Math 325 Problem Set 8 Solutions

28. Show that if $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous on D with $f(x) \geq 0$ for every $x \in D$, then the function $g : D \to \mathbb{R}$ defined by $g(x) = \sqrt{f(x)}$ is <u>also</u> continuous on D.

We can use our characterization of continuity using sequences to leverage our previous work in order to solve this problem. We kow that g is continuous on D if for every $c \in D$ and for every sequence $x_n \in D$ with $x_n \to c$ we have $g(x_n) \to g(c)$. That is, we wish to show, for any such sequence, that $g(x_n) = \sqrt{f(x_n)} \to \sqrt{f(c)} = g(c)$. But we know, since f is continuous on D, that $f(x_n) \to f(c)$. We also know (from class) that if $a_n \geq 0$ for all n and $a_n \to a$, then $\sqrt{a_n} \to \sqrt{a}$. So since we know that $f(x_n) \to f(c)$ and $f(x_n) \geq 0$ for all n, we have $\sqrt{f(x_n)} \to \sqrt{f(c)}$.

So we know that if $x_n \to c$ and $x_n \in D$ for all n, then $g(x_n) = \sqrt{f(x_n)} \to \sqrt{f(c)} = g(c)$. So g is continuous on D.

29. [Lay, p.207, problem # 21.5] Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at $x = 0$ but is discontinuous <u>everywhere</u> <u>else</u>.

There are many such functions out there; we can build several from our 'standard' example of a function that is continuous <u>nowhere</u>, namely the function $f : \mathbb{R} \to \mathbb{R}$ which is 0 for x rational and 1 for x irrational. If we multiply this by a (continuous) function which is 0 only at a single point (for example, $g(x) = x$?), then their product $h(x) = f(x)g(x)$ will be continuous only at that one point (in our example, at $x = 0$). This is because f is bounded and $g(x) \to 0$ as $x \to 0$, so our problem # 26 implies that $h(x) \to 0$ as $x \to 0$, so h is continuous at 0. But it is continuous at no other point $c \neq 0$, since $g(c) \neq 0$ and so near c we can find (irrational) points where $h(x) = g(x)$ is close to $g(c)$ and (rational) points where $h(x) = g(x) \cdot 0 = 0$ is <u>not</u> close to $g(c) \neq 0$. [That is, choosing $\epsilon = |g(c)|/2 > 0$, since $h(c)$ equals either $g(c)$ or 0 (depending upon whether c is irrational or not), we can find points arbitrarily close to c where $|h(x) - h(c)| > \epsilon$.]

30. Show that if $f : [a.b] \to \mathbb{R}$ has the property that for every $c \in \mathbb{R}$ the equation $f(x) = c$ has either <u>no</u> solutions with $x \in [a, b]$ or <u>exactly</u> <u>two</u> solutions with $x \in [a, b]$, <u>then</u> f <u>cannot</u> be continuous on $[a.b]$.

[Hint: Suppose it is! 'Find' the (exactly two!) points where it takes its maximum, the (exactly two!) points where it takes its minimum, and let the intermediate value theorem get you into trouble...]

Suppose, by way of contradiction, that f is continuous. We know, by the extreme value theorem, that there are $c, d \in [a, b]$ so that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. But since $\alpha = f(c)$ and $\beta = f(d)$ are values achieved by f, our hypothesis tells us that these values are actually achieved <u>twice</u>, so we really have $c_1 < c_2$ so that $f(c_1) = f(c_2) = \alpha$ and we have $d_1 < d_2$ so that $f(d_1) = f(d_2) = \beta$.

But now we can get ourselves into trouble. First, note that $\alpha < \beta$, since otherwise $\alpha \leq f(x) \leq \beta = \alpha$ for every $x \in [a, b]$, so $f(x) = \alpha$ for every x and f is constant (so

$f(x) = \alpha$ has infinitely many solutions. Set $\gamma = (\alpha + \beta)/2$, so $\alpha < \gamma < \beta$. But then the intermediate value theorem tells us that we know where to find solutions to $f(x) = \gamma$; there are solutions between c_i and d_j for any choice of i and j. Depending on the relative positions of the c_i and d_j, that gives us too many solutions! If either $c_1 < d_1 < c_2 < d_2$ or $d_1 < c_1 < d_2 < c_2$, then between any two consecutive numbers there is a solution to $f(x) = \gamma$, giving three solutions, contradicting our hypothesis.

The only other possibilities, though, are $d_1 < c_1 < c_2 < d_2$ or $c_1 < d_1 < d_2 < c_2$, or $c_1 < c_2 < d_1 < d_2$ or $d_1 < d_2 < c_1 < c_2$. But each of these possibilities can be eliminated as well, since f cannot be constant between either the two minima or the two maxima. For example, if $d_1 < c_1 < c_2 < d_2$, then between c_1 and c_2 we must have an e with $f(e) > \alpha$, but then there are also, by IVT, $d_1 < r_1 < c_1$ and $c_2 < r_2 < d_2$ so that $f(r_1) = f(r_2) = f(e) = \delta$, giving three distinct solutions to this equation, a contradiction. A parallel argument applies when $c_1 < d_1 < d_2 < c_2$, choosing an e between d_1 and d_2 with $f(e) < \beta$.

Finally, when $c_1 < c_2 < d_1 < d_2$, we pick an e with $c_1 < e < c_2$ so $f(e) > \alpha$. Then setting $\epsilon = (f(e) + \alpha)/2$, we have $\alpha < \epsilon < f(e) \le \beta$, so by IVT there are solutions to $f(x) = \epsilon$ lying between c_1 and e, between e and c_2, and between c_2 and d_1, again giving too many solutions! A parallel argument applies when $d_1 < d_2 < c_1 < c_2$. [All of these arguments are probably best notice by drawing some representative pictures of graphs of continuous functions in each case.]

So, no matter how that points c_1, c_2, d_1, d_2 are positioned relative to one another, we can always find an η for which $f(x) = \eta$ has at least three solutions. But this is a contradiction, so no continuous function can satisfy our hypotheses.

31. [Lay, p.214, problem # 22.7] Suppose that $f : [a, b] \to [a, b]$ is continuous. Show that there is at least one $c \in [a, b]$ with $f(c) = c$ (such a c is called a fixed point of f).

    [Hint: rewrite the conclusion to say that that some (other) function takes a specific value. [Note: "c" isn't a 'specific' value...]] [Alternate hint: read the statement of problem # 22.8 ?]

We wish to show that $f(x) = x$ has at least one solution. That is, we wish to show that $f(x) - x = 0$ has at least one solution. Let us set $g(x) - x - f(x)$, which is a function $g : [a, b] \to \mathbb{R}$. g is the difference of two continuous functions, so is continuous. Moreover, since $f(a), f(b) \in [a, b]$, we know that $f(a) \ge a$ and $f(b) \le b$. So we know that $g(a) = a - f(a) \le a - a = 0$, and $g(b) = b - f(b) \ge b - b = 0$. So $g(a) \le 0 \le g(b)$, so by the intermediate value theorem, we know that there is a $c \in [a, b]$ with $g(c) = c - f(c) = 0$; that is, $f(c) = c$.

[N.B.: This result is known as the (1-dimensional) Brouwer Fixed Point Theorem. Interesting side note: in his later career, Brouwer's philosophical attitudes towards proof moved away from non-constructive arguments such as this one. However, later, constructive, proofs were found that would have satisfied Brouwer's more restrictive attitudes.]