Math 325 Problem Set 1 Solutions

Problems were due Friday, January 20.

1. [Zorn, p.13 #2] Let $S = \{x \in \mathbb{R} \mid x^2 + x = 0\}$ and $T = \{x \in \mathbb{R} \mid x^2 + x < 5\}$.

   (a) Write S and T as (small) unions of points and/or intervals.

   $x^2 + x = x(x + 1) = 0$ only when $x = 0$ or $x + 1 = 0$ (i.e., $x = -1$). So $S = \{-1, 0\}$.

   $x^2+x = 5$ when $x^2+x-5 = 0$; using the Quadratic Formula, this is when $x = \dfrac{-1 \pm \sqrt{1 - 4 \cdot (-5)}}{2 \cdot 1} = \dfrac{-1 \pm \sqrt{21}}{2}$. So $x^2 + x - 5 = (x - \dfrac{-1 - \sqrt{21}}{2})(x - \dfrac{-1 + \sqrt{21}}{2})$. This quantity is less than 0 when one of the terms is negative and the other is positive. Since
   $x - \dfrac{-1 - \sqrt{21}}{2} = x - \dfrac{1}{2} + \dfrac{\sqrt{21}}{2} > x - \dfrac{1}{2} - \dfrac{\sqrt{21}}{2} = x - \dfrac{-1 + \sqrt{21}}{2}$, this
   then happens when $x - \dfrac{-1 - \sqrt{21}}{2} > 0$ and $x - \dfrac{-1 + \sqrt{21}}{2} < 0$, that is, when $\dfrac{-1 - \sqrt{21}}{2} < x$ and $x < \dfrac{-1 + \sqrt{21}}{2}$. So $x \in T$ precisely when $\dfrac{-1 - \sqrt{21}}{2} < x < \dfrac{-1 + \sqrt{21}}{2}$, so $T = (\dfrac{-1 - \sqrt{21}}{2}, \dfrac{-1 + \sqrt{21}}{2})$.

   (b) Decide whether each of the following statements is true, and (briefly) explain:
   $$S \subseteq \mathbb{N}\ ;\ S \subseteq T\ ;\ T \cap \mathbb{Q} \neq \emptyset\ ;\ -2.8 \in \mathbb{Q} \setminus T\ .$$

   FALSE: Since $-1 \in S$ and $-1 \notin \mathbb{N}$, we have $S \not\subseteq \mathbb{N}$.

   TRUE: Since $0 < 5$, when $x^2 + x = 0$ we also have $x^2 + x < 5$, so everything in S is in T, so $S \subseteq T$.

   TRUE: Since T is a (non-trivial) interval, it contains irrational numbers; for example, $\sqrt{21}/1,000,000$ is in T and is not rational. So $T \cap \mathbb{Q} \neq \emptyset$.

   TRUE: $-2.8 = -14/5 \in \mathbb{Q}$. And $(-2.8)^2 + (-2.8) = (0.2 - 3)^2 + (0.2 - 3) = 0.04 - 1.2 + 9 + 0.2 - 3 = 0.04 - 1 + 6 = 5.04$, so $-2.8 \notin T$. So $-2.8 \in \mathbb{Q} \setminus T$.

   (c) Describe the set $U = \{x \in \mathbb{R} \mid x^2 + x < 0\}$ as a union of intervals.

   As above, $x^2 + x = 0$ when $x = -1$ or $x = 0$. Also, as above, $x^2 + x = x(x + 1) < 0$ when one of $x, x + 1$ is negative and one is positive. Since $x < x + 1$ we need $x < 0$ and $x + 1 > 0$, so $-1 < x < 0$. So $U = (-1, 0)$.

2. [Zorn, p.14, #10] Starting with a set S, we can construct a new set $P(S)$, the power set of S, consisting of all subsets of S. For example, $P(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

   (a) Find $P(\{1, 2, 3\})$.

   A subset of $\{1, 2, 3\}$ has 0, 1, 2, or 3 elements. $\emptyset$ is the only 0-element subset, and $\{1, 2, 3\}$ is the only 3-element subset. The 1-element subsets are $\{1\}$, $\{2\}$, and $\{3\}$. A 2-element set is missing exactly one of $1, 2, 3$; so they are $\{2, 3\}$, $\{1, 3\}$, and $\{1, 2\}$.

   (b) Show that if $S \subseteq T$, then $P(S) \subseteq P(T)$.

   Suppose $U \in P(S)$, so U is a subset of S. So every element of U is an element of S. But since $S \subseteq T$, these elements of U must all be elements of T, as well. So every element of U is an element of T; so $U \subseteq T$ and therefore $U \in P(T)$. SO if $U \in P(S)$ then $U \in P(T)$; this means that $P(S) \subseteq P(T)$.

(c) If we set $N_k = \{1, 2, \ldots, k\}$, explain why $P(N_{11})$ has twice as many elements as $P(N_{10})$.

Every element of $P(N_{11})$ is a subset of N_{11}. If U is such a subset, one of two things must be true: either $11 \in U$ or $11 \notin U$. If $11 \in U$, then we can write $U = \{11\} \cup V$, where V is all of the <u>other</u> elements of U. But then $U \subseteq N_{10}$, so $V \in P(N_{10})$. One the other hand, if $11 \notin U$, then $U \in P(N_{10})$. So elements of $P(N_{11})$ come in two flavors; either they are an element of $P(N_{10})$, or they are an element of $P(N_{10})$ <u>together</u> <u>with</u> the element 11. But there are as many elements of each kind as there are elements in $P(N_{10})$ (since one is precisely such an element and the other is built from it by adding 11). So P_{11} has the elements ob P_{10} plus the same number of elements again; so P_{11} has twice as many elements as P_{10}.

3. [Zorn, p.26, #8] Let L be the (linear) function $L(x) = ax + b$, where a and b are (real) constants and $a \neq 0$.

(a) Explain why L is both one-to-one and onto.

If $L(x) = L(y)$, then $ax + b = ay + b$, so $a(x - y) = ax - ay = b - b = 0$. Then $x - y = \dfrac{1}{a} \cdot 0 = 0$ (since, because $a \neq 0$, it has a multiplicative inverse), so $x - y = 0$, so $x = y$. So we have found that $L(x) = L(y)$ implies $x = y$, so f is one-to-one.

On the other hand, for any $y \in \mathbb{R}$ we can solve $y = L(x)$, since $y = ax + b$ means $ax = y - b$ and so $x = \dfrac{y-b}{a}$. That is, $L(\dfrac{y-b}{a}) = a\dfrac{y-b}{a} + b = (y - b) + b = y$. So the image of L is all of $\mathbb{R}$, so L is onto.

(b) Find a formula for the inverse function $M = L^{-1}$, and show that
$L \circ M(x) = M \circ L(x) = x$ for every $x \in \mathbb{R}$.

We essentially did this above, when we solved $y = L(x)$; $x = \dfrac{y-b}{a}$ can be expressed as a function $M(x) = \dfrac{x-b}{a}$. Then:

$$L(M(x)) = L(\frac{x-b}{a} = a\frac{x-b}{a} + b = (x - b) + b = x, \text{ and}$$

$$M(L(x)) = \frac{L(x) - b}{a} = \frac{(ax + b) - b}{a} = \frac{ax}{a} = x.$$

4. [Zorn, p.26, #10 (part)] Suppose the $f : A \to B$ and $g : B \to C$ are both functions.

(a) Show that if f and g are both one-to-one, then $g \circ f : A \to C$ is one-to-one.

Suppose that $x, yinA$ and $(g \circ f)(x) = (g \circ f)(y)$. Then $f(x), f(y) \in B$ and $g(f(x)) = g(f(y))$ so, since g is one-to-one, we have $f(x) = f(y)$. But then since f is one-to-one, we have $x = y$. So $(g \circ f)(x) = (g \circ f)(y)$ implies that $x = y$; so $g \circ f$ is one-to-one .

(b) Show that if $g \circ f$ is onto, then g is onto.

If $g \circ f$ is onto, then for any $y \in C$ we can find an $x \in A$ so that $(g \circ f)(x) = g(f(x)) = y$. But then if we set $z = f(x) \in B$, then $g(z) = y$. So for any $y \in C$ we can find a $z \in B$ so that $g(z) = y$. So g is onto.