Math 325 Problem Set 4 Solutions

Starred (*) problems were due Friday, September 21.

(*) 17. (Belding and Mitchell, p.36, #20)

(*) (a) Show that if $x, y, c \in \mathbb{R}$, $c > 0$, and $|x - y| < c$, then $|x| < |y| + c$.

This follows from one of our inequalities from the previous problem set:

$|x| - |y| \leq |x - y|$, and so if $|x - y| < c$, then $|x| - |y| \leq |x - y| < c$, so $|x| - |y| < c$. Adding $|y|$ to both sides, we get

$|x| = (|x| - |y|) + |y| < c + |y| = |y| + c$, so $|x| < |y| + c$.

(*) (b) Show that if $x, y \in \mathbb{R}$ and $|x - y| < \frac{|x|}{2}$, then $|y| > \frac{|x|}{2}$.

This follows from the same sort of reasoning: $|x| - |y| \leq |x - y|$, so $|x - y| < \frac{|x|}{2}$ means that $|x| - |y| \leq |x - y| < \frac{|x|}{2}$, and so $|x| - |y| < \frac{|x|}{2}$. Adding $|y| - \frac{|x|}{2}$ to both sides of this inequality gives

$(|x| - |y|) + (|y| - \frac{|x|}{2}) < \frac{|x|}{2} + (|y| - \frac{|x|}{2})$, which simplifies to $\frac{|x|}{2} < |y|$, that is, $|y| > \frac{|x|}{2}$. So:

$|x - y| < \frac{|x|}{2}$ implies that $|y| > \frac{|x|}{2}$.

(*) 20. (Belding and Mitchell, p.22, #2) Show that if a set of real numbers S has a least upper bound α, then this least upper bound is unique. That is, if β is also a least upper bound for S, then $\alpha = \beta$. [Hint: what's the alternative?]

Suppose that α and β both satisfy the conditions for a least upper bound for S, but $\alpha \neq \beta$. Then, by trichotomy, we must have either $\alpha < \beta$ or $\beta < \alpha$.

But if $\alpha < \beta$, then because β is a least upper bound (so no smaller number can be an upper bound) there is an $x \in S$ so that $\alpha < x \leq \beta$. But this means that α is not an upper bound for S (so it certainly can't be a least upper bound!). This is a contradiction, so $\alpha < \beta$ is impossible.

But a symmetric argument eliminates the possiblity that $\beta < \alpha$: since α is a least upper bound for S and $\beta < \alpha$, there is a $y \in S$ so that $\beta < y \leq \alpha$. This means that β is not an upper bound for S; this is a contradiction, so $\beta < \alpha$ is impossible.

So since we must have one of $\alpha = \beta$, $\alpha < \beta$, or $\beta < \alpha$, and $\alpha < \beta$ and $\beta < \alpha$ are both impossible, it must be the case that $\alpha = \beta$. Therefore, two least upper bounds for the same set must be equal to one another.

(*) 22. (Belding and Mitchell, p.23, #4) Let $A = \{a_1, a_2, a_3, \ldots\} = \{a_n \ : \ n \in \mathbb{N}\}$ and $B = \{b_1, b_2, b_3, \ldots\} = \{b_n \ : \ n \in \mathbb{N}\}$ be two sequences of real numbers. Let $C = \{a_n + b_n \ : \ n \in \mathbb{N}\}$, the sequence of their sums.

(*) (a) Show that if A and B have least upper bounds α and β, respectively, then $\alpha+\beta$ is an upper bound for C.

Because α is an upper bound for A, we have $a_n \leq \alpha$ for all n. In the same way, β is an upper bound for B, so $b_n \leq \beta$ for all n. Therefore, $a_n + b_n \leq \alpha+\beta$ for all n, so $c \leq \alpha_\beta$ for all $c \in C$, so $\alpha+\beta$ is an upper bound for the set C.

(*) (b) Find an example showing that $\alpha+\beta$ <u>need</u> <u>not</u> <u>be</u> the least upper bound for C.

One way to do this: if we make $\alpha \in A$ and $\beta \in B$, i.e., $\alpha = a_n$ and $\beta = b_m$ for some $n, m \in \mathbb{N}$ but make $n \neq m$, then $\alpha+\beta$ might <u>not</u> be the largest element of the form $a_k + b_k$, so we might create a 'cap' between C and $\alpha+\beta$. For example,

If $a_1 = 1$ and $a_n = 0$ for $n \geq 1$, while $b_1 = 0$ and $b_n = 1$ for $n \geq 1$, then $A = \{0,1\} = B$, so $\alpha = \beta = 1$, but $a_n + b_n = 1$ for eevery $n \in \mathbb{N}$¡ and so $C = \{a_n + b_n \ : \ n \in \mathbb{N}\} = \{1\}$, which has least upper bound $1 < 2 = 1 + +1+ = \alpha + beta$. So C has least upper bound smaller than $\alpha+\beta$.

A selection of further solutions.

18. A set A is said to be *bounded away from* 0 if there is an $\epsilon > 0$ so that for every $x \in A$ we have $|x| > \epsilon$. Show that A is bounded away from 0 if and only if the set $B = \{\frac{1}{x} \mid x \in A\}$ is bounded.

[N.B. "P if and only if Q" means P implies Q <u>and</u> Q implies P; that is, there are two things to show!]

If A is bounded away from 0, then we have an $\epsilon > 0$ so that $x \in A$ implies $|x| > \epsilon$. but then $|x| > 0$, so $1/|x| > 0$ and $1/\epsilon > 0$, so $1/(\epsilon|x|) > 0$. Then multiplication by $1/(\epsilon|x|)$ will not change the direction of an inequality, so

$$1/|x| = \epsilon/(\epsilon|x|) < |x|/(\epsilon|x|) = 1/\epsilon, \text{ for every } x \in A.$$

So $-1/|x| > -1/\epsilon$, as well, But then since $-|x| \leq x \leq |x|$ for any $x \in \mathbb{R}$ (x <u>equals</u> one of them...), we have $-1/|x| \leq 1/x \leq 1/|x|$ (again, $1/x$ equals one of them), so $-1/\epsilon < -1/|x| \leq 1/x \leq 1/|x| < 1/\epsilon$, and so $-1/\epsilon < 1/x < 1/\epsilon$, for every $x \in A$. So B is bounded below (by $-1/\epsilon$) and bounded above (by $1/\epsilon$), so B is bounded.

For the other direction, if we suppose that $B = \{\frac{1}{x} \mid x \in A\}$ is bounded, then there are N and M so that $M \leq 1/x \leq N$ for every $x \in A$. This statement alone <u>requires</u> that $x \neq 0$, since $1/0$ doesn't make sense and the statement assumes that $1/x$ always does make sense. This direction is a little trickier, since we can't 'just' invert our newly-found inequalities (and get a reversed inequality), because, for example, $a < 0 < b$ implies $1/a < 0 < 1/b$ (and the inequality was <u>not</u> reversed). But we can instead sort of borrow from a previous homework problem...

$M \leq 1/x$ <u>does</u> mean that $-1/x \leq -M$, so since we have $1/x \leq N$, we have $-1/x \leq \max(-M, N)$ and $1/x \leq \max(-M, N)$. But since $1/|x|$ must <u>equal</u> one of these two values ($1/x$ or $-1/x$), and both are $\leq \max(-M, N)$, we can conclude that $0 \leq 1/|x| \leq \max(-M, N) = K$ for every $x \in A$. But now we <u>can</u> invert things! Since $1/|x| > 0$ we have $|x| > 0$, and $0 < 1/|x| \leq K$, so $K > 0$. Then $1/|x| \leq K$ means that $1/K = (1/|x|)(|x|/K) \leq K(|x|/K) = |x|$. So $|x| geq 1/K > 1/(2K) = L > 0$ for every $x \in A$, so there is an $L > 0$ so that $|x| > L$ for every $x \in A$. So A is bounded away from 0.

19. If we set $A = \{x \in \mathbb{R} \mid x^3 < 2\}$, show that A is bounded above, so has a supremum $\alpha = \sup(A)$. Then show (in a manner similar to our classroom demonstrations) that $\alpha^3 < 2$ is not possible. (If you are feeling like doing even more, show that $\alpha^3 > 2$ is also impossible! From that, we can conclude that $\alpha^3 = 2$.)

We showed in class that $f(x) = x^3$ is an increasing function. So if we find a single $a \in \mathbb{R}$ so that $a^3 > 2$, then $x \geq a$ will imply that $x^3 \geq a^3 > 2$, so $x \notin A$. This means that $x \in A$ implies that $x < a$, so A will be bounded above by a. But such an a is readily available; $2^3 = 8 > 2$, so 2 is an upper bound for A.

We therefore have a least upper bound $\alpha = \sup(A)$. To show that $\alpha^3 < 2$ is impossible, suppse that $\alpha^3 < 2$! (We will get ourselves into trouble.) Then $2 - \alpha^3 = \epsilon > 0$. What we show is that (as in class) α could not be an upper bound for A, by finding a $\delta > 0$ so that $(\alpha + \delta)^3 < 2$, so $\alpha + \delta \in A$ and $\alpha < \alpha + \delta$, a contradiction.

To determine δ, we note that $(\alpha + \delta)^3 = \alpha^2 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3$. Since we intend to have $\delta > 0$ and we know, from above, that $\alpha \leq 2$, then

$(\alpha + \delta)^3 = \alpha^3 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3 \leq \alpha^3 + 3 \cdot 2^2\delta + 3cdot2\delta^2 + \delta^3 = \alpha^3 + 12\delta + 6\delta^2 + \delta^3$.

So if we make sure that $12\delta + 6\delta^2 + \delta^3 < \epsilon$, then $(\alpha + \delta)^3 < \alpha^3 + \epsilon = \alpha^3 + (2 - \alpha^3) = 2$, as desired.

There are many ways to arrange this. Perhaps the least tortuous way is to insist, first, that $0 < \delta \leq 1$. Then $12\delta + 6\delta^2 + \delta^3 \leq 12\delta + 6\delta \cdot 1 + \delta \cdot 1^2 = 19\delta$. So to ensure that $12\delta + 6\delta^2 + \delta^3 < \epsilon$ we can <u>also</u> insist that $\delta < \epsilon/19$. <u>So</u> if we set $\delta = \min(1, \epsilon/20)$, then

$(\alpha+\delta)^3 = \alpha^3 + 3\alpha^2\delta + 3\alpha\delta^2 + \delta^3 \leq \alpha^3 + 12\delta + 6\delta^2 + \delta^3 \leq \alpha^3 + 12\delta + 6\delta + \delta = \alpha^3 + 19\delta \leq \alpha^3 + 19\epsilon/20 < \alpha^3 + \epsilon = \alpha^3 + (2 - \alpha^3) = 2$.

So $\alpha + \delta > \alpha$ and $(\alpha + \delta)^3 < 2$, so $\alpha + \delta \in A$, contradicting the choice of $\alpha = \sup(A)$. So $\alpha^3 < 2$ is impossible.

For the extra part: Showing $\alpha^3 > 2$ is impossible proceeds similarly. Setting $\epsilon = \alpha^3 - 2 > 0$, we find a $\delta > 0$ so that $(\alpha - \delta)^3 > 2$, so (by our reasoning at the start of the problem) $\alpha - \delta < \alpha$ is an upper bound for A, so α <u>cannot</u> be the least upper bound for A.

Finding an appropriate δ follows the same line as our argument above. $(\alpha - \delta)^3 = \alpha^3 - 3\alpha^2\delta + 3\alpha\delta^2 - \delta^3 > \alpha^3 - 3\alpha^2\delta - \delta^3$ (since $\alpha > 0$; 0 is not an upper bound for A). But if we insist that $0 < \delta \leq 1$, then $\alpha^3 - 3\alpha^2\delta - \delta^3 \geq \alpha^3 - 3\alpha^2\delta - \delta \cdot 1^2 = \alpha^3 - (3\alpha^2 + 1)\delta$, and we can make $(3\alpha^2 + 1)\delta < \epsilon$ by choosing $0 < \delta < \epsilon/(3\alpha^2 + 1)$. For this δ, we find that $(\alpha - \delta)^3 > 2$, a contradiction.

So both $\alpha^3 < 2$ and $\alpha^3 > 2$ are impossible; this means that $\alpha^3 = 2$.