

Relative homology: we build the singular chain complex of a pair (X, A) , i.e., of a space X and a subspace $A \subseteq X$. Since as abelian groups we can think of $C_n(A)$ as a subgroup of $C_n(X)$ (under the injective homomorphism induced by the inclusion $i : A \rightarrow X$) we can set $C_n(X, A) = C_n(X)/C_n(A)$. Since the boundary map $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ satisfies $\partial_n(C_n(A)) \subseteq C_{n-1}(A)$ (the boundary of a map into A maps into A), we get an induced boundary map $\partial_n : C_n(X, A) \rightarrow C_{n-1}(X, A)$. These groups and maps $(C_n(X, A), \partial_n)$ form a chain complex, whose homology groups are the *singular relative homology groups of the pair* (X, A) . To be a cycle in relative homology, you need to have a representative z with $\partial z \in C_{n-1}(A)$, i.e., you are a chain with boundary in A . To be a boundary, you need $z = \partial w + a$ for some $w \in C_{n+1}(X)$ and $a \in C_n(A)$, i.e., you *cobound* a chain in A ($\partial w = z - a$). Note that the relative homology of the pair (X, \emptyset) is just the ordinary homology of X ; we aren't modding out by anything.

The inclusion i_n and projection p_n maps give us short exact sequences

$$0 \rightarrow C_n(A) \rightarrow C_n(X) \rightarrow C_n(X, A) \rightarrow 0$$

and since the boundary on chains in X restricts to the boundary on A and induces the boundary on (X, A) , i_n and p_n are chain maps. So we get a long exact homology sequence

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(X) \rightarrow \cdots$$

There is also a long exact sequence of a triple (X, A, B) , where by triple we mean $B \subseteq A \subseteq X$. From the short exact sequences $0 \rightarrow C_n(A, B) \rightarrow C_n(X, B) \rightarrow C_n(X, A) \rightarrow 0$ (i.e.,

$$0 \rightarrow C_n(A)/C_n(B) \rightarrow C_n(X)/C_n(B) \rightarrow C_n(X)/C_n(A) \rightarrow 0)$$

we get the long exact sequence

$$\cdots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow H_{n-1}(X, B) \rightarrow \cdots$$

A map of pairs $f : (X, A) \rightarrow (Y, B)$ (meaning that $f(A) \subseteq B$) induces (by postcomposition) a map of relative homology $f_* : H_i(X, A) \rightarrow H_i(Y, B)$, just as with absolute homology. We also get a homotopy-invariance result: if $f, g : (X, A) \rightarrow (Y, B)$ are maps of pairs which are *homotopic as maps of pairs*, i.e., there is a map $(X \times I, A \times I) \rightarrow (Y, B)$ which is f on one end and g on the other, then $f_* = g_*$. The proof is identical to the one given before; the prism map P sends chains in A to chains in A , so induces a map $C_i(X \times I, A \times I) \rightarrow C_{i+1}(X, A)$ which does precisely what we want.

While we are in the neighborhood, we may as well deal with the other main piece of homological algebra that we will find useful; the Five Lemma. Now that we have a way of building long exact sequences, we will soon have ways of building maps between them. So the next result becomes useful.

If we have abelian groups and maps

$$\begin{array}{ccccccc} A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & D_n & \xrightarrow{i_n} & E_n \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \xrightarrow{h_{n-1}} & D_{n-1} & \xrightarrow{i_{n-1}} & E_{n-1} \end{array}$$

and the maps $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then γ is an isomorphism.

The proof is in some sense literally a matter of doing the only thing you can. To show injectivity, suppose $x \in C_n$ and $\gamma x = 0$, then $h_{n-1}\gamma x = \delta h_n x = 0$, so, since δ is injective, $h_n x = 0$. So by the exactness at C_n , $x = g_n y$ for some $y \in B_n$. Then $g_{n-1}\beta y = \gamma g_n y = \gamma x = 0$, so by exactness at B_{n-1} , $\beta y = f_{n-1}z$ for some $z \in A_{n-1}$. Then since α is surjective, $f_{n-1}z = \alpha w$ for some w . Then $0 = g_n f_n w$. But $\beta f_n w = f_{n-1}\alpha w = f_{n-1}z = \beta y$, so since β is injective, $y = f_n w$. So $0 = g_n f_n w = g_n y = x$. So $x = 0$.

For surjectivity, suppose $x \in C_{n-1}$. Then $h_{n-1}x \in D_{n-1}$, so since δ is surjective, $h_{n-1}x = \delta y$ for some $y \in D_n$. Then $\epsilon i_n y = i_{n-1}\delta y = i_{n-1}h_{n-1}x = 0$, so since ϵ is injective, $i_n y = 0$. So by exactness at D_n , $y = h_n z$ for some $z \in C_n$. Then $h_{n-1}\gamma z = \delta h_n z = \delta y = h_{n-1}x$, so $h_{n-1}(\gamma z - x) = 0$, so by exactness at C_{n-1} , $\gamma z - x = g_{n-1}w$ for some $w \in B_{n-1}$. Then since β is surjective, $w = \beta u$ for some $u \in B_n$. Then $\gamma g_n u = g_{n-1}\beta u = g_{n-1}w = \gamma z - x$, so $x = \gamma z - \gamma g_n u = \gamma(z - g_n u)$. So γ is onto.

But the big result that allows us to get our homology machine really running is what is known as *excision*. To motivate it, let's try to imagine that we are trying to generalize Seifert - van Kampen. We start with $X = A \cup B$, and we want to try to express the homology of X in terms of that of A , B , and $A \cap B$. With our new-found tool of long exact homology sequences, we might try to first build a short exact sequence out of the chain complexes $C_*(A \cap B)$, $C_*(A)$, $C_*(B)$, and $C_*(X)$. If we take our cue from the proof of S-vK, we might think of chains in X as sums of chains in A and B , except that we mod out by chains in $A \cap B$. Putting this into action, we might try the sequence

$$0 \rightarrow C_n(A \cap B) \rightarrow C_n(A) \oplus C_n(B) \rightarrow C_n(X) \rightarrow 0$$

where $j_n : C_n(A) \oplus C_n(B) \rightarrow C_n(X)$ is defined as $j_n(a, b) = a + b$. In order to get exactness at the middle term (i.e., image = the kernel of this map, which is $\{(x, -x) : x \in C_n(A) \cap C_n(B)\}$), we set $i_n : C_n(A \cap B) \rightarrow C_n(A) \oplus C_n(B)$ to be $i_n(x) = (x, -x)$, since $C_n(A \cap B) = C_n(A) \cap C_n(B)$! i_n is then injective, and we certainly have that this sequence is exact at the middle term. But, in general, j_n is far from surjective! The image of j_n is the set of n -chains that can be expressed as sums of chains in A and B . Which of course not every chain in X can be; singular simplices in X need not map entirely into either A or B .

We can solve this by replacing $C_n(X)$ with the image of j_n , calling it, say, $C_n^{\{A, B\}}(X)$... [Note: these groups would form a chain complex!] Then we have a short exact sequence, and hence a long exact homology sequence. But it involves a "new" homology group $H_n^{\{A, B\}}(X)$. The point is that, like S-vK, under the right conditions, this new homology is the same as $H_n(X)$!

Starting from scratch, the idea is that, starting with an *open cover* $\{\mathcal{U}_\alpha\}$ of X , we build the *chain groups subordinate to the cover* $C_n^{\mathcal{U}}(X) = \{\sum a_i \sigma_i^n : \sigma_i : \Delta^n \rightarrow X, \sigma_i^n(\Delta^n) \subseteq \mathcal{U}_\alpha \text{ for some } \alpha\} \subseteq C_n(X)$. Since the face of any simplex mapping into \mathcal{U}_α also maps into \mathcal{U}_α , our ordinary boundary maps induce boundary maps on these groups, turning $(C_n^{\mathcal{U}}(X), \partial_n)$ into a chain complex. Our main result is that the inclusion i of these groups into $C_n(X)$ induces an isomorphism on homology. And to show this, we (could) once again use the notion of a chain homotopy.

Theorem: There is a chain map $b : C_n(X) \rightarrow C_n^{\mathcal{U}}(X)$ so that $i \circ b$ and $b \circ i$ are both chain homotopic to the identity. i consequently induces isomorphisms on homology.

But we won't prove it quite that way! Another approach is to use the short exact sequence of chain complexes

$$0 \rightarrow C_n^{\mathcal{U}}(X) \xrightarrow{i} C_n(X) \rightarrow C_n(X)/C_n^{\mathcal{U}}(X) \rightarrow 0$$

to build a long exact homology sequence. Every third group is $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$; if we show that these groups are 0, then i_* will be an isomorphism. And to show this, working back through the definition of homology classes in $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$, we need to show that if $z \in C_n(X)$ with $\partial z \in C_{n-1}^{\mathcal{U}}(X)$ (i.e., z is a relative cycle), then there is a $w \in C_{n+1}(X)$ with $z - \partial w \in C_n^{\mathcal{U}}(X)$ (i.e., z is a relative boundary). In words, if z has boundary a sum of small simplices, then there is a chain z' made of small simplices so that $z - z'$ is a boundary.

And the key to building z' and w is a process known as *barycentric subdivision*. The idea is really the same as for S-vK; we cut our singular simplices up into tiny enough pieces so that (via the Lebesgue number theorem) each piece maps into some \mathcal{U}_α . Unlike S-vK, though, we want to do this in a more structured way, so that the cutting up process is "compatible" with our boundary maps. And the best way to describe this cutting up is through *barycentric coordinates*. Recall that an n -simplex is the set of convex linear combinations $\sum x_i v_i$ with $x_i \geq 0$ and $\sum x_i = 1$. The map which sends an n -simplex to the n -simplex Δ^n is literally the map $\sum x_i v_i \mapsto (x_0, \dots, x_n)$. These are the barycentric coordinates of an n -simplex. Since, formally, all singular simplices are considered to have Δ^n for their domain, we can describe barycentric subdivision by describing how to cut up Δ^n . The idea is to build a process that is compatible with the boundary map, so that the subdivision, when restricted to a sub-simplex, is the subdivision of that sub-simplex. A 1-simplex $[v_0, v_1]$ is subdivided by adding the barycenter $w = (v + 0 + v_1)/2$ as a vertex, cutting $[v_0, v_1]$ into two 1-simplices $[v_0, w], [w, v_1]$. A 2-simplex $[v_0, v_1, v_2]$ will, to be compatible with the boundary map, have its boundary cut into 6 1-simplices;

using the barycenter $(v_0 + v_1 + v_2)/3$ we can cone off each of these 1-simplices to subdivide $[v_0, v_1, v_2]$ into 6 2-simplices. Taking the cue that $2 = (1 + 1)!$, $6 = (2 + 1)!$ is probably no accident, we might expect that an n -simplex will be cut into $(n + 1)!$ n -simplices. Note that this is the number of ways of ordering the vertices of our simplex. And following the “pattern” of our two test cases, where each new simplex was the convex span of vertices chosen as (vertex), (barycenter of a 1-simplex having (vertex) as a vertex), (barycenter of a 2-simplex containing the previous 2 vertices), etc., we are led to the idea that the barycentric subdivision of an n -simplex $[v_0, \dots, v_n]$ is the $(n + 1)!$ n -simplices,

$$[v_{\alpha(0)}, (v_{\alpha(0)} + v_{\alpha(1)})/2, (v_{\alpha(0)} + v_{\alpha(1)} + v_{\alpha(2)})/3, \dots, (v_{\alpha(0)} + \dots + v_{\alpha(n)})/(n + 1)]$$

one for every permutation α of $\{0, \dots, n\}$. And since we want to take into account orientations as well, the natural thing to do is to define the barycentric subdivision of a singular n -simplex $\sigma : [v_0, \dots, v_n] \rightarrow X$ to be

$$S(\sigma) = \sum_{\alpha} (-1)^{\text{sgn}(\alpha)} \sigma|_{[v_{\alpha(0)}, (v_{\alpha(0)} + v_{\alpha(1)})/2, (v_{\alpha(0)} + v_{\alpha(1)} + v_{\alpha(2)})/3, \dots, (v_{\alpha(0)} + \dots + v_{\alpha(n)})/(n + 1)]}$$

where the sum is taken over all permutations of $\{0, \dots, n\}$. This (extending linearly over the chain group) is the subdivision operator, $S : C_n(X) \rightarrow C_n(X)$. A “routine” calculation establishes that $\partial S = S \partial$, i.e., it is a chain map (i.e., it behaves well on the boundary of our simplices). The point to this operator is that all of the subsimplices in the sum are a definite factor smaller than the original simplex. In fact, if the diameter of $[v_0, \dots, v_n]$ is d (the largest distance between points, which will, because it is the convex span of the vertices, be the largest distance between vertices), then every individual simplex in $S(\sigma)$ will have diameter at most $nd/(n + 1)$ (the result of a little Euclidean geometry and induction). So by repeatedly applying the subdivision operator S to a singular simplex, we will obtain a singular chain $S^k(\sigma)$, which is “really” σ written as a sum of tiny simplices, whose singular simplices have image as small as we want. Or put more succinctly, if $\{\mathcal{U}_\alpha\}$ is an open cover of X and $\sigma : \Delta^n \rightarrow X$ is a singular n -simplex, then choosing a Lebesgue number ϵ for the open cover $\sigma^{-1}(\mathcal{U}_\alpha)$ of the compact metric space Δ^n , and choosing a k with $d(n/(n + 1))^k < \epsilon$, we find that $S^k(\sigma)$ is a sum of singular simplices each of which maps into one of the \mathcal{U}_α , i.e., $S^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

In the end, we will choose our needed “small” cycle to be $z' = S^k z$. and to show that their difference is a boundary, we will build a chain homotopy between Id and S^k . And to do that, we define a map $R : C_n(X) \rightarrow C_{n+1}(X \times I)$; when followed by the projection-induced map $p_\# : C_{n+1}(X \times I) \rightarrow C_{n+1}(X)$, we get a map $T : C_n(X) \rightarrow C_{n+1}(X)$, and show that $\partial T + T \partial = I - S$. Then we set $H = \sum T S^j$, where the sum is taken over $j = 0, \dots, k - 1$. Once we define T (!), we will have $\partial H_k + H_k \partial = \sum \partial T S^j + T S^j \partial = \sum (\partial T + T \partial) S^j = \sum (S^j - S^{j+1}) = I - S^k$ (since the last sum telescopes). And defining R , is, formally, just another particular sum. Setting up some notation, thinking of $\Delta^n \times I$, as before, as having vertices $\{v_0, \dots, v_n\}$ on the 0-end and $\{w_0, \dots, w_n\}$ on the 1-end, $N = \{0, \dots, n\}$, $\Pi(Q)$ = the group of permutations of Q , and $\sigma' = \sigma \times I : \Delta^n \times I \rightarrow X \times I$, we have

$$R(\sigma) = \sum_{A \subseteq N} \sum_{\pi \in \Pi(N \setminus A)} \{(-1)^{|A|} (-1)^{\text{sgn}(\pi)} \prod_{j \in N \setminus A} (-1)^j\} \sigma'|_{[v_{i_0}, \dots, v_{i_j}, (w_{i_0} + \dots + w_{i_j})/(j+1), (w_{i_0} + \dots + w_{i_j} + w_{\pi(i_{j+1})})/(j+2), \dots, (w_{i_0} + \dots + w_{i_j} + w_{\pi(i_{j+1})} + \dots + w_{\pi(i_n)})/(n+1)]}$$

where we sum over all non-empty subsets of $\{0, \dots, n\}$ (with the induced ordering on vertices from the ordering on $\{0, \dots, n\}$). Intuitively, this map “interpolates” between the simplex $[v_0, \dots, v_n]$ and the barycentric subdivision on w_0, \dots, w_n , by taking the (signed sums of the) convex spans of simplices on the bottom (0) and simplices on the top (1). Again, a “routine” calculation will establish that $\partial T + T \partial = I - S$, as desired. [At any rate, I verified it for $n=1,2$; the formula for the sign of each simplex was determined by working backwards from these examples.]

And the point to all of these calculations was that if $\{\mathcal{U}_\alpha\}$ is an open cover of X , then the inclusions $i_n : C_n^{\mathcal{U}}(X) \rightarrow C_n(X)$ induce isomorphisms on homology. This gives us two big theorems. The first is

Mayer-Vietoris Sequence: If $X = \mathcal{U} \cup \mathcal{V}$ is the union of two open sets, then the short exact sequences $0 \rightarrow C_n(\mathcal{U} \cap \mathcal{V}) \rightarrow C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \rightarrow C_n^{\{\mathcal{U}, \mathcal{V}\}}(X) \rightarrow 0$, together with the isomorphism above, give the long exact sequence

$$\cdots \rightarrow H_n(\mathcal{U} \cap \mathcal{V}) \xrightarrow{(i_{\mathcal{U}*}, -i_{\mathcal{V}*})} H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \xrightarrow{j_{\mathcal{U}*} + j_{\mathcal{V}*}} H_n(X) \xrightarrow{\partial} H_{n-1}(\mathcal{U} \cap \mathcal{V}) \rightarrow \cdots$$

And just like Seifert - van Kampen, we can replace open sets by sets A, B having neighborhoods which deformation retract to them, and whose intersection deformation retracts to $A \cap B$. For example, subcomplexes $A, B \subseteq X$ of a CW-complex, with $A \cup B = X$ have homology satisfying a long exact sequence

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{(i_{A*}, -i_{B*})} H_n(A) \oplus H_n(B) \xrightarrow{j_{A*} + j_{B*}} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \cdots$$

And this is also true for reduced homology; we just augment the chain complexes used above with the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$, where the first non-trivial map is $a \mapsto (a, -a)$ and the second is $(a, b) \mapsto a + b$.

And now we can do some meaningful calculations! An n -sphere S^n is the union $S_+^n \cup S_-^n$ of its upper and lower hemispheres, each of which is contractible, and have intersection $S_+^n \cap S_-^n = S_0^{n-1}$ the equatorial $(n-1)$ -sphere. So Mayer-Vietoris gives us the exact sequence $\cdots \rightarrow \tilde{H}_k(S_+^n) \oplus \tilde{H}_k(S_-^n) \rightarrow \tilde{H}_k(S^n) \rightarrow \tilde{H}_{k-1}(S_0^{n-1}) \rightarrow \tilde{H}_{k-1}(S_+^n) \oplus \tilde{H}_{k-1}(S_-^n) \rightarrow \cdots$ hhtp, i.e., $0 \rightarrow \tilde{H}_k(S^n) \rightarrow \tilde{H}_{k-1}(S_0^{n-1}) \rightarrow 0$ i.e., $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$ for every k and n . So by induction,

$$\tilde{H}_k(S^n) \cong \tilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z}, & \text{if } k=n \\ 0, & \text{otherwise} \end{cases}$$

And this, in turn, let's us prove a fairly sizable theorem:

Brouwer Fixed Point Theorem: For every n , every map $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$ has a fixed point.

Proof: If $f(x) \neq x$ for every x , then is with the $n = 2$ case that you may have seen before, we can construct a retraction $r : \mathbb{D}^n \rightarrow \partial \mathbb{D}^n = S^{n-1}$ by setting $r(x) =$ the (first) point past $f(x)$ where the ray from $f(x)$ to x meets $\partial \mathbb{D}^n$. This function is continuous, and is the identity on the boundary. So from our of your problem sets, the inclusion-induced map $i_* : H_{n-1}(S^n) \rightarrow H_{n-1}(\mathbb{D}^n)$ is injective. But this is impossible, since the first group is \mathbb{Z} and the second is 0 .

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\text{cl}_X(B) \subseteq \text{int}_X(A)$, then for every k the inclusion-induced map $H_k(X \setminus B, A \setminus B) \rightarrow H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\text{int}_X(A) \cup \text{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \rightarrow H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$.] A proof of this second result follows from a relative version of the barycentric subdivision process;