Solutions

Name:

Math 423/823 Exam 1

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Solve for $z = x + yi$:

$$(1+i)(2+i) = (5+2i)z$$

$$2+2i+i+i^2 = (5+2i)z$$

$$1+3i = (5+2i)z$$

$$(1+3i)(5-2i) = (5+2i)(5-2i)z$$

$$5+15i-2i-6i^2 = (25+10i-10i-4i^2)z$$

$$11+13i = (29)z$$

$$z = \frac{11+13i}{29} = \frac{11}{29} + \frac{13}{29}i$$

2. (15 pts.) Sketch the collection of points which satisfy the equation

$$\mathrm{Re}(z \cdot \overline{z} + 4zi) = 5$$

$z = x + yi$

$z\overline{z} = x^2 + y^2$

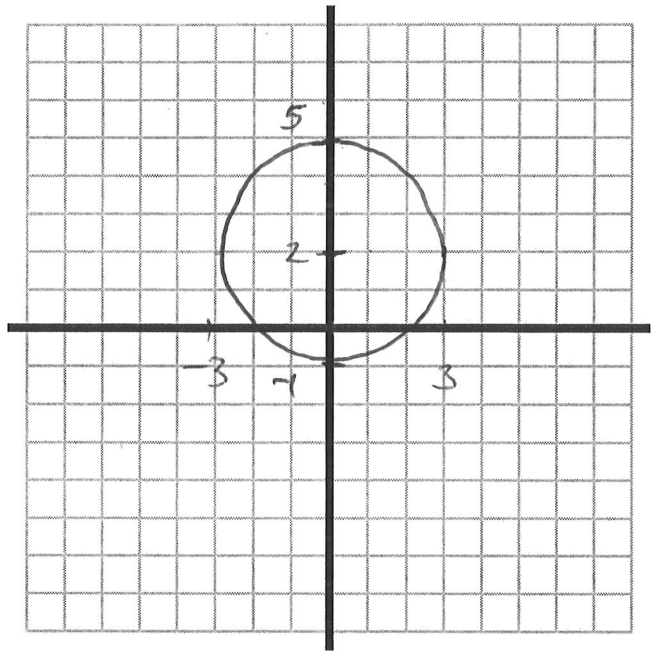
$4zi = 4xi + 4yi^2$

$= 4xi - 4y$

$\mathrm{Re}(z\overline{z} + 4zi)$

$= \mathrm{Re}(x^2 + y^2 + 4xi - 4y)$

$= x^2 + y^2 - 4y = 5$



$x^2 + (y^2 - 4y + 4) = 5 + 4$

$x^2 + (y-2)^2 = 9 = 3^2$

circle with radius 3 and center $(0,2)$

3. (20 pts.) Find the 3-rd roots of $z = 2i$ (i.e, the w for which $w^3 = z$).

[You can write them in either $w = x + yi$ or $w = re^{i\theta}$ form.]

$$w^3 = z = 2i = 2e^{\frac{\pi}{2}i}$$

$$\Rightarrow w = 2^{1/3} e^{\frac{\pi}{6}i} \text{ and } w = 2^{1/3} e^{\overbrace{\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)}^{\frac{5\pi}{6}} i} \text{ and}$$

$$w = 2^{1/3} e^{\underbrace{\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)}_{\frac{3\pi}{2}} i}$$

$$\boxed{w = 2^{1/3} e^{\frac{\pi}{6}i},\ 2^{1/3} e^{\frac{5\pi}{6}i},\ 2^{1/3} e^{\frac{3\pi}{2}i}}$$

In Cartesian coordinates

$$e^{\frac{\pi}{6}i} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$e^{\frac{5\pi}{6}i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$e^{\frac{3\pi}{2}} = -i$$

so

$$\boxed{w = \frac{1}{2^{2/3}}(\sqrt{3}+i),\ \frac{1}{2^{2/3}}(-\sqrt{3}+i),\ -2^{1/3}i}$$

4. (10 pts. each) Find each of the indicated limits (or show that it doesn't exist):

(a): $\lim_{z \to i} \dfrac{z^3 - z + 2i}{z^2+1} = (*)$ plug in i: $i^3 - i + 2i = -i - i + 2i = \underline{0}$.

$i^2 + 1 = -1 + 1 = \underline{0}$

$z^3 - z + 2i = (z-i)(z^2 + iz - 2)$
$z^2 + 1 = (z-i)(z+i)$

So $(*) = \lim_{z \to i} \dfrac{z^2 + iz - 2}{z+i} = \dfrac{i^2 + i \cdot i - 2}{i+i} = \dfrac{-1-1-2}{2i} = \dfrac{-4}{2i} = \underline{2i}$

OR: L'Hôpital! (indeterminate form $0/0$):

$(z^3 - z + 2i)' = 3z^2 - 1$

$(z^2+1)' = 2z$

so $(*) = \dfrac{3(i)^2 - 1}{2(i)} = \dfrac{-3-1}{2i} = \dfrac{-4}{2i} = \underline{2i}$

(b): $\lim_{z \to \infty} e^{-z^2}$

as $z \to \infty$, $-z^2 \to \infty$ But $e^{(\text{far from } 0)}$ can be anywhere.

E.g., $z = x \xrightarrow{\in \mathbb{R}} \infty$, then $e^{-z^2} = e^{-x^2} \longrightarrow 0$

But $z = iy \longrightarrow \infty$ (i.e. $y \in \mathbb{R}$, $y \to \infty$) then

$e^{-z^2} = e^{-(iy)^2} = e^{-(-y^2)} = e^{y^2} \longrightarrow \underline{\infty}$

So e^{-z^2} can approach no single value as $z \to \infty$, so the limit does not exist.

5. (15 pts.) Show that the function $f(z) = (\overline{z})^3$ is analytic at <u>no</u> value of z.

[Hint: You can argue directly, or use the fact that $g(z) = z^3$ <u>is</u> analytic.]

$\overline{z} = (x-iy)$ so $(\overline{z})^3 = (x-iy)^3$

$= x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3$

$= x^3 - 3x^2yi - 3xy^2 - (-i)y^3$

$= (x^3 - 3xy^2) + (y^3 - 3x^2y)i$

But then $= u + iv$

$u_x = 3x^2 - 3y^2$ $\quad u_y = -6xy$

$v_x = -6xy$ $\quad v_y = 3y^2 - 3x^2$

So $u_x = v_y \iff 3x^2 - 3y^2 = 3y^2 - 3x^2 \iff 6x^2 = 6y^2$
$\iff x^2 = y^2 \iff ~~y~~ = \pm x$. But

$u_y = -v_x \iff -6xy = 6xy \iff 12xy = 0 \iff xy = 0$
$\iff x = 0$ or $y = 0$

(and $~~y~~ = \pm x$ $\Rightarrow$) $\iff x = y = 0$ So f can be diff'ble only at $(0,0)$. So f is analytic nowhere. (= diff'ble in an entire nbhd of a pt.)

OR: we know that $g(z) = z^3$ is entire. So if $f(z)$ were analytic anywhere, so is $h(z) = f(z) + g(z) = z^3 + \overline{z}^3$. But $z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$, so $h(z) = 2(x^3 - 3xy^2)$ is always real. But an analytic function that is real is constant. But $h(z)$ certainly isn't: $h(x+0i) = 2x^3$. So $f(z)$ can't be analytic... anywhere.

6. (15 pts.) Find an entire function $f(x, y) = u(x, y) + iv(x, y)$ for which

$$u(x, y) = x^2 + 2xy - y^2$$

[That is, find a harmonic conjugate of $u(x, y)$.]

want $v(x,y)$ so that

$$v_x = -u_y = -(2x-2y) = -2x+2y$$
$$v_y = u_x = 2x+2y$$

$v_x = 2x-2y$, then $v = \int v_x\,dx = \int -2x+2y\,dx$
$$= -x^2 + 2xy + g(y)$$

then $v_y = +2x + g'(y) = 2x+2y \Rightarrow g'(y) = 2y$

$\Rightarrow g(y) = \int 2y\,dy = y^2 + \text{const}$ (want const $= 0$)

So $v(x,y) = -x^2 + 2xy + y^2$ works.

Note: $f(z) = (x^2+2xy-y^2) + (-x^2+2xy+y^2)i$ is actually a familiar function: $(x^2-y^2)+(2xy)i = (x+iy)^2 = z^2$

and $(2xy) + (y^2-x^2)i = \dfrac{(2xy)i + (y^2-x^2)i^2}{i} = \dfrac{(x^2-y^2)+(2xy)i}{i}$
$$= \frac{z^2}{i} = -iz^2$$

so $f(z) = z^2 - iz^2 = (1-i)z^2$!

Math 423/823 Exam 2 Solutions

1. (20 pts.) Find all values of $z \in \mathbb{C}$ for which $\sin(z) = i$.

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, so $\sin(z) = i$ means that
$e^{iz} - e^{-iz} = 2i^2 = -2$, so $e^{iz} - e^{-iz} + 2 = 0$, so $(e^{iz})^2 - 1 + 2e^{iz} = 0$,
or, setting $w = e^{iz}$, $w^2 + 2w - 1 = 0$.

Solving this equation, we have $w = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$.

So our solutions consist of the solutions to $e^{iz} = -1-\sqrt{2}$ and $e^{iz} = -1+\sqrt{2} = \sqrt{2}-1$, that is, the families of values $z = -i\log(-1-\sqrt{2})$ and $z = -i\log(\sqrt{2}-1)$.

So our solutions are:
$z = -i(\ln(1+\sqrt{2}) + i(\pi + 2k\pi)) = (2k+1)\pi - i\ln(1+\sqrt{2})$ for any integer k, and
$z = -i(\ln(\sqrt{2}-1) + i(2k\pi)) = 2k\pi - i\ln(\sqrt{2}-1)$ for any integer k.

[Alternative solutions included using the expression for $\arcsin z$, and solving the pair of equations $\sin x \cosh y = 0$, $\cos x \sinh y = 1$.]