Math 325 Problem Set 8 Solutions

Starred (*) problems were due Friday, October 26.

(*) 47. (Belding and Mitchell, p.100, #10 (sort of)) If $a \in D$ and $f : D \to \mathbb{R}$ is differentiable at a and $f'(a) > 0$, show that there is a $\delta > 0$ $x \in (a, a+\delta)$ implies that $f(x) > f(a)$ and $x \in (a-\delta)$ implies that $f(x) < f(a)$.

Because we know that $f'(a) > 0$, we can use $\epsilon = f'(a) > 0$ in the limit definition of the derivative to show that there is a $\delta > 0$ so that $0 < |x-a| < \delta$ implies that $|\dfrac{f(x)-f(a)}{x-a} - f'(a)| < f'(a)$, which means that $-f'(a) < \dfrac{f(x)-f(a)}{x-a} - f'(a) < f'(a)$, so $0 < \dfrac{f(x)-f(a)}{x-a} < 2f'(a)$.

The most relevant part of this is that $0 < |x-a| < \delta$ (i.e., $x \in (a-\delta, a)$ or $x \in (a, a+\delta)$) implies that $\dfrac{f(x)-f(a)}{x-a} > 0$. So if $x \in (a-\delta, a)$ then $x-a < 0$, and so $f(\dfrac{x)-f(a)}{x-a} > 0$ means that $f(x) - f(a) < 0$, so $f(x) < f(a)$. On the other hand, so $x \in (a, a+\delta)$ then $x - a > 0$, and so $\dfrac{f(x)-f(a)}{x-a} > 0$ means that $f(x) - f(a) > 0$, and so $f(x) > f(a)$. This establishes our needed results.

(*) 48. Show that if $a \in D$, $f, g : D \to \mathbb{R}$ are both differentiable and $x = a$, $f(a) = g(a)$, and $f(x) \leq g(x)$ for all $x \in D$, then $f'(a) = g'(a)$.

[What's the alternative? The previous problem can help!]

The alternative is that either $f'(a) > g'(a)$ or $f'(a) < g'(a)$. That is, setting $h(x) = f(x) - g(x)$, we have either $h'(a) > 0$ or $h'(a) < 0$. Note that $h(a) = 0$.

But from the previous problem, if $h'(a) > 0$, then there is a $\delta > 0$ so that $x \in (a, a+\delta)$ implies that $h(x) > h(a)$, so $h(x) = f(x) - g(x) > 0$, so $f(x) > g(x)$ for all $x \in (a, a+\delta)$. But this contradicts our hypothesis that $f(x) \leq g(x)$ for all $x \in D$, so $h'(a) > 0$ is impossible.

But if $h'(a) < 0$, then setting $k(x) = -h(x) = g(x) - f(x)$ then $k'(a) = -h'(a) > 0$. Note, again, that $k(a) = 0$. Then by the previous problem there is a $\delta > 0$ so that $x \in (a-\delta, a)$ implies that $k(x) < k(a)$, so $g(x) - f(x) < 0$, so $g(x) < f(x)$ for all $x \in (a-\delta, a)$. But, again, this contradicts our hypothesis that $f(x) \leq g(x)$ for all $x \in D$, so $h'(a) < 0$ is impossible.

So we must have $h'(a) = 0$, and therefore $f'(a) = g'(a)$.

(*) 50. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous, and f is differentiable at $x = 0$, with $f(0) = f'(0) = 0$. Show that $h(x) = f(x)g(x)$ is also differentiable at $x = 0$ and $h'(0) = 0$.

[Note that since we do not know that g is differentiable at $x = 0$, we cannot use the product rule (even if we knew what that was)....]

What we need to show is that the difference quotient,

$$\frac{h(x)-h(0)}{x-0} = \frac{f(x)g(x)-f(0)g(0)}{x-0} = \frac{f(x)g(x)}{x}$$

must be close to 0 so long as $x-0=x$ is small enough. That is, given an $\epsilon>0$ we need to produce a $\delta>0$ so that $0<|x-0|=|x|<\delta$ implies that $|\frac{f(x)g(x)}{x}|<\epsilon$.

But what we know is that $f'(0)=0$, so we know that we can make $\frac{f(x)-f(0)}{x-0}=\frac{f(x)}{x}$ small, and that g is continuous (at $x=0$), so we can make $|g(x)-g(0)|$ small. But this means that $g(x)$ cannot get big; in particular, as we have taken advantage of several times, there is a $\delta>0$ so that $|x-0|=|x|<\delta$ implies that $|g(x)-g(0)|<11$, so $g(0)-11<g(x)<g(0)+11$, so $|g(x)|<$max$\{|g(0)-11|,|g(0)+11|\}=N$. [This, formally, is because if $g(x)\geq 0$, then $|g(x)|=g(x)<g(a)+11=|g(a)+11|\leq N$, while if $g(x)\leq 0$, then $|g(x)|=-g(x)<-(g(a)-11)=|g(a)-11|\leq N$; so, no matter which case we are in, we have $|g(x)|\leq N$.] So, so long as $|x-0|<\delta$, we have (*) $=$ $|\frac{f(x)g(x)}{x}|=|\frac{f(x)}{x}|\cdot|g(x)|<N|\frac{f(x)}{x}|$. If we ensure that this is less than ϵ, then we will have controlled (*).

But this is something we can do! Given $\epsilon>0$, we can find a $\delta'>0$ so that

$0<|x-0|=|x|<\delta'$ implies that $|\frac{f(x)}{x}|<\frac{\epsilon}{N}$. Then setting $\delta_0=$min$\{\delta,\delta'\}$, we have $0<|x|<\delta$ implies $|g(x)|<N$ and $|\frac{f(x)}{x}|<\frac{\epsilon}{N}$, so $|\frac{h(x)-h(0)}{x-0}-0|=|\frac{f(x)}{x}|\cdot|g(x)|<\frac{\epsilon}{N}\cdot N=\epsilon$, as desired.

A selection of further solutions.

51. As almost none of us learn, the angle sum formula for tangent is

$$\tan(a+h)=\frac{\tan a+\tan h}{1-\tan a\tan h}$$

Use this to show directly from the ("limit as $h\to 0$" definition) that the derivative of $f(x)=\tan x$ is what you were told it is in calculus class.

[If you want something extra to do, derive this angle sum formula from the angle sum formulas for $\sin x$ and $\cos x$ (for fun!).]

We can go straight at this problem: starting from the angle sum formula, we can compute the difference quotient

$$\frac{\tan(a+h)-\tan a}{h}=\frac{1}{h}(\frac{\tan a+\tan h}{1-\tan a\tan h}-\tan a)$$

$$=\frac{1}{h}\cdot\frac{\tan a+\tan h-(\tan a)(1-\tan a\tan h)}{1-\tan a\tan h}=\frac{1}{h}\cdot\frac{\tan a+\tan h-\tan a+\tan^2 a\tan h}{1-\tan a\tan h}$$

$$=\frac{1}{h}\cdot\frac{\tan h+\tan^2 a\tan h}{1-\tan a\tan h}=\frac{\tan h}{h}\cdot\frac{1+\tan^2 a}{1-\tan a\tan h}=\frac{\sin h}{h}\cdot\frac{1}{\cos h}\cdot\frac{1+\tan^2 a}{1-\tan a\tan h}$$

But! As $h \to 0$, $\dfrac{\sin h}{h} \to 1$, $\cos h \to 1$, and $\tan h = \dfrac{\sin h}{\cos h} \to \dfrac{0}{1} = 0$; the first is a computation from class (or use L'Hôpital!), and the second and third are because $\sin x$ and $\cos x$ are continuous at $x = 0$. Putting these all together, we have, as $h \to 0$,

$$\frac{\tan(a+h) - \tan a}{h} = \frac{\sin h}{h} \cdot \frac{1}{\cos h} \cdot \frac{1 + \tan^2 a}{1 - \tan a \tan h}$$
$$\longrightarrow 1 \cdot \frac{1}{1} \cdot \cdot \frac{1 + \tan^2 a}{1 - (\tan a)(0)} = 1 + \tan^2 a = \sec^2 a$$

(since $1 + \tan^2 a = \dfrac{\cos^2 a + \sin^2 a}{\cos^2 a} = \dfrac{1}{\cos^2 a} = \sec^2 a$).

So, from the difference quotient, $f(x) = \tan x$ has $f'(a) = \sec^2 a$, so $f'(x) = \sec^2 x$, just like your calculus instructor told you....

We can <u>prove</u> the angle sum formula for $\tan x$ by combining the angle sum formulas for $\sin x$ and for $\cos x$:

$\tan(a+h) = \dfrac{\sin(a+h)}{\cos(a+h)} = \dfrac{\sin a \cos h + \cos a \sin h}{\cos a \cos h - \sin a \sin h}$. Dividing top and bottom by $\cos a \cos h$ makes this

$$\tan(a+h) = \frac{\frac{\sin a}{\cos a}\frac{\cos h}{\cos h} + \frac{\cos a}{\cos a}\frac{\sin h}{\cos h}}{\frac{\cos a}{\cos a}\frac{\cos h}{\cos h} - \frac{\sin a}{\cos a}\frac{\sin h}{\cos h}} = \frac{(\tan a)(1) + (1)(\tan h)}{(1)(1) - (\tan a)(\tan h)} = \frac{\tan a + \tan h}{1 - \tan a \tan h}, \text{ as desired.}$$

46. (Belding and Mitchell, p.99, #3(b)) Show that if $0 \in D$ and $f : D \to \mathbb{R}$ is continuous at $a = 0$, then $g : D \to \mathbb{R}$ given by $g(x) = xf(x)$ is differentiable at $a = 0$.

Since $g(0) = 0 \cdot f(0) = 0$, what we want to show is that the limit

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{xf(x) - 0}{x} = \lim_{x\to 0} \frac{xf(x)}{x} \text{ exists.}$$

But since $\dfrac{xf(x)}{x} = f(x)$ for $x \neq 0$, we have

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{xf(x)}{x} = \lim_{x\to 0} f(x) = f(0)$$

since f is continuous at $x = 0$ by hypothesis. So $g'(0) = f(0)$, and in particular g is differentiable at $x = 0$.