Math 423/823 Exercise Set 5 Solutions

17. [BC#2.26.7] Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D and consider the families of level curves

    $\mathcal{U} = \{\{(x,y) : u(x,y) = c_1\} : c_1 \in \mathbb{C}\}$ and $\mathcal{V} = \{\{(x,y) : v(x,y) = c_2\} : c_2 \in \mathbb{C}\}$.

    Show that wherever they meet, the curves in $\mathcal{U}$ are orthogonal to the curves in $\mathcal{V}$. That is, the slopes of the two curves, at a point of intersection, are negative reciprocals.

    [Hint: for each curve treat it as implicitly defining y as a function of x and use the multivariate chain rule to, e.g., differentiate both sides of $u(x, y(x)) = c_1$ w.r.t. x.]

    With the notation above, suppose the level curve $u(x,y) = c_1$ is given (in a little neighborhood of a point) by the function $y = g(x)$. Then using the multivariable chain rule, since $u(x, g(x)) = c_1$ is the constant function, we have

    $$0 = \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}g'(x), \text{ so } g'(x) = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}.$$

    An analogous argument, supposing that $v(x,y) = c_2$ defines $y = h(x)$ as a function of x, gives

    $$h'(x) = -\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}.$$

    But since f is analytic, we have $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. So

    $$g'(x) \cdot h'(x) = (-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}) \cdot (-\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}) = (-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}) \cdot (\frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial x}) = -1,$$

    so the two level curves have negative reciprocal slopes, so have orthogonal tangent lines, as desired.

18. [BC#3.29.12] For $z = x + yi$, write $\mathrm{Re}(e^{1/z})$ in terms of x and y. Explain why this function is harmonic in every domain D that does not contain 0.

    $1/z = \overline{z}/|z|^2 = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$, so
    $\mathrm{Re}(e^{1/z}) = \mathrm{Re}(e^{u+iv}) = e^u\cos(v) = e^{\frac{x}{x^2+y^2}}\cos\Big(-\dfrac{y}{x^2+y^2}\Big) = e^{\frac{x}{x^2+y^2}}\cos\Big(\dfrac{y}{x^2+y^2}\Big)$

    But we know that $h(z) = 1/z$ is analytic except at $z = 0$ (since it is the quotient of 1 and z, both analytic everywhere, and the denominator z is non-zero except at $z = 0$), and $g(z) = e^z$ is entire, so their composition $f(z) = g(h(z)) = e^{1/z}$ is analytic except where g is not, i.e, except at $z = 0$. But then the real part of $f(z)$ is consequently harmonic everywhere that $f(z)$ is analytic. So $e^{\frac{x}{x^2+y^2}}\cos\Big(\dfrac{y}{x^2+y^2}\Big)$, is harmonic on any domain that does not contain $z = 0$.

19. [BC#3.31.5] Show that:

(a): the set of values of $\log(i^{1/2})$ is $\left(n+\frac{1}{4}\right)\pi i$ for n any integer, and that the same is true for $\frac{1}{2}\log(i)$

Since $i = e^{\pi i/2}$, the two values of $i^{1/2}$ are $e^{\pi i/4}$ and $e^{5\pi i/4}$. So the values of $\log(i^{1/2})$ are

$$\log(e^{\pi i/4}) = \ln|e^{\pi i/4}| + i\arg(e^{\pi i/4}) = \ln(1) + (\frac{\pi}{4} + 2k\pi)i = (\frac{\pi}{4} + 2k\pi)i$$

and

$$\log(e^{5\pi i/4}) = \ln|e^{5\pi i/4}| + i\arg(e^{5\pi i/4}) = \ln(1) + (\frac{5\pi}{4} + 2k\pi)i = (\frac{\pi}{4} + (2k+1)\pi)i$$

So the one set of values is $\frac{\pi}{4}$ plus even multiples of π, and the other is $\frac{\pi}{4}$ plus odd multiples of π, so the complete set of values is $\frac{\pi}{4}$ plus <u>all</u> multiples of π.

[Actually, most of your solutions were better than this one....]

On the other hand, $\log(i) = \ln(e^{\pi i/2}) + i\arg(e^{\pi i/2}) = (\pi/2 + 2k\pi)i$, so $\frac{1}{2}\log(i) = \frac{1}{2}(\pi/2 + 2k\pi)i = (\pi/4 + k\pi)i$, as well.

(b): the set of values of $\log(i^2)$ is <u>not</u> the same as the set of values for $2\log(i)$.

In a similar vein, $\log(i^2) = \log(e^{\pi i}) = \arg(e^{\pi i}) = (\pi + 2k\pi)i =$ all odd multiples of πi.

But $2\log(i) = 2(\pi/2 + 2k\pi)i = (\pi + 4\pi)i$ consists of <u>every</u> <u>other</u> odd multiple of πi, so the two sets of values are not the same.

20. For $z = x + yi$, does 1^z always equal 1 ?

By definition, $a^z = e^{z\log(a)}$. So $1^z = e^{z\log(1)}$. But $\log(1) = \ln|1| + i\arg(1) = 0 + i(0 + 2k\pi) = 2k\pi i$, depending on which branch of $\arg(z)$ we choose. So $1^z = e^{2k\pi iz}$ which, depending on our choice of k, need not equal 1. For $k = 0$, $1^z = e^{0z} = e^0 = 1$, but for, e.g., $k = 1$, $1^i = e^{2\pi ii} = e^{-2\pi} \neq 1$. So depending upon which value of $\log(1)$ that we choose, 1^z need not equal 1.

Shorter, pithier solution: $1^{1/2}$ should be <u>allowed</u> to be -1, under any reasonable definition of exponentials, so, no.