Math 325 Problem Set 3

Starred (*) problems are due Friday, September 14.

(*) 11. Show that the maximum of two numbers $x, y \in \mathbb{R}$ can be computed by

$$\max(x,y) = \frac{x+y+|x-y|}{2} .$$

[That is, if $x \leq y$ then $\dfrac{x+y+|x-y|}{2} = y$, while if $y \leq x$ then it equals x.]

Find a similar formula which gives the minimum of x and y.

Given $x, y \in \mathbb{R}$, by trichotomy we know that either $x < y$, $x = y$, or $x > y$. We look at each case separately.

If $x < y$, then $\max(x,y) = y$. But then $x - y < 0$, so $|x-y| = -(x-y) = y - x$, and so $\dfrac{x+y+|x-y|}{2} = \dfrac{x+y+(y-x)}{2} = \dfrac{2y}{2} = y$, so the two quantities agree.

If $x = y$, then $\max(x,y) = x = y$. But then $x - y = 0$, so $|x-y| = 0$, and so $\dfrac{x+y+|x-y|}{2} = \dfrac{x+y+0}{2} = \dfrac{2x}{2} = x = y$, and so the two quantities again agree.

Finally, if $x > y$, then $\max(x,y) = x$. But then $x - y > 0$, so $|x-y| = x - y$, and so $\dfrac{x+y+|x-y|}{2} = \dfrac{x+y+(x-y)}{2} = \dfrac{2x}{2} = x$, so the two quantities again agree.

So, for any choice of x and y, we find that $\max(x,y) = \dfrac{x+y+|x-y|}{2}$, which is what we wished to show.

A formula fo $\min(x,y)$ can be found similarly; we want the exact opposite result (for $x < y$ versus $x > y$), so we want the exact opposite (i.e., negative) result to occur with $|x-y|$. We can do this by subtracting $|x-y|$ intead of adding it; so

$$\min(x,y) = \frac{x+y-|x-y|}{2}$$

which we can verify by the same "case analysis".

(*) 14. (Belding and Mitchell, p.36, #17) Use the triangle inequality to establish that for every $x, y \in \mathbb{R}$ we have

(*) (a) $|x| - |y| \leq |x-y|$

If we start by making what we want look 'more' like the triangle inequality, we would like to show that $|x| \leq |x-y| + |y|$. but since $(x-y) + y = x$, we can express this as $|(x-y)+y| \leq |x-y| + |y|$. But this does look exactly like the triangle inequality, just with different names... Making this a bit more formal, we know that $a, b \in \mathbb{R}$ implies that $|a+b| \leq |a| + |b|$. But then if we have $x, y \in \mathbb{R}$, then $a = x - y$ and $b = y$ are real numbers and so we have, by the triangle inequality, $|a+b| \leq |a| + |b|$, that is, $|(x-y)+y| \leq |x-y| + |y|$. This means that $|x| \leq |x-y| + |y|$, so $|x| - |y| \leq |x-y|$. So for every $x, y \in \mathbb{R}$ we have $|x| - |y| \leq |x-y|$, as desired.

(*) (b) $|x| - |y| \leq |x + y|$

This looks close to the previous problem, except that $x+y = x = (-y)$. But $|-y| = |y|$ for every $y \in \mathbb{R}$, since $y \geq 0$ means that $-y \leq 0$ and so $|y| = y = -(-y) = |-y|$, while if $y \leq 0$ then $-y \geq 0$ and so $|y| = -y = |-y|$. But by problem (a) we know that for every $x, y \in \mathbb{R}$ we have $x, -y \in \mathbb{R}$ and so $|x| - |-y| \leq |x - (-y)|$, that is, $|x| - |y| \leq |x - (-y)| = |x + y|$, as desired.

(*) (d) $\Big||x| - |y|\Big| \leq |x - y|$

We know that $\Big||x| - |y|\Big|$ is equal to <u>either</u> $|x| - |y|$ <u>or</u> $-(|x| - |y|) = |y| - |x|$. If we can show that <u>both</u> of these numbers are $\leq |x - y|$, then now matter which value the left-hand side takes, we will have $\Big||x| - |y|\Big| \leq |x - y|$.

But! '$|x| - |y| \leq |x - y|$ for every $x, y \in \mathbb{R}$' is precisely part (a) above. So what we want to be true work in the case that $\Big||x| - |y|\Big| = |x| - |y|$. But by exchanging the roles of x and y, we know, by (a), that for every $x, y \in \mathbb{R}$ we have $|y| - |x| \leq |y - x|$. But as part of part (b) we showed that $|y| = |-y|$ which means that for every $x, y \in \mathbb{R}$ we have $|x - y| = |-(x - y)| = |y - x|$. Putting these two facts together we get: for $x, y \in \mathbb{R}$ we have $|y| - |x| \leq |y - x| = |x - y|$, so $|y| - |x| \leq |x - y|$.

So no matter which value $\Big||x| - |y|\Big|$ has, it is $\leq |x - y|$. SO for all $x, y \in \mathbb{R}$ we have $\Big||x| - |y|\Big| \leq |x - y|$, as desired.

(*) 15. (a) Show that if $B \subseteq \mathbb{R}$ is *bounded*, and $A \subseteq B$, then A is bounded.

Since B is bounded, it has both an upper and a lower bound, so there are $N, M \in \mathbb{R}$ with $x \leq N$ for every $x \in B$, and $M \leq x$ for every $x \in B$. But sincce $A \subseteq B$, if $x \in A$ then $x \in B$ (and so $x \leq N$ and $M \leq x$. So $x \leq N$ for every $x \in A$, so N is an uppoer bound for A. Also, $M \leq x$ for evvey $x \in A$, so M is a lower bound for A. <u>So</u>, A has both an upper and a lower bound, so A is bounded!

(*) (b) If $S \subseteq \mathbb{R}$, then we define the set $|S|$ as $|S| = \{|s| \ : \ s \in S\}$. Show that if S is bounded, then $|S|$ is bounded.

As above, since S is bounded, there are $N, M \in \mathbb{R}$ so that $x \leq N$ and $M \leq x$ for every $x \in S$. But now if we pick $y \in |S|$, then $y = |x|$ for some $x \in S$. So $y = |x| \geq 0$ for every $y \in |S|$, so 0 is a lower bound for $|S|$.

But we also know that either $y = |x| = x$ (if $x \geq 0$) or $y = |x| = -x$ (if $x \leq 0$). If $y = x$, then $y \leq N$ since $x \in S$ so $x \leq N$. But if $y = -x$, then $y \leq -M$, since $x \in S$ so $M \leq x$, so $x \geq M$, so $-x \leq -M$ (since negating both sides of an inequality reverses the inequality). So for any $y \in |S|$ we have either $y \leq N$ or $y \leq -M$. So if we set $K = \max(N, -M)$, then $N \leq K$ and $-M \leq K$, and so no matter which one of $y \leq N$ or $y \leq -M$ is true, we can conclude that $y \leq N \leq K$ or $y \leq -M \leq K$, so $y \leq K$ in both cases. So for every $y \in |S|$ we have $y \leq K$¡ so K is an upper bound for $|S|$.

So since $|S|$ has both an upper (K) and a lower (0) bound, $|S|$ is bounded.

A selection of further solutions

10. Show, by induction, that the (ordinary) triangle inequality extends to show that for any $n \geq 2$ we have

$$|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k| \; .$$

Arguing by induction, our base case is $n = 2$, where $x_1 + x_2| \leq |x_1| + |x_2|$ is true, because this is the triangle inequality that we established in class.

If we then assume that for some $n \geq 2$ we have $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ when $a_1, \ldots, a_n \in \mathbb{R}$, then if we have $a_1, \ldots, a_{n+1} \in \mathbb{R}$, then $|\sum_{k=1}^{n+1} a_k| = |(\sum_{k=1}^{n} a_k) + a_{n+1}|$. but $\sum_{k=1}^{n} a_k$ is a real number, so the ordinary triangle inequality tells us that $|(\sum_{k=1}^{n} a_k) + a_{n+1}| \leq |(\sum_{k=1}^{n} a_k)| + |a_{n+1}|$. Then our inductive hypothesis tells us that $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$, and so, putting everything together,

$$|\sum_{k=1}^{n+1} a_k| = |(\sum_{k=1}^{n} a_k) + a_{n+1}| \leq |(\sum_{k=1}^{n} a_k)| + |a_{n+1}| \leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k| \; .$$

So $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ implies that $|\sum_{k=1}^{n+1} a_k| \leq \sum_{k=1}^{n+1} |a_k|$. This gives us our inductive step, and so, by induction, we have $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ for any $n \geq 2$.

16. For each of the following sets, either show that it is bounded (and find bounds), or explain why it isn't. [You can appeal to results from calculus in your answers.]

(a) $A = \left\{ \sum_{k=1}^{n} \frac{1}{k} \;:\; n \in \mathbb{N} \right\}$

Every term in each sum is greater then 0, so each sum is greater than 0, so 0 is a lower bound for the set. But since $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent series, calculus tells us that the partial sums tend to ∞ as $n \to \infty$, and so for any $N \in \mathbb{R}$ there is a partial sum $\sum_{k=1}^{n} \frac{1}{k} > N$ (you may recall a certain 'useless' fact from class about when this first exceeds 100...), and so no number can be an upper bound for the set, so A is <u>not</u> bounded from above. So A is not bounded.

(b) $B = \left\{ \sum_{k=1}^{n} \frac{1}{2^k} \;:\; n \in \mathbb{N} \right\}$

Again, each term in a sum is positive, so each sum is posisitve, so 0 is a lower bound. In this case, though, the related infinite series is $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} \sum_{k=0}^{i} nfty(\frac{1}{2})^k$ is a geometric series, which converges (by calculus) to $\frac{1}{2} 1/(1 - \frac{1}{2}) = 1$.

Since the terms being added are positive, each element of B is larger than the previous one, so the infinite sum is larger than them all. So the limit, 1, is an upper bound for all of the elements. [An alternative proof: use induction to show that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} < 1$, for every n .] So since B has both an upper (1, or anything larger than that!) bound and a lower 0, or anything further to the left than that!) bound, B is bounded.