Dynamical Oceanography Assignment 3

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Question a.

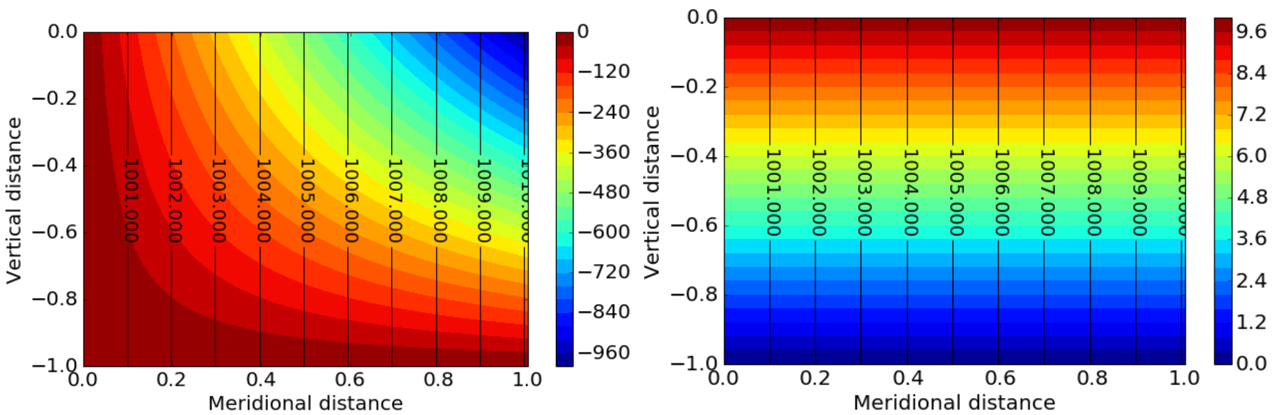
Thermal wind balance relates the density ρ to the stream function ψ according to the following derivation (use $u = -\frac{\partial \psi}{\partial y}$):

$$\begin{aligned} f_0 \frac{\partial u}{\partial z} &= \frac{g}{\rho_0} \frac{\partial \rho}{\partial y} \\ -\frac{\partial \rho}{\partial y} &= \frac{f_0 \rho_0}{g} \frac{\partial^2 \psi}{\partial z \partial y} \\ \rho &= -\frac{f_0 \rho_0}{g} \frac{\partial \psi}{\partial z} + \rho_c(z) \end{aligned}$$

with $\rho_c(z)$ an integration constant independent of y but possibly dependent of z. Considering the given zonal flow $\bar{\psi}$, we get:

$$\begin{aligned} \rho(y,z) &= -\frac{f_0 \rho_0}{g}(-U y \alpha) + \rho_c(z) \\ \rho(y,z) &= \frac{f_0 \rho_0 U \alpha}{g} y + \rho_c(z) \end{aligned}$$

Now, say $\rho_c(z)$ is constant (e.g. $\rho_c = \rho_0$), then we can plot the $\psi(y,z)$ and $\rho(y,z)$. This is displayed in Fig. 1. The values used are $\rho_0 = 1000$ kg m^{-3}, $g = 9.81$ m s^{-2}, $f = 10^{-4}$, $\alpha = 1$ and $U = 10$ m s^{-1}.



(a) (b)

Figure 1: Left figure: stream function (colors) and density (black contours). Right figure: Zonal velocity (colors) and density (black contours).

Concerning related balances, thermal wind balance is of course important to determine ρ. The quasi-geostrophic stratified potential vorticity equation reads the following:

$$\left(\frac{\partial}{\partial t} + u^0 \frac{\partial}{\partial x} + v^0 \frac{\partial}{\partial y}\right)\left(\nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial \psi}{\partial z}\right) + \beta y\right) = 0$$

Filling in $\bar{\psi}$ and that the $\mathbb{O}(1)$ meridional velocity v^0 is zero, we get that all terms drop and that this vorticity balance perfectly holds. It is important to note that this vorticity balance is a consequence of the stratified quasi-geostrophic model. There we allowed ρ to be non-constant. This in particular leads to the equations given in Eqn. 8.20a-f, which in turn leads to the thermal wind balance (Eqn. 8.24a-b) in the $\mathbb{O}(1)$ and to Eqns 8.26a-c in the $\mathbb{O}(\epsilon)$ approximation.

Question b.

Start with the given equation for ψ and fill in $\psi = \bar{\psi} + \phi$:

$$\frac{D}{dt}(\nabla^2\bar{\psi} + \nabla^2\phi + \frac{1}{S}(\frac{\partial^2\bar{\psi}}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}) + \beta y) = 0$$

We know that for $\alpha = 0$, $\bar{\psi} = -Uy$. This means that $\nabla^2\bar{\psi} = \frac{\partial^2\bar{\psi}}{\partial x^2} + \frac{\partial^2\bar{\psi}}{\partial y^2} = 0 + \frac{\partial}{\partial y}(-U) = 0$ and the same way $\frac{\partial^2\bar{\psi}}{\partial z^2} = 0$. This leaves (only writing the zeroth orders for the velocities, higher order terms can be neglected):

$$\begin{aligned}
\frac{D}{dt}(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2} + \beta y) &= 0\\
(\frac{\partial}{\partial t} + u^0\frac{\partial}{\partial x} + v^0\frac{\partial}{\partial y} + w^0\frac{\partial}{\partial z})(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2} + \beta y) &= 0
\end{aligned}$$

We know that $u^0 = -\frac{\partial\bar{\psi}}{\partial y} = U$ and $v^0 = -\frac{\partial\bar{\psi}}{\partial x} = 0$. Furthermore, assume $w^0 << U$, so that we can neglect it. This results in:

$$\begin{aligned}
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2} + \beta y) &= 0\\
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2}) + (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\beta y &= 0\\
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2}) + \beta\frac{\partial}{\partial t}y &= 0\\
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2}) + \beta v &= 0\\
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\phi + \frac{1}{S}\frac{\partial^2\phi}{\partial z^2}) + \beta\frac{\partial\phi}{\partial x} &= 0
\end{aligned}$$

Where we used that $v = \frac{\partial\psi}{\partial x} = \frac{\partial\bar{\psi}}{\partial x} + \frac{\partial\phi}{\partial x} = 0 + \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x}$. The latter is the equation that is asked for.

Question c.

To determine the dispersion relation of Rossby waves, we write ϕ by separating variables:

$$\phi = \Psi(x, y, t)\Phi(z)$$

Filling this into the equation from question (b), gives:

$$\begin{aligned}
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\Phi\nabla_H^2\Psi + \frac{\Psi}{S}\frac{\partial^2\Phi}{\partial z^2}) + \beta\Phi\frac{\partial\Psi}{\partial x} &= 0\\
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla_H^2\Psi + \frac{\Psi}{S\Phi}\frac{\partial^2\Phi}{\partial z^2}) + \beta\frac{\partial\Psi}{\partial x} &= 0
\end{aligned}$$

To make this equality hold, there must be a certain χ such that the following two conditions hold:

$$\begin{aligned}
\frac{1}{S\Phi}\frac{\partial^2\Phi}{\partial z^2} &= -\chi, \text{ and:}\\
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla_H^2\Psi - \chi\Psi) + \beta\frac{\partial\Psi}{\partial x} &= 0
\end{aligned}$$

The first equation results in a solution of Φ like (analogous to the harmonic oscillator):

$$\Phi(z) = A_1\cos(z\sqrt{\chi S}) + A_2\sin(z\sqrt{\chi S})$$

However, we know that $\frac{D}{dt}\frac{\partial\psi}{\partial z} = 0$ at $z = -1$ or 0. This reduces to $\Phi(0) = \Phi(-1) = 0$, so that $A_1 = 0$ (because $\Phi(0) = A_1\cos(z\sqrt{\chi S}) = 0$) and $\chi = \chi_n = \frac{n^2\pi^2}{S}$ for $n \in \mathbb{N}$ (because then $\Phi(-1) = -A_2\sin(\sqrt{\chi S}) = 0$). Now inserting a wave-like solution $\Psi = \Psi_0 e^{i(kx+ly-\sigma t)}$ into the second equation gives us:

$$\begin{aligned}
(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla_H^2\Psi - \chi_n\Psi) + \beta\frac{\partial\Psi}{\partial x} &= 0 \\
\Psi(-i\sigma + Uik)(-k^2 - l^2 - \chi_n) + \beta ik\Psi &= 0 \\
-(-i\sigma + Uik)(k^2 + l^2 + \chi_n) &= -\beta ik \\
\sigma(k^2 + l^2 + \chi_n) &= -\beta k + Uk(k^2 + l^2 + \chi_n) \\
\sigma &= Uk - \frac{\beta k}{k^2 + l^2 + \chi_n}
\end{aligned}$$

The latter is the sought dispersion relation.

Question d.

Concerning barotropic Rossby waves, we need to choose $n = 0$, resulting in:

$$\sigma = Uk - \frac{\beta k}{k^2 + l^2}$$

Now, stationary Rossby waves exist when the phase speed is zero, meaning $\sigma/k = 0$. This is the case when:

$$\begin{aligned}
0 &= \frac{\sigma}{k} \\
0 &= U - \frac{\beta}{k^2 + l^2} \\
U &= \frac{\beta}{k^2 + l^2} \\
k^2 + l^2 &= \frac{\beta}{U}
\end{aligned}$$

Even more specific, we are considering *zonal* stationary Rossby waves, which means that their wave vector does not have a meridional component ($l = 0$). This means that:

$$k = \pm\sqrt{\frac{\beta}{U}}$$

As this is possible these waves exist. The resulting dimensionless wavelength is then (as λ is always positive):

$$\lambda = \frac{2\pi}{|k|} = 2\pi\sqrt{\frac{U}{\beta}}$$