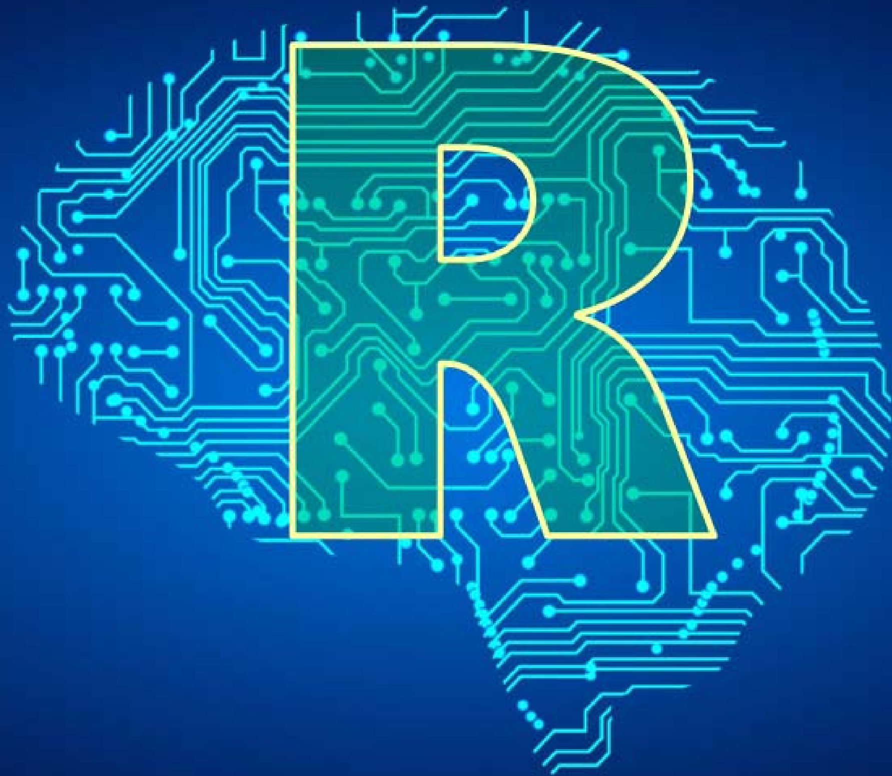
台灣人工智慧學校

參數估計



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參數估計 - 大綱

- **參數估計** (parameter estimation)
  (利用樣本統計量及其抽樣分配來對母體參數進行推估, 以瞭解母體的特性)

- **主題1**
  - **點估計** (動差法、最大概似法、最小平方法)
    - 評斷準則: 不偏性、有效性、一致性、最小變異不偏性、充份性。
  - **區間估計**

- 主題2
  - 貝式定理
  - 貝式估計法

概似函數 (The Likelihood Function)

1. Suppose the sample are iid from a distribution with density function $f(X|\theta)$, where θ is a parameter.

2. The **likelihood function** is the conditional probability of observing the sample, given θ

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) .$$

   (a) The parameter could be a vector of parameters, $\theta = (\theta_1, \cdots, \theta_p)$.

   (b) The likelihood function regards the data as a function of the parameter θ.

   (c) The **log likelihood** function

$$l(\theta) = \log(L(\theta)) = \sum_{i=1}^{n} \log f(x_i|\theta) .$$

最大概似估計法
Maximum Likelihood Estimation (MLE)

The method of maximum likelihood was introduced by **R.A. Fisher** (1890-1962, English statistician).

(a) By maximizing the likelihood function $L(\theta)$ with respect to θ, we are looking for the most likely value of θ given the sample data.

(b) Θ: parameter space of possible values of θ.

(c) If the $\max L(\theta)$ exists and it occurs at a unique point $\hat{\theta} \in \Theta$, then $\hat{\theta}$ is called maximum likelihood estimator of θ.

$$\frac{\partial L(\theta)}{\partial \theta} = 0 \quad 且 \quad \frac{\partial^2 L(\theta)}{\partial \theta^2} < 0$$

點估計步驟：

1. 抽取代表性樣本
2. 選擇一個較佳的樣本統計量當估計式
3. 計算估計式的估計值
4. 以該估計值推論母體參數並作決策

MLE of (μ, σ^2) from a normal population

$$X_1, \ldots, X_n \sim \text{i.i.d. } N(\mu, \sigma^2). \qquad f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

The probability density function for a sample of n independent identically distributed (iid) normal random variables (the likelihood) is

$$f(x_1, \ldots, x_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right),$$

$$\mathcal{L}(\mu, \sigma) = f(x_1, \ldots, x_n \mid \mu, \sigma)$$

$$\log(\mathcal{L}(\mu, \sigma)) = (-n/2)\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$0 = \frac{\partial}{\partial\mu}\log(\mathcal{L}(\mu, \sigma)) = 0 - \frac{-2n(\bar{x}-\mu)}{2\sigma^2}. \quad \Rightarrow \quad \boxed{\hat{\mu} = \bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}.} \qquad E[\widehat{\mu}] = \mu$$

https://en.wikipedia.org/wiki/Maximum_likelihood_estimation

MLE of (μ, σ^2) from a normal population

$$0 = \frac{\partial}{\partial \sigma} \log\left(\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2}{2\sigma^2}\right)\right)$$

$$= \frac{\partial}{\partial \sigma}\left(\frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2}{2\sigma^2}\right)$$

$$= -\frac{n}{\sigma} + \frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2}{\sigma^3}$$

$$\boxed{E\left[\widehat{\sigma}^2\right] = \frac{n-1}{n}\sigma^2.}$$

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2. \qquad \mu = \widehat{\mu} \quad \Rightarrow \quad \boxed{\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}$$

The maximum likelihood estimator (MLE) for $\theta = (\mu, \sigma^2)$ is

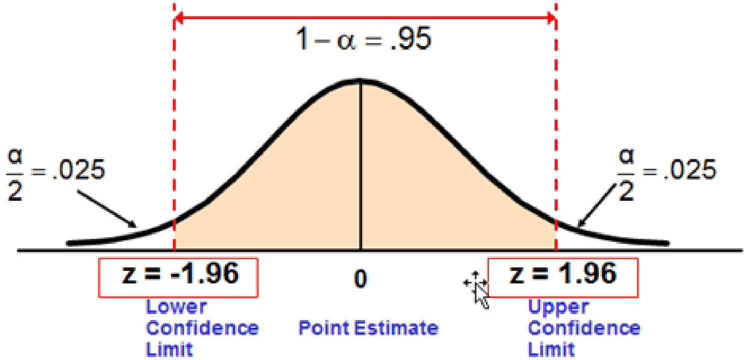
$$\boxed{\hat{\mu} = \bar{x} = \sum_{i=1}^n \frac{x_i}{n}.} \qquad \boxed{\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}$$

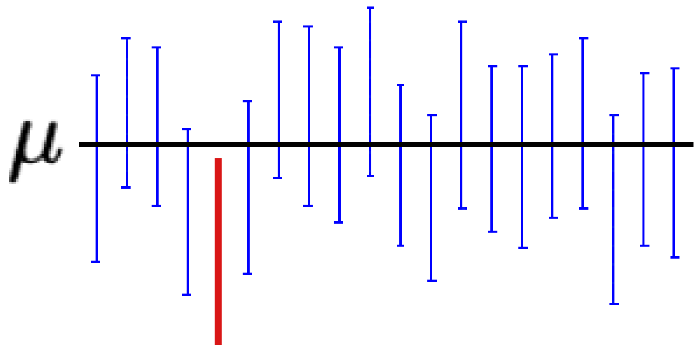
區間估計
(Interval Estimation)

- 區間估計是先對未知的母體參數求**點估計值**，然後在一信賴水準 (Confidence Level) 下，導出一個上下區間，此區間稱為信賴區間 (Confidence Interval)，信賴水準是指該區間**包含**母體參數的可靠度。
- 95% 信賴區間表示，做100 次信賴區間，區間約包含母體參數約95 次。

Interval Estimate of Population Mean

$$\bar{X} \sim N(\mu, \sigma^2/n) \quad \Rightarrow \quad Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \quad \Rightarrow \quad P(-z \le Z \le z) = 1-\alpha = 0.95.$$





$$0.95 = P\left(-1.96 \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le 1.96\right)$$

$$= P\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right).$$

A 95% confidence interval indicates that 19 out of 20 samples (95%) from the same population will produce confidence intervals that contain the population parameter.

範例: 老年人看電視的時間

根據行政院主計處調查，台灣地區15歲以上的人口中，以老年人(65歲以上)看電視的時間最長。現在新立傳播公司計畫推出老年人的電視節目，因此想要了解老年人看電視的時間，以決定電視節目的數量。新立公司於是採隨機抽樣法**抽取台北市100位老人**調查看電視的時數，結果得知，每星期看電視的**平均時間為 21.2小時**。假設根據過去數次調查的資料，已知每星期看電視時間的**標準差為8小時**，問在**95%信賴水準**下，每星期**看電視平均時間的信賴區間**為何？

信賴水準為95%，$\bar{X}$=21.2小時，σ =8小時， n =100

$\bar{X}$ 的抽樣分配為常態分配 $N(\mu,\sigma_{\bar{X}}^2)$ ⇨ $P(|\bar{X}-\mu|\le 1.96\sigma_{\bar{X}})=0.95$

$$\sigma_{\bar{X}}=\frac{\sigma}{\sqrt{n}}=\frac{8}{\sqrt{100}}=0.8$$

在 $1-\alpha$ 信賴水準下，母體平均數的信賴區間為

$$\bar{X}\pm Z_{\alpha/2}\sigma_{\bar{X}}$$

$$\bar{X}\pm Z_{\alpha/2}\sigma_{\bar{X}}=21.2\pm 1.96\times 0.8 \Rightarrow 19.632\le\mu\le 22.768$$

可推論：「老年人每星期平均看電視的時間在 19.632~22.768小時之間，而此一區間的可信度(信賴水準)為95%。」

參數估計 - 大綱

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貝氏定理 (Bayes' Theorem)

$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)\times P(A)}{P(B)}}$$

$$後驗機率 = \frac{可能性\times先驗機率}{標准化常量}$$

- $P(A|B)$: 已知在事件 B 發生的情況下事件 A 發生的機率。
  (稱作 A 的事後機率或後驗機率)(posterior probability)。
- $P(A)$, $P(B)$: A, B 的事前機率或先驗機率 (prior probability)。
  $(P(A)\neq 0,\ P(B)\neq 0)$
- $P(B|A)$: 已知 A 發生後，B 的條件機率。
  (稱作概似函數 likelihood function)。

例子: 假設有兩個甕，第一個甕裡面有 3 顆紅球，第二個甕裡面有 2 顆紅球和 1 顆白球。我們隨機選擇一個甕，然後從中抽出 2 顆球。假設結果是 2 顆紅球，留在甕裡的那顆球是紅球的機率是多少？(https://ccjou.wordpress.com/)

樣本空間 $\Omega = \{r_1, r_2, r_3, r_4, r_5, w_1\}$。
令 $U_1 = \{r_1, r_2, r_3\}$ 和 $U_2 = \{r_4, r_5, w_1\}$。
A: 從一個甕中抽出 2 顆紅球之事件。

$$\begin{aligned} P(U_1|A) &= \frac{P(A|U_1)P(U_1)}{P(A|U_1)P(U_1)+P(A|U_2)P(U_2)} \\ &= \frac{1\cdot\frac{1}{2}}{1\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}} = \frac{3}{4}. \end{aligned}$$

Bayes' Theorem

1. If A and B are events and $P(B) > 0$, then
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$
2. The distributional form of Bayes' Theorem for continuous random variables is
$$f_{X|Y=y}(x) = \frac{f_{Y|X=x}(y)f_X(x)}{f_Y(y)} = \frac{f_{Y|X=x}(y)f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X=x}(y)f_X(x)\ dx}$$

1. In the **frequentist approach** to statistics, the parameters of a distribution are considered to be fixed but unknown constants.
2. The **Bayesian approach** views the unknown parameters of a distribution as random variables.
   (a) In Bayesian analysis, probabilities can be computed for parameters as well as the sample statistics.
   (b) Bayes' Theorem allows one to revise the prior belief about an unknown parameter based on observed data.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

3. Suppose that X has the density $f(x|\theta)$.

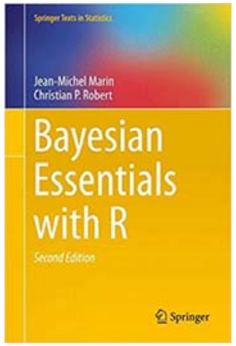
(a) $f_\theta(\theta)$: the pdf of the prior distribution of θ.

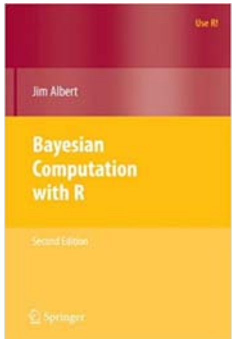
(b) The conditional density of θ given the sample observations $x_1, \cdots, x_n$ is called the posterior density

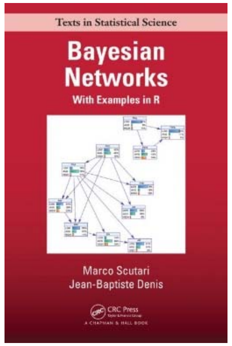
$$f_{\theta|x}(\theta) = \frac{f(x_1, \cdots, x_n|\theta) f_\theta(\theta)}{\int f(x_1, \cdots, x_n|\theta) f_\theta(\theta)\, d\theta}.$$

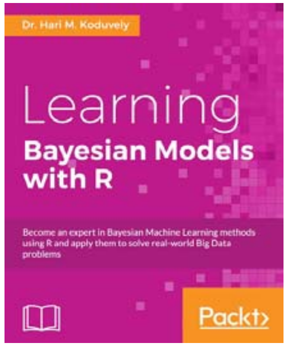
(c) The posterior distribution summarizes our modified belief about the unknown parameters, taking into account the observed data.

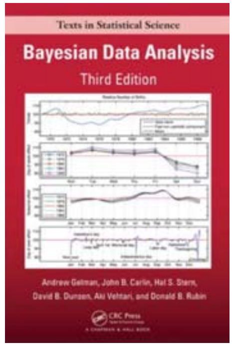
(d) One is interested in computing posterior quantities such as posterior means, posterior modes, posterior standard deviations.











Bayes Estimator for the Mean of a Normal Distribution

$X_1, X_2, \ldots, X_n$ be a random sample $X_1, \ldots, X_n \sim$ i.i.d. $N(\mu, \sigma^2)$.

μ is unknown and σ^2 is known.

prior distribution for μ is normal with mean μ_0 and variance σ_0^2

$$f(\mu) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(\mu-\mu_0)^2/(2\sigma_0^2)} = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(\mu^2-2\mu_0+\mu_0^2)/(2\sigma_0^2)}$$

The joint probability distribution of the sample

$$f(x_1, x_2, \ldots, x_n | \mu) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)\left(\sum x_i^2 - 2\mu\sum x_i + n\mu^2\right)}$$

the joint probability distribution of the sample and μ is

$$f(x_1, x_2, \ldots, x_n, \mu) = \frac{1}{(2\pi\sigma^2)^{n/2}\sqrt{2\pi}\sigma_0} e^{-(1/2)\left[\left(1/\sigma_0^2+n/\sigma^2\right)\mu^2-\left(2\mu_0/\sigma_0^2+2\sum x_i/\sigma^2\right)\mu+\sum x_i^2/\sigma^2+\mu_0^2/\sigma_0^2\right]}$$

$$= e^{-(1/2)\left[\left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma^2/n}\right)\mu^2-2\left(\frac{\mu_0}{\sigma_0^2}+\frac{\bar{x}}{\sigma^2/n}\right)\mu\right]} h_1(x_1, \ldots, x_n, \sigma^2, \mu_0, \sigma_0^2)$$

Bayes Estimator for the Mean of a Normal Distribution

$$f(x_1, x_2, \ldots, x_n, \mu) = e^{-(1/2)\left[\left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma^2/n}\right)\mu^2 - 2\left(\frac{\mu_0}{\sigma_0^2}+\frac{\bar{x}}{\sigma^2/n}\right)\mu\right]} h_1(x_1,\ldots,x_n,\sigma^2,\mu_0,\sigma_0^2)$$

$$\Rightarrow \quad f\left(x_1, x_2, \ldots, x_n, \mu\right) = e^{-(1/2)\left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma^2/n}\right)\left[\mu - \left(\frac{(\sigma^2/n)\,\mu_0}{\sigma_0^2+\sigma^2/n}+\frac{\bar{x}\sigma_0^2}{\sigma_0^2+\sigma^2/n}\right)\right]^2} h_2(x_1,\ldots,x_n,\sigma^2,\mu_0,\sigma_0^2)$$

$h_i(x_1,\ldots,x_n,\sigma^2,\mu_0,\sigma_0^2)$ is a function of the observed values and the parameters σ^2, μ_0, and σ_0^2.

because $f(x_1,\ldots,x_n)$ does not depend on μ,

$$\Rightarrow \quad f\left(\mu|x_1,\ldots,x_n\right) = e^{-(1/2)\left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma^2/n}\right)\left[\mu - \left(\frac{\left(\sigma^2/n\right)\mu_0+\sigma_0^2\bar{x}}{\sigma_0^2+\sigma^2/n}\right)\right]^2} h_3\left(x_1,\ldots,x_n,\sigma^2,\mu_0,\sigma_0^2\right)$$

a normal probability density function

posterior mean $\dfrac{\left(\sigma^2/n\right)\mu_0+\sigma_0^2\,\bar{x}}{\sigma_0^2+\sigma^2/n}$

posterior variance $\left(\dfrac{1}{\sigma_0^2}+\dfrac{1}{\sigma^2/n}\right)^{-1} = \dfrac{\sigma_0^2\left(\sigma^2/n\right)}{\sigma_0^2+\sigma^2/n}$