Andrew Casson, On the Poincaré and Andrews-Curtis Conjectures

Poincaré Conj: Every closed simply-connected 3-manifold is homeomorphic to S^3.

So easy to state, still unknown ⇝ we don't really know much about 3-manifold topology?

- promising approach to a proof: G. Perelman; differential geometric → Ricci flow

Andrews-Curtis Conjecture: (~1965) [Unstable version]

If $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is a (balanced) presentation of the trivial group, then P can be reduced to the trivial presentation $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$ by a sequence of the following moves:

(1) $r_k \longleftrightarrow r_k^{-1}$, $r_i \longleftrightarrow r_i$ $(i \neq k)$

(2) $r_k \longleftrightarrow r_k r_l$ $(l \neq k)$, $r_i \longleftrightarrow r_i$ $(i \neq k)$

(3) $r_k \longleftrightarrow w r_k w^{-1}$ $(w \in \langle x_1, \ldots, x_n \rangle)$, $r_i \longleftrightarrow r_i$ $i \neq k$

Stable version: allow also:

(4) $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \longleftrightarrow \langle x_1, \ldots, x_{n+1} \mid r_1, \ldots, r_n, x_{n+1} \rangle$

Examples:

(1) Akbulut-Kirby examples:

$$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^{n+1}y^{-n} \rangle \quad (n \in \mathbb{Z})$$

(initially studied $n=4$)

- they are unstably AC reducible to the trivial presentation for $n=0,1$, (and with more effort) 2
- unknown (stably or unstably) for $n \geq 3$.

(2) Miller-Schupp:

$\langle x, y \mid x^2 y x^{-3} y^{-1},\ w \rangle$ is the trivial group, for any word w for which the abelianization is trivial (i.e., w has total exponent ± 1 in y).

[in fact $\langle x, y \mid x^n y x^{n+1} y^{-1},\ w \rangle$ works, too.]

(Idea: $x^{2^k 3^k} = w x^{2^k 3^k} w^{-1} = x^{2^{k-1} 3^{k+1}}$ (b/c $x\, x^{\text{blah}} x^{-1} = x^{\text{blah}}$, $y\, x^{\text{blah}} y^{-1} = x^{\text{blah} \cdot 3/2}$)

$w = y^2 x y^{-3} x^{-1}$ is not known to be AC reducible.

Topological motivations : smooth 4-dim'l Poincaré Conj.

Potential counterexamples to the AC conjecture can be used to build potential counterexamples to 4-d Poincaré.

$P = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle \rightsquigarrow K^2 =$ Cayley complex

$K^2 \hookrightarrow \mathbb{R}^5 \qquad W = N(K^2) \subseteq \mathbb{R}^5$

It is easy to show that P= presentation of trivial group

$\Rightarrow W$ is 1-connected, $\chi(W) = 1$ (from balanced presentation)

$\Rightarrow W$ is contractible

$\Sigma = \partial W$ is 1-connected (b/c W has a 2-dim'l spine)

~~and~~ and $\Sigma \simeq S^4$, C^∞-manifold

If P is stably AC reducible to the trivial presentation, then

$\Sigma \cong S^4$ (diffeomorphic)

The idea: W has a handle decomposition

$$W = B^5 \cup \underbrace{h_1^1 \cup \dots \cup h_n^1}_{\text{1-handles}} \cup \underbrace{h_1^2 \cup \dots \cup h_n^2}_{\text{2-handles}}$$

$$\partial(\mathbb{B}^5 \cup h_1^1 \cup \cdots h_n^1) \cong \#(S^1 \times S^3) = M$$

2-handles attach by loops representing the relators in M.

(Un-Stable) AC moves correspond to handle slides, and change of path to the base point.

AC reducibility $\Rightarrow$ can slide handles to cancelling pairs, homotopically, but b/c in a 4-manifold, can isotope to true cancelling pairs, $\Rightarrow$ ~~~~ $W \cong \mathbb{B}^5$.

Stabilization $\Longleftrightarrow$ adding a cancelling pair of handles.

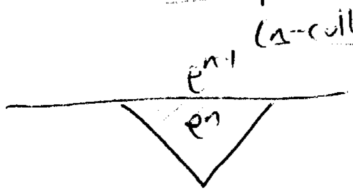
Akbulut-Kirby examples: all lead to the standard S^4 (Gompf). Proof uses introduction of a cancelling pair of a 2- and a 3-handle.

Andrew Casson, Lecture 2

JHC Whitehead (1939) Complexes $L \subseteq K$ { simplicial / CW

$K \searrow L$ (collapses) if $K \setminus L = e^n \cup e^{n-1}$ (with e^{n-1} a face of e^n)

(n-collapses) (e^{n-1} = "free face" of e^n)



In particular, $L \simeq K$; also write $L \nearrow K$ (expands)

$K \nearrow\searrow L$ if $\exists$ sequence of expansions and collapses starting with K & ending with L. "K deforms to L".

(WLOG: all expansions occur first)

K n-deforms to L $K \overset{n}{\nearrow\searrow} L$ if the expansions and collapses can be chosen to have dimension at most n.

$$\boxed{K \nearrow\searrow L \Longrightarrow K \simeq L}$$

Whitehead: the converse is not true, but

$K \nearrow\searrow L \Longleftrightarrow K \simeq L$ by a homotopy equivalence h with Whitehead torsion $\tau(h) = 0$ in $Wh(\pi_1(K))$

Conj: G torsion-free $\Rightarrow Wh(G) = 0$.

h a h.e. with $\tau(h) = 0$ is called a simple homotopy equiv..

$K^k \nearrow\!\!\searrow L^l \Longrightarrow K^k \overset{n}{\nearrow\!\!\searrow} L^l$ provided ~~max{k,l}~~ $n \geq \max\{k+1, l+1, 4\}$

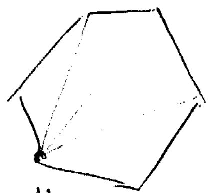
Question: If $K^2 \nearrow\!\!\searrow L^2$, is $K^2 \overset{3}{\nearrow\!\!\searrow} L^2$?

This can be thought of as a generalization of the AC conj:

Th (P. Wright): Stable AC conj hold $\Longleftrightarrow$ for all contractible K^2, $K^2 \overset{3}{\nearrow\!\!\searrow} *$.

Analogue of AC conj is not true for presentations of non-triv groups (cannot get from one to other by AC moves) - first example, the trefoil knot group, done by Dunwoody (1976).

Sketch proof: $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$, from this, build a "simplicial" complex K, by building the Cayley complex and triangulating the 2-cells (however you want to do this)

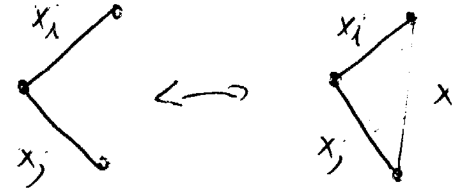


Introduce new gen's for added one-cells, get a new presentation, that is AC equivalent to P, having all relators of length 3.

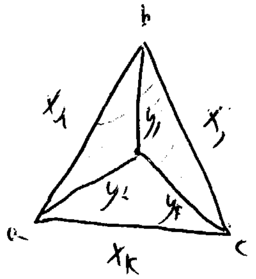
Idea: show $\left.\begin{array}{l}\text{3-expansions}\\ \text{3-collapses}\end{array}\right\} \longleftrightarrow$ stable AC moves.

For such ("triangulated") presentations, stable AC equiv is gen'd by:

(1) 2-expand/2-collapse



(2) tetrahedral moves:



$\longleftrightarrow$ ~~replace~~

replace one of the triangles by $\Delta(a,b,c)$

(add a 3-simplex, collapse along the replaced face)

provided: it is distinct from the other two!

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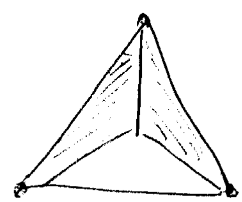


Q: Are the stable & unstable AC conjectures equivalent?

Parallel Q in 3-mflds: All Heegaard splittings of S^3! are standard.

(i.e. if Heegaard splitting of S^3 is stably equiv to a standard one, then it is unstably equiv to standard). True, by Waldhausen.

Also related to

Zeeman Conjecture: If K^2 is a contractible polyhedron, then $K^2 \times I \searrow *$. (allowing further subdivision of $K^2 \times I$)

Zeeman C. $\Longrightarrow$ AC Conj (since $K^2 \nearrow K^2 \times I$)

$\Longrightarrow$ Poincaré Conj.

(K^2 = Spine of $\Sigma^3 \setminus \mathring{B}^3$; $K^2 \times I \searrow *$ $\Longrightarrow$

Gillman - Rolfsen: Poincaré Conj $\Longleftrightarrow$ Zeeman Conj for "special spines" of 3-manifolds.

(non-mfld pts of K have nbhds $\cong$ Y $\times I$)

Matveev: The Stable AC Conj. $\Longleftrightarrow$ Zeeman Conj for "special" polyhedra that <u>don't</u> embed in 3-manifolds.

$P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ presents the trivial group $\Longleftrightarrow$ it has a <u>trivialization</u> $t_1, \ldots, t_n \in \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ ← free group

s.t. $t_i(x_1, \ldots, y_n, 1, \ldots, 1) = 1$

(i.e. t_i = product of conjugates of the y_i's)

<u>and</u> $t_i(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_i \in \langle x_1, \ldots, x_n \rangle$

Ex:

For $\langle x_1,\dots,x_n \mid x_1,\dots,x_n\rangle$, $t_i = y_i$ is a trivialization

Defn: A trivialization $t_1,\dots,t_n \in \overset{F}{\overbrace{\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle}}$ is good if $\{x_1,\dots,x_n,t_1,\dots,t_n\}$ is a basis for F.

Thm: $\langle x_1,\dots,x_n \mid r_1,\dots,r_n\rangle$ is (unstably) AC reducible $\iff$ it has a good trivialization.

Pf:

($\Rightarrow$): $\langle x_1,\dots,x_n \mid x_1,\dots,x_n\rangle \xrightarrow{\text{AC moves}} \langle x_1,\dots,x_n \mid r_1,\dots,r_n\rangle$

Show by induction having a good trivialization is preserved.

Base case is above.

Show having a good trivialization is invariant under AC moves
— there is a "lazy" way to do this..

($\Leftarrow$): $F_{2n} = \langle x_1,\dots,x_n,y_1,\dots,y_n\rangle$ $\quad F_n = \langle x_1,\dots,x_n\rangle$

$\pi: F_{2n}\to F_n \quad \pi(x_i)=x_i,\ \pi(y_i)=1.$

$\pi_p: F_{2n}\to F_n \quad \pi_p(x_i)=x_i,\ \pi_p(y_i)=r_i$

$\varepsilon: F_n\to F_{2n} \quad \varepsilon(x_i)=x_i$

$\pi_1: F_{2n}\to F_n \quad \pi_1(x_i)=x_i,\ \pi_1(y_i)=x_i$

Have a good trivialization; then have a commuting diagram

$\alpha : F_{2n} \to F_{2n}$ $\qquad$ $\alpha(x_i) = x_i$ $\quad$ $\alpha(y_i) = t_i$

$$\begin{tikzcd}
 & F_n \arrow[dl, "\varepsilon"'] \arrow[dr, "\varepsilon"] \arrow[d, phantom, "\circlearrowleft"] & \\
F_{2n} \arrow[rr, dash, "\alpha"] \arrow[dr, "\pi \times \pi_1"'] & \circlearrowleft & F_{2n} \arrow[dl, "\pi \times \pi_p"] \\
 & F_{2n} \times F_{2n} &
\end{tikzcd}$$

<u>Lemma</u>: Any $\alpha : F_{2n} \to F_{2n}$ with $\alpha \circ \varepsilon = \varepsilon$, $\pi \circ \alpha = \pi$ is a product of "Andrews-Curtis" generators (moves)

Andrew Casson, Lecture 3

Andrews-Curtis Conjecture   unstable → stable

So if you're going to try to <u>disprove</u> it, it is (logically) easier to <u>disprove</u> the unstable one.

If you're going to try to <u>prove</u> it, it is easier to prove the stable one.

~~Proving~~ Disproving the unstable ACC:

Build an <u>invariant</u> of balanced presentations of trivial group!

(Miasnikov.... ~ use free <u>solvable</u> groups as a start.)

We have lots of potential counterexamples...

Miller-Schupp $\langle x, y \mid x^{n+1} y x^{-n} y^{-1}, w \rangle$, $\deg_y w = 1$

~~Ex~~ $|w| \sim 2n$.

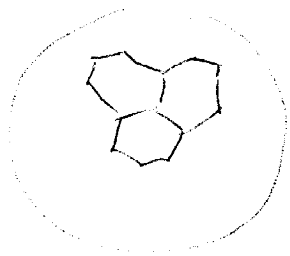
Mantra: to get trivial group, relators tend to need <u>large</u> overlaps with cyclic conjugates of itself and/or other relators.

First relator <u>does</u>: $x^n // x^{-n}$. But we can choose the second relator "randomly", so that it <u>doesn't</u>

(whatever they mean?

Q: Is it true that all trivializations "rescale" the "standard" one? and <u>none</u> is "good"?

van Kampen diagrams for $x = 1$



(this approach is not amenable to stabilization; stabilization would destroy the combinatorial properties of the two relators, that we are trying to exploit.

Recall:

Thm: $\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ is unstably AC trivializable $\Longleftrightarrow$ it has a <u>good</u> trivialization:

$f_1, \dots, f_n \in F_{2n} = \langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ s.t.

$f_i(x_1, \dots, x_n, 1, \dots, 1) = 1$, $f_i(x_1, \dots, x_n, r_1, \dots, r_n) = x_i$ (<u>triv'n</u>)

$x_1, \dots, x_n, f_1, \dots, f_n$ = <u>basis</u> for F_{2n} (<u>good</u>).

Can change f_i's by mult by elts of normal sg <u>gen'd</u> by commutators of y_i's and of the $y_i r_i^{-1}$'s, without changing fact that you <u>have</u> a trivialization. But you <u>can</u> easily hide a <u>good</u> trivialization this way.

$P = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ presentation of trivial grp

if we do <u>not</u> insist that we keep the defect of the presentation constant, then we <u>can</u> trivialize P by what look like AC moves ...

$\langle x_1,\dots,x_n \mid r_1,\dots,r_n\rangle \longrightarrow \langle x_1,\dots,x_n \mid r_1,\dots,r_n,\overbrace{1,\dots,1}^{n}\rangle$

$\underset{AC}{\sim} \langle x_1,\dots,x_n \mid r_1,\dots,r_n, x_1,\dots,x_n\rangle$

(b/c each x_i is a prod of conj's of r_j's = sequence of AC moves applied to the relator $\underline{1}$.

$\underset{AC}{\sim} \langle x_1,\dots,x_n \mid 1,\dots,1, x_1,\dots,x_n\rangle \underset{AC}{\sim} \langle x_1,\dots,x_n \mid x_1,\dots,x_n,1,\dots,1\rangle$

works for $(n-1)$ "illegal" relators too (x_n can be recovered from $x_1,\dots,x_{n-1},r_1,\dots,r_n$ by AC moves applied to $r_1,\dots,r_n$

(homologically, x_n appears only once)

$F = F_{3n-1} = \langle x_1,\dots,x_n, y_1,\dots,y_n, z_1,\dots,z_{n-1}\rangle$

Recall:

$F_{2n} = \langle x_1,\dots,x_n,y_1,\dots,y_n\rangle$ $\quad F_n = \langle x_1,\dots,x_n\rangle$

$\pi: F_{2n}\to F_n \quad \pi(x_i)=x_i,\ \pi(y_i)=1$

$\pi_p: F_{2n}\to F_n \quad \pi_p(x_i)=x_i,\ \pi_p(y_i)=r_i$

$\pi_{triv}: F_{2n}\to F_n \quad x_i\mapsto x_i,\ y_i\mapsto x_i$

$\varepsilon: F_n\to F_{2n}$

$x_i\mapsto x_i$

$$\begin{tikzcd}
 & F_n \arrow[dl, "\varepsilon"'] \arrow[d, "\varepsilon"] \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "\varepsilon"] & & \\
F_{2n} \arrow[dr, "\pi\times\pi_p"'] & F_{3n-1} \arrow[l, "\alpha"'] \arrow[r, dashed, "\beta"] \arrow[d, "\pi\alpha\times\pi_p\alpha"] & F_{3n-1} \arrow[r, "\gamma"] \arrow[dl, "\pi\alpha\times\pi_{triv}\alpha"] & F_{2n} \arrow[dll, bend left, "\pi\times\pi_p"] \\
 & F_n\times F_n & &
\end{tikzcd}$$

$\alpha: x_i\mapsto x_i,\ y_i\mapsto y_i,\ z_i\mapsto 1.$

β = iso making bottom Δ commute

γ : kills $\alpha\beta(z_i)$'s

The data of this ~~data~~ diagram is completely captured by the middle iso β. In fact:

<u>Thm</u>: The unstable ACC holds for all n-gen presentations of the trivial group $\iff$ for any $\xi_1, \dots, \xi_{n-1} \in F_{3n-1} = \langle x_1, y_1, z_1, \dots, z_{n-1} \rangle$ <u>such that</u> $\xi_i \in \ker(\pi \times \pi_{triv})$ and

$x_1, \dots, x_n, \xi_1, \dots, \xi_{n-1}$ are part of a basis of F_{3n-1} <u>we have</u> $\exists\ \beta \in \mathrm{Aut}(F_{3n-1})$ s.t. $\xi_i = \alpha(z_i)$ <u>and</u> $\pi\beta = \pi$, $\pi_{triv}\beta = \pi_{triv}$.

[Part of a basis: Whitehead's algorithm will check this]

$$G_n = \{\alpha \in \mathrm{Aut}(F_{3n-1}) : \alpha\varepsilon = \varepsilon,\ \pi\alpha = \pi,\ \pi_{triv}\alpha = \pi_{triv}\}$$

Question is really about finding the G_n-orbits of $z_1, \dots, z_{n-1}$.

<u>Ex</u>: $n=2$. Looking for G_2-orbits of z_1.

<u>Some</u> <u>examples</u>

$\langle a, b \mid abab^{-1}a^{-1}b^{-1}, a^3b^{-2} \rangle$

$\widetilde{AC}\ \langle a, b \mid a^2bab^{-1}a^{-1}b, a^3b^2 \rangle$

$\widetilde{AC}\ \langle a, b \mid a^2bab^{-1}a^{-1}b, ababa^{-1}b^{-1} \rangle$

$\widetilde{AC}\ \langle a, b \mid ababa^{-1}b^{-1}, a^2bab \rangle$

$\widetilde{AC}\ \langle a, b \mid a^2bab, a^2b \rangle$

$\widetilde{AC}\ \langle a, b \mid a^2b, ab \rangle$

$\widetilde{AC}\ \langle a, b \mid ab, a \rangle \quad \widetilde{AC} \quad \langle a, b \mid a, b \rangle$

relator is replaced by (conj of one) · (conj of other)

Can one use a computer to find examples of AC related presentations?

Idea: attempts to classify

$\{$presentations$\}$ / AC relns and symmetries

Symmetry
(1) cyclic conjugation and/or inversion of relators
(2) permutation of relators
(3) inversions of some or all generators
(4) permutation of generators

Havas, Ramsay: on web page, U of Queensland. (appeared in IJAC)

Miasnikov: $\exists$ no counterexample to unstable 2-gen AC conj with relators having (total?) length ≤ 12.

Havas-Ramsay: Every presentation $\langle x,y \mid r,s \rangle$ of trivial group with $|r|+|s| \leq 13$ is AC reducable to either the trivial presentation or to Akbulut-Kirby presentation

$$\langle x,y \mid xyxy^{-1}x^{-1}y^{-1}, x^4y^{-3} \rangle .$$

Short list of AC equiv classes with $|r|+|s| \leq 14$.

That we can find the pres's of the trivial group is not new; nearly all perfect (= trivial abelianization?) presentations with $|r|+|s| = 14$ present the trivial group

Andrew Casson, Lecture 4

Andrews Curtis equiv classes of pres's of $\langle x,y \mid r,s\rangle$ of non-triv perfect groups $|r|,|s| \le 7$ (modulo symmetries) fall into a few (≤ 5?) classes.
(all are the binary icosahedral group)

(have small #, need equiv classes are unions (unknown) of ones computed.)

For $|r|,|s| \le 8$ have reduced to a rather larger list of classes of perfect groups
(no attempt made to decide which to decide which perfects are trivial?)

Approach: Consider chains of AC moves, no limit on length of chain, but limited lengths of relators to 25 each.
(all such chains where explored.)

HW: $\langle x,y \mid x^2y^{-1}xy^{-1}x^{-1}y,\ x^2y^2x^{-1}y^{-2}\rangle$ is AC trivial

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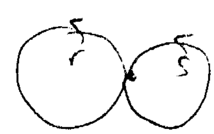
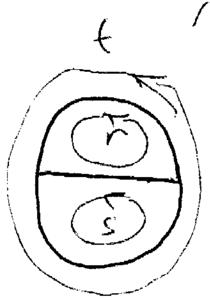
Moves $(r,s) \longrightarrow (r,\ u^{-1}r^{\pm 1}uv^{-1}s^{\pm 1}v) = (r,t)$

special case $uv^{-1}=1$



[don't really use this very much...]

(look for common segments of pair of relators)



Θ-moves (inverse is often a "barbell" move)

Sometimes one insists that both move and its inverse is a Θ-move.

Followed a fairly straightforward branching algorithm, with a ceiling imposed. This is inefficient, b/c there may be several paths to the same presentation.

$\Rightarrow$ Keep a list of presentations visited to prune the branching tree. Very memory intensive.

(Lookups are not bad: keep a hash table (with lex ordering of relators?)

Ranked pres by complexity, always worked on pres of lowest complexity that had not been explored before.

Poincaré Conjecture:

Stallings' "How not to prove the Poincaré Conjecture".

Thm: (Papakyriakopoulos, Stallings, Waldhausen, Jaco):

The Poincaré Conjecture holds for manifolds of Heegaard genus $\leq g$
$\iff$ $\forall$ surjection $\rho : \pi_1(\Sigma_g) \twoheadrightarrow F_g \times F_g$, $\exists$
automorphisms $\alpha : \pi_1(\Sigma_g) \circlearrowleft$, $\beta : F_g \times F_g \circlearrowleft$ s.t.

$$\begin{tikzcd} \pi_1(\Sigma_g) \arrow[r, "\alpha"] \arrow[d, "\rho"] \arrow[dr, phantom, "\circlearrowleft"] & \pi_1(\Sigma_g) \arrow[d, "\pi"'] \\ F_g \times F_g \arrow[r, "\beta"'] & F_g \times F_g \end{tikzcd}$$

$= \langle x_1, \dots, x_g, y_1, \dots, y_g \mid \prod [x_i, y_i] = 1 \rangle$

standard surj $= (S_1 \times S_2)$ $\quad S_1$ kills y_i's, $\quad S_2$ kills x_i's

$$\pi \alpha = \beta \rho .$$

[follows from: all Heegaard splittings of S^3 are standard.] ///

Andrew Casson, Lecture 5

S_g = closed orientable surface of genus g $\quad \Gamma_g = \pi_1(S_g) = \langle x_1, \dots, x_g, y_1, \dots, y_g \mid \prod_{i=1}^{g} [x_i, y_i] \rangle$

Standard surjection $\quad \Gamma_g \xrightarrow{\pi} F_g \times F_g = \langle x_1, \dots, x_g \rangle \times \langle y_1, \dots, y_g \rangle$

Thm: The Poincaré Conj. for mflds of Heegaard genus $\leq g$ is true
$\iff$ $\forall$ surjection $\rho : \Gamma_g \to F_g \times F_g$, $\exists$ automorphisms
$\alpha : \Gamma_g \to \Gamma_g$, $\beta : F_g \times F_g \to F_g \times F_g$ s.t.

$$\begin{tikzcd} \Gamma_g \arrow[r, "\alpha"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & \Gamma_g \arrow[d, "\pi"] \\ F_g \times F_g \arrow[r, "\beta"'] & F_g \times F_g \end{tikzcd}$$

commutes.

Problem: ~~deciding~~ Determining whether or not a collection $u_1, \dots, u_g, v_1, \dots, v_g$ generates $F_g \times F_g$ is undecideable.

So goal: replace surjectivity hypothesis by something more manageable.

First step: we can (almost) replace β by Id.

Lemma:
$$\begin{tikzcd} \Gamma_g \arrow[r] \arrow[d, "\pi"'] & \Gamma_g \arrow[d, "\pi"] \\ F_g \times F_g \arrow[r, "\beta"'] & F_g \times F_g \end{tikzcd}$$

β lifts to an auto of Γ_g?

Look at maps on level of H_2

$K(F_g \times F_g, 1) = (\vee_g S^1)^2 \quad H_2 = \mathbb{Z}^g \otimes \mathbb{Z}^g = \mathbb{Z}^{g^2}$

$K(\Gamma_g, 1) = \Sigma_g \quad H_2 = \mathbb{Z}$.

β lifts $\iff \beta_*(\pi_*[\Sigma_g]) = \pm \pi_*[\Sigma_g]$ in $H_2 \cong \mathbb{Z}^{g^2}$.

So can reformulate algebraic reformulation of PC:

$PC_{\leq g}$ true $\iff$ $\forall$ surj $\rho : \Gamma_g \to F_g \times F_g$ s.t. $\rho_*[\Sigma_g] = \pm \pi_*[\Sigma_g]$

$\exists$ aut $\alpha : \Gamma_g \to \Gamma_g$ s.t.

$$\begin{tikzcd} \Gamma_g \arrow[rr, "\alpha"] \arrow[dr, "\rho"'] & & \Gamma_g \arrow[dl, "\pi"] \\ & F_g \times F_g & \end{tikzcd}$$

commutes.

Pf. of Lemma $\beta : F_g \times F_g \to F_g \times F_g$ $g \geq 2$ $\Rightarrow$ β either preserves the factors or reverses them.

(If $\beta(x_i) = y_i$, $\beta(y_i) = x_i$ $\forall i$ can build a lift by hand, so focus on preserving factors). So $\beta = (\beta_1, \beta_2)$, $\beta_i \in Aut(F_g)$

Set $G = \{ \alpha \in Aut(\Gamma_g) : \pi\alpha = (\beta_1, \beta_2)\pi$ some $\beta_1, \beta_2 \}$

$G \to Aut(F_g) \times Aut(F_g)$

Using handle slides, can show that $(1, \beta_2) \in$ image of G.
(& flips)

(Think: $\pi : \Gamma_g \to F_g \times F_g$ represents the standard Heegaard splitting of S^3, the standard maps from inclusion $(\Sigma_g, H_g) \hookrightarrow H_g$.)

H_2 hypothesis $\Longrightarrow$ $\beta_2 \in IA_g = \ker(Aut\, F_g \to GL_g(\mathbb{Z}))$

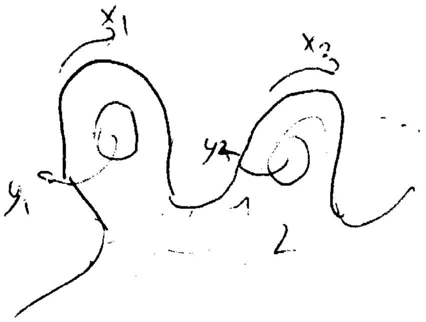
$\{ \beta_2 : (1, \beta_2) \in \text{image of } G \} \trianglelefteq Aut(F_g)$.

(Nielsen)

Classical result: IA_g is normally gen'd by a single element.

So it is enough to lift that one element.

↳ ~~slides one handle over another~~



$x_1 \longmapsto x_2^{-1} x_1 x_2$

$x_i \longmapsto x_i \quad i \neq 1$

$y_j \longmapsto y_j \quad \forall j$

[So this is really a $(\beta_1, 1)$ but argument is symmetric

Then just show that this lifts!

Reformulation: $\Sigma_g = S_g \setminus \mathring{D}^2 \quad \pi_1(\Sigma_g) = \langle x_1, y_1, \dots, x_g, y_g \rangle$

standard map $\pi : \pi_1(\Sigma_g) \longrightarrow F_g \times F_g$ (via $\pi_1(\Sigma_g) \to \pi_1(S_g) \xrightarrow{\bar{\pi}} F_g \times F_g$)

$K = \mathrm{Ker}(\pi_1(\Sigma_g) \xrightarrow{\pi} \pi_1(F_g \times F_g)) \qquad \pi_1(\Sigma_g)/K \cong F_g \times F_g$

normally gen'd by commutators of x_i's with y_j's.

Thm: $P.C_{S_g}$ is true $\iff$

$\forall$ homomorphism $\varphi : \pi_1(\Sigma_g) \longrightarrow \pi_1(\Sigma_g)$ with

$\varphi(\partial \Sigma_g) \equiv \partial \Sigma_g$ modulo $K' = [K, K]$, $\exists$ homom

$\psi : \pi_1(\Sigma_g) \longrightarrow \pi_1(\Sigma_g)$ s.t. $\psi \equiv \varphi$ modulo K (ie., $\pi\psi = \pi\varphi$)

and $\psi(\partial \Sigma_g) = \partial \Sigma_g$.

Rmks: $\psi(\partial\Sigma_g) = \partial\Sigma_g \iff \psi$ is an auto

$\psi \equiv \varphi$ modulo $K \iff \pi\psi = \pi\varphi : \pi_1(\Sigma_g) \to F_g \times F_g$.

Note: To check $\phi(\partial\Sigma_g) \equiv \partial\Sigma_g$ mod K', need $\phi(\partial\Sigma_g) \in K$ (i.e. projects trivially to $F_g \times F_g$; checkable) and need to check its value in $K/K' \cong \mathbb{Z}[F_g \times F_g]^{g^2}$, bases $[x_i, y_i]$, checkable!
(as a left module)

Lemma: For a homom $\phi : \pi_1(\Sigma_g) \to \pi_1(\Sigma_g)$, TFAE:

(1) $\pi\phi : \pi_1(\Sigma_g) \to F_g \times F_g$ is onto and $\phi(\partial\Sigma_g) \equiv \partial\Sigma_g$ modulo $[K, \pi_1(\Sigma_g)]$

(2) $\exists$ homom $\psi : \pi_1(\Sigma_g) \to \pi_1(\Sigma_g)$ with $\psi \equiv \phi$ mod K and $\psi(\partial\Sigma_g) \equiv \partial\Sigma_g$ modulo K'.

Pf:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] & \pi_1(\Sigma_g) \arrow[r, "\pi"] & F_g \times F_g \arrow[r] & 1 \\
 & & \pi_1(\Sigma_g) \arrow[u, "\phi"] \arrow[ur, dashed, "\partial"] \arrow[r, two heads, "\text{standard}"'] & \Gamma_g \arrow[u, two heads, "\rho"'] &
\end{tikzcd}$$

with $\rho_*([\Sigma_g]) = \pi_*[\Sigma_g]$

(2) $\Rightarrow$ (1) ψ is induced by a map on the surface, $\psi : \Sigma_g \to \Sigma_g$
$\psi(\partial\Sigma_g) \equiv \partial\Sigma_g$ modulo K'.

Can represent a product of commutators as image of the bdry of a punctured surface Σ_k, have $\psi_k : \Sigma_k \longrightarrow \Sigma_g$ $(\psi_k)_*(\pi_1(\Sigma_k)) \leq K$

$\underbrace{\psi \#_\partial \psi_k}$ $: \Sigma_g \# \Sigma_k \longrightarrow \Sigma_g$ maps $\partial(\Sigma_g \#_\partial \Sigma_k)$ to $\partial\Sigma_g$ homeo

$\Rightarrow$ has degree 1 $\Rightarrow$ <u>onto</u> on level of π_1.

$\therefore\ \pi\psi$ maps $\pi_1(\Sigma_g)$ onto $\pi_1(\Sigma_g)$.

(1) $\Rightarrow$ (2) argument is a homological calculation

-calculate in K/K'.

Questions: ~~Given $\phi : \Sigma_g \to \Sigma_g$ s.t.~~

(1) Given $\phi : \pi_1(\Sigma_g) \to \pi_1(\Sigma_g)$ s.t. $\phi(\partial\Sigma_g) \equiv \partial\Sigma_g$ modulo K'.

Is $\phi \equiv \psi$ (modulo K) s.t. $\psi(\partial\Sigma_g) \equiv \partial\Sigma_g$ modulo $\underline{\underline{K''}}$?

(2) Is φ as above $\underset{K}{\equiv} \psi$ with $\psi(\partial\Sigma_g) \equiv \partial\Sigma_g$ modulo $[K', \pi_1(\Sigma_g)]$

(3) " " " " $\underset{K}{\equiv} \psi$ s.t. $\psi_* : H_1(\Sigma_g ; \mathbb{Z}[F_g \times F_g]) \longrightarrow K/K'$ is onto?

twisted coeffs

Lemma: (2) & (3) $\Longrightarrow$ (1).