Math 971 Algebraic Topology

Homework # 2 Solutions

A continuous surjection from a compact space to a Hausdorff space is (a closed map hence) a quotient map. (Which makes building induced maps a whole lot more straightforward.)

If $\gamma : I \to X$ is a path in X beginning at x_0 and ending at x_1, then the induced change of basepoint isomorphism $\widehat{\gamma} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is $\widehat{\gamma}([\eta]) = [\overline{\gamma} * \eta * \gamma]$ (to get the basepoints to work out right).

(p.38, # 5): Every map $\gamma : S^1 \to X$ is homotopic to a constant $\Leftrightarrow$ every map $\gamma : S^1 \to X$ extends to a map $\Gamma : \mathbb{D}^2 \to X$ $\Leftrightarrow$ $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$.

We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). We will think of S^1 as the unit circle in $\mathbb{R}^2 = \mathbb{C}$, and $\mathbb{D}^2$ as the unit disk in $\mathbb{R}^2$.

(1) $\Rightarrow$ (2): Given $\gamma : S^1 \to X$, by (1), there is a homotopy $H : S^1 \times I \to X$ with $H(z, 0) = \gamma(z)$ and $H(z, 1) = x_0$ for some $x_0 \in X$. If we define a map $h : S^1 \times I \to \mathbb{D}^2$ by $h(z, t) = h((x, y), t) = (1 - t)z = ((1 - t)x, (1 - t)y)$. As a function of 3 variables, it is continuous, so restricting domain and range it is cts. The map h factors through the quotient space $Z = (S^1 \times I)/(S^1 \times \{1\})$, since on the 1 end h is 0. The resulting map $\overline{h} : Z \to \mathbb{D}^2$ is a cts bijection from (quotient of compact, hence) compact to Hausdorff, so it is a homeomorphism. H also factors through Z, since on the 1 end H is constant; call the resulting map $\overline{H}$. Then $\Gamma = \overline{H} \circ \overline{h}^{-1} : \mathbb{D}^2 \to X$ is the required map extending γ.

(2) $\Rightarrow$ (3): Given an element $[\gamma] \in \pi_1(X, x_0)$, $\gamma : (I, \partial I) \to (X, x_0)$ factors through the (quotient) map $f : I \to S^1$ given by $f(t) = e^{2\pi i t}$, to give a map $g : S^1 \to X$. By hypothesis, this map extends to a map $G : \mathbb{D}^2 \to X$. If we define a map $K : I \times I \to \mathbb{D}^2$ by $K(t, s) = (s, 0) + (1 - s)e^{2\pi i t} = (s + (1 - s)\cos(2\pi t), (1 - s)\sin(2\pi t))$, then $K(t, 0) = f(t)$ and $K(t, 1) = (1, 0)$. Then $H = G \circ K : I \times I \to X$ has $H(t, 0) = \gamma(t)$ and $H(t, 1) = G(1, 0) = x_0$, and $H(0, s) = H(1, s) = K(1, 0) = x_0$. So H represents a homotopy, rel basepoint, from γ to the constant map. So $\pi_1(X, x_0) = \{1\}$.

(3) $\Rightarrow$ (1): Given $\gamma : S^1 \to X$, composing with the (quotient) map f above gives a based loop $g = \gamma \circ f : (I, \partial I) \to (X, x_0)$, where $x_0 = \gamma(1, 0)$. By hypothesis, this map is null-homotopic, so there is a map $H : I \times I \to X$ with $H(t, 0) = g(t)$, and $H(t, 1) = H(0, s) = H(1, s) = x_0$ for all $t, s \in I$. This map factors through the (quotient) map $f \times Id : I \times I \to S^1 \times I$ to give an induced map $\overline{H} : S^1 \times I \to X$ with $\overline{H}(z, 0) = \gamma(z)$ and $\overline{H}(z, 1) = x_0$. So γ is homotopic to a constant map.

X is simply-connected $\Leftrightarrow$ all maps $S^1 \to X$ are homotopic to one another:

($\Rightarrow$): Given two maps $g, h : S^1 \to X$, composing them with the map $p : I \to S^1$, $p(t) = (\cos(2\pi t), \sin(2\pi t)$ gives us a pair of based loops γ, η, based at $g(1, 0) = x_0$ and $h(1, 0) = x_1$ respectively. By hypothesis, each one represents the trivial element in $\pi_1(X, x_\epsilon)$, so there are homotopies $G, H : I \times I \to X$ between these loops and their respective constant maps. Because these maps are constant on $I \times \partial I$, they factor through the map f above to induce maps $G', H' : S^1 \times I \to X$ with restriction to $S^1 \times \{1\}$ the (appropriate) constant map. Since X is 0-connected, there is a path $\delta : I \to X$ with $\delta(0) = x_0$ and $\delta(1) = x_1$.

Then defining $K : S^1 \times I \to X$ by $K(x,t) = \delta(t)$ we have a continuous map (since $K^{-1}(\mathcal{U}) = S^1 \times \delta^{-1}(\mathcal{U})$). And finally, defining $R : S^1 \times I \to X$ by

$$R(x,t) = \begin{cases} G'(x,3t) & \text{, if } t \le 1/3 \\ K(x,3t-1) & \text{, if } 1/3 \le t \le 2/3 \\ H'(x,3-3t) & \text{, if } t \ge 2/3 \end{cases} \qquad \text{defines a homotopy from } g \text{ to } h\ .$$

($\Leftarrow$): We wish to show both that X is path-connected and $\pi_1(X) = \{1\}$. For path connected, given $x_0, x_1 \in X$ for the constant maps $g, h : S^1 \to X$ constant at these points, the hypothesis implies that there is a homotopy $H : S^! \times I \to X$ between them. Then the path $\gamma : I \to X$ given by $\gamma(t) = H((1,0),t)$ has $\gamma(0) = H((1,0),0) = g(1,0) = x_0$ and $\gamma(1) = H((1,0),1) = h(1,0) = x_1$. So X is path connected. And since every map $g : S^1 \to X$ is homotopic to any constant map, by (1) $\Rightarrow$ (2) $\Rightarrow$ (3) above, $\pi_1(X,x_0) = \{1\}$ for every x_0, so X is 1-connected. So X is simply-connected.

(p.39, # 20): If $H : X \times I \to X$ is a cts homotopy from $H(x,0) = x$ to $H(x,1) = x$, then the loop defined by $\gamma(t) = H(x_0,t)$ represents an element in the center of $\pi_1(X,x_0)$.

By Lemma 1.19 of the text, the change of basepoint isomorphism $\widehat{\gamma} : \pi_1(X,x_0) \to \pi_1(X,x_0)$ given by $\widehat{\gamma}[\eta] = [\overline{\gamma} * \eta * \gamma]$ satisfies $H_{0*} = \widehat{\gamma} \circ H_{1*}$. But since $H_0 = H_1 = Id$, so their induced homomorphisms are Id, we have $Id = \widehat{\gamma} \circ Id$, so $\widehat{\gamma} = Id$. But this means that for all η, $[\overline{\gamma} * \eta * \gamma] = [\eta]$, so $[\gamma][\eta] = [\eta][\gamma]$ for every $[\eta] \in \pi_1(X,x_0)$. So $[\gamma]$ commutes with every element of $\pi_1(X,x_0)$, so it is central.

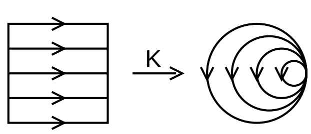
(p.53, # 8): Compute $\pi_1(X)$ where X is obtained from two copies of the torus $S^1 \times S^1$ by identifying the circle $S^1 \times \{x_0\}$ on one with the corresponding circle on the other.

A cheap way to do this is to identify X as a product space itself. X is the quotient space of $S^1 \times S^1 \times \{1,2\}$ where we identify $(x,x_0,1)$ with $(x,x_0,2)$. But this is the same as taking the product of S^1 with the quotient Z of $S^1 \times \{1,2\}$ where we identify $(x_0,1)$ with $(x_0,2)$. But Z is a bouquet of two circles; giving each copy of S^1 a cell structure with vertex x_0 and one 1-cell, Z then has one vertex and two 1-cells, which is what a bouquet of two circles is. Then we have that $\pi_1(Z) =< a,b \mid >= F(a,b)$ is free on two generators, so $\pi_1(X) = \pi_1(S^1 \times Z) \cong \pi_1(S^1) \times \pi_1(Z) = \mathbb{Z} \times F(a,b)$.

Or if you prefer a cell structure approach, each torus can be given a cell structure with one 0-cell, two 1-cells (one of which, with the vertex, is the circle $S^1 \times \{x_0\}$), and one 2-cell whose boundary spells out the commutator of the two 1-cells. X therefore has one 0-cell, three 1-cells (since one from each torus have been identified), and two 2-cells. Thinking of this as gluing two 2-cells to a bouquet of 3 circles, whose boundaries map to $[a,b]$ and $[b,c]$, we have $\pi_1(X) \cong < a,b,c \mid aba^{-1}b^{-1}, bcb^{-1}c^{-1} >$.

The motivated student can verify that these two groups are in fact isomorphic!

(2)==>(3):



#8: