Continued fractions: doing natural things with the Euclidean algorithm

If $(a,b)=1$ then the E.A.

$$\cancel{b=r_1q_1+r_1} \quad a = bq_1 + r_1 \qquad \frac{a}{b} = q_1 + \frac{r_1}{b}$$

$$\cancel{q_1 =} \; b = r_1 q_2 + r_2 \qquad \frac{b}{r_1} = q_2 + \frac{r_2}{r_1}$$

$$\Longrightarrow \quad \frac{r_1}{b} = \frac{1}{q_2 + \frac{r_2}{r_1}}$$

$$\frac{a}{b} = q_1 + \frac{1}{q_2 + \frac{r_2}{r_1}}$$

$$r_1 = r_2 q_3 + r_3 \qquad \frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2}$$

$$\frac{a}{b} = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{r_3}{r_2}}}$$

when this process terminates, $r_n = 0$, we have

$$\frac{a}{b} = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{\ddots + \frac{1}{q_n}}}}$$

This is what the quotients in the E.A "mean"!

Notation: $\frac{a}{b} = \langle q_1, q_2, \dots, q_n \rangle = [q_1, \dots, q_n]$

(Lovett's notation) ← more usual notation

This gets more interesting, though, if we try this with <u>any</u> real number x, ie. "compute" $\gcd(x,1)$ using E.4!

$$x = 1 \cdot a_0 + r_0 \qquad 0 \le r_0 < 1 \qquad a_0 = \lfloor x \rfloor = \text{greatest integer}$$

$$1 = r_0 a_1 + r_{\cancel{2}1} \qquad 0 \le r_1 < r_0$$

$$r_0 = r_1 a_2 + r_2$$

fractional part

ie. $$\frac{x}{1} = \frac{a_0}{1} + \frac{r_0}{1}$$

$$\frac{1}{r_0} = a_1 + \frac{r_1}{r_0} \qquad a_1 = \left\lfloor \frac{1}{r_0} \right\rfloor \qquad \cancel{r_1 = r_0\left(\frac{1}{r_0} - \left[\frac{1}{r_0}\right]\right)}$$

$$\frac{r_1}{r_0} = \frac{1}{r_0} - \left\lfloor \frac{1}{r_0} \right\rfloor$$

$$\frac{r_0}{r_1} = a_2 + \frac{r_2}{r_1}$$

$$a_2 = \left\lfloor \frac{r_0}{r_1} \right\rfloor = \left\lfloor \frac{1}{r_1/r_0} \right\rfloor$$

$$x = \left\lfloor \frac{x}{1} \right\rfloor + \frac{r_0}{1} = a_0 + \frac{r_0}{1} = a_0 + \frac{1}{1/\left(\frac{r_0}{1}\right)}$$

$$= a_0 + \frac{1}{a_1 + r_1/r_0} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\left(1/\frac{r_1}{r_0}\right)}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + r_2/r_1}}$$

§ The basic procedure:

Given $x \in \mathbb{R}$ set $a_0 = \lfloor x \rfloor$

$\cancel{X_0} = x - a_0 \geq 0$ if $x_0 = 0$ stop. (note: $r_0 < 1$)

If $r_0 > 0$ set $\cancel{\frac{1}{r_0}} = [a_1 + [x_1$ $\left(\cancel{x_1 = \lfloor \frac{1}{r_0} \rfloor} \right.$

$a_1 = \lfloor \frac{1}{r_0} \rfloor \geq 1$, $x_1 = \frac{1}{r_0} - \lfloor \frac{1}{r_0} \rfloor < 1$.

Continue:

so long as $x_n \geq 0$, set $a_{n+1} = \lfloor \frac{1}{x_n} \rfloor$, $x_{n+1} = \frac{1}{x_n} - \lfloor \frac{1}{x_n} \rfloor$

Then

$$x = a_0 + x_0 = a_0 + \cfrac{1}{1/x_1} = a_0 + \cfrac{1}{a_1 + x_1}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{(1/x_1)}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + x_2}}$$

So $x = \langle a_0, a_1, \ldots, a_{n-1}, a_n + x_n \rangle = \langle a_0, a_1, \ldots, a_n, \frac{1}{x_n} \rangle$

The numbers $\langle a_0, \ldots, a_n \rangle$ are rational #s (Pf; induction!), which are called the (n^{th}) ~~partial quotients~~ convergent of x. a_n = n^{th} partial quotient

$$\sqrt{3} = 1 + (\sqrt{3}-1) = 1 + \frac{2}{\sqrt{3}+1}$$
$$= 1 + \frac{1}{}$$

$\sqrt{3} = 1 + \sqrt{3}-1 \qquad a_0 = 1 \qquad x_0 = \sqrt{3}-1$

$$\frac{1}{x_0} = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2} = a_1 + x_1$$

$$\frac{1}{x_1} = \frac{2}{\sqrt{3}-1} = \sqrt{3}+1 = 2 + (\sqrt{3}-1) = a_0 + x_2$$

$\underline{\underline{x_2 = x_0}}$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cancel{2}\,(\sqrt{3}-1)}}}}$$

$$= \langle 1, 1, 2, 1, 2, 1, 2, \cancel{\tfrac{\sqrt{3}+1}{2}} \ldots, 1, 2, \frac{\sqrt{3}+1}{2} \rangle$$

$x=\sqrt{7}$

$\sqrt{7} = 2 + (\sqrt{7}-2) = a_0 + x_0$

$\frac{1}{x_0} = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} = a_1 + x_1$

$\frac{1}{x_1} = \frac{3}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2}$

$\frac{1}{x_2} = \frac{2}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3}$

$\frac{1}{x_3} = \frac{3}{\sqrt{7}-2} = \sqrt{7}+2 = 4 + \sqrt{7}-2$

$\langle 2,1,1,1,4,\frac{\sqrt{7}+2}{3} \rangle = \langle 2,1,1,1,4,1,1,1,4,\frac{\sqrt{7}+2}{3} \rangle$

<u>Some theory</u>: A continued fraction $\langle a_0, \dots, a_n \rangle$ is called <u>simple</u> if $a_i \in \mathbb{Z}$ all i, ~~$a_i \geq 1$ for $a_i \geq 1$~~
$a_i \geq 1$ for $i \geq 1$.

<u>Note</u>: $\langle a_0, \dots, a_n \rangle = \langle a_0, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n} \rangle = a_0 + \frac{1}{\langle a_1, \dots, a_n \rangle}$

So, e.g. $\langle a_0, \dots, a_n \rangle = \langle a_0, \dots, a_{n-1}, a_n - 1, 1 \rangle$

<u>But</u> that's the <u>only</u> way to get equality!

<u>Prop</u>: If the simple cont fractions

$$\langle a_0, \dots, a_j \rangle = \langle b_0, \dots, b_n \rangle \quad \text{have}$$

<u>DO</u>

$a_j, b_n > 1$, then $j = n$ and $a_i = b_i$ for all i.

<u>Pf</u>: Set

~~$y_i = \langle b_i, \dots, b_n \rangle = b_i + \frac{1}{\langle b_{i+1}, \dots, b_n \rangle} = b_i + \frac{1}{y_{i+1}}$~~

~~$b_i \geq 1$ for $i = 0, \dots, n-1$~~

~~and $y_n = b_n > 1$. So $y_i > 1$ for all i.~~

The basic idea:

$$\langle a_0, \dots, a_j \rangle = a_0 + \frac{1}{\langle a_1, \dots, a_j \rangle} \quad \text{with } 0 < \frac{1}{\langle a_1, \dots, a_j \rangle} < 1$$

so $a_0 = \lfloor \langle a_0, \dots, a_j \rangle \rfloor$

and $b_0 = \lfloor \langle b_0, \dots, b_n \rangle \rfloor$ so $a_0 = b_0$ <u>so</u>

$$\frac{1}{\langle a_1, \dots, a_j \rangle} = \frac{1}{\langle b_1, \dots, b_n \rangle} \quad \text{so} \quad \langle a_1, \dots, a_j \rangle = \langle b_1, \dots, b_n \rangle$$

which are <u>shorter</u>. So induction $\Rightarrow$ the rest!

... $j = 0$ or $n = 0$.