Math 856 Problem Set 4

Starred (*) problems to be handed in any time. Target date: Thursday, December 1

21. [Lee, p.101, problem 5-8] Let $p : E \to M$ be a smooth n-dimensional vector bundle and $X_1, \dots, X_k$ be linearly independent smooth sections of E defined over an open subset $U \subseteq M$. Show that for every $a \in U$ there is a neighborhood V of a and smooth sections $Y_{k+1}, \dots, Y_n$ defined over V so that $(X_1, \dots, X_k, Y_{k+1}, \dots, Y_n)$ forms a local frame for E over $U \cap V$.

(Hint: if $v_1, \dots, v_n$ form a basis for $\mathbb{R}^n$, then why is it that if you wiggle the first k vectors a little bit, you still have a basis?)

(*) 22. Show that if M is a smooth manifold with boundary, then $\partial M \subseteq M$ is an embedded submanifold. [So it has a unique smooth structure making the inclusion $\iota : \partial M \to M$ a smooth embedding.]

23. (a) Show that an immersion from one n-manifold to another is an open map.

(b) Show that if M and N are n-manifolds, M is compact, N is connected, and $F : M \to N$ is an immersion, then F is onto.

(*) 24. If $S \subseteq M$ is a **closed**, embedded submanifold, $U \supseteq S$ is an open neighborhood of S, and $f : S \to \mathbb{R}$ is a smooth function, show that there is a smooth function $F : M \to \mathbb{R}$ with $F|_S = f$ and $\mathrm{supp}(F) \subseteq U$.

25. Let $X_1, \dots, X_k$ be linearly independent vector fields on a manifold M with Riemannian metric $\langle \cdot, \cdot \rangle$. Show that the Gram-Schmidt procedure can be applied to the vector fields all at once, to give a collection of orthonormal vector fields $Y_1, \dots, Y_k$.

26. [Lee, p.289, problem 11-17] We defined the metric induced by a Riemannian metric $\langle \cdot, \cdot \rangle_a$ to be the infimum of lengths of piecewise smooth curves from p to q. Show that this cannot be a <u>minimum</u>: For $M = \mathbb{R}^2 \setminus (0,0)$ with the usual Riemannian metric from $\mathbb{R}^2$, show that no curve in M from $(-1,0)$ to $(1,0)$ has length equal to the distance between the two points.

(*) 27. [Lee, p.286, problem 11-3] Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Show that there is a canonical (i.e., basis-independent) isomorphism between $V^* \otimes W$ and the space $Hom(V, W)$ of linear maps from V to W.