Math 417 Problem Set 5 Solutions

Starred (*) problems were due Friday, February 26.

(*) 37. (Gallian, p.119, # 10) Find an element of S_{10} that has the largest order of any element in S_{10}.

We've seen that the order of a permutation is determined by its disjoint cycle structure: it is the least common multiple of the lengths of its disjoint cycles. If we include cycles of length one, then the sum of the lengths of the cycles in a disjoint cycle representation of an element of S_{10} is 10. So the question really is: how do we choose positive integers which sum to 10 (called, as it happens, a *partition* of 10) so that their least common multiple is as large as possible?

I don't really know a good answer to this question, except to start building some! A 1-cycle is likely to be wasted, we will proabably do better to add one to another cycle? A 2-cycle is wasted if there is a also another even-length cycle - the 2 will contribute nothing to the lcm. But, e.g, a partition (2,3,5) adds up to 10, and has lcm 30, so we can build a element of order 30. Can we do better than that? With a total of two disjoint cycles, no: calculus (!) tells us that $k(10-k)$ (which is as large as the lcm could be) has maximum $(5)(5)=25$. If we use 4 (or more) disjoint cycles, then at least one of them has to be a 1- or 2-cycle, which does not bode well... These really comes down to an analysisof partitions of 9 and 8; no partition of 9 will beat $(2,3,5)$, and a partition of 8 using 3 or more integers won't beat it, either.

With 3 disjoint cycles, at least one of them is a 1-, 2-, or 3-cycle (since $10/3<4$. With our analysis of two disjoint cycles above, that would mean that the best order we could achieve would be either $(1)(9/2)^2=81/4<21$, $(2)(8/2)^2=32$, or $(3)(7/2)^2=157/4=39.25$. But we can't actually achieve these large bounds: a (2,4,4) cycle structure actually has order 4, and (to get integers) $(3,3,4)$ is order 12, so you need to move on to $(3,2,5)$ which is our current candidate.

Consequently, with high confidence, anyway, a cycle structure of $(2,3,5)$ gives the highest order. An example of such a permutation would be., for example,

$$(1,2)(3,4,5)(6,7,8,9,10)\ .$$

How can we really be sure this is best? The argument above really amounted to saying that if the number of disjoint cycles got too high, or one of the cycles was too small, then we were either really working with an element of S_9 (with a 1-cycle attached) or an element of S_8 (with a 2-cycle attached), and asserting that S_9 couldn't produce an element of order larger than 30, and S_8 couldn't produce an element with order larger than 30 or odd order larger than 15. These could be verified by a argument along the same lines that we have already been developing....

(*) 39. Show that if $\alpha\in S_n$ has $|\alpha|$ odd, then α is an even permutation!

Since $|\alpha|$ is odd, when we write α as a product of disjoint cycles

$$\alpha=(a_{1,1},\ldots,a_{1,k_1})\cdots(a_{m,1},\ldots,a_{m,k_m})\ ,$$

the lcm of $k_1,\ldots k_m$ must be odd. But this implies that each k_i is odd; if one of the k_i were even them all of its multiples would be even, and so the lcm would be even.

But this means that α is a product of (disjoint) cycles of odd length. But an odd-length cycle is an even permutation! So each cycle can be written as a product of an even number of 2-cycles. This means that their product, α can be written as a a sum of even numbers of 2-cycles, and so can be written as the sum of an even number of 2-cycles. So α is even.

[N.B.: Note that this means that the alternating group A_n contains all of the elements of S_n with odd order...]

(*) 42. (Gallian, p.122, # 69) Show that every element of S_n can be written as a product of transpositions of the form $(1, k)$ for $2 \leq k \leq n$. (Assume that $n > 1$ so that you don't have to worry about the philosophical challenges of $S_1 = \{()\}$...)

[Hint: why is it enough to show that this is true for transpositions?]

We have shown in class that every permutation $\alpha \in S_n$ can be written as a product of transpositions $\alpha(a_1, b_1) \cdots (a_k, b_k)$. If we show that every transposition can be written as a product of transpositions $(1, k)$, then by writing each (a_i, b_i) this way, and then multiplying these representations together, we will write α as a product of (products of transpositions of the form $(1, k)$), and so it will be a product of such transpositions.

And to show that any transposition (a, b) can be written this way, we can start by asking: Is either of a or b equal to 1? If yes, then $(a, b) = (1, b)$, or $(a, b) = (a, 1) = (1, a)$, and so if <u>is</u> a transposition of the form $(1, k)$. If no, then both $(1, a)$ and $(1, b)$ are 'real' transpositions, and then we can start taking products of these:

$(1, a)(1, b) = (1, b, a)$, and so

$$(1, b)(1, a)(1, b) = (1, b)(1, b, a) = (1)(b, a) = (b, a) = (a, b),$$

and so (a, b) can be written as a sum of transpositions $(1, k)$, as desired.

A selection of further solutions

40. (Gallian, p.120, #32) If $\beta = (1, 2, 3)(1, 4, 5)$, express β^{99} as a product of disjoint cycles.

Since $(1, 2, 3)(1, 4, 5) = (1, 4, 5, 2, 3)$ (by carrying out the composition), β is a 5-cycle, so it has order 5. So $\beta^5 = e$, and so $\beta^{99} = \beta^{5 \cdot 19 + 4} = (\beta^5)^{19}\beta^4 = e^{19}\beta^4 = \beta^4 = \beta^{-1}$. And since the inverse of a cycle is the cycle written in the opposite order, we have $\beta^{99} = (3, 2, 5, 4, 1) = (1, 3, 2, 5, 4)$ (if you like writing your cycles so that they start with the smallest element moved...).

41. (Gallian, p.121, #48) Show that in the symmetric group S_7, there is <u>no</u> element $x \in S_7$ so that $x^2 = (1, 2, 3, 4)$. On the other hand, find two distinct elements $y \in S_7$ so that $y^3 = (1, 2, 3, 4)$.

$(1, 2, 3, 4)$ is a 4-cycle, and so has order 4. If there were an element $x \in S^7$ with $x^2 = (1, 2, 3, 4)$, then $x^8 = (x^2)^4 = (1, 2, 3, 4)^4 = e$, and so x must have order dividing 8. But if it is smaller than 8, then either x, or x^2, or x^4 must be e. But $x = e$ means $x^2 = e$, $x^2 = e$ means, well, $x^2 = e$, and $x^4 = e$ means $e = x^4 = (x^2)^2 = (1, 2, 3, 4)^2 = (1, 2, 3, 4)(1, 2, 3, 4) = (1, 3)(2, 4)$, all of which are contradictions. So we must have

$|x| = 8$. But in S_7 this is not possible! The order of x can be determined by the lengths of is disjoint cycles, an inorder for this to be 8 one of the cycles has to have order a multiple of 8 (since $8 = 2^3$ and the lcm comes by taking the maximum powers for each prime in the factorization of the cycle orders). But this in turn means that a cycle has to have <u>length</u> at least 8, which is S_7 is impossible.

On the other hand, $y^3 = (1,2,3,4)$ requires $y^{12} = e$, and so y must have order dividing 12. But this can be achieved by having $|y| = 4$ <u>or</u> 12. And both of these are possible. If $|y| = 4$, then $y^3 = y^{-1}$, and so we can arrange this by setting $y^{-1} = (1,2,3,4)$, so $y = (4,3,2,1) = (1,4,3,2)$; this, as it happens, has order 4 ! And $y^3 = (1,4,3,2)^3 = (1,2,3,4)$, as desired. On the other hand, we can also find a y with $|y| = 12 = 3 \cdot 2^2$ by taking a product of a 4-cycle (think $(1,4,3,2)$!) and a disjoint 3-cycle (which is <u>why</u> we needed to be working in S_7 !) like $(5,6,7)$. The point is that $y = (1,4,3,2)(5,6,7)$ has $y^3 = (1,4,3,2)^3(5,6,7)^3 = (1,2,3,4)() = (1,2,3,4)$ (since disjoint cycles commute).

So both $y = (1,4,3,2)$ and $y = (1,4,3,2)(5,6,7)$ have $y^3 = (1,2,3,4)$.