Solutions

**Name:**

# Math 423/823 Final Exam

**Show all work.** How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Trigonometry tells us that $\arg(z+|z|) = \frac{1}{2}\arg(z)$ (see figure below).

   Use $z = e^{\frac{i\pi}{4}}$, thinking of $z+|z|$ in both rectangular and polar coordinates, to show that

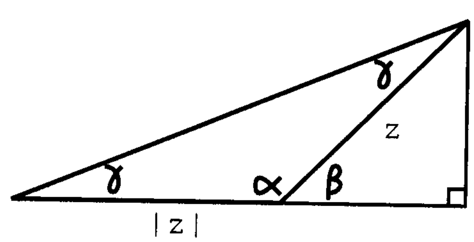
$$\tan(\frac{\pi}{8}) = \frac{1}{\sqrt{2}+1} = \sqrt{2}-1\,.$$

$z = e^{\frac{i\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \qquad |z| = 1 \qquad x+yi$

$z+|z| = \left(\frac{\sqrt{2}}{2}+1\right) + \frac{\sqrt{2}}{2} i = re^{i\theta} \qquad \theta = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$

$\tan\theta = \tan\frac{\pi}{8} = \frac{y}{x} = \boxed{\frac{\sqrt{2}/2}{\sqrt{2}/2+1}} = \frac{\sqrt{2}}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{2}+2} = \frac{\sqrt{2}}{\sqrt{2}+2} = \frac{1}{1+\sqrt{2}}$

$= \frac{\sqrt{2}-1}{\sqrt{2}-1}\cdot\frac{1}{1+\sqrt{2}} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$




$\alpha+\beta = \pi = \alpha + 2\gamma, \qquad \text{so } \gamma = \frac{\beta}{2}.$

2. (15 pts.) Show that if $|z| = 1$ (and $z \neq -1$), then $w = \dfrac{z}{(z+1)^2}$ is <u>real</u> (i.e., $\mathrm{Im}(w) = 0$).

$$\boxed{z\bar{z} = 1}$$

$$w - \bar{w} = \frac{z}{(z+1)^2} - \overline{\left(\frac{z}{(z+1)^2}\right)} = \frac{z}{(z+1)^2} - \frac{\bar{z}}{(\bar{z}+1)^2}$$

$$= \frac{z(\bar{z}+1)^2 - \bar{z}(z+1)^2}{((z+1)(\bar{z}+1))^2} = \frac{z(\bar{z}^2 + 2\bar{z} + 1) - \bar{z}(z^2 + 2z + 1)}{(|z+1|^2)^2}$$

$$= \frac{(z\bar{z})\bar{z} + 2z\bar{z} + z - ((\bar{z}z)z + 2\bar{z}z + \bar{z})}{(|z+1|)^4}$$

$$= \frac{\cancel{\bar{z}} + \cancel{2|z|^2} + \cancel{z} - (\cancel{z} + \cancel{2|z|^2} + \cancel{\bar{z}})}{(|z+1|)^4} = \frac{0}{|z+1|^4} = 0$$

So $w = \bar{w}$ so $w$ is <u>real</u>.

Faster! $$\frac{1}{w} = \frac{(z+1)^2}{z} = \frac{z^2 + 2z + 1}{z} = z + 2 + \frac{1}{z} = z + 2 + \bar{z}$$
$$= (z + \bar{z}) + 2 = 2\mathrm{Re}(z) + 2 \text{ is } \underline{\text{real}}$$

Or: $|z| = 1$ so $z = e^{i\theta}$ some $\theta$; then

$$\frac{z}{(z+1)^2} = \frac{e^{i\theta}}{(e^{i\theta}+1)^2} = \frac{e^{i\theta}}{e^{2i\theta} + 2e^{i\theta} + 1}$$

since $z\bar{z} = 1$, $\bar{z} = \dfrac{1}{z}$

$$= \frac{1}{e^{i\theta} + 2 + e^{-i\theta}} = \frac{1}{(e^{i\theta} + e^{-i\theta}) + 2} = \frac{1}{2\cos\theta + 2}$$, which is <u>real</u>.

3. (15 pts.) Show that if

$f(z) = f(x+yi) = u(x,y) + iv(x,y)$ and $g(z) = g(x+yi) = p(x,y) + iq(x,y)$ both satisfy the Cauchy-Riemann equations at $z = 0$, then $h(z) = f(z)g(z)$ <u>also</u> satisfies the CR-equations at $z = 0$.

[There is nothing at all special about 0; it was chosen for notational convenience.]

$h(z) = f(z)g(z) = (u+iv)(p+iq) = (up - vq) + i(vp + uq)$

we know that at $z=0$, $u_x = v_y \quad p_x = q_y$

$u_y = -v_x \quad p_y = -q_x$

then $(*) = (up - vq)_x = (up)_x - (vq)_x = (u_x p + u p_x) - (v_x q + v q_x)$

$(vp + uq)_y = (vp)_y + (uq)_y = v_y p + v p_y + u_y q + u q_y$

$= u_x p + v(-q_x) + (-v_x)q + u p_x$

$= u_x p + u p_x - (v q_x + v_x q) = (*)$

$(**) = (up - vq)_y = (up)_y - (vq)_y = (u_y p + u p_y) - (v_y q + v q_y)$

$(vp + uq)_x = (vp)_x + (uq)_x = v_x p + v p_x + u_x q + u q_x$

$= \cancel{u_y} - u_y p + v(q_y) + v_y q + u(-p_y)$

$= v_y q + v q_y - (u_y p + u p_y) = -(**)$

So CR-eqn hold!

4. (20 pts.) Show that setting $z = e^{it}$, we can rewrite $\dfrac{\cos 5t}{\cos t}$ as

$$z^4 - z^2 + 1 - z^{-2} + z^{-4}\,.$$

Use this to find the value of $\displaystyle\int_0^{2\pi} \frac{\cos 5t}{\cos t}\,dt$ by converting to an integral over the unit circle $C(t) = e^{it}$, $0 \le t \le 2\pi$.

$\cos t = \frac{1}{2}(e^{it} + e^{-it})$ $\quad \cos 5t = \frac{1}{2}(e^{i5t} + e^{-i5t}) = \frac{1}{2}\left((e^{it})^5 + (e^{-it})^5\right)$

so $\dfrac{\cos 5t}{\cos t} = \dfrac{\frac{1}{2}(z^5 + z^{-5})}{\frac{1}{2}(z + z^{-1})} = \dfrac{z^{-5}(z^{10}+1)}{z^{-1}(z^2+1)}$

$$= z^{-4}\left(\frac{\cancel{(z^2+1)}(z^8 - z^6 + z^4 - z^2 + 1)}{\cancel{(z^2+1)}}\right) = z^{-4}\left(z^8 - z^6 + z^4 - z^2 + 1\right)$$

$$= \boxed{z^4 - z^2 + 1 - z^{-2} + z^{-4}}$$

with $z = e^{it}$, $dz = ie^{it}dt = iz\,dt$, so $dt = \dfrac{dz}{iz}$, then

$$\int_0^{2\pi} \frac{\cos 5t}{\cos t}\,dt = \int_C \left(z^4 - z^2 + 1 - z^{-2} + z^{-4}\right)\frac{dz}{iz}$$ Laurent polynomial! analytic except at $z=0$.

$$= 2\pi i\left(\frac{1}{i}\operatorname*{Res}_{z=0}\left(z^3 - z + z^{-1} - z^{-3} + z^{-5}\right)\right) = 2\pi i\left(\frac{1}{i}(1)\right) = \boxed{2\pi}$$

5. (15 pts.) Find the Laurent series expansion of the function $f(z) = \frac{z^3}{(z-1)^2}$ centered at $z = 0$, valid for $1 < |z| < \infty$.

$$\frac{1}{(z-1)^2} = \frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=1}^{\infty} n z^{n-1}$$

for $|z| < 1$

So

$$\frac{1}{\left(\frac{1}{z}-1\right)^2} = \sum_{n=1}^{\infty} n\left(\frac{1}{z}\right)^{n-1} = \sum_{n=1}^{\infty} n z^{1-n} \quad \text{for } \left|\frac{1}{z}\right| < 1 \text{ i.e., } 1 < |z|$$

$$\| $$

$$\frac{z^2}{z^2\left(\frac{1}{z}-1\right)^2} = \frac{z^2}{(1-z)^2} = \frac{z^2}{(z-1)^2}.$$

So

$$\frac{z^3}{(z-1)^2} = z\,\frac{z^2}{(z-1)^2} = z\sum_{n=1}^{\infty} n z^{1-n} = \boxed{\sum_{n=1}^{\infty} n z^{2-n}}$$

$$= \sum_{n=-\infty}^{-1} |n|\, z^{2+n} = \boxed{\sum_{m=-\infty}^{1} |m-2|\, z^m}$$

6. (20 pts.) Find the value of

$$\int_C \frac{dz}{(z^2+1)(2z+5)}, \quad = \int_C f(z)\,dz$$

where $C$ is the boundary of the 'diamond' $S = \{(x+iy : |x|+|y| \leq 2\}$, traversed counterclockwise (see figure below).

The singularities of $f(z) = \dfrac{1}{(z^2+1)(2z+5)}$ are at $z=i$, $z=-i$, and $z=-5/2$. $i$ and $-i$ are inside the curve.

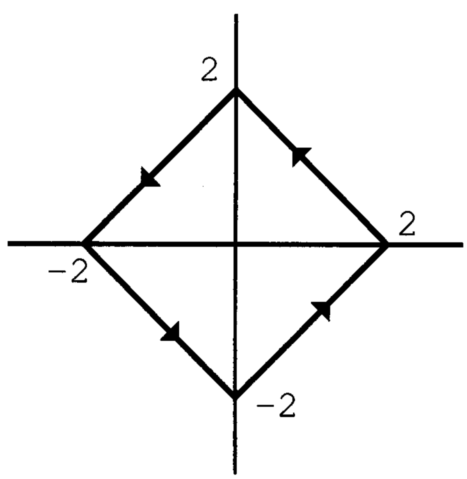
So

$$\int_C f(z)\,dz = 2\pi i \left( \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right)$$

$$= 2\pi i \left( \operatorname*{Res}_{z=i} \frac{\frac{1}{(z+i)(2z+5)}}{(z-i)} + \operatorname*{Res}_{z=-i} \frac{\frac{1}{(z-i)(2z+5)}}{(z+i)} \right)$$

$$= 2\pi i \left( \frac{1}{2i(5+2i)} + \frac{1}{(-2i)(5-2i)} \right) = \frac{2\pi i}{2i}\left( \frac{1}{5+2i} - \frac{1}{5-2i} \right)$$

$$= \pi \left( \frac{(5-2i)-(5+2i)}{(5+2i)(5-2i)} \right) = \pi \left( \frac{-2i-2i}{25+4} \right) = \boxed{-\frac{4\pi i}{29}}$$



## Some potentially useful formulas

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\arcsin(z) = -i \log(iz + \sqrt{1 - z^2})$$

$$\arctan z = \frac{i}{2} \log\left(\frac{i - z}{i + z}\right)$$

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \text{ , for } |z| < 1$$

$$\frac{1}{(1 - z)^2} = \frac{d}{dz}\left(\frac{1}{1 - z}\right)$$

$$\frac{d}{dz}\Big(\log(1 - z)\Big) = \frac{-1}{1 - z}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} z^{\cancel{n}\,2n+1}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{\cancel{n}\,2n}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} z^{\cancel{n}\,2n+1}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{\cancel{n}\,2n}$$

$$\frac{1}{z^2 + 1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \text{ , for } |z| < 1$$