Nathan Dunfield, Laminations and groups of homeos of S^1.

M^3 closed, orientable, irreducible, with π_1 infinite

Q: when is $\pi_1(M)$ a subgroup of $\mathrm{Homeo}(S^1)$?

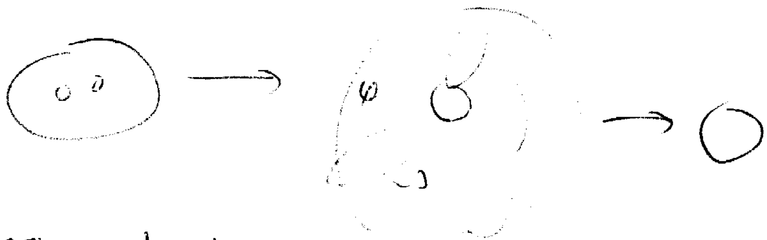
A: Often but not always.

To fully understand $\pi_1(M)$, must understand actions of $\pi_1(M)$ on interesting spaces ($\mathbb{C}^n$, manifolds, $\mathbb{H}^3$, etc)

Start w/ simplest mfld, S^1.

There are common codim-1 objects (foliations and laminations) which give rise to faithful actions on S^1.

Ex Bundle / S^1



$\pi_1(M_\varphi) = \langle \pi_1(\Sigma), t \mid t^{-1} g t = \phi_*(g) \ \forall g \in \pi_1(\Sigma) \rangle$

$\tilde{M}_\varphi \cong \tilde{\Sigma} \times \mathbb{R}$ with coords (p, r)

$\pi_1(\Sigma) \triangleleft \pi_1(M_\varphi)$ acts fixing r coord.

t acts via $(p, r) \longrightarrow (\tilde{\Phi}^{-1}(p), r+1)$

Fix a metric on Σ, then $\widehat{\Sigma} = \mathbb{H}^2 \subseteq \mathbb{H}^2 \cup S^1_\infty$.

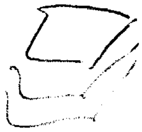
Set $\widetilde{M}_\phi = \mathbb{H}^2 \times \mathbb{R}$, it has a $\pi_1(M_\phi)$ action.

Projection. $p : S^1_\infty \times \mathbb{R} \longrightarrow S^1_{univ}$ to the universal circle

Then $g \in \pi_1(M)$ acts on S^1_{univ} by

$$\begin{tikzcd} S^1_{univ} \arrow[r, "i"] & S^1_\infty \times \{0\} \arrow[d, "g"] \\ & S^1_\infty \times \{r\} \arrow[ul, "p"] \end{tikzcd}$$

Foliation: partition of M into surfaces (the leaves) locally like

$\mathbb{R}^3 \supseteq$  leaves $\mathbb{R}^2 \times \{r\}$ $r \in I$.

Taut: There exists a loop in M, transverse to foliation, intersecting every leaf.

Taut implies $\pi_1(M)$ infinite, $\widehat{M} \cong \mathbb{R}^3$

If M is atoroidal, $\pi_1(M)$ is Gromov hyperbolic (Gabai-Kazez)
(Calegari)

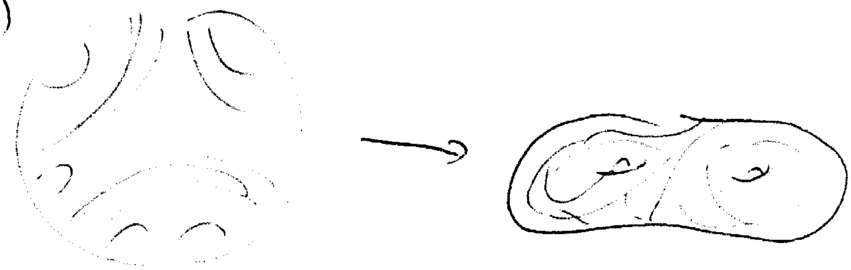
W. Thurston (1997) If M has a taut foliation then there exists an action of $\pi_1(M)$ on S^1_{univ}. If M is atoroidal, then the action is faithful.

Lamination: union of leaves locally a product, fills a closed set.

Essential lamination: [I know the def'n...]

Ex: ϕ a ψ-Anosov homeo of Σ^2. Let λ be ~~the~~ an invariant geodesic lamination.

$(\hat{\Sigma}, \hat{f})$



(Σ, λ)

In M_φ have essential lamination

$$\Lambda = (\lambda \times I) / (p,1) \sim (\varphi(p), 0)$$

The complementary regions to Λ are finite-sided polygon bundles over S^1 (basically, solid tori)

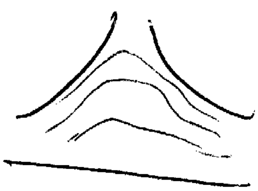
Thm [Calegari–Dunfield] Let M be an atoroidal 3-mfld containing an ess lam Λ. Suppose

- $M \setminus \Lambda$ has solid torus guts
- Λ is tight

Then $\pi_1(M)$ acts faithfully on S^1.

Tight: Leaf space in universal cover is Hausdorff.

No:



Ex: $\Lambda \subseteq M_\varphi$ suspension lamination.

<u>Pf Sketch</u>: Reduce to case compl. regions of Λ are ideal polygon bdles (filling)

Look at $\hat{\Lambda} \subseteq \tilde{M}$. By Gabai-Kazez, $(\hat{M}, \hat{\Lambda}) \cong (\mathbb{R}^2, \lambda) \times \mathbb{R}$

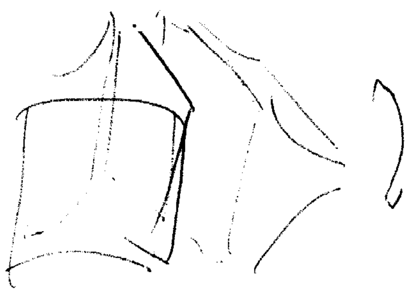
As compl regions of $\hat{\Lambda}$ are idea polygons $\times \mathbb{R}$, the lamination of λ looks like a geodesic lam of $\mathbb{H}^2$.

Idea: Make $\pi_1(M)$ act on $(\mathbb{R}^2, \lambda)$ so that

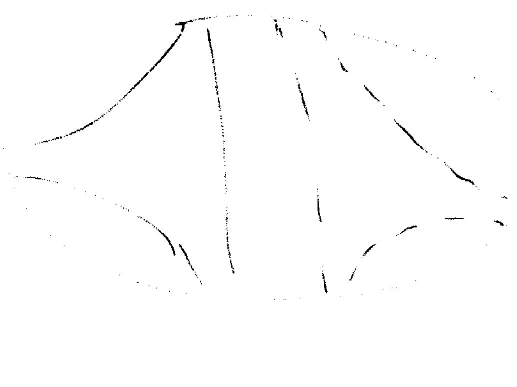
$(\hat{M}, \Lambda) \to (\mathbb{R}^2, \lambda)$ is equivariant

For the action on circle, go to S^1_∞ from $(\mathbb{R}^2, \lambda)$.

The main issue: There are many ways to flatten $(\hat{M}, \hat{\Lambda})$ to $(\mathbb{R}^2, \lambda)$.



flip which way?



Solution: downstairs, orient core curves of components of $M \setminus \Lambda$
now flatten & lifted orientation "points up".

Since the orientations are equivariant, the flattening becomes essentially canonical.
(this uses tight)

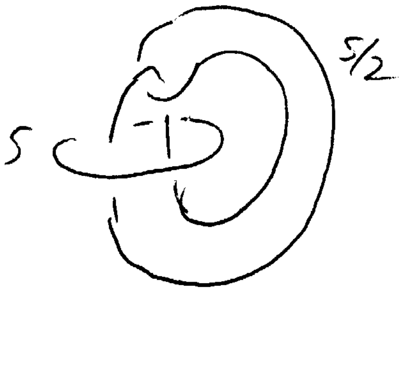
Use atoroidal to get action to be faithful.

M^3 closed, or'ble, irred, with infinite π_1.

Q: When is $\pi_1(M)$ a subgroup of $\mathrm{Homeo}(S^1)$?

A often, but not always

Thm [Calegari - Dunfield] Let W be the Weeks mfld. Then $\pi_1(W)$ does not act faithfully on S^1. In fact, any ~~homo~~ homom $\pi_1(W) \to \mathrm{Homeo}(S^1)$ has image $< \mathbb{Z}_5$.



Cor: W does not have ~~ess~~ taut foln, a tight ess lam, or a Ψ Anosov flow.

(Don't need to say 'w/ solid torus guts', b/c, by Agol, vol is too low to have anything else.

Questions. M^3 closed, orible, irred., atoroidal, with π_1 infinite

Q: Does M have a finite cover N with $\pi_1(N) < \text{Homeo}(S^1)$
(Yes, if virt 1^{st} Betti # conj true...)

Q. Suppose $\pi_1(M)$ acts faithfully on S^1, preserving a lamination.
Does this tell us something else interesting about $\pi_1(M)$?

Q Do lattices of rank 1.5 groups act faithfully on S^1?
(Navas showed there are no C^2-actions.)