Math 423/823 Exercise Set 2 Solutions

5. [BC#1.8.1 sort of] Find the exponential form ($re^{i\theta}$) of the following numbers:

   (a) $z = \dfrac{i}{-2-2i}$ (b) $z = (\sqrt{3}-i)^6$

(a) $z = \dfrac{i}{-2-2i} = \dfrac{i(-2+2i)}{(-2-2i)(-2+2i)} = \dfrac{-2-2i}{4+4} = \dfrac{-1-i}{4}$

So $|z| = (1/4)\sqrt{(-1)^2+(-1)^2} = \sqrt{2}/4$ and $\mathrm{Arg}(z) = \arctan((-1)/(-1)) = \pi/4 - \pi = -3\pi/4$, since $(-1,-1)$ is in the third quadrant. So $z = (\sqrt{2}/4)e^{-3\pi i/4}$.

(b) $z = (\sqrt{3}-i)^6$ can be snuck up upon. $w = \sqrt{3}-i$ has $|w| = \sqrt{3+1} = 2$, and $\arg(w) = \arctan(-1/\sqrt{3}) = -\pi/6$. So $|z| = |w|^6 = 2^6 = 64$, and $\arg(z) = 6\arg(w) = 6(-\pi/6) = -\pi$. So $\mathrm{Arg}(z) = -\pi + 2\pi = \pi$. So $z = 64e^{i\pi}$.

6. [BC#1.8.8] Show that for complex numbers z_1 and z_2 we have $|z_1| = |z_2|$ if and only if there are complex numbers c_1 and c_2 so that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$.

For ($\Leftarrow$), since $|\overline{c_2}| = |c_2|$, we have
$|z_1| = |c_1c_2| = |c_1|\cdot|c_2| = |c_1|\cdot|\overline{c_2}| = |c_1\overline{c_2}| = |z_2|$.

For ($\Rightarrow$), if we write $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $r_1 = |z_1| = |z_2| = r_2$. So $z_1/z_2 = e^{i\theta_1}/e^{i\theta_2} = e^{i(\theta_1-\theta_2)} = e^{i\theta}$.

In essence, the key to what we want is the fact that $x = \dfrac{x+y}{2} + \dfrac{x-y}{2}$ and $y = \dfrac{x+y}{2} - \dfrac{x-y}{2}$. So if we set $\theta = \dfrac{\theta_1+\theta_2}{2}$ and $\psi = \dfrac{\theta_1-\theta_2}{2}$, then $\theta_1 = \theta+\psi$ and $\theta_2 = \theta-\psi$.

So if we set $c_1 = r_1e^{i\theta}$ and $c_2 = e^{i\psi}$, then $\overline{c_2} = e^{-i\psi}$ and $c_1c_2 = r_1e^{i(\theta+\psi)} = r_1e^{i\theta_1} = z_1$ and $c_1\overline{c_2} = r_1e^{i(\theta-\psi)} = r_1e^{i\theta_2} = r_2e^{i\theta_2} = z_2$, as desired.

(Note: technically, this argument fails if z_1 or z_2 is 0 (i.e., has modulus 0), but then both are 0, and we can set $c_1 = 0$ and $c_2 =$anything!)

7. [BC#1.10.3] Find all of the roots indicated:

   (a) $(-1)^{1/3}$ (b) $8^{1/6}$

(a) $(-1)^3 = -1$, and so one cube root of -1 is $-1 = e^{i\pi}$. So the set of all cube roots are $e^{i\pi}$, $e^{i(\pi-2\pi/3)} = e^{i(\pi-2\pi/3)} = e^{i\pi/3}$, and $e^{i(\pi-4\pi/3)} = e^{-i\pi/3}$

[Or, if you prefer, $-1 = e^{i\pi}$, so one cube root is $e^{i\pi/3}$ and work from there.]

(b) $8^{1/6} = (2^3)^{1/6} = \sqrt{2} = \sqrt{2}e^{0i}$ is one root, and so the set of six are $\sqrt{2}e^{0i}$, $\sqrt{2}e^{i\pi/3}$, $\sqrt{2}e^{i2\pi/3}$, $\sqrt{2}e^{i\pi} = -\sqrt{2}$, $\sqrt{2}e^{i4\pi/3} = \sqrt{2}e^{-i2\pi/3}$, and $\sqrt{2}e^{i5\pi/3} = \sqrt{2}e^{-i\pi/3}$

8. Show that $(z - z_0)(z - \overline{z_0}) = z^2 - 2\text{Re}(z_0)z + |z_0|^2$, which has real coefficients. Use this and the results of Problem #7 to express the polynomial $p(x) = x^6 - 8$ as a product of linear and quadratic polynomials with real coefficients.

   $(z - z_0)(z - \overline{z_0}) = z^2 - zz_0 - z\overline{z_0} + z_0\overline{z_0} = z^2 - z(z_0 + \overline{z_0}) + |z_0|^2$, But if $z_0 = z_0 + iy_0$, then $z_0 + \overline{z_0} = 2x_0 = 2\text{Re}(z_0)$, so
   $(z - z_0)(z - \overline{z_0}) = z^2 - (2\text{Re}(z_0))z + |z_0|^2$. And since 1, $2\text{Re}(z_0)$ and $|z_0|^2$ are real, this quadratic polynomial has real coefficients.

   The polynomial $p(x) = x^6 - 8$ has roots the sixth roots r_i of 8, which were found in problem #7(b). Writing $x^6 - 8 = (x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5)(x - r_6)$, and collecting conjugate pairs together, we have

   $(x - \sqrt{2}e^{i\pi/3})(x - \sqrt{2}e^{-i\pi/3}) = x^2 - 2\text{Re}(\sqrt{2}e^{i\pi/3})x + 2 = x^2 - 2\sqrt{2}\cos(\pi/3)x + 1 = x^2 - 2\sqrt{2}(1/2)x + 1 = x^2 - \sqrt{2}x + 2$, and
   $(x - \sqrt{2}e^{i2\pi/3})(x - \sqrt{2}e^{-i2\pi/3}) = x^2 - 2\text{Re}(\sqrt{2}e^{i2\pi/3})x + 2 = x^2 - 2\sqrt{2}\cos(2\pi/3)x + 1 = x^2 - 2\sqrt{2}(-1/2)x + 1 = x^2 + \sqrt{2}x + 2$.

   Putting it all together, we get
   $x^6 - 8 = (x - \sqrt{2})(x + \sqrt{2})(x^2 - \sqrt{2}x + 2)(x^2 + \sqrt{2}x + 2)$

   As a check, we can multiply things together! As a shortcut,
   $(x + \sqrt{2})(x^2 - \sqrt{2}x + 2) = x^3 + (\sqrt{2})^3$, and
   $(x - \sqrt{2})(x^2 + \sqrt{2}x + 2) = x^3 - (\sqrt{2})^3$, whose product is $x^6 - (\sqrt{2})^6 = x^6 - 8$.