Math 417 Problem Set 9 Solutions

Starred (*) problems were due Friday, November 9.

(*) 58. If $\varphi : G \to H$ is a surjective homomorphism and $N \leq G$ is a normal subgroup of G, show that $\varphi(N) \leq H$ is a normal subgroup of H. Show, on the other hand, that if φ is not surjective, then $\varphi(N)$ need not be a normal subgroup (hint: G is a normal subgroup of G !).

If $h \in H$ and $x \in \varphi(N)$, we need to show that $hxh^{-1} \in \varphi(N)$. Since $x \in \varphi(N)$, we know that $x = \varphi(y)$ for some $y \in N$. And since φ is surjective, we know that there is $g \in G$ so that $\varphi(g) = h$. Then $hxh^{-1} = \varphi(g)\varphi(y)\varphi(g)^{-1} = \varphi(g)\varphi(y)\varphi(g^{-1}) = \varphi(gyg^{-1})$. But! Since $y \in N$ and $g \in G$, we have $gyg^{-1} \in N$, since N is normal. This means that $hxh^{-1} = \varphi(gyg^{-1})$ is the image under φ of something in N, and so $hxh^{-1} \in \varphi(N)$. So the conjugate of anything in $\varphi(N)$ lies in $\varphi(N)$, so $\varphi(N)$ is a normal subgroup of H.

However, if φ is not surjective, this need not be true. Probably the quickest way to show this is to use the identity map for φ (or more exactly, the inclusion map). For example, In $H = S_3$, $G = \{e_H, (1,2)\}$ is a subgroup, but not a normal subgroup (since, e.g., $(1,3)(1,2)(1,3) = (2,3) \neq (1,2)$). But the inclusion map $\iota : G \to H$ sending x to x is an injective homomorphism, but not a surjective one, and the normal subgroup $N = G \leq G$ is taken by φ to $G \leq H$, which is not a normal subgroup of H.

We can build more elaborate examples, as well. For example, the map $\mathbb{Z}_8 \to S_8$ sending k to $(1,2,3,4,5,6,7,8)^k$ is a homomorphism, and $2\mathbb{Z}_8$ is a normal subgroup of $\mathbb{Z}_8$, but (you can check!) $\varphi(2\mathbb{Z}_8) = \langle (1,2,3,4,5,6,7,8)^2 \rangle = \langle (1,3,5,7)(2,4,6,8) \rangle$ is not a normal subgroup of S_8.

(*) 60. (Gallian, p.222, # 42) Show that if $N, K \leq G$ are normal subgroups of G and $K \leq N$, then N/K is a normal subgroup of G/K, and $(G/K)/(N/K) \cong G/N$. [This is the "Third Isomorphism Theorem" of Emmy Noether. One approach: start by looking at the 'natural' map $G \to G/N$.]

$G/K = \{gK \ : \ g \in G\}$ is a group under multiplication of cosets, and $N/K = \{nK \ : \ n \in G\}$ is a subset of G/K. We start by showing it is a subgroup: $e \in N$ and so $eK = K \in N/K$ (the identity element of G/K). If $aK, bK \in N/K$, then $a, b \in N$ and so $(ab) \in N$ and $(aK)(bK) = (ab)K \in N/K$. Finally, if $a \in N$ then $a^{-1} \in N$ and so $(aK)^{-1} = a^{-1}K \in N/K$. So N/K is closed under multiplication and inversion, and contains the identity, so N/K is a subgroup of G/K.

Even more, since N is normal in G, N/K is normal, since if $nK \in N/K$ and $gK \in G/K$, then $(gK)(nK)(gK)^{-1} = (gng^{-1})K \in N/K$, since $gng^{-1} \in N$.

Consequently, $(G/K)/(N/K)$ is a group. We have a 'natural' (surjective) homomorphism $\varphi_1 : G/K \to (G/K)/(N/K)$, with kernel N/K. But we also have a 'natural' (surjective) homomorphism $\varphi_2 : G \to G/K$, with kernel K. Composing these two homomorphisms, we get a (surjective!) homomorphism $\psi : G \to (G/K)/(N/K)$.

The first isomorphism theorem then tells us that the induced homomorphism $\overline{\psi}$: $G/\ker(\psi) \to (G/K)/(N/K)$ will be an isomorphism; the only question is, what is $\ker(\psi)$?

To figure that out, start with $g \in G$ with $\psi(g) = e$ in $(G/K)/(N/K)$. This means $\varphi_1(\varphi_2(g)) = \varphi_1(gK) = (gK)(N/K) = e_{(G/K)/(N/K)}$. That is, $(gK)(N/K) = N/K$, so $gK \in N/K$, which means $g \in N$. So (!) $\ker(\psi) \subseteq N$. But conversely, if $g \in N$, then $gK \in N/K$, and so $\psi(g) = (gK)(N/K) = (N/K)$, so $g \in \ker(\psi)$, so $N \subseteq \ker(\psi)$. Together these give $\ker(\psi) = N$, and so

$\overline{\psi} : G/N \to (G/K)/(N/K)$ given by $\overline{\psi}(gN) = (gK)(N/K)$ is an isomorphism!

[N.B. The suggested approach will also work: The surjection $G \to G/N$ can be used to build a (surjective) homomorphism $G/K \to G/N$ given by $gK \mapsto gN$. Then we can show that the kernel of <u>this</u> homomorphism is N/K, yielding an isomorphism $(G/K)/(N/K) \to G/N$ (i.e., built in the opposite direction!). The diligent student can verify that this map is the inverse of the one built above....]

(*) 63. (Gallian, p.202, # 37) If H is a normal subgroup in G and G is finite, and $g \in G$, show that the order of gH in G/H divides the order of g in G.

The quickest approach is to use the fact that if $x^n = e$ in a group then the order of x divides n. Translating that into the language of our problem, since what we want is that $|gH|$ divides $|g|$, this means tht we want gH to play the role of x, and $|g|$ to play the role of n. So it is enough to establish that $(gH)^{|g|} = e$ in G/H.

But this is true: since $g^{|g|} = e_G$, we have

$$(gH)^{|g|} = (gH)(gH)\cdots(gH) = (g \cdot g \cdots g)H = (g^{|g|})H = e_G H = H = e_{G/H}$$

in G/H. So the order of gH divides the order of g.

A selection of further solutions.

61. If G is a group, show that $H = \{(g, g) \ : \ g \in G\}$ is a normal subgroup of $G \oplus G \Leftrightarrow G$ is <u>abelian</u>; when H is normal, show that $(G \oplus G)/H$ is isomorphic to G.

[Hint: how would you build a homomorphism $G \oplus G \to G$ so that H would be the kernel? Note that at this point in the problem you can <u>assume</u> that G is abelian!]

If H is normal, then for $(x, y) \in G \oplus G$ we have, for every $g \in G$, $(x, y)(g, g)(x, y)^{-1} = (x, y)(g, g)(x^{-1}, y^{-1}) = (xgx^{-1}, ygy^{-1}) \in H$, so $(xgx^{-1}, ygy^{-1}) = (h, h)$ for some h, so $xgx^{-1} = h$ and $ygy^{-1} = h$. But then setting $y = e_G$ we have $ygy^{-1} = ege^{-1} = g$, so $h = g$ in this case (and for every choice of g !), and <u>so</u> for any $x \in G$ we have $xgx^{-1} = g$. So $xg = gx$ for every $x \in G$ and $g \in G$, so G is abelian.

On the other hand, if G is abelian, then $xg = gx$ for every $g \in G$ and $x \in G$, and so for each $(g, g) \in H$ and $(x, y) \in G \oplus G$ we have $(x, y)(g, g)(x, y)^{-1} = (x, y)(g, g)(x^{-1}, y^{-1}) = (xgx^{-1}, ygy^{-1}) = (xx^{-1}g, yy^{-1}g) = (g, g) \in H$. So H is a normal subgroup.

To prove the final assertion, following the hint we want to build a surjective homomorphism $\varphi : G \oplus G \to G$ so that $\varphi(g,h) = e_G$ if and only if $g = h$, i.e., $gh^{-1} = e_G$. This suggests that we try $\varphi(g,h) = gh^{-1}$! Since we are assuming that $H = \{(g,g) \ : \ g \in G\}$ is normal, we know that G is abelian, and then φ is a homomorphism: $\varphi(g,h)\varphi(u,v) = (gh^{-1})(uv^{-1}) = (gu)(v^{-1}h^{-1}) = (gu)(hv)^{-1} = \varphi(gu,hv) = \varphi((g,h)(u,v))$, where we used that G is commutative to rearrange terms. So φ is a homomorphism, and $\varphi(g,e_G) = ge_G^{-1} = ge_G = g$ so φ is surjective. Finally, $\varphi(g,h) = gh^{-1} = e_G$ if and only if $g = h$, so $\varphi(g,h) = e_G$ if and only if $(g,h) = (g,g) \in H$, so $Ker(\varphi) = H$. Then by our result from class, we know that φ induces an injective homomorphism $\overline{\varphi} : (G \oplus G)/H \to G$ with image equal to G =image of φ, so $\overline{\varphi}$ is an isomorphism from $(G \oplus G)/H$ to G.

64. Show that in the symmetric group S_n, every commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ is an element of the subgroup A_n = the alternating group. Show, in addition, that every 3-cycle (a,b,c) can be written as a commutator $\alpha\beta\alpha^{-1}\beta^{-1}$. Conclude that every element of A_n can be written as a product of commutators.

Whatever they are, α can be expressed as a product of some number r of transpositions $\alpha = \tau_1 \cdots \tau_r$, and then $\alpha^{-1} = \tau_r \cdots \tau_1$ (since $\tau_i^{-1} = \tau_i$ is also a product of r transpositions). Similarly, $\beta = \sigma_1 \cdots \sigma_m$ is a product of m transpositions, and $\beta^{-1} = \sigma_m \cdots \sigma_1$. Then

$$\alpha\beta\alpha^{-1}\beta^{-1} = \tau_1 \cdots \tau_r \sigma_1 \cdots \sigma_m \tau_r \cdots \tau_1 \sigma_m \cdots \sigma_1$$

is a product of $2r + 2m$ transpositions. In particular, it is a product of an even number of transpositions, and so is an even permutation, and so $\alpha\beta\alpha^{-1}\beta^{-1} \in A_n$.

A 3-cycle can be expressed as a commutator of two 2-cycles, in fact; a little experimenting shows that $(a,b,c) = (a,b)(a,c)(a,b)(a,c) = (a,b)(a,c)(a,b)^{-1}(a,c)^{-1}$.

Finally, we have seen (in a previous problem set) that every element of A_n can be written as a product of 3-cycles. Since every 3-cycle can be expressed as a commutator, every element of A_n can then be expressed as a product of commutators.