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The Geometry and Topology of Three-Manifolds

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This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TeX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

CHAPTER 13

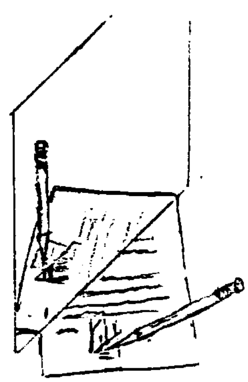
Orbifolds

As we have had occasion to see, it is often more effective to study the quotient manifold of a group acting freely and properly discontinuously on a space rather than to limit one's image to the group action alone. It is time now to enlarge our vocabulary, so that we can work with the quotient spaces of groups acting properly discontinuously but not necessarily freely. In the first place, such quotient spaces will yield a technical device useful for showing the existence of hyperbolic structures on many three-manifolds. In the second place, they are often simpler than three-manifolds tend to be, and hence they often give easy, graphic examples of phenomena involving three-manifolds. Finally, they are beautiful and interesting in their own right.

13.1. Some examples of quotient spaces.

We begin our discussion with a few examples of quotient spaces of groups acting properly discontinuously on manifolds in order to get a taste of their geometric flavor.

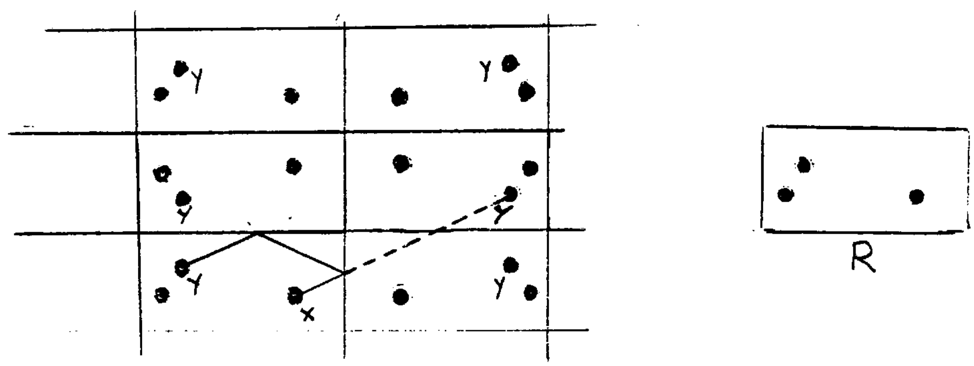
EXAMPLE 13.1.1 (A single mirror). Consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ by reflection in the $y-z$ plane. The quotient space is the half-space $x \geq 0$. Physically, one may imagine a mirror placed on the $y-z$ wall of the half-space $x \geq 0$. The scene as viewed by a person in this half-space is like all of $\mathbb{R}^3$, with scenery invariant by the $\mathbb{Z}_2$ symmetry.



13.2

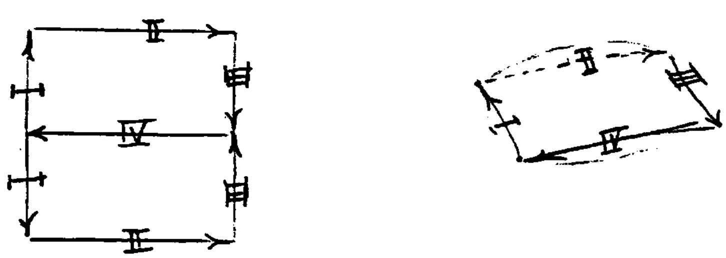
Example 13.1.2 (A barber shop). Consider the group G generated by reflections in the planes $x = 0$ and $x = 1$ in $\mathbb{R}^3$. G is the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$. The quotient space is the slab $0 \le x \le 1$. Physically, this is related to two mirrors on parallel walls, as commonly seen in a barber shop.

Example 13.1.3 (A billiard table). Let G by the group of isometries of the Euclidean plane generated by reflection in the four sides of a rectangle R. G is isomorphic to $D_\infty \times D_\infty$, and the quotient space is R. A physical model is a billiard table. A collection of balls on a billiard table gives rise to an infinite collection of balls on $\mathbb{R}^2$, invariant by G. (Each side of the billiard table should be one ball diameter larger than the corresponding side of R so that the *centers* of the balls can take any position in R. A ball may intersect its images in $\mathbb{R}^2$.)

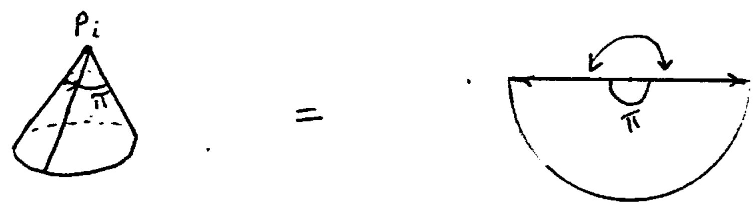


Ignoring spin, in order to make ball x hit ball y it suffices to aim it at any of the images of y by G. (Unless some ball is in the way.)

Example 13.1.4 (A rectangular pillow). Let H be the subgroup of index 2 which preserves orientation in the group G of the preceding example. A fundamental do- 13.3
main for H consists of two adjacent rectangles. The quotient space is obtained by identifying the edges of the two rectangles by reflection in the common edge.

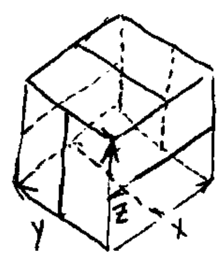


Topologically, this quotient space is a sphere, with four distinguished points or singular points, which come from points in $\mathbb{R}^2$ with non-trivial isotropy ($\mathbb{Z}_2$). The sphere inherits a Riemannian metric of 0 curvature in the complement of these 4 points, and it has curvature $K_{p_i} = \pi$ concentrated at each of the four points p_i. In other words, a neighborhood of each point p_i is a cone, with cone angle $\pi = 2\pi - K_{p_i}$.

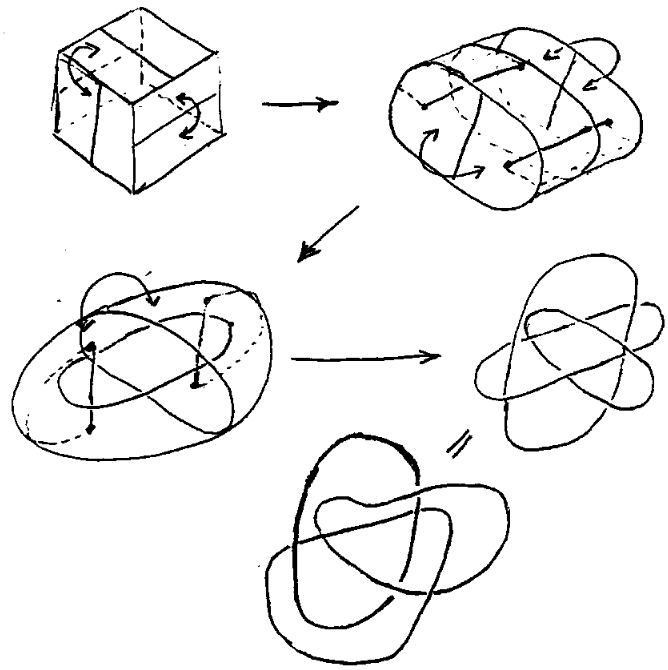


Exercise. On any tetrahedron in $\mathbb{R}^3$ all of whose four sides are congruent, every geodesic is simple. This may be tested with a cardboard model and string or with strips of paper. Explain.

Example 13.1.5 (An orientation-preserving crystallographic group). Here is one more three-dimensional example to illustrate the geometry of quotient spaces. Con- 13.4
sider the 3 families of lines in $\mathbb{R}^3$ of the form $(t, n, m+\frac{1}{2})$, $(m+\frac{1}{2}, t, n)$ and $(n, m+\frac{1}{2}, t)$ where n and m are integers and t is a real parameter. They intersect a cube in the unit lattice as depicted.



Let G be the group generated by $180°$ rotations about these lines. It is not hard to see that a fundamental domain is a unit cube. We may construct the quotient space by making all identifications coming from non-trivial elements of G acting on the faces of the cube. This means that each face must be folded shut, like a book. In doing this, we will keep track of the images of the axes, which form the singular locus.



13.5

As you can see by studying the picture, the quotient space is S^3 with singular locus consisting of three circles in the form of Borromean rings. S^3 inherits a Euclidean structure (or metric of zero curvature) in the complement of these rings, with a cone-type singularity with cone angle π along the rings.

In these examples, it was not hard to construct the quotient space from the group action. In order to go in the opposite direction, we need to know not only the quotient space, but also the singular locus and appropriate data concerning the local behavior of the group action above the singular locus.

13.2. Basic definitions.

An *orbifold** O is a space locally modelled on $\mathbb{R}^n$ modulo finite group actions. Here is the formal definition: O consists of a Hausdorff space X_O, with some additional structure. X_O is to have a covering by a collection of open sets $\{U_i\}$ closed under finite intersections. To each U_i is associated a finite group Γ_i, an action of Γ_i on an open subset $\tilde{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\varphi_i : U_i \approx \tilde{U}_i/\Gamma_i$. Whenever $U_i \subset U_j$,

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*This terminology should not be blamed on me. It was obtained by a democratic process in my course of 1976-77. An orbifold is something with many folds; unfortunately, the word "manifold" already has a different definition. I tried "foldamani," which was quickly displaced by the suggestion of "manifolded." After two months of patiently saying "no, not a manifold, a manifol*dead*," we held a vote, and "orbifold" won.

there is to be an injective homomorphism

$$f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$$

and an embedding

$$\tilde{\varphi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$$

equivariant with respect to f_{ij} (i.e., for $\gamma \in \Gamma_i$, $\tilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\varphi}_{ij}(x)$) such that the 13.6
diagram below commutes.†

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\varphi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i"] & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] \arrow[r, phantom, "\subset"] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}$$

We regard $\tilde{\varphi}_{ij}$ as being defined only up to composition with elements of Γ_j, and f_{ij} as being defined up to conjugation by elements of Γ_j. It is not generally true that $\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should exist an element $\gamma \in \Gamma_k$ such that $\gamma\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$.

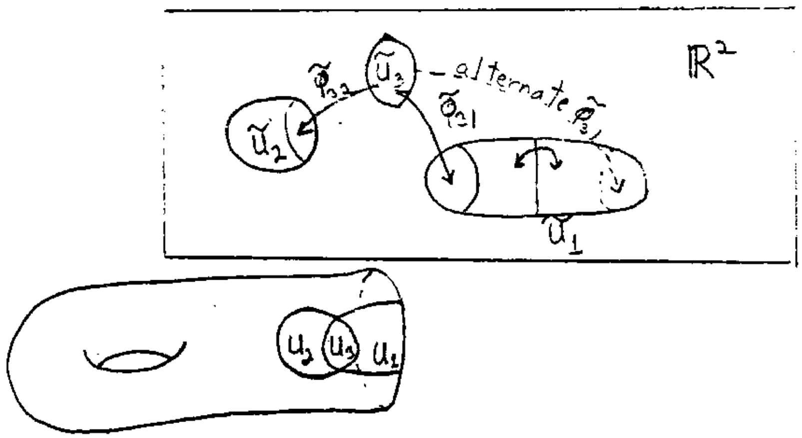
Of course, the covering $\{U_i\}$ is not an intrinsic part of the structure of an orbifold: two coverings give rise to the same orbifold structure if they can be combined consistently to give a larger cover still satisfying the definitions.

A $\mathcal{G}$-orbifold, where $\mathcal{G}$ is a pseudogroup, means that all maps and group actions respect $\mathcal{G}$. (See chapter 3).

Example 13.2.1. A closed manifold is an orbifold, where each group Γ_i is the trivial group, so that $\tilde{U} = U$.

Example 13.2.2. A manifold M with boundary can be given an orbifold structure mM in which its boundary becomes a "mirror." Any point on the boundary has
a neighborhood modelled on $\mathbb{R}^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection in a hyperplane. 13.7

†The commutative diagrams in Chapter 13 were made using Paul Taylor's diagrams.sty package (available at ftp://ftp.dcs.qmw.ac.uk/pub/tex/contrib/pt/diagrams/). —SL



Proposition 13.2.1. *If M is a manifold and Γ is a group acting properly discontinuously on M, then M/Γ has the structure of an orbifold.*

Proof. For any point $x \in M/\Gamma$, choose $\tilde{x} \in M$ projecting to x. Let I_x be the isotropy group of $\tilde{x}$ (I_x depends of course on the particular choice $\tilde{x}$.) There is a neighborhood $\tilde{U}_x$ of $\tilde{x}$ invariant by I_x and disjoint from its translates by elements of Γ not in I_x. The projection of $U_x = \tilde{U}_x/I_x$ is a homeomorphism. To obtain a suitable cover of M/Γ, augment some cover $\{U_x\}$ by adjoining finite intersections. Whenever $U_{x_1} \cap \ldots \cap U_{x_k} \neq \emptyset$, this means some set of translates $\gamma_1\tilde{U}_{x_1} \cap \ldots \cap \gamma_k\tilde{U}_{k_k}$ has a corresponding non-empty intersection. This intersection may be taken to be

$$\widetilde{\overbrace{U_{x_1} \cap \cdots \cap U_{x_k}}},$$

with associated group $\gamma_1 I_{x_1}\gamma_1^{-1} \cap \cdots \cap \gamma_k I_{x_k}\gamma_k^{-1}$ acting on it. □

The orbifold mM arises in this way, for instance: it is obtained as the quotient space of the $\mathbb{Z}_2$ action on the double dM of M which interchanges the two halves. 13.8

Henceforth, we shall use the terminology M/Γ to mean M/Γ as an orbifold.

Note that each point x in an orbifold O is associated with a group Γ_x, well-defined up to isomorphism: in a local coordinate system $U = \tilde{U}/\Gamma$, Γ_x is the isotropy group of any point in $\tilde{U}$ corresponding to x. (Alternatively Γ_x may be defined as the smallest group corresponding to some coordinate system containing x.) The set $\Sigma_O = \{x|\Gamma_x \neq \{1\}\}$ is the *singular locus* of O.

We shall say that O is a manifold when $\Sigma_O = \emptyset$. Warning. It happens much more commonly that the underlying space X_O is a topological manifold, especially in dimensions 2 and 3. Do not confuse properties of O with properties of X_O.

The singular locus is a closed set, since its intersection with any coordinate patch is closed. Also, it is nowhere dense. This is a consequence of the fact that a non-trivial homeomorphism of a manifold which fixes an open set cannot have finite order. (See Newman, 1931. In the differentiable case, this is an easy exercise.)

When M in the proposition is simply connected, then M plays the role of universal covering space and Γ plays the role of the fundamental group of the orbifold M/Γ, (even though the underlying space of M/Γ may well be simply connected, as in the examples of §13.1). To justify this, we first define the notion of a covering orbifold.

DEFINITION 13.2.2. A *covering orbifold* of an orbifold O is an orbifold $\tilde{O}$, with a projection $p : X \to X_O$ between the underlying spaces, such that each point $x \in X_O$ has a neighborhood $U = \tilde{U}/\Gamma$ (where $\tilde{U}$ is an open subset of $\mathbb{R}^n$) for which each 13.9
component v_i of $p^{-1}(U)$ is isomorphic to $\tilde{U}/\Gamma_i$, where $\Gamma_i \subset \Gamma$ is some subgroup. The isomorphism must respect the projections.

Note that the underlying space $X_{\tilde{O}}$ is not generally a covering space of X_O.

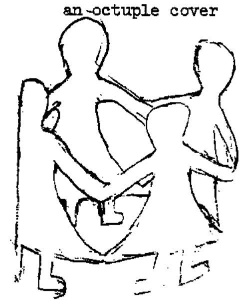
As a basic example, when Γ is a group acting properly discontinuously on a manifold M, then M is a covering orbifold of M/Γ. In fact, for any subgroup $\Gamma' \subset \Gamma$, M/Γ' is a covering orbifold of M/Γ. Thus, the rectangular pillow (13.1.4) is a two-fold covering space of the billiard table (13.1.3).

Here is another explicit example to illustrate the notion of covering orbifold. Let S be the infinite strip $0 \leq x \leq 1$ in $\mathbb{R}^2$; consider the orbifold mS. Some covering spaces of S are depicted below.


mS

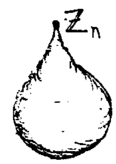

a triple cover

a quadruple cover


an octuple cover


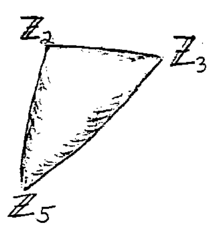
Definition 13.2.3. An orbifold is *good* if it has some covering orbifold which is a manifold. Otherwise it is *bad*.

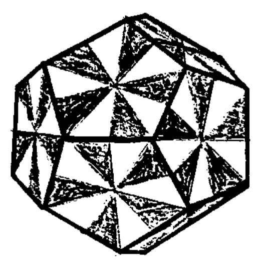
The teardrop is an example of a bad orbifold. The underlying space for a teardrop is S^2. Σ_O consists of a single point, whose neighborhood is modelled on $\mathbb{R}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts by rotations.



By comparing possible coverings of the upper half with possible coverings of the lower half, you may easily see that the teardrop has no non-trivial connected coverings.

Similarly, you may verify that an orbifold O with underlying space $X_O = S^2$ having only two singular points associated with groups $\mathbb{Z}_n$ and $\mathbb{Z}_n$ is bad, unless $n = m$. The orbifolds with three or more singular points on S^2, as we shall see, are always good. For instance, the orbifold below is S^2 modulo the orientation-preserving symmetries of a dodecahedron.





13.11

PROPOSITION 13.2.4. *An orbifold O has a universal cover $\tilde{O}$. In other words, if $* \in X_O - \Sigma_O$ is a base point for O,*

$$\tilde{O} \xrightarrow{p} O$$

is a connected covering orbifold with base point $\tilde{*}$ which projects to $*$, such that for any other covering orbifold

$$\tilde{O}' \xrightarrow{p'} O$$

with base point $\tilde{*}'$, $p'(\tilde{*}') = *$, there is a lifting $q : \tilde{O} \to \tilde{O}'$ of p to a covering map of $\tilde{O}'$.

$$\begin{tikzcd} \tilde{O} \arrow[dd, "p"'] \arrow[dr, "q"] & \\ & \tilde{O}' \arrow[dl, "p'"] \\ O & \end{tikzcd}$$

The universal covering orbifold $\tilde{O}$, in some contexts, is often called the universal branched cover. There is a simple way to prove 13.2.4 in the case Σ_O has codimension 2 or more. In that case, any covering space of O is determined by the induced covering space of $X_O - \Sigma_O$ as its metric completion. Whether a covering Y space of $X_O - \Sigma_O$ comes from a covering space of O is a local question, which is expressed algebraically by saying that $\pi_1(Y)$ maps to a group containing a certain obvious normal subgroup of $\pi_1(X - \Sigma_O)$.

When O is a good orbifold, then it is covered by a simply connected manifold, M. It can be shown directly that M is the universal covering orbifold by proving that every covering orbifold is isomorphic to M/Γ', for some $\Gamma' \subset \Gamma$, where Γ is the group of deck transformations of M over O.

PROOF OF 13.2.4. One proof of the existence of a universal cover for a space X goes as follows.

Consider pointed, connected covering spaces

$$\tilde{X}_i \xrightarrow{p_i} X.$$

For any pair of such covering spaces, the component of the base point in the fiber product of the two is a covering space of both. 13.12

$$\begin{tikzcd} & \tilde{X}_3 \arrow[dl] \arrow[dr] & \\ \tilde{X}_1 \arrow[dr] & & \tilde{X}_2 \arrow[dl] \\ & X & \end{tikzcd}$$

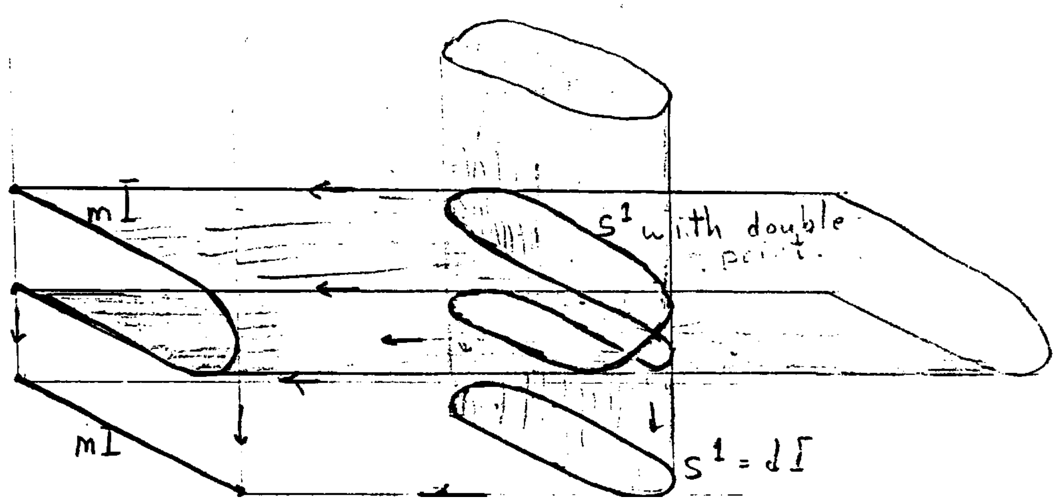
(Recall that the fiber product of two maps $f_i : X_i \to X$ is the space $X_1 \times_X X_2 = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$.)

If X is locally simply connected, or more generally, if it has the property that every $x \in X$ has a neighborhood U such that every covering of X induces a trivial covering of U (that is, each component of $p^{-1}(U)$ is homeomorphic to U), then one can take the inverse limit over some set of pointed, connected covering spaces of X which represents all isomorphism classes to obtain a universal cover for X.

We can follow this same outline with orbifolds, but we need to refine the notion of fiber product. The difficulty is best illustrated by example. Two covering maps

$$S^1 = dI \xrightarrow{f_2} m_i \quad \text{and} \quad m_i \to m_i$$

are sketched below, along with the fiber product of the underlying maps of spaces.



(This picture is sketched in $\mathbb{R}^3 = \mathbb{R}^2 \times_{\mathbb{R}_1} \mathbb{R}^2$.) The fiber product of spaces is a circle but with a double point. In the definition of fiber product of orbifolds, we must eliminate such double points, which always lie above Σ_O.

To do this, we work in local coordinates. Let $U \approx \tilde{U}/\Gamma$ be a coordinate system. 13.13
We may suppose that U is small enough so in every covering of O, $p^{-1}(U)$ consists

of components of the form $\tilde{U}/\Gamma'$, $\Gamma' \subset \Gamma$. Let

$$O_i \xrightarrow{p_i} O$$

be covering orbifolds ($i = 1, 2$), and consider components of $p_i^{-1}(U)$, which for notational convenience we identify with $\tilde{U}/\Gamma_1$ and $\tilde{U}/\Gamma_2$. Formally, we can write $\tilde{U}/\Gamma_1 = \{\Gamma_1 y \mid y \in \tilde{U}\}$. [It would be more consistent to use the notation $\Gamma_1 \backslash \tilde{U}$ instead of $\tilde{U}/\Gamma_1$]. For each pair of elements γ_1 and $\gamma_2 \in \Gamma$, we obtain a map

$$f_{\gamma_1,\gamma_2} : \tilde{U} \to \tilde{U}/\Gamma_1 \times \tilde{U}/\Gamma_2,$$

by the formula

$$f_{\gamma_1,\gamma_2} y = (\Gamma_1\gamma_1 y, \Gamma_2\gamma_2 y).$$

In fact, f_{γ_1,γ_2} factors through

$$\tilde{U}/\gamma_1^{-1}\Gamma_1\gamma_1 \cap \gamma_2^{-1}\Gamma_2\gamma_2.$$

Of course, f_{γ_1,γ_2} depends only on the cosets $\Gamma_2\gamma_1$ and $\Gamma_2\gamma_2$. Furthermore, for any $\gamma \in \Gamma$, the maps f_{γ_1,γ_2} and $f_{\gamma_1\gamma,\gamma_2\gamma}$ differ only by a group element acting on $\tilde{U}$; in particular, their images are identical so only the product $\gamma_1\gamma_2^{-1}$ really matters. Thus, the "real" invariant of f_{γ_1,γ_2} is the double coset

$$\Gamma_1\gamma_1\gamma_2^{-1}\Gamma_2 \in \Gamma_1\backslash\Gamma/\Gamma_2.$$

(Similarly, in the fiber product of coverings X_1 and X_2 of a space X, the components are parametrized by the double cosets $\pi_1 X_1 \backslash \pi_1 X / \pi_1 X_2$.) The fiber product of $\tilde{U}/\Gamma_1$ and $\tilde{U}/\Gamma_2$ over $\tilde{U}/\Gamma$, is defined now to be the disjoint union, over elements γ representing double cosets $\Gamma_1\backslash\Gamma/\Gamma_2$ of the orbifolds $\tilde{U}/\Gamma_1 \cap \gamma^{-1}\Gamma_2\gamma$. We have shown above how this canonically covers $\tilde{U}/\Gamma_1$ and $\tilde{U}/\Gamma_2$, via the map $f_{1,\gamma}$. This definition agrees with the usual definition of fiber product in the complement of Σ_O. These locally defined patches easily fit together to give a fiber product orbifold $O_1 \times_O O_2$. As in the case of spaces, a universal covering orbifold $\tilde{O}$ is obtained by taking the inverse limit over some suitable set representing all isomorphism classes of orbifolds. $\square$

The universal cover $\tilde{O}$ of an orbifold O is automatically a regular cover: for any preimage of $\tilde{x}$ of the base point $*$ there is a deck transformation taking $\tilde{*}$ to $\tilde{x}$. 13.14

DEFINITION 13.2.5. The *fundamental group* $\pi_1(O)$ of an orbifold O is the group of deck transformations of the universal cover $\tilde{O}$.

The fundamental groups of orbifolds can be computed in much the same ways as fundamental groups of manifolds. Later we shall interpret $\pi_1(O)$ in terms of loops on O.

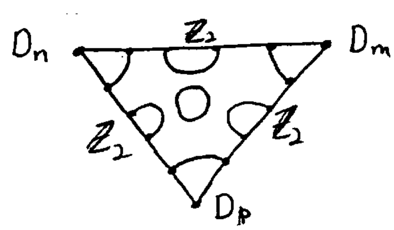
Here are two more definitions which are completely parallel to definitions for manifolds.

DEFINITION 13.2.6. *An orbifold with boundary* means a space locally modelled on $\mathbb{R}^n$ modulo finite groups and $\mathbb{R}^n_+$ modulo finite groups.

When X_O is a topological manifold, be careful not to confuse ∂X_O with ∂O or $X_{\partial O}$.

DEFINITION 13.2.7. A *suborbifold* O_1 of an orbifold O_2 means a subspace $X_{O_1} \subset X_{O_2}$ locally modelled on $\mathbb{R}^d \subset \mathbb{R}^n$ modulo finite groups.

Thus, a triangle orbifold has seven distinct "closed" one-dimensional suborbifolds, up to isotopy: one S^1 and six mI's.



Note that each of the seven is the boundary of a suborbifold with boundary (defined in the obvious way) with universal cover D^2. 13.15

13.3. Two-dimensional orbifolds.

To avoid technicalities, we shall work with differentiable orbifolds from now on.

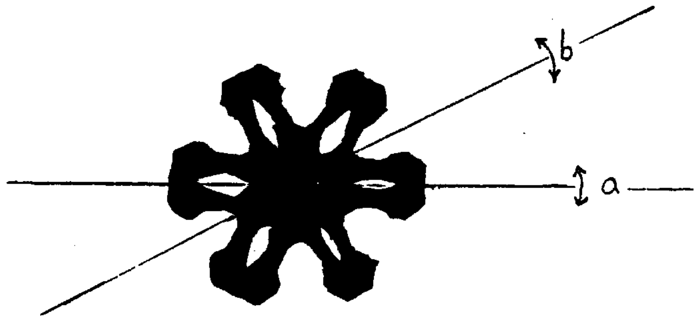
The nature of the singular locus of a differentiable orbifold may be understood as follows. Let $U = \tilde{U}/\Gamma$ be any local coordinate system. There is a Riemannian metric on $\tilde{U}$ invariant by Γ: such a metric may be obtained from any metric on $\tilde{U}$ by averaging under Γ. For any point $\tilde{x} \in \tilde{U}$ consider the exponential map, which gives a diffeomorphism from the ϵ ball in the tangent space at $\tilde{x}$ to a small neighborhood of $\tilde{x}$. Since the exponential map commutes with the action of the isotropy group of $\tilde{x}$, it gives rise to an isomorphism between a neighborhood of the image of $\tilde{x}$ in O, and a neighborhood of the origin in the orbifold $\mathbb{R}^n/\Gamma$, where Γ is a finite subgroup of the orthgonal group O_n.

PROPOSITION 13.3.1. *The singular locus of a two-dimensional orbifold has these types of local models:*

(i) *The mirror: $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection in the y-axis.*
(ii) *Elliptic points of order n: $\mathbb{R}^2/\mathbb{Z}_n$, with $\mathbb{Z}_n$ acting by rotations.*
(iii) *Corner reflectors of order n: $\mathbb{R}^2/D_n$, with D_n is the dihedral group of order $2n$, with presentation*

$$\langle a, b : a^2 = b^2 = (ab)^n = 1 \rangle.$$

The generators a and b correspond to reflections in lines meeting at angle π/n. 13.16



PROOF. These are the only three types of finite subgroups of O_2. ☐

It follows that the underlying space of a two-dimensional orbifold is always a topological surface, possibly with boundary. It is easy to enumerate all two-dimensional orbifolds, by enumerating surfaces, together with combinatorial information which determines the orbifold structure. From a topological point of view, however, it is not completely trivial to determine which of these orbifolds are good and which are bad.

We shall classify two-dimensional orbifolds from a geometric point of view. When G is a group of real analytic diffeomorphisms of a real analytic manifold X, then the elementary properties of (G, X)-orbifolds are similar to the case of manifolds (see §3.5). In particular a developing map

$$D : \tilde{O} \to X$$

can be defined for a (G, X)-orbifold O. Since we do not yet have a notion of paths in O, this requires a little explanation. Let $\{U_i\}$ be a covering of O by a collection of open sets, closed under intersections, modelled on $\tilde{U}_i/\Gamma_i$, with $\tilde{U}_i \subset X$, such that the 13.17
inclusion maps $U_i \subset U_j$ come from isometries $\tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j$. Choose a "base" chart $\tilde{U}_0$. When $U_0 \supset U_{i_1} \subset U_{i_2} \supset \cdots \subset U_{i_{2n}}$ is a chain of open sets (a simplicial path in the one-skeleton of the nerve of $\{U_i\}$), then for each choice of isometries of the form

$$\tilde{U}_0 \overset{\gamma_0 \tilde{\varphi}_{i_1,0}}{\longleftarrow} \tilde{U}_{i_1} \overset{\gamma_2' \tilde{\varphi}_{i_1,i_2}}{\longrightarrow} \tilde{U}_{i_2} \longleftarrow \cdots \longrightarrow \tilde{U}_{i_{2n}}$$

one obtains an isometry of $\tilde{U}_{i_{2n}}$ in X, obtained by composing the transition functions (which are globally defined on X). A covering space $\tilde{O}$ of O is defined by the covering $\{(\varphi, \varphi(\tilde{U}_i))\} \subset G \times X$, where φ is any isometry of $\tilde{U}_i$ obtained by the above construction.

These are glued together by the obvious "inclusion" maps, $(\varphi, \varphi\tilde{U}_i) \hookrightarrow (\psi, \psi\tilde{U}_j)$ whenever $\psi^{-1} \circ \varphi$ is of the form $\gamma_j \circ \tilde{\varphi}_{ij}$ for some $\gamma_j \in \Gamma_j$.

The reader desiring a picture may construct a "foliation" of the space $\{(x, y, g) \mid x \in X,\ y \in X_O,\ g$ is the germ of a G-map between neighborhoods of x and $y\}$. Any leaf of this foliation gives a developing map.

Proposition 13.3.2. *When G is an analytic group of diffeomorphisms of a manifold X, then every (G, X)-manifold is good. A developing map*

$$D : \tilde{O} \to X$$

and a holonomy homomorphism

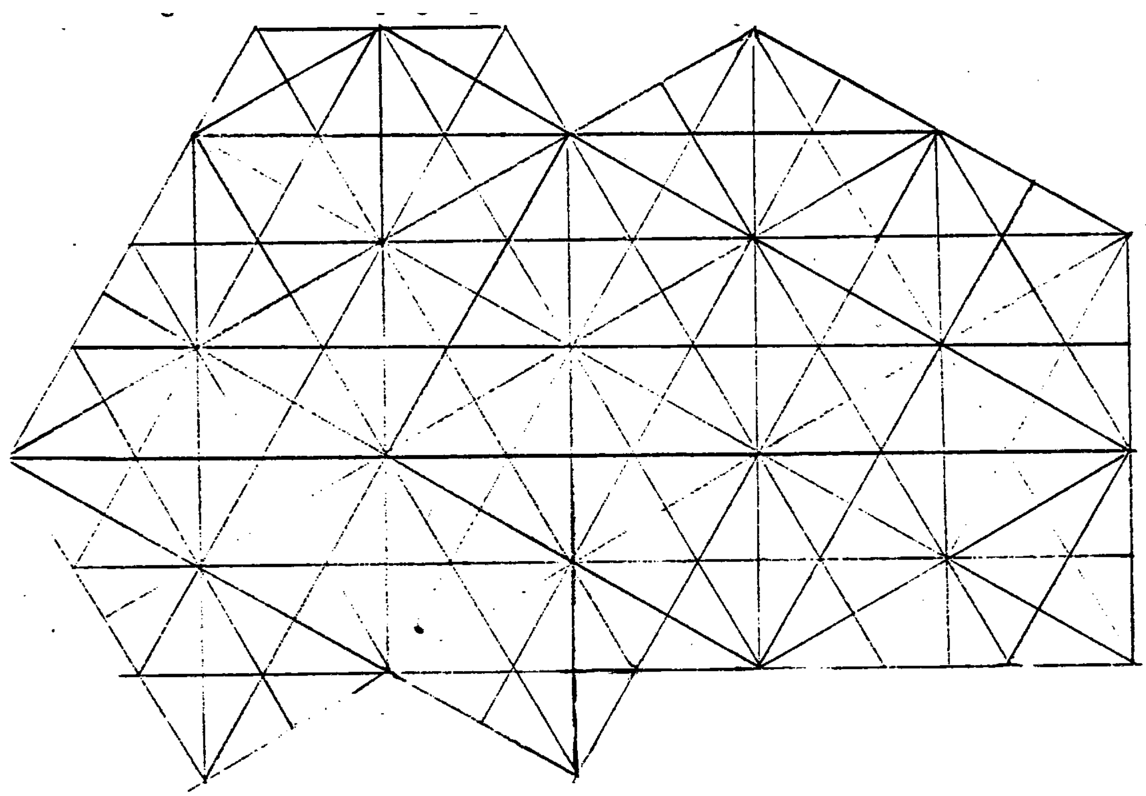
$$H : \pi_1(O) \to G$$

are defined.

If G is a group of isometries acting transitively on X, then if O is closed or metrically complete, it is complete (i.e., D is a covering map). In particular, if X is simply connected, then $\tilde{O} = X$ and $\pi_1(O)$ is a discrete subgroup of G. 13.18

Proof. See §3.5. □

Here is an example. $\triangle_{2,3,6}$ has a Euclidean structure, as a 30°, 60°, 90° triangle. The developing map looks like this:



Here is a definition that will aid us in the geometric classification of two-dimensional orbifolds.

Definition 13.3.3. When an orbifold O has a cell-division of X_O such that each open cell is in the same stratum of the singular locus (i.e., the group associated to the interior points of a cell is constant), then the *Euler number* $\chi(O)$ is defined by 13.19
the formula

$$\chi(O) = \sum_{c_i} (-1)^{\dim(c_i)} \frac{1}{|\Gamma(c_i)|},$$

where c_i ranges over cells and $|\Gamma(c_i)|$ is the order of the group $\Gamma(c_i)$ associated to each cell. The Euler number is not always an integer.

The definition is concocted for the following reason. Define the *number of sheets* of a cover to be the number of preimages of a non-singular point.

Proposition 13.3.4. *If $\tilde{O} \to O$ is a covering map with k sheets, then*

$$\chi(\tilde{O}) = k\chi(O).$$

Proof. It is easily verified that the number of sheets of a cover can be computed by the ratio

$$\#\text{ sheets } = \sum_{\tilde{x} \ni p(\tilde{x}) = x} \left(|\Gamma_x| \quad / \, |\Gamma_{\tilde{x}}| \right),$$

where x is *any* point. The formula [???] for the Euler number of a cover follows immediately. ☐

As an example, a triangle orbifold Δ_{n_1,n_2,n_3} has Euler number $\frac{1}{2}\left(\sum(1/n_i) - 1\right)$: here $+1$ comes from the 2-cell, three $-\frac{1}{2}$'s from the edges, and $1/(2n_i)$ from each vertex. 13.20

Thus, $\Delta_{2,3,5}$ has Euler number $+1/60$. Its universal cover is S^2, with deck transformations the group of symmetries of the dodecahedron. This group has order $120 = 2/(1/60)$. On the other hand, $\chi(\Delta_{2,3,6}) = 0$ and $\chi(\Delta_{2,3,7}) = -1/84$. These orbifolds cannot be covered by S^2.

The general formula for the Euler number of an orbifold O with k corner reflectors of orders $n_1, \ldots, n_k$ and l elliptic points of orders $m_1, \ldots, m_l$ is

13.3.4. $$\chi(O) = \chi(X_O) - \tfrac{1}{2}\sum(1 - 1/n_i) - \sum(1 - 1/m_i).$$

Note in particular that $\chi(O) \le \chi(X_O)$, with equality if and only if O is the surface χ_O or if $O = m\chi_O$.

If O is equipped with a metric coming from invariant Riemannian metrics on the local models $\tilde{U}$, then one may easily derive the Gauss-Bonnet theorem,

13.3.5. $$\int_O K\,dA = 2\pi\chi(O).$$

One way to prove this is by excising small neighborhoods of the singular locus, and applying the usual Gauss-Bonnet theorem for manifolds with boundary. For O to have an elliptic, parabolic or hyperbolic structure, $\chi(O)$ must be respectively positive, zero or negative. If O is elliptic or hyperbolic, then area $(O) = 2\pi|\chi(O)|$.

THEOREM 13.3.6. *A closed two-dimensional orbifold has an elliptic, parabolic or hyperbolic structure if and only if it is good.*

An orbifold O has a hyperbolic structure if and only if $\chi(O) < 0$, and a parabolic structure if and only if $\chi(O) = 0$. An orbifold is elliptic or bad if and only if $\chi(O) > 0$. 13.21

All bad, elliptic and parabolic orbifolds are tabulated below, where

$$(n_1, \dots, n_k; m_1, \dots, m_l)$$

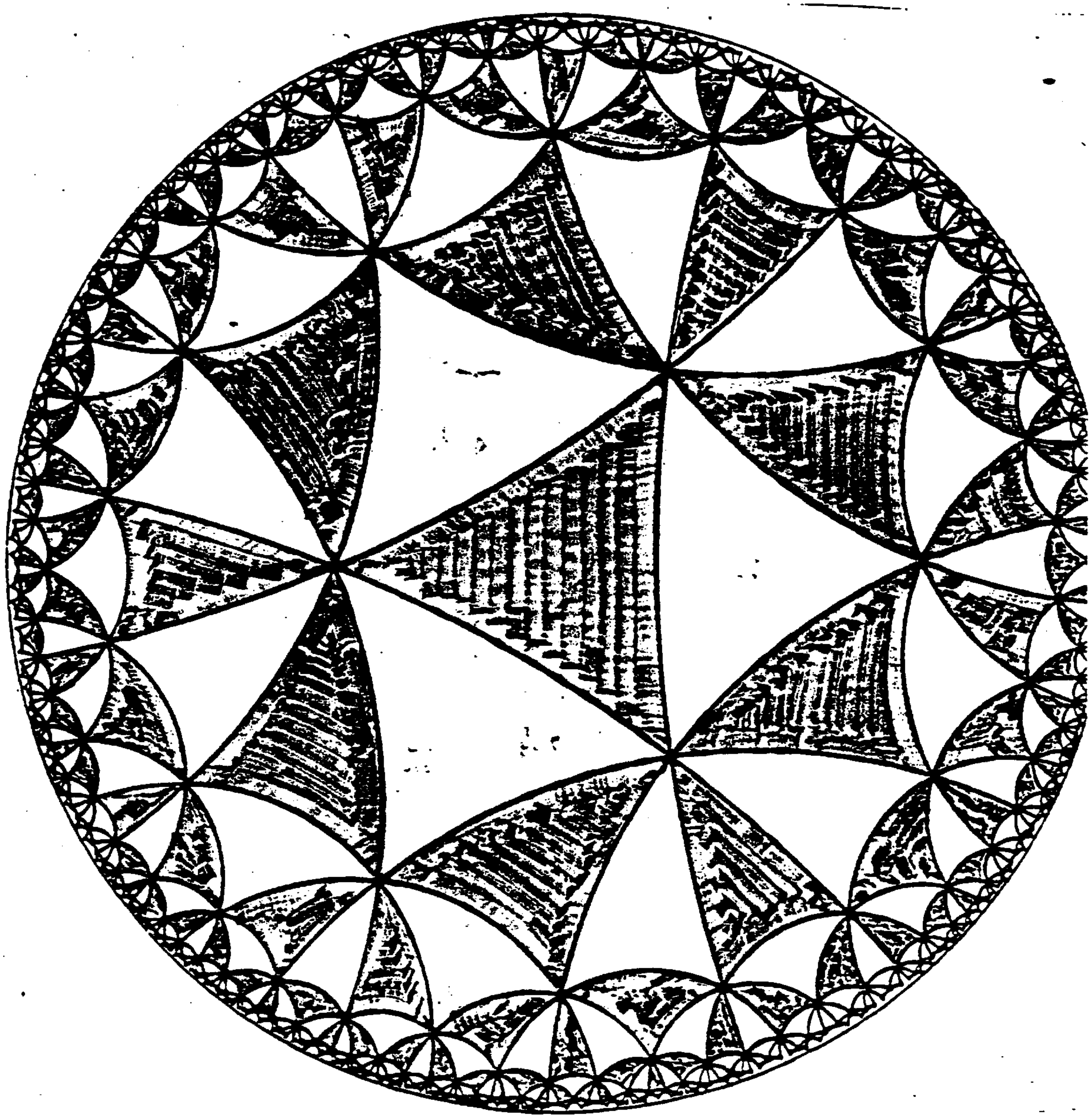
denotes an orbifold with elliptic points of orders $n_1, \dots, n_k$ (in ascending order) and corner reflectors of orders $m_1, \dots, m_l$ (in ascending order). Orbifolds not listed are hyperbolic.

- Bad orbifolds:
  - $X_O = S^2$: (n), (n_1, n_2) with $n_1 < n_2$.
  - $X_O = D^2$: $(\,;n)$, $(\,;n_1, n_2)$ with $n_1 < n_2$.
- Elliptic orbifolds:
  - $X_O = S^2$: $(\ )$, (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.
  - $X_O = D^2$: $(\,;)$, $(\,;n, n)$, $(\,;2, 2, n)$, $(\,;2, 3, 3)$, $(\,;2, 3, 4)$, $(\,;2, 3, 5)$, $(n;)$, $(2;m)$, $(3;2)$.
  - $X_O = \mathbb{P}^2$: $(\ )$, (n).
- Parabolic orbifolds:
  - $X_O = S^2$: $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$, $(2, 2, 2, 2)$.
  - $X_O = D^2$: $(\,;2, 3, 6)$, $(\,;2, 4, 4)$, $(\,;3, 3, 3)$, $(\,;2, 2, 2, 2)$, $(2;2, 2)$, $(3;3)$, $(4;2)$, $(2;2;)$.
  - $X_O = \mathbb{P}^2$: $(2, 2)$.
  - $X_O = T^2$: $(\ )$
  - $X_O =$ Klein bottle: $(\ )$
  - $X_O =$ annulus: $(\,;)$
  - $X_O =$ Möbius band: $(\,;)$

13.21.a

The universal covering space of $D^2_{(;4,4,4)}$ and $S^2_{(4,4,4)} \cdot \pi_1(D^2_{(;4,4,4)})$ is generated by reflections in the faces of one of the triangles. The full group of symmetries of this tiling of H^2 is $\pi_1(D^2_{(;2,3,8)})$.

This picture was drawn with a computer by Peter Oppenheimer.



PROOF. It is routine to list all orbifolds with non-negative Euler number, as in the table. We have already indicated an easy, direct argument to show the orbifolds listed as bad are bad; here is another. First, by passing to covers, we only need consider the case that the underlying space is S^2, and that if there are two elliptic 13.22

points their orders are relatively prime. These orbifolds have Riemannian metrics of curvature bounded above zero,

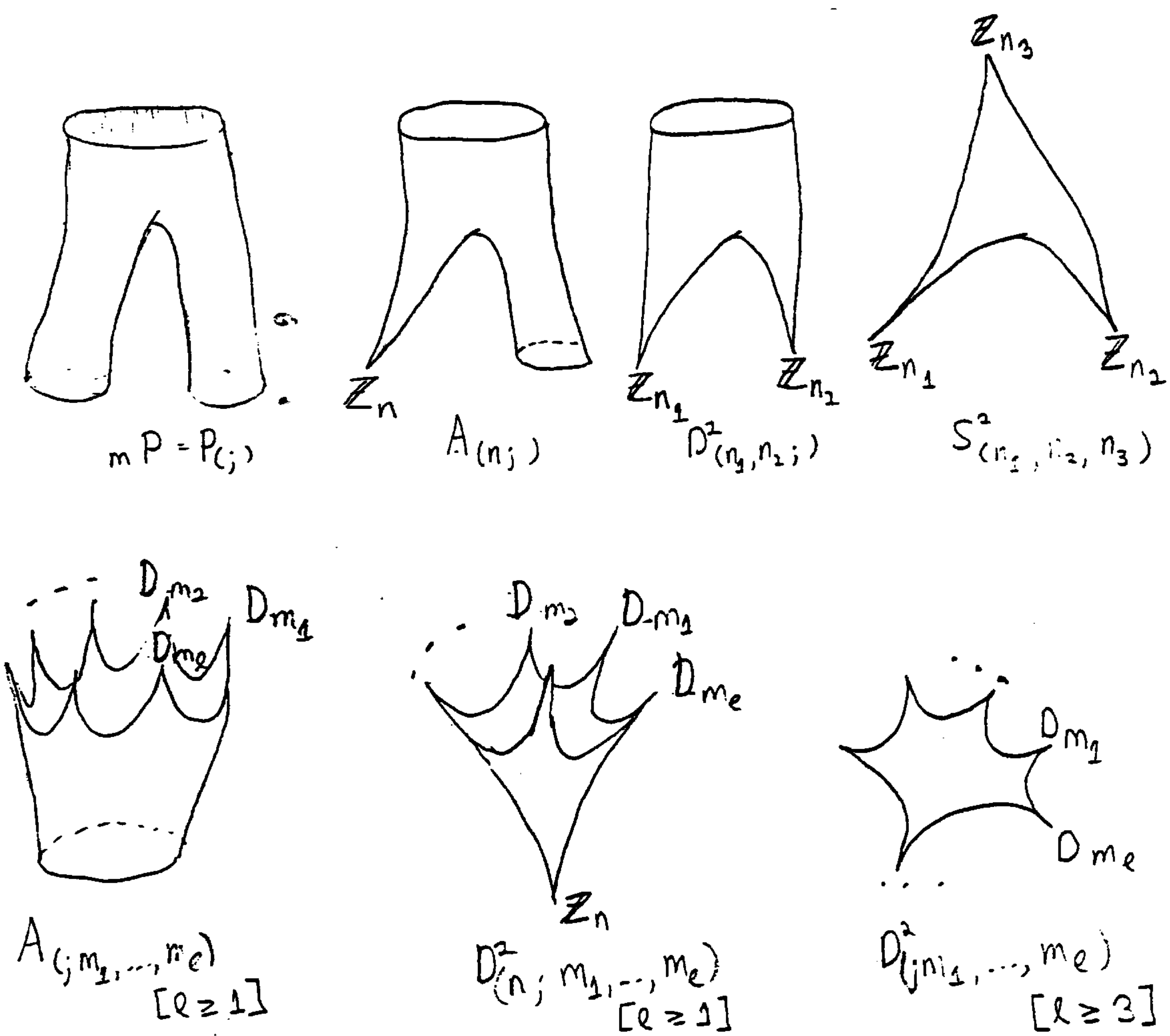
which implies (by elementary Riemannian geometry) that any surface covering them must be compact. But the Euler number is either $1 + 1/n$ or $1/n_1 + 1/n_2$, which is a rational number with numerator > 2.

Since no connected surface has an Euler number greater than 2, these orbifolds must be bad.

QUESTION. What is the best pinching constant for Riemannian metrics on these orbifolds?

All the orbifolds listed as elliptic and parabolic may be readily identified as the quotient of S^2 or E^2 modulo a discrete group. The 17 parabolic orbifolds correspond to the 17 "wallpaper groups." The reader should unfold these orbifolds for himself, 13.23
to appreciate their beauty. Another pleasant exercise is to identify the orbifolds associated with some of Escher's prints.

Hyperbolic structures can be found, and classified, for orbifolds with negative Euler characteristics by decomposing them into primitive pieces, in a manner analogous to our analysis of Teichmüller space for a surface (§5.3). Given an orbifold O with $\chi(O) < 0$, we may repeatedly cut it along simple closed curves and then "mirror" these curves (to remain in the class of closed orbifolds) until we are left with pieces of the form below. (If the underlying surface is unoriented, then make the first cut so the result is oriented.)



13.24

The orbifolds mP, $A_{(n;)}$ and $D_{(n_1,n_2;)}$ (except the degenerate case $A_{(2,2;)}$) and $S^2_{(n_1,n_2,n_3)}$ have hyperbolic structures paremetrized by the lengths of their boundary components. The proof is precisely analogous to the classification of shapes of pants in §5.3; one decomposes these orbifolds into two congruent "generalized triangles" (see §2.6).