Math 325 Problem Set 5 Solutions

16. Show that if $a, b \in \mathbb{R}$ and $0 < a \le b$, then $\sqrt{a} \le \sqrt{b}$. [Suppose not.....]

If not, then $\sqrt{a} > \sqrt{b}$. But since $\sqrt{a}$ means the positive square root, we have $\sqrt{a}, \sqrt{b} > 0$ (since either equal to 0 implies a or b is 0). Then $\sqrt{a} > \sqrt{b}$ implies $a = \sqrt{a}\sqrt{a} > \sqrt{a}\sqrt{b}$, and $\sqrt{a}\sqrt{b} > \sqrt{b}\sqrt{b} = b$, so $a > \sqrt{a}\sqrt{b} > b$, so $a > b$, a contradiction. So $\sqrt{a} > \sqrt{b}$ is impossible; so we must have $\sqrt{a} \le \sqrt{b}$, as desired.

17. [Lay, p.173, # 17.15] Show that as $n \to \infty$ we have

(a): $\sqrt{n+1} - \sqrt{n} \to 0$.

$a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \dfrac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$ But then $|a_n - 0| = |a_n| = a_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{\sqrt{n}} < \epsilon$ so long as $n > \dfrac{1}{\epsilon^2}$, so, for any $\epsilon > 0$, choosing an $N > \dfrac{1}{\epsilon^2}$ gives $n \ge N$ implies $|a_n - 0| < \epsilon$, so $a_n \to 0$.

(b): $\sqrt{n^2+1} - n \to 0$.

$b_n = \sqrt{n^2+1} - n = \sqrt{n^2+1} - \sqrt{n^2} = \dfrac{(\sqrt{n^2+1} - \sqrt{n^2})(\sqrt{n^2+1} + \sqrt{n^2})}{\sqrt{n^2+1} + \sqrt{n^2}} = \dfrac{(n^2+1) - n^2}{\sqrt{n^2+1} + \sqrt{n^2}} = \dfrac{1}{\sqrt{n^2+1} + \sqrt{n^2}}$ But then $|b_n - 0| = |b_n| = b_n = \dfrac{1}{\sqrt{n^2+1} + \sqrt{n^2}} < \dfrac{1}{\sqrt{n^2}} = \dfrac{1}{n} < \epsilon$ so long as $n > \dfrac{1}{\epsilon}$, so, for any $\epsilon > 0$, choosing an $N > \dfrac{1}{\epsilon}$ gives $n \ge N$ implies $|b_n - 0| < \epsilon$, so $b_n \to 0$.

(c): $\sqrt{n^2+n} - n \to \dfrac{1}{2}$.

$c_n = \sqrt{n^2+n} - n = \sqrt{n^2+n} - \sqrt{n^2} = \dfrac{(\sqrt{n^2+n} - \sqrt{n^2})(\sqrt{n^2+n} + \sqrt{n^2})}{\sqrt{n^2+n} + \sqrt{n^2}} = \dfrac{(n^2+n) - n^2}{\sqrt{n^2+n} + \sqrt{n^2}} = \dfrac{n}{\sqrt{n^2+n} + \sqrt{n^2}} = \dfrac{1}{\sqrt{1+\frac{1}{n}} + 1}$ But since $1overn \to 0$ as $n \to \infty$, $\sqrt{1 + \dfrac{1}{n}} \to \sqrt{1+0} = \sqrt{1} = 1$, so $\sqrt{1 + \dfrac{1}{n}} + 1 \to 1 + 1 = 2$, so $c_n = \dfrac{1}{\sqrt{1+\frac{1}{n}} + 1} \to \dfrac{1}{2}$, as desired.

18. [Lay, p.180, # 18.7] Define the sequence $(a_n)_{n=1}^{\infty}$ by $a_1 = \sqrt{6}$, and, for $n > 1$,

$$a_n = \sqrt{6 + a_{n-1}}$$

(so, e.g, $a_2 = \sqrt{6+\sqrt{6}}$, $a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$, etc.). Show that the sequence is monotone and bounded, and determine what it converges to.

$a_2 = \sqrt{6+\sqrt{6}} > \sqrt{6+\sqrt{4}} = \sqrt{8} > \sqrt{6} = a_1$, so the sequence starts by increasing. We claim: the sequence is increasing: $a_{n+1} \ge a_n$ for every $n \ge 1$. This we can show by induction: the initial step was just demonstrated, and if we suppose that $a_{n+1} \ge a_n$, then $a_{n+2} = \sqrt{6 + a_{n+1}} \ge \sqrt{6 + a_n} = a_{n+1}$, where the inequality in the middle follows from $6 + a_{n+1} \ge 6 + a_n > 0$ (by Problem # 16 above), which in turn is true since $a_{n+1} \ge a_n$, by our inductive hypothesis. So $a_{n+1} \ge a_n$ implies that $a_{n+2} \ge a_{n+1}$, giving our inductive step. So $(a_n)_{n=1}^{\infty}$ is increasing, by induction.

To show that it is bounded above, what we would like to argue (by induction!) is that if $a_n \le M$, then $a_{n+1} \le M$. But what we do know is that $a_{n+1} = \sqrt{6 + a_n} \le \sqrt{6 + M}$, essentially by the same argument used above. So if we know that (well, $a_1 \leq M$ and) $\sqrt{6+M} \leq M$, we could finish an inductive argument. But numbers like this are all over the place: for example, $\sqrt{6+10} = \sqrt{16} = 4 \leq 10$, and $a_1 = \sqrt{6} \leq \sqrt{9} = 3 \leq 10$, so $M = 10$ works. That is, $a_1 \leq 10$ and $a_n \leq 10$ implies that $a_{n+1} \leq \sqrt{6+10} = 4 \leq 10$, so by induction $a_n \leq 10$ for all $n \in \mathbb{N}$. [Lots of other upper bounds work, as well...]

This gives us that $(a_n)_{n=1}^{\infty}$ is increasing and bounded above, so $a_n \to L$ for some $L \in \mathbb{R}$. But then $a_{n+1} \to L$ as well, but $a_{n+1} = \sqrt{6+a_n} \to \sqrt{6+L}$, by our limit theorems. So since limits are unique, $L = \sqrt{6+L}$. Solving for L, we find that $L^2 = 6+L$, so $L^2 - L - 6 = (l+2)(L-3) = 0$, so $L = -2$ or $L = 3$. But since $a_1 = \sqrt{6} \geq 0$ and the sequence is increasing, $a_n \geq 0$ for all n, so $L \geq 0$. So we must have $L = 3$, so $a_n \to 3$.

19. [Lay, p.180, # 18.5 (sort of)] Show by example that, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are both monotone sequences, then the following conclusions need <u>not</u> be true:

    (a): $(c_n)_{n=1}^{\infty}$, where $c_n = a_n + b_n$, is monotone.

    (b): $(d_n)_{n=1}^{\infty}$, where $d_n = a_n b_n$, is monotone.

    What additional hypotheses (if any), will make one or both of these conclusions <u>true</u>?

Some experimentation will probably convince you that if a_n and b_n are both monotone increasing, then $a_n + b_n$ is, too: $a_{n+1} \geq a_n$ and $b_{n+1} \geq b_n$ implies $a_{n+1} + b_{n+1} \geq a_n + b_n$. Similarly, if both are decreasing, then their sum will be decreasing. But if one is increasing and the other is decreasing, then more or less anything can happen; $(a_{n+1} + b_{n+1}) - (a_n + b_n) = (a_{n+1} - a_n) + (b_{n+1} - b_n)$, so whether the sum increases or decreases comes down to whether the increasing sequence is increasing <u>faster</u> than the decreasing sequence is decreasing, or not. And we can tailor the sequences so that sometimes the increasing sequence wins, and sometimes the decreasing sequence wins. For example:

$a_n = -23n$ always goes down by 23, but $b_n = n^2$ intially goes up more slowly (by 3, then 5, then 7, then...), so $a_n + b_n = n^2 - 23n$ initially decreases. But for n large, b_n increases by $2n+1$ (which will be larger than 23), so later the sum is increasing. A more stark example can be built where $a_n + b_n$ alternates increasing and decreasing: $a_n = (-1)^n - 2n$ alternates standing still and dropping by 4 (the sequence is $-3, -3, -7, -7, -11, -11, \ldots$) while $b_n = (-1)^n + 2n$ alternates standing still and going up by 4 (it is $1, 5, 5, 9, 9, \ldots$). Their sum is then $a_n + b_n = 2(-1)^n$ which alternates between 2 and -2. [If you don't like that the sequences stand still, you can tweak this example so that they both change by a little bit, just not enough to overcome the other sequence's contribution.] [Lots of other sequences work fine, as well.]

For products, much the same thing can be done. If both are monotone the same way <u>and</u> <u>each</u> <u>never</u> <u>changes</u> <u>sign</u>, then $a_n b_n$ will be monotone; $a_{n+1} - a_n$ and $b_{n+1} - b_n$ each always keep the same sign implies that $a_{n+1}b_{n+1} - a_n b_n = (a_{n+1} - a_n)b_{n+1} + (b_{n+1} - b_n)a_n$ .... well, the differences all have the same sign, so if the two <u>sequences</u> both have the same sign as well, then $a_{n+1}b_{n+1} - a_n b_n$ will always have the same sign.

But if we break any of these conditions, we can make the product misbehave. For example, if $a_n = n$ and $b_n = 5 - n$, then $a_n b_n$ is the sequence that starts $4, 6, 6, 4, 0, -6, -14, \ldots$ which starts by increasing but then goes the other way. A more sophisticated example might be built by setting $a_n = 1 + \epsilon_n$ and $b_n = 1 - \delta_n$ where ϵ_n and δ_n decrease to 0, then $a_n b_n = (1+\epsilon_n)(1-\delta_n)$. If we arrange for the $epsilon_n$ to remain constant for awhile while the δ_n decrease, then the sequence $a_n b_n$ increases; doing the opposite at another time makes the product $a_n b_n$ decrease. So the product will not be monotone.

[Again, lots of other sequences work fine, as well.]