Math 856 Homework 2

Starred **(*)** problems are due Friday, October 9

(*) 9: [Lee, problem 2-1 (part)] Using the charts on the circle S^1 given by steriographic projection, compute the local coordinate representations of the functions

$$f_n : S^1 \to S^1 \text{ given by } f_n(z) = z^n \text{ (in complex coordinates)}$$

and use this to demonstrate that each f_n is C^∞.

10: We know that if $C, D \subseteq M$ are disjoint closed sets of the smooth manifold M, then there exists a smooth function $f : M \to [0,1]$ with $C \subseteq f^{-1}(0)$ and $D \subseteq f^{-1}(1)$. But we can in fact make these containments *equalities*:

(a) Show that it suffices to build a smooth function $g : M \to [0,1]$ with $C = g^{-1}(0)$.

(b) Build a countable cover $\{U_i\}$ of $M \setminus C$ by open sets of the form $h_i^{-1}(B(x_i, 1)$ for a collection of coordinate charts $h_i = (x^1, \ldots, x^n)$ with image containing $B(x_i, 2)$. Build C^∞ functions $g_i : M \to \mathbb{R}$ which are > 0 in U_i and $= 0$ on $M \setminus U_i$. Note that $\overline{U_i}$ is compact; for each i, let

$$\alpha_i = \sup_{x \in \overline{U_i}; j \le i; m \le i; k_1, \ldots k_m \le n} \left\{ \frac{\partial^m g_j}{\partial x^{k_1} \cdots \partial x^{k_m}}(x) \right\} .$$

Show that the function $g = \sum g_i/(\alpha_i 2^i)$ is C^∞ and $C = g^{-1}(0)$.

11: [Lee, problem 2-6] For M a (smooth) manifold, let $C(M)$ denote the set of continuous functions from M to $\mathbb{R}$, thought of as an algebra (i.e., a ring and a vector space over $\mathbb{R}$) with scalar multiplication by $\mathbb{R}$, and pointwise addition and multiplication. Let $C^\infty(M)$ be the subalgebra of smooth functions. If $F : M \to N$ is continuous, let $F^* : C(N) \to C(M)$ be given by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Show that F is smooth $\Leftrightarrow F^*(C^\infty(N)) \subseteq C^\infty(M)$.

(c) Suppose F is a homeomorphism. Show that F is a diffeomorphism $\Leftrightarrow F^* : C^\infty(N) \to C^\infty(M)$ is an isomorphism.

(*) 12: [Lee, problem 2-17] Find an example of a (non-closed: it can't be done if the set is closed! This is what the Tietze Extension Theorem says...) subset A of a smooth manifold M, and a smooth function $f : A \to \mathbb{R}$ which admits **no** extension to a smooth function $\tilde{f} : M \to \mathbb{R}$.

(Recall that f is called smooth if for every $x \in A$ there is a neighborhood $x \in \mathcal{U}$ and a smooth extension of $f|_{A \cap \mathcal{U}}$ to the neighborhood $\mathcal{U}$.

13. Giving $M_1 \times M_2$ the product smooth structure, show that $f : N \to M_1 \times M_2$ is smooth $\Leftrightarrow$ the maps $p_1 \circ f : N \to M_1$, $p_2 \circ f : N \to M_2$ are smooth, where p_1, p_2 are the projections onto the first and second factors, respectively. Show, moreover, that the product smooth structure is the only smooth structure on $M_1 \times M_2$ with this property.

(*) 14. For $a \in M$, let $\mathcal{F}_a \subseteq C^\infty(M)$ denote the smooth functions satisfying $f(a) = 0$. and let $L : \mathcal{F}_a \to \mathbb{R}$ be a linear operator satisfying $L(fg) = 0$ for all $f, g \in \mathcal{F}_a$. Show that there is a unique derivation $X \in T_a M$ satisfying $X|_{\mathcal{F}_a} = L$.

(N.B. This leads to still another characterization of tangent vectors, has the vector space of linear maps $X : \mathcal{F}_a/W \to \mathbb{R}$, where $W = \mathcal{F}_a^2 =$ the ideal generated by products fg for $f, g \in \mathcal{F}_a$.)

15. Using our 'standard' charts on the sphere S^n (domains are the $2n$ open hemispheres, and maps ignore the coordinate whose sign is being restricted), show that the map $F : S^3 \to S^2$ given by

$$F(x, y, z, w) = (2xz + 2yw, 2yz - 2xw, x^2 + y^2 - z^2 - w^2)$$

is smooth (here we write points of S^3 as points of $\mathbb{R}^4$ satisfying $x^2 + y^2 + z^2 + w^2 = 1$). [Check also that it maps into S^2 !]

[N.B.: This is a fairly famous map, known as the *Hopf map.* Writing points in $\mathbb{R}^4$ as pairs of complex numbers (z_1, z_2), this map can be expressed as $F(z_1, z_2) = (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2)$. (This was stolen straight from Wikipedia.) If $F(z_1, z_2) = F(w_1, w_2)$, then $z_1 = \lambda w_1$ and $z_2 = \lambda w_2$ for some complex number λ with $|\lambda| = 1$. This implies that every point inverse is a 'round' circle.]