# Math 971 Algebraic Topology
## April 5, 2005

In the end, the big result that allows us to get our homology machine really running is what is known as *excision*. To motivate it, let's try to imagine that we are trying to generalize Seifert - van Kampen. We start with $X = A \cup B$, and we want to try to express the homology of $X$ in terms of that of $A$, $B$, and $A \cap B$. With our new-found tool of long exact homology sequences, we might try to first build a short exact sequence out of the chain complexes $C_*(A \cap B), C_*(A), C_*(B)$, and $C_*(X)$. If we take our cue from the proof of S-vK, we might think of chains in $X$ as sums of chains in $A$ and $B$, except that we mod out by chains in $A \cap B$. Putting this into action, we might try the sequence

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(X) \to 0$$

where $j_n : C_n(A) \oplus C_n(B) \to C_n(X)$ is defined as $j_n(a, b) = a + b$ . In order to get exactness at the middle term (i.e., image = the kernel of this map, which is $\{(x, -x) : x \in C_n(A) \cap C_n(B)\}$), we set $i_n : C_n(A \cap B) \to C_n(A) \oplus C_n(B)$ to be $i_n(x) = (x, -x)$ , since $C_n(A \cap B) = C_n(A) \cap C_n(B)$ ! $i_n$ is then injective, and we certainly have that this sequence is exact at the middle term. But, in general, $j_n$ is far from surjective! The image of $j_n$ is the set of $n$-chains that can be expressed as sums of chains in $A$ and $B$. Which of course not every chain in $X$ can be; singular simplices in $X$ need not map entirely into either $A$ or $B$.

We can solve this by replacing $C_n(X)$ with the image of $j_n$, calling it, say, $C_n^{\{A,B\}}(X)$ ... [Note: these groups would form a chain complex!] Then we have a short exact sequence, and hence a long exact homology sequence. But it involves a "new" homology group $H_n^{\{A,B\}}(X)$ . The point is that, like S-vK, under the right conditions, this new homology is the same as $H_n(X)$ !

Starting from scratch, the idea is that, starting with an *open cover* $\{\mathcal{U}_\alpha\}$ of $X$ (or, more generally, with a collections of subspaces $A_\alpha$ whose interiors $\mathcal{U}_\alpha$ cover $X$), we build the *chain groups subordinate to the cover* $C_n^{\mathcal{U}}(X) = \{\sum a_i \sigma_i^n : \sigma_i : n : \Delta^n \to X, \sigma_i^n(\Delta^n) \subseteq \mathcal{U}_\alpha$ for some $\alpha\} \subseteq C_n(X)$ . Since the face of any simplex mapping into $\mathcal{U}_\alpha$ also maps into $\mathcal{U}_\alpha$, our ordinary boundary maps induce boundary maps on these groups, turning $(C_n^{\mathcal{U}}(X), \partial_n)$ into a chain complex. Our main result is that the inclusion $i$ of these groups into $C_n(X)$ induces an isomophism on homology. And to show this, we (could) once again use the notion of a chain homotopy.

**Theorem:** There is a chain map $b : C_n(X) \to C_n^{\mathcal{U}}(X)$ so that $i \circ b$ and $b \circ i$ are both chain homotopic to the identity. $i$ consequently induces isomorphisms on homology.

But we won't prove it quite that way! Another approach is to use the short exact sequence of chain complexes

$$0 \to C_n^{\mathcal{U}}(X) \xrightarrow{i} C_n(X) \to C_n(X)/C_n^{\mathcal{U}}(X) \to 0$$

to build a long exact homology sequence. Every third group is $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$ ; if we show that these groups are 0, then $i_*$ will be an isomorphism. And to show this, working back through the definition of homology classes in $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$, we need to show that if $z \in C_n(X)$ with $\partial z \in C_{n-1}^{\mathcal{U}}(X)$ (i.e., $z$ is a relative cycle), then there is a $w \in C_{n+1}(X)$ with $z - \partial w \in C_n^{\mathcal{U}}(X)$ (i.e., $z$ is a relative boundary). In words, if $z$ has boundary a sum of small simplices, then there is a chain $z'$ made of small simplices so that $z - z'$ is a boundary.

And the key to building $z'$ and $w$ is a process known as *barycentric subdivision*. The idea is really the same as for S-vK; we cut our singular simplices up into tiny enough pieces so that (via the Lebesgue number theorem) each piece maps into some $\mathcal{U}_\alpha$ . Unlike S-vK, though, we want to do this in a more structured way, so that the cutting up process is "compatible" with our boundary maps. And the best way to describe this cutting up is through *barycentric coordinates*. Recall that an $n$-simplex is the set of convex linar combinations $\sum x_i v_i$ with $x_i \geq 0$ and $\sum x_i = 1$ . The map which sends an $n$-simplex to the $n$-simplex $\Delta^n$ is literally the map $\sum x_i v_i \mapsto (x_0, \dots, x_n)$ . These are the barycentric coordinates of an $n$-simplex. Since, formally, all singular simplices are considered to have $\Delta^n$ for their domain, we can describe barycentric subdivision by describing how to cut up $\Delta^n$. The idea is to build

a process the is compatible with the boundary map, so that the subdivision, when restricted to a sub-simplex, is the subdivision of that sub-simplex. A 1-simplex $[v_0, v_1]$ is subdivided by adding the barycenter $w = (v + 0 + v_1)/2$ as a vertex, cutting $[v_0, v_1]$ into two 1-simplices ,$[v_0, w], [w, v_1]$ . A 2-simplex $[v_0, v_1, v_2]$ will, to be compatible withe boundary map, have its boundary cutinto 6 1-simplices; using the barycenter $(v_0 + v_1 + v_2)/3$ we can cone off each of these 1-simplices to subdivide $[v_0, v_1, v_2]$ into 6 2-simplices. Taking the cue that $2 = (1 + 1)!$ , $6 = (2 + 1)!$ is probably no accident, we might expect that an $n$-simplex will be cut into $(n + 1)!$ $n$-simplices. Note that this is thew number of ways of ordering the vertices of our simplex. And following the "pattern" of our two test cases, where each new simplex was the convex span of vertices chosen as (vertex) , (barycenter of a 1-simplex having (vertex) as a vertex), (barycenter of a 2-simplex containing the previous 2 vertices), etc., we are led to the idea that the barycentric subdivision of an $n$-simplex $[v_0, \dots, v_n]$ is the $(n + 1)!$ $n$-simplices,
$$[v_{\alpha(0)}, (v_{\alpha(0)} + v_{\alpha(1)})/2, (v_{\alpha(0)} + v_{\alpha(1)} + v_{\alpha(2)})/3, \dots, (v_{\alpha(0)} + \cdots v_{\alpha(n)})/(n+1)]$$
one for every permutation $\alpha$ of $\{0, \dots, n\}$ . And since we want to take into account <u>orientations</u> as well, the natural thing to do is to define the barycentric subdivision of a singular $n$-simplex $\sigma : [v_0, \dots, v_n] \to X$ to be
$$S(\sigma) = \sum_\alpha (-1)^{\text{sgn}(\alpha)} \sigma|_{[v_{\alpha(0)}, (v_{\alpha(0)} + v_{\alpha(1)})/2, (v_{\alpha(0)} + v_{\alpha(1)} + v_{\alpha(2)})/3, \dots, (v_{\alpha(0)} + \cdots v_{\alpha(n)})/(n+1)]}$$
where the sum is taken over all permutations of $\{0, \dots, n\}$ . This (extending linearly over the chain group) is the subdivision operator, $S : C_n(X) \to C_n(X)$ . A "routine" calculation establishes that $\partial S = S \partial$ , i.e., it is a chain map (i.e., it behaves well on the boundary of our simplices). The point to this operator is that all of the subsimplices in the sum are a definite <u>factor</u> smaller than the original simplex. In fact, if the diameter of $[v_0, \dots, v_n]$ is $d$ (the largest distance between points, which will, because it is the convex span of the vertices, be the largest distance between vertices), then every individual simplex in $S(\sigma)$ will have diameter at most $nd/(n+1)$ (the result of a little Euclidean geometry and induction). So by <u>repeatedly</u> applying the subdivision operator $S$ to a singular simplex, we will obtain a singular chain $S^k(\sigma)$, which is "really" $\sigma$ written as a sum of tiny simplices, whose singular simplices have image as small as we want. Or put more succinctly, if $\{\mathcal{U}_\alpha\}$ is an open cover of $X$ and $\sigma : \Delta^n \to X$ is a singular $n$-simplex, then choosing a Lebesgue number $\epsilon$ for the open cover $\sigma^{-1}(\mathcal{U}_\alpha)$ of the compact metric space $\Delta^n$, and choosing a $k$ with $d(n/(n+1))^k < \epsilon$, we find that $S^k(\sigma)$ is a sum of singular simplices each of which maps into one of the $\mathcal{U}_\alpha$, i.e., $S^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

In the end, we will choose our needed "small" cycle to be $z' = S^k z$. and to show that their difference is a boundary, we will build a chain homotopy between $Id$ and $S^k$. And to do <u>that,</u> we define a map $R : C_n(X) \to C_{n+1}(X \times I)$; when followed by the projection-induced map $p_\# : C_{n+1}(X \times I) \to C_{n+1}(X)$, we get a map $T : C_n(X) \to C_{n+1}(X)$, and show that $\partial T + T \partial = I - S$ . Then we set $H = \sum T S^j$, where the sum is taken over $j = 0, \dots k - 1$. Once we define $T$ (!) , we will have $\partial H_k + H_k \partial = \sum \partial T S^j + T S^j \partial = \sum (\partial T + T \partial) S^j = \sum (S^j - S^{j+1}) = I - S^k$ (since the last sum telescopes). And defining $R$, is, formally, just another particular sum. Setting up some notation, thinking of $\Delta^n \times I$ , as before, as having vertices $\{v_0, \dots v_n\}$ on the 0-end and $\{w_0, \dots, w_n\}$ on the 1-end, $N = \{0, \dots, n\}$, $\Pi(Q) = $ the group of permutations of $Q$, and $\sigma' = \sigma \times I : \Delta^n \times I \to X \times I$), we have
$$R(\sigma) = \sum_{A \subseteq N} \sum_{\pi \in \Pi(N \setminus A)} \left\{ (-1)^{|A|} (-1)^{\text{sgn}(\pi)} \prod_{j \in N \setminus A} (-1)^j \right\}$$
$$\sigma'|_{[v_{i_0}, \dots, v_{i_j}, (w_{i_0} + \cdots w_{i_j})/(j+1), (w_{i_0} + \cdots w_{i_j} + w_{\pi(i_{j+1})})/(j+2), \dots, (w_{i_0} + \cdots w_{i_j} + w_{\pi(i_{j+1})} + \cdots w_{\pi(i_n)})/(n+1)]}$$
where we sum over all <u>non-empty</u> subsets of $\{0, \dots n\}$ (with the induced ordering on vertices from the ordering on $\{0, \dots, n\}$). Intuitively, this map "interpolates" between the simplex $[v_0, \dots v_n]$ and the barycentric subdivision on $w_0, \dots, w_n$, by taking the (signed sums of the) convex spans of simplices on the bottom (0) and simplices on the top (1). Again, a "routine" calculation will establish that $\partial T + T \partial = I - S$ , as desired. [At any rate, I verified it for n=1,2; the formula for the sign of each simplex was determined by working backwards from these examples.]

And the point to all of these calculations was that if $\{\mathcal{U}_\alpha\}$ is an open cover of $X$, then the inclusions $i_n : C_n^{\mathcal{U}}(X) \to C_n(X)$ induce isomorphisms on homology. This gives us two big theorems. The first is

**Mayer-Vietoris Sequence**: If $X = \mathcal{U} \cup \mathcal{V}$ is the union of two open sets, then the short exact sequences
$0 \to C_n(\mathcal{U} \cap \mathcal{V}) \to C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \to C_n^{\{\mathcal{U}, \mathcal{V}\}}(X) \to 0$   , together with the isomorphism above, give the long exact sequence

$$\cdots \to H_n(\mathcal{U} \cap \mathcal{V}) \overset{(i_{\mathcal{U}*}, -i_{\mathcal{V}*})}{\to} H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \overset{j_{\mathcal{U}*} + j_{\mathcal{V}*}}{\to} H_n(X) \overset{\partial}{\to} H_{n-1}(\mathcal{U} \cap \mathcal{V}) \to \cdots$$

And just like Seifert - van Kampen, we can replace open sets by sets $A, B$ having neigborhoods which deformation retract to them, and whose intersection deformation retracts to $A \cap B$. For example, subcomplexes $A, B \subseteq X$ of a CW-complex, with $A \cup B = X$ have homology satisfying a long exact sequence

$$\cdots \to H_n(A \cap B) \overset{(i_{A*}, -i_{B*})}{\to} H_n(A) \oplus H_n(B) \overset{j_{A*} + j_{B*}}{\to} H_n(X) \overset{\partial}{\to} H_{n-1}(A \cap B) \to \cdots$$

And this is also true for reduced homology; we just augment the chain complexes used above with the short exact sequence    $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$ , where the first non-trivial map is $a \mapsto (a, -a)$ and the second is $(a, b) \mapsto a + b$ .

And now we can do some meaningful calculations! An $n$-sphere $S^n$ is the union $S^n_+ \cup S^n_-$ of its upper and lower hemispheres, each of which is contractible, and have intersection $S^n_+ \cap S^n_- = S^{n-1}_0$ the equatorial $(n-1)$-sphere. So Mayer-Vietoris gives us the exact sequence
$\cdots \to \widetilde{H}_k(S^n_+) \oplus \widetilde{H}_k(S^n_-) \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to \widetilde{H}_{k-1}(S^n_+) \oplus \widetilde{H}_{k-1}(S^n_-) \to \cdots$     , i.e,
$0 \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to 0$    i.e., $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for every $k$ and $n$. So by induction,

$$\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z}, & if\text{k=n} \\ 0, & otherwise \end{cases}$$

There is another piece of homological algebra that we will find useful ; the Five Lemma. It allows us to <u>compare</u> the information contained in two long exact sequences.

**Five Lemma:** If we have abelian groups and maps

$$
\begin{array}{ccccccccc}
A_n & \overset{f_n}{\to} & B_n & \overset{g_n}{\to} & C_n & \overset{h_n}{\to} & D_n & \overset{i_n}{\to} & E_n \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow \\
A_{n-1} & \overset{f_{n-1}}{\to} & B_{n-1} & \overset{g_{n-1}}{\to} & C_{n-1} & \overset{h_{n-1}}{\to} & D_{n-1} & \overset{i_{n-1}}{\to} & E_{n-1}
\end{array}
$$

where the rows are exact, and the maps $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then $\gamma$ is an isomorphism.

The proof is in some sense literally a matter of doing the only thing you can. To show injectivity, suppose $x \in C_n$ and $\gamma x = 0$, then $h_{n-1}\gamma x = \delta h_n x = 0$, so, since $\delta$ is injective, $h_n x = 0$. So by the exactness at $C_n$, $x = g_n y$ for some $y \in B_n$. Then $g_{n-1}\beta y = \gamma g_n y = \gamma x = 0$, so by exactness at $B_{n-1}$, $\beta y = f_{n-1} z$ for some $z \in A_{n-1}$. Then since $\alpha$ is surjective, $f_{n-1} z = \alpha w$ for some $w$. Then $0 = g_n f_n w$ . But $\beta f_n w = f_{n-1} \alpha w$ $f_{n-1} z = \beta y$, so since $\beta$ is injective, $y = f_n w$ . So $0 = g_n f_n w = g_n y = x$. So $x = 0$.

For surjectivity, suppose $x \in C_{n-1}$. Then $h_{n-1} x \in D_{n-1}$, so since $\delta$ is surjective, $h_{n-1} x = \delta y$ for some $y \in D_n$. Then $\epsilon i_n y = i_{n-1} \delta y = i_{n-1} h_{n-1} x = 0$, so since $\epsilon$ is injective, $i_n y = 0$. So by exactness at $D_n$, $y = h_n z$ for some $z \in C_n$. Then $h_{n-1} \gamma z = \delta h_n z = \delta y = h_{n-1} x$, so $h_{n-1}(\gamma z - x) = 0$, so by exactness at $C_{n-1}$, $\gamma z - x = g_{n-1} w$ for some $w \in B_{n-1}$. Then since $\beta$ is surjective, $w = \beta u$ for some $u \in B_n$. Then $\gamma g_n u = g_{n-1} \beta u = g_{n-1} w = \gamma z - x$, so $x = \gamma z - \gamma g_n u = \gamma(z - g_n u)$ . So $\gamma$ is onto.

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\text{cl}_X(B) \subseteq \text{int}_X(A)$, then for every $k$ the inclusion-induced map $H_k(X \setminus B, A \setminus B) \to H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\text{int}_X(A) \cup \text{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \to H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$ .] To prove the second statement, we know that the inclusion $C_n^{\{A, B\}}(X) \to C_n(X)$ induce isomorphisms on homology, as does $C_n(A) \to C_n(A)$, so, by the five lemma, the induced map $C_n^{\{A, B\}}(X)/C_n(A) \to C_n(X)/C_n(A) = C_n(X, A)$ induces isomorphisms on homology. But the inclusion

$C_n(B) \to C_n^{\{A,B\}}(X)$ induces a map $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \to C_n^{\{A,B\}}(X)/C_n(A)$ which is an isomorphism of chain groups; a basis for $C_n^{\{A,B\}}(X)/C_n(A)$ consists of singular simplices which map into $A$ or $B$, but don't map into $A$, i.e., of simplices mapping into $B$ but not $A$, i.e., of simplices mapping into $B$ but not $A \cap B$. But this is the <u>same</u> as the basis for $C_n(B, A \cap B)$ !

With these tools, we can start making some <u>real</u> homology computations. First, we show that if $\emptyset \neq A \subseteq X$ is "nice enough", then $H_n(X, A) \cong \widetilde{H}_n(X/A)$ . The definition of nice enough, like Seifert - van Kampen, is that $A$ is closed and has an open neighborhood $\mathcal{U}$ that deformation retracts to $A$ (think: $A$ is the subcomplex of a CW-complex $X$). Then using $\mathcal{U}.X \setminus A$ as a cover of $X$, and $\mathcal{U}/A, (X \setminus A)/A$ as a cover of $X/A$, we have

$$\widetilde{H}_n(X/A) \overset{(1)}{\cong} H_n(X/A, A/A) \overset{(2)}{\cong} H_n(X/A, \mathcal{U}/A) \overset{(3)}{\cong} H_n(X/A \setminus A/A, \mathcal{U}/A \setminus A/A) \overset{(4)}{\cong} H_n(X \setminus A, \mathcal{U} \setminus A) \overset{(5)}{\cong} H_n(X, A)$$

Where (1),(2) follow from the LES for a pair, (3),(5) by excision, and (4) because the restriction of the quotient map $X \to X/A$ gives a homeomorphism of pairs.

Second, if $X, Y$ are $T_1$, $x \in X$ and $y \in Y$ each have neighborhoods $\mathcal{U}, \mathcal{V}$ which deformation retract to each point, then the one-point union $Z = X \vee Y = (X \coprod Y)/(x = y)$ has $\widetilde{H}_n(Z) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$; this follows from a similar sequence of isomorphisms. Setting $z =$ the image of $\{x, y\}$ in $Z$, we have

$$\widetilde{H}_n(Z) \cong H_n(Z, z) \cong H_n(Z, \mathcal{U} \vee \mathcal{V}) \cong H_n(Z \setminus z, \mathcal{U} \vee \mathcal{V} \setminus z) \cong H_n([X \setminus x] \coprod [Y \setminus y], [\mathcal{U} \setminus x] \coprod [\mathcal{V} \setminus y]) \cong$$
$$H_n(X \setminus x, \mathcal{U} \setminus x) \oplus H_n(Y \setminus y, \mathcal{V} \setminus y) \cong H_n(X, x) \oplus H_n(Y, y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

By induction, we then have $\widetilde{H}_n(\vee_{i=1}^k X_i) cong \oplus \_i = 1^k \widetilde{H}_n(X_i)$

We have so far introduced two homologies; simplicial, $H_*^\Delta$, whose computation "only" required some linear algebra, and singular, $H_*$, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For $\Delta$-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every $X$. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology <u>could</u> have been built for simplicial homology, including relative homology (for a sub-$\Delta$-complex $A$ of $X$), and a SES of chain groups, giving a LES sequence for the pair,

$$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X, A) \to H_{n-1}^\Delta(A) \to \cdots$$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on $k$-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on $k$ using the Five Lemma applied to the diagram

$$
\begin{array}{ccccccccc}
H_{n+1}^\Delta(X^{(k)}, X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)}, X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^{(k)}, X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)}, X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)})
\end{array}
$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because we can, in each case, identify these groups. $H_n(X^{(k)}, X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$

Some topological results with homological proofs: if $\mathbb{R}^n \cong \mathbb{R}^m$, via $h$, then $n = m$ . This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, \mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial \mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial \mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n - 1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$ .

More generally, we can establish a result which is known as *invariance of domain*, a result which is useful in both topology and analysis.

**Invariance of Domain:** If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

Note it is enough to proof this for our favorite open set, which in this context will be $\mathcal{V} = (-1, 1)^n \subseteq \mathbb{R}^n$, since given any open $\mathcal{U}$ and $x \in \mathcal{U}$, we can find an injective linear map $h : (-1, 1)^n \to \mathcal{U}$ taking $0$ to $x$. If we can show that $f \circ h$ has open image, then $f(x) \in f \circ h(\mathcal{V}) \subseteq f(\mathcal{U})$ shows that $f(x)$ has an open neighborhood in $f(\mathcal{U})$ . Since $x$ is arbitrary, $f(\mathcal{U})$ is open.

This in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of $f$ with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.

Our next goal is to show that, when both make sense, simplicial and singular homology are isomorphic. In fact, the inclusion of the simplicial chain groups into the singular ones induces an isomorphism on homology.