Math 971 Algebraic Topology

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We typically think of building a Δ-complex X inductively. The *0-simplices* (i.e., points), or *vertices*, form the 0-skeleton $X^{(0)}$. n-simplices $\sigma^n = [v_0, \dots v_n]$ attach to the $(n-1)$-skeleton to form the n-skeleton $X^{(n)}$; the restriction of the attaching map to each face of σ^n is, by definition, an $(n-1)$-simplex in X. The attaching map is (by induction) really determined by a map $\{v_0, \dots, v_n\} \to X^{(0)}$, since this determines the attaching maps for the 1-simplices in the boundary of the n-simplex, which gives 1-simplices in X, which then give the attaching maps for the 2-simplices in the boundary, etc. Note that the reverse is not true; the vertices of two different n-simplices in X can be the same. For example, think of the 2-sphere as a pair of 2-simplices whose boundaries are glued by the identity.

The final detail that we need before defining (simplicial) homology groups is the notion of an *orientation* on a simplex of X. Each simplex σ^n is determined by a map $f : \{v_0, \dots, v_n\} \to X^{(0)}$; an orientation on σ^n is an (equivalence class of) the ordered $(n+1)$-tuple $(f(v_0), \dots f(v_n)) = (V_0, \dots, V_n)$. Another ordering of the same vertices represents the same orientation if there is an *even* permutation taking the entries of the first $(n+1)$-tuple to the second. This should be thought of as a generalization of the right-hand rule for $\mathbb{R}^3$, interpreted as orienting the vertices of a 3-simplex. Note that there are precisely two orientations on a simplex.

Now to define homology! We start by defining *n-chains*; these are (finite) formal linear combinations of the (oriented!) n-simplices of X, where $-\sigma$ is interpreted as σ with the opposite (i.e., other) orientation. Adding formal linear combinations formally, we get the n-th *chain group* $C_n(X) = \{\sum n_\alpha \sigma_\alpha : \sigma_\alpha$ an oriented n-simplex in $X\}$. We next define a *boundary operator* $\partial : C_n(X) \to C_{n-1}(X)$, whose image will be the $(n-1)$-chains that are the "boundaries" of n-chains. We define it on the basis elements $\sigma_\alpha = \sigma$ of $C_n(X)$ as $\partial\sigma = \sum(-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}$, where $\sigma : [v_0, \dots, v_n] \to X$ is the characteristic map of σ_α . $\partial\sigma$ is therefore an alternating sum of the faces of σ. The pont that really make this definition go is that we need *oriented* simplices, so that we know what the i-th face of σ is (the one oppoisite the i-th vertex). We then extend the definition by linearity to all of $C_n(X)$. When a notation indicating dimension is needed, we write $\partial = \partial_n$. We define $\partial_0 = 0$.

This definition is cooked up to make the maxim "boundaries have no boundary" true; that is $\delta_{n-1} \circ \delta_n = 0$, the 0 map. This is because, for any simplex $\sigma = [v_0, \dots v_n]$,

$$\delta \circ \delta(\sigma) = \delta(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]})$$
$$= (\sum_{j<i} (-1)^j (-1)^i \sigma|_{[v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_n]}) + (\sum_{j>i} (-1)^{j-1} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]})$$

The distinction between the two pieces is that in the second part, v_j is actually the $(j-1)$-st vertex of the face. Switching the roles of i and j in the second sum, we find that the two are negatives of one another, so they sum to 0, as desired.

And this little calculation is all that it takes to define homology groups! What this tells us is that $\mathrm{im}(\delta_{n+1}) \subseteq \ker(\delta_n$ for every n. $\ker(\delta_n = Z_n(X)$ are called the *n-cycles* of X; they are the n-chains with 0 (i.e., empty) boundary. They form a (free) abelian subgroup of $C_n(X)$. $\mathrm{im}(\delta_{n+1} = B_n(X)$ are the *n-boundaries* of X; they are, of course, the boundaries of $(n+1)$-chains in X. The n-th homology group of X, $H_n(X)$ is the quotient $Z_n(X)/B_n(X)$; it is an abelian group.

A key observation is that the boundary maps δ_n are linear, that is, they are homomorphisms between the free abelian groups $\delta_n : C_n(X) \to C_{n-1}(X)$. Consequently, they can be expressed as (integer-valued) matrices Δ_n. Row reducing Δ_n (over the integers!) allows us to find a basis $v_1, \dots, v_k$ for $Z_n(X)$ (clearing denomenators to get vectors over $\mathbb{Z}$). Then since $\Delta_n\Delta_{n+1} = 0$, the columns of Δ_{n+1} are in the kernel of Δ_n, so can be expressed as linear combinations of the v_i . These combinations can be determined by row reducing the augmented matrix $(v_1 \cdots v_k | \Delta_{n+1})$. This will row reduce to

$\left(\begin{array}{c|c} I & C \\ 0 & 0 \end{array}\right)$, and C basically describes the boundaries $B_n(X)$ in terms of the basis $v_1, \dots, v_k$. The homology group $H_n(X)$ is then the *cokernel* of C, i.e., $\mathbb{Z}^k/\mathrm{im}C$. Note that C will have integer entries, since we know that the columns of Δ_{n+1} can be expressed as integer linear combinations of the v_i, and, being a basis, there is only one such expression.

For example, the Klein bottle K has a Δ-complex structure with 2 2-simplices, 3 1-simplices, and 1 0-simplex; we will call them $f_1 = [0,1,2], f_2 = [1,2,3], e_1 = [0,2] = [1,3], e_2 = [1,0] = [2,3], e_3 = [1,2]$, and $v_1 = [0] = [1] = [2] = [3]$. Computing, we find $\partial_2 f_1 = \partial[0,1,2] = [1,2] - [0,2] + [0,1] = e_3 - e_1 - e_2$, $\partial_2 f_2 = e_2 - e_1 + e_3$, $\partial_1 e_1 = \partial_1 e_2 = \partial_1 e_3 = 0$ and $\partial_i = 0$ for all other i (as well). So we have the chain complex
$\cdots \to 0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$

and all of the boundary maps are 0, except for ∂_2, which has the matrix $\begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$. This matrix is injective, so $\ker \partial_2 = 0$, so $H_2(K) = 0$, on the other hand, $H_1(K) = \mathrm{coker}(\partial_2)$, and applying column operations we can transform the matrix for ∂_2 to $\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 0 \end{pmatrix}$, which implies that the cokernel is $\mathbb{Z} \oplus \mathbb{Z}_2$, since $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis for $\mathbb{Z}^3$. Finally, $H_0(K) = \mathbb{Z}$, since $\partial_1, \partial_0 = 0$, and all higher homology groups are also 0, for the same reason.

As another example, the topologist's dunce hat has a Δ-structure with 1 2-simplex, 1 1-simplex, and 1 0-simplex. The boundary maps, we can work out (starting from $C_2(X)$), are $(1), (0)$, and (0), so $H_2(X) = H_1(X) = 0$, and $H_0(X) = \mathbb{Z}$. all higher groups are also 0.

These homology groups are, in the end, fairly routine to calculate from a Δ-complex structure. But there is one very large problem; the calculations <u>depend</u> on the Δ structure! This is not a group defined from the space X; it is defined from the space and a Δ structure on it. A priori, we don't know that if we chose a different structure on the same space, that we would get isomorphic groups! We should really denote our groups by $H_i^{\Delta}(X)$, to acknowledge this dependence on the structure.

But we don't *want* a group that depends on this structure. We want groups that just depend on the topological space X, i.e., which are topological invariants. In really turns out that these groups $H_i^{\Delta}(X)$ <u>are</u> topological invariants, but we will need to take a very roundabout route to show this. What we will do now is to define another sequence $H_i(X)$ of groups, the *singular homology groups*, which their definition makes apparent from the outset that they are topological invariants. But this definition will also make it very unclear how to really compute them! Then we will show that for Δ-complexes these two sequences of groups are really the same. In so doing, we will have built a sequence of topological invariants that for a large class of spaces are fairly routine to compute. Then all we will need to show is that they also capture useful information about a space (i.e., we can prove useful theorems with them!).

And the basic idea behind defining them is that, with simplicial homology, we have already done all of the hard work. What we do is, as before, build a sequence of (free) abelian groups, the chain groups $C_n(X)$, and boundary maps between them, with consecutive maps composing to 0. Then, as before, the homology groups are kernels mod images, i.e., cycles mod boundaries. And, as before, the basis elements for each of our chain groups $C_n(X)$ will be the n-simplices in X. But now X is <u>any</u> topological space. So how do we get n-simplices in such a space? We do the only thing we can; we *map* them in.

More precisely, we work with *singular n-chains*, that is, formal (finite) linear combinations $\sum a_i\sigma_i$, where $a_i \in BbbZ$ and the σ_i are *singular simplices*, that is, (continuous) maps $\sigma_i : \Delta^n \to X$ from the (standard) n-simplex into X. The boundary maps are really exactly as before; they are the alternating sum of the restrictions of σ_i to the $n+1$ faces of Δ^n . (Formally, we must precompose these face maps with the (orientation-preserving) linear isomorphism from the standard $(n-1)$-simplex to each of the faces, preserving the ordering of their vertices.) The same proof as before (except that we interpret the faces as restrictions of the map σ_i, instead of as physical faces) shows that the composition of two successive boundaries are 0, and so all of the machinery is in place to define the *singular homology groups* $H_i(X)$ as the kernel of ∂_i modulo the image of $\partial_{i+1} = Z_i(X)/B_i(X)$. They are, by their definition, groups defined using the topological space X as input, and so are topological invariants of X. The elements are equivalence classes of i-cycles, where $z_1 \sim z_2$ if $z_1 - z_2 = \partial w$ for some $(i+1)$-chain w . We say that z_1 and z_2 are *homologous*.

The fun comes when you try to compute them. $C_n(X) = \{\sum a_i\sigma_i : a_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ is continuous$\}$ is typically a <u>huge</u> group, since there will be immense numbers of maps $\Delta^n \to X$. About the only space for which this is not true is the one-point space $*$; then there are, for each n, exactly one (distinct) map $\sigma_n : \Delta^n \to *$; the constant map. Therefore each face of Δ^n gives the same restriction map σ^{n-1}, and so the boundary maps can be dirctly computed (the depend on the parity of the number $n+1$ of faces an n-simplex has). We find that $\partial_{2n} = Id$ and $\partial_{2n-1} = 0$. so in computing homology groups, we either have kernel everything ($\partial_i = 0$) and image everything ($\partial_{i+1} = Id$) or kernel nothing ($\partial_i = Id$) and image nothing ($\partial_{i+1} = 0$), so in both cases $H_i(*) = 0$. Except for $i = 0$; then $\partial_0 = 0$ (by definition) and $\partial_1 = 0$, so $H_0(*) = \mathbb{Z}$. But other than this example (and, well, OK, spaces with the discrete topology; it's the same calculation as above for every point!), computing singular homology from the definition is quite a chore! so we need to build some labor-saving devices, namely, some theorems to help us break the problem of computing these groups into smaller, more managable pieces.

First set of managable pieces: if we decompose X into its path components, $X = \bigcup X_\alpha$, then $H_i(X) \cong \bigoplus H_i(X_\alpha)$ for every i. This is because every singular simplex, since Δ^i is path-connected, maps into some X_α . So $C_i(X) \cong \bigoplus C_i(X_\alpha)$. Since the boundary of a simplex mapping into X_α consists of simplices in X_α, the boundary maps respect the decomposistions of the chain groups, so $B_i(X) \cong \bigoplus B_i(X_\alpha)$ and $Z_i(X) \cong \bigoplus Z_i(X_\alpha)$, and so the quotients are $H_i(X) \cong \bigoplus H_i(X_\alpha)$.

So, if we wish to, we can focus on computing homology groups for path-connected spaces X. For such a space, $H_0(X) \cong \mathbb{Z}$, generated by any map of a 0-simplex (= a point) into X. This is because any pair of 0-simplices are homologous; given any two points $x, y \in X$, there is a path $\gamma : I \to X$ from x to y, This path can be interpreted as a singular 1-simplex, and $\partial\gamma = y - x$. So $H_0(X)$ is generated by a single point $[x]$. No multiple of this point is null-homologous, because for any 1-chain $\sum n_i\sigma_i$, the sum of the coefficients of its boundary is 0 (since this is true for each singular 1-simplex), and any 0-chain $\sum n_i[x_i]$ is homologous to $(\sum n_i)[x]$ by the above argument.

Perhaps the most important property of the fundamental group is that a continuouos map between spaces induces a homomorphism between groups. (This implied, for instance, that homeomorphic spaces have isomorphic π_1). The same is true for homology groups, for essentially the same reason. Given a map $f : X \to Y$, there is an induced map $f_\# : C_n(X) \to C_n(Y)$ defined by postcomposition; for a singular simplex σ, $f_\#(\sigma) = f \circ \sigma$, and we extend the map linearly. Since $f \circ (g|_A) = (f \circ g)|_A$ (postcomposition commutes with restriction of the domain), $f_\#$ commutes with ∂ : $f_\#(\partial\sigma) = \partial(f_\#(\sigma))$. A homomorphism between chain complexes (i.e., a sequence of such maps, one for each chain group) which commutes with the boundaries maps in this way, is called a *chain map*. A chain map, such as $f_\#$, therefore, takes cycles to cycles, and boundaries to boundaries, and so $f_\# : Z_i(X) \to Z_i(Y)$ (which is linear, hence a homomorphism) induces a homomorphism $f_* : H_i(X) \to H_i(Y)$ by $f_*[z] = [f_\#(z)]$. Since it is defined by composition with singular simplices, it is immediate that, for a map $g : Y \to Z$, $(g \circ f)_* = g_* \circ f_*$. And since the identity map $I : X \to X$ satisfies $I_\# = Id$, so $I_* = Id$, homeomorphic spaces have isomorphic homology groups.

Another important property of π_1 is that homotopic maps give the same induced map (after correcting for basepoints). This is also true for homology; if $f \sim g : X \to Y$, then $f_* = g_*$. The proof, however, is not quite as straightforward as for homotopy. And it requires some new technology; the chain homotopy. A chain homotopy H between the chain complexes $f_\#, g_\# : C_*(X) \to C_*(Y)$ is a sequence of

homomorphisms $H_i : C_i(X) \to C_{i+1}(Y)$ satisfying $H_{i-1}\partial_i + \partial_{i+1}H_i = f_\# - g_\# : C_i(X) \to C_i(Y)$. The existence of H implies that $f_* = g_*$, since for an i-cycle z (with $\partial_i(z) = 0$) we have
$f_*[z] - g_*[z] = [f_\#(z) - g_\#(z)] = [H_{i-1}\partial_i(z) + \partial_{i+1}H_i(z)] = [H_{i-1}(0) + \partial_{i+1}(w)] = [\partial_{i+1}(w)] = 0.$
And the existence of a homotopy between f and g implies the existence of a chain homotopy between $f_\#$ and $g_\#$. This is because the homotopy gives a map $H : X \times I \to Y$, which induces a map $H_\# : C_{i+1}(X \times I) \to C_{i+1}(Y)$. Then we pull, from our back pocket, a *prism map* $P : C_i(X) \to C_{i+1}(X \times I)$; the composition $H_\# \circ P$ will be our chain homotopy. The prism map takes a (singular) i-simplex σ and sends it to a sum of singular $(i+1)$-simplices in $X \times I$. and the way we define it is to take the i-simplex Δ^i, and taking it to $\Delta^i \times I$ (i.e., a *prism*), and thinking of this as a sum of $(i+1)$-simplices. Using the map $\sigma \times Id : \Delta^i \times I \to X \times I$ restricted to each of these $(i+1)$-simplices yields the prism map. Now, there are many ways of decomposing a prism into simplices, but we need to be careful to choose one which restricts well to each of the <u>faces</u> of Δ^i, in order to get the chain homotopy property we require. In the end, what this requires is that the decomposition, when restricted to any face of Δ^i (which we think of as a copy of Δ^{i-1}), is the same as the decomposition we would have applied to a prism over an $(i-1)$-simplex.