Math 325 Problem Set 7 Solutions

24. ("An accumulation point of the set of accumulation points is an accumulation point.") Show that, for a subset D of $\mathbb{R}$, if b_i, for $i \in \mathbb{N}$ are all accumulation points of D, and if $b_i \to b$ as $i \to \infty$, then b is also an accumulation point of D. [Note: it is enough to show that for every $n \in \mathbb{N}$ there is a $c_n \in D$ <u>with</u> $c_n \neq b$ and $|c_n - b| < \dfrac{1}{n}$; you will probably need to have a little care to make sure that the c_n you build, based on what we are given, is not equal to b !]

We wish to show that there is a sequence of points $x_n \in D$ with $x_n \neq b$ for all n so that $x_n \to b$. First, if $b_i = b$ for any i, we are done; b_i is an accumulation point of D. So we can start by assuming that $b_i \neq b$ for every i.

Given an n, it is enough for us to find an $x_n \in D$ with $x_n \neq b$ and $|x_n - b| < \frac{1}{n}$, since then $x_n \to b$. But we know that we can find a b_i with $|b_i - b| < \frac{1}{2n}$, and then, because b_i is an accumulation point of D, there is an $x_n \in D$ with $x_n \neq b_i$ and $|x_n - b_i| < \frac{1}{2n}$. Then by the triangle inequality, we have $|x_n - b| \leq |x_n - b_i| + |b_i - b| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$, as desired.

But that's not good enough! Because we might have $x_n = b$. To fix this, we need to pick x_n <u>closer</u> to b_i than b_i is to b. (Then we can't have gotten back to b; the triangle inequality also gives $|x_n - b| \geq |b - b_i| - |b_i - x_n| > |b - b_i| - |b - b_i| = 0$, so $|x_n - b| > 0$.) So, instead, pick x_n so that $x_n \in D$ <u>and</u> $|x_n - b_i| < |b_i - b| < \frac{1}{2n}$. Then we get both $|x_n - b| < \frac{1}{n}$ and $x_n \neq b$, as desired.

25. [Lay, p.198, problem # 20.13] (The Squeeze Play Theorem for functions): Show that *if* $f, g, h : D \to \mathbb{R}$ are functions, c is an accumulation point of D, $f(x) \leq g(x) \leq h(x)$ for all x in D <u>and</u> $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, *then* $\lim_{x \to c} g(x) = L$.

What we wnt to show is that for every $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |x - c| < \delta$ implies that $|g(x) - L| < \epsilon$. That is, we want $-\epsilon < g(x) - L < \epsilon$, i.e., we want $L - \epsilon < g(x) < L + \epsilon$.

But by the same reasoning we know that there are $\delta_1, \delta_2 > 0$ so that $0 < |x - c| < \delta_1$ implies that $L - \epsilon < f(x) < L + \epsilon$, and $0 < |x - c| < \delta_2$ implies that $L - \epsilon < h(x) < L + \epsilon$. Together with $f(x) \leq g(x) \leq h(x)$, we then find that if $0 < |x - c| < \delta_1$ <u>and</u> $0 < |x - c| < \delta_2$ [that is, if $0 < |x - c| < \delta = \min(\delta_1, \delta_2)$] we have

$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon.$

So we have that $0 < |x - c| < \delta$ implies that $L - \epsilon < g(x) < L + \epsilon$, that is, $|g(x) - L| < \epsilon$, as desired.

[Alternatively, we could have used the squeeze play theorem for <u>sequences</u> to show that, since $f(x_n) \leq g(x_n) \leq h(x_n)$, then $x_n \to c$ (with $x_n \neq c$ for all n) implies that $f(x_n) \to L$ and $h(x_n) \to L$, so $g(x_n) \to L$.]

26. [Lay, p.198, problem # 20.15] Show that if $f, g : D \to \mathbb{R}$ are functions and c is an accumulation point of D, and if $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le M$ for all $x \in D$ and some constant M, then $\lim_{x\to c}(fg)(x) = 0$.

We wish to show that for any $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |x - c| < \delta$ implies that $|f(x)g(x) - 0| = |f(x)| \cdot |g(x)| < \epsilon$. But since $|g(x)| \le M$ for every $x \in D$, we know that $|f(x)g(x) - 0| = |f(x)| \cdot |g(x)| \le M|f(x)|$, so we can make this less than ϵ so long as $|f(x)| < \epsilon/M$. But since $\epsilon/M > 0$, we do know that there is a $\delta > 0$ so that $0 < |x-c| < \delta$ implies that $|f(x) - 0| = |f(x)| < \epsilon/M$. So using this same δ, we have $0 < |x - c| < \delta$ implies that $|f(x)g(x) - 0| = |f(x)| \cdot |g(x)| < M\epsilon/M = \epsilon$, as desired.

There is one situation where, technically, this doesn't work; $M = 0$ has us dividing by 0, which isn't allowed! But then $|g(x)| \le 0$, implying that $g(x) = 0$ for every x, so $f(x)g(x) = 0$ for every x, so our finction is the constant function 0, which converges to 0.

27. [Lay, p.208, problem # 21.10] If $f : D \to \mathbb{R}$ is a function which is continuous at $c \in D$, then the function $g = |f| : D \to \mathbb{R}$ defined as $g(x) = |f(x)|$ is also continuous at c. Show by example, however, that the opposite is not true: it is possible for $|f|$ to be continuous at c while f itself is discontinuous at c.

We wish to show that for every $c \in D$ we have that $|f(x)| \to |f(c)|$ as $x \to c$. We could use most any of our equivalent versions of continuity for this, here we will use the version with sequences; we show that $x_n \to c$ implies that $|f(x_n)| \to |f(c)|$. That is, we want $|f(x_n)| - |f(c)| \to 0$. But the triangle inequality gives us that

$|f(x_n)| = |f(c) + (f(x_n) - f(c)| \le |f(c)| + |f(x_n) - f(c)|$,

so $|f(x_n) - f(c)| \ge |f(x_n)| - |f(c)|$. But by the same token,

$|f(c)| = |f(x_n) + (f(c) - f(x_n)| \le |f(x_n)| + |f(c) - f(x_n)| = |f(x_n)| + |f(x_n) - f(c)|$, so $|f(x_n) - f(c)| \ge |f(c)| - |f(x_n)|$. So we have

$-|f(x_n) - f(c)| \le |f(x_n)| - |f(c)| \le |f(x_n) - f(c)|$, and so since $|f(x_n) - f(c)| \to 0$ as $n \to \infty$ [by the continuity of f (!); $x_n \to c$ implies $f(x_n) \to f(c)$], the squeeze play theorem again tells us that $|f(x_n)| - |f(c)| \to 0$, as desired.

Building a function f so that $|f|$ is continuous but f isn't amounts, really, to picking your favorite continuous function $g : D \to \mathbb{R}$ which is continuous and non-negative, and making f equal to g or $-g$ depending upon some criterion of your choosing. So, for example, the function $f : \mathbb{R} \to \mathbb{R}$ which is 1 for $x \ge 0$ and -1 for $x < 0$ has $|f(x)| = 1$ for all x, so is continuous at $x = 0$, but f itself is not; as $x \to 0$ with $x < 0$, $f(x) = -1 \not\to f(0) = 1$. A more inventive example is to set $f(x) = 1$ for x rational and $f(x) = -1$ for x irrational. Then $|f|(x) = 1$ is continuous everywhere, but f itself is continuous nowhere!