Math 971 Algebraic Topology

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Building universal coverings: If a space X is path connected, locally path connected, and semi-locally simply connected (S-LSC), then it has a universal covering; we describe a general construction. The idea is that a covering space should have the path lifting and homotopy lifting properties, and the universal cover can be characterized as the only covering space for which *only* null-homotopic loops lift to loops. So we build a space and a map which <u>must</u> have these properties. We do this by making a space $\widetilde{X}$ whose points are (equivalence classes of) $[\gamma]$ based paths $\gamma : (I, 0) \to (X, x_0)$, where two paths are equivalent if they are homotopic rel endpoints! The projection map is $p([\gamma]) = \gamma(1)$. The S-LSCness is what guarantees that this is a covering map; choosing $x \in X$, a path γ_0 from x_0 to x, and a neighborhood $\mathcal{U}$ of x guaranteed by S-LSC, paths from x_0 to points in $\mathcal{U}$ are based equivalent to $\gamma * \gamma_0 * \eta$ where γ is a based loop at x_0 and η is a path in $\mathcal{U}$. But by simple connectivity, a path in $\mathcal{U}$ is determined up to homotopy by its endpoints, and so, with γ fixed, these paths are in one-to-one correspondence with $\mathcal{U}$. So $p^{-1}(\mathcal{U}$ is a disjoint union, indexed by $\pi_1(X, x_0)$, of sets in bijective correspondence with $\mathcal{U}$. The appropriate topology on $\widetilde{X}$, essentially given as a basis by triples $\gamma*, gamma_0, \mathcal{U}$ as above, make p a covering map. Note that the inverse image of the basepoint x_0 is the equivalence classes of <u>loops</u> at x_0, i.e., $\pi_1(X, x_0)$. A path γ lifts to the path of paths $[\gamma_t]$, where $\gamma_t(s) = \gamma(ts)$, and so the only loop in X which lifts to a loop in $\widetilde{X}$ has endpoint $[\gamma] = [c_{x_0}]$, i.e., $[\gamma] = 1$ in $\pi_1(X, x_0)$. This implies that $p_*(\pi_1(\widetilde{X}, [c_{x_0}])) = \{1\}$, so $\pi_1(\widetilde{X}, [c_{x_0}]) = \{1\}$. However, nobody in their right minds would go about building $\widetilde{X}$ in this way, in general! Before describing how to do it "right", though, we should perhaps see why we should want to?

One reason for the importance of the universal cover is that it gives us a unified approach to building <u>all</u> connected covering spaces of X. The basis for this is the *deck transformation group* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$. These homeomorphisms, by definition, permute each of the point inverses of p. In fact, since h can be thought of as a lift of the projection p, by the lifting criterion h is determined by which point in the inverse image of the basepoint x_0 it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give the map h, and the opposite inclusion to ensure it is a bijection (because its inverse exists)]. These two groups are in general *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$; this can be seen by following the change of basepoint isomorphism down into $G = \pi_1(X, x_0)$. As we have seen, paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 correspondence with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $p_*(\pi_1(X, x_0)$; so deck transformations are in 1-to-1 correspondence with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of H in G*, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore in 1-to-1 correspondence with the group $N(H)/H$ under $h \mapsto$ the coset represented by the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$. And since h is essentially built by lifting paths, it follows quickly that this map is a homomorphism, hence an isomorphism.

In particular, applying this to the universal covering space $p : \widetilde{X} \to X$, since in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$, its deck transformation group is isomorphic to $\pi_1(X, x_0)$. For example, this gives the quickest possible proof that $\pi_1(S^1) \cong \mathbb{Z}$, since $\mathbb{R}$ is a contractible covering space, whose deck transformations are the translations by integer distances. Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action <u>is</u> the projection map p. The evenly covered property of p implies that X does have the quotient topology under this action.

So every space it X the quotient of its universal cover (if it has one!) by its fundamental group $G = \pi_1(X, x_0)$, realized as the group of deck transformations. And the quotient map is the covering projection. So $X|cong\widetilde{X}/G$. In general, a quotient of a space Z by a group action G need not be a covering map; the action must be *properly discontinuous*, which means that for every point $z \in Z$, there is a neighborhood $\mathcal{U}$ of x so that $g \neq 1 \Rightarrow \mathcal{U} \cap g\mathcal{U} = \emptyset$ (the group action carries sufficiently small neighborhoods off of themselves). The evenly covered neighborhoods provide these for the universal cover. And conversely, the quotient of a space by a p.d. group action is a covering space.

But! Given $G = \pi_1(X, x_0)$ and its action on a universal cover $\widetilde{X}$, we can, instead of quotienting out by G, quotient out by any subgroup H of G, to build $X_H = \widetilde{X}/H$. This is a space with fundamental group H, having $\widetilde{X}$ as universal covering. And since the quotient (covering) map $p_G : \widetilde{X} \to X = \widetilde{X}/G$ factors through $\widetilde{X}/H$, we get an induced map $p_H : \widetilde{X}/H \to X$, which is a covering map; open sets with trivial inclusion-induced homomorphism lift homeomorphically to $\widetilde{X}$, hence homeomorphically to $\widetilde{X}/H$; taking lifts to each point inverse of $x \in X$ verifies the evenly covering property for p_H . So every subgroup of G is the fundamental group of a covering of X.

We can further refine this to give the *Galois correspondence*. Two covering spaces $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ are *isomorphic* if there is a homeomorphism $h : X_1 \to X_2$ with $p_1 = p_2 \circ h$. Choosing basepoints x_1, x_2 mapping to $x_0 \in X$, this implies that, if $h(x_1) = x_2$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(h_*(\pi_1(X_1, x_1))) = p_{2*}(\pi_1(X_2, x_2))$. On the other hand, our homeomorphism h need not take our chosen basepoints to one another; if $h(x_1) = x_3$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$. But $p_{2*}(\pi_1(X_2, x_2))$ and $p_{2*}(\pi_1(X_2, x_3))$ are isomorphic, via a change of basepoint isomorphism $\widehat{\eta}$, where η is a path in X_2 from x_2 to x_3. But such a path projects to X has a loop at x_0, and since the change of basepoint isomorphism is by "conjugating" by the path η, the resulting groups $p_{2*}(\pi_1(X_2, x_2))$ and $p_{2*}(\pi_1(X_2, x_3))$ are conjugate, by $p_2 \circ \eta$. So, without reference to basepoints, isomorphic coverings give, under projection, conjugate subgroups of $\pi_1(X, x_0)$. But conversely, given covering spaces X_1, X_2 whose subgroups $p_{1*}(\pi_1(X_1, x_1))$ and $p_{2*}(\pi_1(X_2, x_2))$ are conjugate, lifting a loop γ representing the conjugating element to a loop $\widetilde{\gamma}$ in X_2 starting at x_2 gives, as its terminal endpoint, a point x_3 with $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$ (since it conjugates back!), and so, by the lifting criterion, there is an isomorphism $h : (X_1, x_1) \to (X_2, x_3)$. So conjugate subgroups give isomorphic coverings. Thus, for a path-connected, locally path-connected, semi-locally simply-connected space X, the image of the induced homomorphism on π_1 gives a one-to-one correspondence between [isomorphism classes of (connected) coverings of X] and [conjugacy classes of subgroups of $\pi_1(X)$].

So, for example, if you have a group G that you are interested in, you know of a (nice enough) space X with $\pi_1(X) \cong G$, and you know enough about the covering of X, then you can gain information about the subgroup structure of G. For example, and in some respects as motivation for all of this machinery!, a free group $F(\Sigma)$ is π_1 of a bouquet of circles X. Any covering space $\widetilde{X}$ of X is a union of vertices and edges, so is a graph. Collapsing a maximal tree to a point, $\widetilde{X}$ is $\simeq$ a bouquet of circles, so has free π_1. So, every subgroup of a free group is free. (That is a lot shorter than the original, group-theoretic, proof...) A subgroup H of index n in $F(\Sigma)$ corresponds to a n-sheeted covering $\widetilde{X}$ of X. If $|\Sigma| = m$, then $\widetilde{X}$ will have n vertices and nm edges. Collapsing a maximal tree, having $n-1$ edges to a point, leaves a bouquet of $nm - n + 1$ circles, so $H \cong F(nm - n + 1)$. For example, for $m = 3$, index n subgroups are free on $2n + 1$ generators, so every free subgroup on 4 generators has infinite index in $F(3)$. Try proving that directly!

Kurosh Subgroup Theorem: If $H < G_1 * G_2$ is a subgroup of a free product, then H is (isomorphic to) a free product of a collection of conjugates of subgroups of G_1 and G_2 and a free froup. The proof is to build a space by taking 2-complexes X_1 and X_2 with π_1's isomorphic to G_1, G_2 and join their basepoints by an arc. The covering space of this space X corresponding to H consists of spaces that cover X_1, X_2 (giving, after basepoint considerations, the conjugates) connected by a collection of arcs (which, suitably interpreted, gives the free group).

Residually finite groups: G is said to be residually finite if for every $g \neq 1$ there is a finite group F and a homomorphism $\varphi : G \to F$ with $\varphi(g) \neq 1$ in F. This amounts to saying that $g \notin$ the (normal) subgroup $\ker(\varphi)$, which amounts to saying that a loop corresponding to g does not lift to a loop in the finite-sheeted covering space corresponding to $\ker(\varphi)$. So residual finiteness of a group can be verified by building coverings of a space X with $\pi_1(X) = G$. For example, free groups can be shown to be residually finite in this way.

Ranks of free (sub)groups: A free group on n generators is isomorphic to a free group on m generators $\Leftrightarrow n = m$; this is because the abelianizations of the two groups are $\mathbb{Z}^n, \mathbb{Z}^m$. The (minimal) number of generators for a free group is called its *rank*. Given a free group $G = F(a_1, \ldots a_n)$ and a collection of words $w_1, \ldots w_m \in G$, we can determine the rank and ndex of the subgroup it H they generate by building the corresponding cover. The idea is to start with a bouquet of m circles, each subdivided and labelled to spell out the words w_i. Then we repeatedly identify edges sharing on common vertex if they are labelled precisely the same (same letter *and* same orientation). This process is known as *folding*. One can inductively show that the (obvious) maps from these graphs to the bouquet of n circles X_n both have image H under the induced maps on π_1; the graphs are in fact homotopy equivalent, and the map for the unfolded graph factors through the one for the folded graph. We continue until there is no more folding to be done; the resulting graph X is what is known (in combinatorics) as a *graph covering*; the map to X_n is locally injective. If this map is a covering map, then our subgroup H has finite index (equal to the degree of the covering) and we can compute the rank of H (and a basis!) from this index as above. If not, then the map is not locally surjective at some vertices; if we graft trees onto these vertices, we can extend the map to an (infinite-sheeted) covering map without changing the homotopy type of the graph. H therefore has infinite index in G, and its rank can be computed from $H \cong \pi_1(X)$. An example of this procedure is given below.

Postscript: why care about covering spaces? The preceding discussion probably makes it clear that covering places play a central role in (combinatorial) group theory. It also plays a role in embedding problems; a common scenario is to have a map $f : Y \to X$ which is injective on π_1, and we wish to know if we can lift f to a finite-sheeted covering so that the lifted map $\widetilde{f}$ is homotopic to an embedding. Information that is easier to obtain in the case of an embedding can then be passed down to gain information abut the original map f. And covering spaces underlie the theory of analytic continuation in complex analysis; starting with a domain $D \subseteq \mathbb{C}$, what analytic continuation really builds is an (analytic) function from a covering space of D to $\mathbb{C}$. For example, the logarithm is really defined as a map from the universal cover of $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$. The various "branches" of the logarithm refer to which sheet in this cover you are in.

Homology theory: