Math 971 Algebraic Topology

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The isomorphism between simplicial and singular homology provides very quick proofs of several results about singular homology, which would other would require some effort:

If the Δ-complex X has no simplices in dimension greater than n, then $H_i(X) = 0$ for all $i > n$.

This is because the simplicial chain groups $C_i^\Delta(X)$ are 0, so $H_i^\Delta(X) = 0$.

If for each n, the Δ-complex X has finitely many n-simplices, then $H_n(X)$ is finitely generated for every n.

This is because the simplicial chain groups $C_n^\Delta(X)$ are all finitely generated, so $H_n^\Delta(X)$, being a quotient of a subgroup, is also finitely generated. [We are using here that the number of generators of a subgroup H of an *abelian* group G is no larger than that for G; this is not true for groups in general!]

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A, B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex K whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to K, with boundary a (for the exact same reasons as above) closed orientable surface.

Another: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h, then $n = m$. This is because we can arrange, by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n - 1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$.

More generally, we can establish a result which is known as *invariance of domain*, which is useful in both topology and analysis.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

We will approach this through the **Brouwer-Jordan Separation Theorem:** an embedded $(n-1)$-sphere in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two path components. And for this we need to do a slightly unusual homology calculation:

For $k < n$ and $h : I^k \to S^n$ an embedding of a k-cube in to the n-sphere, $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$ for all i.

Here $I = [-1, 1]$. The proof proceeds by induction on k. For $k = 0$, $S^n \setminus h(I^k) \cong \mathbb{R}^n$, and the result follows. Now suppose the result os true for all embeddings of $C = I^{k-1}$, but is false for some embedding $h : I^k \to S^n$ and some i. Then if we divide the cube along its last coordinate, say, as $I^{k-1} \times [-1, 0] = C \times [-1, 0]$ and $C \times [0, 1]$, we can set $A = S^n \setminus h(C \times [-1, 0])$, $B = S^n \setminus h(C \times [0, 1])$, $A \cup B = S^n \setminus h(C \times \{0\})$, and $A \cap B = S^n \setminus h(I^k)$. These sets are all open, since the image under h of the various sets is compact, hence closed. By hypothesis, $A \cup B = S^n \setminus h(C \times \{0\})$ has trivial reduced

homology, while $A \cap B = S^n \setminus h(I^k)$ has non-trivial reduced homology in some dimension i. Then the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_{i+1}(A \cup B) \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \widetilde{H}_i(A \cup B) \to \cdots$$

reads $0 \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to 0$ so $\widetilde{H}_i(A \cap B) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$, so at least one of the groups on the right must be non-trivial, as well. WOLOG $\widetilde{H}_i(B) = \widetilde{H}(S^n \setminus h(C \times [0,1])) \neq 0$. Note that, in fact, the inclusion-induced map $\widetilde{H}(A \cap B)\widetilde{H}_i(B)$ is (injective and) non-zero.

Now we continue inductively, cutting $C \times [0,1]$ in two along the last coordinate as $C \times [0,1/2], C \times [1/2,1]$ and repeat the same argument. We fnd that $\widetilde{H}_i(S^n \setminus h(C \times [a,b])) \neq 0$. Continuing inductively, we find a sequence of nested intervals $I_n = [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ whose lengths tend to zero (so $a_n, b_n \to x_0 \in I$ as $n \to \infty$), and injective inclusion-induced maps

$$0 \neq \widetilde{H}_i(S^n \setminus h(I^n)) \to \cdots \to \widetilde{H}_i(S^n \setminus h(C \times I_n)) \to \widetilde{H}_i(S^n \setminus h(C \times I_{n+1}))$$

all of which have an inclusion-induced map to $\widetilde{H}_i(S^n \setminus h(C \times \{x_0\}) = 0$. So there is a non-trivial element $[z] \in \widetilde{H}_i(S^n \setminus h(I^n))$ which remains non-zero in all $\widetilde{H}_i(S^n \setminus h(C \times I_n))$, but is zero in $\widetilde{H}_i(S^n \setminus h(C \times \{x_0\}))$. consequently, $z\partial w$ for some chain $w = \sum a_j \sigma_j^{i+1} \in C_{i+1}(S^n \setminus h(C \times \{x_0\}))$. Each singular simplex, however, is a map $\sigma_j^{i+1} : \Delta^{i+1} \to S^n \setminus h(C \times \{x_0\})$, and so has compact image. But the sets $S^n \setminus h(C \times I_n)$ form a nested open cover of $S^n \setminus h(C \times \{x_0\})$, and so of $\sigma_j^{i+1}(\Delta^{i+1})$, and so there is an n_j with $\sigma_j^{i+1}(\Delta^{i+1}) \subseteq S^n \setminus h(C \times I_{n_j})$. Then setting $N = \max\{n_j\}$, we have $\sigma_j^{i+1} : \Delta^{i+1} \to S^n \setminus h(C \times I_N)$ for every j, so $w \in C_{i+1}(S^n \setminus h(C \times I_N))$, so $0 = [z] \in \widetilde{H}_i(S^n \setminus h(C \times I_N))$, a contradiction. So $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$, and our inductive step is proved.

One immediate consequence of this is that if $h : S^k \to S^n$ is an embedding of the k-sphere into the n-sphere, then thinking of S^k as the union of its upper and lower hemispheres, D_+^k, D_-^k, each of which is homeomorphic to I^k, we have $D_+^k \cap D_-^k = S^{k-1}$, the equatorial $(k-1)$-sphere, and so by Mayer-Vietoris we have

$$\to \widetilde{H}_{i+1}(S^n \setminus h(D_-^k)) \oplus \widetilde{H}_{i+1}(S^n \setminus h(D_+^k) \to \widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \to \widetilde{H}_i(S^n \setminus h(S^k)) \to \widetilde{H}_i(S^n \setminus h(D_-^k)) \oplus \widetilde{H}_i(S^n \setminus h(D_+^k$$

i.e., $\widetilde{H}_i(S^n \setminus h(S^k)) \cong \widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \cong \cdots \cong \widetilde{H}_{i+k}(S^n \setminus h(S^0)) \cong \widetilde{H}_{i+k}(S^{n-1})$, since $S^0 = 2$ points, and so $S^n \setminus S^0 \cong S^{n-1} \times \mathbb{R} \simeq S^{n-1}$. So $\widetilde{H}_i(S^n \setminus h(S^k)) = 0$ unless $i + k = n - 1$ (i.e., $i = n - k - 1$), when it is $\mathbb{Z}$.

In particular, $\widetilde{H}_0(S^n \setminus h(S^{n-1})) = \mathbb{Z}$, so every embedded S^{n-1} in S^n has two complementary path-components A, B. With a little work, one can show that $\overline{A} \cap \overline{B} = h(S^{n-1})$, so the $(n-1)$-sphere is the frontier of each complementary component. [Removing a point from S^n to get $\mathbb{R}^n$ does not change the conclusion (for $n > 1$); a point does not disconnect an open subset of S^n.]

When $n = 2$, the Jordan Curve Theorem (as it is then called) has the additional consequence that the closure of each complementary region is a compact 2-disk, each having the embedded circle $h(S^1)$ as its boundary. This stronger result does not extend to higher dimensions, without putting extra restrictions on the embedding. This was shown by Alexander (shortly after publishing an incorrect proof without restrictions) for $n = 3$; these examples are known as the Alexander horned spheres.

To prove Invariance of Domain, let $\mathcal{U} \subseteq \mathbb{R}^n \subseteq S^n$ be an open set, and $f : \mathcal{U} \to \mathbb{R}^n \hookrightarrow S^n$ be injective and continuous. It suffices to show, for every $x \in \mathcal{U}$, that there is an open neighborhood $\mathcal{V}$ with $f(x) \subseteq \mathcal{V} \subseteq f(\mathcal{U})$. Since $\mathcal{U}$ is open, there is an open ball B^n centered at x whose closure D^n is contained in $\mathcal{U}$. f is then an embedding of $\partial D^n = S^{n-1}$ into S^n, and of $D^n \cong I^n$ into S^n. By our calculations above, $S^n \setminus f(S^{n-1})$ has two path components A, B; being an open set and contained in a locally path-connected space, these are also the connected components of the complement. But our calculations above also show that $S^n \setminus f(D^n)$ is path-connected, hence connected, and $f(B^n)$, being the image of a connected set, is connected. Since $f(B^n) \cup (S^n \setminus f(D^n)) = S^n \setminus f(S^{n-1} = A \cup B)$, it follows that $f(B^n) = A$ and $S^n \setminus f(D^n) = B$ (or vice versa). In particular, $f(B^n)$ is open, forming an open subset of $f(\mathcal{U})$ containing $f(x)$, as desired.

This in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \ldots, x_m) =$

$(x_1, \ldots, x_m, 0, \ldots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.