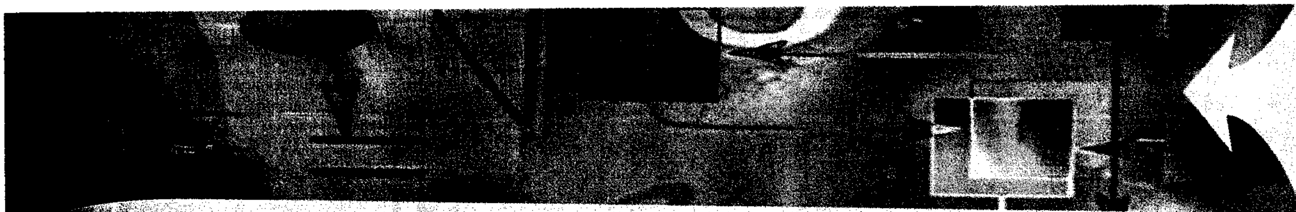
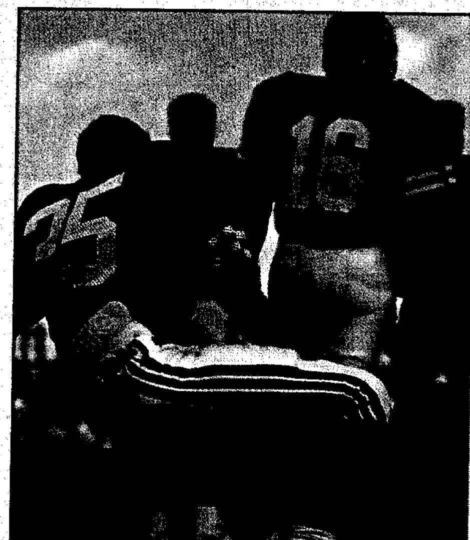


10

Probability

Run That Ball!

In the opening game of the season, the local college's quarterback suffered a broken collarbone, an injury that would cause him to miss most of the season. This football team runs an option offense. In the option offense, the quarterback is more exposed because he has the option of running the ball himself, handing the ball to a running back, or passing to a receiver. Fans noticed that quarterback injuries have occurred more frequently since the option offense was installed 2 years earlier. At the boosters club meeting the following week, the coach was asked if his quarterbacks would continue getting hurt. The coach responded, "Four quarterbacks in the country were injured on Saturday. One was an option quarterback and the others were standard drop-back passers. Just because you run the option does not mean that you are going to get hurt. There is risk in any sport."



Jeffrey Blackman/Index Stock Imagery

The coach's reasoning seems plausible: his quarterback was injured, but the quarterbacks of three other teams were injured as well. On closer inspection, however, we see a fallacy in his reasoning. The NCAA consists of just over 100 division 1-A football teams. Of these, only 4 were using the option offense. Thus, based on the results of the first week of play, the probability that one of the option quarterbacks would be injured is $\frac{1}{4}$, or 25%. By comparison, only 3 of approximately 100, or about 3%, of the rest of the teams in the country lost a quarterback. Looked at this way, the option offense seems hazardous to a quarterback's health.

The Human Side of Mathematics

DAVID BLACKWELL (1919–) grew up in Centralia, Illinois. As a student, Blackwell did not care for algebra and trigonometry, but geometry excited him. He entered college at the age of 16 with the goal of earning a bachelor's degree and becoming an elementary school teacher. He received his A.B., A.M., and Ph.D. in Statistics at the University of Illinois. He was the seventh African-American to earn a Ph.D. in Mathematics in the United States. In 1941, he was nominated for a fellowship at the Institute for Advanced Study at Princeton. The position included an honorary membership in the faculty at nearby Princeton University, but the university objected to the appointment of a black man as a faculty member. The director of the institute insisted on appointing Blackwell and eventually won out. At the Institute, Blackwell met the brilliant John von Neumann, who asked Blackwell about his Ph.D. thesis. Blackwell relates that "He [von Neumann] listened for ten minutes and he started telling me about my thesis." From Princeton, Blackwell went on to teach for 10 years at Howard University and then moved to the University of California at Berkeley. He has been a prolific researcher and writer and has made important contributions to probability, statistics, game theory, and set theory. In 1965, he became the first African-American named to the National Academy of Sciences. In 1979, he won the John von Neumann Theory Prize, which is awarded each year to a scholar who has made fundamental, sustained contributions to theory in operations research and the management sciences.

Source: www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Blackwell.html

In addition to being an accomplished research

MARILYN VOS SAVANT (1946–) was asked the following question in her "Ask Marilyn" column in the September 1990 issue of *Parade* magazine:

Source: Timothy White/Courtesy of Parade Magazine

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind each of the other doors, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He then says to you, 'Do you want to pick door number 2?' Is it to your advantage to switch your choice?" Ms. vos Savant (who just happened to have the world's highest tested IQ), replied, "Yes, you should switch. The first door has a one-third chance of winning, but the second door has a two-thirds chance."

There was a very strong response to this column. Many professional mathematicians and statisticians (complete with Ph.D.s) informed her that she was wrong. However, vos Savant's answer was correct and she explained the reasoning behind her answer in her next column. The response this time was even more intense than before. The letters included one from a deputy director of the Center for Defense Information and another from a research statistician from the National Institutes of Health. Of the letters from the public, 92% disputed her answer, compared with 65% of the letters from universities. However, as one writer, a Ph.D. from the Massachusetts Institute of Technology put it, "You are indeed correct. My colleagues at work had a ball with this problem, and I dare say that most of them—including me at first—thought you were

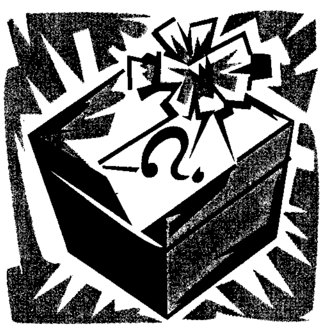
OBJECTIVES

In this chapter you will learn how to

1. compute probabilities of events in situations in which all outcomes have the same chance of occurring;
2. compute probabilities when partial information is available;
3. use expected values to find the true cost of lottery tickets, insurance premiums and similar items; and
4. determine the odds in favor of or against an event.

10.1 Computing Probabilities in Simple Experiments

INITIAL PROBLEM



Following a wedding, the attendants for the groom loaded the wedding gifts into a van and took them to the reception hall. After they had taken all the presents into the hall, they noticed that three of the presents did not have gift cards from the senders. They returned to the van and found the three cards, but there was no way to tell which card went with which gift. Slightly flustered, the attendants decided to arbitrarily put a card with each of the untagged gifts. What are the chances that at least one of those gifts was paired with the correct card?

A solution of this Initial Problem is on page 644.

Tidbit

The chances of being struck by lightning are approximately 1 in 700,000. Each year in the United States, about 300 people are struck and injured by lightning. Roy Cleveland Sullivan, a "human lightning conductor" and a forest ranger from Virginia, was struck by lightning 7 times between 1942 and 1983. He suffered injuries to his toes, arms, legs, chest, and stomach. His death in 1983, however, was not caused by lightning.

Most people have an intuitive understanding of probability as it relates to chance events. For instance, when we hear statements such as, "The probability of getting struck by lightning is greater than the probability of winning the lottery," we know that our chances of winning a major lottery must be nearly nonexistent, since the likelihood of getting struck by lightning is very small. When it comes to a more sophisticated understanding of what probability actually *is*, or to determining the probability of a complex action, an intuitive understanding is not enough. We need a more precise way to think about probability. In this section, you will learn about the language, concepts, and rules used in the mathematical discussion of probability.

INTERPRETING PROBABILITY

Probability is the mathematics of chance, and the terminology used in probability theory is often heard in daily life. For example, you may hear on the radio "The probability of precipitation today is 80%." This statement should be interpreted as meaning that on 80% of past days that had atmospheric conditions like those of today, it rained at some time. The intuitive interpretation might be "carry an umbrella."

To further illustrate the use of probability in everyday language, an article about test results in medical science may state that a patient has a 6-in-10 chance of improving if treated with drug X. We interpret this statement to mean that if 100 patients had the same symptoms as the patient being treated, $\frac{6}{10} \times 100 = 60$ of them improved when they were given drug X.

EXAMPLE 10.1 An advertisement for one of the state lottery games says, "The chances of winning the lottery game 'Find the Winning Ticket' are 1 in 150,000." How should you interpret this statement?

SOLUTION This statement means that only 1 of every 150,000 lottery tickets printed is a winning ticket. ■

SAMPLE SPACES AND EVENTS

Tidbit

Even though the founding of probability theory as a science is usually attributed to Blaise Pascal in the mid-1600s, the concept of sample space was not formulated until the 20th century by R. von Mises.

To study probability in a mathematically precise way, we need special terminology and notation. Making an observation or taking a measurement of some act, such as flipping a coin, is called an **experiment**. An **outcome** is one of the possible results of an experiment, such as getting a head when flipping a coin. The set of all the possible outcomes is called the **sample space**. Finally, an **event** is any collection of the possible outcomes; that is, an event is a subset of the sample space. These concepts are illustrated in Example 10.2.

EXAMPLE 10.2 List the sample space and one possible event for each of the following experiments.

a. ***Experiment:*** A standard six-sided die has 1, 2, 3, 4, 5, and 6 dots, respectively, on its faces, as shown in Figure 10.1.

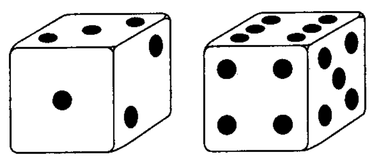


Figure 10.1

Roll one such die and record the number of dots showing on the top face.

b. ***Experiment:*** Toss a coin three times and record the results in order.

c. ***Experiment:*** Spin a spinner like the one in Figure 10.2 twice and record the colors of the regions where it comes to rest in order.

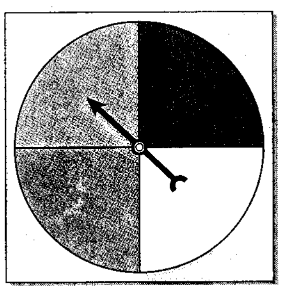


Figure 10.2

d. ***Experiment:*** Roll two standard dice and record the number of dots appearing on the top of each.

SOLUTION

a. ***Sample Space:*** There are six possible outcomes: $\{1, 2, 3, 4, 5, 6\}$, where numerals represent the number of dots showing on the top of the die.

Event: The set $\{2, 4, 6\}$ is the event of getting an even number of dots, and $\{2, 3, 5\}$ is the event of getting a prime number of dots. Examples of other possible events are $\{3\}$ = the event the result of the roll is a 3, and $\{5, 6\}$ = the event the result is a number greater than 4.

b. ***Sample Space:*** Use three-letter sequences to represent the possible outcomes. For example, HHH represents tossing three heads. First, we list all possible outcomes. *Note:* HTH represents tossing a head first, a tail second, and a head third.

Event: The sample space of eight elements has many subsets. Any one of its subsets is an event. For example, the subset {HTH, HTT, TTH, TTT} is the event of getting a tail on the second coin, since it contains all possible outcomes matching that condition. Not all events have such simple descriptions, however.

c. ***Sample Space:*** Using pairs of letters to represent the outcomes (colors in this experiment), we have the sample space shown below. Note how the outcomes are listed in a meaningful way.

RR	*YR*	*GR*	*BR*
RY	*YY*	*GY*	*BY*
RG	*YG*	*GG*	*BG*
RB	*YB*	*GB*	*BB*

Event: One possible event is that the two colors match, which can be represented as {*RR*, *YY*, *GG*, *BB*}. The event that at least one of the spins is green can be represented in set notation as {*RG*, *YG*, *GG*, *BG*, *GR*, *GY*, *GB*}.

d. ***Sample Space:*** We use ordered pairs to represent the possible outcomes, namely, the number of dots on the faces of the two dice. For example, the ordered pair (1, 3) represents one dot showing on the first die and three dots showing on the second. The sample space consists of the 36 outcomes listed next. Again, notice that the possible outcomes are listed in a meaningful way.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Event: The event of getting a total of seven dots on the two dice is {(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)}. The event of getting more than nine dots is {(6, 4), (5, 5), (4, 6), (6, 5), (5, 6), (6, 6)}. Many other events could have been chosen as examples. ■

Timely Tip

Sets are often represented by listing the members of the set in braces. The set of positive integers strictly between 3 and 9, for example, is represented in set notation as {4, 5, 6, 7, 8}. We often use a capital letter to refer to a set, so we might write A = {4, 5, 6, 7, 8}. A set with no members, such as the set of positive integers strictly between 4 and 5, is called an empty set, and is represented as { } or as ∅. A subset of a set A is one whose members all belong to A. For example, the set {6, 8} is a subset of {4, 5, 6, 7, 8}.

Tidbit

Cubical dice marked equivalently to modern dice have been found in Egyptian tombs dated before 2000 B.C. and in Chinese excavations dating to 600 B.C.

EXPERIMENTAL PROBABILITY

One way to find the probability of an event is to conduct a series of experiments. For example, if we wanted to know the probability of "heads" landing face up when a coin is tossed, we could toss a coin repeatedly. To find the probability of heads landing face up in this experiment, we would record the number of times heads appears face up when the coin is tossed and divide that number by the total number of times the coin is tossed. This leads to the definition of **experimental probability**, which is the relative frequency with which an event occurs in a particular sequence of trials.

Experiments have shown that a coin tossed hundreds of times will land "heads up" approximately 50% of the times it is tossed. Based on such experiments, we say that the probability of a coin landing "heads up" is $\frac{1}{2}$ (or 0.5, or 50%). Note that experimental probabilities may differ from one set of observations to another. One person tossing a coin 1000 times could get heads 507 times while another person performing the same experiment 1000 times might get heads only 480 times.

The probability of an event can be reported as a fraction, a decimal, or a percentage, as illustrated above. This number must be between zero and one, inclusive, if expressed as a fraction or a decimal; it must be between 0% and 100%, inclusive, if expressed as a percentage. The greater the probability, the more likely the event is to occur. Conversely, the smaller the probability, the less likely the event is to occur. An event with a probability of 0 is expected never to occur. An event with probability 1 is a "sure thing" and is expected to happen every time the experiment is repeated.

Tidbit

The French naturalist Georges-Louis Leclerc de Buffon (1707–1788) tossed a coin 4040 times, obtaining 2048 heads (50.69% heads). Around 1900, the English statistician Karl Pearson tossed a coin 24,000 times, obtaining 12,012 heads (50.05%). While imprisoned by the Germans during World War II, the English mathematician John Kerrich tossed a coin 10,000 times, obtaining 5067 heads (50.67%).

EXAMPLE 10.3 An experiment consists of tossing two coins 500 times and recording the results. Table 10.1 gives the observed results and the experimental probability of each outcome. Let E be the event of getting a head on the first coin. Find the experimental probability of E.

Table 10.1

Outcome	Frequency	Experimental Probability
HH	137	$\frac{137}{500}$
HT	115	$\frac{115}{500}$
TH	108	$\frac{108}{500}$
TT	140	$\frac{140}{500}$
	Total: 500	*Total:* $\frac{500}{500} = 1.00$

SOLUTION The event E is {HH, HT}. From the table, we see that a head showed on the first coin $137 + 115 = 252$ times. Thus, the experimental probability of E is $\frac{137 + 115}{500} = \frac{252}{500} = 0.504$. ■

THEORETICAL PROBABILITY

The advantage of finding a probability experimentally is that it can be done by simply performing experiments. The disadvantage is that the experimental probability will depend on the particular set of repetitions of the experiment and hence may need to be recomputed when more experiments are performed. In addition, rare outcomes may not appear at all in the list of actual observations, which may lead us to conclude erroneously that the probability of an outcome is 0. Yet another disadvantage to using experimental probabilities is that performing the necessary experiments may not be as simple as we have been assuming and, even if each repetition of the experiment is simple, the process of repeating the experiment many times may be time-consuming or tedious.

In many cases, we can determine what fraction of the time an event is likely to occur without actually performing experiments. For example, you already knew that a coin is going to land heads up about $\frac{1}{2}$ of the time and that a die is going to land showing three dots about $\frac{1}{6}$ of the time. Maybe you knew these facts because of your experience with coins and dice. On the other hand, you may have realized that the symmetry of a coin implies that it *should be* equally likely for the coin to land heads up or tails up. Likewise, the symmetry of the die implies that all six sides *should be* equally likely to be showing when the die is tossed.

Definition

PROBABILITY OF AN EVENT WITH EQUALLY LIKELY OUTCOMES

Suppose that all the outcomes in the sample space S are equally likely to occur. Let E be an event. Then the **probability of event E**, denoted $P(E)$, is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}.$$

There are two things to notice about this definition. First, if you consider only one outcome, its probability is 1 divided by the number of outcomes in the entire sample space. For example, the sample space for one die has 6 outcomes, so the probability of any one particular face showing is $\frac{1}{6}$. The second thing to notice is that the probability of any event is a number from 0 to 1 because the number of outcomes in any event is less than or equal to the number of outcomes in the sample space. An event containing no outcomes has probability zero, and an event that contains all the outcomes in the sample space has probability 1. Of course, it seems silly to discuss the probability of the event containing no outcomes, but in some problems, it might not be immediately obvious that an event actually cannot happen.

We cannot be sure that a real-world coin or die is perfectly balanced, so we cannot be sure that the outcomes in the sample space are equally likely. Thus, when we apply the definition, we are computing **theoretical probabilities**. In fact, theoretical probabilities work very well for real-world problems. When we wish to make it clear that we are dealing with theoretical probabilities of an ideal coin or an ideal die, we refer to them as a **fair coin** or a **fair die**. Example 10.4 illustrates how to assign theoretical probabilities.

EXAMPLE 10.4 An experiment consists of tossing two fair coins. Find the theoretical probabilities for each outcome and for E, the event of getting a head on the first coin, as defined in Example 10.3, and for the event of getting at least one head.

SOLUTION There are four outcomes in the sample space: HH, HT, TH, and TT. If the coins are fair, all outcomes should be equally likely to occur, and each outcome should occur $\frac{1}{4}$ of the time. Hence, we make the assignments listed in Table 10.2.

Table 10.2

Outcome	Theoretical Probability
HH	$\frac{1}{4} = 0.25$
HT	$\frac{1}{4} = 0.25$
TH	$\frac{1}{4} = 0.25$
TT	$\frac{1}{4} = 0.25$

As in Example 10.3, the event E is {HH, HT}. Each outcome is equally likely, and there are two outcomes in E and four outcomes in the sample space S. Thus, the theoretical probability of E is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{2}{4} = \frac{1}{2} = 0.5.$$

Notice that this theoretical probability is close to, but not exactly equal to, the experimental probability of 0.504 calculated in Example 10.3. Also, notice that this theoretical probability was calculated without tossing any coins.

Now let's calculate the probability of getting at least one head. In this case, $E =$ {HH, HT, TH}. The theoretical probability of this new event, E, is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{3}{4}.$$

In other words, $P(E) = 0.75$; so theoretically we expect to get at least one head approximately 75% of the time when tossing two coins. ■

EXAMPLE 10.5 We toss two fair dice. Let A be the event of getting 7 dots, B the event of getting 8 dots, and C the event of getting at least 4 dots. Find the theoretical probabilities $P(A)$, $P(B)$, and $P(C)$.

SOLUTION In Example 10.2(d), we found that the sample space for this experiment has 36 possible outcomes. Since the outcomes obtained by tossing two fair dice are equally likely, we can compute the theoretical probabilities $P(A)$, $P(B)$, and $P(C)$ using the definition. Notice that of all the possible outcomes, there are 6 ways to get a total of 7 dots; that is, $A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$. Similarly, there are 5 outcomes in B and 33 outcomes in C. The theoretical probabilities of A, B, and C are shown in Table 10.3. ■

Table 10.3

Event	Number of Outcomes	Probability
A	6	$P(A) = \frac{6}{36} = \frac{1}{6}$
B	5	$P(B) = \frac{5}{36}$
C	33	$P(C) = \frac{33}{36} = \frac{11}{12}$

EXAMPLE 10.6 A jar contains four marbles: one red, one green, one yellow, and one white (Figure 10.3). If we draw two marbles from the jar, one after the other, without replacing the first one drawn, what is the probability of each of the following events?

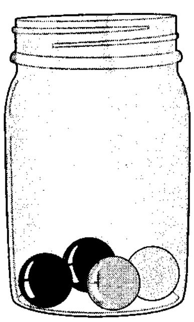


Figure 10.3

A: One of the marbles is red.

B: The first marble is red or yellow.

C: The marbles are the same color.

D: The first marble is not white.

E: Neither marble is blue.

SOLUTION The sample space consists of the following outcomes. RG, for example, means that the first marble is red and the second marble is green.

RG	GR	YR	WR
RY	GY	YG	WG
RW	GW	YW	WY

The sample space has 12 possible outcomes. Since there is exactly one marble of each color, we assume that all the outcomes are equally likely. Now we can use the number of outcomes in each event to determine the probability of the event.

$$A = \{RG, RY, RW, GR, YR, WR\}, \text{ so } P(A) = \frac{6}{12} = \frac{1}{2}.$$

$E =$ the entire sample space, S, because the jar has no blue marbles, so

$$P(E) = \frac{12}{12} = 1.$$ ■

MUTUALLY EXCLUSIVE EVENTS

The **union** of two events $A \cup B$ refers to all outcomes that are in one, or the other, or both events. The **intersection** of two events $A \cap B$ refers to outcomes that are in both events. In words, the union of the two events, $A \cup B$, corresponds to *A or B* happening, while the intersection of the two events, $A \cap B$, corresponds to *A and B* happening. Notice that the event B in Example 10.6 could be represented as the union of two events corresponding to getting a red marble on the first draw (L) or getting yellow on the first draw (M). That is, if we let $L = \{RG, RY, RW\}$ and $M = \{YR, YG, YW\}$, then $B = L \cup M$. On the other hand, notice that $L \cap M = \varnothing$ because it is not possible for the first marble to be both red and yellow. Events such as L and M, which have no outcome in common, are said to be **mutually exclusive**.

If we compute $P(L \cup M)$, $P(L)$, and $P(M)$, we find $P(L \cup M) = P(B) = \frac{6}{12} = \frac{1}{2}$, while $P(L) = \frac{3}{12} = \frac{1}{4}$ and $P(M) = \frac{3}{12} = \frac{1}{4}$. Notice that $P(L \cup M) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Thus, $P(L \cup M) = P(L) + P(M)$. This result, which holds true for any two mutually exclusive events, is summarized next.

> **PROBABILITY OF MUTUALLY EXCLUSIVE EVENTS**
>
> If L and M are mutually exclusive events, then
>
> $$P(L \cup M) = P(L) + P(M).$$

Another example of mutually exclusive events follows.

EXAMPLE 10.7 A card is drawn from a standard deck (Figure 10.4). Let A be the event the card is a face card. Let B be the event the card is a black 5. Find and interpret $P(A \cup B)$.

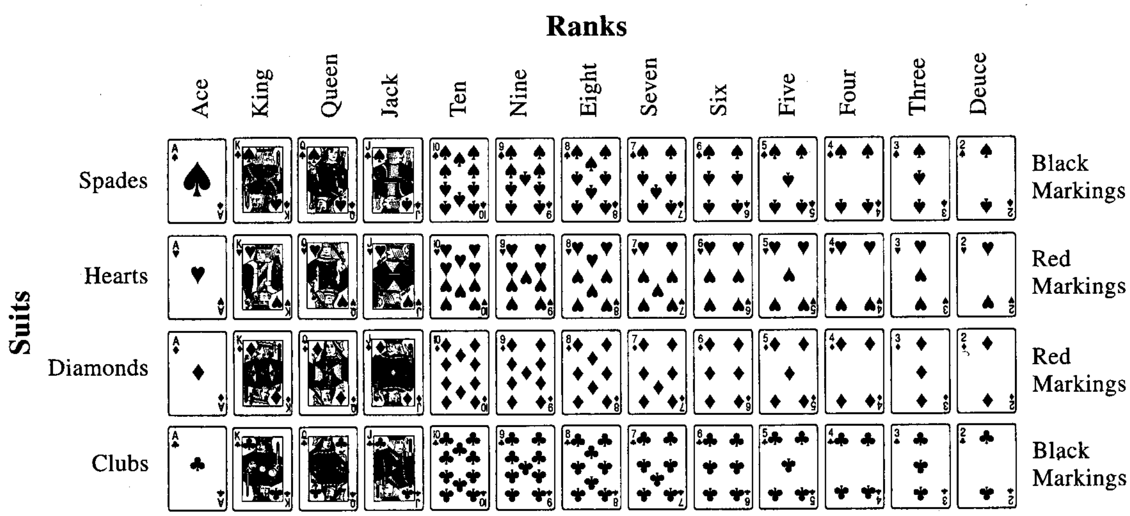


Figure 10.4

SOLUTION The sample space for this experiment consists of 52 outcomes, as shown in Figure 10.4. Each of these outcomes is equally likely. The face cards are the kings, queens, and jacks, so event A has 12 outcomes consisting of the three face cards in each of the four suits. Thus, $P(A) = \frac{12}{52}$. Event B has two outcomes because there are two black fives in the deck. Thus, $P(B) = \frac{2}{52}$. Events A and B are mutually exclusive because they have no outcomes in common. In other words, it is not possible for the card drawn to be *both* a face card *and* a black five. Therefore, we may use the result for mutually exclusive events to calculate $P(A \cup B)$.

$$P(A \cup B) = P(A) + P(B) = \frac{12}{52} + \frac{2}{52} = \frac{14}{52} = \frac{7}{26}.$$

$P(A \cup B)$ means the probability that the one card drawn is a face card *or* a black five. ■

We can use this interpretation to calculate $P(A \cup B)$ another way and verify the result we just obtained. $A \cup B$ consists of all outcomes for which the card drawn is a face card or a black five. We could circle those cards in Figure 10.4 or simply list them as follows, where KS means the king of spades, QH means the queen of hearts, and so on.

$$A \cup B = \{\text{KS, KH, KD, KC, QS, QH, QD, QC, JS, JH, JD, JC, 5S, 5C}\}.$$

Counting the outcomes in $A \cup B$, we see that there are 14 ways that the card drawn could be a face card or a black five. Since the sample space has 52 outcomes, we have $P(A \cup B) = \frac{14}{52} = \frac{7}{26}$, which agrees with what we found by using the formula $P(A) + P(B)$. In Example 10.7, we were able to find $P(A \cup B)$ by finding $P(A) + P(B)$ in this case because events A and B were mutually exclusive. As we will see later, if events A and B are *not* mutually exclusive, then it is *not* true that $P(A \cup B) = P(A) + P(B)$.

COMPLEMENT OF AN EVENT

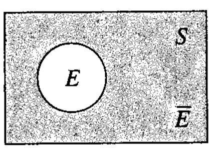


Figure 10.5

Sometimes we are just as interested in the likelihood of an event *not* taking place as we are in the likelihood that it does take place. The set of outcomes in a sample space S, but not in an event E, is called the **complement of the event *E*** (Figure 10.5). We write the complement of E as $\overline{E}$ (read "not E"). Since $S = E \cup \overline{E}$ and $E \cap \overline{E} = \varnothing$, it follows that $P(E) + P(\overline{E}) = P(S)$. But $P(S) = 1$, so $P(E) + P(\overline{E}) = 1$. Thus, we have the following result.

PROBABILITY OF AN EVENT AND ITS COMPLEMENT

The relationship between the probability of an event E and the probability of its complement $\overline{E}$ is given by

$$P(E) = 1 - P(\overline{E}) \text{ and } P(\overline{E}) = 1 - P(E).$$

The preceding equations can be used to find $P(E)$ whenever $P(\overline{E})$ is known or vice versa. In Example 10.6, D was the event that the first marble is not white. Therefore, $P(\overline{D})$ is the probability that the first marble *is* white, so $P(\overline{D}) = \frac{3}{12}$, or $\frac{1}{4}$. Using the complement to find the probability of D, we have $P(D) = 1 - P(\overline{D}) = 1 - \frac{1}{4} = \frac{3}{4}$, as we found directly in Example 10.6.

EXAMPLE 10.8 Carolan and Mary are playing a number-matching game. Carolan chooses a whole number from 1 to 4 but does not tell Mary, who then guesses a number from 1 to 4. Assume that all numbers are equally likely to be chosen by each player.

a. What is the probability that the numbers Carolan and Mary choose are equal?

b. What is the probability that the numbers they choose are unequal?

SOLUTION The sample space can be represented as ordered pairs of numbers from 1 to 4, the first being Carolan's number, the second Mary's number. As shown next, the sample space contains 16 outcomes.

(1, 1)	(1, 2)	(1, 3)	(1, 4)
(2, 1)	**(2, 2)**	(2, 3)	(2, 4)
(3, 1)	(3, 2)	**(3, 3)**	(3, 4)
(4, 1)	(4, 2)	(4, 3)	**(4, 4)**

a. Let E be the event that the numbers chosen by Carolan and Mary are equal. Outcomes for E are shown in boldface. Assuming that all outcomes are equally likely, we have $P(E) = \frac{4}{16} = \frac{1}{4}$.

b. The event that the numbers chosen are unequal is $\overline{E}$. Hence, the probability that the numbers are unequal is $1 - P(E) = 1 - \frac{1}{4} = \frac{3}{4}$. We can also find this result directly by counting the outcomes *not* in E. There are 12 of them. Thus, $P(\overline{E}) = \frac{12}{16} = \frac{3}{4}$. ■

The next example demonstrates how probabilities may be calculated when an experiment and sample space are represented visually.

EXAMPLE 10.9 Figure 10.6 shows a diagram of a sample space S for an experiment with equally likely outcomes. Events A, B, and C are indicated, with their equally likely outcomes represented by points. Find the probability of each of the following events: $S, \varnothing, A, B, C, A \cup B, A \cap B, A \cup C, \overline{C}$.

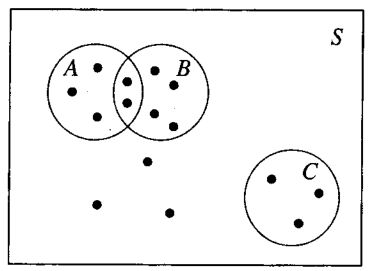


Figure 10.6

SOLUTION We can tabulate the number of outcomes in each event and use those numbers to determine the probabilities. For example, the number of outcomes in A is 5 and the number of outcomes in S is 15. Thus, $P(A) = \frac{5}{15} = \frac{1}{3}$. The probabilities of the remaining events can be calculated in a similar manner, as shown in Table 10.4.

Table 10.4

Event E	Number of Outcomes in E	$P(E) = \dfrac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S}$
S	15	$\frac{15}{15} = 1$
$\varnothing$	0	$\frac{0}{15} = 0$
A	5	$\frac{5}{15} = \frac{1}{3}$
B	6	$\frac{6}{15} = \frac{2}{5}$
C	3	$\frac{3}{15} = \frac{1}{5}$
$A \cup B$	9	$\frac{9}{15} = \frac{3}{5}$
$A \cap B$	2	$\frac{2}{15}$
$A \cup C$	8	$\frac{8}{15}$
$\overline{C}$	12	$\frac{12}{15} = \frac{4}{5}$

■

PROPERTIES OF PROBABILITY

In Example 10.9, the event $A \cup B$ had 9 outcomes. Three of these outcomes were in A, but not in B, four of these outcomes were in B but not in A, and two of these outcomes were in *both* A and B. We saw that $P(A \cup B) = \frac{9}{15}$. Notice that $P(A) = \frac{5}{15}$ and $P(B) = \frac{6}{15}$, so in this case $P(A \cup B)$ was *not* equal to $P(A) + P(B) = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$. Why? Because events A and B are not mutually exclusive; that is, $A \cap B$ is not the empty set. In fact, there are two outcomes in $A \cap B$, as shown in Figure 10.6. Although $P(A \cup B) \neq P(A) + P(B)$, there is a relationship between these three probabilities. Notice that $P(A) + P(B) - P(A \cap B) = \frac{5}{15} + \frac{6}{15} - \frac{2}{15} = \frac{9}{15}$, which is the same value we found for $P(A \cup B)$.

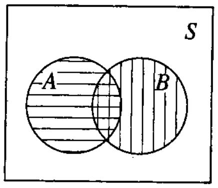


Figure 10.7

This result is true for all events A and B in any sample space. In Figure 10.7, observe how the region $A \cap B$ is shaded twice, once from shading A and once from shading B. Thus, for any sets A and B, to find the number of elements in $A \cup B$ we can find the sum of the number of elements in A and B, but we must subtract the number of elements in $A \cap B$ so that these elements are not counted twice. Hence, the number of elements in $A \cup B$ equals the number of elements in A plus the number of elements in B minus the number of elements in $A \cap B$. This gives us the following relationship

PROBABILITY OF THE UNION OF TWO EVENTS

If A and B are any two events from a sample space, S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

EXAMPLE 10.10 A card is drawn from a standard deck. Let A be the event the card is a face card and B be the event the card is a heart. Find and interpret $P(A \cup B)$.

SOLUTION The sample space contains 52 equally likely outcomes. Event A has 12 outcomes, so $P(A) = \frac{12}{52}$. Event B has 13 outcomes because there are 13 hearts in the deck, so $P(B) = \frac{13}{52}$. However, A and B are not mutually exclusive because there are three cards in the deck that are face cards *and* hearts, namely the king of hearts, the queen of hearts, and the jack of hearts; that is, $P(A \cap B) = \frac{3}{52}$. To find $P(A \cup B)$, we use the result stated prior to this example.

$$\begin{aligned} P(A \cup B) = P(A) + P(B) - P(A \cap B) &= \frac{12}{52} + \frac{13}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &= \frac{11}{26}. \end{aligned}$$

Notice that in Example 10.10, we subtracted $P(A \cup B)$ to avoid counting twice the hearts that are face cards, once as hearts and once as face cards. The equation $P(A \cup B) = \frac{11}{26}$ means the probability that the card drawn is a face card, a heart, or both is $\frac{11}{26}$. If we performed this experiment many times, we would expect to get a face card or a heart or a face card that is also a heart in approximately $\frac{11}{26}$ of the experiments. ■

We can summarize our observations about the properties of probability as follows.

PROPERTIES OF PROBABILITY

For sample space S and events A and B,

1. For any event A, $0 \leq P(A) \leq 1$.
2. $P(\varnothing) = 0$.
3. $P(S) = 1$.
4. If events A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.
5. If A and B are any events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
6. If $\overline{A}$ is the complement of event A, then $P(\overline{A}) = 1 - P(A)$.

EXAMPLE 10.11 An experiment consists of spinning the spinner shown in Figure 10.8 and recording the number on which it lands.

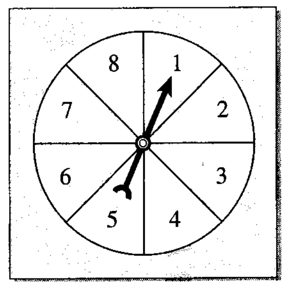


Figure 10.8

Define four events as follows:

A: the spinner lands on an even number
B: the spinner lands on a number greater than 5
C: the spinner lands on a number less than 3
D: the spinner lands on a number other than 2

a. Find $P(A)$, $P(B)$, $P(C)$, and $P(D)$.
b. Find and interpret $P(A \cup B)$ and $P(A \cap B)$.
c. Find and interpret $P(B \cup C)$ and $P(B \cap C)$

SOLUTION

a. The sample space has 8 outcomes, which can be represented in set notation as $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Because each numbered section of the spinner is the same size (that is, has the same central angle measure), these 8 outcomes are equally likely.

Event $A = \{2, 4, 6, 8\}$, so $P(A) = \frac{4}{8} = \frac{1}{2}$.
Event $B = \{6, 7, 8\}$, so $P(B) = \frac{3}{8}$.
Event $C = \{1, 2\}$, so $P(C) = \frac{2}{8} = \frac{1}{4}$.
Event $D = \{1, 3, 4, 5, 6, 7, 8\}$, so $P(D) = \frac{7}{8}$.

It may be easier to calculate $P(D)$ by working with $\overline{D}$. The only way event D will *not* occur is if the spinner lands on 2; that is, event $\overline{D} = \{2\}$, and thus, $P(\overline{D}) = \frac{1}{8}$. That means $P(D) = 1 - P(\overline{D}) = 1 - \frac{1}{8} = \frac{7}{8}$, which is the same result we obtained by counting the outcomes in D.

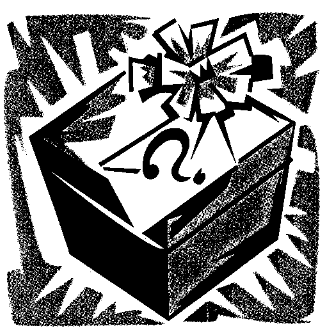
b. $A \cup B$ means the event the spinner lands on a number that is even *or* greater than 5 *or* both. Since it is possible for a number to be *both* greater than 5 and also even, A and B are not mutually exclusive events. To find $P(A \cup B)$, therefore, we can first find $P(A \cap B)$. Then we can calculate the probability by using the result $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. $A \cap B$ means the event in which the spinner lands on a number that is even *and* greater than 5. This event can happen in two ways; that is, $A \cap B = \{6, 8\}$. Therefore, $P(A \cap B) = \frac{2}{8}$. Now we have enough information to calculate $P(A \cup B)$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{8} + \frac{3}{8} - \frac{2}{8} = \frac{5}{8}.$$

This result means that about $\frac{5}{8}$ of the time this experiment is performed, the spinner lands on a number that is even or greater than five (or both). The spinner might be expected to land on an even number greater than five $\frac{2}{8}$, or $\frac{1}{4}$, of the time.

c. $B \cup C$ means the event the spinner lands on a number that is greater than 5 *or* less than 3. It is impossible for a number to satisfy both conditions, so B and C are mutually exclusive events. Thus, $P(B \cup C) = P(B) + P(C) = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$. We can verify this result by counting the outcomes in $B \cup C$. Listing the numbers that are greater than 5 or less than 3, we have $B \cup C = \{6, 7, 8, 1, 2\}$. Thus, $P(B \cup C) = \frac{5}{8}$. On the other hand, $B \cap C$ means the event the spinner lands on a number that is greater than 5 *and* also less than 3. Since this cannot happen, $B \cap C = \varnothing$ and $P(B \cap C) = 0$. Thus, $P(B \cup C) = \frac{5}{8}$ means that when the experiment is performed, we expect that the spinner will land on a number greater than 5 or less than 3 about $\frac{5}{8}$ of the time. ■

SOLUTION OF THE INITIAL PROBLEM



Following a wedding, the attendants for the groom loaded the wedding gifts into a van and took them to the reception hall. After they had taken all the presents into the hall, they noticed that three of the presents did not have gift cards from the senders. They returned to the van and found the three cards, but there was no way to tell which card went with which gift. Slightly flustered, the attendants decided to arbitrarily put a card with each of the untagged gifts. What are the chances that at least one of those gifts was paired with the correct card?

SOLUTION Let E be the event that at least one gift receives the correct card. We will indicate the three gifts by the letters A, B, and C, and their respective cards by a, b, and c. We list all the possible combinations of gifts and cards in Table 10.5. Each line of the table indicates one way in which gifts can be matched with cards. For example, the entry (B, c) means that gift B receives the card that belongs with gift C.

Table 10.5

(A, a)	(B, b)	(C, c)
(A, a)	(B, c)	(C, b)
(A, b)	(B, a)	(C, c)
(A, b)	(B, c)	(C, a)
(A, c)	(B, a)	(C, b)
(A, c)	(B, b)	(C, a)

There are six outcomes in the sample space corresponding to the six rows in the table, and only the fourth and fifth lines correspond to all the gifts receiving the wrong cards. In the other four cases, at least one card is matched with the correct gift. We conclude that $P(E) = \frac{4}{6} = \frac{2}{3}$.

PROBLEM SET 10.1

Problems 1 through 4

Calculating probabilities often requires that you perform operations with fractions. The following problems are designed to help you brush up on fractions. Perform the given operations by hand. If the result is a fraction, express it in lowest terms.

1. a. $\frac{1}{8} + \frac{3}{4}$ **b.** $\frac{1}{3} + 1 - \frac{11}{15}$

c. $\frac{3}{10} \cdot \frac{2}{9}$ **d.** $\dfrac{\frac{4}{7}}{\frac{3}{2}}$

2. a. $\frac{2}{3} + \frac{1}{5}$ **b.** $\frac{5}{8} + 1 - \frac{1}{12}$

c. $\frac{5}{8} \cdot \frac{4}{15}$ **d.** $\dfrac{\frac{3}{5}}{\frac{2}{3}}$

3. a. $\dfrac{\frac{8}{15}}{\frac{11}{45}}$

b. $\frac{1}{4} \cdot \frac{2}{5} + \frac{3}{4} \cdot \frac{3}{5}$

c. $\frac{1}{3}(240) + \frac{2}{5}(-50)$

4. a. $\dfrac{\frac{2}{7}}{1 - \frac{2}{7}}$

b. $\frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3}$

c. $\frac{3}{8}(500) + \frac{7}{16}(80) + \frac{3}{16}(-1200)$

5. According to the weather report, there is a 20% chance of snow in the county tomorrow. Which of the following statements would be an appropriate interpretation of this statement?

a. Out of the next 5 days, it will snow 1 of those days.

b. Out of the next 24 hours, snow will fall for 4.8 hours.

c. Of past days when conditions were similar, 1 of 5 had some snow in the county.

d. It will snow on 20% of the area of the county tomorrow.

6. The doctor says, "There is a 40% chance that your problem will get better without surgery." Which of the following statements would be an appropriate interpretation of this statement?

a. You can expect to feel 40% better.

b. In the future, you will feel better on 2 of every 5 days.

c. Among you and the next four other patients with the same problem, two will get better without surgery.

d. Among patients with symptoms similar to yours who have participated in research studies of nonsurgical treatments, about 40% got better.

7. List the elements of the sample space and one possible event for each of the following experiments.

a. A quarter is tossed, and the result is recorded.

b. A single die with faces labeled *A*, *B*, *C*, *D*, *E*, *F* is rolled, and the letter on the top face is recorded.

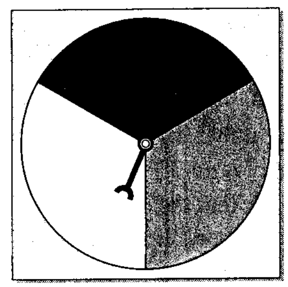
c. A telephone number is selected at random from a telephone book, and the fourth digit is recorded.

8. List the elements of the sample space and one possible event for each of the following experiments.

a. A \$20 bill is obtained from an automatic teller machine, and the right-most digit of the serial number is recorded.

b. Some white and black marbles are placed in a jar, mixed, and a marble is chosen without looking. The color of the marble is recorded.

c. The following "Red–Blue–Yellow" spinner is spun once, and the color is recorded. (All central angles in the spinner are 120°.)



9. An experiment consists of drawing a slip of paper from a bowl in which there are 10 slips of paper labeled *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*, and *J* and recording the letter on the paper. List each of the following:

a. The sample space

b. The event that a vowel is drawn

c. The event that a consonant is drawn

d. The event that a letter between *B* and *G* (excluding *B* and *G*) is drawn

e. The event that a letter in the word ZOOLOGY is drawn

10. An experiment consists of drawing a ping-pong ball out of a box in which 12 balls were placed, each marked with a number from 1 to 12, inclusive, and recording the number on the ball. List the following:

a. The sample space

b. The event that an even number is drawn

c. The event that a number less than 8 is drawn

d. The event that a number divisible by 2 and 3 is drawn

e. The event that a number greater than 12 is drawn

11. An experiment consists of tossing four coins and noting whether each coin lands with a head or a tail showing. List each of the following:
    a. The sample space
    b. The event that the first coin shows a head
    c. The event that three of the coins show heads
    d. The event that the fourth coin shows a tail
    e. The event that the second coin shows a head and the third coin shows a tail

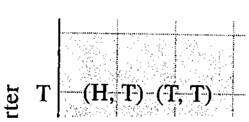
12. An experiment consists of flipping a coin and rolling an eight-sided die and noting whether the coin lands with a head or a tail showing and which number faces up on the die. The eight faces on each die are labeled 1, 2, 3, 4, 5, 6, 7, and 8 as shown.

Eight-Sided Die

List each of the following:
    a. The sample space
    b. The event that a 2 faces up on the die
    c. The event that the coin lands with a head showing
    d. The event that the coin lands with a tail showing and an odd number faces up on the die
    e. The event that the coin lands with a head showing or a 7 faces up on the die

Problems 13 and 14

One way to find the sample space of an experiment involving two parts is to plot the possible outcomes of one part of the experiment horizontally and the outcomes of the other part vertically, then fill in the pairs of outcomes in a rectangular array. For example, suppose that an experiment consists of tossing a dime and a quarter. The sample space could be plotted as:



13. Use the method just described to construct the sample space for the experiment of tossing a coin and rolling a four-sided die with faces labeled 1, 2, 3, and 4.

14. Use the method just described to construct the sample space for the experiment of tossing a coin and drawing a marble from a jar containing purple, green, and yellow marbles.

15. A standard six-sided die is rolled 60 times with the following results.

Outcome	Frequency
1	10
2	9
3	10
4	12
5	8
6	11

    a. Find the experimental probability of the following events.
        1. Getting a 4
        2. Getting an odd number
        3. Getting a number greater than 3
    b. Based on the experimental probability in (a), if the die is rolled 250 times, how many times would you expect to get an even number?

16. A dropped thumbtack will land with the point up or the point down. The results for tossing a thumbtack 60 times are as follows.

Outcome	Frequency
Point up	42
Point down	18

    a. What is the experimental probability that the thumbtack lands

17. An experiment consists of tossing three fair coins.
- **a.** List the outcomes in the sample space and the theoretical probability for each outcome in a table.
- **b.** Find the theoretical probability for the event of getting at least one head.
- **c.** Find the theoretical probability for the event of getting exactly two heads.

18. A jar contains three marbles: one red, one green, and one yellow. An experiment consists of drawing a marble from the jar, noting its color, placing it back in the jar, mixing, and drawing a second marble.
- **a.** List the outcomes in the sample space and the theoretical probabilities for each outcome in a table.
- **b.** Find the theoretical probability for the event of getting at least one red marble.
- **c.** Find the theoretical probability for the event of getting no red marbles.

Problems 19 and 20

Refer to Example 10.2(d), which gives the sample space for the experiment of rolling two standard dice. Assume the dice are fair, and give the theoretical probabilities of the listed events.

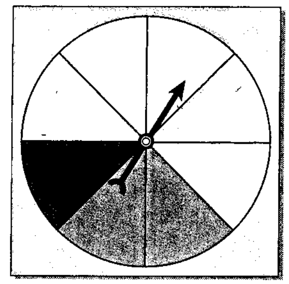
19. **a.** Getting a 4 on the second die
- **b.** Getting an even number on each die
- **c.** Getting a total of at least 7 dots
- **d.** Getting a total of 15 dots

20. **a.** Getting a 5 on the first die
- **b.** Getting an even number on one die and an odd number on the other die
- **c.** Getting a total of no more than 7 dots
- **d.** Getting a total greater than 1

21. An experiment consists of rolling two standard dice and noting the numbers that show on the top faces. Assume the dice are fair.
- **a.** List the elements in the sample space.
- **b.** Find the theoretical probability of the event that the product of the two numbers is even.
- **c.** Find the theoretical probability of the event that the product of the two numbers is odd.
- **d.** Find the theoretical probability of the event that the product of the two numbers is a multiple of 5.

22. An experiment consists of rolling an eight-sided die and a standard six-sided die and noting the numbers that show on the top faces. Assume the dice are fair.
- **a.** List the elements in the sample space.
- **b.** Find the theoretical probability of the event that the sum of the two numbers is greater than 6.
- **c.** Find the theoretical probability of the event that the sum of the two numbers is less than 7.
- **d.** Find the theoretical probability of the event that the product of the two numbers is a multiple of 5.

23. Refer to the following spinner.
- **a.** What is the probability of the spinner landing on yellow?
- **b.** Explain why the probability of getting white is the same as the probability of getting blue.



24. Refer to the preceding spinner.
- **a.** What is the probability of the spinner landing on white?
- **b.** Explain why the probability of getting red is less than the probability of getting any other color.

Problems 25 and 26

Two twelve-sided dice having the numbers 1–12 on their faces are rolled and the numbers facing up are added. Assume the dice are fair and find the probabilities for the listed events.

Twelve-Sided Die

25. **a.** The total is 5.
- **b.** The total is a perfect square.
- **c.** The total is a prime number.

26. **a.** The total is 11.
- **b.** The total is a multiple of 7.
- **c.** The total is even or 19.

27. A six-sided die is constructed that has two faces marked with 2s, three faces marked with 3s, and one face marked with a 5. If this die is rolled once, find the following probabilities:

a. Getting a 2
b. Not getting a 2
c. Getting an odd number
d. Not getting an odd number

28. A 12-sided die is constructed that has three faces marked with 1s, two faces marked with 2s, three faces marked with 3s, and four faces marked with 4s. If this die is rolled once, find the following probabilities:

a. Getting a 4
b. Not getting a 4
c. Getting an odd number
d. Not getting an odd number

29. A couple planning their wedding decides to randomly select a month in which to marry. If T is the event the month is 30 days long, and Y is the event the month ends in the letter y, find and interpret $P(T)$, $P(Y)$, and $P(T \cup Y)$.

30. A family planning a vacation randomly selects one of the states in the United States as their destination. If O is the that event the state borders the Pacific Ocean and N is the event that the state's name contains the word "New," find and interpret $P(O)$, $P(N)$, and $P(O \cup N)$.

31. In a class consisting of girls and boys, 6 of the 14 girls and 7 of the 11 boys have done their homework. A student is selected at random. Consider the following events.

G: The student is a girl.
D: The student has done his or her homework.

a. Find $P(G)$ and $P(D)$.
b. Find and interpret $P(G \cup D)$ and $P(G \cap D)$.
c. Are events G and D mutually exclusive? Explain.

32. For an experiment in which a fair coin is tossed and a fair standard die is rolled, consider the following events.

H: The coin lands heads up.
F: The die shows a number greater than 4.

a. Find $P(H)$ and $P(F)$.
b. Find and interpret $P(H \cup F)$ and $P(H \cap F)$.
c. Are events H and F mutually exclusive?

33. Consider the sample space for the experiment in Example 10.2(c) and the following events.

A: getting a green on the first spin
B: getting a yellow on the second spin

a. List the sample space and find $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$.
b. Verify that the equation $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ holds for the probabilities in part (a).

34. Suppose a jar contains 20 marbles, numbered 1 through 20, with each odd-numbered marble colored red, and each even-numbered marble colored black. A marble is drawn from the jar and its color and number are noted.

a. List the sample space.
b. Consider the following events.

A: getting a black marble
B: getting a number divisible by 3

Find $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$.

c. Verify that the equation $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ holds for the probabilities in part (b).

35. a. Suppose 45% of people have blood type O. Let E be the event that a person has type O blood. Describe $\overline{E}$ and find $P(\overline{E})$.

b. Approximately 8% of babies are born left-handed. Let F be the event that a baby is left-handed. Describe $\overline{F}$ and find $P(\overline{F})$.

36. Based on research conducted after the 1989 Loma Prieta earthquake, U.S. Geological Survey (USGS) results indicate that there is a 62% probability of at least one quake of magnitude 6.7 or greater striking the San Francisco Bay region before 2032. Describe the complement of this event and give its probability.

37. In Example 10.6, a jar contains four marbles: one red, one green, one yellow, and one white. Two marbles are drawn from the jar, one after another, without replacing the first one drawn. Let A be the event the first marble is green, let B be the event the first marble is green and the second marble is white, and let C be the event the second marble is red.

a. Are events B and C mutually exclusive? Explain.
b. Are events A and C mutually exclusive? Explain.
c. Describe in words the complement of event A.
d. Find the probability of the event A and the event $\overline{A}$.
e. Verify that the equation $P(\overline{A}) = 1 - P(A)$ holds for the probabilities you found in (d).

38. A card is drawn from a standard deck. Consider the sample space in Figure 10.4. Let A be the event the card is a diamond, let B be the event the card is a club, and let C be the event the card is a jack, queen, or king.

a. Are events A and B mutually exclusive? Explain.
b. Are events A and C mutually exclusive? Explain.
c. Describe in words the complement of event A.
d. Find the probability of the event A and the event $\overline{A}$.
e. Verify that the equation $P(\overline{A}) = 1 - P(A)$ holds for the probabilities you found in (d).

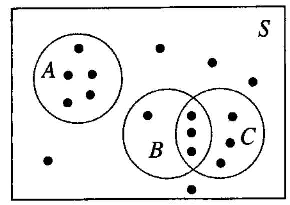
39. Consider the experiment of randomly placing one car (C) and two goats (G) behind three curtains so that one object is behind each curtain. The results are recorded in order.

a. List all possible outcomes in the sample space.
b. Let E be the event the car is hidden behind curtain number 1. List the outcome(s) of the sample space that correspond to event E.
c. Describe $\overline{E}$ and list the outcome(s) of the sample space that correspond to $\overline{E}$.
d. Find $P(E)$ and $P(\overline{E})$.

40. Consider the experiment of randomly placing one silver dollar (D) and three rocks (R) inside four drawers so that one object is in each drawer. The results are recorded in order.

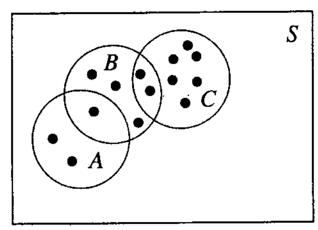
a. List all possible outcomes in the sample space.
b. Let E be the event the silver dollar is hidden in the first drawer. List the outcome(s) of the sample space that correspond to event E.
c. Describe $\overline{E}$ and list the outcome(s) of the sample space that correspond to $\overline{E}$.
d. Find $P(E)$ and $P(\overline{E})$.

41. Consider the sample space S, as shown, for an experiment with equally likely outcomes. Events A, B, and C are indicated. Outcomes are represented by points. Find the probability of each of the following events.



a. A **b.** B **c.** C **d.** S
e. $A \cup B$ **f.** $B \cap C$ **g.** $A \cap C$ **h.** $\overline{A}$
i. $\overline{B}$ **j.** $\overline{C}$

42. Consider the sample space S, as shown, for an experiment with equally likely outcomes. Events A, B, and C are indicated. Outcomes are represented by points. Find the probability of each of the following events.



a. A **b.** B **c.** C **d.** S
e. $\mathrm{A} \cup \mathrm{B}$ **f.** $\mathrm{B} \cap \mathrm{C}$ **g.** $\mathrm{A} \cap \mathrm{C}$ **h.** $\overline{A}$
i. $\overline{B}$ **j.** $\overline{C}$

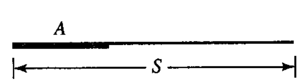
Extended Problems

Problems 43 and 44

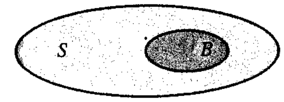
When the probability of an event is proportional to a measurement such as length or area, the probability is determined as follows. Let A be an event that can be measured as a length or as an area. Let $m(A)$ and $m(S)$ represent the measure of the event A and of the sample space S, respectively. Then

$$P(A) = \frac{m(A)}{m(S)}.$$

For example, in the following figure marked (a), if the length of S is 12 cm and the length of A is 4 cm, then $P(A) = \frac{4}{12} = \frac{1}{3}$. Similarly, in the figure marked (b), if the area of region B is 10 square centimeters and the area of the region S is 60 square centimeters, then $P(B) = \frac{10}{60} = \frac{1}{6}$.



(a)



(b)

43. A bus travels between Albany and Binghamton, a distance of 100 miles. Suppose the bus breaks down. Answer the questions below to find the probability $P(A)$ that the bus breaks down within 10 miles of either city.

a. The road from Albany to Binghamton is the sample space. What is $m(S)$?

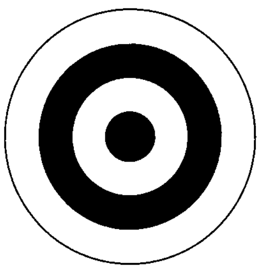
b. Event A is that part of the road within 10 miles of either city. What is $m(A)$?

c. Find $P(B)$.

44. The following dartboard is made up of circles with radii of 1, 2, 3, and 4 inches. Suppose a dart hits the board randomly.

a. Find the probability the dart hits the bull's eye. (*Hint:* The area of a circle with radius r is πr^2.)

b. Find the probability the dart hits the outer ring.



45. Search parties have the difficult job of deciding exactly where to search for a hiker lost in the woods. Many factors must be taken into account, such as the terrain and the hiker's age, fitness, and personality. The selection of specific areas to search is based on assigning a probability to each area. A newer method of assigning probabilities is called **Trail-Based Probability of Areas.** To use this method, the search party has to assume that the hiker was following an established trail from a known point of origin. Research the Trail-Based Probability of Areas assignment procedure by searching keywords "search and rescue trail-based probability of areas" on the Internet or go to www.sarinfo.bc.ca/Trailpoa.htm for more information. How are probabilities assigned to areas? How is the procedure different for a single-trail search or a multiple-trail search? Summarize your findings in a report.

46. The terms *10-year flood, 50-year flood, 100-year flood*, and *500-year flood* describe the estimated probability of a flood happening in any given year. A 10-year flood is defined as a flood that has a 1 in 10 chance, or a 10% probability, of occurring in any given year. A 50-year flood is defined as a flood that has a 1 in 50 chance, or a 2% probability, of occurring in any given year. How are these probabilities determined? What is the probability that a 100-year and a 500-year flood will both occur in any given year? Floods are classified in this way primarily to determine flood insurance rates in areas where floods can occur. For a home insured against floods, what is the probability a 100-year flood will occur during a 30-year mortgage payoff period? The Missouri River has had six 100-year floods since 1945. Which other rivers in the United States have experienced frequent 100-year or 500-year floods? Research floods by using search keywords "100-year flood" on the Internet or go to http://water.usgs.gov/pubs/FS/FS-229-96/ for more information. Write a report of your findings.

47. A classic problem in geometric probability is the **Buffon Needle Problem.** If a needle of a certain length, 2 inches for example, is dropped at random on a floor made of planks wider than the needle, what is the probability the needle will fall across a crack between two planks? To simulate the experiment without a planked floor, use a large piece of paper and draw parallel lines 3 inches apart across the whole paper.

a. Drop a needle onto the "floor" from a consistent height (about 5 feet). Repeat the experiment 60 times, recording whether the needle falls across a crack. Compute the experimental probability for the event.

b. Repeat part (a) with a longer needle (but a needle that is also shorter than the distance between the "planks").

c. Divide the length of the longer needle by the length of the shorter one (measure the lengths carefully). Next, divide the probability you found in part (b) by the probability from part (a). How do the two compare? Is a relationship or generalization suggested?

48. In our examples and problems, we have generally assumed that coins and dice were "fair"; that is, we assumed each face was equally likely to appear on top. Conduct the following experiment with a die that is possibly not fair.

a. Find a wooden cube from a set of children's blocks or a hardware store, and number its faces 1 through 6.

b. Roll this "die" 100 times, record the results, and calculate the experimental probability of each number on your die.

c. Hollow out the center of the face marked with a "6"; this can be done with a drill. Make the hollowed region about $\frac{1}{2}$ inch deep and over about half the face. Be sure you do not cut an edge.

d. Roll the hollowed die 100 times, record the results, and compute the experimental probability of each number on your die.

e. Compare your results from part (d) with those from part (b). To what could you attribute any differences?

49. If you drop a thumbtack, it will land either with the point up or with the point down. Are these events equally likely? Conduct an experiment to find out. Drop a handful of thumbtacks 20 times and record the number of tacks that land with the point up and the number that land with the point down. Calculate the experimental probability that a tack will land with the point up and the experimental probability that the tack will land with the point down. Do the events appear to be equally likely? Conduct the experiment again. Do your results of the second experiment differ significantly from the results of the first experiment? Based on your experimental results, how many tacks would land with the point down if you dropped 1000 tacks? Write a short essay that explains your experiment, and summarize your results in a table.

50. At the beginning of this chapter, the three-door problem was defined as follows. Suppose that a contestant on a game show is given a choice of three doors. Behind one door is a car, and behind each of the other two doors are goats. The contestant randomly picks door number 1, but does not open it, and the host, who knows what is behind the doors, opens door number 3 to reveal a goat. The host then says to the contestant, "Do you want to keep the door you selected or switch to door number 2?"

Conduct a simulation to determine if it is to the contestant's advantage to switch from the original door choice to the other door. For this simulation, three people are needed: a contestant, a host, and a data recorder. Gather the following materials: three paper cups (labeled 1, 2, and 3), one penny, and one standard die. The contestant must not look while the host rolls the die until a 1, 2, or 3 is rolled. The penny will be placed under the corresponding cup. The contestant then rolls the die until a 1, 2, or 3 is rolled and selects the corresponding cup but does not look under it. The host deliberately lifts a cup with no penny underneath from the two cups not selected by the contestant.

a. After the host lifts a cup with no penny underneath, suppose the contestant does not want to switch to the other cup and wants to stay with the original selection. Simulate this "stay" strategy 50 times. Instead of asking the contestant whether he or she wants to switch, the contestant lifts the cup originally selected. The contestant wins if the penny is under the cup and loses if there is no penny. The data recorder should use a table to keep track of the number of wins and losses.

b. After the host lifts a cup with no penny underneath, suppose the contestant does want to switch to the other cup. Simulate this "switch" strategy 50 times. Instead of asking the contestant whether he or she wants to switch, the contestant lifts the other cup that was not selected. The contestant wins if the penny is under the cup and loses if there is no penny. The data recorder should use a table to keep track of the number of wins and losses.

c. Calculate $P(\text{win})$ for the situation when the contestant "stays." Calculate $P(\text{win})$ for the situation in which the contestant "switches."

d. Consider the experimental probabilities calculated in part (c), and explain whether it is to the contestant's advantage to switch his or her choice.

51. The four human blood types are O, A, B, and AB. For each blood type, there are two Rh factors: positive and negative.

a. What percent of the human population falls into each of the eight categories? Search the Internet using keywords "blood types" or go to the American Red Cross's website at www.redcross.org/home/ for more information. Make a table listing each blood type and Rh factor combination and the percentage of the population that falls into each category.

b. What is the probability that one person, selected at random, will have A-positive blood? Answer the same question for each of the other seven classifications of blood.

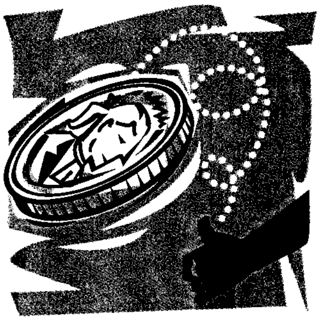
c. In a movie theater containing 450 people, approximately how many people will have each blood type?

d. Survey at least 50 people and keep track of how many people fall into each category. Calculate the percentage of people in each blood category. Do your survey results match the Red Cross's percentages?

e. Research the history of blood type categorization. When were the differences in blood types first recognized? People with blood type A can receive what types of blood? Answer the same question for blood types B, AB, and O. For a person with blood type A, what it the probability that he or she can receive blood from a randomly selected person? Answer the same question for persons with blood type B, AB, and O.

10.2 Computing Probabilities in Multistage Experiments

INITIAL PROBLEM



A friend who likes to gamble makes you the following wager: He bets you \$2 against \$1 that if you toss a coin repeatedly, you will get a total of two tails before you can get a total of three heads. You reason that two heads are as likely as two tails, and you have a one-out-of-two chance of getting a third head after getting two heads. Should you take the bet?

A solution of this Initial Problem is on page 663.

In section 10.1, probability was defined in terms of the relative frequency of given events. Relative frequency is a simple concept, but it requires care in determining the number of ways in which a certain event can occur. It also requires carefully keeping track of the sequence of actions that may make up an event. We will now introduce two tools to help with these difficulties. The first helps us visualize the possible outcomes of an experiment, and the second helps us count the possible outcomes in situations in which it is not practical (or even possible) to do a tally.

TREE DIAGRAMS

For some experiments, it is difficult to list all possible outcomes. The list may be too long to write down, or it may not be clear what pattern to follow in constructing the list. Therefore, we will now examine an alternative method for depicting the sample space of an experiment.

A **tree diagram** is a visual aid that can be used to represent the outcomes of an experiment. The simplest tree diagrams involve experiments in which only one action is taken. For example, consider drawing one ball from a box containing three balls: a red (R), a white (W), and a blue (B). The following steps show how to draw a tree diagram for this experiment.

CONSTRUCTING A ONE-STAGE TREE DIAGRAM

Starting from a single point

1. Draw one branch for each outcome in the experiment, and
2. Place a label at the end of the branch to represent each outcome.

Since the sample space S has three outcomes R, W, and B, there are three right-hand endpoints in the one-stage tree diagram in Figure 10.9.

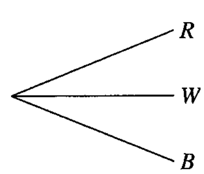


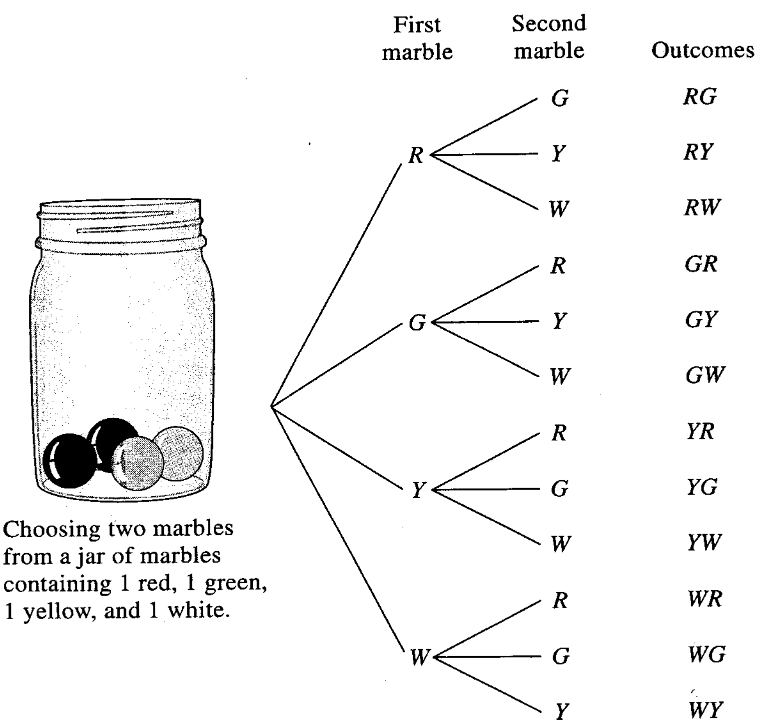
Figure 10.9

Two-stage tree diagrams may be used to represent experiments that consist of a sequence of two actions. To draw a two-stage tree diagram, follow these steps.

CONSTRUCTING A TWO-STAGE TREE DIAGRAM

1. Draw a one-stage tree diagram for the outcomes of the first action. The branches in this part of the tree are called **primary branches.**
2. Starting at the end of each branch of the tree in step 1, draw a one-stage tree diagram for each outcome of the second action. These branches are called **secondary branches.**

For example, suppose an experiment consists of drawing two marbles, one at a time, from a jar of four marbles without putting the first-selected marble back in the jar. This experiment may be represented by the two-stage tree diagram in Figure 10.10.



Choosing two marbles from a jar of marbles containing 1 red, 1 green, 1 yellow, and 1 white.

Figure 10.10

The two-stage tree diagram in Figure 10.10 shows that there are 12 possible outcomes in the sample space because there are 12 right-hand endpoints on the tree. The outcome corresponding to each endpoint is represented by a pair of letters to the right of the endpoint. For example, the outcome of choosing first the red marble and then the green marble corresponds to the uppermost primary and secondary branches and is represented by *RG*.

FUNDAMENTAL COUNTING PRINCIPLE

Rather than count all the possible outcomes by counting the number of endpoints in the tree diagram for the preceding experiment, we can compute the number of outcomes in the sample space by making an observation about the tree diagram. Notice that the four primary branches correspond to the four possible colors of the first marble. Attached to each primary branch are three secondary branches corresponding to the three possible colors of the second marble. Because the same number of secondary branches is connected to each primary branch, the total number of outcomes can be found by multiplying the number of primary branches by the number of secondary branches attached to each primary branch; that is, there are $4 \times 3 = 12$ possible outcomes.

The counting procedure developed in the previous paragraph suggests the following general method for determining the number of outcomes in an experiment.

FUNDAMENTAL COUNTING PRINCIPLE

If an event or action A can occur in r ways, and, for each of these r ways, an event or action B can occur in s ways, the number of ways events or actions A and B can occur, in succession, is rs.

Multiplication may also be used to determine the number of outcomes when there are three, four, or more events or actions. The next example shows how the Fundamental Counting Principle may be applied to more than two actions.

EXAMPLE 10.12 A person ordering a one-topping pizza may choose from three sizes (small, medium, and large), two crusts (white and wheat), and five toppings (sausage, pepperoni, bacon, onions, and mushrooms). Apply the Fundamental Counting Principle to find the number of possible one-topping pizzas.

SOLUTION Because 3 sizes and 2 crusts are possible, there are $3 \times 2 = 6$ combinations of size and crust: (small, white), (medium, white), (large, white), (small, wheat), (medium, wheat), (large, wheat). Each of those 6 combinations can be covered with any one of 5 different toppings, which gives $6 \times 5 = 30$ different selections. ■

The number of different possible selections could have been found in one step by finding the product $3 \times 2 \times 5 = 30$. A tree diagram for the process of ordering a pizza would be a three-stage diagram, since there are three actions to be taken. The first stage would have 3 branches corresponding to the various sizes, the second would have 2 branches corresponding to the two types of crust, and the third stage would have 5 branches corresponding to the five choices of topping. This three-stage tree diagram would have 30 endpoints representing the 30 different possible one-topping pizzas.

Next, we will see how to apply the Fundamental Counting Principle to compute the probability of an event.

Tidbit

An ancient game of chance played by the Greeks and Romans was called "Knucklebones" and used the anklebones of sheep as playing pieces. Values were assigned to the four distinctive sides of these bones. One player tossed two or more bones and observed how they fell. As in modern dice games, the numerical values on the sides of the bones determined the outcome of a toss. Over time, players revised the game and gave us the six-sided die in use today. Even now, dice are often referred to as "bones."

EXAMPLE 10.13 Find the probability of getting a sum of 11 when tossing a pair of fair dice.

SOLUTION Because we have two dice, each with six faces, there are six ways the first die can come up and six ways the second die can come up. Thus, by the Fundamental Counting Principle, there are $6 \times 6 = 36$ possible outcomes for the experiment of rolling two dice. This is the same number of possible outcomes we determined in Example 10.2(d) in Section 10.1. Of the 36 possible outcomes, there are two ways of tossing an 11, namely, (5, 6) and (6, 5). Therefore, the probability of tossing an 11 is $\frac{2}{36}$, or $\frac{1}{18}$. ■

The next example shows how the Fundamental Counting Principle may be used to find the probability of an event having many outcomes.

EXAMPLE 10.14 Suppose two cards are drawn from a standard deck of 52 cards. Use the two-stage tree diagram partially shown in Figure 10.11 to find the probability of getting a pair.

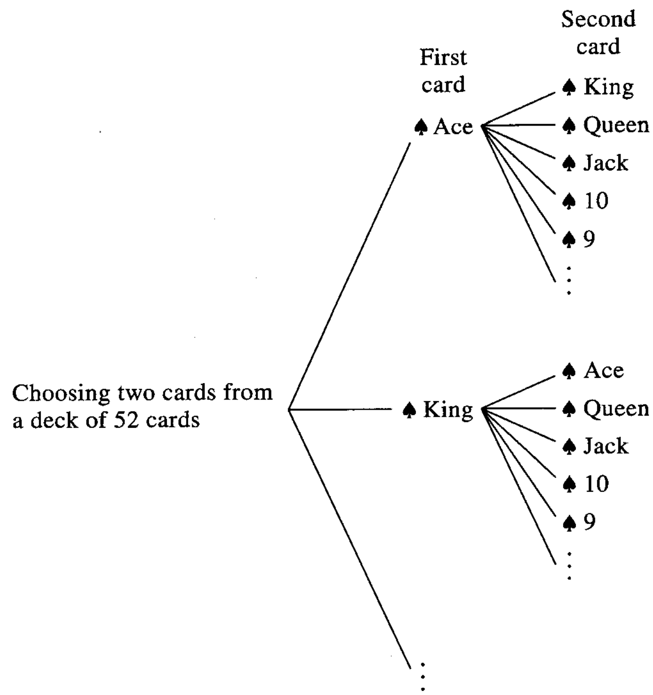


Figure 10.11

SOLUTION The tree diagram in Figure 10.11 represents all the ways to draw two cards from the deck and keeps track of which card is drawn first and which card is drawn second. In Figure 10.11, there are 52 primary branches. There are 51 secondary branches attached to each primary branch, although not all branches are shown. Thus, by the Fundamental Counting Principle, there are $52 \times 51 = 2652$ possible outcomes in the sample space, which means that 2652 two-card hands are possible.

In Figure 10.12, we construct another tree diagram containing only those outcomes that result in drawing a pair.

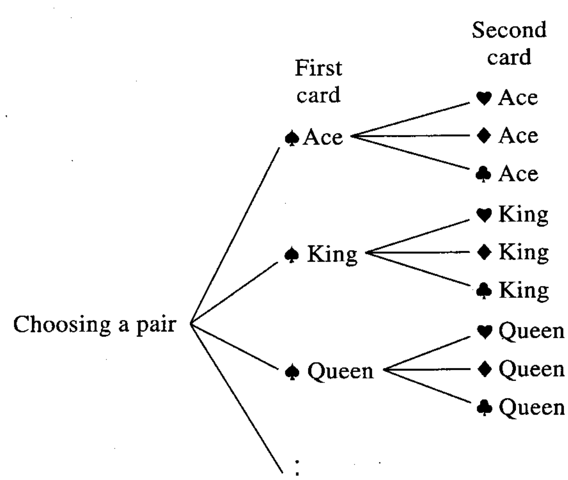


Figure 10.12

In Figure 10.12, there are again 52 primary branches, but now there are only three secondary branches attached to each primary branch because once the first card has been selected, only three cards in the deck match it. By the Fundamental Counting Principle, there are $52 \times 3 = 156$ outcomes represented in Figure 10.12. If S is the sample space associated with drawing two cards in succession (Figure 10.11) and E is the event that those cards form a pair (Figure 10.12), then the number of outcomes in S is 52×51 and the number of outcomes in E is 52×3. Thus, the probability of drawing a pair is given by

$$P(E) = \frac{\text{number of ways of drawing a pair}}{\text{number of ways of drawing two cards}} = \frac{52 \times 3}{52 \times 51} = \frac{156}{2652} = \frac{3}{51} = \frac{1}{17}. \blacksquare$$

PROBABILITY TREE DIAGRAMS AND THEIR PROPERTIES

In addition to helping us to display and count outcomes, tree diagrams may be used to determine probabilities in multistage experiments. For this purpose, it is often useful to assign a probability to each branch of the tree diagram. Tree diagrams that are labeled with the probabilities of events are called **probability tree diagrams**, as shown in Figure 10.13.

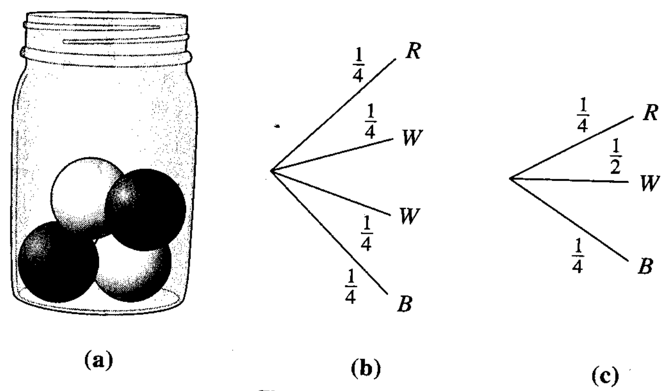
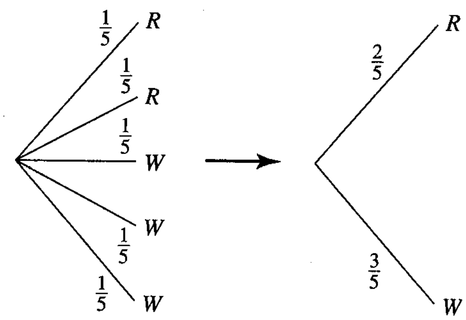


Figure 10.13

Here a container holds four balls: one red, two white, and one blue [Figure 10.13(a)]. Because there are four balls, we draw a tree with four branches, one for each ball. We label each branch with probability $\frac{1}{4}$ since each of the 4 outcomes is equally likely to occur. The resulting probability tree diagram is shown in Figure 10.13(b). Two ways to get a white ball are possible, each with probability $\frac{1}{4}$. To find the probability of drawing a white ball, we add the probabilities on the two branches labeled W. Thus, $P(W) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. To simplify the tree diagram, combine the two branches labeled W into one branch. However, that branch must be labeled with probability $\frac{1}{2}$, the sum of the probabilities on the two branches we combined [Figure 10.13(c)]. Note that the new, simplified probability tree diagram still shows that $P(W) = \frac{1}{2}$.

EXAMPLE 10.15 Draw a probability tree diagram that represents the experiment of drawing one ball from a container holding two red balls and three white balls. Combine branches where possible.

SOLUTION Figure 10.14 shows two one-stage probability tree diagrams.



(a) (b)

Figure 10.14

Figure 10.14(a) has one branch for each ball. Figure 10.14(b) is a simplified version of the first tree diagram and was created by combining all branches for R and all branches for W in Figure 10.14(a). ■

Note that in the preceding example, the event of getting a red ball consisted of getting one red ball or the other red ball. That is, it could be considered as the union of two events: the event of getting one particular red ball and the event of getting the other red ball. These two events were mutually exclusive (you could not get *both* red balls by drawing just one ball), so the probability of the union is the sum of the probabilities. The idea of adding probabilities together, as illustrated in the example, is one that we will use frequently in determining probabilities. It is stated in general terms next.

ADDITIVE PROPERTY OF PROBABILITY TREE DIAGRAMS

If an event, E, is the union of events $E_1, E_2, \ldots, E_n$, where each pair of events is mutually exclusive, then

$$P(E) = P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n).$$

The probabilities of the events $E_1, E_2, \ldots, E_n$ can be viewed as those associated with the corresponding branches in a probability tree diagram.

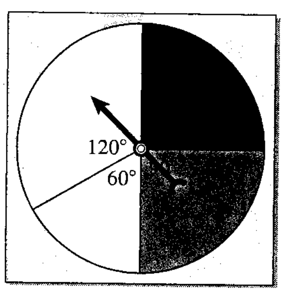


Figure 10.15

The additive property of probability tree diagrams as just stated is an extension of the property $P(A \cup B) = P(A) + P(B)$, where A and B are mutually exclusive events. In Example 10.15, the experiment had two possible outcomes, R and W, but those outcomes had unequal probabilities, $\frac{2}{5}$ and $\frac{3}{5}$, respectively. It is often the case that outcomes in a sample space are not equally likely.

EXAMPLE 10.16 Draw a probability tree for the experiment of spinning the spinner in Figure 10.15 and recording the color on which it lands. Determine the probability of the spinner landing on W or on G.

SOLUTION The experiment has four possible outcomes: R, G, W, and Y. These outcomes are mutually exclusive because the spinner cannot land on two different colors at the same time. Using the central angle for each portion of the spinner to determine the probabilities, we find that $P(R) = \frac{90}{360} = \frac{1}{4}$, $P(G) = \frac{90}{360} = \frac{1}{4}$, $P(W) = \frac{120}{360} = \frac{1}{3}$, and $P(Y) = \frac{60}{360} = \frac{1}{6}$. See the probability tree diagram in Figure 10.16.

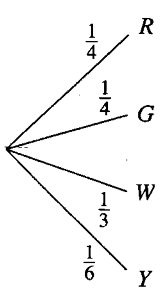


Figure 10.16

Because $P(W) = \frac{1}{3}$ and $P(G) = \frac{1}{4}$, the probability of spinning W or G is $P(W \text{ or } G)$, which may be expressed as $P(W \cup G)$. Using the Additive Property of Probability Tree Diagrams, we have

$$P(W \text{ or } G) = P(W \cup G) = P(W) + P(G) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$ ■

Notice that although the probabilities in Figure 10.16 are unequal, their sum is one: $\frac{1}{4}+\frac{1}{4}+\frac{1}{3}+\frac{1}{6}=\frac{3}{12}+\frac{3}{12}+\frac{4}{12}+\frac{2}{12}=\frac{12}{12}=1$. The sum of the probabilities of all the possible outcomes of an experiment is always 1, whether or not the outcomes are equally likely. In terms of the probability tree diagram, this means that the probabilities on all branches emanating from one point will always add up to 1.

In the next example, a marble is drawn from a jar and replaced. This action is referred to as **drawing with replacement**. If the marble is not replaced, then the process is called **drawing without replacement**. In Example 10.14, two cards were drawn from a standard deck without replacement. The next example describes an experiment in which two marbles are drawn with replacement.

EXAMPLE 10.17 A jar contains three marbles, two black and one red (Figure 10.17). A marble is drawn and replaced, and then a second marble is drawn. What is the probability that both marbles are black? Assume that the marbles are equally likely to be drawn.

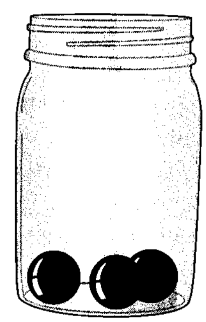


Figure 10.17

SOLUTION A probability tree diagram for this experiment is shown in Figure 10.18(a).

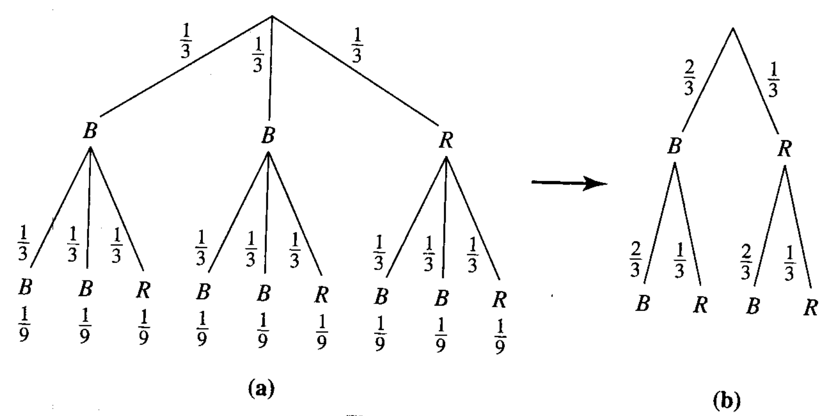
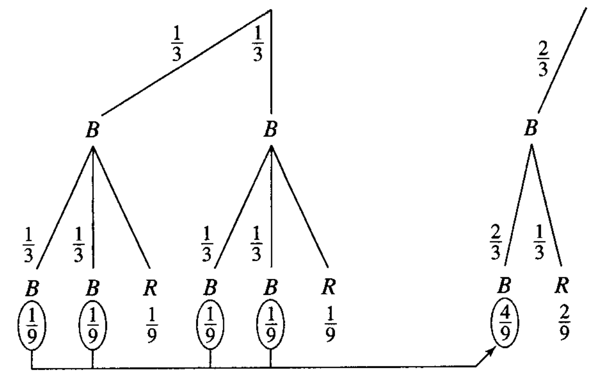


Figure 10.18

Notice that the diagram is drawn vertically here rather than horizontally; either orientation is acceptable. Figure 10.18(b) shows how the number of branches in Figure 10.18(a) can be reduced by combining branches that represent the same type of outcome and by adjusting probabilities accordingly. Notice also in Figure 10.18(a) that we labeled the end of each secondary branch of the tree diagram with a probability. Because each of the 9 outcomes in the sample space is equally likely, each outcome has a probability of $\frac{1}{9}$.

Now we will see how to assign a probability to the end of each secondary branch of the simplified probability tree diagram in Figure 10.18(b). To do this, we will look at just the left portion of Figure 10.18(a), as shown in Figure 10.19.

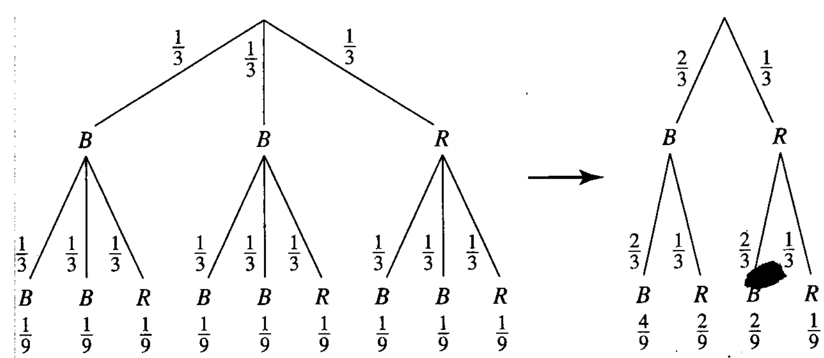


(a) (b)

Figure 10.19

That is, we will look at the portion of the tree diagram that represents getting a black marble on the first draw. Because two black marbles are in the jar, we will examine two primary branches of Figure 10.18(a). Each of the circled probabilities in Figure 10.18(a) corresponds to one way of drawing a black marble on the first draw and a black marble on the second draw. The portion of the tree diagram shown in Figure 10.19(a) tells us that the probability of drawing a black marble first and a black marble second is $P(BB) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{4}{9}$. Because the simplified tree diagram in Figure 10.19(b) represents the same experiment as the diagram in Figure 10.19(a), we label the leftmost secondary branch of that diagram with this probability, which again indicates that $P(BB) = \frac{4}{9}$. In a similar way, we may use the probabilities in Figure 10.19(a) to determine the probability of drawing a black marble first and a red marble second. That is, $P(BR) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$. The branch corresponding to the outcome BR in the simplified tree diagram is thus labeled with a probability of $\frac{2}{9}$ in Figure 10.19(b).

Similarly, we can simplify the rightmost portion of Figure 10.18(a) to obtain the simplified probability tree diagram, including probabilities for the ends of all secondary branches, shown in Figure 10.20(b).



(a) (b)

Figure 10.20

Using either probability tree diagram, we now see that $P(BB) = \frac{4}{9}$. ■

Notice in Figure 10.20(b) that $P(BB) = \frac{4}{9}$ is at the end of connected primary and secondary branches labeled with probabilities $\frac{2}{3}$ and $\frac{2}{3}$, and $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. This multiplicative procedure works for all branches of the probability tree diagram in Figure 10.20(b) (Verify this).

The process of multiplying the probabilities along a series of branches in a probability tree diagram to find the probability at the end of a branch is based on the Fundamental Counting Principle. This technique is summarized next.

MULTIPLICATIVE PROPERTY OF PROBABILITY TREE DIAGRAMS

Suppose an experiment consists of a sequence of simpler experiments that are represented by branches of a probability tree diagram. The probability of the sequence of simpler experiments is the product of all the probabilities on its branches.

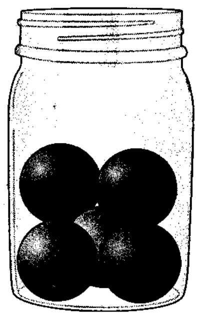


Figure 10.21

EXAMPLE 10.18 A jar contains three red balls and two green balls (Figure 10.21). A two-stage experiment is performed. First, a coin is tossed. If the coin lands heads, a red ball is added to the jar. If the coin lands tails, a green ball is added to the jar. In the second step of the experiment, a ball is chosen from the jar. What is the probability that a red ball is chosen?

SOLUTION The probability tree diagram for the first stage of this experiment, namely, tossing a coin and adding the right color ball to the jar, is shown in Figure 10.22(a). The second stage of the probability tree diagram, namely, choosing a ball from the jar, is shown in Figure 10.22(b).

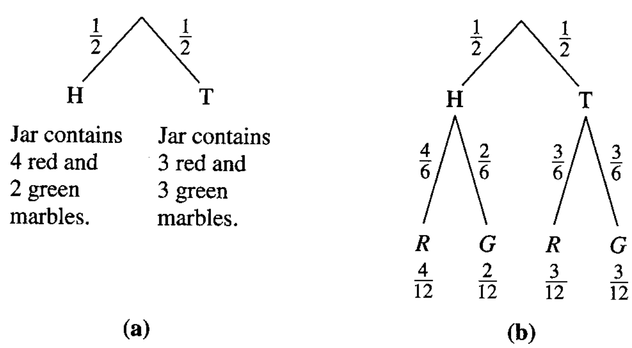


Figure 10.22

The probabilities are different on the left and right pairs of branches for the second stage in Figure 10.22(b) because the contents of the jar are different after the coin is tossed. The probability at the end of each branch is the product of the probabilities along the two branches leading to the end. The probability that the red ball will be chosen is found by adding the probabilities at the end of the branches labeled R, so $P(R) = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$. ■

In Example 10.17, two marbles were drawn *with replacement* from a jar. The next example reconsiders that same experiment, but in this case, the marbles are drawn *without replacement.*

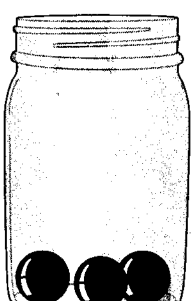


Figure 10.23

EXAMPLE 10.19 A jar contains three marbles, two black and one red (Figure 10.23). A marble is drawn and not replaced. Then a second marble is drawn. What is the probability that both marbles drawn are black?

SOLUTION We will create a two-stage probability tree for the experiment. The first stage of the tree diagram (Figure 10.24) will be identical to the first stage of the tree diagram that was shown in Figure 10.18(b).

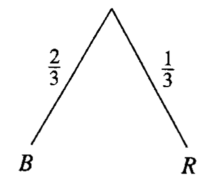


Figure 10.24

The second stage of the tree diagram will look different from Figure 10.18(b) because the selected marble is not put back into the jar. No matter what color marble was drawn first, there are now just two marbles left in the jar. To construct the next stage of the probability tree diagram, we consider the number and colors of the remaining marbles. If a black marble was selected first, for example, that leaves one red and one black marble in the jar, so the probability of selecting a black on the second draw is $\frac{1}{2}$, as is the probability of selecting a red. The completed probability tree diagram is shown in Figure 10.25.

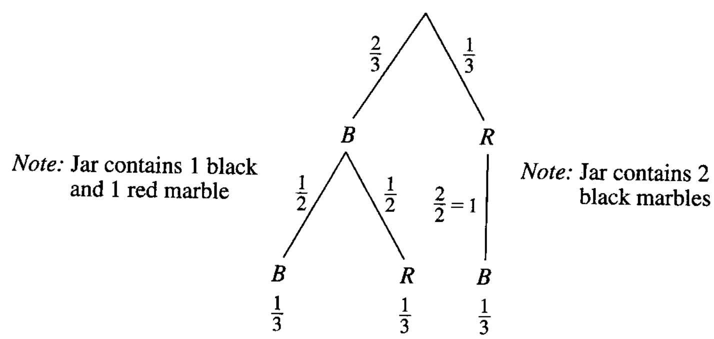


Figure 10.25

The probabilities at the ends of each secondary branch were found by multiplying the probabilities on the branches leading to that end. Notice that the right side of the tree diagram has just one secondary branch leading to an outcome of B because if the red marble is chosen first, the two remaining marbles are black. Thus, a black marble *must* be chosen next and a probability of 1, or 100%, is assigned to this branch. Using the probably tree diagram, we see the probability that both marbles selected are black is $P(BB) = \frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$, which is the probability that appears at the end of that path in the tree diagram. Recall that when the marbles were selected *with replacement*, as in Example 10.17, the probability that both marbles were black was $P(BB) = \frac{4}{9}$. ■

The next example presents another experiment in which selections are made without replacement.

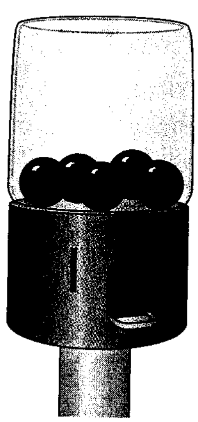


Figure 10.26

EXAMPLE 10.20 A jar contains three red gumballs and two green gumballs (Figure 10.26). An experiment consists of drawing gumballs one at a time from the jar, without replacement, until a red gumball is obtained. Find the probability of each of the following events:

A = the event that only one draw is needed to get a red gumball.

B = the event that exactly two draws are needed to get a red gumball.

C = the event that exactly three draws are needed to get a red gumball.

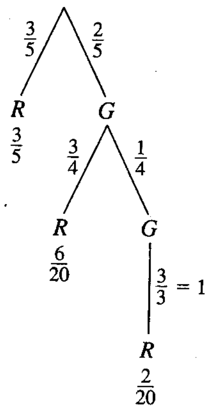


Figure 10.27

SOLUTION We will use a probability tree diagram (Figure 10.27). Notice that this tree diagram has three stages in some places because it might take as many as three draws to get a red gumball. From the probability tree diagram, we see that

$P(A) = \frac{3}{5}$ (a red gumball is drawn on the first draw),
$P(B) = \frac{2}{5} \times \frac{3}{4} = \frac{6}{20} = \frac{3}{10}$ (a green gumball is followed by a red), and
$P(C) = \frac{2}{5} \times \frac{1}{4} \times 1 = \frac{2}{20} = \frac{1}{10}$ (two green gumballs are followed by a red). ■

Recall that in every probability tree diagram, the sum of the probabilities at the ends of the branches is 1. This sum is the probability of the entire sample space. In Figure 10.27, $P(S) = \frac{3}{5} + \frac{6}{20} + \frac{2}{20} = \frac{12}{20} + \frac{6}{20} + \frac{2}{20} = 1$. Summing the probabilities at the ends of the branches serves as one way to a check that the computations along each branch are correct.

APPLYING PROPERTIES OF PROBABILITY TREE DIAGRAMS

In the next example, we will use both the additive and multiplicative properties of probability tree diagrams to calculate probabilities in a multistage experiment.

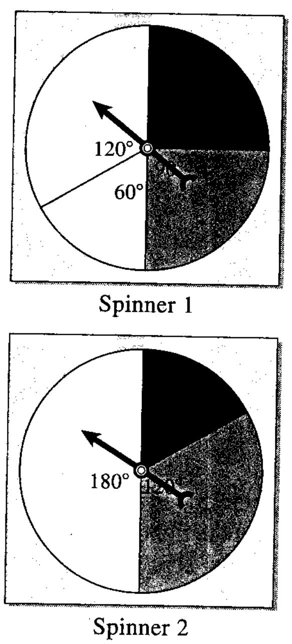


Figure 10.28

EXAMPLE 10.21 Both spinners shown in Figure 10.28 are spun. Find the probability that the two spinners stop on the same color.

SOLUTION Using the central angles on spinner 1 to calculate probabilities, we have $P(W) = \frac{120}{360} = \frac{1}{3}$, $P(R) = P(G) = \frac{90}{360} = \frac{1}{4}$, and $P(Y) = \frac{60}{360} = \frac{1}{6}$. For spinner 2, we have $P(W) = \frac{180}{360} = \frac{1}{2}$, $P(R) = \frac{60}{360} = \frac{1}{6}$, and $P(G) = \frac{120}{360} = \frac{1}{3}$. Next, we draw an appropriate probability tree diagram (Figure 10.29).

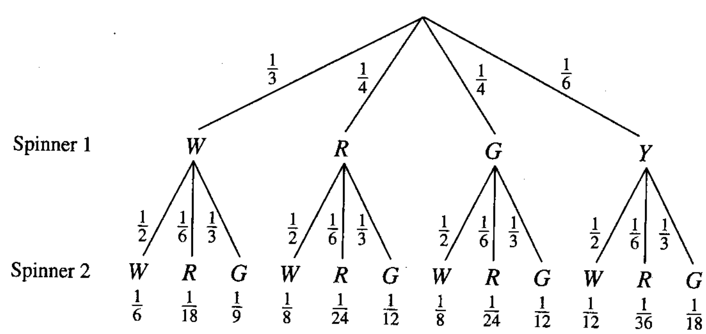


Figure 10.29

The desired event, that the colors on the spinners match, is $\{WW, RR, GG\}$. To determine the probability of this event, we first find the probabilities of each event WW, RR, and GG separately. By the multiplicative property of probability tree diagrams, we multiply probabilities on the branches of the tree diagram to get $P(WW) = \frac{1}{3} \times \frac{1}{2} =$

In summary, the probability of a complex event such as the one described in Example 10.21 can be found as follows.

Algorithm

DETERMINING THE PROBABILITY OF A COMPLEX EVENT

1. Construct an appropriate probability tree diagram.
2. Assign probabilities to each branch in the diagram.
3. Multiply the probabilities along individual branches to find the probability of the outcome at the end of each branch.
4. If necessary, add the probabilities of the relevant outcomes at the ends of each branch.

SOLUTION OF THE INITIAL PROBLEM

A friend who likes to gamble makes you the following wager: He bets you \$2 against \$1 that if you toss a coin repeatedly, you will get a total of two tails before you can get a total of three heads. You reason that two heads are as likely as two tails, and you have a one-out-of-two chance of getting a third head after getting two heads. Should you take the bet?

SOLUTION One way to think about the problem is as an experiment with several stages. The experiment can be represented by a probability tree diagram based on tossing a coin as many times as needed. The branches of the tree we construct will end whenever the particular sequence of outcomes indicates the end of a game (two tails or three heads, whichever comes first). You will win the game if 3 heads appear first and you will lose the game if 2 tails appear first.

Because we assume the coin is a fair coin, the outcomes at each stage will be equally likely, and we can assign $\frac{1}{2}$ as the probability along each branch. Finally, we label the end of each branch to indicate the probability of that particular sequence ending with your winning or losing the bet (Figure 10.30).

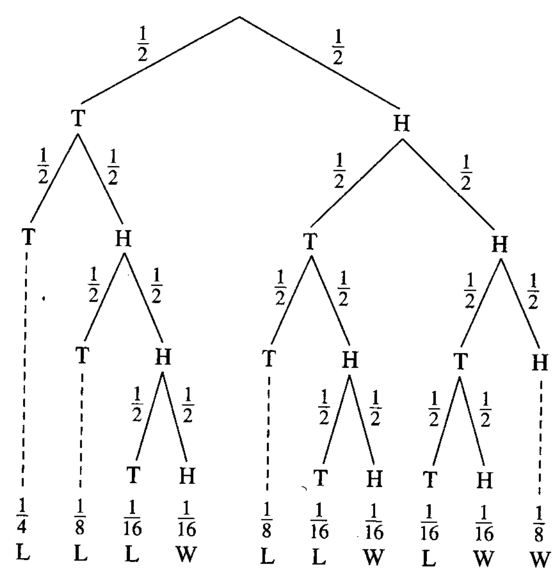


Figure 10.30

An "L" at the end of a branch indicates that outcome results in your losing the bet, and a "W" at the end of a branch indicates that you win with that outcome. When we add the probabilities at the ends of each branch labeled L, we see that the probability the game ends as a loss is $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{11}{16}$, while the probability is only $\frac{5}{16}$ that you will win. Because the chances of losing are more than twice the chance of winning, a payoff of two dollars against our one-dollar bet does not seem like a good deal. If you only want to play when the game is "fair," you should not take the bet.

PROBLEM SET 10.2

Problems 1 and 2

Draw one-stage tree diagrams to represent the possible outcomes of the given experiments.

1. a. Toss one dime and observe whether the coin lands heads or tails.
   b. Pull a dollar bill from your wallet, and note the last digit in the serial number.

2. a. Draw a marble from a bag containing red, green, black, and white marbles, and observe the color.
   b. Open a yearlong wall calendar and note the month.

Problems 3 and 4

Draw two-stage tree diagrams to represent the possible outcomes of the given experiments.

3. a. Toss a coin twice, and observe on each toss whether the coin lands heads or tails.
   b. Select paint colors for an historic home from navy, stone, peach or blue and then select trim colors from light gray, rosedust, or ivory.

4. a. Draw a marble from a box containing yellow and green marbles and observe the color. Then draw a marble from a box containing yellow, red, and blue marbles and observe the color.
   b. Build a computer and printer package, choosing a computer from Dell, Apple, or Hewlett Packard and a printer from Epson, Brother, or Hewlett Packard.

5. Different branches on a tree diagram need not have

6. Tree diagrams need not be finite. For example, consider the experiment of tossing a coin until it lands heads. This usually takes only a few tosses (in fact, two on average), but it could take any number of tosses. Draw at least three stages of the tree diagram representing the possible outcomes of this experiment to show the pattern.

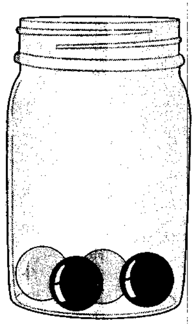
7. For your vacation, you will travel from your home to New York City, then to London. You may travel to New York City by car, train, bus, or plane, and from New York to London by ship or plane.
   a. Draw a tree diagram to represent all possible travel arrangements.
   b. How many different travel arrangements are possible?
   c. Apply the Fundamental Counting Principle to find the number of possible travel arrangements. Does your answer agree with your result in part (b)?

8. Suppose that a frozen yogurt dessert can be ordered in three sizes (small, medium, large), two flavors (vanilla, chocolate), and with any one of four topping options (plain, sprinkles, hot fudge, chocolate chips).
   a. Draw a tree diagram to represent all possible yogurt desserts.
   b. How many different desserts are possible?
   c. Apply the Fundamental Counting Principle to find the number of possible desserts. Does your answer agree with your result in part (b)?

10. An experiment consists of selecting one card from a standard deck, flipping a coin, and spinning a three-color spinner. How many outcomes are possible for the following? Use the Fundamental Counting Principle.
    a. selecting the card
    b. tossing the coin
    c. spinning the spinner
    d. conducting the experiment

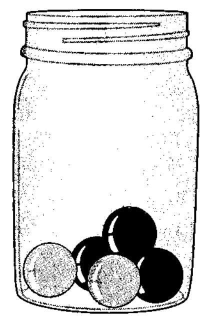
11. One marble is selected from each of the following containers.
    a. Draw a one-stage probability tree diagram and find the probability of drawing the blue marble.

    b. Draw a one-stage probability tree diagram and find the probability of drawing the blue marble. Find the probability of drawing a yellow marble. Find the probability of drawing the blue marble or a yellow marble.



12. One marble is selected from each of the following containers.
    a. Draw a one-stage probability tree diagram and find the probability of drawing the blue marble.

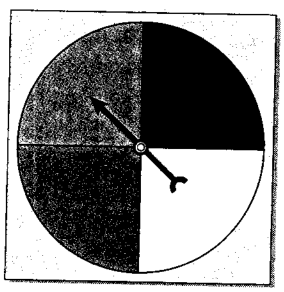
    b. Draw a one-stage probability tree diagram and find the probability of drawing a red marble. Find the probability of drawing a yellow marble. Find the probability of drawing a red marble or a yellow marble.



13. Refer to the container from problem 11(b). A marble is drawn and replaced, and then a second marble is drawn.
    a. Draw a two-stage probability tree diagram to represent this experiment.
    b. What is the probability that both marbles selected are yellow?
    c. What is the probability that the second marble selected is blue?
    d. What is the probability that both selected marbles are the same color?
    e. Repeat the problem if a marble is drawn and *not* replaced, and then a second marble is drawn.

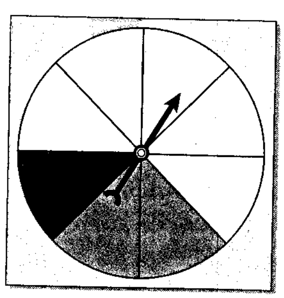
14. Refer to the container from problem 12(b). A marble is drawn and replaced, and then a second marble is drawn.
   a. Draw a two-stage probability tree diagram to represent this experiment.
   b. What is the probability that both marbles selected are red?
   c. What is the probability that the second marble selected is green?
   d. What is the probability that both selected marbles are the same color?
   e. Repeat the problem if a marble is drawn and *not* replaced, and then a second marble is drawn.

15. The following spinner is spun twice.



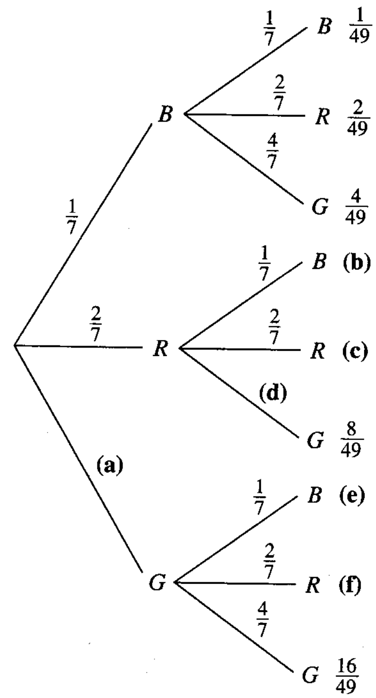
   a. Draw a two-stage probability tree diagram to represent this experiment.
   b. Find the probability the spinner lands on yellow both times.
   c. Find the probability the spinner lands on red on the second spin.
   d. Find the probability the spinner lands on blue and then green or lands on green and then blue.

16. The following spinner is spun twice.



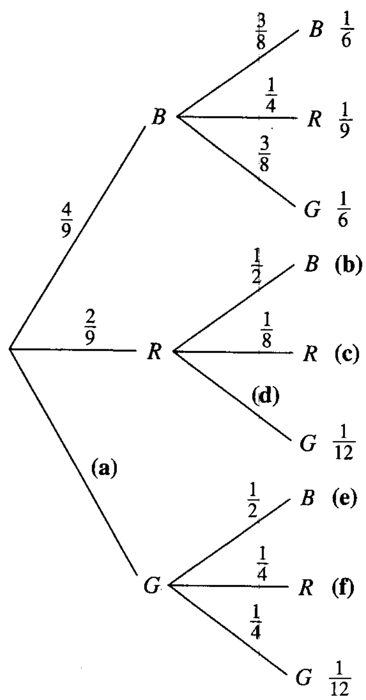
   a. Draw a two-stage probability tree diagram to represent this experiment.
   b. Find the probability the spinner lands on yellow both times.
   c. Find the probability the spinner lands on red on the second spin.
   d. Find the probability the spinner lands on blue and then yellow or lands on yellow and then red.

17. Consider the following two-stage probability tree diagram for the experiment of drawing two marbles from a box. The diagram is unfinished. Fill in the missing probabilities for parts (a) through (f) to complete the diagram.



   g. How many outcomes are in the sample space?
   h. Were the marbles drawn with replacement or without replacement? Explain.
   i. Can you tell how many marbles of each type were in the box? Explain.
   j. Find the probability of getting two marbles that are the same color.

18. Consider the following two-stage probability tree diagram for the experiment of drawing two marbles from a box. The diagram is unfinished. Fill in the missing probabilities for parts (a) through (f) to complete the diagram.



g. How many outcomes are in the sample space?

h. Were the marbles drawn with replacement or without replacement? Explain.

i. Can you tell how many marbles of each type were in the box? Explain.

j. Find the probability of getting two marbles that are the same color.

19. A fair coin is flipped four times.

a. Use the Fundamental Counting Principle to find the number of possible outcomes.

b. Find the probability of getting four heads.

c. Find the probability of getting exactly two heads.

d. Find the probability of getting exactly three tails.

20. A bowl contains three marbles (red, blue, green). A box contains four numbered tickets (1, 2, 3, 4). One marble is selected at random, and then a ticket is selected at random.

a. Use the Fundamental Counting Principle to find the number of possible outcomes.

b. Find the probability that the green marble is selected.

c. Find the probability that the ticket numbered 2 is selected.

d. Find the probability that the red marble and the ticket numbered 3 are selected.

21. A box of 20 chocolates contains three different varieties: nut-filled, nougat, and caramel, but all the chocolates appear identical on the outside. Near the nutrition information, the package reads "This box contains 10% nut-filled, 30% caramel, and 60% nougats." Suppose you select two chocolates.

a. How many stages does this experiment have?

b. How many chocolates of each type are in the box?

c. Create the probability tree diagram for this experiment.

d. Find the probability of selecting two nut-filled chocolates.

e. Find the probability of selecting a caramel and a nut-filled chocolate.

f. Find the probability of selecting a nougat or a caramel.

22. While planning the landscaping for your front yard, you select tulip bulbs at a nursery. Your plan is to plant 10 red and 8 white tulips in one flower bed and to plant 2 purple tulips in a small pot near your door. On the way home from the nursery, the bulbs roll out of their bags in the trunk and are mixed up. You cannot predict the color of the tulips just by looking at the bulbs. Suppose you select 2 bulbs to plant in the pot.

a. How many stages does this experiment have?

b. Create the probability tree diagram for this experiment.

c. How many outcomes are there in the sample space?

d. Find the probability that both bulbs are purple.

e. Find the probability of selecting a red and a white bulb.

f. Find the probability of selecting a purple or a red bulb.

23. A pinochle deck contains 48 cards. The cards are arranged in the usual four suits. Each suit contains two of each of the following cards: 9, 10, jack, queen, king, and ace. Suppose two cards are drawn without replacement from a standard pinochle deck.

a. How many outcomes are possible for this experiment?

b. How many outcomes correspond to the event that both cards are face cards?

c. What is the probability that both cards are face cards?

d. What is the probability of getting a pair?

e. What is the probability of getting an identical pair of cards?

24. Consider the experiment of drawing two cards without replacement from a standard deck of 52 cards.

a. How many outcomes are possible for this experiment?

b. How many outcomes correspond to the event that the cards are both face cards?

c. What is the probability that both cards are face cards?

d. What is the probability of getting a pair?

e. Find the probability of drawing two cards that have the same suit.

25. A light bulb is selected from box 1 and another from box 2. In box 1, 30% of the bulbs are defective. In box 2, 45% of the bulbs are defective. Each of the selected bulbs is recorded as defective or not defective.

a. Draw a probability tree diagram for this experiment.

b. Find the probability that both bulbs are defective.

c. Find the probability that the first bulb is defective and the second is not defective.

26. While still half asleep, you randomly select a black sock from your drawer. After you remove that sock, the drawer contains two white socks and four more black socks. Without replacement you continue to randomly select one sock at a time from your drawer until another black sock is selected. Draw a probability tree diagram to represent this experiment. Find the probability of each of the following events.

a. Exactly one draw is needed to get another black sock.

b. Exactly two draws are needed to get another black sock.

c. Exactly three draws are needed to get another black sock.

27. A game at a carnival consists of throwing darts at balloons. Eight balloons are arranged in such a way that the player will always pop one of them. The popped balloon is replaced after each dart is thrown. Two stars are hidden behind the balloons. If the player pops a balloon that reveals a star, he wins a prize. A player pays 50¢ for three darts. Assuming that skill is not involved, find the probability that the player

a. wins a prize (gets a star) after just one shot.

b. wins in exactly two shots.

c. wins in exactly three shots.

d. does not win.

28. As a Back-to-School promotion, a cereal manufacturer distributes 350,000 boxes of cereal that contain prizes. Fifty boxes contain a certificate for a free desktop computer. Fifty thousand boxes contain a certificate for a dictionary. The rest of the boxes contain a CD spelling program. Suppose you select two boxes.

a. Find the probability that you win two computers.

b. Find the probability that you win a computer and a CD spelling program.

c. Find the probability that you win at least one CD spelling program.

d. Find the probability that you win two dictionaries.

29. Each individual letter of the word MISSISSIPPI is placed on a piece of paper, and all 11 pieces of paper are placed in a bowl. Two letters are selected at random from the bowl without replacement. Find the probability of

a. selecting the two Ps.

b. selecting the same letter in both selections.

c. selecting two consonants.

30. Four families get together for a barbecue. Every member of each family puts his or her name in a hat for a prize drawing. The Martell family has three members, the Werner family four, the Borschowa family four, and the Griffith family six. Two names are drawn from the hat. Find the probability of

a. selecting two members of the Borschowa family.

b. selecting two members from the same family.

c. selecting two members from different families.

31. A pair of dice is constructed so that each die is marked with a 1 on one side, a 2 on two sides, and a 3 on three sides. The dice are rolled. Find the probability that

a. two 3s are rolled.

b. the same number appears on each die.

c. two odd numbers are rolled.

32. A pair of dice is constructed as in problem 31. The two dice are rolled. What is the probability that

a. neither die shows a 2?

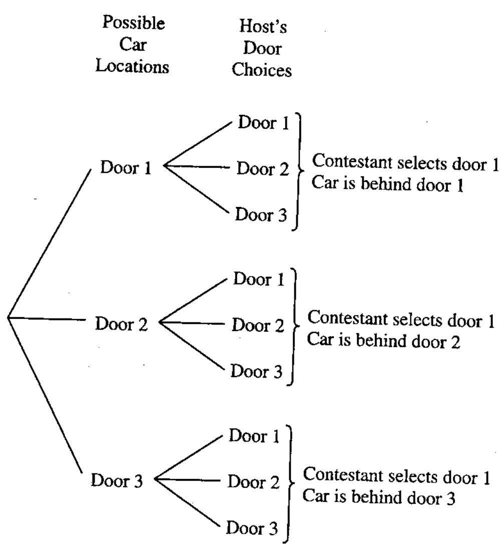
b. the numbers showing on the two dice are different?

c. one die shows an odd number and the other shows an even number?

Extended Problems

33. Consider the three-door problem, which was introduced at the beginning of this chapter. Suppose a contestant on a game show is given the choice of three doors. Behind one door is a car, and behind each of the other two doors are goats. Assume the car and the goats have been randomly placed. The contestant picks door 1, and the host, who knows what is behind the doors, opens another door to reveal a goat. We assume that the host will always reveal a goat, and that when the host has a choice of revealing the goat behind either of the two doors the contestant did not choose, the host will make the choice randomly.

   a. In the following, the first stage represents the possible car locations. The car could be behind door 1, door 2, or door 3. The second stage represents the host's possible selections. Throughout, we assume that the contestant selects door 1. Assign a probability to each branch of the following tree diagram.



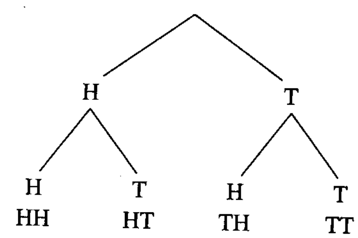
   b. Find the probability that the car is behind door 1.
   c. Find the probability that the car is behind door 2 and the host opens door 3.
   d. Find the probability that the car is behind door 3 and the host opens door 3.
   e. Find the probability that the host opens door 2.
   f. Find the probability that the host opens door 2 or 3.

34. The study of probability theory is generally considered to have begun around 1654 with the correspondence between the mathematicians Blaise Pascal and Pierre de Fermat involving several problems concerning dice games. Write a brief report on the history of probability and the major mathematicians who contributed to its development.

35. One of the most important and interesting counting devices in algebra and probability is known as **Pascal's triangle**. Several rows in the triangle are shown next.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

The top row, containing only the number 1, is generally referred to as the "0" row, so that the entries in the "first" row are 1 and 1. Entries in the "second" row are 1, 2, and 1. One interpretation of Pascal's Triangle relates to the results of **binomial experiments**. A binomial experiment is one that has two possible outcomes, such as tossing a coin and observing either a head or a tail. Now suppose you toss a coin two times and count the number of heads. The tree diagram for this experiment is shown below.

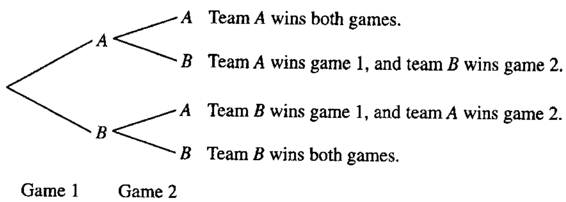


Notice that you can get two heads in one way, exactly one head in two ways, and no heads in one way. The numbers of ways of obtaining two heads, one head, or no heads are given in the "second" row of Pascal's triangle.

1	2	1
two heads	one head	no heads
(one way)	(two ways)	(one way)

   a. Draw the tree diagram for the experiment of tossing three coins. Show that the numbers of ways of obtaining three, two, one, or no heads are given by the entries of the third row of Pascal's triangle.
   b. Draw the tree diagram for the experiment of tossing four coins. Find the numbers of ways of obtaining four, three, two, one, or no heads. In which row of Pascal's triangle are these values found?
   c. Study the entries in Pascal's triangle. There is a way to generate each row of numbers in the triangle by using the entries in the row above it. Determine the relationship and give the entries in the fifth, sixth, and seventh rows of Pascal's Triangle without constructing a tree diagram.

36. Do you have a favorite baseball team? Maybe you enjoy women's or men's college basketball. Who is your favorite team's biggest rival? If your team were to be matched up with a rival team in a series of games, who would most likely win? Consider the following two-stage tree diagram for a two-game match up.



Search the Internet to find the site for your favorite team. All professional teams have websites. Sites for a college team can be found at that school's website. Tally the number of wins your team had when matched with a rival team over the course of one entire season. Use this information to calculate the experimental probability that your team will win when they play the rival team.

a. Suppose these two teams meet in a best-three-out-of-five-game series. Draw the probability tree diagram for this experiment, incorporating the experimental probability that you just calculated. Determine the probability that your team will win in 3 games, in 4 games, or in 5 games.

b. Suppose these teams meet in a best-four-out-of-seven-game series. Draw the probability tree diagram for this experiment and determine the probability your team will win in 4 games, in 5 games, in 6 games, or in 7 games.

Problems 37 through 40: Ordered Samples with Replacement

The idea behind the Fundamental Counting Principal can be used to count outcomes in more complex cases. For example, to find the number of three-letter identification codes that can be formed using the vowels A, E, I, O, and U, we think of the process of forming identification codes as consisting of three stages. A first letter is chosen, then a second letter is chosen, and finally the third letter is chosen. In each case, there are five letters to select from. By the Fundamental Counting Principle, there are $5 \times 5 \times 5 = 5^3 = 125$ identification codes.

An identification code, such as the one just described, is called an **ordered sample**, since IOU is not the same as UOI. Because the same letter may be used over again (for example, EEE is an acceptable code), we are forming **ordered samples with replacement**. The general rule for the number of ordered samples with replacement is given next.

NUMBER OF ORDERED SAMPLES WITH REPLACEMENT OF k OBJECTS FROM AMONG n OBJECTS

The number of ways to arrange k objects chosen from a set of n distinct objects using each object any number of times is

$$\underbrace{n \times n \times \cdots \times n}_{k \text{ factors}} = n^k$$

For example, in creating the three-letter identification code by choosing from the five vowels, we had $k = 3$ letters to choose from $n = 5$ vowels. Thus, there were $n^k = 5^3 = 125$ possible codes.

37. A standard die is tossed nine times and the results are recorded after each toss. In how many different ways could the results be recorded?

38. Eight well-trained athletes are competing in a triathlon (running, swimming, and biking). Awards will be given to athletes with the best times in each of the individual events. In how many ways can the awards be given?

39. A four-letter identification code can be formed from the letters A, B, C, D, E, and F. Assume letters can be repeated.

a. How many codes can be formed?

b. Find the probability that the code begins with the letter A.

c. Find the probability that the code uses the same letter all four times.

40. Suppose a seven-digit telephone number can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, and digits can be repeated.

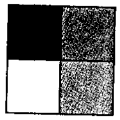
a. How many telephone numbers can be formed?

b. How many telephone numbers can be formed that do not start with the number 0?

c. Find the probability the telephone number begins with the digits 917.

d. Find the probability the telephone number begins and ends with an odd digit.

Problems 41 through 45:
Permutations and Factorial Notation



Suppose you have four small square tiles in the colors red, green, yellow, and blue. One way to group the four small tiles into one large square on the wall is shown above. First, one of the tiles is placed at the upper left. Second, one tile is placed at the upper right. Third, one tile is placed at the lower left. Finally, the last remaining tile is placed at the lower right. By the Fundamental Counting Principal, there are $4 \times 3 \times 2 \times 1 = 24$ ways to group the four colored tiles into one large square. Any 1 of the 24 arrangements of the squares is called a **permutation**. Notice, in the case of the tiles, there is one tile of each color, so repeating a color is not allowed. In general, if there is a collection of k distinct objects and you want to know in how many ways you can arrange them in some order without repetition, the total number of permutations of those objects is computed as follows.

> **NUMBER OF PERMUTATIONS OF *k* OBJECTS**
>
> The number of ways to arrange k distinct objects using each object exactly once is
>
> $$k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1.$$
>
> This number is written $k!$ ($k!$ is read as "**k factorial**").
>
> As a special case 0! is defined to equal 1. The factorial operation cannot be applied to negative integers.

Therefore, in the example of the large square made up of four tiles, we had $k = 4$ different tiles to be arranged. The total number of possible arrangements (permutations), as we saw already, was $k! = 4 \times 3 \times 2 \times 1 = 24$.

41. Evaluate each of the following.

a. $\frac{12!}{9!}$ **b.** $\frac{11!}{7!}$ **c.** $\frac{12!}{8!4!}$ **d.** $\frac{14!}{3!11!}$

Hint: Write out what the numerators and denominators represent and you may be able to simplify your calculation.

42. The birthdays for all 20 students in a math class are written down next to their names on the class roster. In how many different orders can the birthdays be recorded?

43. A student has five errands to complete. In how many different orders can all five errands be done one at a time?

44. Six students will make their final reports on the last day of class.

a. In how many orders can the reports be scheduled for presentation?

b. If Lorna and Eli are two of the students, find the probability Lorna gives her report first and Eli gives his last.

c. Find the probability Lorna gives her report first or second and Eli gives his report third.

45. Five friends attend a movie together and sit in the same row.

a. In how many ways can the five friends sit in a row?

b. If Sasha and Lamar are two of the friends, and they sit down in a random order, find the probability Sasha and Lamar sit next to each other.

Problems 46 through 51:
Ordered Samples without Replacement

Factorial notation is also helpful in counting the number of arrangements of objects, even when not all of the objects are used. The number of ordered arrangements that can be formed using three of the seven letters A, B, C, D, E, F, and G is $7 \times 6 \times 5 = 210$, by the Fundamental Counting Principle. However, notice that this result can be written in factorial notation as: $7 \times 6 \times 5 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{7!}{4!}$. Each arrangement is called an **ordered sample without replacement**. In general, we have the following rule.

> **NUMBER OF ORDERED SAMPLES WITHOUT REPLACEMENT OF *k* OBJECTS FROM AMONG *n* OBJECTS**
>
> The number of ways to arrange k objects chosen from a set of n distinct objects using each object at most once is
>
> $$\frac{n!}{(n-k)!}.$$
>
> This number is sometimes written ${}_nP_k$ and read "the number of permutations of n things taken k at a time."

Therefore, for example, we just calculated the number of permutations of 7 letters taken 3 at a time, and we found that ${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$.

46. Find each of the following and explain what you have found in each case.

a. ${}_4P_2$ **b.** ${}_9P_5$ **c.** ${}_{15}P_{11}$ **d.** ${}_8P_3$

47. How many ways can the chairperson, vice-chairperson, and secretary of a committee be selected from a committee of 10 people?

48. Four academic departments have 8, 12, 15, and 10 members. Each department will select a delegate and an alternate for a conference on teaching. In how many ways can the group of delegates and alternates be selected?

49. A state senate has 18 Republicans and 12 Democrats. The senate leadership consists of a leader and a "whip" from each of the parties. In how many ways can the senate leadership be selected?

50. Twelve horses are entered in a race.

a. How many different finishes are possible among the first three finishers?

b. If Harlo's Pride and Jabba Boy are two of the horses, find the probability that they finish in first and second place.

51. A class of 15 students is selecting members of the class to be responsible for three separate tasks, and no student will be responsible for more than one task. There are 8 girls and 7 boys in the class.

a. In how many ways can the selections be made?

b. Find the probability all the tasks will be completed by girls.

Problems 52 through 59: Combinations: Choosing k Objects from among n Objects

The order in which objects are chosen may not matter. For example, a poker hand is a collection of five cards chosen from the standard deck of 52 cards. Once the cards are in your hand, the order does not matter. If you rearrange the cards, you have the same hand you started with. Unordered samples without replacement are called **combinations**. When selecting 2 letters from the vowels A, E, I, O, and U, there are 5×4 or $\frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$ ordered samples. Since each pair, say AE and EA, appears twice as an ordered pair of vowels, we would need to divide by $2 = 2!$ to find the number of distinct unordered pairs. Thus, there are 10 unordered pairs, or combinations, of the vowels: AE, AI, AO, AU, EI, EO, EU, IO, IU, and OU. We generalize this idea next.

NUMBER OF WAYS TO CHOOSE k OBJECTS FROM AMONG n OBJECTS

The number of ways to choose k objects from a set of n distinct objects is

$$\frac{n!}{k!(n-k)!}.$$

This number is written ${}_nC_k$ or $\binom{n}{k}$ and read "the number of combinations of n things taken k at a time."

52. Find each of the following and explain what you have found in each case.

a. ${}_4C_2$ **b.** ${}_9C_5$ **c.** ${}_{15}C_{11}$ **d.** ${}_8C_3$

53. Twenty computers are produced each day by a manufacturer, and several are selected for testing. In how many ways can four of the computers be selected for testing?

54. Five students will be selected from a class of 18 to work on a special class project. In how many ways can the five students be chosen?

55. A catering service offers six appetizers, eight main courses, and five desserts. In how many ways can a banquet committee select two appetizers, four main courses, and two desserts?

56. The game of bridge is one of the most popular in the world. A bridge hand consists of 13 cards from a standard 52-card deck.

a. How many different bridge hands are possible?

b. What is the probability a bridge hand contains all 13 diamonds?

c. What is the probability a bridge hand contains 4 hearts, 2 diamonds, 3 spades, and 4 clubs?

57. A poker hand is formed by choosing 5 cards from among the 52 cards in the standard deck.

a. How many different poker hands are possible?

b. How many different poker hands consist of all hearts?

c. What is the probability of being dealt a flush (a hand in which all cards have the same suit)?

d. What is the probability of being dealt a hand consisting of three kings and two aces?

58. For a five-card poker hand, find the probability of being dealt

a. four aces.

b. no aces.

c. a pair of aces and no other cards that match.

d. an ace, king, queen, jack, and ten.

59. Lotteries have become commonplace in the United States. Millions of people play lotteries every week, and many states count on money from the lotteries to fund education and other public services. To play the Delaware LOTTO, a player selects six numbers from the numbers 1 to 38. All six numbers must match to win the jackpot. However, the order in which the numbers are chosen does not matter.

a. What is the probability of matching all six numbers?

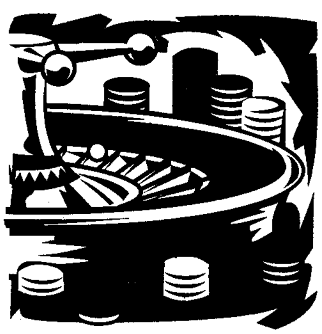
b. A \$500 prize is awarded in the Delaware LOTTO if any five numbers match. What is the probability of matching exactly five of the numbers?

c. A \$25 prize is awarded in the Delaware LOTTO if any four numbers match. What is the probability of matching exactly four of the numbers?

d. Research lottery games from three other states. Calculate the probability of winning the jackpot. In which state would you have the best chance of winning the jackpot if you purchased one ticket? On the Internet, search keywords "state lotteries." Summarize your results in a table.

10.3 Conditional Probability, Expected Value, and Odds

INITIAL PROBLEM



American roulette wheels have 38 slots numbered 00, 0, and 1 through 36 (Figure 10.31).

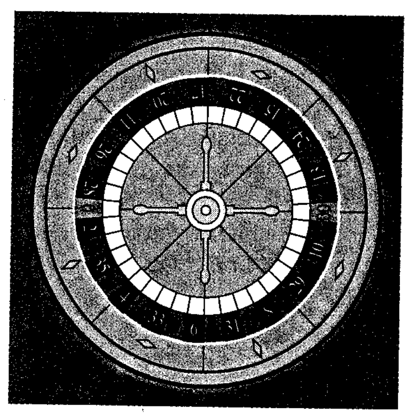


Figure 10.31

You place a bet on a specific number by putting your wager on the numbered square on the roulette cloth or layout. Bets may also be placed on more than one number or on combinations of numbers. The wheel is spun in one direction and the ball is rolled in the opposite direction in a surrounding sloped bowl. When the ball slows sufficiently, it drops down into the numbered slots and bounces along until coming to rest on the winning number. If you bet on the winning number, the croupier (the manager of the table) leaves your bet on the layout and adds to it 35 times as much as you bet. If you chose a number other than the winning number, your wager and the other losing bets are gathered in with a rake. If you bet \$100 on one number, what is your expected gain or loss?

A solution of this Initial Problem is on page 685.

In the solution to the Initial Problem of the previous section, we concluded that a certain game wasn't "fair," although we didn't define what we meant by a fair game in mathematical terms. In this section, we will discuss several additional properties of probability that will be used to analyze complex events and that will allow us to determine when a game is fair.

CONDITIONAL PROBABILITY

Sometimes a condition is imposed that forces us to focus on a portion of the sample space, called the **conditional sample space**; that is, certain information is known about the experiment that affects the possible outcomes. Such "given" information is illustrated in the next example.

EXAMPLE 10.22 An experiment consists of tossing three fair coins. Suppose that you know the first coin came up heads. Describe the sample space.

SOLUTION When tossing three coins, eight outcomes are possible: $S = \{$HHH, HHT, HTH, THH, HTT, THT, TTH, TTT$\}$. The condition "the first coin is heads" produces the following smaller sample space: $\{$HHH, HHT, HTH, HTT$\}$. We call this smaller sample space the conditional sample space for the condition "the first coin is heads." ■

In Example 10.22, the original sample is reduced to those outcomes that satisfy the given condition, namely H for the first coin. Consider the following example using this conditional sample space.

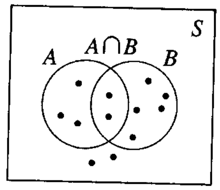
Let A be the event that exactly two tails appear among the three coins, and B be the event that the first coin tossed is heads. Therefore, $A = \{$HTT, THT, TTH$\}$ and $B = \{$HHH, HHT, HTH, HTT$\}$. Event B is the same as the conditional sample space in Example 10.22. Suppose the first coin landed heads up and we wish to find the probability that, in addition, two tails appear. We say that we want to find the probability of two tails appearing *given* that the first coin was a head. The probability of A *given* B, represented by the notation $P(A \mid B)$ is called a conditional probability, and it means the probability of event A occurring within the conditional sample space B.

In this example, A can occur in only one way, namely HTT, within the set B, which contains four elements. Thus, the probability of A given B is $P(A \mid B) = \frac{1}{4}$. Notice that $A \cap B = \{\text{HTT}\}$, which means that we can express $P(A \mid B)$ as follows.

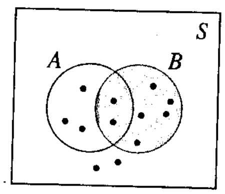
$$P(A \mid B) = \frac{1}{4} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{P(A \cap B)}{P(B)}.$$

That is, $P(A \mid B)$ is the relative frequency of the event A within the conditional sample space B. This suggests the following definition, which provides us with a technique for calculating conditional probabilities.

> ***Definition***
>
> **CONDITIONAL PROBABILITY**
>
> Suppose A and B are events in a sample space S and that $P(B) \neq 0$. The **conditional probability that the event *A* occurs, given that the event *B* occurs**, or briefly the **probability of *A* given *B***, denoted **$P(A \mid B)$**, is
>
> $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$



(a)



(b)

Figure 10.32

A diagram can be used to illustrate the definition of conditional probability. A sample space S of 12 equally likely outcomes is shown in Figure 10.32(a). The conditional sample space consisting of just the outcomes for which B occurs is shaded in Figure 10.32(b). Using Figure 10.32(a), we see that the probability of A and B occurring is $P(A \cap B) = \frac{2}{12}$ and $P(B) = \frac{7}{12}$. Thus, $\frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{12}}{\frac{7}{12}} = \frac{2}{7}$. From Figure 10.32(b), we also see that $P(A|B) = \frac{2}{7}$, since two of the seven outcomes in B are in A. Thus, $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

The next example illustrates conditional probability in the case of outcomes that are not equally likely.

EXAMPLE 10.23 Suppose we have two jars of marbles. The first jar contains two white marbles and one black marble. The second jar contains one white marble and two black marbles (Figure 10.33). A fair coin is tossed. If the coin lands heads up, then a marble is drawn from the first jar. If the coin lands tails up, then a marble is drawn from the second jar. Find the probability that the coin landed heads up, given that a black marble was drawn.

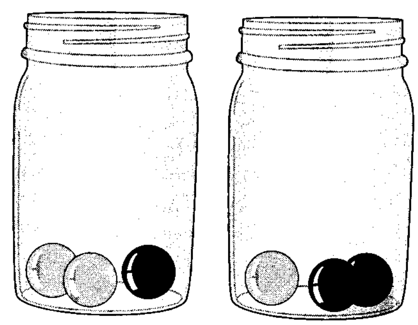


Figure 10.33

SOLUTION First, we construct a probability tree diagram to describe the experiment (Figure 10.34).

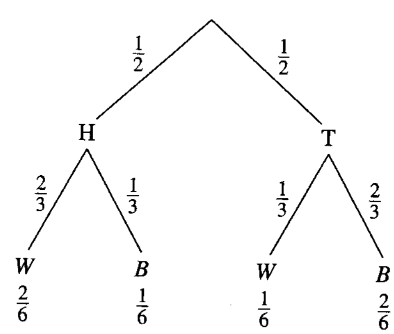


Figure 10.34

From the tree diagram, we can read the sample space and probabilities. For example, the leftmost path down the tree gives us the outcome HW (that is, the coin is heads, then a white marble is drawn) in the sample space, with probability $P(\text{H}W) = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$. But we do not want to consider the entire sample space. Instead, we want to consider only those paths that end with a black marble being drawn.

Figure 10.35 is a copy of Figure 10.34, but we emphasize the paths that end in a black marble by using thicker line segments. We also circled the path showing a black marble following a heads-up coin from the first jar.

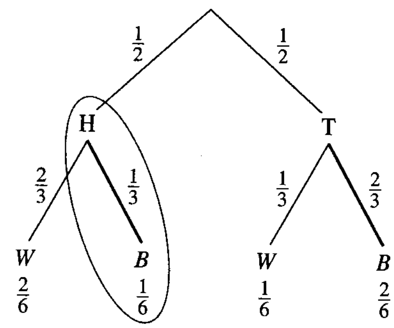


Figure 10.35

The sum of the probabilities at the ends of the two emphasized paths leading to a black marble is $\frac{1}{6} + \frac{2}{6} = \frac{3}{6}$. Thus, the probability of a black marble in this experiment is $P(B) = \frac{3}{6} = \frac{1}{2}$. The probability of a head and a black marble, which is circled, is $P(\mathrm{H} \cap B) = \frac{1}{6}$. In this example, $P(\mathrm{H}|B)$ can be interpreted verbally as "the probability that the coin lands heads up, given that a black marble is selected" and is calculated as follows:

$$P(\mathrm{H}|B) = \frac{P(\mathrm{H} \cap B)}{P(B)} = \frac{P(\text{heads and a black marble})}{P(\text{a black marble})} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}. \qquad \blacksquare$$

The conditional probability we obtained in Example 10.23 may seem unrelated to everyday life. However, many common life and work experiences involve conditional probabilities. The probability of severe weather is dependent on related factors such as air pressure, winds, time of year, and other conditions. Decisions based on such related information require more subtle analysis than most people realize. For example, the significance of medical tests is interpreted differently depending on whether a positive or negative result is obtained. The next example reinforces this point.

EXAMPLE 10.24 Suppose a test can detect the presence of a viral infection, but the test is not 100% accurate. Assume $\frac{1}{4}$ of the population is infected, and the other $\frac{3}{4}$ is not. Further assume that 90% of those infected test positive and 80% of uninfected persons test negative. Testing positive means the test indicates the presence of the viral infection. Testing negative means that the test indicates that there is no infection.

a. What is the probability that the test gives correct results?

b. Given that a person's test is positive, what is the probability that the person is actually infected?

SOLUTION This is a two-stage experiment similar to the one presented in Example 10.23. In the first stage of this experiment, a person is chosen at random, and that person may be either infected or uninfected. (The testing takes place during the second stage of the experiment, and the result is either positive or negative.) We create the two-stage probability tree diagram shown in Figure 10.36. We have indicated two paths that correspond to incorrect test results, that is, to false positives and false negatives.

Tidbit

The now-common faith in medical tests dates back to only the early 1900s, when Wasserman, Neisser, and Bruck developed a test to locate the causative agent of syphilis in the blood.

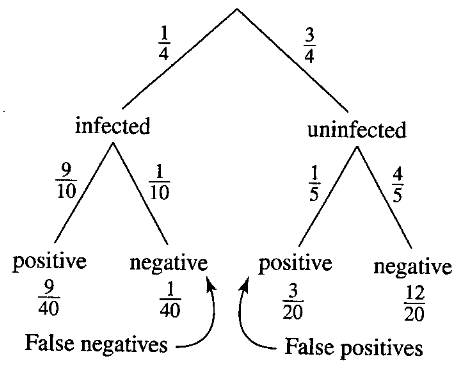


Figure 10.36

a. The test yields a correct result whenever an infected person has a positive test result or an uninfected person has a negative test result. Thus, we are interested in finding *P*(infected and positive result ***or*** uninfected and negative result). This probability can be determined by multiplying down the two paths in the tree and adding the results as follows.

$$P[(\text{infected and positive result}) \text{ or } (\text{uninfected and negative result})] =$$
$$P(\text{infected and positive result}) + P(\text{uninfected and negative result})$$
$$= \frac{1}{4} \times \frac{9}{10} + \frac{3}{4} \times \frac{4}{5}$$
$$= \frac{33}{40} = 82.5\%.$$

In other words, 82.5% of the time, the test results correctly indicate whether a person does or does not have the infection.

b. Now we want to determine the probability that the person is infected *given* that his or her test is positive. Thus, we are seeking the conditional probability $P(\text{infected} \mid \text{test positive})$. To find this conditional probability, we focus on the branches of the tree diagram corresponding to the event that "the test is positive". Those branches are thicker in Figure 10.37. The part of the tree that corresponds to "the person is infected and the test is positive," that is, the path through "infected" *and* "positive," is circled in that figure.

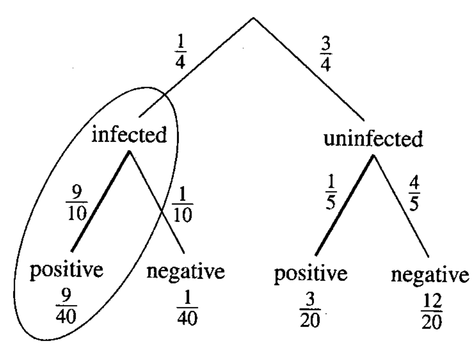


Figure 10.37

Two of these branches represent a positive test result, so $P(\text{test positive}) = \frac{9}{40} + \frac{3}{20} = \frac{15}{40}$. The circled path shows us that $P(\text{infected and test positive}) = \frac{9}{40}$. Therefore, we can use the formula for conditional probability, and we find that

$$P(\text{infected} \mid \text{test positive}) = \frac{P(\text{infected and test positive})}{P(\text{test positive})} = \frac{\frac{9}{40}}{\frac{15}{40}} = \frac{9}{15} = 0.60.$$

Thus, there is a 60% chance that the person is infected, given that person tested positive. ■

Notice that even though the test produces correct results more than 80% of the time, the probability of a person being infected given that he or she tested positive is only 60%. This result is typical. In people with no symptoms or reason to think they are at risk, a positive test result for a rare disease is often a false positive, causing a lot of needless worry.

The next example is another illustration of conditional probability.

EXAMPLE 10.25 Smiley's Candy Company has production lines in two different locations, one in Bay City and the other in nearby Springfield. A recent quality-control check at the main distribution center found that a small number of chocolates from each facility failed to meet standards. Results of that inspection appear in Table 10.6. Assume the results in the table are representative of the entire candy production.

Table 10.6

SMILEY'S CHOCOLATE COMPANY—QUALITY CONTROL CHECK

	Chocolates from Bay City Facility	Chocolates from Springfield Facility
Chocolates that meet standards	212	137
Chocolates that fail to meet standards	4	7

Suppose that a customer purchased a box of Smiley's chocolates and found a piece of candy that failed to meet the company's usual high standards. Given that the piece of candy is substandard, what is the probability that it came from Smiley's Bay City factory?

SOLUTION This is another two-stage experiment. The first stage corresponds to a piece of candy coming from the Bay City Facility or the Springfield Facility, and the second stage corresponds to the candy meeting or not meeting standards.

We will solve this problem in two ways: by using the information in the table and by constructing a tree diagram.

Method 1: Solution by Table

We must find the conditional probability that a piece of candy came from Bay City *given* that it is substandard, that is, we want to find $P(\text{Bay City} \mid \text{substandard})$. To calculate this probability, we will use the definition of conditional probability.

$$P(\text{Bay City} \mid \text{substandard}) = \frac{P(\text{Bay City } \textit{and} \text{ substandard})}{P(\text{substandard})}$$

We will use the data in the table to calculate this probability in three steps.

STEP 1: Calculate P(Bay City *and* substandard).

This is the probability that a single piece of candy came from Bay City and fails to meet standards. Using the results in the table, we see that 4 substandard chocolates from Bay

City were discovered during the quality-control check. The total number of chocolates inspected was $212 + 4 + 137 + 7 = 360$, so P(Bay City *and* sub-standard) $= \frac{4}{360}$.

STEP 2: Calculate P(substandard).

This is the probability that a single piece of candy fails to meet standards. We see in the table that a total of $4 + 7 = 11$ pieces of chocolate failed to meet standards—that is, 11 chocolates out of a total of 360 inspected chocolates. Thus, $P(\text{substandard}) = \frac{11}{360}$.

STEP 3: Divide to find the conditional probability.

Now that we have the two needed probabilities, we use the definition of conditional probability to find

$$P(\text{Bay City} \mid \text{substandard}) = \frac{P(\text{Bay City } \textit{and} \text{ substandard})}{P(\text{substandard})} = \frac{\frac{4}{360}}{\frac{11}{360}} = \frac{4}{360} \times \frac{360}{11} = \frac{4}{11}.$$

Thus, the probability that the substandard piece of chocolate found by the customer came from Bay City is $\frac{4}{11} = 0.363636\ldots$, or about 36%.

Notice that this solution can be found by simply comparing the numbers in the bottom row of Table 10.6. A total of 11 substandard candies were found and 4 of the substandard candies came from the Bay City Facility, so $P(\text{Bay City} \mid \text{sub-standard}) = \frac{4}{11}$. This method works only when the table is representative of the entire candy production. In particular, since 216 (that is, $212 + 4$) candies from Bay City were inspected and a total of 360 candies were inspected, then 60% (that is, $\frac{216}{360} = 0.60$) of the inspected candies came from Bay City. It must also be true that 60% of *all* the company's candies come from Bay City.

Method 2: Solution by Tree Diagram

Taking another approach, we will solve the problem using a tree diagram in two steps.

STEP 1: Construct a probability tree diagram.

The primary branches in the tree will represent a single piece of candy coming from Bay City or from Springfield. The secondary branches will correspond to a piece of candy meeting or not meeting standards.

Because the results in the table are representative of the entire candy production, probabilities for the tree diagram may be calculated from the information given in the table. For example, the probability that a piece of candy came from Bay City is given by $P(\text{Bay City}) = \frac{212 + 4}{360} = \frac{216}{360}$ because 216 of the 360 chocolates inspected came from Bay City. Similarly, the probability that a piece of candy fails to meet standards given that it came from the Bay City facility is $P(\text{substandard} \mid \text{Bay City}) = \frac{4}{216}$ because 4 of the 216 chocolates from Bay City failed to meet standards. The completed tree diagram is shown next in Figure 10.38.

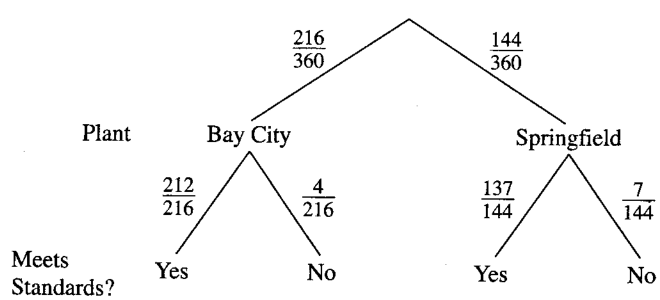


Figure 10.38

STEP 2: Find the conditional probability.
Now we can use the probability tree diagram and the definition of conditional probability to recalculate the desired probability.

$$\begin{aligned} P(\text{Bay City} \mid \text{substandard}) &= \frac{P(\text{Bay City and substandard})}{P(\text{substandard})} \\ &= \frac{\frac{216}{360} \cdot \frac{4}{216}}{\frac{216}{360} \cdot \frac{4}{216} + \frac{144}{360} \cdot \frac{7}{144}} \\ &= \frac{\frac{4}{360}}{\frac{4}{360} + \frac{7}{360}} \\ &= \frac{\frac{4}{360}}{\frac{11}{360}} \\ &= \frac{4}{11} \end{aligned}$$

Thus, the probability that the piece of candy came from Bay City given that it fails to meet standards is $\frac{4}{11}$, which agrees with our earlier calculation. ■

INDEPENDENT EVENTS

If a pair of dice is rolled, the probability of getting a "1" on the first die is $\frac{1}{6}$, and the probability of getting a "1" on the second die is also $\frac{1}{6}$. The probability of getting a "1" on the first die *and* a "1" on the second die is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. To convince yourself that this result is correct, recall from Section 10.1 that there are 36 possible outcomes when rolling two dice, one of which is (1, 1). These two events are called **independent events,** in the sense that one event does not influence the other. In other words, getting a "1" on the first die does not make it any more or less likely that you will get a "1" on the second die.

Now suppose that a jar contains 8 white marbles and 2 red marbles. If we mix the marbles and choose one, the probability of getting a red marble is $\frac{2}{10} = \frac{1}{5}$. If we replace the marble and remix, then we have the same situation as when we started. Obtaining a red marble on the first draw does not affect our chance of getting a red marble on the second draw. These events are said to be independent. The probability of getting red marbles on both the first *and* second draws will be $\frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$. The following statement summarizes this special property of independent events.

PROBABILITY OF INDEPENDENT EVENTS

When two events are independent, the probability of both occurring equals the product of their probabilities. For independent events A and B,

$$P(A \cap B) = P(A) \times P(B).$$

EXAMPLE 10.26 Consider drawing 2 marbles from the jar containing 8 white marbles and 2 red marbles, but *not replacing* the first marble before drawing the second. Find the following probabilities:

a. The probability that the first marble is red

b. The probability that the second marble is red

c. The probability that both marbles are red.

SOLUTION We construct a probability tree diagram for the experiment (Figure 10.39).

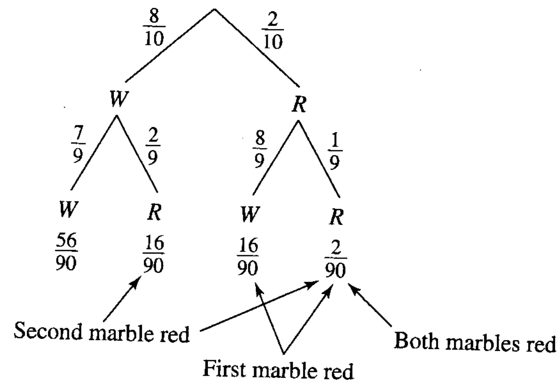


Figure 10.39

From the tree diagram, we see that

a. $P(\text{first marble is red}) = P(RW) + P(RR) = \frac{16}{90} + \frac{2}{90} = \frac{18}{90} = \frac{1}{5}$.

b. $P(\text{second marble is red}) = P(RR) + P(WR) = \frac{2}{90} + \frac{16}{90} = \frac{18}{90} = \frac{1}{5}$.

c. $P(\text{both marbles are red}) = P(RR) = \frac{2}{90} = \frac{1}{45}$. ■

Notice that in Example 10.26 the probability of both events happening is not the product of the probabilities of the individual events, which shows that these two events are not independent. That is, $P(\text{first marble is red}) \times P(\text{second marble is red}) = \frac{1}{5} \times \frac{1}{5} \neq \frac{1}{45} = P(\text{both marbles are red})$.

Recognizing that two events are independent makes it much easier to compute the probability that both happen. It also makes the computation of conditional probabilities easier, as we illustrate next. If A and B are independent and if $P(B) > 0$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B)}{P(B)} = P(A).$$

In other words, $P(A|B) = P(A)$ if A and B are independent. This result is consistent with our intuitive understanding. If two events are independent, then the occurrence of one of these events does not affect the probability that the other will occur.

EXAMPLE 10.27 A student name is chosen at random from the college enrollment list and the student is interviewed. Let A be the event the student regularly eats breakfast and B the event the student has a 10:00 A.M. class.

a. Explain *in words* what is meant by each of the following probabilities: $P(A \cap B)$, $P(A|B)$, and $P(\overline{A})$.

b. If $P(A) = \frac{1}{4}$, $P(B) = \frac{3}{7}$, and $P(A \cap B) = \frac{1}{5}$, are events A and B independent?

SOLUTION

a. $P(A \cap B)$ means the probability that the student both regularly eats breakfast *and* has a 10:00 A.M. class.

$P(A|B)$ means the probability that the student regularly eats breakfast *given* that that student has a 10:00 A.M. class.

$P(\overline{A})$ means the probability that the student does not regularly eat breakfast.

b. As stated earlier, if events A and B are independent, then $P(A \cap B) = P(A) \times P(B)$. In this example, $P(A) \times P(B) = \frac{1}{4} \times \frac{3}{7} = \frac{3}{28}$, which is not the same as $P(A \cap B) = \frac{1}{5}$. Thus, these events are not independent. We might also look at $P(A|B)$. Recall that $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{5}}{\frac{3}{7}} = \frac{1}{5} \times \frac{7}{3} = \frac{7}{15}$. If events A and B are independent, then $P(A|B) = P(A)$. But in this case, $P(A|B) = \frac{7}{15}$ and $P(A) = \frac{1}{4}$. This result again shows that the events are not independent. ■

EXPECTED VALUE

Sometimes the possible outcomes of a probability experiment are numbers, such as the number of dots showing on a die. In other cases, the possible outcomes of an experiment may not actually be numbers, but numbers can be assigned to the outcomes. For example, suppose Bob wins \$1 from Jennifer if he chooses a higher card from the deck than Jennifer does; otherwise he loses \$1 to Jennifer. We can assign $+1$ to the event that Bob chooses the higher card (and Bob wins) and assign -1 to the event that Jennifer chooses the higher card (and Bob loses). For experiments with numerical outcomes, it is useful to know what the average numerical outcome should be for many repetitions of the experiment. This number is called the expected value.

Suppose you are asked to play a game in which you win \$3 if you toss a three or greater on a standard die and lose \$5 if you toss a one or a two. Let's see if you should play this game. You win \$3 on $\frac{4}{6} = \frac{2}{3}$ of the tosses and lose \$5 on $\frac{2}{6} = \frac{1}{3}$ of the tosses. You should expect to win \$3 two times out of three (on average) and lose \$5 the other one-third of the time. Thus, on average over the long run, in three tosses you should expect to win \$3 + \$3 = \$6 and lose \$5, yielding a net profit of \$1 in three plays. On a per-game basis, the analysis of your expected return looks like this:

$$\$3 \times \frac{2}{3} + (-\$5) \times \frac{1}{3} = \$2 + \left(-\$1\frac{2}{3}\right) = \$\frac{1}{3}, \text{ or about } 33¢.$$

On average, you would expect to win about \$0.33 per game.

In general, to find the expected value of an experiment with a numerical outcome (or with an associated numerical outcome), multiply each possible numerical outcome by its probability, and add all of the products. More formally, we have the following definition.

> ***Definition***
>
> **EXPECTED VALUE**
>
> Suppose that the outcomes of an experiment are numbers (values) $v_1, v_2, \ldots v_n$, and the outcomes have probabilities $p_1, p_2, \ldots, p_n$, respectively. The **expected value**, E, of the experiment is the sum of the products
>
> $$E = (v_1 \times p_1) + (v_2 \times p_2) + \cdots + (v_n \times p_n).$$

The expected value of an experiment can be a positive number, a negative number, or 0. In the dice game described earlier, your expected winnings were \$0.33 per game. In this case, the game is slightly biased in your favor. Often games of chance played in casinos are biased in favor of the casino (called "the house"), meaning a player's expected winnings per game will be negative. If the expected value of a game of chance is exactly 0, we say that the game is **fair**. We use the definition of expected value in the next example.

EXAMPLE 10.28 An experiment consists of rolling a fair die and noting the number on top of the die. Compute the expected value of one roll of the die.

SOLUTION The possible outcomes are the whole numbers 1 through 6, and each of these outcomes has the probability $\frac{1}{6}$. The computation of the expected value can be organized by putting the necessary information into a table, as shown in Table 10.7.

Tidbit

The first maritime insurance companies were established in Italy and Holland in the 14th century. These companies calculated the risks, since larger risks made for larger insurance premiums. For shipping by sea, premiums amounted to about 12% to 15% of the cost of the goods.

Table 10.7

Value	1		2		3		4		5		6		
Probability	$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		
Product	$\frac{1}{6}$	+	$\frac{2}{6}$	+	$\frac{3}{6}$	+	$\frac{4}{6}$	+	$\frac{5}{6}$	+	$\frac{6}{6}$	=	$\frac{21}{6}$
													Expected value

Therefore, we see that the expected value is $E = \frac{21}{6} = \frac{7}{2} = 3.5$. In other words, you should expect to average 3.5 dots per toss. ■

Expected values are used in determining premiums for insurance and in determining the price to play games of chance. The next example shows a simplified version of how insurance companies use expected values.

EXAMPLE 10.29 Suppose an insurance company has compiled yearly automobile claims for drivers age 16 through 21, as shown in Table 10.8. How much should the company charge as its average premium in order to break even on its costs for claims?

Table 10.8

Amount of Claim (nearest $2000)	Probability
0	0.80
$2000	0.10
4000	0.05
6000	0.03
8000	0.01
10,000	0.01

SOLUTION We can think of Table 10.8 as giving us the probabilities for various numerical outcomes of an experiment in which the outcomes are the dollar amounts of the claims. Then we compute the expected value of that experiment. The information is again organized in a table (Table 10.9).

Table 10.9

Value	0		2000		4000		6000		8000		10000		
Probability	0.80		0.10		0.05		0.03		0.01		0.01		
Product	0	+	200	+	200	+	180	+	80	+	100	=	760
													Expected value

The expected value is $760, which is the average value of a claim. Thus, the average automobile premium should be set at $760 per year for the insurance company to break even on its claims costs. ■

Tidbit

The Multi-State Lottery Association (MUSL) is a nonprofit, government-benefit association operated by 26 members, including 24 states, the District of Columbia, and the U.S. Virgin Islands. The largest MUSL lottery is Powerball. A jackpot winner in Powerball must match 5 out of 53 numbers and also match the powerball, which is 1 out of 42 numbers. The odds of winning are 1 to 120,526,770. The largest jackpot of 314.9 million was won by Andrew "Jack" Wittaker on Christmas Day in 2002 when he correctly picked the numbers 5-14-16-29-53 and the powerball, which was 7.

Tidbit

"Long" odds do not mean the probability of winning is zero. According to Pennsylvania lottery officials, the odds of winning the $1 million jackpot were 1 to 1.44 million. However, Donna Goeppert of Bethlehem, PA, beat those odds. In 2005, she won $1 million with a $20 lottery scratch-off ticket, not once but twice in the same year!

ODDS

The chances of an event occurring are often expressed in terms of odds rather than as a probability. For example, state lotteries usually describe the odds of a player's winning. In the case of equally likely outcomes, the **odds in favor** of an event compare the total number of outcomes favorable to the event to the total number of outcomes unfavorable to the event. The odds in favor of getting a 4 when tossing a six-sided die are 1:5 (read "1 to 5"), because one side has a 4 and the other five sides have different numbers.

Although odds are based on probability, we represent them differently. The odds of an event are customarily written as a ratio of whole numbers, with a colon between the two numbers. If the odds in favor of an event E are $a{:}b$, then the **odds against** E are $b{:}a$. Thus, the odds *against* getting a 4 when tossing a die are 5:1. Often the words "in favor" are not explicitly included, but are understood, so usually "the odds of the event" means "the odds in favor of the event."

EXAMPLE 10.30 Suppose a card is randomly drawn from a well-shuffled standard deck. What are the odds in favor of drawing a face card?

SOLUTION There are 12 face cards in the deck and there are 40 other cards. Thus, the odds in favor of drawing a face card are 12:40, which can be simplified to 3:10. ■

We can compute the probability of an event given the odds in favor of the event. If the odds in favor of an event E are $a{:}b$, then for each a outcomes in E, there are b outcomes in the complement of E and, hence, for each a outcomes in E, there are $a + b$ outcomes in the sample space. Thus, $P(E) = \frac{a}{a+b}$. For example, in Example 10.30 we found that the odds in favor of drawing a face card are 3:10, so the probability of drawing a face card is $\frac{3}{3+10} = \frac{3}{13}$. Because the odds ratio 3:10 was simplified from 12:40, the sum $3 + 10$ does not give us the total number of outcomes in the experiment. This technique for determining probability is summarized next.

COMPUTING PROBABILITY FROM THE ODDS

If the odds in favor of an event E are $a{:}b$, then the probability of E is given by

$$P(E) = \frac{a}{a+b}.$$

EXAMPLE 10.31 Find $P(E)$ given the following odds.

a. The odds in favor of E are 3:7.

b. The odds against E are 5:13.

SOLUTION

a. $P(E) = \frac{3}{3+7} = \frac{3}{10}$.

b. If the odds against E are 5:13, then the odds in favor of E are 13:5 Therefore,

$$P(E) = \frac{13}{5+13} = \frac{13}{18}.$$ ■

It is also possible to determine the odds in favor of an event directly from the probability. If there are $a + b$ equally likely outcomes with a of the outcomes favorable to E and b of the outcomes unfavorable to E, then $P(E) = \frac{a}{a+b}$ and $P(\overline{E}) = \frac{b}{a+b}$. Notice that the odds in favor of E are $a{:}b = \frac{\frac{a}{a+b}}{\frac{b}{a+b}} = \frac{P(E)}{P(\overline{E})}$. For instance, the odds in favor of

Tidbit

Powerball officials were baffled and suspicious when 110 people won \$100,000–\$500,000 in the March 30, 2005, lottery. All winners had guessed the same 5 out of 6 numbers correctly. At first, officials doubted the legitimacy of the wins and believed a scam had been committed because the probability of so many winners was quite small. However, the mystery was solved when some winners reported they had chosen "lucky numbers" printed on a fortune enclosed in fortune cookies distributed to Chinese restaurants throughout the U.S. by Won Ton Foods. Five of those 6 lucky numbers matched the 6 winning Powerball numbers.

rolling a 4 on a die are 1:5. We know the probability of rolling a 4 is $\frac{1}{6}$, and the probability of *not* rolling a 4 (the complement) is $\frac{5}{6}$. Thus, the odds in favor of rolling a 4 are $\frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{6} \times \frac{6}{5} = \frac{1}{5}$, or 1:5 as we found earlier.

This result, which is summarized next, can be used to define odds using probabilities both for outcomes that are not equally likely as well as equally likely outcomes.

THE ODDS IN FAVOR OF AN EVENT AND THE ODDS AGAINST AN EVENT

Odds in favor of an event: $E = P(E):P(\overline{E})$

Odds against an event: $E = P(\overline{E}):P(E)$

If the probability of an event occurring is $\frac{1}{3}$, then the probability the event doesn't occur (the complement) is $\frac{2}{3}$. The odds in favor of the event are $\frac{1}{3}:\frac{2}{3} = \frac{1}{3} \div \frac{2}{3} = \frac{1}{2} = 1:2$. In a similar manner, the odds against the event are 2:1. If the probability of an event is $\frac{1}{2}$, then the probability of the complement of the event is also $\frac{1}{2}$. In this case, the odds in favor of the event are 1:1, meaning that the event is just as likely to occur as not.

EXAMPLE 10.32 Find the odds in favor of the event E, where E has the following probabilities.

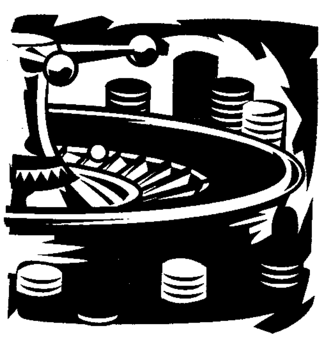
a. $\frac{1}{4}$ **b.** $\frac{3}{5}$

SOLUTION

a. Odds in favor of $E = \dfrac{\frac{1}{4}}{1 - \frac{1}{4}} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3} = 1:3$

b. Odds in favor of $E = \dfrac{\frac{3}{5}}{1 - \frac{3}{5}} = \dfrac{\frac{3}{5}}{\frac{2}{5}} = \frac{3}{5} \times \frac{5}{2} = \frac{3}{2} = 3:2$ ■

SOLUTION OF THE INITIAL PROBLEM



American roulette wheels have 38 slots numbered 00, 0, and 1 through 36 (Figure 10.40). You place a bet on a specific number by putting your wager on the numbered square on the roulette cloth or layout. Bets may also be placed on more than one number or on combinations of numbers. The wheel is spun in one direction and the ball is rolled in the opposite direction in a surrounding sloped bowl. When the ball slows sufficiently, it drops down into the numbered slots and bounces along until coming to rest on the winning number. If you bet on the winning number, the croupier (the manager of the table) leaves your

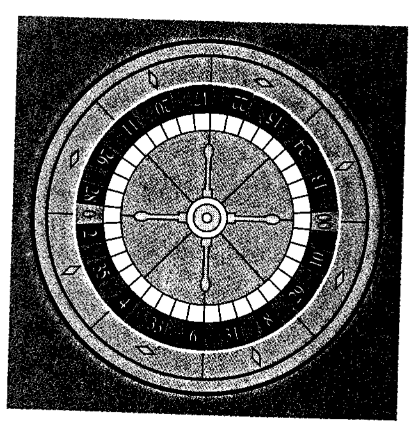


Figure 10.40

bet on the layout and adds to it 35 times as much as you bet. If you chose a number other than the winning number, your wager and the other losing bets are gathered in with a rake. If you bet \$100 on one number, what is your expected gain or loss?

SOLUTION This problem describes a probability experiment with numerical outcomes. The probability of winning is $\frac{1}{38}$, since there are 38 equally likely outcomes, namely, 00, 0, 1, 2, 3, . . ., 36. If you win, you win \$3500. The probability of losing is $\frac{37}{38}$. If you lose, you lose \$100 so we will assign this outcome a value of $-\$100$. We put this information into a table as shown in Table 10.10 and compute the expected value by multiplying the assigned values and their probabilities.

Table 10.10

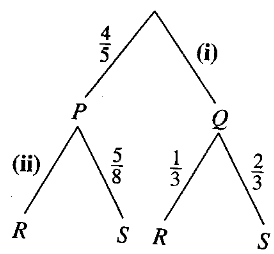
Value	−100		3500		
Probability	$\frac{37}{38}$		$\frac{1}{38}$		
Products	$\frac{-3700}{38}$	+	$\frac{3500}{38}$	=	$\frac{-200}{38}$
					Expected value

Since the expected value is $\frac{-200}{38} \approx -\$5.26$, for every \$100 bet, you should expect to lose \$5.26 even though you lose \$100 on some bets and win \$3500 on others.

PROBLEM SET 10.3

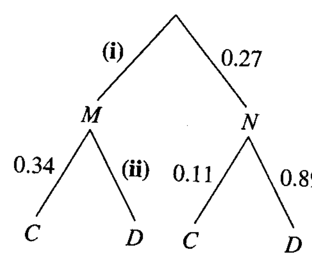
1. An experiment consists of rolling two standard dice and noting the numbers showing on the two dice.
   - **a.** How many outcomes are in the sample space?
   - **b.** Let M be the event that the first die shows a multiple of 3. List the outcomes in M.
   - **c.** Let O be the event that the second die shows an odd number. List the outcomes in O.
   - **d.** List the outcomes in $M \cap O$.
   - **e.** Find $P(M)$, $P(O)$, and $P(M \cap O)$.
   - **f.** Find and interpret $P(M|O)$ and $P(O|M)$.
2. An experiment consists of drawing two cards, with replacement, from a pile of cards containing only the five cards 9, 10, J, Q, and K of diamonds.
   - **a.** How many outcomes are in the sample space?
   - **b.** Let F be the event that the first card is a face card. List the outcomes in F.
   - **c.** Let S be the event that the second card is a 10 or J. List the outcomes in S.
   - **d.** List the outcomes in $F \cap S$.
   - **e.** Find $P(F)$, $P(S)$, and $P(F \cap S)$.
   - **f.** Find and interpret $P(F|S)$ and $P(S|F)$.
3. A jar contains five white balls and three green balls. Two balls are drawn, in order, without replacement. Find the following:
   - **a.** the probability the second ball is green.
   - **b.** the probability the first ball is white.
   - **c.** the probability the first ball is white and the second ball is green.
   - **d.** the probability the second ball is green given the first ball is white.
   - **e.** the probability the first ball is white given the second ball is green.
   - **f.** the probability the first ball is green given the second ball is green.
4. Two standard dice are rolled. Find the following:
   - **a.** the probability that at least one of the dice is a five.
   - **b.** the probability the sum is eight.
   - **c.** the probability that at least one of the dice is a five and the sum is eight.
   - **d.** the probability that one of the dice is a five given the sum is eight.
   - **e.** the probability that the sum is eight given that at least one of the dice is a five.

5. Consider the following incomplete probability tree diagram.



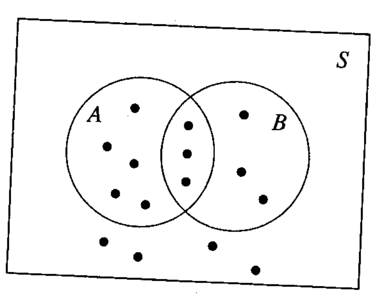
a. Find (i) and (ii), the missing probabilities, in the tree diagram.
b. Find $P(R \cap P)$, $P(S \cap P)$, and $P(P)$.
c. Find $P(R \cap Q)$, $P(S \cap Q)$, and $P(R)$.
d. Find and interpret $P(S|P)$.
e. Find and interpret $P(Q|R)$.

6. Consider the following incomplete probability tree diagram.



a. Find (i) and (ii), the missing probabilities, in the tree diagram.
b. Find $P(C \cap M)$, $P(D \cap M)$, and $P(M)$.
c. Find $P(C \cap N)$, $P(D \cap N)$, and $P(D)$.
d. Find and interpret $P(C|M)$.
e. Find and interpret $P(N|D)$.

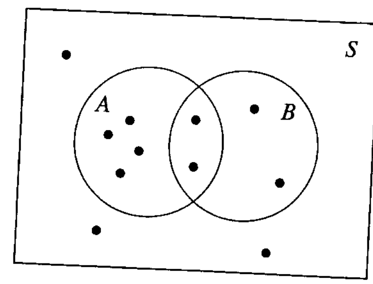
7. The diagram shows a sample space S of equally-likely outcomes and events A and B. Each outcome is represented by a dot in the figure.



Find the following probabilities.

a. $P(A)$ b. $P(B)$ c. $P(A \cap B)$
d. $P(A|B)$ e. $P(B|A)$

8. The diagram shows a sample space S of equally likely outcomes and events A and B. Each outcome is represented by a dot in the figure.



Find the following probabilities.

a. $P(\overline{A})$
b. $P(B)$
c. $P(\overline{A} \cap B)$
d. $P(\overline{A}|B)$
e. $P(B|\overline{A})$

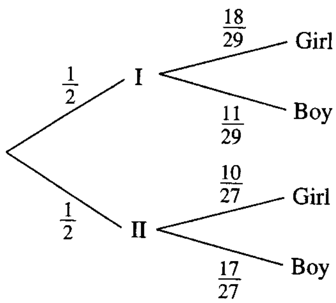
9. If L is the event that a person is left-handed, and M is the event that a person is male, then state *in words* what probabilities are expressed by each of the following:

a. $P(L|M)$
b. $P(M|L)$
c. $P(L \cap M)$
d. $P(\overline{L} \cap \overline{M})$
e. $P(\overline{L}|M)$
f. $P(\overline{L}|\overline{M})$

10. A student's name is chosen at random from a college registration list and the student is interviewed. If H is the event that the student completes his or her homework each night, and G is the event that the student gets good grades, then state *in words* what probabilities are expressed by each of the following:

a. $P(H \cap G)$
b. $P(H \cap \overline{G})$
c. $P(G|H)$
d. $P(\overline{G}|H)$
e. $P(G|\overline{H})$
f. $P(\overline{G}|\overline{H})$

11. There are two fifth-grade classes in a school. Class I has 18 girls and 11 boys while class II has 10 girls and 17 boys. The principal will randomly select one fifth-grade class and one student from that class to represent the fifth grade on the student council. A probability tree diagram for this experiment is given next.



Find each of the following:

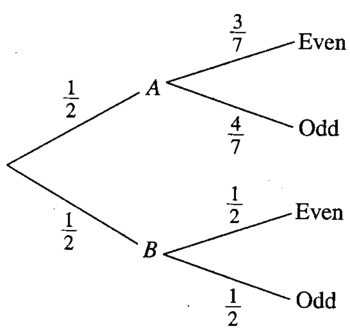
a. P(a girl is selected)

b. P(a boy is selected)

c. P(a girl is selected given that the student came from class II)

d. P(a boy is selected given that the student came from class I)

12. Box A contains 7 cards numbered 1 through 7, and box B contains 4 cards numbered 1 through 4. A box is chosen at random and a card is drawn. It is then noted whether the number on the card is even or odd. A probability tree diagram for this experiment is given next.



Find the following probabilities:

a. P(the number is even)

b. P(the number is odd)

c. P(the number is even given that it came from

13. A triple test is a blood test, which can be offered to a woman who is in her 15th to 22nd week of pregnancy, to screen for such fetal abnormalities as Down syndrome. A positive triple test would indicate that the fetus might have a birth defect. The table below contains triple test results from a large sample of women. Assume that the results in the table are representative of the population as a whole.

Triple Test Results	Fetus with Down Syndrome	Fetus without Down Syndrome
Positive	5	208
Negative	1	2386

a. Given a positive test result, what is the probability that the fetus has Down syndrome?

b. Given a positive test result, what is the probability that the fetus does not have Down Syndrome?

c. Given that a fetus has Down syndrome, what is the probability that the test is negative?

14. In a fuse factory, machines A, B, and C manufacture 20%, 45%, and 35%, respectively, of the total fuses. Of the outputs for each machine, A produces 5% defectives, B produces 2% defectives, and C produces 3% defectives. A fuse is drawn at random.

a. Given that a fuse is defective, what is the probability it came from machine A?

b. Given that a fuse is defective, what is the probability it came from machine B?

c. Given that a fuse is defective, what is the probability it came from machine C?

15. A random sample of 400 adults are classified according to sex and highest education level completed as shown next.

Education	Female	Male
Elementary school	64	53
High school	99	87
College	28	39
Graduate school	14	16

If a person is picked at random from this group, find the probability that

a. the person is female given that the person has a graduate degree.

b. the person is male given that the person has a high school diploma as his or her highest level of education completed.

c. the person does not have a college degree given that the person is female.

16. A company would like to find out how the number of defective items produced varies between the day, evening, and night shifts. The following table shows the results of a sample of items taken from each shift.

	Day	Evening	Night
Defective	24	28	47
Nondefective	279	224	165

If an item is picked at random, find the probability that

a. the item is defective, given that it came from the night shift.

b. the item is not defective, given that it came from the day shift.

c. the item was produced by the night shift, given that it was not defective.

17. Suppose a screening test for a certain virus is 95% accurate for both infected and uninfected persons. If 10% of the population is infected and one person is selected at random, find the following:

a. the probability that the test result is positive.

b. the probability of a false positive test result.

c. the probability that a person is infected, given that the test result is positive.

d. the probability that the result is positive, given that the person is infected.

18. Suppose a screening test for a certain virus is 90% accurate for both infected and uninfected persons. If 2% of the population is infected, find the following:

a. the probability that the test result is negative.

b. the probability of a false negative test result.

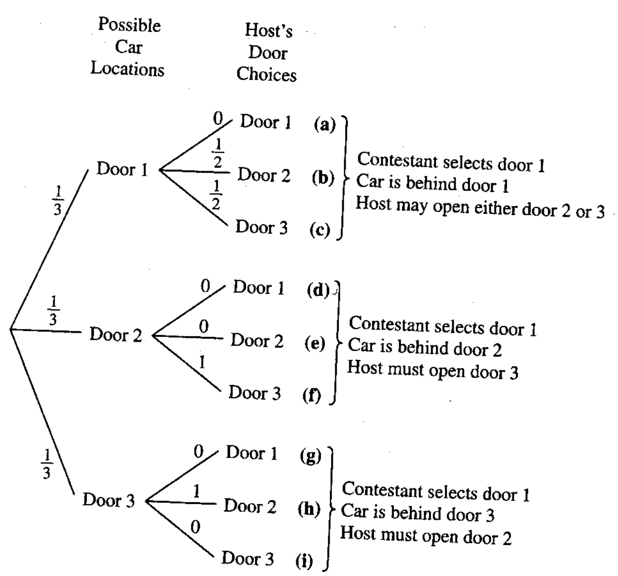
c. the probability that the person is uninfected, given that the test result is negative.

d. the probability that the test result is negative, given that the person is uninfected.

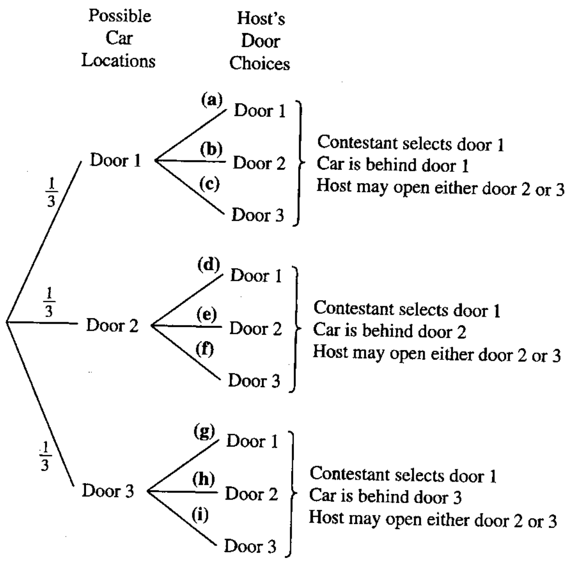
Problems 19 through 24

These problems relate to the three-door problem introduced at the start of this chapter.

19. Suppose a contestant on a game show is given the choice of three doors. Behind one door is a car, and behind each of the other two doors are goats. Assume the car and the goats have been randomly placed. The contestant picks door 1, and the host, who knows what is behind the doors, opens another door to reveal a goat. We assume that the host will always reveal a goat, and that when the host has a choice of revealing the goat behind either of the two doors the contestant did not choose, the host will make the choice randomly. The problem can be represented by the probability tree diagram below. Fill in the missing probabilities.



20. In problem 19, the assumption has been made that the host will always open a door to reveal a goat. Suppose we allow the case in which, after the contestant selects a door, the host randomly selects a different door. In this case, the host might open a door to reveal the car. Fill in the missing probabilities in the following tree diagram.



21. The contestant selects door 1 in problem 19. Suppose the host opens door 2.

a. Find P(Car is behind door 1 | Host opens door 2).

b. Find P(Car is behind door 3 | Host opens door 2).

c. Considering the probabilities found in parts (a) and (b), is it to the contestant's advantage to switch doors?

22. Consider the tree diagram from problem 20. The contestant selects door 1. Suppose the host opens door 2.

a. Find P(Car is behind door 1 | Host opens door 2).

b. Find P(Car is behind door 3 | Host opens door 2).

c. Considering the probabilities found in parts (a) and (b), is it to the contestant's advantage to switch doors?

23. The contestant selects door 1 in problem 19. Suppose the host opens door 3.

24. Consider the tree diagram from problem 20. The contestant selects door 1. Suppose the host opens door 3.

a. Find P(Car is behind door 1 | Host opens door 3).

b. Find P(Car is behind door 2 | Host opens door 3).

c. Considering the probabilities found in parts (a) and (b), is it to the contestant's advantage to switch doors?

25. Two classes at a university are studying modern Latin-American fiction. Twenty of the 25 students in the first class speak Spanish, and 12 of the 18 students in the second class speak Spanish. If a student is selected at random from each of the 2 classes, what is the probability that both students speak Spanish? Show how you calculate your answer. What probability property did you use? Explain.

26. Two assembly lines are producing ink cartridges for a desktop printer. Five percent of the cartridges produced by the first assembly line are defective, while 10% of those produced from the second assembly line are defective. If a cartridge is selected randomly from each line, what is the probability that neither cartridge will be defective? Show how you calculate your answer. What probability property did you use? Explain.

27. Suppose you set your compact disc player to randomly play the 11 tracks on a CD. Tracks 1, 4, and 5 are your favorites. You listen to two songs.

a. Find the probability that the second song is one of your favorites.

b. Find the probability that the second song is one of your favorites, given that the first song was one of your favorites.

c. Are these events independent? Explain your reasoning.

28. Suppose the random-track-selection feature on your CD player is malfunctioning so that once a track is played, the same track is twice as likely to be selected next. After setting your compact disc player to randomly play the five tracks on a CD, you listen to two songs.

a. Find the probability track 3 plays first

29. Suppose a standard die is rolled twice. Let A be the event a 3 occurs on the first roll, let B be the event that the sum of the two rolls is 7, and let C be the event that the same number is rolled both times.

a. Find $P(A)$, $P(B)$, $P(A \cap B)$, and determine if events A and B are independent.

b. Find $P(A)$, $P(C)$, $P(A \cap C)$, and determine if events A and C are independent.

c. Find $P(B)$, $P(C)$, $P(B \cap C)$, and determine if events B and C are independent.

30. In a box of computer chips, two are defective and five are not defective. Two chips are selected, without replacement. Let A be the event that a defective chip is chosen first, let B be the event that a nondefective chip is chosen second, and let C be the event that both chips are defective. Use the definition of independent events to

a. Find $P(A)$, $P(B)$, $P(A \cap B)$, and determine if events A and B are independent.

b. Find $P(A)$, $P(C)$, $P(A \cap C)$, and determine if events A and C are independent.

c. Find $P(B)$, $P(C)$, $P(B \cap C)$, and determine if events B and C are independent.

31. Complete the table below, in which the values in the first column shows the number of girls possible in a family with four children, and the second shows the probabilities for those outcomes. Assume that boys and girls are equally likely and the births are independent. Then find the expected number of girls in a family of four.

Number of Girls	Probability
0	
1	
2	
3	
4	

32. Suppose you play a game in which two fair standard dice are rolled. If the numbers showing on the dice are different, you lose \$2. If the numbers showing are the same, you win \$2 plus the dollar value of the sum of the dice. Complete the next table, in which the values in the first column are the outcomes for the rolls of the two dice. The second column has the probabilities for those outcomes, and the third column has the payoff values for each outcome. What is the expected value of the game?

Numbers Showing	Probability	Payoff
Different		
1, 1		
2, 2		
3, 3		
4, 4		
5, 5		
6, 6		

33. In a lottery, there are 50 prizes of \$10, 10 prizes of \$15, 5 prizes of \$30, and 1 prize of \$50. Suppose that 1000 tickets are sold.

a. What is a fair price to pay for one ticket?

b. What should the price of one ticket be if, on the average, people lose \$0.50?

c. If all 1000 tickets are sold, what can the lottery expect to gain if a ticket costs \$2?

34. A church conducts a drawing to raise money for the building fund. One thousand tickets are placed in a box. On each ticket is placed one of the following dollar amounts: \$0, \$5, \$10, \$50, or \$200. The table below shows the numbers of each type of ticket that will be placed in the box. Each participant will pay \$3 to draw one ticket from the box.

Number of Tickets	Dollar Amount
1	200
4	50
10	10
20	5
965	0

a. Find the expected value of one ticket.

b. If all the tickets are sold, how much money will the church make for the building fund?

c. If you buy a single ticket, what is the probability that you will not win any money?

35. For visiting a resort (and listening to a sales presentation), you will receive one gift. The probability of receiving each gift and the manufacturer's suggested retail value are as follows:

gift A, 1 in 52,000 ($9272.00)
gift B, 25,736 in 52,000 ($44.95)
gift C, 1 in 52,000 ($2500.00)
gift D, 3 in 52,000 ($729.95)
gift E, 25,736 in 52,000 ($26.99)
gift F, 3 in 52,000 ($1000.00)
gift G, 180 in 52,000 ($44.99)
gift H, 180 in 52,000 ($63.98)
gift I, 160 in 52,000 ($25.00)

Find the expected value of your gift. Round to the nearest cent.

36. According to a publisher's records, 20% of the children's books published break even, 30% lose $1000, 25% lose $10,000, and 25% earn $20,000. When a book is published, what is the expected income for the book?

37. Suppose you and a friend play a game. Two standard dice are rolled and the numbers showing on each die are multiplied. If the product is even, your friend gives you a quarter, but if the product is odd, you must give your friend one dollar.
   a. What is the expected value of the game for you? Round to the nearest cent.
   b. What is the expected value of the game for your friend? Round to the nearest cent.
   c. How could you change the amount you pay your friend so that the expected value of the game for you is $0.05?

38. At a carnival, you play a dice game in which you roll two standard dice. If you roll a total of 7, then you win $1. If you roll double 6s, you lose $5. If you roll any other combination, you win $0.25.
   a. What is the expected value of the game?
   b. If the carnival wants to make sure that the player loses $0.10 on average, how should the payoff for rolling a total of 7 be adjusted?

39. Suppose that the probability of an event is $\frac{1}{5}$.
   a. What are the odds in favor of the event?
   b. What are the odds against the event?

40. Suppose the probability of an event is $\frac{7}{19}$.
   a. What are the odds in favor of the event?
   b. What are the odds against the event?

41. If the odds against an event are 2 to 1, what is the probability of the event?

42. If the odds in favor of an event are 13:11, what is the probability of the complement of the event?

43. Suppose three coins are tossed.
   a. What are the odds in favor of getting all heads?
   b. What are the odds against getting only one head?
   c. If event T is defined as getting exactly two tails, then what are the odds for $\overline{T}$?

44. Suppose two standard dice are rolled.
   a. What are the odds in favor of getting a sum of 6?
   b. What are the odds against getting a 3 on the second die?
   c. If event L is defined as getting a total of at least 9, what are the odds in favor of the complement of L?

Extended Problems

45. In an effort to fight an apparent growth in the use of illegal drugs, many companies, professional sports teams, and schools have established drug-testing programs. Research the most common forms of

46. Two major trials in the latter half of the 1990s focused the attention of the nation on DNA testing: the 1995 murder trial of O. J. Simpson and the 1998–1999 impeachment trial of President Bill

47. Lottery games can be attractive, especially when there is a chance, no matter how small, of winning millions of dollars. The Multi-State Lottery Association sponsors many lottery games in which the top prize, which can be millions of dollars, grows until there is a winner. Powerball is one such game. Research the Powerball game. How is the game played? What are the odds in favor of winning the top prize? Are other prizes awarded in this game? What are the odds of winning each of the other prizes? Calculate the expected value of the game. Summarize your results in a short report and be sure to include a table of prizes, odds, and probabilities. Would you conclude that this game is fair? Use search keyword "Powerball" on the Internet or go to www.powerball.com/pb_odds.asp for more information.

48. Many states run lottery games and sell "scratch-off" tickets. The payoff for most of these types of games is very small in comparison to the millions that could be won in other lottery games. For example, the Delaware Lottery sponsors the "Silver 6's" game in which there are "six chances to win a smooth top prize of $600!"

Scientific Games, Alpharetta, GA

a. Search the Internet for lottery information in several states that offer scratch-off lottery tickets. Use search keywords "state lottery." Compare the odds of winning a $10 prize for several different scratch-off games, and determine in which state a player would have the highest probability of spending $1 and winning $10.

b. Suppose your friend buys a scratch-off ticket and wins $15. She exclaims, "I won $15 this week and $10 two weeks ago! I must be pretty lucky!" Out of curiosity, you ask her how many tickets she buys each week. She says she buys 20 tickets every week. When you flip the ticket over, you notice that the odds of winning $10 are 1 to 25, and the odds of winning $15 are 1 to 75. Write a short essay about how you would explain your friend's "luck" to her.

Problems 49 through 52

Blaise Pascal was a mathematician, scientist, writer, and philosopher in the 17th century. In 1654, prolific gambler Antoine Gombauld wrote a letter to Pascal asking for help on two problems. Gombauld had two favorite bets, which he made at even odds. One was that he could roll at least one 6 in four rolls of a single die. The second was that he could roll at least one pair of 6s in 24 rolls of a pair of dice. He was successful with the first bet, but not with the second, and he asked Pascal to explain it to him.

49. **a.** Find the probability of rolling a 6 in a single roll of a die.

b. Find the probability of *not* rolling a 6 in a single roll of a die.

c. Find the probability of *not* rolling a 6 in four rolls of a die.

d. Find the probability of rolling at least one 6 in four rolls of a die.

50. **a.** Find the probability of rolling a pair of 6s in a single roll of a pair of dice.

b. Find the probability of *not* rolling a pair of 6s in a single roll of a pair of dice.

c. Find the probability of *not* rolling a pair of 6s in 24 rolls of a pair of dice.

d. Find the probability of rolling at least one pair of 6s in 24 rolls of a pair of dice.

51. **a.** What are the odds in favor of rolling at least one 6 in four rolls of a single die?

b. What are the odds in favor of rolling at least one pair of 6s in 24 rolls of a pair of dice?

52. Write a letter to Gombauld and explain why betting that he could roll at least one pair of 6s in 24 rolls of a pair of dice is a bad bet.

CHAPTER 10 REVIEW

Key Ideas and Questions

The following questions review the main ideas of this chapter. Write your answers to the questions and then refer to the pages listed by number to make certain that you have mastered these ideas.

1. What is the difference between an outcome and an event? pg. 634 How can all the elements in a sample space be listed? pgs. 634–635
2. How are experimental probabilities determined? pgs. 635–636 How are theoretical probabilities of events with equally likely outcomes calculated? pgs. 636–637
3. What is meant by the union of two events and the intersection of two events? pg. 639 What does it mean to say that two events are mutually exclusive? pg. 639 How is the probability of mutually exclusive events calculated? pg. 639
4. What is the complement of an event? pg. 640 How are the probabilities of an event and the complement of an event related? pg. 640
5. How do you compute the probability the union of two events that are not mutually exclusive? pg. 642 What are the properties of probability? pg. 643
6. What types of experiments can be represented by one-stage tree diagrams? pg. 652 What types of experiments can be represented by two-stage tree diagrams? pg. 652
7. What is the Fundamental Counting Principal? pg. 654 Under what condition can the probabilities of events in a probability tree diagram be added? pg. 657
8. For the experiment of drawing two marbles from a jar, how do drawing with replacement and drawing without replacement affect probability? pgs. 658–661
9. For what reason would you multiply probabilities along a series of branches in a probability tree diagram? pg. 660
10. How do you describe the meaning of conditional probability? pg. 674 How do you compute conditional probability? pg. 674
11. What does it mean to say two events are independent? pg. 680 If A and B are independent, then what is the conditional probability of A given B? pg. 681
12. What is the interpretation of the expected value of an experiment? pg. 682 How do you calculate the expected value of an experiment? pg. 682
13. What is the difference between the odds in favor of an event and the odds against an event? pg. 684 How can the probability of an event be determined given the odds in favor of the event? pg. 684

Vocabulary

Following is a list of key vocabulary for this chapter. Mentally review each of these terms, write down the meaning of each one in your own words, and use it in a sentence. Then refer to the page number following each term to review any material that you are unsure of before solving the Chapter 10 Review Problems.

SECTION 10.1

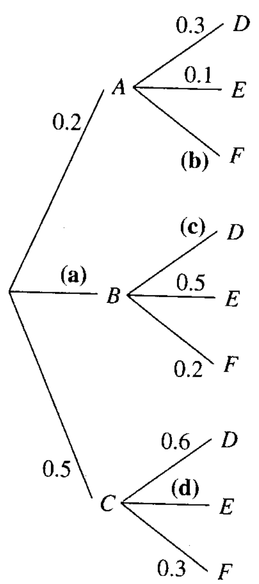
SECTION 10.2

Tree Diagram 652
One-Stage Tree Diagram 652
Two-Stage Tree Diagram 652–653
Primary Branches 653
Secondary Branches 653
Fundamental Counting
Additive Property of Probability Tree Diagrams 657
Drawing with Replacement 658
Drawing without Replacement 658
Multiplicative Property

CHAPTER 10 REVIEW PROBLEMS

1. a. For event A, $P(A)$ must be at least ________ and no more than ________.
   b. What is the probability of a "sure thing"?
   c. What is the probability of an "impossible" event?
   d. If $P(A \cup B) = P(A) + P(B)$, then what must be true about events A and B?
   e. If $P(A|B) = P(A)$, then what must be true about events A and B?

2. Fill in the missing probabilities in the following probability tree diagram.



3. Consider the probability tree diagram in problem 2. Find each of the following probabilities.
   a. $P(A \cap D)$
   b. $P(E)$
   c. $P(B|F)$
   d. $P(\overline{C})$
   e. $P(E|A)$

4. An experiment consists of tossing a fair coin three times in succession and noting whether the coin shows heads or tails each time.
   a. List the sample space for this experiment.
   b. List the outcomes that correspond to getting a tail on the second toss.
   c. Find the probability of getting three tails.
   d. Find the probability of getting a tail on the first coin and a head on the third coin.
   e. Find the probability of getting at least one head.

5. An experiment consists of tossing a fair coin four times in succession and noting whether the coin shows heads or tails each time.
   a. List the sample space for this experiment.
   b. List the outcomes in the event that more heads appear than tails.
   c. Determine the probability that more heads appear than tails.
   d. Determine the probability of getting at most one tail.

6. A student used the laundry room in the dormitory once a week. A particular clothes dryer is often broken. In fact, out of the 30 times the student has used the laundry room, this particular clothes dryer was broken 12 times. What is the experimental probability that the clothes dryer will be broken the next time the student uses the laundry room?

7. Suppose that a jar has four coins: a penny, a nickel, a dime, and a quarter. You remove two coins at random without replacement. Let A be the event that you remove the quarter. Let B be the event that you remove the dime. Let C be the event that you remove less than 12 cents.
   a. List the sample space.
   b. Draw a probability tree diagram.
   c. Which pair(s) of these events is (are) mutually exclusive? Explain.
   d. Find the probabilities of events A, C, and $\overline{B}$.
   e. Compute and interpret $P(A \cup B)$ and $P(B \cup C)$.

8. Suppose that prizes are put into 100,000 boxes of cereal. One box will have a grand prize worth \$10,000. One hundred boxes will have prizes worth \$20. All the other boxes will have prizes that are worth \$0.25. Suppose you buy a box of cereal.
   a. What is the sample space of this experiment?
   b. What is the probability that you will win the grand prize?
   c. What is the probability that you will win one of the \$20 prizes?
   d. What is the probability that you will win at least \$20?
   e. What is the expected value of the box if one box of cereal costs \$2.45?

9. A fast-food restaurant requires that employees wear a uniform. Employees have three color options for shirts (red, yellow, white) and four color options for slacks (black, gray, navy, khaki). Suppose a shirt and a pair of slacks are each selected at random.

a. How many uniform combinations are possible?

b. Construct the probability tree diagram for the experiment of selecting a uniform.

c. Find the probability the employee is wearing black slacks and a shirt that is not white.

10. a. What is the probability of drawing three aces in a row from a thoroughly shuffled deck if the cards are drawn with replacement?

b. What is the probability of drawing three aces in a row from a thoroughly shuffled deck if the cards are drawn without replacement?

11. A pediatric dental assistant randomly sampled 200 patients and classified them according to whether or not they had at least one cavity on their last checkup and according to the type of tooth decay preventive measures they used. The information is presented in the following table.

	At Least One Cavity	No Cavities
Brush only	69	2
Brush and floss only	34	11
Brush and tooth sealants only	22	13
Brush, floss, and tooth sealants	3	46

If a patient is picked at random from this group, find the probability that

a. the patient had at least one cavity.

b. the patient brushes only.

c. the patient had no cavities, given he or she brushes, flosses, and has tooth sealants.

d. the patient brushes only, given that he or she had at least one cavity.

12. A jar contains four marbles: two red and two blue. Marbles will be drawn one at a time, without replacement, until two marbles of the same color have been drawn.

a. Draw a probability tree diagram to represent this experiment.

b. What is the probability that the first marble drawn is blue?

c. What is the probability that the second marble drawn is blue?

d. What is the probability that three drawings are needed and the final marble drawn is blue?

e. What is the probability that only two drawings are necessary?

13. Suppose there are three urns, each containing six balls. The first urn has three red and three white balls. The second urn has four red and two white. The third urn has five red and one white. Two of the urns are selected randomly and their contents are mixed. Then four balls are drawn without replacement. What is the probability that all four balls are red?

14. Suppose two standard dice are rolled. Let F be the event of getting a 4 on the first die. Let O be the event of getting an odd number on the second die. Find and interpret $P(F \cup O)$ and $P(F \cap O)$. Are events F and O independent? Explain.

15. A small college has two psychology classes. The first class has 25 students, 15 of whom are female, and the other class has 18 students, 8 of whom are female. One of the classes is selected at random, and two students are randomly selected from that class for an interview. If both of the students are female, what is the probability they both came from the first class?

16. The probability is 0.6 that a student will study for a true/false test. If the student studies, she has a 0.8 probability of getting an A. If she does not study, she has a 0.3 probability of getting an A. Make a probability tree diagram for this experiment. What is the probability that she gets an A? If she gets an A, what is the conditional probability that she studied?

17. Suppose an experiment consists of drawing two cards from a standard deck of cards and recording their suits. Let A be the event the first card is a heart. Let B be the event the second card is a heart.
    a. If the first card is not replaced before the second card is drawn, find the probability that both cards are hearts. Are events A and B independent? Explain.
    b. If the first card is replaced, and the cards are shuffled before the second card is drawn, find the probability that both cards are hearts. Are events A and B independent? Explain.

18. A box contains five \$1 bills, four \$5 bills, three \$10 bills, two \$20 bills, and one \$100 bill.
    a. If you draw one bill out of the box at random, what is the expected value of the outcome?
    b. Would you be willing to pay \$20 for a chance to randomly pull a bill out of the box in case (a)? Explain.

19. A claim for towing costs an insurance company either \$30 or \$55. The probability of a customer making a claim for \$30 in a year is 0.12 and for \$55 in a year is 0.08. The insurance company wishes to make \$10 per year per claim to cover administrative charges. How much should it charge for a premium?

20. A company receives two shipments of computer equipment. One shipment has 20 computers (4 of which have some operational defect), and the other shipment has 12 printers (2 of which have a defect). If 4 computers and 4 printers are selected at random and paired up as units, what is the probability that at least 1 of the 4 units will have a defect?

21. A local school is selling raffle tickets to raise money. The prize is a trip for two to Hawaii. Suppose you purchase one ticket and notice, when you read the ticket stub, that 2000 tickets were printed. Assume all tickets were sold.
    a. What is the probability that you will win the prize?
    b. What is the probability that you will not win the prize?
    c. What are the odds in favor of you winning the prize?
    d. What are the odds against you winning the prize?

22. At the Tillamook Cheese Factory in Tillamook, Oregon, blocks of cheese are sealed in plastic wrappers. Sometimes the machine does not seal the plastic completely, and the cheese must be rewrapped. Suppose 23% of the cheese blocks packaged during the morning shift must be rewrapped and 12% of the cheese blocks from the afternoon shift must be rewrapped.
    a. If one block of cheese is selected from each shift, what is the probability that neither block must be rewrapped?
    b. If one block of cheese is selected from each shift, what is the probability that exactly one block must be rewrapped?
    c. If the morning shift produces 55% of the blocks of cheese while the afternoon shift produces 45%, and a block of cheese is selected and needs to be rewrapped, what is the probability it came from the morning shift?

23. A golf course makes a profit of \$900 on fair-weather days but loses \$250 on each day of bad weather. The probability for bad weather is 0.35. Find the expected value in terms of profit or loss to the golf course on a single day.