Joy of Numbers – Take Home Test # 1 Comments

1. Every odd integer can be written either as $4n + 1$ or $4n + 3$ for some integer n. Why is this true? We will call these two kinds of numbers $(4, 1)$-numbers and $(4, 3)$-numbers. Do some experiments with multiplying numbers of different kinds together. What can you conclude? Give a formal proof about what happens when two $(4, 3)$-numbers are multiplied together.

   This question was a bit less well-written than it should have been. Many of you (correctly) showed that for any of these numbers, the result of multiplication will be odd. Somewhat fewer noted that two $(4, 3)$ numbers, when multiplied together, always resulted in a $(4, 1)$ number. This can in fact be shown true in general: multiplying $4n+3$ and $4m+3$ together, we get

   $(4n + 3)(4m + 3) = 4(4nm) + 4(3n) + 4(3m) + 9 = 4(4nm + 3n + 3m + 2) + 1$

   which is a $(4, 1)$ number!

2. Explain how to show, using only hand computation (but with the least computation that you can muster), that $N = 1013$ is a prime number.

   As (I think) all of you showed, since $31^2 < 1013 < 32^2$, from what we learned in class it is enough to test to see if primes smaller than 32 are factors of 1013. These primes are 2,3,5,7,11,13,17,19,23,29, and 31. Most of you did not show the work (although, again, the problem statement did not really compel you to do that!) used to carry out these tests; since none of these numbers is a factor of 1013, we can conclude that 1013 is prime!

3. A triple of positive integers (a, b, c) (which we will assume is written in increasing order $a \le b \le c$) is called a *Pythagorean triple* if $a^2+b^2 = c^2$. [You may know that a triangle built with these side lengths will be a right triangle.] The most famous such triple is probably $(3, 4, 5)$. The goal here is to find more! One way to do this is to suppose that c is a fixed amount, say 1 (!), larger than b, so $c = b+1$, and then ask *"When is $c^2 - b^2 = (b+1)^2 - b^2$ equal to a^2, i.e., a perfect square?* Use this to find Pythagorean triples with $a = 5, 7, 9, 11$. Describe, if you can, a general procedure to keep going.

   By subtracting b^2 from $c^2 = (b + 1)^2$ for large enough values of b, we can discover the numbers which will yield 5^2 through 11^2. A more systematic approach, adopted by many of you, was to note that we wanted $c^2 = b^2 = 2b + 1 - a^2$, which we can solve for b for! This yields $b = 12, 24, 40, 60$.

   To find a general pattern, one could actually solve for b; for any odd a, $b = \dfrac{a^2 - 1}{2}$ will yield the b term of a Pythagorian triple containing a as its smallest value. [It was noted by those who took this road that b is an integer, since $a^2 - 1$ is even.]

   Several of you noticed other patterns to the values of b, that would be good enough to find their values: the successive differences in the b-values increases by 4 each time! Alternatively, several of you noted that the n-th b, b_n is equal to $a_n + a_{n-1} + b_{n-1}$, again giving a way to build all values. Another noted that, if we count starting with $a_2 = 5$, then $b_n = na_n + n$ for all of the values we found, a pattern which does continue for higher values!

4. We will learn that, in some sense, finding factors of a number is a much more challenging problem than knowing that the number is composite! But if the numbers have a particular form, we can use our knowledge about factoring polynomials to find factors of the number. And factoring polynomials can sometimes be done by finding roots of the polynomial.

   By factoring the polynomial $Q(h) = h^2 + 6h + 5$, show that there are very few values of n for which $N = n^2 + 6n + 5$ is prime; what are they?

   The polynomial $R(h) = h^2 + h + 5$ <u>cannot</u> be factored into lower degree polynomials with integer coefficients. Does this mean that $N = n^2 + n + 5$ is always prime? What does some experimentation tell you?

Since $h^2 + 6h + 5 = (h + 1)(h + 5)$, we can actually determine precisely which of these numbers are prime. If prime, the factorization we've found must not be 'real', i.e., either $h+1$ or $h+5$ must be 1 or -1. This leads to $h = 0, -2, -4$ or -6 as the only values which <u>could</u> be prime. As I think all of you showed, all of these values do give primes. What we also know, though, is that these are the only values that can give primes!

While $h^2 + h + 5$ does not factor as a polynomial (the quadratic formula gives complex roots), it does take on both prime and composite values. $0^2 + 0 + 5 = 5$ is prime, while (as I think all of you found) $4^2 + 4 + 5 = 25$ is composite. Whether or not $n^2 + n + 5$ is 'usually' prime or not, is a bit harder to determine...