Math 971 Algebraic Topology

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We typically think of building a Δ-complex X inductively. The *0-simplices* (i.e., points), or *vertices*, form the 0-skeleton $X^{(0)}$. n-simplices $\sigma^n = [v_0, \dots v_n]$ attach to the $(n-1)$-skeleton to form the n-skeleton $X^{(n)}$; the restriction of the attaching map to each face of σ^n is, by definition, an $(n-1)$-simplex in X. The attaching map is (by induction) really determined by a map $\{v_0, \dots, v_n\} \to X^{(0)}$, since this determines the attaching maps for the 1-simplices in the boundary of the n-simplex, which gives 1-simplices in X, which then give the attaching maps for the 2-simplices in the boundary, etc. Note that the reverse is not true; the vertices of two different n-simplices in X can be the same. For example, think of the 2-sphere as a pair of 2-simplices whose boundaries are glued by the identity.

The final detail that we need before defining (simplicial) homology groups is the notion of an *orientation* on a simplex of X. Each simplex σ^n is determined by a map $f : \{v_0, \dots, v_n\} \to X^{(0)}$; an orientation on σ^n is an (equivalence class of) the ordered $(n+1)$-tuple $(f(v_0), \dots f(v_n)) = (V_0, \dots, V_n)$. Another ordering of the same vertices represents the same orientation if there is an *even* permutation taking the entries of the first $(n+1)$-tuple to the second. This should be thought of as a generalization of the right-hand rule for $\mathbb{R}^3$, interpreted as orienting the vertices of a 3-simplex. Note that there are precisely two orientations on a simplex.

Now to define homology! We start by defining *n-chains*; these are (finite) formal linear combinations of the (oriented!) n-simplices of X, where $-\sigma$ is interpreted as σ with the opposite (i.e., other) orientation. Adding formal linear combinations formally, we get the n-th *chain group* $C_n(X) = \{\sum n_\alpha \sigma_\alpha : \sigma_\alpha$ an oriented n-simplex in $X\}$. We next define a *boundary operator* $\partial : C_n(X) \to C_{n-1}(X)$, whose image will be the $(n-1)$-chains that are the "boundaries" of n-chains. We define it on the basis elements $\sigma_\alpha = \sigma$ of $C_n(X)$ as $\partial\sigma = \sum(-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}$, where $\sigma : [v_0, \dots, v_n] \to X$ is the characteristic map of σ_α . $\partial\sigma$ is therefore an alternating sum of the faces of σ. The pont that really make this definition go is that we need *oriented* simplices, so that we know what the i-th face of σ is (the one oppoisite the i-th vertex). We then extend the definition by linearity to all of $C_n(X)$. When a notation indicating dimension is needed, we write $\partial = \partial_n$.

This definition is cooked up to make the maxim "boundaries have no boundary" true; that is $\delta_{n-1} \circ \delta_n = 0$, the 0 map. This is because, for any simplex $\sigma = [v_0, \dots v_n]$,

$$\delta \circ \delta(\sigma) = \delta(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]})$$
$$= (\sum_{j<i} (-1)^j (-1)^i \sigma|_{[v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n]}) + (\sum_{j>i} (-1)^{j-1} (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n]})$$

The distinction between the two pieces is that in the second part, v_j is actually the $(j-1)$-st vertex of the face. Switching the roles of i and j in the second sum, we find that the two are negatives of one another, so they sum to 0, as desired.

And this little calculation is all that it takes to define homology groups! What this tells us is that $\mathrm{im}(\delta_{n+1}) \subseteq \ker(\delta_n$ for every n. $\ker(\delta_n = Z_n(X)$ are called the *n-cycles* of X; they are the n-chains with 0 (i.e., empty) boundary. They form a (free) abelian subgroup of $C_n(X)$. $\mathrm{im}(\delta_{n+1} = B_n(X)$ are the *n-boundaries* of X; they are, of course, the boundaries of $(n+1)$-chains in X. The n-th homology group of X, $H_n(X)$ is the quotient $Z_n(X)/B_n(X)$; it is an abelian group.