Math 325 Problem Set 4 Solutions

12. [Lay, p.164, # 16.7(f) (sort of)] Show that if $0 \leq x < 1$ then for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ so that $x^n < \epsilon$. [Hint: Suppose not! Then look at lower bounds for $A = \{x^n : n \in \mathbb{N}\}$.] Conclude that for <u>every</u> $m \in \mathbb{N}$ with $m \geq n$ we have $|x^m| = x^m < \epsilon$, so $x^n \to 0$ as $n \to \infty$.

Suppose not! Suppose there is an $\epsilon > 0$ so that $\epsilon \leq x^n$ for all n. Then the set $A = \{x^n : n \in \mathbb{N}\}$ is bounded below by ϵ, so it has a greatesst lower bound λ. We then have $0 < \epsilon \leq \lambda$, so $0 < \lambda$, and if $\lambda < \mu$ then there is an n with $x^n < \mu$ (since μ cannot be a lower bound for A. But since $0 < x < 1$ we have $1/x > 1$ (since otherwise $1/x \leq 1$ so either $x < 0$ (which is false) or, multiplying through by $x > 0$ $1 \leq 1 \cdot x = x$, which is also false). So $\mu = \lambda/x > \lambda$,m so $x^n < \lambda/x$ for som n, implying that $x^{n+1} < \lambda$ (by multiplying by $x > 0$ on both sides), a contradiction. So our supposition is false; there must be an n so that $x^n < \epsilon$.

13. [Lay, p.165, # 16.13] (The 'Squeeze Play' Theorem) Suppose that $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(c_n)_{n=1}^{\infty}$ are sequences with $a_n \leq b_n \leq c_n$ for all n. Suppose further that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Show that $\lim_{n\to\infty} b_n = L$.

We want, for $\epsilon > 0$, and N so that $n \geq N$ implies $|b_n - L| < \epsilon$; that is $-\epsilon < b_n - L < \epsilon$, i.e., $L - \epsilon < b_n < L + \epsilon$.

What we know is that there is an N_1 so that $n \geq N_1$ implies that $|a_n - L| < \epsilon$, which (as above) gives $L - \epsilon < a_n$. There is also an N_2 so that $n \geq N_2$ implies that $|c_n - L| < \epsilon$, which gives $c_n < L + \epsilon$.

So if we take $N = \max\{N_1, N_2\}$, then $n \geq N$ implies $L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$, so $L - \epsilon < b_n < L + \epsilon$, yielding $|b_n - L| < \epsilon$.

So for every $\epsilon > 0$ there is an N so that $n \geq N$ implies $|b_n - L| < \epsilon$, so $b_n \to L$ as $n \to \infty$, as desired.

14. Show, from the definition of limit (i.e., no limit theorems!) that

(a) $\displaystyle\lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}$

We have $|a_n - L| = |\dfrac{2n+1}{3n+2} - \dfrac{2}{3}| = |\dfrac{(3)(2n+1) - (2)(3n+2)}{(3)(3n+2)} = \dfrac{6n+3-6n-4}{9n+6}| = |\dfrac{-1}{9n+6}| = \dfrac{1}{9n+6} < \dfrac{1}{9n}$,

since $9n + 6 > 9n > 0$, so $\dfrac{1}{9n+6} < \dfrac{1}{9n}$.

So we can show that, for a given $\epsilon > 0$, then $|a_n - L| < \epsilon$, provided that $\dfrac{1}{9n} < \epsilon$, which we can arrange by making n large enough; we need $n > \dfrac{1}{9\epsilon}$, but since we can find an $N \in \mathbb{N}$ so that $N > \dfrac{1}{9\epsilon}$, then $n \geq N$ implies that $n \geq N > \dfrac{1}{9\epsilon}$, so $\dfrac{1}{9n} < \epsilon$, so $|a_n - L| = \dfrac{1}{9n+6} < \dfrac{1}{9n} < \epsilon$, as desired.

(b) $\displaystyle\lim_{n\to\infty} \frac{n^2+n-2}{2n^2+n-1} = \frac{1}{2}$

$|b_n - M| = |\dfrac{n^2+n-2}{2n^2+n-1} - \dfrac{1}{2}| = |\dfrac{2(n^2+n-2)-(1)(2n^2+n-1)}{(2)(2n^2+n-1)}|$
$= |\dfrac{2n^2+2n-4-2n^2-n+1)}{(2)(2n^2+n-1)}| = |\dfrac{n-3}{4n^2+4n-2}|$.

But if $n \geq 3$ then $n-3 \geq 0$ and $4n^2+4n-2 \geq 4n^2+12 = 2 = 4n^2+10 \geq 10 > 0$, so $|\dfrac{n-3}{4n^2+4n-2}| = \dfrac{n-3}{4n^2+4n-2} < \dfrac{n}{4n^2+4n-2} < \dfrac{n}{4n^2} = \dfrac{1}{4n}$, since $0 \leq n-3 < n$ and $4n-2 \geq 0$ so $0 < 4n^2 \leq 4n^2+4n-2$; making the numerator a larger positive number, and making the denomenator a smaller positive number, yields a larger quotient.

So, as before, for any $\epsilon > 0$, we can make $|b_n - M| < \epsilon$ (when $n \geq 3$) by making $\dfrac{1}{4n} < \epsilon$; this we can do when $n \geq N$ for an integer $N \in \mathbb{N}$ with $N \geq 3$ and $\dfrac{1}{4N} < \epsilon$, i.e., $N > \dfrac{1}{4\epsilon}$. (Choosing an $M \in \mathbb{N}$ with $M > \dfrac{1}{4\epsilon}$ and setting $N = M+3$, for example, works.)

So for every $\epsilon > 0$ we can find an N so that $n \geq N$ implies $|\dfrac{n^2+n-2}{2n^2+n-1} - \dfrac{1}{2}| < \epsilon$, so $\dfrac{n^2+n-2}{2n^2+n-1} \to \dfrac{1}{2}$.

15. Show that if $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\displaystyle\lim_{n\to\infty} (-1)^n a_n = L$, <u>then</u> $L = 0$.

The alternative is that $L > 0$ or $L < 0$. If for the sake of clarity we call $(-1)^n a_n = b_n$, then $b_n \to L$ and $L > 0$ implies, from class, that eventually $b_n > 0$, since eventually $|b_n - L| < L/2$, so $-L/2 < b_n - L$, so $0 < L/2 = L - L/2 < b_n$. So eventually every term in the sequence must be positive. But the terms in this sequence with odd index are $(-1)a_n = -a_n \leq 0$, a contradiction.

But $b_n \to L$ and $L < 0$ leads to a similar contradiction; eventually $|b_n - L| < -L/2$, so $b_n = L < -L/2$ and $b_n < L - L/2 = L/2 < 0$. But since the terms with even index are $(-1)^2 a_n = a_n/geq$, this again is a contradiction.

So since $L > 0$ and $L < 0$ both lead to a contradiction, by trichotomy the only possibility we have is that $L = 0$, as desired.