Math 856 Differential Topology

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Differential topology is about introducing concepts and methods from calculus to the realm of topological spaces. That is, we wish to use notions of differentiation and integration in a topological setting. There are (at least) two reasons for doing so. The first is, essentially, waste not want not; lots of people have put in a lot of effort into developing the tools of analysis, why shouldn't topologists want to take advantage of all of that body of work? Any tool that we can bring to bear to better understand topological spaces helps us, well, understand topological spaces better. The other reason is that by figuring out how to introduce analysis into topology, we will have extended the range of applicability of these concepts. Experience has also shown that the topological point of view can, in hindsight, provide a more natural setting for many problems of analysis. It can also provide a natural framework for explaining some of its results; Stokes' Theorem is perhaps the first and most well-known, but certainly not the only such result that we may encounter in our study. As with nearly any branches of mathematics, once you figure out how to reconcile the immediate difficulties in introducing one subject to another (analysis to topology, or topology to analysis?), you discover innumerable ways in which they open up new avenues of exploration, and neither subject is ever the same again. The goal of this course is to explore the ways in which we can bring analysis and topology together, and some of the ways in which anaysis helps to illuminate the study of topology.

Our first task is to determine *which* topological spaces we can reasonably introduce such concepts and methods to.

A basic principle in topology is that a topological space is explored through its continuous functions/continuous maps, both in and out of the space. Calculus as we usually encounter it applies to functions between Euclidean spaces $\mathbb{R}^n$. We have derivatives, partial derivatives, integrals, and multipe integrals, and many variations, depending upon what domain or range/codomain we choose for our functions. So if we want to be able to introduce the idea of a "differentiable" map, the simplest tack to take is to look at topological spaces which "behave" like Euclidean spaces. Differentiability is a local property; a (partial) derivative of a function at a point (much less whether you have one, i.e., are differentiable) depends only on the values of the function near that point. Of course the notion of "local" is in some sense what a topology on a space is designed to describe; open neighborhoods of a point x are precisely the sets describing which points are "near" x. So on a most basic level, the topological spaces most naturally to introduce calculus to are those in which the the points have open neighborhoods which "look like" the spaces that we know how to do calculus on, namely, Euclidean spaces. This motivates our first definition.

A *topological manifold* M of dimension n is a Hausdorff, second countable space with the property that for every $x \in M$ there is an open neighborhood U of x which is homeomorphic to $\mathbb{R}^n$.

The shorthand for the last property is that M is *locally Euclidean*. The other two properties, Hausdorffness and second countability, are designed, really, to make the topologists job easier. One occasionally encounters situations in which a locally Euclidean space is either not Hausdorff or not second countable, but they are very much the exception rather than the rule. And being able to assume both conditions when someone starts tossing the term "manifold" around certainly make proving theorems a lot easier. Surely this isn't the first time that you have encountered hypotheses being imposed for the purpose of making theorems easier to prove? At any rate, any subset of a Euclidean space is both Hausdorff and second countable in the subspace topology; most (all?) manifolds we will meet can (with effort) be interpreted as such a subspace.

Some standard examples: Euclidean space $\mathbb{R}^n$ itself. Spheres S^n = the points at unit distance in $\mathbb{R}^{n+1}$; given a point, $x \in S^n$ at least one of its coordinates x_i is non-zero. Then the set of points $y \in S^n$ whose i-th coordinate has the same sign as x form a locally Euclidean neighborhood of x; the homeo to the unit ball in $\mathbb{R}^n$ is given by projection onto the other coordinates. Cartesian products of manifolds are manifolds; take the Cartesian product of neighborhood in each as your local models. Open subsets of manifolds are manifolds. These basic building blocks already let you build a wide variety of examples.

Once we are confortable with the setting, manifolds, into which we will ultimately introduce differentiability, we are left with actually *doing* it. It turns out that in order to do so in a meaningful way, we have to introduce additional "structure"; simply having a topological manifold won't be enough.

On the face of it, once we have a space M which locally "looks like" Euclidean space, we can seemingly define differentiability at a point for any function $f : M \to \mathbb{R}$. Given a point $x \in M$, we have, by definition, a neighborhood U of X and a homeomorphism $h : U \to \mathbb{R}^n$. This is, at least, enough to *describe* a function for which differentiability makes sense, namely the composition of h^{-1} with the restriction of f to U; $f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R}$. So as a first approximation, we could say that f is differentiable at x if $f \circ h^{-1}$ is differentiable at $h(x)$.

There is only one problem with this. Surely if we are generalizing the notion of diferentiability to more general spaces we don't want to *change* what functions $g : \mathbb{R}^n \to \mathbb{R}$ we wish to consider to be differentable. But, technically, our first attempt at a definition just did. Consider the function $f(x) = |x|$, which we are all, presumably, willing to agree is not differentiable at 0. But we can treat the domain $\mathbb{R} = M$ as a 1-dimensional manifold, where $U = \mathbb{R}$ and the homeomorphism $h(x) = x^{1/3}$ serves as the proof for each $x \in M$ that M is locally Euclidean. But then in testing whether or not f is differentiable at 0, we can just check that $f \circ h^{-1}(x) = |x^3|$ is in fact differentiable at 0. Which it is; the derivative is 0.

What went wrong? Nothing. Unless you don't want to change the notion of differentiability... The point is, our definition of differentiability mentions both a neighborhood U of x (which won't, in the end, really affect things) and a specific homeomorphism $h : U \to \mathbb{R}^n$. The function f and the point x wasn't enough to define differentiability; we also needed a *chart* (U, h), that is, a specific description for *how* to identify a neighborhood of x with $\mathbb{R}^n$. And whether or not we decide f is differentiable depends on which chart we pick. (In our example above, if we chose the identity map to define our chart, we would have decided that $f(x) = |x|$ is not differentible at 0.) So, in order to *unambiguously* decide if a function is differentiable, we need to restrict which pairs (h, U) we are willing to allow ourselves to use as charts. This special collection of charts is the extra structure that we need.

What is the basic idea? We wish to find some way to ensure that if one of two charts (U,h) and (V,k) with $x \in U,V$ tells us that f is differentiable at x, then the other chart *must* do so, as well. That is, we wish to guarantee that $f\circ h^{-1}$ is differentiable at $h(x)$ iff $f\circ k^{-1}$ is differentiable at $k(x)$. And how to do this? The Chain Rule to the rescue! The thing which connects $f\circ h^{-1}$ to $f\circ k^{-1}$ is a *transition map* $k\circ h^{-1}$; $f\circ h^{-1} = (f\circ k^{-1})\circ(k\circ h^{-1})$. This equality holds on $h(U\cap V)$, which is the image under a homeo of an open subset of U containing x, so is an open subset of $\mathbb{R}^n$ contining $h(x)$. And if $k\circ h^{-1}$ is differentiable, then Chain Rule tell us that $f\circ k^{-1}$ differentiable at $k(x)$ implies $f\circ h^{-1}$ differentiable at $h(x)$. The reverse implication follows from knowing that $h\circ k^{-1}$ is differentiable.

This leads us to our basic construction. A $C^{(k)}$ *atlas* $\mathcal{A}$ on a topological manifold M is a collection (U_i,h_i) of charts on M so that (1) $\bigcup U_i = M$ and (2) for every i,j with $U_i\cap U_j\neq\emptyset$, $h_i\circ h_j^{-1} : h_i(U_i\cap U_j)\to h_j(U_i\cap U_j)$ is $C^{(k)}$, that is, has continuous partial derivatives through order k. Note that notationally, by reversing the roles of i and j, we are also insisting that $h_j\circ h_i^{-1}$ be $C^{(k)}$. Given a $C^{(k)}$ atlas $\mathcal{A}$ on a manifold M, we can then unambiguously define differentiatible functions, or $C^{(m)}$ functions for any $m\leq k$, $f : M\to\mathbb{R}$, by requiring that $f\circ h_i^{-1} : h(U_i)\to\mathbb{R}$ is $C^{(m)}$, for every i. More generally, given atlases on manifolds M,N, we can define a map $f : M\to N$ to be *differentiable* by requiring that $k_j\circ f\circ h_i^{-1}$ is differentiable for every k_j in the atlas for N and h_i in the atlas for M.

It will be useful to introduce some notation at this point, so that we don't have to keep writing " $h\circ k^{-1}$ is $C^{(k)}$ "; we will say that h and k are " $C^{(k)}$-related " if $h\circ k^{-1}$ and $k\circ h^{-1}$ are both $C^{(k)}$.

A $C^{(k)}$ atlas is enough to be able to define $C^{(m)}$ functions for $m\leq k$, but from a philosophical point of view, some atlases are better than others. If $f : M\to\mathbb{R}$ is a $C^{(m)}$ function and (h,U) is a chart on M, and $V\subseteq U$ is open, then $f\circ(h|_V)^{-1} : h(V)\to\mathbb{R}$, as the restriction of $f\circ h^{-1}$, is $C^{(m)}$. In fact, $h|_V$ is $C(k)$-related to every chart on M (if we started with a $C^{(k)}$ atlas), and so it doesn't hurt to add $h|_V$ to our atlas; it won't alter what functions we will call $C^{(m)}$. But it *might* actually help! We re all no doubt familiar with ϵ-δ arguments where we keep shrinking δ (effectively, shrinking the neighborhood of some point x) in order to make better things happen. The same will be true here; we will want to shink the domains of charts in order to make good things happen. It would be nice if such domains were already part of our atlas. So, we do the natural thing; just toss them in. And while we're at it, we might as well toss in everything that we can for free (without changing what we'll call a smooth map). This turns out to be everything which is $C^{(k)}$-related to *everything* already in our atlas. This is also the *largest* $C^{(k)}$ atlas which contains our original atlas. Such an atlas is called a *maximal* atlas.

A $C^{(k)}$ **structure** on a manifold M, $0\leq k\leq\infty$, is a maximal $C^{(k)}$ atlas on M. M, together with a $C^{(k)}$ structure, will be called a $C^{(k)}$ manifold. A $C^{(0)}$ manifold is "just" a manifold; a $C^{(0)}$ structure is a collection of homeomorphisms from the sets of an open cover of M to $\mathbb{R}^n$ (that the transition maps are $C^{(0)}$, i.e., continuous, is automatic). In general we will content ourselves to study $C^{(\infty)}$ structures on manifolds, but it is important to know that there are other possible choices. (When an author never needs anything beyond a second derivative, they will often talk only about $C^{(2)}$ manifolds, for example. It is a fact (see, e.g., Hirsch, *Differential Topology*, p.51) that for every $1 \leq r \leq s \leq \infty$, a $C^{(r)}$ structure $\mathcal{A}$ on a manifold M *contains* a $C^{(s)}$ structure $\mathcal{B} \subseteq \mathcal{A}$; that is, $\mathcal{A}$ contains an atlas which is $C^{(s)}$-compatible. But we will likely not use this result.)

Our standard examples of manifolds above also provide some standard examples of smooth manifolds; one merely needs to verify that the charts that we built are $C^{(\infty)}$-related, so that the have an atlas, and then wave our magic wand to 'build' the corresponding maximal atlas. Restriction to an open set and Cartesian product both preserve smoothness, so we have several general approaches to building smooth manifolds at our fingertips.

Just as in topology we have a notion, homeomorphism, which allows us to treat two spaces as essentially the "same", there is a corresponding notion of same in the smooth setting. Two $C^{(k)}$ manifolds $(M, \mathcal{A}), (N, \mathcal{B})$ are *diffeomorphic* if there is a $C^{(k)}$ bijection $f : M \to N$ with $C^{(k)}$ inverse. Just as with a homeomorphism, a diffeomorphism induces a bijection between charts of M and N, via $h : U \to \mathbb{R}^n$, for $U \subseteq M$, is taken to $h \circ f^{-1} : f(U) \to \mathbb{R}^n$. Because f^{-1} is $C^{(k)}$, this map is $C^{(k)}$, hence is in the (maximal) atlas $\mathcal{B}$.

Just as in the "standard" definition of topology, the field of differential topology can be most succintly described as the study of the properties of smooth manifolds that are invariant under diffeomorphism (i.e., are defined in terms of the smooth structure). You will have learned in the homework that a given manifold can have many different smooth structures, menaing that the atlases defining them are distinct. (Note that the maximality requirement in fact implies that distinct atlases are disjoint.) But in many cases these atlases can still define the 'same' smooth structure, that is, they are diffeomorphic. In particular, up to diffeomorphism, $\mathbb{R}$,$\mathbb{R}^2$,$\mathbb{R}^3$, and $\mathbb{R}^n$ for $n \geq 5$ all have unique differentiable structure. It was a major breakthrough of the mid-1980's that $\mathbb{R}^4$ was discovered to have more than one smooth structure; it in fact has uncountably many non-diffeomorphic smooth structures. Every 2-manifold has a unique smooth structure up to diffeo; the same is true for 3-manifolds, as well (Moise, 1950's). But there actually exist 4-manifolds which posess *no* smooth structure. This was first discovered as a result of work of Freedman and Donaldson (for which both received the Fields Medal in 1986). Freedman showed that simply-connected (meaning every map of a circle into M extends to a map of a disk) topological 4-manifolds were determined up to homeo by their 'intersection pairing on second homology' (whatever that is), and further, every unimodular symmetric bilinear pairing has a corresponding manifold. This, by the way, implies the topological 4-dimensional Poincaré conjecture. Donaldson, on the other hand, showed that for simply-connected *smooth* 4-manifolds, certain intersection pairings could not arise. His work essentially involved PDE's on 4-manifolds. In particular, the pairing "E8" could not occur. So the 4-manifold "E8", which Freedman's work shows exists, has no smooth structure.

On the other hand, there are manifolds which have 'too many' smooth structures, i.e., admit multiple structures which are not diffeomorphic to one another. $\mathbb{R}^4$ is the most famous these days, but it turns out that most spheres have this property, as well. In the late 1950's John ('Jack') Milnor showed that S^7 has more than one smooth structure; it was later shown that it has exactly 28 non-diffeomorphic structures. S^{31} has more than 16 million! And in case you think these structures are really wierd things that you are never likely to meet, the 28 structures in S^7 arise on the links of singularities of algebraic surfaces, specifically, as the intersection of the solutions (in $\mathbb{C}^5$) to the equation

$$a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0$$

with a small sphere centered at the origin, for $k = 1, \dots, 28$, gives all 28 exotic 7-spheres (source: the Wikipedia entry for 'exotic sphere'). While wandering the web, I found an assertion (by Ron Stern) that 'all known 4-manifolds have infinitely many distinct smooth structures', but I am not sure how to interpret that...

So what do you do when you have a smooth structure? Start building smooth maps! We know how to identify a smooth map $f : M^n \to N^m$; we must have $h \circ f \circ k^{-1} : K(V) \to h(U)$ smooth for every pair of charts on M and N. Note that it is enough, though, to verify this for charts in a pair of atlases contained in the smooth structures for M and N; the compatibility of every other chart in our smooth structure with those of the atlases will guarantee smoothness of $h \circ f \circ k^{-1}$ over the entire maximal atlas. (Note also that this does *not* contradict what you've shown in one of your homework problems!) So, for example, to verify that some function $f : S^5 \to S^8$ (using the standard smooth structures!) is smooth, it suffices to use an atlas consisting to two charts on each (the stereographic projections from the poles), so smoothness can be verified by examining only 4 functions from $\mathbb{R}^5$ to $\mathbb{R}^8$. Actually verifying that such functions *are* smooth we are going to mostly leave to the same slightly fuzzy realm one encounters in calculus: if it is built up out of functions that we "know" are smooth, then it is smooth wherever it is defined.

One thing that can help us in things is to recognize that smoothness is local. This is just like in topology, where continuity is local; if $f : M \to N$ is a map such that for every $x \in X$ there is a chart (h, U) for M with $x \in U$ and a chart (k, V) for N with $f(x) \in V$, and $h \circ f \circ k^{-1}$ is smooth (where it is defined), then f is smooth. This is simply because the h's and the k's form atlases for M and N, respectively. But if you turn it around it can be thought of as a prescription for building a smooth function, by patching together smooth functions defined on open sets; if $\mathcal{O}$ is an open cover of M, and for each $U \in \mathcal{O}$ we have a smooth map $f_U : U \to N$ such that $f_U = f_V$ on $U \cap V$ for every $U, V \in \mathcal{O}$, then the map $f : M \to N$ defined by '$f(x) = f_U(x)$ if $x \in U$' is smooth. This is the direct analogue of the Gluing Lemma from topology. Of course, in topology, one more often wants to glue together maps defined on *closed* sets, rather than open sets; it is less messy. But in the smooth setting things aren't nearly so nice; on $\mathbb{R}$ the function $f(x) = |x|$ can be obtained by gluing together two smooth functions, but it is not smooth (using the standard smooth structures!) Question: are there *other* smooth structures on $\mathbb{R}$ for which f *is* smooth?

We also have many of the standard results. The composition of two smooth maps is smooth; this is essentially just because the corresponding result is true for maps between Euclidean spaces. The sum, difference, and product of two smooth maps $M \to \mathbb{R}$ are all smooth; again, this is basically because this is true for maps from $\mathbb{R}^n$ to $\mathbb{R}$. And the quotient is smooth so long as the denomenator is never zero. And a map into a Cartesian product of smooth manifolds (using the product smooth structure) is smooth iff the map into each factor is smooth (i.e., the composition with projection onto each factor is smooth). This last fact you have probably already had to use, since to decide on the smoothness of $h \circ f \circ k^{-1} : K(V) \to h(U) \subseteq \mathbb{R}^m$, you had to look at each of the m coordinate functions (projecting onto each coordinate factor $\mathbb{R}$). But some things *don't* work; for example the maximum $\max\{f, g\}$ of two smooth functions (mapping to $\mathbb{R}$) *need not* be smooth; $h(x) = |x|$, for example, can be defined as the maximum of the functions $f(x) = x$ and $g(x) = -x$.

There will be many situations in the material to come where we will want to assemble information obtained locally into a single smooth map $f : M \to N$. To do so, we will introduce the notion of a *partition of unity*; this is a way of writing the function $f(x) = 1$ as a sum of smooth functions.