## Math 856 Homework 2

Target date for starred **(*)** problems to be handed in: Friday, September 25

**(*) 7:** [Lee, problem 2-1 (part)] Using the charts on the circle $S^1$ given by steriographic projection, compute the local coordinate representations of the functions

$$f_n : S^1 \to S^1 \text{ given by } f_n(z) = z^n \text{ (in complex coordinates)}$$

and use this to demonstrate that each $f_n$ is $C^\infty$.

**(*) 8:** Show that a function $f : M^n \to N^m$ is $C^\infty \Leftrightarrow g \circ f : M^n \to \mathbb{R}$ is $C^\infty$ for *every* $C^\infty$ function $g : N^m \to \mathbb{R}$. (Hint: you might need to use the technology of bump functions to do this?)

**9:** Let $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Suppose that $f : \mathbb{H}^n \to \mathbb{R}$ is a function for which every $x \in \mathbb{H}^n$ has an open neighborhood $U_x$ of $x$ such that $f$ extends to a $C^\infty$ function on $U_x$. Show that $f$ extends to a $C^\infty$ function on a neighborhood of $\mathbb{H}^n$ in $\mathbb{R}^n$. (Short version: show that if $f$ is locally smooth, then $f$ is smooth. A well-chosen partition of unity might help?)

**10:** We know that if $C, D \subseteq M$ are disjoint closed sets of the smooth manifold $M$, then there exists a smooth function $f : M \to [0,1]$ with $C \subseteq f^{-1}(0)$ and $D \subseteq f^{-1}(1)$. But we can in fact make these containments *equalities*:

(a) Show that it suffices to build a smooth function $g : M \to [0,1]$ with $C = g^{-1}(0)$.

(b) Build a countable cover $\{U_i\}$ of $M \setminus C$ by open sets of the form $h_i^{-1}(B(x_i, 1)$ for a collection of coordinate charts $h_i = (x^1, \ldots, x^n)$ with image containing $B(x_i, 2)$. Build $C^\infty$ functions $g_i : M \to \mathbb{R}$ which are $> 0$ in $U_i$ and $= 0$ on $M \setminus U_i$. Note that $\overline{U_i}$ is compact; for each $i$, let

$$\alpha_i = \sup_{x \in \overline{U_i}; j \leq i; m \leq i; k_1, \ldots k_m \leq n} \{\frac{\partial^m g_j}{\partial x^{k_1} \cdots \partial x^{k_m}}(x)\} .$$

Show that the function $g = \sum g_i/(\alpha_i 2^i)$ is $C^\infty$ and $C = g^{-1}(0)$ .

**11:** [Lee, problem 2-6] For $M$ a (smooth) manifold, let $C(M)$ denote the set of continuous functions from $M$ to $\mathbb{R}$, thought of as an algebra (i.e., a ring and a vector space over $\mathbb{R}$) with scalar multiplication by $\mathbb{R}$, and pointwise addition and multiplication. Let $C^\infty(M)$ be the subalgebra of smooth functions. If $F : M \to N$ is continuous, let $F^* : C(N) \to C(M)$ be given by $F^*(f) = f \circ F$.

(a) Show that $F^*$ is a linear map.

(b) Show that $F$ is smooth $\Leftrightarrow F^*(C^\infty(N)) \subseteq C^\infty(M)$ .

(c) Suppose $F$ is a homeomorphism. Show that $F$ is a diffeomorphism $\Leftrightarrow F^* : C^\infty(N) \to C^\infty(M)$ is an isomorphism.

**(*) 12:** [Lee, problem 2-17] Find an example of a (non-closed: it can't be done if the set is closed!) subset $A$ of a smooth manifold $M$, and a smooth function $f : A \to \mathbb{R}$ which admits **no** extension to a smooth function $\tilde{f} : M \to \mathbb{R}$.

(Note that $f$ is called smooth if for every $x \in A$ there is a neighborhood $x \in \mathcal{U}$ and a smooth extension of $f|_{A \cap \mathcal{U}}$ to the neighborhood $\mathcal{U}$.