Math 325 Problem Set 2 Solutions

4. [Lay, p.115, # 11.5] We define $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. Show that for every pair of real numbers $x, y \in \mathbb{R}$, $|x| \cdot |y| = |xy|$.

There are, essentially, nine cases, depending on the sign of each of x and y. But if either of x or y is 0, then $xy = 0$, so $|xy| = 0$; but then also either $|x| = 0$ or $|y| = 0$, so $|x| \cdot |y| = 0$. This deals with five of the caes!

If $x > 0$ and $y > 0$, then $xy > 0$ and $|x| = x$, $|y| = y$, so $|xy| = xy = |x| \cdot |y|$.

If $x > 0$ and $y < 0$, then $xy < 0$ and $|x| = x$, $|y| = -y$, so
$|xy| = -(xy) = (x)(-y) = |x| \cdot |y|$.

If $x < 0$ and $y > 0$, then $xy < 0$ and $|x| = -x$, $|y| = y$, so
$|xy| = -(xy) = (-x)(y) = |x| \cdot |y|$.

If $x < 0$ and $y < 0$, then $xy > 0$ and $|x| = -x$, $|y| = -y$, so
$|xy| = xy = (-x)(-y) = |x| \cdot |y|$.

So in every case, we find that $|xy| = |x| \cdot |y|$, so the result holds for any pair of real numbers.

5. [Lay, p.127, # 12.6(a)] Show that the least upper bound of a set S is unique; that is, if S is bounded from above, and if α and β both satisfy the properties required so be the supremum of S, then $\alpha = \beta$.

Suppose that $\alpha = \beta$ is <u>false</u>. Then it must be the case that either $\alpha < \beta$ or $\alpha > \beta$.

But if $\alpha < \beta$, then since β is a <u>least</u> upper bound, α cannot be an upper bound (there is an $x \in S$ so that $\alpha < x$). But since α is a supremum, it must in particular be an upper bound!, a contradiction. so $\alpha < \beta$ is impossible.

But by a symmetric argument, if $\alpha > \beta$ then since α is a least upper bound, β cannot be an upper bound ($\beta < \alpha$ implies that there is an $x \in S$ with $\beta < x$). So $\beta < \alpha$ is also impossible.

So $\alpha \neq \beta$ leads, in all cases, to a contradiction, so it must be the case that $\alpha = \beta$.

6. [Lay, p.127, # 12.3,12.4(g,h)]
Find the supremum (= lub) and infimum (= glb) of each of the following sets:

(α) $\left\{\dfrac{n}{n+1} \;:\; n \in \mathbb{N}\right\} = A$

Writing out a few terms suggests that the elements of the set get larger as n increases, and calculus tells us that they limit on 1. So we would assert that sup(A)=1 and inf(A)=1/2= the 'first' element of the set.

Verifying these can be done by noting that $1/2 \in A$, and for every $n \geq 1$, $1/2 \geq n/(n+1)$, since $n \geq 1$ implies that $2n = n + n \geq n + 1$, so $n \geq (n+1)/2$ [since $1/2 > 0$], so $n/(n+1) \geq 1/2$ [since $1/(n+1) > 0$].

sup(A)= 1, since $0 < 1$ implies that $n < n+1$ for all $n \geq 1$, so $n/(n+1) < 1$ for all n (showng that 1 is an upper bound), and if $x < 1$ then $1 - x > 0$, so $(n+1)(1-x) > 1$ for some n (by a result from class), so $1 - x > 1/(n+1)$, so $x < 1 - 1/(n+1) = n/(n+1)$, showing that x cannot be an upper bound for A.

(β) $\left\{(-1)^n(1+\frac{1}{n}) \ : \ n \in \mathbb{N}\right\} = B$

Again, writing out a few terms convinces us that the odd-numbered terms are negative and increase from -2 towards -1, and the even-numbered terms decrease from $3/2$ towards 1. So we assert that $\sup(B) = 3/2$ and $\inf(B) = -2$.

Verifying this can be done by noting that $3/2$ and -2 are in B, and then showing that, if $n \geq 1$ is odd, then $-2 \leq (-1)^n(1+1/n) < 0 < 3/2$ and if $n \geq 1$ is even then $-2 < 0 < (-1)^n(1+1/n) \leq 3/2$. This is because (using our knowledge of the sign of $-1)^n$, these assert that $0 < 1 + 1/n \leq 2$ for n odd and $0 < 1 + 1/n \leq 3/2$ for n even. These in turn follow from (muliplying by $n > 0$ and $2n > 0$, respectively) $0 < n+1 \leq 2n$ and $0 < 2n+2 \leq 3n$, which assert that $1 \leq n$ and $2 \leq n$ respectively.

7. For subsets $A, B \subseteq \mathbb{R}$, we define their 'sum' $A + B = \{a + b \ : \ a \in A, b \in B\}$.

   Show that if A and B are both bounded from above, then
   $$\mathrm{lub}(A+B) = \mathrm{lub}(A) + \mathrm{lub}(B) \ .$$
   [Hint: show that lub(A)+lub(B) is an upper bound! Then worry about whether there might be a smaller one...]

Some of you pointed out, in a burst of honesty, that this result can be found in the textbook... [All that I noticed was that it <u>wasn't</u> in the exercise sets.] The idea is that since $a \leq \mathrm{lub}(A)$ and $b \leq \mathrm{lub}(B)$ for every $a \in A$ and $b \in B$, we then know that $x = a + b \leq \mathrm{lub}(A) + \mathrm{lub}(B)$ for every $a \in A$ and $b \in B$, i.e., for every $x \in A + B$. So $\mathrm{lub}(A) + \mathrm{lub}(B)$ is an upper bound for $A + B$.

To so that it is the <u>least</u> upper bound, we suppose we are given a number $\mu < \mathrm{lub}(A) + \mathrm{lub}(B) = \alpha + \beta$. From this what we want to do (at least, this is one approach) is to construct a pair of numbers less than α and β (to use that fact that these are suprema). If we set $(\alpha + \beta) - \mu = \epsilon > 0$, then we can 'split' this excess between α and β, setting $\alpha' = \alpha - \epsilon/2 < \alpha$ and $\beta' = \beta - \epsilon/2 < \beta$.

Then $\alpha' + \beta' = (\alpha + \beta) - \epsilon = \mu$, and by the properties of the suprema, we know thazt there is an $a \in A$ and $b \in B$ with $\alpha' < a$ and $\beta' < b$, so $\mu = \alpha' + \beta' =< \alpha' + b < a + b$ with $a + b \in A + B$. So μ is not an upper bound for $A + B$, showing that $\alpha + \beta = \sup(A + B)$, as desired.