**Math 325 Problem Set 7 Solutions**

Starred (*) problems were due Friday, October 19.

(*) 38. (Belding and Mitchell, p.89, #10(a)) Show that if $a, b \in \mathbb{R}$, $a < b$, and $f : (a, b) \to \mathbb{R}$ is *uniformly* continuous, then $f$ is bounded: there are $M, N \in \mathbb{R}$ so that $M \leq f(x) \leq N$ for every $x \in (a, b)$.

Hint: the argument that we gave in class sort of works, but we can't 'start' at $a$ (it's not in the domain)!. 'Start' in the middle, instead!

Starting from $c = \dfrac{a+b}{2}$, we have $a < b$ implies that $a < c < b$, so $c \in (a, b)$ . Since $f$ is uniformly continuous, for $\epsilon = 1 > 0$ there is a $\delta > 0$ so that for any $x, y \in (a, b)$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < 1$. Then by the Archimedean Property, since $\delta > 0$ and $b - a > 0$, there is an $n \in \mathbb{N}$ so that $n\delta > b - a$.

Then for any $x \in (a, b)$ we have $|x - c| < (b - a)/2$, since either $a < x \leq c$ or $c \leq x < b$ and $|c - a| = |b - c| = (b - a)/2$, so $x$ lies in an interval of length $(b - a)/3$ with one endpoint equal to $c$. Therefore $|x - c| < (b - a)/2 < (n\delta)/2 = n(\delta/2)$, and so we can pick $n + 1$ equally spaced points $x_0 = x, x_1, \ldots, x_n = c$ each pair being a distance $|x_{i+1} - x_i| = |x - c|/n < \delta/2$ apart. Therefore, $|f(x_{i+1}) - f(x_i| < 1$, and so, by the triangle inequality and induction (as we did in class), we have $|f(x) - f(c)| = |f(x_0) - f(x_n)| < n \cdot 1 = n$. Therefore, $M = f(c) - n < f(x) < f(c) + n + N$. Since this is true for <u>every</u> $x \in (a, b)$, we have found constants $M$ and $N$ so that $M < f(x) < N$ for every $x \in (a, b)$, so $f$ is bounded.

(*) 40. (Belding and Mitchell, p., #5, the other part) If $a, b \in \mathbb{R}$ and $f, g : (a, b) \to \mathbb{R}$ are uniformly continuous, show that $fg : (a, b) \to \mathbb{R}$ is uniformly continuous. Show, by example, on the other hand that $f, g : \mathbb{R} \to \mathbb{R}$ both uniformly continuous does <u>not</u> imply that $fg : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

[Hint: for the first, try to follow the 'usual' continuity argument for products; problem #38 will help! for the second, you don't need to get very creative...]

Since $f$ and $g$ are both uniformly continuous on $(a, b)$, by problem #38 we know that both $f$ and $g$ are bounded, and so there are $M, N \in \mathbb{R}$ so that $|f(x)| \leq M$ and $|g(x)| \leq N$ for all $x \in (a, b)$. Also, since they are uniformly continuous, for each $\epsilon > 0$ there are $\delta_1, \delta_2 > 0$ so that (this next comes <u>after</u> running throguh the argument, and coming back to "fix" the bounds that we need to get our final statement!) $x, y \in (a, b)$ and $|x - y| < \delta_1$ implies that $|f(x) - f(y)| < \epsilon/(2N)$, and $|x - y| < \delta_2$ implies that $|g(x) - g(y)| < \epsilon/(2M)$.

Then if we set $\delta = min(\delta_1, \delta_2) > 0$, then $x, y \in (a, b)$ and $|x - y| < \delta$ implies that $|x - y| < \delta_1$ so $|f(x) - f(y)| < \epsilon/(2N)$, and $|x - y| < \delta_2$ so $|g(x) - g(y)| < \epsilon/(2M)$, and then

$$|(fg)(x) - (fg)(y)| = |f(x)g(x) - f(y)g(y)||f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| = |f(x)| \cdot |g(x) - g(y)| + |g(y) \cdot |f(x) - f(y)| \leq M \cdot |g(x) - g(y)| + N \cdot |f(x) - f(y)| < M \cdot \epsilon/(2M) + N \cdot \epsilon/(2N) = \epsilon/2 + \epsilon/2 = \epsilon\ .$$

So $x, y \in (a, b)$ and $|x - y| < \delta$ implies that $|(fg)(x) - (fg)(y)| < \epsilon$, and so $fg$ is uniformly continuous.

[This is not how the argument was discovered! The last set of inequalities was carried out with the intention of finding out what the quantities $\epsilon/(2M), \epsilon/(2N)$ needed to be in order to get a nice quantity at the end... Formally, we should also insist that $N > 0$ and $M > 0$ by replacing either of them which is (well, the only other possibility is really) 0 with $1 > 0$, if necessary...]

On the other hand, this fails if we replace $(a, b)$ with $\mathbb{R}$, since then we can't rely on the boundedness of $f$ and $g$. As an example, both $f(x) = x$ and $g(x) = x$ are uniformly continuous ($\delta = \epsilon$ will always work), but $(fg)(x) = x^2$, as a function from $\mathbb{R}$ to $\mathbb{R}$, is not uniformly continuous, since $|(fg)(x) - (fg)(y)| = |x^2 - y^2| = |x - y| \cdot |x + y| < \epsilon$ requires that $|x - y| < \epsilon/|x + y|$, but for any $\delta > 0$, we can pick $y$ so large that $x = y + \delta/2$ will have $|x - y| = \delta/2$, but $|x^2 - y^2| = x^2 - y^2 = (y + \delta/2)^2 - y^2 = 2y\delta/2 + (\delta/2)^2 > y\delta > \epsilon$ (by choosing $y > \epsilon/delta$. So no $\delta$ will work for every $x$ and $y$, so $fg$ is not uniformly continuous.

(*) 44. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there is a $P > 0$ so that $f(x + P) = f(x)$ for every $x \in \mathbb{R}$. Show that if $f$ is periodic and continuous, then $f$ is uniformly continuous.

[Hint: you can always insist that your $\delta$'s be smaller than some particular positive number (why?).]

Because $f$ is continuous, then as a function from $[0, 2P]$ to $\mathbb{R}$, $g = f|_{[0,2P]}$ is continuous and therefore uniformly continuous. We can therefore, for every $\epsilon > 0$, find a $\delta > 0$ so that $x, y \in [0, 2P]$ and $|x - y| < \delta_0$ implies that $|g(x) - g(y)| = |f(x) - f(y)| < \epsilon$. Then if we set $\delta = min(\delta_0, P) > 0$, then $x, y \in [0, 2P]$ and $|x - y| < \delta$ implies that $|x - y| < \delta_0$, as well, so $|g(x) - g(y)| = |f(x) - f(y)| < \epsilon$, as well.

But now if $x, y \in \mathbb{R}$ and $|x - y| < \delta$, then $|x - y| < P$ (so $x - P < y < x + P$), and so if we assume, WOLOG, that $x \leq y$ and find the $n \in \mathbb{Z}$ so that $nP \leq x < (n + 1)P$, then $nP < x \leq y < x + P < (n + 2)P$. Then $x - nP, y - nP \in [0, 2P]$ and $|(x - nP) - (y - nP)| = |x - y| < \delta$, and so

$|g(x - nP) - g(y - nP)| < \epsilon$. Therefore $|x - y| < \delta$ implies that $|(x - nP) - (y - nP)| < \delta$ and so $|f(x) - f(y)| = |f(x - nP) - f(y - nP)| = |g(x - nP) - g(y - nP)| < \epsilon$, where for the first equality we used the periodicity of $f$. So $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. So $f$ is uniformly continuous.

**A selection of further solutions.**

41. (Belding and Mitchell, p.89, #14) Let $f$ and $g$ be uniformly continuous on $\mathbb{R}$. Prove that their composition $f \circ g$ is also uniformly continuous on $\mathbb{R}$ .

Given $\epsilon > 0$, since $f$ is uniformly continuous, there is an $\eta > 0$ so that $x, y \in \mathbb{R}$ and $|x - y| < \eta$ implies that $|f(x) - f(y)| < \epsilon$. But then, because $g$ is uniformly continuous, there is a $\delta > 0$ so that $x, y \in \mathbb{R}$ and $|x - y| < \delta$ implies that $|g(x) - g(y)| < \eta$.

Therefore, if $x, y \in \mathbb{R}$ and $|x - y| < \delta$, then ($g(x), g(y) \in \mathbb{R}$ and) $|g(x) - g(y)| < \eta$, and so $|f(g(x)) - f(g(y))| = |(f \circ g)(x) - (f \circ g)(y)| < \epsilon$. So for every $\epsilon > 0$ we have found our $\delta > 0$ that works everywhere, and so $f \circ g$ is uniformly continuous.

43. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and define a new function $g : [a, b] \to \mathbb{R}$ by $g(x) = lub\{f(t) \ : \ a \leq t \leq x\}$. Show that $g$ is also continuous!

[Hint: that least upper bound is (always) achieved!]

Note first that $g$ is an increasing (well, non-decreasing) function, since $x \leq y$ implies that $[a, x] \subseteq [a, y]$ and so $\{f(t) \ : \ t \in [a, x]\} \subseteq \{f(t) \ : \ t \in [a, y]\}$ and larger sets have larger l.u.b.'s.

We want to show that for every $c \in [a, b]$ and $\epsilon > 0$, we wish to find a $\delta > 0$ so that $x \in [a, b]$ and $|x - c| < \delta$ implies that $|g(x) - g(c)| < \epsilon$. But we know that $g(c) = lub\{f(t) \ : \ a \leq t \leq c\}$, and since $f$ is continuous on $[a, c]$, the Extreme Value Theorem tells us that $g(c) = f(x_c)$ for some $x_c \in [a, c]$.

Probably the cleanest way to proceed is to consider two possibilities: either $x_c = c$ or $x_c < c$. If $x_c = c$, then using the continuity of $f$, for any $\epsilon > 0$, there is a $\delta > 0$ so that $|x - c| < \delta$ (and $x \in [a, b]$) implies that $|f(x) - f(c)| < \epsilon$, and so $f(c) - \epsilon < f(x) < f(c) + \epsilon$. This means that for $t \in [a, x]$ we either have $t \in [a, c]$ and so $f(t) \leq f(x_c) = f(c)$, or ($x > c$ and) $t \in [c, x]$ and so $|t - c| = t - c < x - c = |x - c| < \delta$ and so $f(t) < f(c) + \epsilon$, so in any case $f(t) < f(c) + \epsilon$ for all $t \in [a, x]$ and so $g(x) \leq f(c) + \epsilon = g(c) + \epsilon$. On the other hand, $f(x) > f(c) - \epsilon = g(c) - \epsilon$, so $g(x) > g(c) + \epsilon$. Therefore, in the case that $x_c = c$, so $g(c) = f(c)$, we have $|x - c| < \delta$ implies that $g(c) - \epsilon < g(x) \leq g(c) + \epsilon$, and so $|g(x) - g(c)| \leq \epsilon$. (The annoyance of the $\leq$ that we get can be eliminated by starting whole argument again and pretendint we were working with $\epsilon/2$, instead.) So we have established that continuity of $g$ at $c$ holds, when $x_c = c$.

On the other hand, if $x_c < c$, then $c - x_c = \eta > 0$, and so $x < c$ and $c - x < \eta$ implies that for $t \in [x, c]$ we have $f(t) \leq f(x_c)$ and so $g(x) = f(x_c)$ (since $x_c \in [a, x]$) and so $g(x) = g(c)$, when $c - \eta < x < c$. On the other hand, for any $\epsilon > 0$ there is a $\delta_0 > 0$ so that $|x - c| < \delta_0$ implies that $|f(x) - f(c)| < \epsilon$, and so $c \leq x < c + \delta_0$ implies that $f(x) < f(c) + \epsilon \leq f(x_c) + \epsilon = g(c) + \epsilon$. So $t \in [a, x]$ implies that either $t \in [a, c]$, so $f(t) \leq g(c)$ (since $g(c)$ is a l.u.b.), or $t \in [c, x]$ so $|t - c| < \delta$ and so $f(t) < g(c) + \epsilon$. Therefore $g(x) \leq g(c) + \epsilon$. Therefore, $g(x) = g(c)$ for $x \in (c - \eta, c]$, and $g(c) \leq g(x) \leq g(c) + \epsilon$ for $x \in [c, c + \delta_0)$. So setting $\delta = min(\eta, delta_0)$ gives that $|x - c| < \delta$ implies either $x \leq c$ and $g(x) = g(c)$, so $|g(x) - g(c)| = 0 < \epsilon$, or $x \geq c$ and so $g(c) \leq g(x) \leq g(c) + \epsilon$, so $|g(x) - g(c)| \leq \epsilon$. So in either case, $|x - c| < \delta$ implies that $|g(x) - g(c)| \leq \epsilon$, giving continuity of $g$ at $c$ in the case that $x_c < c$.

So in every case, $g$ is continuous at $c$ for every $c \in [a, b]$, and so $g$ is continuous.