# Math 423/823 Exercise Set 8

Due Thursday, April 21

29. [BC#4.52.10] Suppose that $w = f(z)$ is an *entire* function, and there is a (real) $A > 0$ so that for every $z \in \mathbb{C}$ we have $|f(z)| \leq A|z|$. Show that there is a (complex) $a$ so that $f(z) = az$ for all $z$.

For any $z_0$ we have, for $|z - z_0| = R$, $R = |z - z_0| \geq |z| - |z_0|$, so $|z| \leq |z_0| + R$, so $|f(z)| \leq A(|z_0| + R)$ for any $R$, and for any $z$ on the circle of radius $R$ centered at $z_0$, so $|f(z)| \leq A(|z_0| + R)$ for any $z$ on and inside of the circle (since it would lie on a circle of even smaller radius).

But then by Cauchy's Inequality, $|f''(z_0)| \leq \dfrac{2A(|z_0| + R)}{R^2}$, for any radius $R$; letting $R \to \infty$, the expression on the righthand side of this inequality goes to 0, so $|f''(z_0)| = 0$ for every $z_0$, so $f''(z_0) = 0$, so $f''(z)$ is the zero function. So $f''(z)$ is entire and its Taylor series centered at $z = 0$ is the zero series. Integrating term-by-term, the power series for $f'(z)$ is the constant series, so $f'(z) = a$ for som constant $a$. Integrating this power series term-by-term, we find that $f(z) = az + b$ for some constants $a$ and $b$. But $|b| = |f(0)| \leq A|0| = 0$, so $|b| = 0$, so $b = 0$, so $f(z) = az$ for some constant $a$, as desired.

30. [BC#5.62.4] Find the Laurent series expansions centered at $z = 0$ for the function
$f(z) = \dfrac{1}{z^2(1-z)}$ valid for (a) $0 < |z| < 1$, and (b) $1 < |z| < \infty$ .

(a): For $0 < |z| < 1$ we have $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$,

so $f(z) = \dfrac{1}{z^2(1-z)} = \dfrac{1}{z^2}\dfrac{1}{(1-z)} = \dfrac{1}{z^2}\displaystyle\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \sum_{n=-2}^{\infty} z^n$ .

(b) For $1 < |z| < \infty$ we have $0 < |\frac{1}{z}| < 1$ and so $f(z) = \dfrac{1}{z^3((1/z)-1)} = -\dfrac{1}{z^3}\dfrac{1}{1-(1/z)} =$
$-\dfrac{1}{z^3}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}(-1)z^{-(n+3)} = \sum_{n=-\infty}^{-3}(-1)z^n$ .

31. [BC#5.62.8] (a) If $a$ is real and $|a| < 1$, show how to derive the Laurent series expansion $\dfrac{a}{z-a} = \displaystyle\sum_{n=1}^{\infty}\frac{a^n}{z^n}$ valid for $|a| < |z| < \infty$ .

(b) Setting $z = e^{i\theta}$ in the equation from (a), set the real and imaginary parts of each side equal to one another to show that

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$

for any real $a$ with $|a| < 1$ and any real $\theta$.

(a): This is roughly the same as the previous problem; if $|a| < |z| < \infty$ then $0 < |\frac{a}{z}| < 1$.

Then: $\dfrac{a}{z-a} = \dfrac{a}{z(1-(a/z))} = \dfrac{a}{z}\dfrac{1}{1-(a/z)} = \dfrac{a}{z}\sum_{n=0}^{\infty}\left(\dfrac{a}{z}\right)^n = \sum_{n=1}^{\infty}\left(\dfrac{a}{z}\right)^n = \sum_{n=1}^{\infty}\dfrac{a^n}{z^n}$ .

(b): Setting $z = e^{i\theta} = \cos\theta + i\sin\theta$, then $|z| = 1$ so for $|a| < 1$ we have $|a| < |z| < \infty$ so the results of part (a) apply. Then we have $1/z = \overline{z} = \cos\theta - i\sin\theta$, so

$$\frac{a}{z-a} = \frac{a}{(\cos\theta + i\sin\theta) - a} = \frac{a}{(\cos\theta - a) + i\sin\theta}$$

$$= \frac{a[(\cos\theta - a) - i\sin\theta]}{[(\cos\theta - a) + i\sin\theta][(\cos\theta - a) - i\sin\theta]} = \frac{a[(\cos\theta - a) - i\sin\theta]}{[(\cos\theta - a)^2 + (\sin\theta^2]}$$

$$= \frac{a[(\cos\theta - a) - i\sin\theta]}{a^2 - 2a\cos\theta + 1} = \frac{a(\cos\theta - a)}{a^2 - 2a\cos\theta + 1} - i\frac{a\sin\theta]}{a^2 - 2a\cos\theta + 1}$$

$$= \frac{a(\cos\theta - a)}{a^2 - 2a\cos\theta + 1} - i\frac{a\sin\theta]}{a^2 - 2a\cos\theta + 1}$$

But setting $z = e^{i\theta}$, we have $z^{-n} = e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$, so

$$\sum_{n=1}^{\infty}\frac{a^n}{z^n} = \sum_{n=1}^{\infty}a^n z^n = \sum_{n=1}^{\infty}a^n\cos(n\theta) - i\sin(n\theta) = \sum_{n=1}^{\infty}a^n\cos(n\theta) - i\sum_{n=1}^{\infty}a^n\sin(n\theta)$$

<u>provided</u> both of these last series converge, which they do, absolutely, by comparison with the series $\sum_{n=1}^{\infty}a^n$.

So equating the real and imaginary parts of these two expressions, we have

$\sum_{n=1}^{\infty}a^n\cos(n\theta) = \dfrac{a(\cos\theta - a)}{a^2 - 2a\cos\theta + 1}$ and $\sum_{n=1}^{\infty}a^n\sin(n\theta) = \dfrac{a\sin\theta]}{a^2 - 2a\cos\theta + 1}$, as desired.

32. [BC#6.71.2(part)] Use the Residue Theorem to evaluate the integral

$$\int_C z^2 e^{\frac{1}{z}}\, dz\ ,$$

where $C(t) = 3e^{it}$, $0 \le t \le 2\pi$.

$f(z) = z^2 e^{\frac{1}{z}}$ is analytic everywher except at $z = 0$, since $e^z$ and $z^2$ are entire. Since 0 lies inside of the circle $C$, by the Residue Theorem we have

$$\int_C z^2 e^{\frac{1}{z}}\, dz = (2\pi i)\mathrm{Res}_{z=0}f(z).$$

But since $e^z = \sum_{n=0}^{\infty}\dfrac{1}{n!}z^n$ for all $z$, $e^{1/z} = \sum_{n=0}^{\infty}\dfrac{1}{n!}z^{-n} = \sum_{n=-\infty}^{0}\dfrac{1}{|n|!}z^n$, so $f(z) = z^2\sum_{n=-\infty}^{0}\dfrac{1}{|n|!}z^n = \sum_{n=-\infty}^{0}\dfrac{1}{|n|!}z^{n+2} = \sum_{n=-\infty}^{2}\dfrac{1}{|n-2|!}z^n$ is the Laurent series for $f(z)$ for $0 < |z| < \infty$.

So $\mathrm{Res}_{z=0}f(z)$ = the coefficient of $z^{-1}$ in this series expenasion $= \dfrac{1}{3!} = \dfrac{1}{6}$, so $\int_C z^2 e^{\frac{1}{z}}\, dz = (2\pi i)/6 = \pi i/3$.