Math 314 Matrix Theory

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Vector equations $x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ can be useful in understand solutions to linear systems. One reason for this is that addition and scalar multiplication are so well-beahved (because these operations are caried out component by component):

$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ $\quad (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ $\quad c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ $\quad \mathbf{u}+\mathbf{0}=\mathbf{u}+(0,\ldots,0)=\mathbf{u}$

$(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ $\quad (cd)\mathbf{u}=c(d\mathbf{u})$ $\quad 1\mathbf{u}=\mathbf{u}$ $\quad$ with $-\mathbf{u}=(-1)\mathbf{u}$, $\mathbf{u}+(-\mathbf{u})=(0,\ldots,0)$

The *span* of a collection of vectors is the collection of all linear combinations of the vectors;

$$\mathbf{Sp}(\mathbf{v_1},\ldots,\mathbf{v_n}) = \{x_1\mathbf{v_1}+\cdots+x_n\mathbf{v_n} : x_1,\ldots x_n \in \mathbf{R}\}$$

Then a vector equation $x_1\mathbf{v_1}+\cdots+x_n\mathbf{v_n}=\mathbf{b}$ has a solution precisely when $\mathbf{b} \in \mathbf{Sp}(\mathbf{v_1},\ldots,\mathbf{v_n})$

We can therefore understand linear systems (via vector equations) better, by understanding what the span of the column vectors of the coefficient matrix might look like. This will be a point of view that we will continue to develop throughout the course.

Matrix equations:

There is still a third point of view that we will aproach systems of equations from: matrix multiplication. We can interpret a linear combination of vectors, $x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$, as a product of the matrix $A = \begin{bmatrix} a_{11} & \cdots & \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, which we denote $A\mathbf{x}$.

In this notation a system of equations has a very compact form: $A\mathbf{x}=\mathbf{b}$.

This is really just a new notation for systems, but it will turn out to be remarkably useful. One reason for its utility is that the matrix product is *linear* (in the vector term): $\quad A(\mathbf{u}+\mathbf{v})=A\mathbf{u}+A\mathbf{v}$ $\quad A(c\mathbf{u}=c(A\mathbf{u})$

With this new notation, our basic goal becomes: understand the solutions $\mathbf{x}$ to the equation $A\mathbf{x}=\mathbf{b}$.

Another example of how these different perspectives give different ways to view the same result:

If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $v_1,\ldots,v_n$ are the column vectors of A, and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, then the system of equations $\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$ has a solution for *every* $b_1,\ldots,b_m$ $\Leftrightarrow$ $A\mathbf{x}=\mathbf{b}$ has a solution for every $\mathbf{b}$ $\Leftrightarrow$ every $\mathbf{b}$ is a linear combination of $v_1,\ldots,v_n$ $\Leftrightarrow$ $\mathbf{Sp}(v_1,\ldots,v_n)=\mathbf{R}^m$.

These, in turn, are true $\Leftrightarrow$ after row reducing A to RREF, every row has a pivot in it. To see this, note that if $(A|\mathbf{b})$ is row reduced, then a pivot in every row means that there is no row $(0\cdots 0|1)$, (because we can't have the row of 0's in the coefficient matrix), so the system is consistent, so there is a solution. Conversely, if the RREF does *not* have a pivot in every row, then its bottom row will be a row of 0's. Then if we start with the inconsistent system $\left(\begin{array}{c|c} & 0 \\ RREF & \vdots \\ & 0 \\ & 1 \end{array}\right)$ and reverse the row reduction steps, we will arrive at $(A|\mathbf{b})$ (for *some* $\mathbf{b}$), which we know to be inconsistent, so this equation has no solution. So if *every* system has a solution, then there must be a pivot in every row.