Math 325 Problem Set 6 Solutions

Starred (*) problems were due Friday, October 12.

(*) 31. (Belding and Mitchell, p.80, #2) Show that if $f : (a,b) \to \mathbb{R}$ is a continuous function, then the function $g : (a,b) \to \mathbb{R}$ given by $g(x) = |f(x)|$ is also continuous. (You should argue directly from ϵ's and δ's.)

We start by knowing that for any $c \in (a,b)$ and given an $\epsilon > 0$, we can always find a $\delta > 0$ so that $|x-a| < \delta$ and $x \in (a,b)$ implies that $|f(x) - f(a)| < \epsilon$. What we wish to show is that for $c \in (a,b)$ and $\epsilon > 0$, there is a $\delta' > 0$ so that $|x-a| < \delta'$ and $x \in (a,b)$ implies that $|g(x) - g(a)| < \epsilon$.

But! $|g(x) - g(a)| = \Big||f(x)| - |f(a)|\Big|$, and, by the 'reverse' triangle inequality,

$$\Big||A| - |B|\Big| \le |A - B| \text{ for any } A, B \in \mathbb{R}.$$

[Recall that the proof is fairly short: $|A| = |B + (A-B)| \le |B| + |A-B|$ implies $|A| - |B| \le |A-B|$, which is half of what we need; $|B| = |A + (B-A)| \le |A| + |B-A| = |A| + |A-B|$ provides the other half.] So $|g(x) - g(c)| = \Big||f(x)| - |f(c)|\Big| \le |f(x) - f(c)|$ for any $x, c \in (a,b)$. $|f(x) - f(c)| < \epsilon$ automatically implies that $\Big|g(x) - g(c)\Big| < \epsilon$.

So, given $\epsilon > 0$, choose $\delta > 0$ so that $|x-c| < \delta$ and $x \in (a,b)$ implies that $|f(x) - f(c)| < \epsilon$. Then $|x-c| < \delta$ implies that $|g(x) - g(c)| \le |f(x) - f(c)| < \epsilon$, as well. So for every $\epsilon > 0$ we can find $\delta > 0$ so that $|x-c| < \delta$ and $x \in (a,b)$ implies that $|g(x) - g(c)| < \epsilon$. So g is continuous at $x = c$ for every $c \in (a,b)$.

(*) 33. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for every $x \in \mathbb{Q}$. Show that $f(x) = 0$ for every $x \in \mathbb{R}$.

[Hint: what's the alternative? Remember that rational numbers are 'everywhere'!]

Suppose not. Suppose that there is an $a \in \mathbb{R}$ with $f(a) \ne 0$. Then $|f(a)| = \epsilon > 0$, and so, since f is continuous at a, there is a $\delta > 0$ so that $|x-a| < \delta$ implies that $|f(x) - f(a)| < \epsilon = |f(a)|$. In particular we know that $|x-a| < \delta$ implies that $|f(a) - |f(x)| \le |f(a) - f(x)| = |f(x) - f(a)| < |f(a)|$ (by the reverse triangle inequality, from a previous problem) so $0 = |f(a)| - |f(a)| < |f(x)|$, so $0 < |f(x)$. In particular (again!) we have that $|x-a| < \delta$ implies that $f(x) \ne 0$.

But this is impossible. No matter what a and $\delta > 0$ are, we know that there is an $x \in \mathbb{Q}$ so that $|x-a| < \delta$. So, by hypothesis, $f(x) = 0$. But the above says that, for a particular choice of $\delta > 0$, every such x has $f(x) \ne 0$. Therefore, the assumption we made, that there is an $a \in \mathbb{R}$ with $f(a) \ne 0$, must be false. So $f(x) = 0$ for every $x \in \mathbb{R}$.

(*) 36. (Belding and Mitchell, p.89, #9) Use the intermediate value theorem to show that any positive number $a \in \mathbb{R}$, $a > 0$ has an n-th root, that is, for any $n \in \mathbb{N}$, there is some real number $x \geq 0$ such that $x^n = a$.

[The textbook provides an outline that you could follow.]

Because $f(x) = x^n$, f$\mathbb{R} \to \mathbb{R}$ is continuous (it ia a polynomial), we can show that there is a $c \in \mathbb{R}$ so that $f(c) = c^n = a$ by showing that there are real numbers $u < v$ so that $f(u) = u^n < a$ and $f(v) = v^n > a$, since then by the intermediate Value Theorem, since f is continuous on the interval $[u, v]$ and a lies between $f(u)$ and $f(v)$, there is a $c \in [u, v]$ with $f(c) = c^n = a$.

Finding a $u \in \mathbb{R}$ with $f(u) = u^n < a$ is fairly quick, since $a > 0$ and so $0^n = 0 < a$, so we can take $u = 0$. To find a v with $v^n > a$, we can rely on the fact that, for any $n \in \mathbb{N}$, x^n is 'usually' larger than x. In particular, if $x \geq 1$, then by induction, for every $n \in \mathbb{N}$ we have $x^n \geq x$. The base case $n = 1$ is the (true) statement that $x \geq x$, while $x^n \geq x$ implies that $x^{n+1} = x \cdot x^n \geq 1 \cdot x^n = x^n \geq x$ (where the first inequality uses $x \geq 1$ and multiplication by $x^n > 0$ preserves inequalities).

This means that for any $a > 0$ we have $(a+1) > a$ and $a+1 > 1$ so $(a+1)^n \geq a+1 > a$, so setting $v = a+1$ we have $v^n > a$. So $f(u) = f(0) = 0^n = 0 < a < a+1 \leq (a+1)^n = f(a+1) = f(v)$, so IVT implies that there is a $c \in [0, a+1]$ so that $f(c) = c^n = a$. So for every positive integer $n \in \mathbb{N}$, every positive real number a has a (positive) n-th root c (with $c^n = a$).

A selection of further solutions.

34. Using the problem #33 above, show that if $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous functions, and $f(x) = g(x)$ for every $x \in \mathbb{Q}$, then $f = g$ (i.e., $f(x) = g(x)$ for every $x \in \mathbb{R}$).

['A continuous function is <u>determined</u> by its values on the rational numbers.']

This has a fairly quick proof. If f and g are both continuous, then $h(x) = f(x)-g(x)$ is also continuous, and our hyppothesis iplies that $h(x) = f(x)-g(x) = 0$ for every $x \in \mathbb{Q}$. Our previous problem therefore tells us that $h(x) = 0$ for every x, so $f(x) = g(x)$ for every $x \in \mathbb{R}$. So $f = g$.

32. Using the previous problem #31 (and a problem from a previous problem set!), show that if $f, g : (a, b) \to \mathbb{R}$ are continuous functions, then the function $M : (a, b) \to \mathbb{R}$ given by $M(x) = \max\{f(x), g(x)\}$ is also continuous.

From a previous problem, we know that treating $f(x)$ and $g(x)$ as real numbers, we have

$$M(x) = \max\{f(x), g(x)\} = \frac{(f(x) + g(x) + |f(x) - g(x)|}{2}$$
$$= \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|) .$$

But since f and g are continuous, we know that $f(x)-g(x)$ is continuous [the difference of two continuous functions is continuous], and so by the previous problem, $|f(x)-g(x)|$ is continuous. Then $f(x) + g(x) + |f(x) - g(x)|$ is continuous [the sum of continuous functions is continuous], and so $M(x) = \max\{f(x), g(x)\} = \frac{1}{2}(f(x)+g(x)+|f(x)-g(x)|)$ is continuous [a constant multiple of a continuous function is continuous], as desired.