Math 325 Problem Set 6 Solutions

Problems were due Friday, March 3.

20. [Zorn, p.99, #9] Suppose $(a_n)_{n=1}^{\infty}$ is a sequence and $L \in \mathbb{R}$. Show that if $a_{n_k} \to L$ for every <u>monotonic</u> subsequence of $(a_n)_{n=1}^{\infty}$, then $a_n \to L$.

    Suppose, instead, that $(a_n)_{n=1}^{\infty}$ does <u>not</u> converge to L. Then there is an $\epsilon > 0$ so that the statement "eventually $|a_n - L| < \epsilon$ <u>fails</u>. So for some $\epsilon > 0$ we have, for every N, a number $n_N \geq N$ with $|a_{n_N} - L| \geq \epsilon$. But then if we start with $n_1 \geq 1$, then choose $n_2 \geq n_1 + 1$, and then $n_3 \geq n_2 + 1$ (i.e., keep trying N equal to $n_k + 1$), we can then keep finding $n_1 < n_2 < n_3 < \ldots < n_k < \ldots$ with $|a_{n_k} - L| \geq \epsilon$.

    But! This gives us a new sequence $(a_{n_k})_{k=1}^{\infty}$, which happens to be a subsequence of the original sequence. By our result from class, this new sequence must have a monotonic subsequence $(a_{n_{k_r}})_{r=1}^{\infty}$. This is too many subscripts! But as a subsequence of a subsequence, it is a subsequence $(b_m)_{m=1}^{\infty}$ of our original sequence. But since it is a subsequence of the $a_n k$, we have $|b_m - L| \geq \epsilon$ for <u>every</u> m. Consequently, this sequence cannnot converge to L. But! it is a monotonic subsequence of $(a_n)_{n=1}^{\infty}$, and so our hypothesis says that it <u>must</u> converge to L (!). This is a contradiction, so the statement that $a_n \not\to L$ must be false; so $a_n \to L$.

21. Show *directly* (i.e., without quoting "Cauchy implies convergent" and "convergent implies Cauchy") that if a_n and b_n are Cauchy sequences, then so are the sequences $c_n = a_n + b_n$ and $d_n = a_n b_n$. [Hint: for the second, you will need to use Cauchy implies bounded?]

    For the first, we want to show that, given an $\epsilon > 0$, for n and m large enough, we have $|(a_n + b_n) - (a_m + b_m)| < \epsilon$. But (*) $= |(a_n + b_n) - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)| \leq |a_n - a_m| + |b_n - b_m|$, by the triangle inequality. We can therefore make (*) small by making <u>both</u> $|a_n - a_m|$ and $|b_n - b_m|$ small enough.

    <u>So</u>, given $\epsilon > 0$, we can choose N_1 and N_2 so that $n, m \geq N_1$ implies that $|a_n - a_m| < \epsilon/2$, and $n, m \geq N_2$ implies that $|b_n - b_m| < \epsilon/2$. Then setting $N = \max\{N_1, N_2\}$ we have $n, m \geq N$ implies that both results hold, so $|(a_n + b_n) - (a_m + b_m)| \leq |a_n - a_m| + |b_n - b_m| < \epsilon/2 + \epsilon/2 = \epsilon$. So $(a_n + b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

    For the second, we want to show that , given an $\epsilon > 0$, for n and m large enough, we have $|a_n b_n - a_m b_m| < \epsilon$. But $|a_n b_n - a_m b_m| = |(a_n - a_m) b_n + a_m (b_n - b_m)| \leq |(a_n - a_m) b_n| + |a_m (b_n - b_m)| = |a_n - a_m| \cdot |b_n| + |a_m| \cdot |b_n - b_m|$. We can make this small by making both $|a_n - a_m|$ and $|b_n - b_m|$ small enough, <u>provided</u> neither $|b_n|$ nor $|a_m|$ can get too big.

    But! We know that we can do this; from class, we know that every Cauchy sequence is bounded. So there are numbers $Q_1, Q_2 \in \mathbb{R}$ so that $|a_m| \leq Q_1$ for every m, and $|b_n| \leq Q_2$ for every n.

Then, given $\epsilon > 0$, we can choose N_1 and N_2 so that $n, m \geq N_1$ implies that $|a_n - a_m| < \epsilon/[2(Q_2+1)]$, and $n, m \geq N_2$ implies that $|b_n - b_m| < \epsilon/[2(Q_1+1)]$. Then setting N=max$\{N_1, N_2\}$ we have $n, m \geq N$ implies that both results hold, so

$|a_n b_n - a_m b_m| \leq |a_n - a_m| \cdot |b_n| + |a_m| \cdot |b_n - b_m| \leq Q_2|a_n - a_m| + Q_1|b_n - b_m| < Q_2\epsilon/[2(Q_2+1)] + Q_1\epsilon/[2(Q_1+1)] < \epsilon/2 + \epsilon/2 = \epsilon$. So $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

22. A sequence a_n is called *contractive* if for some constant $0 < k < 1$ we have $|a_{n+2} - a_{n+1}| < k|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$. Show that every contractive sequence is Cauchy (and therefore converges).

[Hint: By induction, $|a_{n+2} - a_{n+1}| < k^n|a_2 - a_1|$, and $\displaystyle\sum_{r=m+1}^{n} k^r$ is something (from calculus!) we know the exact value of...]

For a contractive sequence, we claim that $|a_{n+2} - a_{n+1}| < k^n|a_2 - a_1|$ for every $n \geq 1$. The base case, $n = 1$, is $|a_3 - a_2| < k|a_2 - a_1|$, which is our hypothesis with $n = 1$. Then if we assume (by induction) that $|a_{n+2} - a_{n+1}| < k^n|a_2 - a_1|$, then $|a_{(n+1)+2} - a_{(n+1)+1}| = |a_{n+3} - a_{n+2}| < k|a_{n+2} - a_{n+1}| < k(k^n|a_2 - a_1|) = k^{n+1}|a_2 - a_1|$, so $|a_{(n+1)+2} - a_{(n+1)+1}| < k^{n+1}|a_2 - a_1|$, which completes the inductive step.

With this, we can now study (if we suppose that $n \geq m$; if not, reverse roles of n and m !) $|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{m+1} - a_m)|$, which by a previous problem set, we know is $\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m| = \displaystyle\sum_{r=m}^{n-1} |a_{r+1} - a_r|$. To show that $(a_n)_{n-1}^{\infty}$ is Cauchy, it is then enough to show that given $\epsilon > 0$, we can find an N so that $n, m \geq N$ implies that $\displaystyle\sum_{r=m}^{n-1} |a_{r+1} - a_r| < \epsilon$ (since this sum is greater than $|a_n - a_m|$).

But! we can do this. Since $|a_{r+1} - a_r| < k^{r-1}|a_2 - a_1|$, it is (again) enough to show that $\displaystyle\sum_{r=m}^{n-1} k^{r-1}|a_2 - a_1| < \epsilon$, that is to show that $\displaystyle\sum_{r=m}^{n-1} k^{r-1} < \epsilon/|a_2 - a_1|$ (for n and m large enough). But! from calculus, we know that

$\displaystyle\sum_{r=m}^{n-1} k^{r-1} < \sum_{r=m}^{\infty} k^{r-1} = k^{m-1}\sum_{r=0}^{\infty} k^r = k^{m-1}\frac{1}{1-k}$ (it's a geometric series!). So $\displaystyle\sum_{r=m}^{n-1} k^{r-1} < k^{m-1}\frac{1}{1-k} < \epsilon/|a_2 - a_1|$ <u>provided</u> that $k^m < \dfrac{k(1-k)\epsilon}{|a_2 - a_1|}$. Then because $0 < k < 1$, we can arrange this to happen so long as m is large enough: there is a (first) N so that $k^N < \dfrac{k(1-k)\epsilon}{|a_2 - a_1|}$ (since $k^n \to 0$ as $n \to \infty$), and then $m \geq N$ implies that $k^m \leq k^N < \dfrac{k(1-k)\epsilon}{|a_2 - a_1|}$.

Then for this choice of N we have that if $n, m \geq N$ (and $n \geq m$) we have $|a_n - a_m| \leq \sum_{r=m}^{n-1} k^{r-1}|a_2 - a_1| < k^{m-1}\frac{1}{1-k}|a_2 - a_1| < k^N \frac{1}{k(1-k)}|a_2 - a_1| < \frac{k(1-k)\epsilon}{|a_2 - a_1|}\frac{1}{k(1-k)}|a_2 - a_1| = \epsilon$. So (!) the sequence $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence!

23. [Zorn, p.144, #2 (sort of)] Use calculus to 'determine' the following limits, then use the $\epsilon - \delta$ definition of limit to prove that you are correct:

(α) $\lim_{x\to 1} 2x + 3$

Calculus tells us that the limit should be $2 \cdot 1 + 3 = 5$. To prove this, we want to make $|2x + 3 - 5| = |2x - 2| = 2|x_1|$ small by making $|x - 1|$ small enough (but non-zero). This we can do: given an $\epsilon > 0$, we can choose $\delta = \epsilon/2$; then $0 < |x - 1| < \delta$ implies that $|x - 1| < \epsilon/2$, so $|(2x + 3) - 5| = 2|x - 1| < 2(\epsilon/2) = \epsilon$. So $2x + 3 \to 5$ as $x \to 1$.

(β) $\lim_{x\to 2} \frac{1}{2x + 3}$

Calculus tells us that the limit should be $\frac{1}{2 \cdot 2 + 3} = \frac{1}{7}$. To prove this, we want to make

$|\frac{1}{2x+3} - \frac{1}{7}| = |\frac{7 - (2x + 3)}{7(2x + 3)}| = |\frac{4 - 2x}{7(2x + 3)}| = \frac{2}{7}\frac{|x - 2|}{|2x + 3|}$

small, by making $|x - 2|$ small enough. Again, this we can do, by <u>also</u> making sure that $|2x + 3|$, which is in the denomenator, is never <u>too</u> small. But if we insist, for example, that $|x - 2| < 1$, then $-1 < x - 2 < 1$, so $1 < x < 3$, so $2 < 2x < 6$, and so $5 < 2x + 3 < 9$, so $5 < |2x + 3|$, so $\frac{1}{|2x + 3|} < \frac{1}{5}$.

So if $|x - 2| < 1$, then $|\frac{1}{2x+3} - \frac{1}{7}| = \frac{2}{7}\frac{|x - 2|}{|2x + 3|} < \frac{2}{7}\frac{|x - 2|}{5} = \frac{2}{35}|x - 2| < |x - 2|$. So given an $\epsilon > 0$, if we pick $\delta = \min\{\epsilon, 1\}$, then $0 < |x - 2| < \delta$ implies that $|x - 2| < \epsilon$ <u>and</u> $|x - 2| < 1$, so $|\frac{1}{2x+3} - \frac{1}{7}| < |x - 2| < \epsilon$. So $\frac{1}{2x + 3} \to \frac{1}{7}$ as $x \to 2$.