has a's and b's (as in the figure). But fundamentally (no pun intended) it is what we want, just without some constant maps (the vertical sides that we want for H) inserted. But since a loop followed by the constant map is homotopic to the loop, this is something that we can fix. A formal approach involves grafting on some auxiliary homotopies to the one we have built, and using the fact that the resulting domain is still homeomorphic to $I \times I$. (Writing this homeo explicitly is tedious but not hard.)

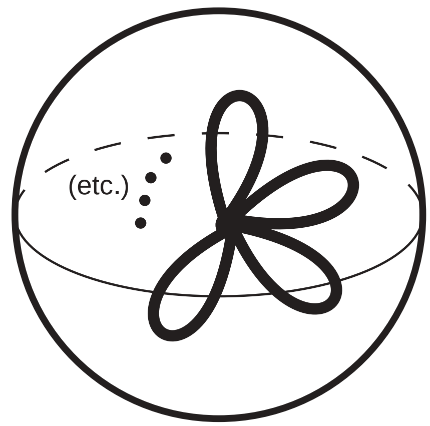
Actually, in the end, it appears I used a pair of "constant" (i.e., ignore the last factor) homotopies, $(t, s) \longmapsto (a * b)(t)$ and $(t, s) \longmapsto (b * a)(t)$, together with homeos that map the upper right and lower left portions of the last figure to a standard rectangle $I \times I$, and pasting things together with the Pasting Lemma to assure continuity. And as Susan (H.) has pointed out, if we shave a bit off of that picture, we get a function (represented by the last picture) that, with patience, we can really write down:

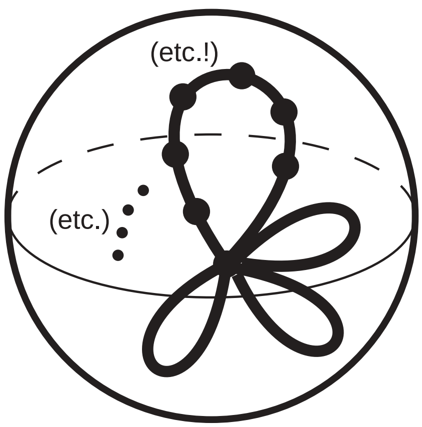
$$H(t, s) = \begin{cases} a(t), & \text{if } t + s \leq 1/2 \\ b(t), & \text{if } t \geq s + 1/2 \\ b(t), & \text{if } s \geq t + 1/2 \\ a(t), & \text{if } s + t \geq 3/2 \\ K(s + t - \frac{1}{2}, s - t + \frac{1}{2}) & \text{otherwise} \end{cases}$$



v=1,e=0,f=1

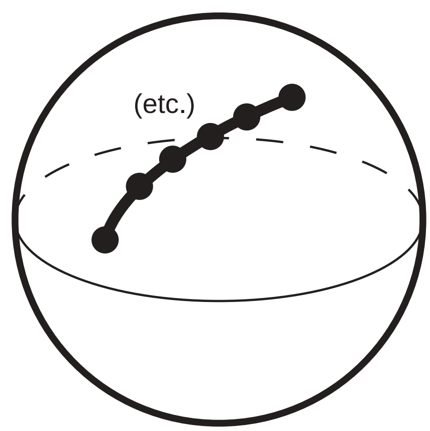
v=1,e=n,f=n+1

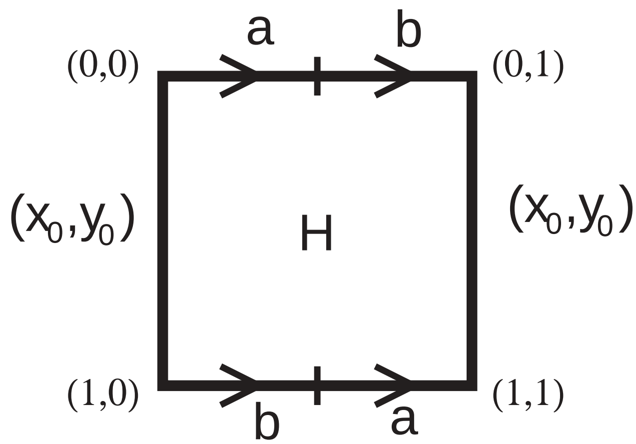


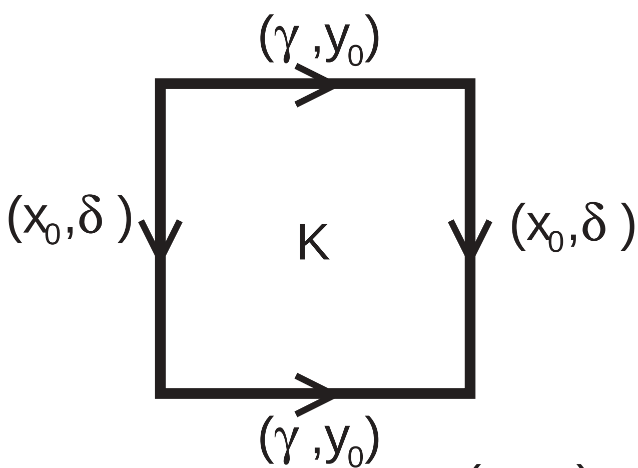


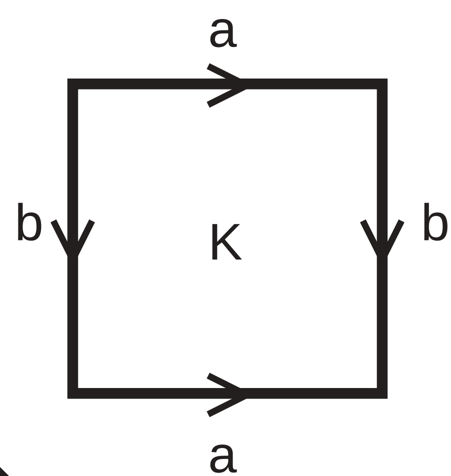
v=1+m,e=n+m,f=n+1

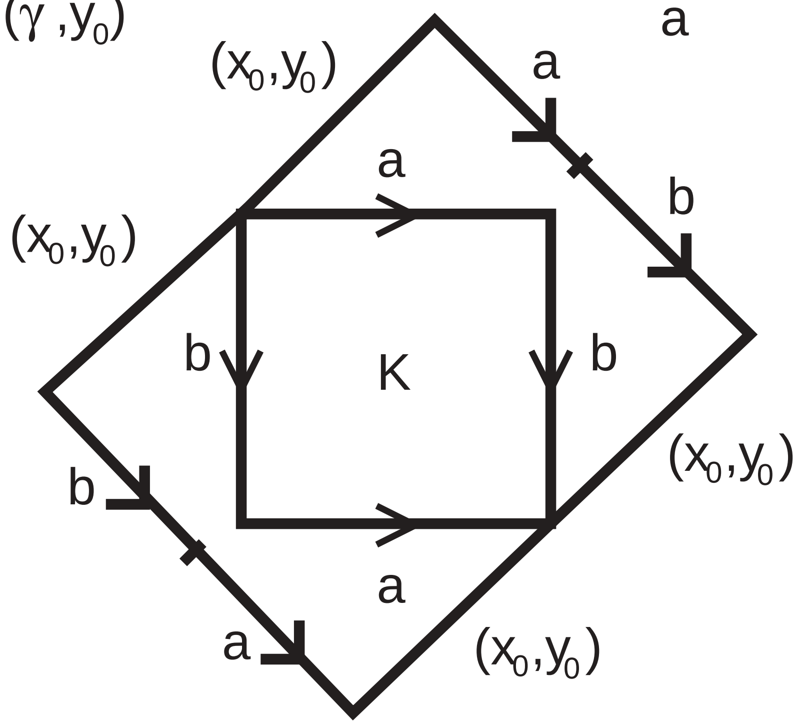
v=1+m,e=m,f=1




a
b
(0,0)
(0,1)
(x_0,y_0)
H
(x_0,y_0)
(1,0)
(1,1)
b
a


(γ,y_0)
(x_0,δ)
K
(x_0,δ)
(γ,y_0)


a
b
K
b
a


(x_0,y_0)
a
a
b
(x_0,y_0)
b
K
b
(x_0,y_0)
b
a
a
(x_0,y_0)