Math 971 Algebraic Topology

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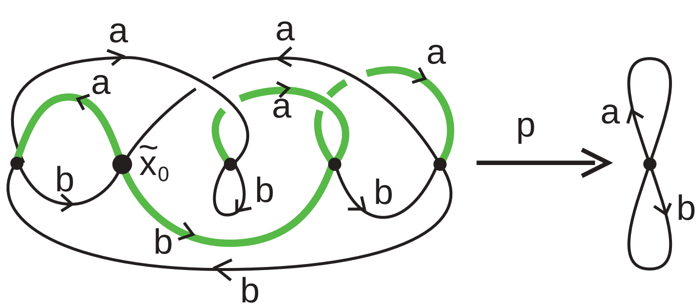
The **proof** of the homotopy lifting property follows a pattern that we will become very familiar with: we lift maps a little bit at a time. For every $x \in X$ there is an open set $\mathcal{U}_x$ evenly covered by p . For each fixed $y \in Y$, since I is compact and the sets $H^{-1}(\mathcal{U}_x)$ form an open cover of $Y \times I$, then since I is compact, the Tube Lemma provides an open neighborhood $\mathcal{V}$ of y in Y and finitely many $p^{-1}\mathcal{U}_x$ whose union covers $\mathcal{V} \times I$.

To define $\widetilde{H}(y,t)$, we (using a Lebesgue number argument) cut the interval $\{y\} \times I$ into finitely many pieces, the ith mapping into $\mathcal{U}_{x_i}$ under H. $\widetilde{f}(y)$ is in one of the evenly covered sets $\mathcal{U}_{x_1\alpha_1}$, and the restricted map $p^{-1} : \mathcal{U}_{x_1} \to \mathcal{U}_{x_1\alpha_1}$ following H restricted to the first interval lifts H along the first interval to a map we will call $\widetilde{H}$. We then have lifted H at the end of the first interval = the beginning of the second, and we continue as before. In this way we can define $\widetilde{H}$ for all (y,t) . To show that this is independent of the choices we have made along the way, we imagine two ways of cutting up the interval $\{y\} \times I$ using evenly covered neighborhoods $\mathcal{U}_{x_i}$ and $\mathcal{V}_{w_j}$, and take intersections of both sets of intervals to get a common refinement of both sets, covered by the intersections $\mathcal{U}_{x_i} \cap \mathcal{V}_{w_j}$, and imagine building $\widetilde{H}$ using the refinement. At the start, at $\widetilde{f}(y)$, we are in $\mathcal{U}_{x_1\alpha_1} \cap \mathcal{V}_{w_1\beta_1}$. Because at the start of the lift $(y,0)$ we lift to the same point, and p^{-1} restricted to this intersection agrees with p^{-1} restricted to each of the two pieces, we get the same lift acroos the first refined subinterval. This process repeats itself across all of the subintervals, showing that the lift is independent of the choices made. This also shows that the lift is unique; once we have decided what $\widetilde{H}(y,0)$, the rest of the values of the $\widetilde{H}$ are determined by the requirement of being a lift. also, once we know the map is well-defined, we can see that it is continuous, since for any y, we can make the same choices across the entire open set V given by the Tube Lemma, and find that $\widetilde{H}$, restricted to $\mathcal{V} \times (a_i - \delta, b_i + \delta)$ (for a small delta; we could wiggle the endpoints in the construction without changing the resulting function, by its well-definedness) is H estricted to this set followed by p^{-1} restriced in domain and range, so this composition is continuous. So $\widetilde{H}$ is locally continuous, hence continuous.

So, for example, if we build a 5-sheeted cover of the bouquet of 2 circles, as before, (after choosing a maximal tree upstairs) we can read off the images of the generators of the fundamental group of the total space; we have labelled each ede by the ereator it traces out downstairs, and for each ede outside of the maximal tree chosen, we read from basepoint out the tree to one end, across the edge, and then back to the basepoint in the tree. In our example, this gives:

$$< ab, aaab^{-1}, baba^{-1}, baa, ba^{-1}bab^{-1}, bba^{-1}b^{-1}| >$$



This is (from its construction) a copy of the free group on 6 letters, in the free group $F(a,b)$. In a similar way, by explicitly building a covering space, we find that the fundamental group of a closed surface of genus 3 is a subgroup of the fundamental group of the closed surface of genus 2.

The cardinality of a point inverse $p^{-1}(y)$ is, by the evenly covered property, constant on (small) open sets, so the set of points of x whose point inverses have any given cardinality is open. Consequently, if X is connected, this number is constant over all of X, and is called the number of *sheets* of the covering $p : \widetilde{X} \to X$. It can also be determined from the fundamenatal groups:

Proposition: If X and $\widetilde{X}$ are path-connected, then the number of sheets of a covering map equals the index of the subgroup $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $G = \pi_1(X, x_0)$.

To see this, choose loops $\{\gamma_\alpha\}$ representing representatives $\{g_\alpha\}$ of each of the (right) cosets of H in G. Then if we lift each of them to loops based at $\widetilde{x}_0$, they will have distinct endpoints; if $\widetilde{\gamma}_1(1) = \widetilde{\gamma}_2(1)$, then by uniqueness of lifts, $\gamma_1 * \overline{\gamma_2}$ lifts to $\widetilde{\gamma}_1 * \overline{\widetilde{\gamma}_2}$, so it lifts to a loop, so $\gamma_1 * \overline{\gamma_2}$ represents an element of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$, so $g_1 = g_2$. Conversely, every point in $p^{-1}(x_0)$ is the endpoint of on of these lifts, since we can construct a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to any such point y, giving a loop $\gamma = p \circ \widetilde{\gamma}$ representing an element $g \in \pi_1(X, x_0)$. But then $g = hg_\alpha$ for some $h \in H$ and α, so γ is homotopic rel endpoints to $\eta * \gamma_\alpha$ for some loop η representing h. But then lifting these based at $\widetilde{x}_0$, by hmotopy lifting, $\widetilde{\gamma}$ is homotopic rel endpoints to $\widetilde{\eta}$, which is a loop, followed by the lift $\widetilde{\gamma}_\alpha$ of γ_α starting at $\widetilde{x}_0$. So $\widetilde{\gamma}$ and $\widetilde{\gamma}_\alpha$ have the same value at 1.

Therefore, lifts of representatives of coset representatives of H in G give a 1-to-1 correspondence, given by $\widetilde{\gamma}(1)$, with $p^{-1}x_0$. In particular, they have the same cardinality.

The path lifting property (because $\pi([0,1], 0) = \{1\}$) is actually a special case of a more general **lifting criterion**: If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, and $f : (Y, y_0) \to (X, x_0)$ is a map, where Y is path-connected and locally path-connected, then there is a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f (i.e., $f = p \circ \widetilde{f}$) $\Leftrightarrow f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, two lifts of f which agree at a single point are equal.

If the lift exists, then $f = p \circ \widetilde{f}$ implies $f_* = p_* \circ \widetilde{f}_*$, so $f_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, as desired. Conversely, if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, then we wish to build the lift of f. Not wishing to waste our work on the special case, we will *use* path lifting to do it! Given $y \in Y$, choose a path γ in Y from y_0 to y and use path lifting in X to lift the path $f \circ \gamma$ to a path $\widetilde{f \circ \gamma}$ with $\widetilde{f \circ \gamma}(0) = \widetilde{x}_0$. Then define $\widetilde{f}(y) = \widetilde{f \circ \gamma}(1)$. Provided we show that this is well-defined and continuous, it is our required lift, since $(p \circ \widetilde{f})(y) = p(\widetilde{f}(y)) = p(\widetilde{f \circ \gamma}(1)) = p \circ \widetilde{f \circ \gamma})(1) = (f \circ \gamma)(1) = f(\gamma(1)) = f(y)$. To show that it is well-defined, if η is any other path from y_0 to y, then $\gamma * \overline{\eta}$ is a loop in Y, so $f \circ (\gamma * \overline{\eta}) = (f \circ \gamma) * \overline{(f \circ \eta)}$ is a loop in X representing an element of $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and so lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$. Consequently, as before, $f \circ \gamma$ and $f \circ \eta$ lift to paths starting at $\widetilde{x}_0$ with the same value at 1. So $\widetilde{f}$ is well-defined. To show that $\widetilde{f}$ is continuous, we use the evenly covered property of p. Given $y \in Y$, and a neighborhood $\widetilde{\mathcal{U}}$ of $\widetilde{f}(y)$ in $\widetilde{X}$, we wish to find a nbhd $\mathcal{V}$ of y with $\widetilde{f}(\mathcal{V}) \subseteq \widetilde{\mathcal{U}}$. Choosing an evenly covered neighborhood $\mathcal{U}_y$ for $f(y)$, choose the sheet $\widetilde{\mathcal{U}_y}$ over $\mathcal{U}_y$ which contains $\widetilde{f}(y)$, and set $\mathcal{W} = \widetilde{\mathcal{U}} \cap \widetilde{\mathcal{U}}_y$. This is open in $\widetilde{X}$, and p is a homeomorphism from this set to the open set $p(\mathcal{W}) \subseteq X$. Then if we set $\mathcal{V}' = f^{-1}(p(\mathcal{W})$ this is an open set containing y, and so contains a path-connected open set $\mathcal{V}$ containing y. Then is for every point $z \in \mathcal{V}$ we build a path γ from y_0 to z by concatenating a path from y_0 to y with a path *in* $\mathcal{V}$ from y to z, then by unique path lifting, since $f(\mathcal{V} \subseteq \mathcal{U}_y$, $f \circ \gamma$ lifts to the concatenation of a path from $\widetilde{x}_0$ to $\widetilde{f}(y)$ and a path *in* $\widetilde{\mathcal{U}}_y$ from $\widetilde{f}(y)$ to $\widetilde{f}(z)$. So $\widetilde{f}(z) \in \widetilde{\mathcal{U}}$.

Because $\widetilde{f}$ is built by lifting paths, and path lifting is unique, the last statement of the proposition follows.

Universal covering spaces: As we shall see, a particularly important covering space to identify is one which is simply connected. One thing we can see from the lifting crierion is that such a covering is essentially unique:

If X is locally path-connected, and has two connected, simply connected coverings $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, then choosing basepoints $x_i, i = 0, 1, 2$, since $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_2)) = \{1\} \subseteq \pi_1(X, x_0)$, the lifting criterion with each projection playing the role of f in turn gives us maps $\widetilde{p}_1 : (X_1, x_1) \to (X_2, x_2)$ and $\widetilde{p}_2 : (X_2, x_2) \to (X_1, x_1)$ with $p_2 \circ \widetilde{p}_1 = p_1$ and $p_1 \circ \widetilde{p}_2 = p_2$. Consequently, $p_2 \circ \widetilde{p}_1 \circ \widetilde{p}_2 = p_1 \circ \widetilde{p}_2 = p_2$ and similarly, $p_1 \circ \widetilde{p}_2 \circ \widetilde{p}_1 = p_2 \circ \widetilde{p}_1 = p_1$. So $\widetilde{p}_1 \circ \widetilde{p}_2 : (X_2, x_2) \to (X_2, x_2)$, for example, is a lift of p_2 to the covering map p_2. But so is the identity map! By uniqueness, therefore, $\widetilde{p}_1 \circ \widetilde{p}_2 = Id$. Similarly, $\widetilde{p}_2 \circ \widetilde{p}_1 = Id$. So (X_1, x_1) and (X_2, x_2) are homeomorphic. So up to homeomorphism, a space can have only one connected, simply-connected covering space. It is known as the *universal covering* of the space X.

Not every (locall path-connected) space X has a universal covering; a (further) necessary condition is that X be *semi-locally simply connected*. The idea is that If $p : \widetilde{X} \to X$ is the universal cover, then for every point $x \in X$, we have an evenly-covered neighborhood $\mathcal{U}$ of x. The inclusion $i : \mathcal{U} \to X$, by definition, lifts to $\widetilde{X}$, so $i_*(\pi_1(\mathcal{U}, x)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}) = \{1\}$, so i_* is the trivial map. Consequently, every loop in $\mathcal{U}$ is null-homotopic in X. This is semi-local simple connectivity; every point has a neighborhood whose inclusion-induced homomorphism is trivial. Not all spaces have this property; the most famous is the Hawaiian earrings $X = \bigcup_n \{x \in \mathbb{R}^2 : ||x - (1/n, 0)|| = 1/n\}$. The point $(0, 0)$ has no such neighborhood. We shall see later that this property is also sufficient to guarantee the existence of a universal cover.

One reason for the importance of the universal cover is that it gives us a unified approach to building <u>all</u> connected covering spaces of X. The basis for this is the *deck transformation group* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$. These homeomorphisms, by definition, permute each of the point inverses of p. In fact, since h can be thought of as a lift of the projection p, by the lifting criterion h is determined by which point in the inverse image of the basepoint x_0 it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give the map h, and the opposite inclusion to ensure it is a bijection (because its inverse exists)]. These two groups are in general *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$; this can be seen by following the change of basepoint isomorphism down into $G = \pi_1(X, x_0)$. As we have seen, paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 correspondence with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $p_*(\pi_1(X, x_0)$; so deck transformations are in 1-to-1 correspondence with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of H in G*, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore in 1-to-1 correspondence with the group $N(H)/H$ under $h \mapsto$ the coset represented by the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$. And since h is essentially built by lifting paths, it follows quickly that this map is a homomorphism, hence an isomorphism.

In particular, applying this to the universal covering space $p : \widetilde{X} \to X$, since in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$, its deck transformation group is isomorphic to $\pi_1(X, x_0)$. Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action <u>is</u> the projection map p. The evenly covered property of p implies that X does have the quotient topology under this action.

So every space it X the quotient of its universal cover (if it has one!) by its fundamental group $\pi_1(X, x_0)$, realized as the group of deck transformations. And the quotient map is the covering projection. In general, a quotient by a group action need not be a covering map; the action must be *free* - the only group element which fixes a point is Id (this s from the uniqueness of lifts) - and *properly discontinuous*