Math 325 Problem Set 8 Solutions

Problems were due Friday, March 17.

28. [Zorn, p.152, # 1] Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for every $x \in \mathbb{Q}$. Show that $f(x) = 0$ for every $x \in \mathbb{R}$.

    Suppose not. Suppose that there is an $a \in \mathbb{R}$ with $f(a) \neq 0$. Then $|f(a)| = \epsilon > 0$, and so, since f is continuous at a, there is a $\delta > 0$ so that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon = |f(a)|$. In particular (borrowing from a previous problem!) we know that $|x - a| < \delta$ implis that $||f(x) - |f(a)|| \leq |f(x) - f(a)| < |f(a)|$ so $-|f(a)| < |f(x)| - |f(a)|$, so $0 < |f(x)$. In particular (again!) we have that $|x - a| < \delta$ implies the $f(x) \neq 0$.

    But this is impossible. No matter what a and $\delta > 0$ are, we know that there is an $x \in \mathbb{Q}$ so that $|x - a| < \delta$. So, by hypothesis, $f(x) = 0$. But the above says that, for a particular choice of $\delta > 0$, every such x has $f(x) \neq 0$. Therefore, the assumption we made, that there is an $a \in \mathbb{R}$ with $f(a) \neq 0$, must be false. So $f(x) = 0$ for every $x \in \mathbb{R}$.

29. Using the problem above, show that if $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous functions, and $f(x) = g(x)$ for every $x \in \mathbb{Q}$, then $f = g$ (i.e., $f(x) = g(x)$ for every $x \in \mathbb{R}$).

    ['A continuous function is <u>determined</u> by its values on the rational numbers.']

    This has a fairly quick proof. If f and g are both continuous, then $h(x) = f(x) - g(x)$ is also continuous, and our hyppothesis iplies that $h(x) = f(x) - g(x) = 0$ for every $x \in \mathbb{Q}$. Our previous problem therefore tells us that $h(x) = 0$ for every x, so $f(x) = g(x)$ for every $x \in \mathbb{R}$. So $f = g$.

30. ['Pasting' continuous functions together.] Show that if $a < b < c$ and if $f : [a, b] \to \mathbb{R}$ and $g : [b, c] \to \mathbb{R}$ are both continuous functions, and $f(b) = g(b)$, then the function $h : [a, c] \to \mathbb{R}$ defined by

    $$h(x) = \begin{cases} f(x) & \text{if } x \leq b \\ g(x) & \text{if } x \geq b \end{cases}$$

    is continuous at $x = b$. Why is it also continouous at every <u>other</u> point in $[a, c]$?

    To establish that h is continuous at $x = b$, we wish to show that for any $\epsilon > 0$ there is a $\delta > 0$ so that $|x - a| < \delta$ implies that $|h(x) - h(a)| < \epsilon$. But since f is countinuous at $x = b$ (which is the right endpoint of its interval of definition), we know that $\lim\limits_{x \to b^-} f(x) = f(b)$, so for our $\epsilon > 0$ above, there is a $\delta_1 > 0$ so that $|x - b| < \delta + 1$ <u>and</u> $x < b$ we have $|f(x) - f(b)| < \epsilon$. Also, since g is continuous at b (which is the left endpoint of its interval of definition), we know that $\lim\limits_{x \to b^+} g(x) = g(b)$, so for our $\epsilon > 0$ above, there is a $\delta_2 > 0$ so that $|x - b| < \delta_2$ <u>and</u> $x > b$ we have $|g(x) - g(b)| < \epsilon$.

    But since $f(b) = h(b) = g(b)$, and $h(x) = f(x)$ when $x < b$ and $h(x) = g(x)$ when $x > b$, we have actually established that if $|x - b| < \delta_1$ and $x < b$ then $|h(x) - h(b)| < \epsilon$, and if $|x-b|<\delta_2$ and $x>b$ then $|h(x)-h(b)|<\epsilon$. Note that if $x=b$, then $|h(x)-h(b)|=|h(b)-h(b)|=0<\epsilon$ automatically. So, if we set $\delta=\min\{\delta_1,\delta_2\}>0$, then $|x-b|<\delta$ implies that $x=b$ or $|x-b|<\delta_1$ and $x<b$ or $|x-b|<\delta_2$ and $x>b$; in every case, we can conclude that $|h(x)-h(b)|<\epsilon$. SO we have found a $\delta>0$ so that $|x-b|<\delta$ implies tht $|h(x)-h(b)|<\epsilon$. So h is continuous at $x=b$.

For every other point $d\in[a,c]$, either $d<b$ or $d>b$. If $d<b$, then $b-d=\delta_1>0$, and so $|x-d|<\delta_1$ implies that $x<b$, so $h(x)=f(x)$. So if we have an $\epsilon>0$, then the continuity of f at $x=d$ implies that there is a $\delta_2>0$ so that $|x-d|<\delta_2$ and $x\in[a,b]$ implies that $|f(x)-f(d)|<\epsilon$. Then, setting $\delta=\min\{\delta_1,\delta_2\}>0$, if $|x-d|<\delta$, then $|x-d|<\epsilon_1$ so $x<b$ and so $x\in[a,b]$, so $f(x)=h(x)$, and so since $|x-d|<\delta_2$, as have $|h(x)-h(d)|=|f(x)-f(d)|<\epsilon$. So h is continuous at $x=d$.

The case of $d>b$ is essentially identical. If $d>b$, then $d-b=\delta_1>0$, and so $|x-d|<\delta_1$ implies that $x>b$, so $h(x)=g(x)$. So if we have an $\epsilon>0$, then the continuity of g at $x=d$ implies that there is a $\delta_2>0$ so that $|x-d|<\delta_2$ and $x\in[a,b]$ implies that $|g(x)-g(d)|<\epsilon$. Then, setting $\delta=\min\{\delta_1,\delta_2\}>0$, if $|x-d|<\delta$, then $|x-d|<\epsilon_1$ so $x>b$ and so $x\in[b,c]$, so $g(x)=h(x)$, and so since $|x-d|<\delta_2$, as have $|h(x)-h(d)|=|g(x)-g(d)|<\epsilon$. So h is continuous at $x=d$.

31. [Zorn, p.154, #10] Suppose that $a<0<b$ and $f:(a,b)\to\mathbb{R}$ is a function that is bounded (i.e., for some $M\in\mathbb{R}$, $|f(x)|\leq M$ for every $x\in(a,b)$). Show that the function $g:(a,b)\to\mathbb{R}$ defined by $g(x)=xf(x)$ is continuous at $x=0$. Show, on the other hand, that for any other $c\in(a,b)$ we have that g is continuous at c if and only if f is continuous at c.

[The last assertion can be attacked using 'general' results we have established, or directly using ϵ's and δ's (your choice!).]

First note that $g(0)=0\cdot f(0)=0$. Because $|f(x)|\leq M$ for every $x\in(a,b)$, we know that $|g(x)|=|xf(x)|=|x|\cdot|f(x)|\leq M|x|$ for every $x\in(a,b)$. [Note that since $|f(x)|\geq 0$, we must have $M\geq 0$. Since (because this is really being written backwards) we will eventually want to divide by M, we would actually like to have $M>0$, so if $M=0$ actually works, then $M=1$ does, too. So in waht follows we will assume that $M>0$.]

So for any $\delta>0$ we have $|x|=|x-0|<\delta$ implies that $|g(x)-g(0)|=|xf(x)-0|=|xf(x)|\leq M|x|<M\delta$. So, given an $\epsilon>0$, if we set $\delta=\epsilon/M$, then $|x-0|<\delta$ implies that $|g(x)-g(0)|=|xf(x)-0|=|xf(x)|\leq M|x|<M\delta=M(\epsilon/M)=\epsilon$, so $|g(x)-g(0)|<\epsilon$. So g is continuous at $x=0$.

On the other hand, if $c\neq 0$, then $h(x)=\dfrac{1}{x}$ and $k(x)=x$ are both continuous at $x=c$. So if $f(x)$ is continuous at $x=c$, then $k(x)f(x)=xf(x)=g(x)$ is continuous at c, since the product of functions continuous at $x-c$ is continuous at c, while if g is continuous at $x=c$ then $h(x)g(x)=\dfrac{1}{x}(xf(x))=f(x)$ is continuous at $x=c$. So g is continuous at $c\neq 0$ if and only if f is continuous at c.

[There is an argument working directly with the ϵ - δ definition of continuity, but it ia much less pleasant...]