# Math 417 Problem Set 1

Starred (*) problems are due Friday, Augsut 31.

(*) 1. Some linear algebra(!) shows that that rotation $R(\theta)$ by angle $\theta$ around the origin, and reflection $M(\theta)$ in the line through the origin making angle $\theta$, are linear transformations, given in matrix terms as multiplication by

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \text{ and } M(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Using matrix multiplication, show that $M(\theta) \circ M(\psi)$ is a rotation, and $M(\theta) \circ R(\psi)$ and $R(\psi) \circ M(\theta)$ are both reflections, and determine which angle they rotate or reflect by.

[You can show yourself that the matrices are correct, since their columns are the vectors that $R(\theta)$ and $M(\theta)$ send the 'standard' basis vectors of $\mathbb{R}^2$ to.]

We can recognize each of these compositions as a rotation or reflection by multiplying the two matrices together and using angle sum/difference formulas for $\sin x$ and $\cos x$. In particular,

$$B(\theta)B(\psi) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta)\cos(2\psi) + \sin(2\theta)\sin(2\psi) & \cos(2\theta)\sin(2\psi) - \sin(2\theta)\cos(2\psi) \\ \sin(2\theta)\cos(2\psi) - \cos(2\theta)\sin(2\psi) & \sin(2\theta)\sin(2\psi) + \cos(2\theta)\cos(2\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta - 2\psi) & \sin(2\psi - 2\theta) \\ \sin(2\theta - 2\psi) & \cos(2\theta - 2\psi) \end{pmatrix} = \begin{pmatrix} \cos(2\theta - 2\psi) & -\sin(2\theta - 2\psi) \\ \sin(2\theta - 2\psi) & \cos(2\theta - 2\psi) \end{pmatrix}$$

$= A(2\theta - 2\psi)$, so $S(\theta) \circ S(\psi)$ is a rotation by angle $2\theta - 2\psi$.

Similarly,

$$B(\theta)A(\psi) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta)\cos(\psi) + \sin(2\theta)\sin(\psi) & -\cos(2\theta)\sin(\psi) + \sin(2\theta)\cos(\psi) \\ \sin(2\theta)\cos(\psi) - \cos(2\theta)\sin(\psi) & -\sin(2\theta)\sin(\psi) - \cos(2\theta)\cos(\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta - \psi) & \sin(2\theta - \psi) \\ \sin(2\theta - \psi) & -\cos(2\theta - \psi) \end{pmatrix}$$

$= B(\theta - \psi/2)$, so $S(\theta) \circ R(\psi)$ is a reflection in the line with angle $\theta - \frac{1}{2}\psi$, and

$$A(\psi)B(\theta) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi)\cos(2\theta) - \sin(\psi)\sin(2\theta) & \cos(\psi)\sin(2\theta) + \sin(\psi)\cos(2\theta) \\ \sin(\psi)\cos(2\theta) + \cos(\psi)\sin(2\theta) & \sin(\psi)\sin(2\theta) - \cos(\psi)\cos(2\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi + 2\theta) & \sin(\psi + 2\theta) \\ \sin(\psi + 2\theta) & -\cos(\psi + 2\theta) \end{pmatrix}$$

$= B(\theta + \psi/2)$, so $R(\psi) \circ S(\theta)$ is a reflection in the line with angle $\theta + \frac{1}{2}\psi$ .

(*) 2. (Gallian, p.38, #14) If we build a rhombus $R$ (a quadrilateral with all four sides having equal length) by gluing two equilateral triangles together along a pair of sides, describe the symmetries of $R$ in terms of rotations and reflections.

Call the vertices, in order, $A, B, C, D$. The centroid of $R$ is where the two diagonals $AC$ and $BD$ meet, and two (opposite) vertices $A, C$ will be farther from the centroid than the other two. So any symmetry must either fix both $A$ and $C$, or swap them. Since it must take the vertices to the vertices, in a one-to-one way, it must also either fix both $B$ and $D$ or swap them. Every possible combination of these options is possible; the identity fixes both pairs, rotation by $\pi$ swaps both pairs, reflection in $AC$ swaps $B$ and $D$ but fixes the other two, and reflection in $BD$ swaps $A$ and $C$ but fixes the other two. Since the symmetry is determined by where it sends the vertices (linearity tells us where every other point goes), there are precisely four symmetries.

(*) 5. (Gallian, p.55, #18) Which elements $x \in D_4$ = the group of symmetries of a regular 4-gon (i.e., square) satisfy $x^2 = e$ ? Which satisfy $x = y^2$ for some $y \in D_4$ ?

[Problem #1 can help you decide what an element $y^2$ can look like...]

Problem #1 tells us that for a reflection $S(\theta)$, $S(\theta) \circ S(\theta)$ is a rotation by angle $2\theta - 2\theta = 0$, i.e., is the identity. On the other hand, for a rotation $R(\theta)$, $R(\theta) \circ R(\theta) = R(2\theta)$ is a rotation by angle $2\theta$ (this is what our understanding of rotations tells us; it can also be verified computationally like in Problem #1). So every reflection $x$ in $D_4$ satisfies $x^2 = e$, and the rotations in $D_4$ that satisfy $x^2 = e$ must be by an angle $\theta$ so that $2\theta$ is a multiple of $2\pi$. So the four reflections $S(0)$, $S(\pi/4)$, $S(\pi/2)$, and $S(3\pi/4)$, and the rotations $R(0) = e$ and $R(\pi)$, are the elements $x$ with $x^2 = e$.

When looking for squares (i.e., $y \in D_4$ with $y = x^2$ for some $x \in D_4$), we now know that all elements $x^2$ are the identity if $x$ is a reflection, or a rotation, by an angle twice the angle of rotation of $x$, if $x$ is a rotation. So $e$ is a square, and any rotation having the rotation by half its angle also in $D_4$, is a square. The only angle which works (other than 0) is $\pi$, so $e$ and $R(\pi)$ are the perfect squares in $D_4$.

A completely different, but perfectly valid, approach to this problem is to list all eight elements $e$, $R(\pi/2)$, $R(\pi)$, $R(3\pi/2)$, $S(0)$, $S(\pi/4)$, $S(\pi/2)$, and $S(3\pi/4)$, and square them! The ones that square to $e$ answer the first part, and the elements you get as squares answers the second part!

**A selection of further solutions**

4. Show that the set $G = \{1, 5, 9, 13\}$ forms a group, with group multiplication being multiplication modulo 16. (One approach: build the 'Cayley' table! This helps to see why some needed properties hold.)

We need to show that $G$ has an identity, inverses, and its multiplication is associative. However, multiplication modulo 16 is associative!, so we know that multiplication of these numbers modulo 16 is associative. If we build the Calyey table, by multiplying all pairs of numbers together and reducing modulo 16, we get:

| * | | 1 | 5 | 9 | 13 |
|---|---|---|---|---|---|
| 1 | | 1 | 5 | 9 | 13 |
| 5 | | 5 | 25 | 45 | 65 |
| 9 | | 9 | 45 | 81 | 117 |
| 13 | | 13 | 65 | 117 | 169 |

i.e.,

| * | | 1 | 5 | 9 | 13 |
|---|---|---|---|---|---|
| 1 | | 1 | 5 | 9 | 13 |
| 5 | | 5 | 9 | 13 | 1 |
| 9 | | 9 | 13 | 1 | 5 |
| 13 | | 13 | 1 | 5 | 9 |

since $117 = 5 + 112 = 5 + 16 \cdot 7$ . From this table we see that 1 is an identity element for the group, and since a 1 appears in every row and column, every element has a right and a left inverse; in particular, $1^{-1} = 1$, $5^{-1} = 13$, $9^{-1} = 9$, and $13^{-1} = 5$. So $G$ has an associative multiplication, an identity, and inverses, so $G$ is a group. The important point, really, is that products of things in $G$ are <u>in</u> $G$! (The set $G$ is "closed" under multiplication.)