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Name: Solutions

Math 325 Exam 2

Show all work! How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (25 pts.) Show that if $(a_n)_{n=1}^\infty$ is a sequence, and if both of the subsequences $(\alpha_n)_{n=1}^\infty = (a_{2n})_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty = (a_{2n+1})_{n=1}^\infty$ converge to the same limit L, then $a_n \to L$ as $n \to \infty$.

Given $\varepsilon>0$ we know that there is $N_1 \in \mathbb{N}$ so that $n \geq N_1 \Rightarrow |\alpha_n - L| = |a_{2n} - L| < \varepsilon$.

There is also an $N_2 \in \mathbb{N}$ so that $n \geq N_2 \Rightarrow |\beta_n - L| = |a_{2n+1} - L| < \varepsilon$.

But then all a_k for k large enough fit into one of these two situations! Formally, set $N = \max\{2N_1, 2N_2+1\}$. Then if $n \geq N$ then either $n = 2m \geq 2N_1$ so $m \geq N_1$ so $|a_n - L| = |a_{2m} - L| < \varepsilon$, or $n = 2m+1 \geq 2N_2+1$ so $m \geq N_2$ so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$.

So in each case, $|a_n - L| < \varepsilon$. So $n \geq N$ implies $|a_n - L| < \varepsilon$. So $a_n \to L$ as $n \to \infty$

The idea, really, is that eventually all of the even-index terms a_{2n} are close to L, and eventually all of the odd index terms a_{2n+1} are close to L, so eventually all of the terms are close to L. The rest is about determining when that eventually is, for $|a_n - L| < \varepsilon$ ....

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2. (25 pts.) Show, directly from the ϵ-δ definition, that $f(x) = 2x^2 - 3x - 5$ is continuous at $x = 7$.

$f(7) = 2\cdot 7^2 - 3\cdot 7 - 5 = 2\cdot 49 - 21 - 5 = 98 - 26 = \underline{72}.$

So we want $\lim_{x\to 7} f(x) = 72$.

Given $\varepsilon > 0$ we want a $\delta > 0$ so that

$|x-7| < \delta$ implies $|f(x) - f(7)| = |(2x^2-3x-5) - 72| < \varepsilon$.

But $|2x^2 - 3x - 5 - 72| = |2x^2 - 3x - 77|$

$= |(2x+11)(x-7)| = |2x+11| \cdot |x-7|$

↑ this we control.

If $|x-7| < 1$, then

$-1 < x-7 < 1$ so $6 < x < 8$, so

$\underset{\overset{\parallel}{23}}{2\cdot 6 + 11} < 2\cdot x + 11 < 2\cdot 8 + 11 = 27$ so $|2x+11| < 27$.

So $|f(x) - f(7)| = |2x+11| \cdot |x-7| < 27|x-7|$

So if we set $\delta = \min\{1, \varepsilon/27\}$ then $|x-7| < \delta \Rightarrow$

$|x-7| < 1 \Rightarrow |2x+11| < 27 \Rightarrow$

$|f(x) - f(7)| \leq |x-7|\cdot|2x+11| < |x-7|\cdot 27 \leq \frac{\varepsilon}{27}\cdot 27 = \varepsilon.$

So f is cts at x=7.

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3. (25 pts.) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous, that $[f(x)]^2 = [g(x)]^2$ for every $x \in \mathbb{R}$, and $g(x) \neq 0$ for every $x \in \mathbb{R}$. Show that <u>either</u> $f(x) = g(x)$ for every $x \in \mathbb{R}$ <u>or</u> $f(x) = -g(x)$ for every $x \in \mathbb{R}$.

Since $g(x) \neq 0$ for all x <u>and</u> g is cts, we must <u>either</u> have $g(x) > 0$ all x <u>or</u> $g(x) < 0$ all x, since $g(a) > 0$ & $g(b) < 0$ means 0 is <u>between</u> $g(a)$ & $g(b)$, so the intermediate value theorem ~~says~~ says that $g(x) = 0$ somewhere between a & b, a contradiction.

Since $(f(x))^2 = (g(x))^2 \neq 0$, we have $f(x) \neq 0$ for all x, too, so we, by the same argument, have $f(x) > 0$ all x <u>or</u> $f(x) < 0$ all x.

But now $f(x)^2 = (g(x))^2$ means $(f(x))^2 - (g(x))^2 = 0$
$= (f(x) - g(x))(f(x) + g(x))$

so for <u>each</u> x, <u>either</u> $f(x) - g(x) = 0$ <u>or</u> $f(x) + g(x) = 0$.

But! If f & g have the same sign, then $f(x) + g(x)$ cannot be 0, so $f(x) - g(x) = 0$ <u>must</u> always be true, we $f(x) = g(x)$ for all x. On the other hand, if f & g have opposite signs, then $f(x) - g(x) = 0$ can never happen, so $f(x) + g(x) = 0$ for all x, so $f(x) = -g(x)$ for all x.

<u>So</u> either $f = g$ <u>or</u> $f = -g$. //

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3. (25 pts.) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous, that $[f(x)]^2 = [g(x)]^2$ for every $x \in \mathbb{R}$, and $g(x) \neq 0$ for every $x \in \mathbb{R}$. Show that <u>either</u> $f(x) = g(x)$ for every $x \in \mathbb{R}$ <u>or</u> $f(x) = -g(x)$ for every $x \in \mathbb{R}$.

A "better" solution! (Inspired by some of yours...)

Since $g(x) \neq 0$ for all x $\quad h(x) = \frac{f(x)}{g(x)}$ is defined <u>and</u> continuous, $h : \mathbb{R} \to \mathbb{R}$. Then $(f(x))^2 = (g(x))^2$ means $h(x)^2 = \frac{f(x)^2}{(g(x))^2} = 1$ for <u>all</u> x,

so $h(x) = 1$ or $h(x) = -1$ for each $x \in \mathbb{R}$.

But h is <u>cts</u> and so if $h(a) = 1$ & $h(b) = -1$, then IVT says $h(x) = 0$ somewhere b/w a & b, since $-1 < 0 < 1$, which is absurd.

$f(b)$? $\quad = f(a)$

So either $h(x) = 1$ for all x (so $f(x) = g(x)$ for all x)

<u>or</u> $h(x) = -1$ for all x (so $f(x) = -g(x)$ for all x) !

4. (25 pts.) Show that if $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are both *uniformly continuous*, then the function $h : I \to \mathbb{R}$ defined by $h(x) = f(x) + 3g(x)$ is also uniformly continuous.

Since f & g are unif cts, given $\varepsilon>0$ there is a $\delta_1>0$ so that $x,y\in I$ and $|x-y|<\delta_1$ implies $|f(x)-f(y)|<\varepsilon$. Also, there is a $\delta_2>0$ so that $x,y\in I$ and $|x-y|<\delta_2$ implies $|g(x)-g(y)|<\varepsilon$.

We want to control $|h(x)-h(y)| = |f(x)+3g(x) - (f(y)+3g(y))|$

$= |(f(x)-f(y)) + 3(g(x)-g(y))| \le |f(x)-f(y)| + 3|g(x)-g(y)|$

(by the triangle inequality).

So back up! Choose $\delta_1>0$ and $\delta_2>0$ so that $|x-y|<\delta_1 \Rightarrow |f(x)-f(y)|<\varepsilon/2$ and $|x-y|<\delta_2 \Rightarrow |g(x)-g(y)|<\varepsilon/6$. Then with $\delta=\min\{\delta_1,\delta_2\}$, we have $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon/2$ and $|g(x)-g(y)|<\varepsilon/6$, so

$|h(x)-h(y)| \le |f(x)-f(y)| + 3|g(x)-g(y)| < \varepsilon/2 + 3(\varepsilon/6)$

$= \varepsilon/2 + \varepsilon/2 = \varepsilon$.

So for all $\varepsilon>0$ we can find a $\delta>0$ so that $x,y\in I$ and $|x-y|<\delta \Rightarrow |h(x)-h(y)|<\varepsilon$. Since δ does not depend on x or y, h is uniformly cts. ///

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As some of you noted, other values also work, e.g. $\varepsilon/4$ and $\varepsilon/4$ instead of $\varepsilon/2$ and $\varepsilon/6$.

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