Old Final solutions

1-1: $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \, (\sec x \tan x \, dx)$

$= \int \sec^2 x (\sec^2 x - 1)(\sec x \tan x \, dx)$ $\quad [u = \sec x \quad du = \sec x \tan x \, dx]$

$= \int u^2(u^2-1)\,du \Big|_{u=\sec x} = \int u^4 - u^2 \, du \Big|_{u=\sec x} = \frac{u^5}{5} - \frac{u^3}{3} + C \Big|_{u=\sec x}$

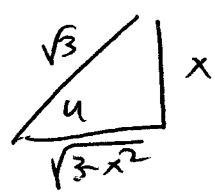
$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.}$

1-2: $\int \frac{x^2 \, dx}{\sqrt{3-x^2}}$ $\quad x = \sqrt{3}\sin u, \; dx = \sqrt{3}\cos u \, du, \; 3 - x^2 = 3\cos^2 u$

$= \int \frac{(\sqrt{3}\sin u)^2 (\sqrt{3}\cos u \, du)}{\sqrt{3\cos^2 u}} \Big|_{x=\sqrt{3}\sin u} = \int 3 \sin^2 u \, du \Big|_{x=\sqrt{3}\sin u}$

$= 3\int \frac{1}{2}(1 - \cos 2u)\, du \Big|_{x=\sqrt{3}\sin u} = \frac{3}{2}\left(u - \frac{1}{2}\sin 2u + c\right)\Big|_{x=\sqrt{3}\sin u}$

$= \frac{3}{2}(u - \sin u \cos u) + C \Big|_{x=\sqrt{3}\sin u}$



$\boxed{= \frac{3}{2}\left(\text{Arcsin}\left(\frac{x}{\sqrt{3}}\right) - \frac{x}{\sqrt{3}}\frac{\sqrt{3-x^2}}{\sqrt{3}}\right) + C}$.

1-3: $\int x^2 e^{3x} \, dx$ $\quad u = x^2, \; dv = e^{3x}dx \quad du = 2x\,dx, \; v = \frac{1}{3}e^{3x}$

$\quad u = x \quad dv = e^{3x} \quad du = dx \quad v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}dx\right) \boxed{= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C.}$

1-4: $\int \frac{2x+3}{x^3+x^2-2}\,dx = \int \frac{2x+3}{(x-1)(x^2+2x+2)}\,dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}\,dx$

$2x+3 = A(x^2+2x+2) + (x-1)(Bx+C)$: $\quad x=1, \; 5 = A(5) \to A = 1$

$x=0, \; 3 = (1)(2) + (-1)(C), \; C = 2-3 = -1, \; x=-1, \; 1 = (1)(1) + (-2)(-B-1)$

$-B-1 = 0, \; B = -1$

$$\int \frac{2x+3}{x^3+x^2-2}\,dx = \int \frac{1}{x-1} - \frac{x+1}{x^2+2x+2}\,dx = \ln|x-1| - \int \frac{x+1}{(x+1)^2+1}\,dx$$

$$= \ln|x-1| - \int \frac{u}{u^2+1}\,du\Big|_{u=x+1} = \ln|x-1| - \int \frac{\frac{1}{2}dv}{v}\Big|_{v=u^2+1}\Big|_{u=x+1}$$

$$= \ln|x-1| - \tfrac{1}{2}\ln|v| + C\Big|_{v=u^2+1}\Big|_{u=x+1} = \ln|x-1| - \tfrac{1}{2}\ln|u^2+1| + C\Big|_{u=x+1}$$

$$\boxed{= \ln|x-1| - \ln|(x+1)^2+1| + C}$$

2. $f(x) = g(x)$: $2x-1 = x^4+x-1$, $x^4 - x = 0 = (x^3-1)x$

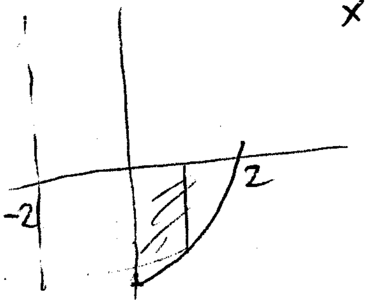
$\Rightarrow x=0$ or $x^3-1=0 \rightarrow x^3=1 \rightarrow x=1$

on $[0,1]$, $2x-1 \geq x^4+x-1$ (check $x=\frac{1}{2}$: $0 = 1-1 > \frac{1}{16}+\frac{1}{2}-1$)

$$\text{Area} = \int_0^1 (2x-1)-(x^4+x-1)\,dx = \int_0^1 -x^4+x\,dx$$

$$= \frac{x^2}{2} - \frac{x^5}{5}\Big|_0^1 = \left(\frac{1}{2}-\frac{1}{5}\right) - (0-0) = \frac{1}{2}-\frac{1}{5} = \frac{5}{10}-\frac{2}{10} = \boxed{\frac{3}{10}}$$

$x^3+7x-22=0$: $(x-2)(x^2+2x+11)=0$ $\quad \underline{x=2}$.

3. 

By shells: (width) (height)

$$\text{Volume} = \int_0^2 2\pi(x-(-2))(x^3+7x-22)\,dx$$

$$= \int_0^2 2\pi(x+2)(x^3+7x-22)\,dx = 2\pi\int_0^2 x^4+2x^3+7x^2+14x-22x-44\,dx$$

$$= 2\pi\int_0^2 x^4+2x^3+7x^2-8x-44\,dx = 2\pi\left(\frac{x^5}{5}+\frac{x^4}{2}+\frac{7x^3}{3}-4x^2-44x\right)\Big|_0^2$$

$$= \boxed{2\pi\left(\left(\frac{2^5}{5}+\frac{2^4}{2}+\frac{7\cdot 8}{3}-4\cdot 4-44\cdot 2\right)-(0)\right)}$$

10. Does the integral $\int_1^\infty \frac{1}{e^x - x}\,dx$ converge or diverge?

(Note: 'Yes' is not considered a correct answer.....)

$\frac{1}{e^x - x}$ looks "like" $\frac{1}{e^x}$ (or $\frac{1}{-x}$ or...)

$$\int_1^\infty \frac{1}{e^x}dx = \int_1^\infty e^{-x}dx = -e^{-x}\Big|_1^\infty$$

$$= \lim_{b\to\infty}\left(-e^{-x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-e^{-b} - (-e^{-1})\right) = \lim_{b\to\infty}\frac{1}{e} - \frac{1}{e^b} \to 0$$

$$= \frac{1}{e} < \infty.$$

$$\lim_{x\to\infty}\frac{\frac{1}{e^x}}{\frac{1}{e^x - x}} = \lim_{x\to\infty}\frac{e^x - x}{e^x} = \lim_{x\to\infty} 1 - \frac{x}{e^x} = 1 - \lim_{x\to\infty}\frac{x}{e^x} = \cancel{\lim_{x\to\infty}\frac{x}{e^x}}$$

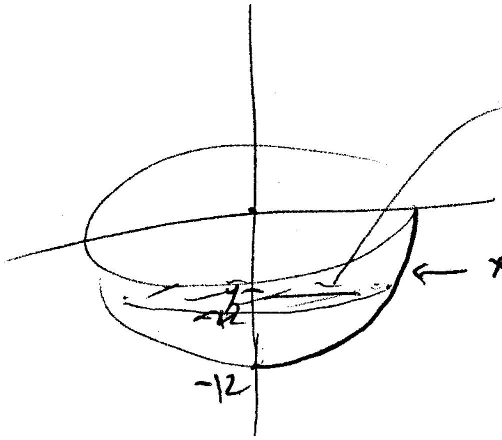
$$= 1 - \lim_{x\to\infty}\frac{1}{e^x} = 1 - 0 = 1 \neq 0, \infty$$

L'Hopital!

So since $\int_1^\infty \frac{1}{e^x}dx$ converges,

$\int_1^\infty \frac{1}{e^x - x}dx$ converges by limit comparison

4.



$\leftarrow x^2+y^2=12^2$

$\text{Area} = A(x) = \pi x^2 = \pi(144-y^2)$

$$\text{work} = \int_{-12}^{0} \underbrace{(-y)}_{\text{distance}} \underbrace{300\,\pi(144-y^2)\,dy}_{\text{mass}}$$

IGNORE ...

$$= 300\pi \int_{-12}^{0} -144y + y^3 \, dy = 300\pi\left(-72y^2 + \frac{y^4}{4}\right)\Big|_{-12}^{0}$$

$$= 300\pi\left((0+0) - \left(-72(-12)^2 + \frac{(-12)^4}{4}\right)\right)$$

$$= 300\pi\left(72\cdot 12^2 - \frac{12^2}{4}\cdot 12^2\right) = 300\pi\cdot 12^2(72-36)$$

$$= \boxed{300\pi\cdot 12^2\cdot 36.}$$

5 (a) $$\lim_{x\to\infty} \frac{x^2-3x^3+9}{4x^2-6x+1} = \lim_{x\to\infty} \frac{1 - \overset{\to -\infty}{3x} + \overset{\to 0}{9/x^2}}{4 - \underset{\to 0}{6\,1/x} + \underset{\to 0}{1/x^2}} \approx \frac{\text{large neg}}{4} \; \boxed{= -\infty}$$

(b) $$\lim_{x\to\infty} \frac{(x^2+1)^x}{(x+1)^{2x}} = L \qquad \ln L = \lim_{x\to\infty} x\ln(x^2+1) - 2x\ln(x+1)$$

$$= \lim x(\ln(x^2+1) - 2\ln(x+1)) = \cancel{\lim x \ln\left(\frac{x^2+1}{(x+1)^2}\right) = \lim \ln\left(\frac{1+(\frac{1}{x})^2}{}\right)}$$

$$= \lim x \ln\left(\frac{x^2+1}{(x+1)^2}\right) = \lim x\ln\left(\frac{1+(\frac{1}{x})^2}{(1+(\frac{1}{x}))^2}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{1+(\frac{1}{x})^2}{(1+(\frac{1}{x}))^2}\right)}{1/x}$$

$$= \lim_{h\to 0^+} \frac{1}{h}\ln\left(\frac{1+h^2}{(1+h)^2}\right) = f'(0), \quad f(x) = \ln\left(\frac{1+x^2}{(1+x)^2}\right)$$

But: $$f'(x) = \left(\frac{1}{\left(\frac{1+x^2}{(1+x)^2}\right)}\right)\left(\frac{(1+x)^2(2x) - (1+x^2)(2(1+x))}{(1+x)^2}\right); \text{ at } x=0,$$

$$f'(0) = \frac{1}{\left(\frac{1}{1^2}\right)}\left(\frac{(1)(0)-(1)(2)}{1^2}\right) = -2. \text{ So } \ln L = -2, \text{ so } \boxed{L = e^{-2}}.$$

6-1: $\sum_{n=1}^{\infty} \frac{(n+1)^{1/2}}{n^2} = \sum a_n$ $\quad b_n = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$ then

$\frac{a_n}{b_n} = \frac{(n+1)^{1/2}}{n^{1/2}} = \left(1+\frac{1}{n}\right)^{1/2} \to 1 \neq 0$ as $n\to\infty$, so since $\sum b_n$ converges (p-series, $p=3/2>1$), $\boxed{\sum a_n \text{ conv}}$ by lim. compar.

6-2: $\sum \frac{n!}{(n^2+n-3)^{3/2}} = \sum a_n$ $\quad b_n = \frac{n!}{(n^2)^{3/2}} = \frac{n!}{n^3}$ then

$\frac{a_n}{b_n} = \left(\frac{n^2}{n^2+n-3}\right)^{3/2} \to 1 \neq 0$ as $n\to\infty$. But $\sum \frac{n!}{n^3} \geq \sum \frac{n(n-1)(n-2)(n-3)}{n^3}$

and $\frac{n(n-1)(n-2)}{n^3} = \left(1-\frac{1}{n}\right)\left(1-2\frac{1}{n}\right) \to 1 \neq 0$ as $n\to\infty$ so $\sum b_n$ diverges by n^{th} term test, so $\boxed{\sum a_n \text{ diverges}}$ by lim compar.

6-3: $\sum \left(\frac{n+3}{3n-5}\right)^n = \sum a_n$ $\quad a_n^{1/n} = \frac{n+3}{3n-5} = \frac{1+3\frac{1}{n}}{3-5\frac{1}{n}} \to \frac{1}{3} < 1$ as $n\to\infty$ so $\boxed{\sum a_n \text{ conv}}$ by the root test.

6-4: $\sum_{n=1}^{\infty} \frac{\ln n}{n^{5/3}} = \sum a_n$. But $\frac{\ln n}{n^{\alpha}} \to 0$ as $n\to\infty$ for any $\alpha>0$, so compare to $b_n = \frac{n^{1/3}}{n^{5/3}} = \frac{1}{n^{4/3}}$.

$\frac{a_n}{b_n} = \frac{\ln n}{n^{1/3}} \to 0$ as $n\to\infty$ and since $\sum b_n$ conv (p-series, $p=4/3>1$) $\boxed{\sum a_n \text{ conv}}$ by limit comparison.

7: $f(x) = (x^2-5)^{5/2}$ centered at $c=3$. $f(3) = (9-5)^{5/2} = 4^{5/2} = 2^5 = 32$.

$f'(x) = \frac{5}{2}(x^2-5)^{3/2}(2x)$ $\quad f'(3) = \frac{5}{2}(4)^{3/2}(6) = 5\cdot 2^3\cdot 3 = 120$

$f''(x) = \frac{5}{2}\left(\frac{3}{2}(x^2-5)^{1/2}(2x)(2x) + 2(x^2-5)^{3/2}\right)$

$f''(3) = \frac{5}{2}\left(\frac{3}{2}4^{1/2}(6)(6) + 2(4)^{3/2}\right) = \frac{5}{\cancel{2}}\left(\cancel{108}+16\right) = 310$ $\quad \overset{62}{124}$

4

$$f'''(x) = \frac{5}{2}\left(\frac{3}{2}\left(\frac{1}{2}(x^2-5)^{-1/2}(2x)(4x^2) + (8x)(x^2-5)^{1/2}\right) + 2\left(\frac{3}{2}\right)(x^2-5)^{1/2}(2x)\right)$$

$$f'''(3) = \frac{5}{2}\left(\frac{3}{2}\left(\frac{1}{2}(4)^{-1/2}(6)(4\cdot 6^2) + (24)(4)^{1/2}\right) + 3(4)^{1/2}(6)\right)$$

$$= \frac{5}{2}\left(\frac{3}{2}(\underbrace{216+48}_{264}) + 36\right) = \frac{5}{2}(\underbrace{3\cdot 132}_{396} + 36) = \frac{5}{2}(\cancel{432}^{216}) = 1080.$$

So $$p_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2 + \frac{f'''(3)}{6}(x-3)^3$$

$$\boxed{= 32 + 120(x-3) + 155(x-3)^2 + 180(x-3)^3}$$

8: $x(t) = t^4$, $y(t) = t^6$ $\quad x'(t) = 4t^3 \quad y'(t) = 6t^5$

$$\text{Length} = \int_0^2 \sqrt{(4t^3)^2 + (6t^5)^2}\,dt = \int_0^2 \sqrt{16t^6 + 32t^{10}}\,dt$$

$$= \int_0^2 t^3 (16 + 32t^4)^{1/2}\,dt \qquad u = 16 + 32t^4 \quad du = 128t^3\,dt$$

$$t^3\,dt = \frac{1}{128}du$$

$t=0 \to u = 16$

$t=2 \to u = 16 + 32\cdot 16 = 33\cdot 16 = 480 + 48 = 528$

$$= \int_{16}^{528} u^{1/2}\frac{1}{128}\,du$$

$$= \frac{1}{128}\frac{2}{3}u^{3/2}\Big|_{16}^{528} = \boxed{\frac{1}{3\cdot 64}\left((528)^{3/2} - 16^{3/2}\right)}$$

5(b), REDUX:

$$\frac{(x^2+1)^x}{(x+1)^{2x}} = \left(\frac{x^2+1}{(x+1)^2}\right)^x = \left(\frac{x^2+1}{x^2+2x+1}\right)^x = \left(1 - \frac{2x}{(x+1)^2}\right)^x$$

$$= \left(1 - \frac{2}{\frac{(x+1)^2}{x}}\right)^x = \left(\left(1 - \frac{2}{\frac{(x+1)^2}{x}}\right)^{\frac{(x+1)^2}{x}}\right)^{\frac{x^2}{(x+1)^2}} = \left(\left(1 - \frac{2}{\text{blah}}\right)^{\text{blah}}\right)^{\left(\frac{x}{x+1}\right)^2}$$

But! blah $\to \infty$ as $x \to \infty$, $\left(1 - \frac{2}{\text{blah}}\right)^{\text{blah}} \to e^{-2}$ as blah $\to \infty$, and $\left(\frac{x}{(x+1)}\right)^2 \to 1$ as $x \to \infty$, so as $x \to \infty$

$$\frac{(x^2+1)^x}{(x+1)^{2x}} = \left(\left(1 + \frac{-2}{\text{blah}}\right)^{\text{blah}}\right)^{\left(\frac{x}{x+1}\right)^2} \to (e^{-2})^1 = e^{-2}$$

6. Find the area inside of the graph of the polar curve

$$r = \sin(\theta) - \cos(\theta)$$

from $\theta = \dfrac{\pi}{4}$ to $\theta = \dfrac{5\pi}{4}$.

What does this curve look like? (Hint: multiply both sides by r.)

Since Area$=\displaystyle\int \frac{1}{2}(f(\theta))^2 \, d\theta$, we have

$$\text{Area} = \frac{1}{2}\int_{\pi/4}^{5\pi/4} (\sin\theta - \cos\theta)^2 \, d\theta = \frac{1}{2}\int_{\pi/4}^{5\pi/4} \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{5\pi/4} 1 - 2\sin\theta\cos\theta \, d\theta = \frac{1}{2}\int_{\pi/4}^{5\pi/4} 1 - \sin(2\theta) \, d\theta = \frac{1}{2}[\theta + \frac{1}{2}\cos(2\theta)]\Big|_{\pi/4}^{5\pi/4}$$

$$= \frac{1}{2}[(5\pi/4 + \frac{1}{2}\cos(5\pi/2)) - (\pi/4 + \frac{1}{2}\cos(\pi/2))] = \frac{1}{2}[(5\pi/4 + \frac{1}{2}) - (\pi/4 + \frac{1}{2})]$$

$$= \frac{1}{2}[(5\pi/4) - (\pi/4)] = \frac{1}{2}[\pi] = \frac{\pi}{2}$$

To see what this curve is, we have $r = \sin(\theta) - \cos(\theta)$, so $r^2 = r\sin(\theta) - r\cos(\theta)$, so $x^2 + y^2 = y - x$, so $(x^2 + x) + (y^2 - y) = 0$, so $(x^2 + x + \frac{1}{4}) + (y^2 - y + \frac{1}{4}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2} = (\frac{1}{\sqrt{2}})^2$

This is a circle, centered at $(-\frac{1}{2}, \frac{1}{2})$, with radius $\frac{1}{\sqrt{2}}$!

7. (15 pts.) A particle is moving around in space; at $t = 0$ its position is $(1, 2, 3)$ and its velocity is $(-1, 0, 2)$. At every time t, the acceleration of the particle is given by the vector

$$\vec{a}(t) = (\sin t, \sin(2t), 1) .$$

What is the particle's position at time $t = \pi$?

$$\vec{v}(t) = (-1,0,2) + \int_0^t (\sin x, \sin(2x), 1)\, dx$$

$$= (-1,0,2) + \left(-\cos x, -\tfrac{1}{2}\cos(2x), x)\Big|_0^t \right)$$

$$= (-1,0,2) + \left(-\cos t - (-1), -\tfrac{1}{2}\cos(2t) - (\tfrac{-1}{2}), t-0 \right)$$

$$= (-1,0,2) + \left(1-\cos t, \tfrac{1}{2} - \tfrac{1}{2}\cos(2t), t\right)$$

$$= \left(-\cos t, \tfrac{1}{2} - \tfrac{1}{2}\cos(2t), 2+t\right)$$

$$\vec{r}(t) = (1,2,3) + \int_0^t \left(-\cos x, \tfrac{1}{2} - \tfrac{1}{2}\cos(2x), 2+x\right) dx$$

$$= (1,2,3) + \left(-\sin x, \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x), 2x + \tfrac{1}{2}x^2)\Big|_0^t\right)\Big|_0^t$$

$$= (1,2,3) + \left(-\sin t, \tfrac{1}{2}t - \tfrac{1}{4}\sin(2t), 2t + \tfrac{1}{2}t^2 \right)$$

$$= \left(1 - \sin t, 2 + \tfrac{1}{2}t - \tfrac{1}{4}\sin(2t), 3 + 2t + \tfrac{1}{2}t^2\right).$$

At $t = \pi$,

$$\vec{r}(\pi) = \left(1 - 0, 2 + \tfrac{1}{2}\pi - \tfrac{1}{4}0, 3 + 2\pi + \tfrac{1}{2}\pi^2\right)$$

$$= \left(1, 2 + \tfrac{\pi}{2}, 3 + 2\pi + \tfrac{\pi^2}{2}\right).$$