Math 325 Problem Set 1 Solutions

1. Working from our axioms for the ordered field $\mathbb{R}$, show that if $x, y \in \mathbb{R}$ and $x < y$, then $x < \dfrac{x+y}{2} < y$.

   Letting $\mathcal{P}$ stnd for the positive reals, we wish to show that, if $y - x \in \mathcal{P}$, then $\dfrac{x+y}{2} - x \in \mathcal{P}$ and $y - \dfrac{x+y}{2} \in \mathcal{P}$.

   But $\dfrac{x+y}{2} - x = \dfrac{x}{2} + \dfrac{y}{2} - x = \dfrac{x}{2} + \dfrac{y}{2} - (\dfrac{x}{2} + \dfrac{x}{2}) = \dfrac{y}{2} - \dfrac{x}{2} = \dfrac{1}{2}(y-x)$

   And since $y - x \in \mathcal{P}$ and $\dfrac{1}{2} \in \mathcal{P}$, their product is, too. So $\dfrac{x+y}{2} - x \in \mathcal{P}$.

   A similar argument establishes the second assertion; in fact,
   $y - \dfrac{x+y}{2} = (\dfrac{y}{2} + \dfrac{y}{2}) - (\dfrac{x}{2} + \dfrac{y}{2}) = \dfrac{y}{2} - \dfrac{x}{2} = \dfrac{1}{2}(y-x)$ as well!, so is also in $\mathcal{P}$.

   [Why is $\dfrac{1}{2} \in \mathcal{P}$? Because otherwise, since $1 > 0$ (below!), we have $2 = 1 + 1 > 0$, so if $\dfrac{1}{2} < 0$ then $1 = 2 \cdot \dfrac{1}{2} < 0$, a contradiction!]

2. [Lay, p. 115, # 11.3 (c,d,f)] Show:

   (α) If $x \neq 0$, then $\dfrac{1}{x} \neq 0$ and $\dfrac{1}{(1/x)} = x$.

   If $\dfrac{1}{x} = 0$, then $0 = 0 \cdot x = \dfrac{1}{x}x = 1$, so $0 = 1$, violating one of our axioms. So $\dfrac{1}{x} \neq 0$.
   Then since $x \cdot \dfrac{1}{x} = 1 = \dfrac{1}{(1/x)} \cdot \dfrac{1}{x}$, both x and $\dfrac{1}{(1/x)}$ work as an inverse to $\dfrac{1}{x}$, so they are equal; $x = \dfrac{1}{(1/x)}$.

   (β) If $xy = xz$ and $x \neq 0$, then $y = z$.

   Since $x \neq 0$, it has an inverse $1/x$, and then $xy = xz$ implies that
   $y = 1 \cdot y = ((1/x)x)y = (1/x)(xy) = (1/x)(xz) = ((1/x)x)z = 1 \cdot z = z$, so $y = z$.

   (γ) $0 < 1$.

   This asserts that $1 - 0 = 1 \in \mathcal{P}$. The only other possibilities are that $-1 \in \mathcal{P}$ or $1 = 0$. But $1 = 0$ is impossible, essentially by the definition of the number 1. And if $-1 \in \mathcal{P}$, then $(-1)(-1) \in \mathcal{P}$, as well. But from class we know that $(-1)(-1) = 1 \cdot 1 = 1$, so if $-1 \in \mathcal{P}$ then $1 \in \mathcal{P}$. But our axioms for $\mathcal{P}$ stated that only one of these two statements could be true. So since $-1 \in \mathcal{P}$ requires that $1 \in \mathcal{P}$ as well, it cannot be true that $-1 \in \mathcal{P}$. So we must have $1 \in \mathcal{P}$, as desired.

3. Working from our axioms for the ordered field $\mathbb{R}$, show that for any $x \in \mathbb{R}$, $x^2 + 1 > 0$.

   We know from a problem above that $1 > 0$. So if we show that $x^2 \geq 0$ for every $x \in \mathbb{R}$, we'll be done: $x^2 + 1 > x^2 + 0 = x^2 \geq 0$, So $x^2 + 1 > 0$.

   [On some level, we need a little more: $a > b$ and $b \geq c$ implies $a > c$, because either $b > c$ so $a > b$ gives $a > c$, or $b = c$, so $a > b$ is the same thing as $a > c$.]

   But for any $x \in \mathbb{R}$, we know that either $x > 0$, $-x > 0$, or $x = 0$. But $x > 0$ implies $x^2 = x \cdot x > 0$, so $x^2 \geq 0$. $-x > 0$ implies that $x^2 = x \cdot x = (-x)(-x) > 0$, so $x^2 \geq 0$. And finally $x = 0$ implies that $x^2 = x \cdot x = 0 \cdot 0 = 0$, so $x^2 \geq 0$. So every possible case leads to the same conclusion, that $x^2 \geq 0$.

   So for every $x \in \mathbb{R}$, $x^2 \geq 0$. This finishes our argument, so $x^2 + 1 > 0$ for every real number x.

   [N.B.: In particular, $x^2 + 1 = 0$ has no solution. This shows that the complex numbers $\mathbb{C}$ cannot support an order '$<$' making $\mathbb{C}$ an ordered field. This is because the complex numbers possess a number, i, satisfying $i^2 + 1 = 0$. So no ordering on $\mathbb{C}$ can satisfy $i^2 + 1 > 0$ (since $0 > 0$ violates trichotomy). So $\mathbb{C}$ cannot be an ordered field.]