Joy of Numbers – Take Home Test # 2 Comments

1. Everyone, I think, went at this one with the Euclidean algorithm, and most everyone ended up with the same final answer of 13:

   $4433221 = 1223456 \cdot 3 + 762853$ $\quad 1223456 = 762853 \cdot 1 + 460603$
   $762853 = 460603 \cdot 1 + 302250$ $\quad 460603 = 302250 \cdot 1 + 158353$
   $302250 = 158353 \cdot 1 + 143897$ $\quad 158353 = 143897 \cdot 1 + 144564$
   $143897 = 14456 \cdot 9 + 137934$ $\quad 14456 = 13793 \cdot 1 + 6634$
   $13793 = 663 \cdot 20 + 533$ $\quad 663 = 533 \cdot 1 + 130$
   $533 = 130 \cdot 4 + 13$ $\quad 130 = 13 \cdot 10 + 0$

2. This problem could be approached using the Euclidean algorithm, to find the gcd (which is 3), recover the gcd as a combination, and multiply everything by 11. Some of you took the approach we used initially in class, finding mutiples of previously discovered combinations that are close together and subtracting, to get closer to 33. This had the advantage that it found the 'smallest' values (probably?) that worked:

   $(-27) \cdot 117 + (7) \cdot 456 = 33.$

   The rest of us tended to find the solution

   $(429) \cdot 117 + (-110) \cdot 456 = 33$, since $(39) \cdot 117 + (-10) \cdot 456 = 3$.

3. Everyone correctly found the divisors of all of the numbers, and everyone, I think, correctly identified the perfect squares as the only numbers having an odd number of divisors. Most I think pointed out that all squares will have an odd number of divisors, and the basic reason for the distinction between the two types of behavior is that the perfect squares a^2 are precisely those which have a factor a, which doesn't 'pair' with another different factor. As some of you put it, when you start with a list of all of the factors of a number, starting from the bottom we can cross them off in pairs; if $n = ab$ with $a < b$ we cross of a and b. As we work up, we either don't meet in the middle at $n = a^2$, in which case we had some number of pairs of factors (hence an even number), or we do meet in the middle, in which case we had some number of pairs of factors, plus one! (I.e., an odd number!)

4. Everyone, I think, correctly asserted that the only possible gcd's were 1,2,3, and 6, and found lots of very different examples. A few of you even pointed out that if you start at $a = 0$ and count up, the gcd's follow a repeating pattern: $6, 1, 2, 3, 2, 1, 6, 1, 2, 3, 2, 1, \ldots$. One way to show that these are the only 4 possible values for the gcd (and give an explanation for the pattern, too!) lies in noting that the first line of the Euclidean algorithm will always (well, if a is bigger than 6, really) read

   $$a + 6 = a \cdot 1 + 6$$

   which implies that anything dividing a and $a + 6$ must also divide 6, so it is the factors of 6 that are possible, as gcd's. In the next step, we would compute the remainder of a on division by 6, which will, starting from $a = 0$, read $0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, \ldots$, i.e., it repeats! The pattern of gcd's comes from this repeating pattern of remainders.