ACTSC 445/845: FINAL PROJECT
DISTORTED RISK MEASURES

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Abstract. Development of robust frameworks to accurately assess and manage risk is essential within quantitative risk management (QRM). Throughout this paper we will be taking a measure theoretic approach to developing the tools needed in constructing the class of distorted risk measures. This family of risk measures allows us to expand on the notion of Coherence and address its concerns by introducing an alternate framework which offers more accurate and reliable risk assessments. The former assumption of coherence has been seen as insufficient, leading to inaccurate risk assessments. As a result, alternative frameworks have been developed to address these criticisms, with distorted risk measures offering a promising alternative through the incorporation of distortion functions.

1. Introduction

Within QRM, the systematic assessment of potential losses in financial markets can be modelled with a probabilistic framework. This begins with treating a loss as a random variable which represents the future value of our position over a fixed period of time. As a result, randomness arises due to the uncertainty associated with market changes and uncertain events which may affect the value of our position.

Definition 1.1. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- Define $X : \Omega \to \mathbb{R}$ to be the random loss associated with a given investment over time.
- Let $\mathcal{X} \supseteq L^\infty$ be a convex cone which we take to represent our set of risks.

Throughout our discussion, we typically take $\mathcal{X} = L^\infty$ to be the space of all bounded random variables.

Definition 1.2. A risk measure, ρ, is a function $\rho : \mathcal{X} \to \mathbb{R}$ that quantifies risk. We can interpret $\rho(X)$ as the capital requirement necessary to safeguard against the risk X over a fixed period.

The goal of any financial institution is to avoid insolvency which can be done so through the use of effective risk measures. However, this requires a firms risk measurement to be sufficiently accurate to withstand various market conditions which may arise.

Two important risk measures we analyze are Value-at-Risk (VaR) and Expected Shortfall (ES). Both have been instrumental in the establishment of risk measure theory and the establishment of regulatory standards in the financial industry.

Example 1.3. For an $\alpha \in (0,1)$ and some random loss $X \sim F$,

(1) $\mathrm{VaR}_\alpha : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\} = F^{\leftarrow}(\alpha)$$

(2) $\mathrm{ES}_\alpha : L^0 \to \mathbb{R}$.

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)du \overset{F \text{ cont.}}{=} \mathbb{E}(X|X > \mathrm{VaR}_\alpha(X))$$

2. Axioms of Risk Measures

The role regulators play in QRM is crucial to the sustainability of our financial markets. Their task of ensuring that financial institutions maintain positions that are deemed "acceptable" from a risk standpoint ties directly into the concept of acceptance sets, which serve as a foundational element in the theory of risk measures.

Definition 2.1. The acceptance set of a risk measure ρ is defined by

$$\mathcal{A}_\rho := \{X \in \mathcal{X} \mid \rho(X) \leq 0\}$$

This set describes positions deemed "acceptible" to a regulator, meaning no additional capital is required to safeguard against the associated risk.

When $\rho(X)$ is positive, we can interpret this as the capital one needs to add to a position X so that it's deemed acceptable from a regulators point of view. This ties directly to properties of risk measures we wish to have.

Cash Invariance:

$$\text{(CI)} \qquad \rho(X + c) = \rho(X) + c, c \in \mathbb{R}$$

Monotonicity:

$$\text{(M)} \qquad X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$$

When we're faced with a position deemed "unacceptable", (CI) characterizes the notion of adding the fixed amount $\rho(X) = c \in (0, \infty)$ to a position, leading to the adjusted loss $\tilde{X} = X - c$. Thus, our position $\rho(\tilde{X}) = \rho(X - \rho(x)) = 0 \in \mathcal{A}_\rho$ is acceptible.

Definition 2.2. The class of risk measures satisfying (CI) and (M) are called monetary risk measures.

Monetary risk measures posses many desirable mathematical properties that lay the foundation to their applications in risk management. For instance, it can be shown that a monetary risk measure is Lipschitz continuous with respect to the infinity norm, which ensures stability and robustness under small changes in the underlying risk.

Lemma 2.3. *A monetary risk measure is lipschitz continuous,*

$$|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty, X, Y \in \mathcal{X}$$

Proposition 2.4. *Suppose ρ is a monetary risk measure with acceptance set $\mathcal{A}_\rho$. Then,*

(1) $\emptyset \in \mathcal{A}_\rho$
(2) $\mathcal{A}_\rho$ *is closed with respect to the* L^∞ *norm,* $\|\cdot\|_\infty$.
(3) $\sup\{m \in \mathbb{R} \mid \mathcal{A}_\rho\} < \infty$
(4) If $X \in \mathcal{A}_\rho, Y \in \mathcal{X}$ *and* $Y \leq X$*, then* $Y \in \mathcal{A}_\rho$.

Proof of 2. Let $(X_n)_{n=1}^\infty \in \mathcal{A}_\rho$ be a sequence such that $\lim_{n\to\infty} X_n = X \in \mathcal{X}$. It suffices to show X is also contained in $\mathcal{A}_\rho$, that is, $\rho(X) \leq 0$. Since $X_n \in \mathcal{A}_\rho$, we have $\rho(X_n) \leq 0, \forall n \in \mathbb{N}$. By Lemma 2.3, we know $|\rho(X) - \rho(X_n)| \leq \|X - X_n\|_\infty \to 0$ as $n \to \infty$. This forces $\lim_{n\to} X_n = X \leq 0$. So it follows that $X \in \mathcal{A}_\rho$. □

Proof of 4. By (M), we know $Y \leq X \Rightarrow \rho(Y) \leq \rho(X) \leq 0$ since $X \in \mathcal{A}_\rho$. Thus, $\rho(Y) \leq 0$, that is, $Y \in \mathcal{A}_\rho$. □

We can see how a monetary risk measure can be fully characterized by its acceptance set.

Theorem 2.5. *(1) Let $\mathcal{A} \subseteq \mathcal{X}$ be any subset satisfying Proposition 2.4.4. Then, we get the monetary risk measure,*

$$\rho_{\mathcal{A}} = \inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}\}$$

(2) For any monetary risk measure ρ,

$$\rho(X) = \rho_{\mathcal{A}_\rho}(X)$$

Example 2.6. Take S_T to be a security at time T with $S_0 = 1$, and let $\mathcal{A} \subseteq \mathcal{X}$ satisfying Proposition 2.4.4. Then, we can define a risk measure from a regulators point of view,

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} \mid X - m \cdot S_T \in \mathcal{A}\}$$

If we consider the set of self-financing portfolios, Π, then

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R}^n \mid X - \pi_T \in \mathcal{A}, \pi_0 = m\}$$

Take the acceptance set $\mathcal{A} = \{X \in \mathcal{X} \mid X \leq 0 \ , \mathbb{P} - a.s.\}$. That is, the regulator only accepts positions ending in profits, not losses. Then $\rho(X)$ is called the superhedging price of X, and is the arbitrage free price of X if the market is complete.

While monetary risk measures form a broad class, there are additional properties that are often desired in a risk measure.

Positive Homogeneity:

$$\rho(\alpha X) = \alpha\rho(X), \alpha \in (0, \infty) \tag{PH}$$

Sub Additivity:

$$\rho(X + Y) \leq \rho(X) + \rho(Y) \tag{SA}$$

(PH) is a desirable property as it guarantees that scaling a position should proportionally scale the risk by the same factor. This is easy to see as converting a positions currency should only change the capital requirement by the exchange rate. However, (PH) lacks the ability to account for changes liquidity risks, where increasing the size of a position may disproportionally increase the risk due to market impact and reduced liquidity.

(SA) is a crucial as it characterizes the notion of diversification. This property ensures the risk of the portfolio is at least as much as the sum of the risk of its constituents, which aligns with the benefits of diversification found in traditional financial theory, particularly under the CAPM framework. We can see that if ρ is (PH) then $\rho(0) = 0$.

Definition 2.7. We say the risk measure ρ is coherent if it satisfies (CI), (M), (PH), and (SA).

Remark 2.8. As we already know, a large debate between the VaR and ES supremacy centres around the fact that ES is coherent while VaR is not, due to VaR lack of (SA). However, ES is not without its flaws; it fails as a global regulatory risk measure due to issues surrounding changes in currency.

As pointed out when analyzing (PH), we used the toy example of scaling risks by a fixed exchange rate. In practice, the assumption of a fixed exchange rate is a gross oversimplification, and is more realistic to be defined as a random variable. Take the randon exchange rate at time T to be R_T, and let ρ be a monetary risk measure. Further, suppose a regulator uses the acceptance set $\mathcal{A}_\rho$, where an institution is solvent if $X \in \mathcal{A}_\rho$.

Then, regulators can apply the same acceptance set calculated based on a foreign currency, and an institution remains solvent when $\frac{R_T}{R_0}X \in \mathcal{A}_\rho$. Thus, we get for $X \in \mathcal{X}$, if $\rho(X) \leq 0$, then $\rho(RX) \leq 0$, for $R_T/R_0 =: R \in \mathcal{X}$, represents Exchange-invariance.

It can be shown that VaR is exchange invariant, but ES is not [?]. This is a major drawback to ES and is an important subject surrounding the VaR and ES debate.

3. Representation Theorem of Risk Measures

We start by recalling the definitions of additivity in the context of measures.

Definition 3.1 (Additivity)**.** The measure $Q : \mathcal{A} \to [0,1]$ is said to be:

(1) *finitely additive* if for any finite collection of disjoint sets, $(A_i)_{i=1}^n \in \mathcal{A}$,

$$Q\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n Q(A_i) \tag{Finite-Add}$$

(2) σ-additive if for any disjoint countable collection of sets, $(A_n)_{n=1}^\infty \in \mathcal{A}$,

$$Q\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty Q(A_n) \tag{σ-Add}$$

Next, we introduce the theorem allowing us to give a general representation of coherent risk measures.

Theorem 3.2. *Representation Theorem of Coherent Risk Measures*
Let $\emptyset \neq \Omega, \mathcal{X} = L^\infty$. Let $\mathcal{M}$ be a collection of finitely additive measures, $Q : \mathcal{A} \to [0,1]$, with $Q(\Omega) = 1, Q \in \mathcal{M}$. Then, a coherent risk measure has the representation,

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q(X), X \in \mathcal{X}$$

where $\mathcal{R} \subseteq \mathcal{M}$.

In this case, $\mathcal{R}$ can be seen as set of generalized scenarios.

Example 3.3. For $\alpha \in (0,1)$, we have

$$\mathrm{ES}_\alpha(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q(X), X \in \mathcal{X}$$

where $\mathcal{R} = \{Q \text{ is a probability measure} : dQ/d\mathbb{P} \leq 1/(1-\alpha)\}$

The issue we currently face is that we can only guarantee $\mathcal{R}$ is a collection of probability measures if Ω is finite, hence $\mathcal{A}$ is finite, ensuring a measure is σ-additive. To correct this, one may consider taking a risk measure to be continuous with respect to $\mathbb{P}$. This, however, would allow for some unwanted scenarios to occur.

Example 3.4. Consider the scenario where $(X_n)_{n=1}^\infty \in \mathcal{X}$ represent a sequence of catastrophic events with diminishing probabilies. We can model this by taking $X_n = n^2 I_{U \geq 1/n}$ for some $U \sim \mathcal{U}_{(0,1)}$. Then, $X_n \to 0$ almost surely, as but X_n becomes increasingly disastrous.

To combat this issue, we can impose an alternative form related to continuity.

Definition 3.5 (Fatou Property)**.** If $X, X_1, X_2, \dots \in \mathcal{X}$, with $\sup_{n\in\mathbb{N}} \|X_n\|_\infty < \infty$, and $X_n \overset{n\to\infty}{\longrightarrow} X$ a.s., then

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n) \tag{FP}$$

This allows us to have the following properties

Proposition 3.6. *Let ρ be a monetary risk measure. Then the following are equivalent.*

(1) ρ has Fatou property
(2) ρ is continuous a.s. from below.

$$X_n \uparrow X \Rightarrow \rho(X_n) \uparrow \rho(X)$$

Proof. Take ρ to be a monetary risk measure.

"⇒": Suppose ρ satisfies the Fatou property, and consider the sequence $(X_n)_{n=1}^\infty \in \mathcal{X}$ such that $X_n \uparrow X \in \mathcal{X}$. By Monotonicity, $\rho(X_n) \leq \rho(X), \forall n \in \mathbb{N}$. By the Fatou property,

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n) = \lim_{n\to\infty} \left(\inf_{k\geq n} X_k \right) = \lim_{n\to\infty} \rho(X_n)$$

So it follows that $\rho(X_n) \uparrow \rho(X)$, as required.

"⇐": Suppose ρ is continuous from below. Let $(X_n)_{n=1}^\infty \in \mathcal{X}$ such that $X_n \to X \in \mathcal{X}$ a.s. Define $Y_n := \inf_{k\geq n} X_k \leq X_n, \forall n \in \mathbb{N}$. Then, $Y_n \uparrow X$, and by continuity from below, $\rho(Y_n) \uparrow \rho(X)$. Since $\rho(Y_n) \leq \rho(X_n)$, we observe

$$\rho(X) = \lim_{n\to\infty} \rho(Y_n) \leq \liminf_{n\to\infty} \rho(X_n)$$

Hence, ρ satisfies the Fatou property, as desired.

□

Now, we obtain the most popular representation of coherent risk measures.

Theorem 3.7. *The coherent risk measure ρ with (FP) has the following representation*

$$\rho(X) = \sup_{Q\in\mathbb{R}} \mathbb{E}^Q(X), X \in \mathcal{X}$$

where $\mathcal{R}$ is a collection of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

4. Constructing the Distorted Risk Measure

We start by introducing a simpler class risk measures determined by the distribution of a random loss.

Definition 4.1. A risk measure ρ is said to be *law-determined* when for random losses $X, Y \in \mathcal{X}$,

$$X \overset{d.}{=} Y \Rightarrow \rho(X) = \rho(Y) \tag{LD}$$

Observe that (LD) emphasises the importance of the probability measure $\mathbb{P}$ which the distribution's of X and Y depend on. In comparison to the properties we have previously studied, knowing how a measure Q is defined is of little importance since it retains the same properties of a similarly defined measure $\mathbb{P}$. From a practical setting, we may not always know the precise mapping of the risk $X : \Omega \to \mathbb{R}$, but rather its distribution. Treating this issue from as a statistical problem, we are able to reduce the overall set of measures which we are studying. Hence, law-determined functionals are sometimes called *statistical functionals*.

The class of distorted risk measures turn out to be of great importance in the study of law-determined risk measures. If fact, VaR and ES are particularly import and play an interesting role. We look at some key properties which are relevant to the study of distorted risk measures.

Definition 4.2. The pair of random losses $(X, Y) \in L^0 \times L^0$ are said to be *comonotonic* if

$$\mathbb{P}(X \leq x, Y \leq y) = \min\{\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)\} \tag{CO}$$

denoted by $X//Y$.

Two losses being comonotonic gives us desirable properties within a financial setting.

Definition 4.3. *Comonotonic Additivity*

$$X//Y \Rightarrow \rho(X+Y) = \rho(X) + \rho(Y),\ X, Y \in \mathcal{X} \tag{CA}$$

We can interpret (CA) within a financial context as $X//Y$ implying they don't hedge eachother. This is a desirable property as it fully characterizes the lack of diversification benefits we would see from comonotonic securities.

Proposition 4.4. *Let ρ be a monetary risk measure satisfying (CA). Then, ρ has (PH).*

Proof. Suppose $X//Y \in \mathcal{X}$, and let $\alpha > 0$. By (CA), $\rho(\alpha(X+Y)) = \rho(\alpha X) + \rho(\alpha Y)$. Since ρ is monetary, by (CI) we get $\rho(\alpha(X+Y)) = \alpha(\rho(X) + \rho(Y))$. Applying (CA) one more time, we conclude $\rho(\alpha(X+Y)) = \alpha\rho(X+Y)$, as required. □

Theorem 4.5. *Let ρ be a monetary risk measure which is also law-determined and comonotonic additive. Then ρ has the following representation.*

$$\rho(X) = \rho_h(X) := \int_0^\infty dh(\bar{F}(x)),\ X \in \mathcal{X}, X \sim F$$

where h is an increasing function on $[0,1]$ such that $h(0) = 0$, and $h(1) = 1$.

Here, ρ_h is called the *distortion risk measure*, and h is the distortion function of ρ_h.

Example 4.6. We can see our beloved risk measures VaR and ES are also distorted risk measures. Take $\alpha \in (0,1)$, and let $X \sim F$ for some distribution F. Consider the distortion function

$$h(x) := \begin{cases} 1 & \text{If } 1-\alpha \leq x \leq 1 \\ 0 & \text{If } 0 < x < 1-\alpha \end{cases}$$

Then, we get the distorted risk measure

$$\rho_h(X) = \int_0^\infty dh(\bar{F}(x)) = \int_0^{F_X^{-1}(\alpha)} dx = \mathrm{VaR}_\alpha(X)$$

The following is a simpler alternative representation.

Theorem 4.7. *Let $X \sim F$ and F^{-1} be continuous on $[0,1]$. Then the distortion risk measure, ρ_h, can be represented by*

$$\rho_h(X) = \int_0^1 VaR_\alpha(X) dh(\alpha)$$

where h is a distortion function on [0,1].

Example 4.8. Continuing Example 4.6, if we consider the distortion function

$$\tilde{h}(x) := \begin{cases} 1 & \text{If } 1-\alpha \leq x \leq 1 \\ \frac{x}{1-\alpha} & \text{If } 0 < x < 1-\alpha \end{cases}$$

Then, we get the distorted risk measure

$$\rho_{\tilde{h}}(X) = \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{VaR}_u du = \mathrm{ES}_{\alpha}(X)$$

Remark 4.9. Due to the properties defining distorted risk measures, they give rise to clear and intuitive economic interpretations. These measures also lend us with considerable advantages surrounding their ease of estimation and computation, making them a popular choice among practitioners and academics. The next section will further emphasise the connections distorted risk measures have with economic decision theory, allowing the risk tolerance behaviours of economic agents to be integrated within our models.

5. Modelling Risk Aversion Through Risk Measures

When looking to incorperate risk aversion into a measure, it is natural to incorperate a concept called *stochastic dominance*. This raises the quesion: what conditions must ρ satisfy in order to retain preferential ordering between risks? Consider the risks $X, Y \in L^1$. We denote $X \succ Y$ to mean we prefer X over Y, and we say that a risk measure preserves stochastic ordering if $X \succ Y \Rightarrow \rho(X) \leq \rho(Y)$.

Definition 5.1 (1^{st} Order Stochastic Dominance (FSD))**.** If $\bar{F}_X(t) \leq \bar{F}_Y(t), \forall t \geq 0$, and $\bar{F}_X(t) < \bar{F}_Y(t)$ for some $t \geq 0$, then we say X has first-order stochastic dominance over Y, $X \succ_{1st} Y$

It's easy to see that all distorted risk measures preserve first-order stochastic dominance due to the distortion function being increasing.

Definition 5.2 (2^{nd} Order Stochastic Dominance (SSD))**.** We say X has second-order stochastic dominance over Y, $X \succ_{2nd} Y$, if

$$\int_x^{\infty} \bar{F}_X(t)dt \leq \int_x^{\infty} \bar{F}_Y(t)dt, \forall x \geq 0$$

This gives us the property known as *strong risk aversion* in economic decision theory.

$$X \succ_{2nd} Y \Rightarrow \rho(X) \leq \rho(Y) \tag{SRA}$$

We can see why (SRA) is a good requirement to have within risk measures due to its consistency with the notion of risk aversion. It turns out that distorted risk measures also retain (SRA), but only when its underlying distortion function is strictly concave.

Theorem 5.3. *A risk measure ρ satisfies (SRA) if for a strictly concave function h, one has the form*

$$\rho(X) = \int_0^{\infty} h(\bar{F}(x))dx$$

6. Current Applications of Distorted Risk Measures: *Regulatory Arbitrage*

We conclude this paper by looking into some of the most recent developments surrounding the use of distorted risk measures within mathematical finance, particularly focusing on the concept of regulatory arbitrage. The notion of an arbitrage opportunity can be thought of as seeing a unicorn, and is more commonly though of within the investment framework as essentially earning risk-free profits. However, arbitrage can be generalized to a broader range of scenarios while staying true to the idea of generating risk-free capital. One such scenario we discuss is regulatory arbitrage.

Definition 6.1. The set of *allocations* of risks $X \in \mathcal{X}$ is defined as

$$\mathbb{A}_n(X) := \left\{ (X_1, \cdots, X_n) \in \mathcal{X}^n \mid \sum_{i=1}^{n} X_i = X \right\}$$

We can leverage allocations of risks to interpret how arbitrage opportunities may arise within risk management. Firms may have incentives to reduce their regulatory capital requirements by splitting its business into n subsidies. We write $X = \sum_{i=1}^{n} X_i$, and compare $\rho(X)$ against $\sum_{i=1}^{n} \rho(X_i)$. So, $\rho(X) - \sum_{i=1}^{n} \rho(X_i)$ represents regulatory arbitrage. Formally,

Definition 6.2. For $X \in \mathcal{X}$, take

$$\Psi_\rho(X) := \inf \left\{ \sum_{i=1}^{n} \rho(X_i) \middle| \, n \in \mathbb{N}, (X_1, \cdots, X_n) \in \mathbb{A}_n(X) \right\}$$

Then, we define the *regulatory arbitrage of* ρ to be

$$\Phi_\rho(X) = \rho(X) - \Psi_\rho(X)$$

One key observation to be made is how $\Psi_\rho(X)$ represents the least amount of capital required according to ρ after dividing the risk of X. It is easy to see that $\Psi_\rho(X) \leq \rho(X)$. An interesting property which also arises from this definition is that $\Psi_\rho = \rho \Leftrightarrow \rho$ is sub-additive.

Definition 6.3. For a risk measure ρ, we can categorize it as follows.

1. If $\forall X \in \mathcal{X}, \Phi_\rho(X) = 0$, then ρ is free of regulatory arbitrage.
2. If $\forall X \in \mathcal{X}, \Phi_\rho(X) < \infty$, then ρ is of limited regulatory arbitrage.
3. If $\exists X \in \mathcal{X}, \Phi_\rho(X) = \infty$, then ρ is of unlimited regulatory arbitrage.
4. If $\forall X \in \mathcal{X}, \Phi_\rho(X) = \infty$, then ρ is of infinite regulatory arbitrage.

Example 6.4. Take VaR_α for some $\alpha \in (0, 1)$, then $\Phi_{\mathrm{VaR}_\alpha}(X) = \infty, \forall X \in \mathcal{X}$. Thus, VaR is of infinite regulatory arbitrage, which can be interpreted as being highly susceptible to manipulation.

The following theorem provides us with a generalized framework to identify risk measures which are of limited regulatory arbitrage.

Theorem 6.5. *If ρ is a distorted risk measure, then*

$$\rho \textit{ is of limited regulatory arbitrage} \Leftrightarrow \rho(X) \geq \mathbb{E}(X), \forall X \in \mathcal{X}$$

Remark 6.6. Theorem 6.5 gives us an intuitive approach to identifying risk measures which may be seen as more desirable from a regulator's perspective. This is due to the fact that a firm could be prevented from giving the false perception that they have sufficient capital reserves. Additionally, the use of distorted risk measures ties nicely with the theory we've already discussed. A risk measure bounded below the mean is also consistent with viewing

(SFA) through the lense of expected utility theory. This implies a distorted risk measure ρ which preserves second-order stochastic dominance tells us it's also of limited regulatory arbitrage.

What is quite interesting though is that ρ is free of regulatory arbitrage if and only if it's coherent - the framework originally looking to move beyond. This begs the question whether we should be considering alternative framework to coherence at all, and if regulators should allow risk measures to deviate from representations seen in Theorems 3.2 and 3.7.

7. Conclusion

Throughout our discussion, we've looked at one of the fundamental tasks within quantitative risk management and centred our focus on the development of an alternative framework to coherence. The proposed framework of distorted risk measures provides us with promising solutions to the concerns surrounding coherence. By starting with an axiomatic approach to constructing risk measures, we could identify concerns expressed by regulators which originally motivated the development of risk measure theory. Through the continuous improvement upon these primitive risk measures - many of which were motivated by commonly accepted financial and economic theories - we eventually developed the notion of coherence as we know it today. As we gained a better understanding of the machinery behind our risk measurement tools, we were able to break down the critical points which sparked the debate between VaR and ES.

After our discussion surrounding the coherent framework, we set out to address the criticisms we pointed out through the development of distorted risk measures. By stripping back down to the skeleton of monetary risk measures, we saw how incorporating properties such as law-determination and comonotic additivity allowed us to arrive to this alternative class of measures. These properties not only provide an intuitive economic interpretations but also align distorted risk measures with key principles in economic decision theory, such as stochastic dominance and risk aversion. By connecting these measures to concepts like first-order and second-order stochastic dominance, we demonstrated how distorted risk measures can effectively model the risk tolerance behaviors of economic agents.

Finally, we took a brief look at the applications of distorted risk measures through the study of regulatory arbitrage. Here, we highlighted how these measures can be used to identify and combat the risks posed by regulatory capital manipulation, providing us with valuable insights into the debate over the coherence framework and its alternatives. This inquiry into regulatory arbitrage expressed the importance of selecting risk measures which not only meet regulatory standards but also conform to the true economic risks faced by financial institutions.

In closing, distorted risk measures offer a robust and desirable alternative framework for assessing market risk. By questioning our initial assumptions surrounding coherence and searching for better results, we saw how we were able to relax the restrictions imposed by sub-additivity while simultaneously gain a deeper appreciation to the results this coherence granted us. As the field of risk management continues to evolve, the integration of distorted risk measures into regulatory frameworks will likely play a crucial role in shaping the future of financial risk management.

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