Note that

$$\left|x-\frac{h_n}{k_n}\right| < \frac{1}{k_n k_{n+1}}$$ is alot better than we deserve

For a randomly chosen x and a random N, we can expect to find an M so that $\left|x-\frac{M}{N}\right|<\frac{1}{2N}$, but we can't expect to get much closer than that.

The point, of course is that the k_n's are not randomly chosen! They are "determined" by x.

* There is also a precise sense in which the $\frac{h_n}{k_n}$'s are the best approx's to x:

If $x\notin\mathbb{Q}$ and $a,b\in\mathbb{Z}$ with $(b\geq 1)$ and $\left|x-\frac{a}{b}\right|<\frac{1}{2b^2}$, then $\frac{a}{b}=\frac{h_n}{k_n}$ for some n.

Pf. Suppose not. Then $|ak_n-bh_n|\geq 1$ for all n. since $k_n\to\infty$ as $n\to\infty$ and $k_0=1$, $\exists\, n$ with $k_n\leq b<k_{n+1}$.

Then from the above we know that

$$|xk_n-h_n|\leq|xb-a|<\left|x-\frac{a}{b}\right||b|<\frac{1}{2b}$$

So $\left|x-\frac{h_n}{k_n}\right|<\frac{1}{2bk_n}$ then

GM

$$\frac{1}{bk_n} \le \frac{|bh_n - ak_n|}{bk_n} = \left|\frac{a}{b} - \frac{h_n}{k_n}\right|$$

$$= \left|\left(\frac{a}{b} - x\right) + \left(x - \frac{h_n}{k_n}\right)\right| \le \left|x - \frac{a}{b}\right| + \left|x - \frac{h_n}{k_n}\right|$$

$$\le \frac{1}{2b^2} + \frac{1}{2bk_n} \quad \text{so}$$

$$\frac{1}{2bk_n} = \frac{1}{bk_n} - \frac{1}{2bk_n} \le \frac{1}{2b^2} \implies 2b^2 \cancel{\le} < 2bk_n$$

$$\implies b \cancel{\le} < k_n$$

$\cancel{\Rightarrow b = k_n}$

Mo

Did: Pell's eqn.

$n<0$, $n=p^2$ cases.

Stated main case, $N < n^2$

then $x^2 - ny^2 = N \implies \frac{x}{y} = \frac{h_m}{k_m}$

some m.

If $x = \langle a_0, \dots, a_n, \overline{b_0, \dots, b_m} \rangle$, then

x is a <u>quadratic irrational</u>.

(i.e. irrational root of $ax^2+bx+c=0$ some $a,b,c \in \mathbb{Z}$.)

<u>Pf.</u> Set $\alpha = \langle \overline{b_0, \dots, b_m} \rangle$ so

$$x = \langle a_0, \dots, a_n, \alpha \rangle \implies$$

for $\langle a_0, \dots, a_n \rangle = \frac{h_n}{k_n}$, $\quad x = \dfrac{\alpha h_n + h_{n-1}}{\alpha k_n + k_{n-1}}$

<u>If</u> $\alpha = p + q\sqrt{r}$, $p, q \in \mathbb{Q}$,

then $$x = \frac{((p+q\sqrt{r})h_n + h_{n-1})(k_n q\sqrt{r} - (pk_n + k_{n-1}))}{(k_n q)^2 r - (pk_n + k_{n-1})^2}$$

$$= (\text{blah}) + (\text{bleh})\sqrt{r}.$$

<u>But</u> $\alpha = \langle \overline{b_0, \dots, b_m} \rangle = \langle b_0, \dots, b_m, \alpha \rangle$

So $$\alpha = \frac{\alpha h_m' + h_{m-1}'}{\alpha k_m' + k_{m-1}'} \implies$$

$$k_m' \alpha^2 + (k_{m-1}' - h_m')\alpha - h_{m-1}' = 0$$

$\implies \alpha = p + q\sqrt{r}$ some p, q, r ///

Converse!

$x = \frac{p+\sqrt{d}}{q}$ x's conjugate $= \frac{p-\sqrt{d}}{q}$

Quadratic formula: x' is the other root of the quadratic having x as root

Thm: If $x = \sqrt{n} + \lfloor\sqrt{n}\rfloor$, then

$x = \langle \overline{a_0, \ldots, a_n} \rangle$ is purely periodic.

Pf: $x' = \lfloor\sqrt{n}\rfloor - \sqrt{n}$ so $-1 < x' < 0$.

Set $x = \langle a_0, \ldots, a_k + x_k \rangle = \langle a_0, \ldots, a_k, \xi_{k+1} \rangle$ with

$\xi_{k+1} = \frac{1}{x_k}$ (and so $a_k = \lfloor \xi_k \rfloor$, $x=$

Then in your HW you essentially show (since $\sqrt{n} = \langle b_0, b_1, \ldots \rangle$ has $b_0 = a_0 - \lfloor\sqrt{n}\rfloor$, $b_i = a_i$ other i)

~~$x_k = \xi_k - a_k = \frac{\sqrt{n} - m_k}{q_k}$ for integers m_k, q_k.~~

~~where~~ $x_k = \frac{\sqrt{n} - m_k}{q_k}$ so $\xi_{k+1} = \frac{q_k}{\sqrt{n} - m_k} = \frac{\sqrt{n} + m_k}{q_{k+1}}$

so $x_{k+1} = \xi_{k+1} - a_{k+1} = \frac{\sqrt{n} - m_{k+1}}{q_{k+1}}$ where

$q_k q_{k+1} = n - m_k^2$ ($\longrightarrow$ defines q_{k+1})

$a_{k+1} = \lfloor \xi_{k+1} \rfloor$

$m_{k+1} = a_{k+1} q_{k+1} - m_k$ ($\longrightarrow$ defines m_{k+1})

Note that

$$\xi_{k+1} = \frac{\sqrt{n} + m_{k-1}}{q_{k+1}} \quad \text{and} \quad \xi_{k+1} = \frac{1}{\xi_k - a_k} = \frac{1}{\frac{\sqrt{n}+m_{k-1}}{q_k} - a_k}$$

$$= \frac{q_k\sqrt{n} - (m_{k-1} - a_k q_k) q_k}{n - (m_{k-1} - a_k q_k)^2}$$

then

$$\xi_k' = \frac{-\sqrt{n} + m_{k-1}}{q_k} \quad \text{and} \quad \frac{1}{\xi_k' - a_k} =$$

$$\frac{1}{\frac{-\sqrt{n}+m_{k-1}}{q_k} - a_k} = \frac{q_k}{(m_{k-1} - a_k q_k) - \sqrt{n}} = \frac{q_k\left((m_{k-1} - a_k q_k) + \sqrt{n}\right)}{(m_{k-1} - a_k q_k)^2 - n}$$

$$= \frac{-q_k\sqrt{n} - (m_{k-1} - a_k q_k) q_k}{(m_{k-1} - a_k q_k)^2 - n} = \xi_{k+1}'$$

But $x = \xi_0$ so $-1 < \xi_0' < 0$, and then by induction

$$-1 < \xi_k' < 0 \implies \xi_k' - a_k < -1 \implies -1 < \xi_{k+1}' < 0$$

So $-1 < \xi_k' < 0$ for all k, so

$$\left\lfloor \frac{-1}{\xi_{k+1}'} \right\rfloor = \lfloor a_k - \xi_k' \rfloor = a_k \quad \text{since} \quad a_k < a_k - \xi_k' < a_k + 1.$$

But now, from HW, we know that the ctd fraction for $\sqrt{n}$, and so for $\sqrt{n} + \lfloor\sqrt{n}\rfloor$, becomes periodic; so for some $m > 0$ $a_{n+s} = a_{n+m+s}$ for ~~some~~ all $s \geq 0$. Let m, n be the smallest such values. Then claim: $n = 0$.

Ie. $\boxed{\xi_n = \xi_{n+m}}$

[ie. m is smallest repeat,
n = ~~smallest~~ earliest pt of repeat.

This is because if $n>0$, then $\xi_n = \xi_{n+m} \Rightarrow \xi_n' = \xi_{n+m}'$

$$\Rightarrow \left\lfloor -\frac{1}{\xi_n'} \right\rfloor = a_{n-1} = a_{n+m-1} = \left\lfloor -\frac{1}{\xi_{n+m}'} \right\rfloor$$

$$\Rightarrow \frac{1}{\xi_{n-1} - a_{n-1}} = \xi_n = \xi_{n+m} = \frac{1}{\xi_{n+m-1} - a_{n+m-1}} = \frac{1}{\xi_{n+m-1} - a_{n-1}}$$

$$\Rightarrow \xi_{n-1} - a_{n-1} = \xi_{n+m-1} - a_{n-1} \Rightarrow \xi_{n-1} = \xi_{n+m-1}$$

contradicting choice of $\underline{\underline{n}}$.

So $n=0$; so $a_{m+s} = a_s$ for all s, so

$$\sqrt{n} + \lfloor \sqrt{n} \rfloor = \langle \overline{a_0, \ldots, a_{m-1}} \rangle = \langle a_0, \overline{a_1, \ldots, a_{m-1}, a_0} \rangle$$

Note $a_0 = 2\lfloor \sqrt{n} \rfloor$, so

$$\sqrt{n} = \langle \lfloor \sqrt{n} \rfloor, \overline{a_1, \ldots, a_{m-1}, 2\lfloor \sqrt{n} \rfloor} \rangle$$

Note that: $\xi_0 = \dfrac{\sqrt{n} + \lfloor \sqrt{n} \rfloor}{1}$ so $m_0 = \lfloor \sqrt{n} \rfloor$, $q_1 = 1$

Then since $\xi_0 = \xi_m = \xi_{cm} = \cdots$ we have

$q_{mt+1} = 1$ for all t.