Cameron Gordon, Surface subgroups of Coxeter groups
(Joint with Darren Long, Alan Reid)

Surface group $= \pi_1(\Sigma)$ Σ = closed orientable, genus ≥ 1
hyp surf group " " " ≥ 2

G is word hyp if G fin gen'd & Cayley graph is δ-hyp,

Q (Gromov): Does every 1-ended word hyp group contain a (hyp) surface group.

E.g. $G = \pi_1$(closed hyp 3-mfld)

Note: G word hyp $\Rightarrow$ $G \not\supseteq \mathbb{Z}+\mathbb{Z}$.

Equivalently:

Q': G word hyp, not virtually free. Does G contain a (hyp) surface group?

Γ finite graph (no loops, no multiple edges)

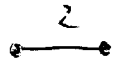
$V(\Gamma) = \{\text{vertices}\} = \{s_1, .., s_n\}$

m = a labelling of Γ, i.e. each edge of Γ with endpts s_i & s_j is labelled $m_{ij} \in \mathbb{Z}$ $m_{ij} \geq 2$ Set $m_{ii} = 1$.

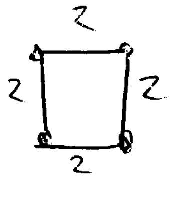
Coxeter group $G = G(\Gamma, \underline{m}) = \{ s_1, \dots, s_m : (s_i s_j)^{m_{ij}} = 1 \}$

In part, $s_i^2 = 1 \ \forall i$.

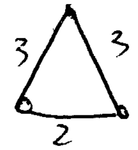
Examples:

 $\cong \mathbb{Z}_2$ 

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$



$\cong (\mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$



$\cong S_4$.

Γ = graph, $V \subseteq V(\Gamma)$ $\quad Sp(V)$ = span of V
= $\cup \{$all edges w/ both endpts in $V\}$

$\Gamma' \subseteq \Gamma$ is full if
$\Gamma' = Sp(V(\Gamma'))$

Bourbaki: $\Gamma' \subset \Gamma$ full, then the natural map
$G(\Gamma', \underline{m}') \longrightarrow G(\Gamma, \underline{m})$ is injective
(image = special subgroup.)

Coxeter group is word hyp $\iff G \not\supset \mathbb{Z} + \mathbb{Z}$
$\iff$ explicit conditions on labelled graph.

$\mathcal{F}$ = { finite Coxeter groups }

$\mathcal{G}$ = smallest class of Coxeter groups s.t.

(1) $\mathcal{F} \subseteq \mathcal{G}$

(2) $G_1, G_2 \in \mathcal{G}$, $G_0 \in \mathcal{F} \Rightarrow G_1 *_{G_0} G_2 \in \mathcal{G}$

($G_0 \hookrightarrow G_i$ special subgroup, $i = 1,2$)

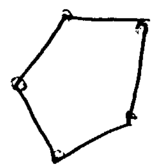
[Note: all groups in $\mathcal{G}$ are virtually free]

Thm 1: G a Coxeter group then either $G \in \mathcal{G}$ or $G \geq$ surface group.

Cor: A 1-ended word hyp Coxeter group $\geq$ hyp surface group.

Pf:

A graph is chordal if $\Gamma \not\supseteq$ full n-cycle, $n \geq 4$ ($\Leftrightarrow$ always $\exists$ chord for such a thing)



Ex: complete graphs

$\mathcal{C}$ = smallest class of graphs s.t.

(1) $K_n \in \mathcal{C}$ $\forall\, n \geq 0$

(2) $\Gamma_1, \Gamma_2 \in \mathcal{C}$, $\Gamma_0 \cong K_n \Rightarrow \Gamma_1 \cup_{\Gamma_0} \Gamma_2 \in \mathcal{C}$

[Note: induction $\Rightarrow$ all such graphs are chordal]

<u>Thm</u> (G. Dirac, 1961): $\Gamma \in \mathcal{G} \iff \Gamma$ is chordal.

($\Longrightarrow$) induction ✓

($\Longleftarrow$) If $\Gamma = K_n$, ✓

<u>If</u> $\Gamma \not\cong$ any K_n, then $\exists\, a,b \in V(\Gamma)$ s.t. <u>no</u> edge b/w a & b

Let $\Gamma' = Sp(V(\Gamma) \setminus \{a,b\})$

Γ' full & separating ($\Gamma \setminus \Gamma'$ disconnected)

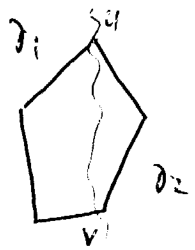
<u>Let</u> $\Gamma_0 \subseteq \Gamma$ be a minimal, full separating subgraph.

Then $\Gamma \setminus \Gamma_0 = \coprod_{i=1}^{k} X_i \quad k \geq 2$.

$\bar{X}_i = X_i \cup V(\Gamma_0)$ (by minimality)

If $\Gamma_0 \not\cong$ any K_n, $\exists\, u,v \in V(\Gamma_0)$ s.t. no edge b/w u,v

Let ∂_i = minimal length path in $\bar{X}_i$ from u to v, $i=1,2$



Then $\partial_1 \cup \partial_2$ is a full n-cycle in Γ, $n \geq 4$ ✱

$\therefore \Gamma_0 \cong K_m$

Γ_0

$\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2 \quad |V(\Gamma_i)| < |V(\Gamma)| \quad i=1,2$

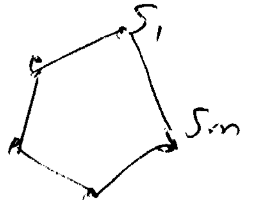
Γ chordal $\Longrightarrow \Gamma_i$ chordal $i=1,2$

$\therefore$ by induction $\Gamma_1, \Gamma_2 \in \mathcal{G}$, $\therefore \Gamma \in \mathcal{G}$.

Prop 1: Γ an n-cycle, $n \geq 4$ then

$G(\Gamma, \underline{m}) \supseteq$ a surface group, hyp unless $n=4$ & $\underline{m}=2$.

Pf:



$G \cong$ gp gen'd by reflection in faces of an n-gon $\subseteq \mathbb{H}^2$ (or $\mathbb{E}^2$ if $n=4$ & $\underline{m}=2$) with angles π/m_i.

$\therefore G \supseteq$ surface group.

Prop 2: K_n a complete graph. Then $G(K_n, \underline{m})$ either is finite or $\supseteq$ a surface group.

Pf. By induction on n. $n = 0,1,2$ ✓ (G finite)

Consider the n proper special subgroups $G(K_{n-1}, \underline{m}') \hookrightarrow G(K_n, \underline{m})$

If one is infinite, then by induction it $\supseteq$ surface group

$\therefore G \supseteq$ surface group.

So assume all $G(K_{n-1}, \underline{m}')$ are finite

Thm (Bourbaki) either G is finite, or

$G =$ Euclidean reflection group $\supseteq \mathbb{Z} \times \mathbb{Z}$, or

$G =$ cocpt hyp reflection group in $\mathbb{H}^N$ where

$N=2$ Δ grps $\therefore$ $\cong$ surf grp

$N=3$ 9 examples $\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\}$ $\cong$ Fuchsian grps, $\therefore$ $\cong$ surf grp

$N=\cancel{5}4$ 5 examples

$N\geq 5$ no examples.

Pf of Thm 1: $G=\Gamma($

If $\Gamma \cong$ full n-cycle, some $n\geq 4$, then

$G \cong$ sfce grp, by Prop 1.

So suppose Γ is chordal. Then $\Gamma\in\mathcal{C}$ by Thm 2

If $\Gamma = K_n$, some $n\geq 0$ ✓ by Prop 2

If not, then $\Gamma \cong \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ where $\Gamma_0 \cong K_n$

By induction may assume result holds for $G_i = G(\Gamma_i, \underline{m}_i')$ $^{i=1,2}$

$G = G_1 *_{G_0} G_2$ $(G_0 = G(K_n, \underline{m}_0))$

If either G_1 or $G_2 \cong$ sfce grp, then so does G.

$\therefore$ may assume $G_1, G_2 \in \mathcal{G}$

By Prop 2, $G_0 \in \mathcal{F}$, $\therefore$ $G\in\mathcal{G}$ //

Q1 : When does $G(\Gamma; \underline{m})$ contain a hyp surface group?

In part,

Q2 ~~When~~ Does $G(\Gamma; 3)$ contain a hyp surface group

$\Longleftrightarrow$ $\Gamma \supseteq$ full n-cycle, $n \geq 5$?

Artin groups: $A(\Gamma; \underline{m}) = \{ \langle s_1, \ldots, s_n \mid \underbrace{s_i s_j \cdots = s_j s_i \cdots}_{m_{ij} \text{ terms}} \rangle$

$$G(\Gamma; \underline{m}) = A(\Gamma; \underline{m}) \big/ \langle\langle s_i^2 : i = 1, \ldots, n \rangle\rangle$$

$$E(\Gamma) \neq \phi \Rightarrow A(\Gamma; \underline{m}) \supseteq \mathbb{Z} \times \mathbb{Z}$$

Q: Which Artin groups $\supseteq$ hyp surface group?

(1) (Servatius-Droms, Servatius, 1989)

$\Gamma \supseteq$ full n-cycle, $n \geq 5 \Rightarrow A(\Gamma; 3) \supseteq$ hyp surface group.

Q: Is converse true?

(2) Γ = a tree. Then $A(\Gamma; \underline{m}) \cong \pi_1(S^3 - \#(\Sigma m)$ torus links$)$

$\not\supseteq$ hyp surface groups

(3) $A(\Gamma,\underline{m})$ irreducible finite type.

If $(\Gamma,\underline{m}) \neq \bullet$, $\bullet\overset{m}{—}\bullet$, ${}^{3}\underset{2}{\triangle}{}^{5}$, then $A(\Gamma,\underline{m}) \supseteq$ hyp surface grp.

Main observation

$$A\left({}^{3}\underset{2}{\triangle}{}^{3}\right) \cong \text{braid group } B_4$$

$B_4/Z(B_4) \supseteq$ Figure-8 knot group $\supseteq$ hyp surface group

$\therefore\ A\left({}^{3}\underset{2}{\triangle}{}^{4}\right) \cong B_{1,3} \subseteq B_4$ (finite index) $\Longrightarrow\ \supseteq$ hyp surface group.

Q: Does $A\left({}^{3}\underset{2}{\triangle}{}^{5}\right) \supseteq$ hyp surface group?

<u>Thm</u>: TFAE

(1) $A(\Gamma,\underline{m})$ is a 3-mfld group

(2) $A(\Gamma,\underline{m})$ is virtually a 3-mfld group

(3) each component of $(\Gamma,\underline{m})$ is a tree or ${}^{3}\underset{2}{\triangle}{}^{3}$

((1) $\equiv$ (3): for $\underline{m}=\underline{2}$: Droms
for $\underline{m}$ even: Hermiller-Meier)

Q: Is $A\left({}^{3}\underset{2}{\triangle}{}^{5}\right)$ coherent?