PMATH 450: LEBESGUE INTEGRATION AND FOURIER ANALYSIS

MARK GAGARINE

Abstract. This is a second course in real analysis, coming in the continuation of PMath 351. There are two main strands of the course: one of them concerns Hilbert spaces and elements of Fourier series for periodic functions (the so-called "harmonic analysis on the circle"). The other strand concerns the study of the Lebesgue measure on the real line and of the Lebesgue integral, which are then used in the treatment of the aforementioned harmonic analysis developments.

Contents

Part 1. Hilbert Space

1. Inner Product Space, Hilbert Space

1.1. May 6, Lecture 1.

Definition 1.1. Let X be a vector space over $\mathbb{R}$ (could also be $\mathbb{C}$). The *inner product* over X is a rule that assigns values $\langle x, y\rangle \in \mathbb{R}, \forall x, y \in X$, which satisfies the following rules.

Bilinearity:

(Bi-Lin 1) $$\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle = \alpha_1\langle x_1, y\rangle + \alpha_2\langle x_2, y\rangle, \forall x_1, x_2, y \in X, \alpha_1, \alpha_2 \in \mathbb{R}$$

(Bi-Lin 2) $$\langle x, \beta_1 y_1 + \beta_2 y_2\rangle = \beta_1\langle x, y_1\rangle + \beta_2\langle x, y_2\rangle, \forall x, y_1, y_2 \in X, \beta_1, \beta_2 \in \mathbb{R}$$

Symmetry:

(Sym) $$\langle x, y\rangle = \langle y, x\rangle, \forall x, y \in X$$

Positive Definite:

(Pos Def) $$\langle x, x\rangle > 0, \forall x \in X$$

The pair $(X, \langle\cdot,\cdot\rangle)$ is called an *inner product space* (ips).

Remark 1.2. (1) **Redundancy in (Bi-Lin):** Bi-Lin 1 + Sym $\Rightarrow$ Bi-Lin 2.

(2) **Rephrasing of (Pos Def):** Observe that $\langle 0_X, y\rangle = \langle y, 0_X\rangle, \forall y \in X$. *Why?*

$$2\cdot 0_X = 0_X \Rightarrow \langle 2\cdot 0_X, y\rangle = \langle 0_X, y\rangle \Leftrightarrow 2\langle 0_X, y\rangle = \langle 0_X, y\rangle \Rightarrow \langle 0_X, y\rangle = 0$$

This allows us to rephrase Pos Def as

(Pos Def′) $$\langle x, x\rangle \geq 0, \forall x \in X, \text{ with equality iff } x = 0_X.$$

This technique of checking if the inner product is 0 is a useful trick for proving identities.

Notation 1.3. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. For all $x \in X$, denote

$$||x|| = \sqrt{\langle x, x\rangle} \in [0, \infty)$$

Proposition 1.4. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.*

(1) ***Cauchy-Schwarz Inequality:***

(C-S) $$|\langle x, y\rangle| \leq ||x||\cdot||y||, \forall x, y \in X.$$

Moreover, C-S holds with equality iff x, y are dependent. That is, either one of $x = 0_X$, $y = 0_X$, or x, y are dependent. That is, $\exists \alpha \in \mathbb{R}$ such that $x = \alpha y$.

(2) ***Norm:*** *The function $||\cdot|| : X \to \mathbb{R}$ in Notation 1.3 is a norm on X.*

Proof. ... □

Definition 1.5 (Hilbert Space)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. When viewed as a normed vector space $(X, ||\cdot||)$ (hence a metric space (X, d) with $d(x,y) = ||x-y||$), if X is complete wrt d, then $(X, \langle\cdot,\cdot\rangle)$ is a *Hilbert Space.*

This comes from the fact that a complete normed vector space is a Banach space, so a Hilbert space is simply a collection within a Banach space.

Example 1.6 (A Hilbert Space)**.** Consider $X = \mathbb{R}^k$ with the *standard inner product.* That is, for $x = \left(x^{(1)}, \cdots, x^{(k)}\right)$, $\langle x, y\rangle = \sqrt{x^{(1)}y^{(1)} + \cdots + x^{(k)}y^{(k)}}$. We get $||x|| = \sqrt{\left(x^{(1)}\right)^2 + \cdots + \left(x^{(1)}\right)^2}$.

Example 1.7 (Not a Hilbert Space)**.**

$$\textbf{Recall: } X = c_{00} = \left\{x = \left(x^{(1)}, \cdots, x^{(k)}, \cdots\right) \mid \exists k_0 \in \mathbb{N} \text{ s.t. } x^{(k)} = 0, \forall k > k_0\right\}.$$

For $x, y \in X$, set $\langle x, y\rangle = \sum_{k=1}^{\infty} x^{(k)}y^{(k)} = \sum_{k=1}^{k_0} x^{(k)}y^{(k)}$. It's easy to see $(c_{00}, \langle\cdot,\cdot\rangle)$ is an inner product space. We denote the norm on c_{00} associated to $\langle\cdot,\cdot\rangle$ as $||\cdot||_2$,

$$||x||_2 = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{k=1}^{\infty}\left[x^{(k)}\right]^2}$$

However, we can find a cauchy sequence in c_{00} that doesn't converge wrt $||\cdot||_2$. Hence $(c_{00}, ||\cdot||_2)$ isn't complete, so $(c_{00}, \langle\cdot,\cdot\rangle)$ is not a Hilbert space.

1.2. **May 8, Lecture 2.** *How do we get a Hilbert space?*

Recall $(c_{00}, ||\cdot||_2)$ in Example 1.7 isn't complete, so how do we get a Hilbert space? We can *complete* $(c_{00}, ||\cdot||_2)$. That is, we embed c_{00} into a larger complete normed vector space Z, such that c_{00} is dense in Z.

Denote $\ell^2 = \left\{x = \left(x^{(1)}, \cdots, x^{(k)}, \cdots\right) \mid \sum_{k=1}^{\infty}\left[x^{(k)}\right]^2 < \infty\right\} \Leftrightarrow \sup\left\{\sum_{k=1}^{n}\left[x^{(k)}\right]^2 | n \in \mathbb{N}\right\}$.

The following is a series of claims showing ℓ^2 is complete, proved in Assignment 1 Q1-3.

Claim 1. For $x, y \in \ell^2$, $\sum_{k=1}^{\infty} x^{(k)}y^{(k)}$ converges absolutely. ... □

So we can put $\langle x, y\rangle = \sum_{k=1}^{\infty} x^{(k)}y^{(k)} = \lim_{n\to\infty}\sum_{k=1}^{n} x^{(k)}y^{(k)}$.

Claim 2. $\left(\ell^2, \langle\cdot,\cdot\rangle\right)$ is an inner product space ... □

Claim 3. ℓ_2 is complete wrt $||\cdot||_2$ ... □

So $\left(\ell^2, \langle\cdot,\cdot\rangle\right)$ is a Hilbert space. We can also see that $c_{00} \subseteq \ell^2$, leading to the final claim.

Claim 4. c_{00} is dense in ℓ^2 (wrt $||\cdot||$) ... □

Example 1.8. Pick $a < b \in \mathbb{R}$, and consider the vector space

$$C([a,b], \mathbb{R}) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

We saw in PMATH 351: $(C([a,b], \mathbb{R}), ||\cdot||_\infty)$ is a Banach space with

$$||f||_\infty = \sup\{|f(x)| \mid x \in [a,b]\} \tag{Sup-Norm}$$

In PMATH 450: For $f, g \in C([a,b],\mathbb{R})$, we denote

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx \in \mathbb{R}.$$

Claim. $(C([a,b],\mathbb{R}), \langle\cdot,\cdot\rangle)$ *is an inner product space.* Observe $\langle f, f\rangle = \int_a^b [f(x)]^2 dx$.

Verifying the inner product space axioms is straight forward, but special attention is needed for $\langle f, f\rangle = 0$ That is, when f is the zero function in $C([a,b],\mathbb{R})$, denoted $\underline{0} : [a,b] \to \mathbb{R}$, where $\underline{0}(x) = 0, \forall x \in [a,b]$.

Recall: (Pos Def′) states $\langle x, x\rangle \geq 0$ with equality iff $x = 0_X$. So we need to check

$$f(x) \text{ continuous}, \int_a^b [f(x)]^2 dx \Rightarrow f(x) = 0, \forall x \in [a,b]$$

□

So we get that $(C([a,b],\mathbb{R}), \langle\cdot,\cdot\rangle)$ is an inner product space. The associated norm is $\|\cdot\|_2$,

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b [f(x)]^2 dx}$$

2. Some Hilbert Space Geometry: *Distance to a closed convex set*

2.1. **May 10, Lecture 3.** *Recap of convex sets and some metrics*

Definition 2.1. Let X be a vector space over $\mathbb{R}$.

(1) The *line segment* between two points x, y in X is denoted by

$$Co(x,y) = \{tx + (1-t)y \mid t \in [0,1]\} \tag{Lin-Seg}$$

(2) The set $A \subseteq X$ is said to be *convex* if the following is satisfied.

$$x, y \in A \Rightarrow Co(x,y) \subseteq A \tag{Convex}$$

Remark 2.2. (1) Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$.
If $A \subseteq X$ is convex, then $\mathrm{Cl}(A)$ is also convex.

(2) Let (X, d) be a metric space, and let $A \subseteq X$ be a closed non-empty set.
Recall, in PMATH 351 we defined the *distance-to-A* function $d_A : X \to \mathbb{R}$,

$$d_A(x) = \inf\{d(x,a) \mid a \in A\} \tag{dist-to-A}$$

Some properties associated with this function included

- $d_A(x) \geq 0 \forall x \in X$, with $d_A(x) = 0 \Leftrightarrow x \in A$.
- d_A is continuous, and contractive to mean

$$|d_A(x) - d_A(y)| \leq d(x,y),\ \forall x, y \in X$$

Combining these facts, we introduce the first theorem of the course.

Theorem 2.3. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Suppose we're given a closed, convex, non-empty set $A \subseteq X$, and the point $x_0 \in X$.*

Then, there exists a unique point $a_0 \in A$ such that

$$\|x_0 - a_0\| = \inf\{\|x_0 - a\| \mid a \in A\} = d_A(x_0)$$

Proof The case where $x_0 \in A$ is clear. Indeed, if $x_0 \in A$, then $d_A(x_0) = 0$, and the unique point $a_0 \in A$ is x_0. So for the rest of the proof assume $x_0 \notin A$, so $d_A(x_0) > 0$.

Denote $\alpha = d_A(x_0) = \inf\{\|x_0 - a\| \mid a \in A\}$. It suffices to show $\exists a_0 \in A$ uniquely determined such that $\|x_0 - a_0\| = \alpha$.

Proof of Uniqueness[1]

Suppose for a contradiction that we have two distinct points $a_1 \neq a_2 \in A$ such that $\|x_0 - a_1\| = \|x_0 - a_2\| = \alpha$. Denote the distance between these points as $\beta = \|a_1 - a_2\|$. Note that $\beta > 0$ since $a_1 \neq a_2$.

Let $a_3 = \frac{1}{2}(a_1 + a_2)$ be the midpoint between our points a_1 and a_2, and observe $a_3 \in A$ since A is assumed to be convex. Question 5c says $\|x_0 - a_3\| = \sqrt{\alpha^2 - \frac{\beta^2}{4}}$

So then $\|x_0 - a_3\| < \alpha = \inf\{\|x_0 - a\| \mid a \in A\}$. A contradiction since the definition of α forces $\|x_0 - a\| \geq \alpha, \forall a \in A$. $\square$

2.2. **May 13, Lecture 4.** We start by completing the proof of Theorem 2.3.

Proof of Existence of a_0. Moving along with the proof we take not of a few facts.

$$\|x_0 - a\| \geq \alpha, \forall a \in A \tag{$\diamondsuit$}$$

$$\forall n \in \mathbb{N}, \exists a_n \in A \text{ s.t } \|x_0 - a_n\| < \alpha + \frac{1}{n}. \tag{$\diamondsuit\diamondsuit$}$$

From ($\diamondsuit\diamondsuit$) we get a sequence $(a_n)_{n=1}^{\infty}$ of points in A. The following are a list of claims finishing off the proof.

Claim 1. $\forall m, n \in \mathbb{N}$ we have

$$\begin{aligned} \|a_n - a_m\|^2 &\leq 4\alpha\left(\frac{1}{m} + \frac{1}{n}\right) + \left(\frac{2}{m^2} + \frac{2}{n^2}\right) \\ &\leq 4\alpha\left(\frac{1}{m} + \frac{1}{n}\right) + \left(\frac{2}{m} + \frac{2}{n}\right) \\ &= (4\alpha + 2)\left(\frac{1}{m} + \frac{1}{n}\right) \end{aligned}$$

[1]We leverage question 5c from Homework 1

Verification. Write $\|a_m - a_n\|^2 = \|(x_0 - a_n) - (x_0 - a_m)\|^2$, and by (Par Law),

$$(☎) \qquad = 2 \cdot \|x_0 - a_m\|^2 + 2 \cdot \|x_0 - a_n\|^2 - \|(x_0 - a_n) - (x_0 - a_m)\|^2$$
$$\leq 2\left(\alpha + \frac{1}{m}\right)^2 + 2\left(\alpha + \frac{1}{n}\right)^2 - 4\alpha^2$$

For the 3^{rd} term in ☎:

$$\begin{aligned} \|(x_0 - a_n) - (x_0 - a_m)\|^2 &= \|2x_0 - (a_n + a_m)\|^2 \\ &= \|2\left(x_0 + \frac{1}{2}(a_n + a_m)\right)\|^2 \\ (\diamondsuit) \qquad &= 2^2\|x_0 + \frac{1}{2}(a_m + a_n)\|^2 \geq 4\alpha^2 \end{aligned}$$

This gives us ☎ is bounded by $4\alpha\left(\frac{1}{m} + \frac{1}{n}\right) + \left(\frac{1}{m^2} + \frac{1}{n^2}\right)$, so we are done with *Claim 1.*

Claim 2. We now claim that $(a_n)_{n=1}^{\infty}$ is a cauchy sequence.

Verification. Let $\epsilon > 0$ be given. We wish to find an $n_0 \in \mathbb{N}$ such that

$$m, n \geq n_0 \Rightarrow \|a_m - a_n\| < \epsilon.$$

Pick n_0 such that $\frac{8\alpha+4}{n_0} < \epsilon^2$. Then $\forall m, n \geq n_0$,

$$\begin{aligned} \|a_m - a_n\|^2 \leq (4\alpha + 2)\left(\frac{1}{m} + \frac{1}{n}\right) &\leq (4\alpha + 2)\left(\frac{1}{m_0} + \frac{1}{n_0}\right) \\ &= \frac{8\alpha + 4}{n_0} < \epsilon^2 \end{aligned}$$

Hence $\|x_n - x_m\| < \epsilon, \forall m, n \geq n_0$ as required, and we are done with *Claim 2.*

Claim 3. We now claim that $(a_n)_{n=1}^{\infty}$ converges to the limit $a_0 \in A$.

Verification. Combining the above claim that $(a_n)_{n=1}^{\infty}$ is cauchy and using the fact since X is a Hilbert space, hence complete, it follows that $a_n \overset{n\to\infty}{\to} a_0 \in X$. But since $a_n \to a_0$, $a_n \in A$, and A is closed, it follows that $a_0 \in A$. So we are done with *Claim 3.*

Claim 4. We now claim the point $a_0 \in A$ in *Claim 3* satisfies $\|x_0 - a_0\| = \alpha$.

Verification. For all $n \in \mathbb{N}$, from $(\diamondsuit)$ we have $\alpha \leq \|x_0 - a_0\| \leq \|x_0 - a_0\| + \|a_n - a_0\|$. Recall from $(\diamondsuit\diamondsuit)$ we have $\|x_0 - a_0\| \leq \alpha + \frac{1}{n}$. So we get

$$\|x_0 - a_0\| + \|a_n - a_0\| < \alpha + \frac{1}{n} + \|a_n - a_0\| \overset{n\to\infty}{\to} \alpha + 0 = \alpha$$

By applying the squeeze theorem we see $\|x_0 - a_0\| = \alpha$, thus completing the proof. □

Remark 2.4. In Theorem (2.3), we note that the completeness of X is an essential part to the hypothesis.

3. Orthogonal Projection onto a Closed Linear Subspace

3.1. May 15, Lecture 5.

Remark 3.1. Note that for the normed vector space $(X, \|\cdot\|)$ over $\mathbb{R}$, if we consider the closed linear subspace $W \subseteq X$, then W is a non-empty, closed, convex set.

Definition 3.2. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. $x, y \in X$ are said to be *orthogonal* to each other when they satisfy $\langle x, y \rangle = 0$, denoted

$$x \perp y$$

If we consider $A, B \subseteq X$, we say that A and B are *orthogonal* to each other to mean that

$$x \perp y, \forall x \in A, y \in B$$

A few observations to point out would be that $x \perp y$ is the same as $y \perp x$, and we have for $x \in X$, $x \perp x \Leftrightarrow x = 0_X$. We also have the special case for $x \in X, B \subseteq X$, we write $x \perp B$ to mean that $x \perp y, \forall y \in B$.

Theorem 3.3. *Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and suppose we're given a closed linear subspace, $W \subseteq X$, and an $x_0 \in X$. Further, define $\alpha := d_W(x_0) = \inf\{\|x_0 - w\| \mid w \in W\}$ and let w_0 be the unique point in W such that $\|x_0 - w_0\| = \alpha$.*

Then,

$$(x_0 - w_0) \perp W \tag{OP Perp}$$

Proof We wish to prove $\langle x_0 - w_0, w \rangle = 0, \forall w \in W$.
Assume for a contradiction that $\exists w_1 \in W$ such that $\langle x_0 - w_0, w_1 \rangle \neq 0$. Note by positive definiteness, we are sure that $w_1 \neq 0_X$, hence $\|w_1\| > 0$. So, the value $\langle x_0 - w_0, w_1 \rangle \in \mathbb{R} \setminus \{0\}$. WLOG, assume it's positive. If it's negative, then replace w_1 by $-w_1$. Observe $\forall t \in \mathbb{R}$, we have $w_0 + t w_1 \in W$, and therefore, $\|x_0 - (w_0 + t w_1)\| \geq d_W(x_0) = \alpha$. Hence,

$$\|x_0 - (w_0 + t w_1)\|^2 \geq \alpha^2 \tag{●}$$

Now, pick a $t \in \mathbb{R}$ and compute

$$\|x_0 - (w_0 + t w_1)\|^2 = \|(x_0 - w_0) + t w_1\|^2 = \langle (x_0 - w_0) - t w_1, (x_0 - w_0) - t w_1 \rangle$$

$$= \langle x_0 - w_0, x_0 - w_0 \rangle - 2\langle x_0 - w_0, t w_1 \rangle + t^2 \langle w_1, w_1 \rangle$$

$$= \underbrace{\|x_0 - w_0\|^2}_{\alpha^2} - 2t\langle x_0 - w_0, w_1 \rangle + t^2 \|w_1\|^2$$

$$= \alpha^2 - t\|w_1\|^2 \left[\underbrace{\frac{2\langle x_0 - w_0, w_1 \rangle}{\|w_1\|^2}}_{=:\delta} - t \right]$$

We thus obtain

$$\|x_0 - (w_0 + t w_1)\|^2 = \alpha^2 - t\|w_1\|^2 \cdot (\delta - t) \tag{■}$$

Since we assume $\langle x_0 - w_0, w_1\rangle > 0$, we get $\delta = 2\langle x_0 - w_0, w_1\rangle/\|w_1\|^2 \in (0, \infty)$. Comparing (●) with (■), we see $0 < t < \delta$ creates a conflict. If $t = \delta/2$, then (■) gives

$$\|x_0 - (w_0\frac{\delta}{2}w_1)\|^2 = \alpha^2 - \frac{\delta^2\|w_1\|}{4} < \alpha^2$$

This is a contradiction with (●). □

Notation: For every $x_0 \in X$, let $P_W(x_0)$ denote $w_0 \in X$ which is the unique point in W who's at minimal distance from x_0, called the *orthogonal projection* of x_0 into W. The name is used in connection to Theorem 3.3 which asserts that $(x_0 - P_W(x_0)) \perp W$.

Definition 3.4. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and take W to be a closed linear subspace of X. The *orthogonal compliment* of W, $W^\perp$, is defined

$$W^\perp := \{x \in X \mid x \perp W\} = \{x \in X \mid x \perp w, \forall w \in W\}$$

Note that $W^\perp$ is also a closed linear subspace. Hence, $W \rightsquigarrow W^\perp$ is an operation we do with closed linear subspaces of X.

3.2. **May 17, Lecture 6.** *Do we know anything about the converse of Theorem (3.3)?*

Remark 3.5. *Converse to Theorem 3.3* Let $(X, \langle\cdot,\cdot\rangle)$, $W \subseteq X$, and $x_0 \in X$ be as in Theorem 3.3. Suppose we found a point, $w_0 \in W$, such that $(x_0 - w_0) \perp W$. Then,

(OP Metric) $$\|x_0 - w\| > \|x_0 - w_0\|, \ \forall w \in W \setminus \{w_0\}$$

Why is this? For $w \neq w_0 \in W$ with $x_0 - w = (x_0 - w_0) + (w_0 - w)$, observe that $x_0 - w_0 \perp w_0 - w$. This is because $w_0 - w \in W$ and $(x_0 - w) \in W$. So,

$$\|x_0 - w\|^2 = \|(x_0 - w_0) + (w_0 - w)\|^2 \overset{Pythag.}{=} \|x_0 - w_0\|^2 + \|w_0 - w\|^2 > \|x_0 - w_0\|^2$$

Since $\|w_0 - w\| > 0$ as $w \neq w_0$. This implies w_0 must be the unique point in W which is at minimal distance from x_0.

Furthermore, we view $P_W(x_0)$ as a function mapping from $X \to X$ which is a contractive linear operator on X (Shown in Question 4 of Homework Assignment 2).

Remark 3.6. Recall our definition (3.4) for the *orthogonal compliment* of W, $W^\perp$. We have that $W^\perp$ is also a closed linear subspace of X!

But is there a formula for $(W_1 \cap W_2)^\perp$*?* For convenience, let's rewrite $W^\perp$ as

$$W^\perp = \bigcap_{w\in W} \underbrace{\{x \in X \mid \langle x, w\rangle = 0\}}_{:= Y_w} = \bigcap_{w\in W} Y_w$$

To check that $W^\perp$ is a closed linear subspace of X, it suffices to show each Y_w is so.

We already know

→ arbitrary intersections of linear subspaces are still linear subspaces, and

→ arbitrary intersections of closed sets are closed.

We are then left to fix a point $w \in W$, and check both

$$\left.\begin{array}{rcl} 0_X \in Y_w & & \\ x, y \in Y_w & \Rightarrow & x + y \in Y_w \\ x \in Y_w, \alpha \in \mathbb{R} & \Rightarrow & \alpha x \in Y_w \end{array}\right\} Y_w \text{ is a vector space of } X$$

$$\left.\begin{array}{c}\text{If } {\scriptstyle (x_n)_{n=1}^{\infty}\in Y_w^{\mathbb{N}}} \\ \text{with } {\scriptstyle x_n \xrightarrow{\|\cdot\|} x\in X}\end{array} \quad \Rightarrow \quad x\in Y_w \quad \right\} Y_w \text{ is closed.}$$

3.3. **May 21, Lecture 7.** In this lecture we introduced a third characterization of $P_W(x_0)$ by re-introducing a concept previously found in a course on Linear Algebra.

Proposition 3.7. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and let $W \subseteq X$ be a closed linear subspace. Then, every $x \in X$ can be decomposed as a sum $x = w + y$, where $w \in W$, and $y \in W^{\perp}$. Moreover, this decomposition is unique.*

Proof. **Existence** Every $x \in X$ can be written as

$$x = \underbrace{P_W(x)}_{w} + \underbrace{x - P_W(x)}_{y}$$

We have $w \in W$ by definition, and $y \perp w$ by OP Perp, hence $y \in W^{\perp}$. So we get $x = w + y$.

Uniqueness Suppose $x \in X$ is written as $x = w + y = w' + y'$ and $w, w' \in W, y, y' \in W^{\perp}$. Observe that $w + y = w' + y' \Rightarrow w - w' = y' - y =: z$. Thus we have $w - w' \in W$, and $y' - y \in W^{\perp}$, since $W, W^{\perp}$ are linear subspaces. So $z \in W \cap W^{\perp} = \{0_X\}$. Hence $z = 0_X$, therefore $w = w'$, and $y' = y$. □

Remark 3.8. Proposition (3.7) and its proof gives us another view on what is the orthogonal projection, $P_W(x)$.

(OP 3) $\quad P_W(x)$ is the w part in the unique decomposition $x = w + y$

Proposition 3.9. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and $W \subseteq X$ be a closed linear subspace. Consider the linear operators $P_W : X \to X$, and $P_{W^{\perp}} : X \to X$. Then,*

$$P_W(x) + P_{W^{\perp}}(x) = x, \forall x \in X$$

(That is, $P_W + P_{W^{\perp}} = I$, where $I : X \to X$, is the idenity linear operator, $I(x) = x, \forall x \in X$)

Proof. ... □

Corollary 3.10. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and $W \subseteq X$ be a closed linear subspace. Then,*

$$\left(W^{\perp}\right)^{\perp} = W$$

4. Orthonormal Basis: *For a seperable, infinite dimensional Hilbert space*

4.1. **May 22, Lecture 8.**

Remark and Notation 4.1. Let X be a vector space over $\mathbb{R}$. Then for any subset, $S \subseteq X$, for the linear span of S we use the notation span(S).

How do we describe span(S)? We use 2 conventions to characterize the span of S.

Convention 1:

$$\text{span}(S) := \left\{ x \in X \;\middle|\; \begin{array}{c} n \in \mathbb{N}, x_1, \cdots, x_n \in S, \alpha_1, \cdots, \alpha_n \in \mathbb{R}, \\ \text{such that } x = \alpha_1 x_1 + \cdots + \alpha_n x_n \end{array} \right\}$$

Convention 2: We can also describe span(S) as the smallest linear subspace of X containing S. That is, we have

$$\begin{array}{rc} i) & \text{span}(S) \subseteq X, \text{and} \\ ii) & \begin{array}{c}\text{If } V\subseteq X \text{ is a linear subspace of } X \\ \text{such that } S\subseteq V, \text{ then span}(S)\subseteq V\end{array} \end{array}$$

Why does (ii) hold? First note if W is a linear subspace such that $W \supseteq S \Rightarrow W \supseteq \text{span}(S)$. Now from PMATH351 we have W closed and $W \supseteq \text{span}(S) \Rightarrow W \supseteq \text{cl}(\text{span}(S)) \supseteq \text{span}(S)$.

Stepping back into real analysis, we now introduce some new notation.

Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. Then for any $S \subseteq X$, we denote

$$\text{clspan}(S) := \text{cl}(\text{span}(S))$$

Alternatively: We can also characterize the clspan of S in a similar fashion to span(S), namely, it's now the smallest *closed* subspace of X containing S.

4.2. **May 24, Lecture 9.**

Proposition 4.2. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. Then the following are equivalent.*

(1) X is infinite dimensional and separable.
(2) One can find an increasing chain of linear subspaces, $X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$, of X such that $\dim(X_n) = n$, and $\cup_{n=1}^{\infty} X_n$ is dense in X.

Proof. (1) $\Rightarrow$ (2)**:** Q5 on Homework Assignment 2
(2) $\Rightarrow$ (1)**:** Q1 on Homework Assignment 3

□

Remark 4.3. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $\forall n \in \mathbb{N}$, its first n elements, $x_1, \cdots, x_n$, form a linear basis in X. Let $S = \{x_1, \cdots, x_n, \cdots\} \subseteq X$, which has no repetitions, to mean $x_i \neq x_j, \forall i \neq j \in \mathbb{N}$, due to the linear independence of $x_1, \cdots, x_n$ when $n > \max\{i, j\}$. **Observe:** that clspan$(S) = X$.

Indeed, $\forall n \in \mathbb{N}$ we have $\text{span}(S) \supseteq \text{span}(x_1, \cdots, x_n) = X_n$. Hence, $\text{span}(S) \supseteq \cup_{n=1}^{\infty} X_n$, and therefore,

$$\text{clspan}(S) = \text{cl}(\text{span}(S)) \supseteq \text{cl}(\cup_{n=1}^{\infty} X_n) = X$$

$$\Rightarrow \text{clspan}(S) = X$$

The sequence $(x_n)_{n=1}^{\infty}$ with this property is said to be a *total sequence* in X.
Warning: This is not saying S is dense, but rather that its span is dense in X.

In the setting up of Proposition (4.2), all X_n's are sure to be closed linear subspaces.

→ In general, for a normed vector space over $\mathbb{R}$, $(X, \|\cdot\|)$, with $V \supseteq X$, we have that

$$\dim(V) < \infty \Rightarrow V \text{ is closed.}$$

→ This is because V is complete in the metric associated to $\|\cdot\|$, which is in turn a consequence of the EVT.

4.3. **May 27, Lecture 10.**

Proposition 4.4. *Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{R}$. Assume X is both separable and infinite dimensional. Let $X_1 \subseteq \cdots X_n \subseteq \cdots$ be linear subspaces in X (as in Proposition 4.2). Then, we can find a sequence, $(\xi_n)_{n=1}^{\infty}$ in X such that*

$$span(\xi_1, \cdots, \xi_n) = X_n, \forall n \in \mathbb{N} \tag{✎}$$

$$\xi_i \perp \xi_j, \forall i \neq j \in \mathbb{N} \tag{✎✎}$$

$$\|\xi_i\| = 1, \forall i \in \mathbb{N} \tag{✎✎✎}$$

We refer to (✎✎) and (✎✎✎) by saying $(\xi_n)_{n=1}^{\infty}$ is an orthonormal sequence in X.

Proof. Question 1 (a) in Homework Assignment 3 provides us with a sequence $(x_n)_{n=1}^{\infty}$ in X such that for every $n \in \mathbb{N}$, the vectors $x_1, \cdots, x_n$ form a linear basis for X_n. We will use Gram-Schmidt to convert the x_n's into an orthonormal sequence. Formally, we proceed by induction on n.

Base Case: Put $\xi_1 = \frac{1}{\|x_1\|}x_1 \in X_1$. Note that $\|x_1\| \neq 0$ since X_1 has dimension 1, of which x_1 forms the basis.

Inductive Step: Suppose for some $n \geq 1$, we have constructed $\xi_1, \cdots, \xi_n$ such that $\|\xi_1\| = \cdots = \|\xi_n\| = 1$, $\xi_i \perp \xi_j, \forall 1 \leq i < j \leq n$, and $\text{span}(\xi_1, \cdots, \xi_n) = X_n = \text{span}(x_1, \cdots, x_n)$. We look at $X_{n+1} = \text{span}(x_1, \cdots, x_n, x_{n+1})$ and put $\eta := x_{n+1} - (t_1\xi_1 + \cdots + t_n\xi_n)$, with $t_i = \langle x_{n+1}, \xi_i\rangle, 1 \leq i \leq n$. The following claims are used complete the proof.

- $\eta \neq 0_X$
- $\langle \eta, \xi_i\rangle = 0, \forall 1 \leq i \leq n$, and
- $\text{span}(\xi_1, \cdots, \xi_n, \eta) = \text{span}(x_1, \cdots, x_n, x_{n+1}) = X_{n+1}$.

Finally, put $\xi_{n+1} = \frac{1}{\|\eta\|}\eta$ since $\|\eta\| \neq 0$ by claim 1, and observe $\|\xi_{n+1}\| = 1$. So we get

$$\langle \xi_{n+1}, \xi_i\rangle = \frac{1}{\|\eta\|}\langle \eta, \xi_i\rangle \overset{claim\ 2}{=} 0, \forall 1 \leq i \leq n.$$

So, $\text{span}(\xi_1, \cdots, \xi_{n+1}) = \text{span}(x_1, \cdots, x_{n+1})$, completing the proof. $\square$

Example 4.5. Consider the space ℓ^2 as in Homework Assignment 1, and for an increasing chain of X_n's, use $X_n = \{(x^{(1)}, \cdots, x^{(n)}, 0, \cdots) \mid x^{(k)} \in \mathbb{R}, 1 \leq k \leq n\}$.

Definition 4.6. Let $(X, \langle\cdot,\cdot\rangle)$ be a separable, infinite dimensional Hilbert space on $\mathbb{R}$. A sequence satisfying Proposition 4.4 is said to be an *orthonormal basis* for $(X, \langle\cdot,\cdot\rangle)$.

5. Coefficients with respect to an Orthonormal Basis

For the remainder of the chapter, we fix a separable, infinite dimensional Hilbert space, $(X, \langle\cdot,\cdot\rangle)$, and an orthonormal basis, $(\xi_n)_{n=1}^{\infty}$, for X.

5.1. **May 30, Lecture 11.**

Definition 5.1. $\forall x \in X$, the sequence $(\langle x, \xi_n\rangle)_{n=1}^{\infty}$ is (for now) called a *sequence of ξ-coefficients of x.*[2]

Remark 5.2. *Given this sequence, can we get x back?*
Let $B := \{\xi_1, \cdots, \xi_n, \cdots\} \subseteq X$, which has no repetitions *(since they're a basis, hence linearly independent).*

Observe that $\text{clspan}(B) = X$. Indeed, in terms of our notation with $X_n = \text{span}(\xi_1, \cdots, \xi_n)$, we get $\text{span}(B) = \cup_{n=1}^{\infty} X_n \Rightarrow \text{clspan}(B) = \text{clspan}(\cup_{n=1}^{\infty} X) = X$. As a concequence, if $z \in X$ such that $z \perp \xi_i, \forall i \in \mathbb{N}$, it follows that $z = 0_X$.
Indeed, $z \perp \xi_n, \forall n \in \mathbb{N} \Rightarrow z \in B^{\perp} \Rightarrow B^{\perp} \overset{HwkA3,Q2}{=} (\text{clspan}(B))^{\perp} = X^{\perp} = \{0_X\}$. So $z = 0_X$.

Proposition 5.3. *If x, x' have the same sequence of ξ-coefficients, then $x = x'$.*

Proof. ... $\square$

[2]This is in relation to *Fourier coefficients*, which we have yet to "discover".

Theorem 5.4. *Rieze-Fisher Theorem*
Take $x \in X$, and define the sequence of ξ-coefficients, $(c_n)_{n=1}^{\infty}$ of x. For every $n \in \mathbb{N}$, let $x_n := c_1\xi_1 + \cdots + c_n\xi_n$. Then,

(1)
$$\forall n \in \mathbb{N}, x_n = P_{X_n}(x)$$

(2)
$$\lim_{n\to\infty} \|x_n - x\| = 0$$

Note that P_{X_n} is the orthogonal projection onto $X_n := \text{span}\{\xi_1, \cdots, \xi_n\}$ as in Proposition 4.4.

Proof of (1) Fix an $n \in \mathbb{N}$ and look at $x_n = \sum_{i=1}^{n} c_i\xi_i$.

Claim 1: $(x - x_n) \perp \xi_j, \forall 1 \leq j \leq n$.

$$\langle x - x_n, \xi_j\rangle = \langle x, \xi_j\rangle - \langle x_n, \xi_j\rangle = c_j - \left\langle \sum_{i=1}^{n} c_i\xi_i, \xi_j \right\rangle$$

$$= c_j - \sum_{i=1}^{n} c_i\langle \xi_i, \xi_j\rangle = c_j - c_j = 0$$

Claim 2: $(x - x_n) \in X_n^{\perp}$.

$$x - x_n \in \{\xi_1, \cdots, \xi_n\}^{\perp} = (\text{span}\{\xi_1, \cdots, \xi_n\})^{\perp} = X_n^{\perp}$$

Claim 3: $x_n = P_{X_n}(x)$.

Decompose $x = x_n + (x - x_n)$, where $x_n \in X_n$ by definition, and $(x - x_n) \in_n^{\perp}$ by claim 2. This is the unique decomposition

$$x = w + y, w \in W, y \in W^{\perp}$$

Here, $W = X_n, w = x_n$, and $y = x - x_n$. Hence, $P_W(x) = x_n$ is the "w" part of the unique decomposition of x.

□

Proof of 2 Pick an $\epsilon > 0$. We want to find $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\|x - x_n\| < \epsilon$.

Since $\cup_{n=1}^{\infty} X_n$ is dense in X, we can find a $y \in \cup_{n=1}^{\infty} X_n$ such that $\|y - x\| < \epsilon$. Because $y \in \cup_{n=1}^{\infty} X_n$, we can pick an n_0 such that $y \in X_{n_0}$. Indeed, $\forall n \geq n_0$, we have

$$y \in X_{n_0} \subseteq X_n \Rightarrow y \in X_n \Rightarrow \epsilon > \|x - y\| \overset{OP-Metric}{\geq} \|x - P_{X_n}(x)\| = \|x - x_n\|$$

Where $x_n = P_{X_n}(x)$ by part 1. □

5.2. **June 1, Lecture 12.**

Remark 5.5. *Interpretation of Rieze-Fisher*

We can rephrase (1) as $x_n := \sum_{n=1}^{\infty} c_i\xi_i$. Then we get $x_n \xrightarrow{\|\cdot\|} x$, that is,

$$\|\cdot\| - \lim_{n\to\infty}\left(\sum_{i=1}^{n} c_i\xi_i\right) = x$$

This equation can be written as the $\|\cdot\|$-convergent series,

$$x = \sum_{n=1}^{\infty} c_n\xi_n \tag{R-F}$$

Proposition 5.6. *Using the framework as above, pick $x \in X$, and let $(c_n)_{n=1}^{\infty}$ be its sequence of ξ-coefficients. Then,*

$$\|x\|^2 = \sum_{i=1}^{\infty} c_i^2 \tag{Parseval}$$

Proof $\forall n \in \mathbb{N}$, let $x_n = \sum_{i=1}^{n} c_i\xi_i$ as in Theorem 5.4. We have $x_n \longrightarrow x$ by Theorem 5.4.2, which implies $_n\| \to \|x\|$. So, $|\|x_n\| - \|x\|| \leq \|x_n - x\|$. Then, we get

$$\|x_n\|^2 = \lim_{n\to\infty} \infty\|x_n\|^2 = \lim_{n\to\infty} \|c_1\xi_1 + \cdots + c_n\xi_n\|^2$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} c_i^2\right] = \sum_{i=1}^{\infty} c_i^2$$

Observe that $\|c_1\xi_1 + \cdots + c_n\xi_n\|^2 = c_1^2 + \cdots + c_n^2$ is a generalization of the Pythagorean Theorem. □

Remark 5.7. *How to interpret Parseval's formula*

Parseval says a vectors sequence of ξ-coefficients $c = (c_1, \cdots, c_n, \cdots) \in \ell^2$, and $\|c\|_{\ell^2}^2 = \|x\|^2$. Thus, we can find a function $\varphi : X \to \ell^2$.

$$\varphi(x) = (\langle x, \xi_1\rangle, \cdots, \langle x, \xi_n\rangle, \cdots) \tag{ℓ^2-iso}$$

Properties of φ:

1. It's immediate to see that φ is linear
2. Parseval says φ is an isomorphic linear map. This gives us the notation for an isomorphic linear map between two Hilbert spaces:
$$\|\varphi(x)\|_{\ell^2} = \|x\|_X, \forall x \in X$$
3. Our map $\varphi : X \to \ell^2$ tells us $\varphi(\xi_n) = e_n$, with $e_n = (0, \cdots, 0, 1, 0, \cdots)$, the standard basis vector.
4. $\varphi : X \to \ell^2$ is bijective, hence its what one calls a Hilbert space isomorphism.

Moral of studying φ: We get $X \approx \ell^2$, an isomorphic hilbert space. Hence, any two infinite dimensional, separable Hilbert spaces are isomorphic to each other ($X \approx \ell^2 \approx Y$). The precise statement $X \approx Y$ is in Homework Assignment 4, Question 4.

Part 2. The Lebesgue Measure

6. Towards the Lebesgue Measure on $[a, b]$

6.1. June 3, Lecture 13.

Remark 6.1. *Where do we proceed from chapters 4 and 5?*
In chapters 4 and 5, we had a Hilbert space, $(X, \langle\cdot,\cdot\rangle)$, with an orthonormal basis, $(\xi_n)_{n=1}^{\infty}$, but we now focus on a special case of X. Start from $X_0 = C([-\pi,\pi],\mathbb{R})$, with the inner product and associated norm found in Example (1.8). Note we have an orthonormal basis provided by trigonometric functions, as found in Question 2 on Homework Assignment 4.

An issue to confront is that $(X_0, \langle\cdot,\cdot\rangle)$ isn't complete. However, we can use a Hilbert space, $(X, \langle\cdot,\cdot\rangle)$, (and an inner product which extends $\langle\cdot,\cdot\rangle$ onto X) which is a completion of X_0. *Is this X a space of functions on $[-\pi,\pi]$?* Almost yes: We will get $X = L^2([-\pi,\pi])$ in the sense of the Lebesgue measure, where $f, g : [-\pi,\pi] \to \mathbb{R}$ are identified when $f(x) = g(x)$, "almost everywhere" on $[-\pi,\pi]$.

Remark 6.2. *What exactly is the Lebesgue measure on the interval $[a, b]$?*
The Lebesgue measure is an advanced version of the notion of length for a set $S \subseteq [a, b]$. So, we must identify a convenient collection of subsets $S \subseteq [a, b]$ of which we know how to assign a length, $\lambda(S)$.

7. The Notion of σ-Algebra

Fix for the chapter a non-empty set, Ω, our "space of points".

7.1. June 5, Lecture 14.

Definition 7.1. We say a collection of sets $\mathcal{A} \subseteq 2^{\Omega}$ is a *σ-algebra* to mean it satisfies the following;

(σ-Alg-1) $$\emptyset \in \mathcal{A}$$

(σ-Alg-2) $$A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A}$$

(σ-Alg-3) $$(A_n)_{n=1}^{\infty} \in \mathcal{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$$

Remark 7.2. Let $\mathcal{A} \subseteq 2^{\Omega}$ be a σ-algebra . Then, we can do any kind of set operations we want, involving countably many sets from $\mathcal{A}$, and the results of these operations still remain in $\mathcal{A}$.

- $\Omega \in \mathcal{A}$
- *Stability under finite intersections.*

For $A_1, \cdots, A_k \in \mathcal{A}, \forall k \in \mathbb{N} \Rightarrow A_1 \cap \cdots \cap A_k \in \mathcal{A}$. This is from taking $A_{k+1} = \cdots = \Omega$, then $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$ by

- *Stability under countable unions.*

...

- *Stability under finite unions.*

...

- *Stability under set difference.*

...

Lemma 7.3. *Let $(\mathcal{A}_i)_{i\in I}$ be a family of σ-algebras of subsets of Ω, and let $\mathcal{A} = \cap_{i\in I}\mathcal{A}_i$. Then, $\mathcal{A}$ is a σ-algebra.*

Proof **σ-Alg-1:** We have $\emptyset \in \mathcal{A}_i, \forall i \in I$, by σ-Alg-1 for $\mathcal{A}_i$. Hence, $\emptyset \in \cap_{i\in}\mathcal{A}_i$.

σ-Alg-2: Pick $A \in \mathcal{A}$, hence $A \in \mathcal{A}_i, \forall i \in I$, and look at $\Omega \setminus A$. We have $\Omega \setminus A \in \mathcal{A}_i, \forall i \in$ by σ-Alg-2 for $\mathcal{A}_i$. This means $\Omega \setminus A \in \cap_{i\in I}\mathcal{A}_i \in \mathcal{A}$.

σ-Alg-3: Pick $(A_n)_{n=1}^{\infty} \in \mathcal{A}$, and look at $\cap_{n=1}^{\infty} A_n$. Then, we have $\cap_{n=1}^{\infty} A_n \in \mathcal{A}_i$ by σ-Alg-3 for $\mathcal{A}_i$ since $(A_n)_{n=1}^{\infty} \in A_i$, for each $i \in I$. So, $\cap_{n=1}^{\infty} \in \mathcal{A}$.

□

Proposition 7.4. *Let $\mathcal{C} \subseteq 2^{\Omega}$. Then, there exists some unique $\mathcal{A}_0 \subseteq 2^{\Omega}$ such that;*

- *$\mathcal{A}_0$ is a σ-algebra, and $\mathcal{A}_0 \supseteq \mathcal{C}$*
- *Whenever $\mathcal{A} \subseteq 2^{\Omega}$ is a σ-algebra such that $\mathcal{A} \supseteq \mathcal{C}$, then $\mathcal{A} \supseteq \mathcal{A}_0$.*

Proof **Existence:** Let $(\mathcal{A}_i)_{i\in I}$ be the family of all σ-algebras which contain the given $\mathcal{C}$. Such σ-algebras are sure to exits, 2^{Ω} is one of them. Let $\mathcal{A}_0 = \cap_{i\in I}\mathcal{A}_i$, which is a σ-algebra by Lemma 7.3. Moreover, from $\mathcal{C} \in \mathcal{A}_i, \forall i \in I$, it follows that $\mathcal{C} \in \cap_{i\in I}\mathcal{A}_i = \mathcal{A}_0$. Hence, (i) is satisfied. On the other hand, let $\mathcal{A} \subseteq 2^{\Omega}$ be a σ-algebra such that $\mathcal{A} \supseteq \mathcal{C}$. Then, $\mathcal{A}$ is counted somewhere in $(\mathcal{A}_i)_{i\in I}$. Hence, $\exists j \in I$ such that $\mathcal{A} = \mathcal{A}_j$. This ensures that $\mathcal{A} = \mathcal{A}_j \supseteq \cap_{i\in I}\mathcal{A}_i = \mathcal{A}_0 \Rightarrow \mathcal{A} \supseteq \mathcal{A}_0$. This verifies (ii).

Uniqueness: Suppose we're given $\mathcal{A}_0' \subseteq 2^{\Omega}$ with the same properties as $\mathcal{A}_0$.

- $\mathcal{A}_0'$ is a σ-algebra, and $\mathcal{A}_0' \supseteq \mathcal{C}$
- Whenever $\mathcal{A} \subseteq 2^{\Omega}$ is a σ-algebra such that $\mathcal{A} \supseteq \mathcal{C}$, it follows that $\mathcal{A} \supseteq \mathcal{A}_0'$

So, $\mathcal{A}_0$ satisfies $(i) + (ii)$, and $\mathcal{A}_0'$ satisfies $(i') + (ii')$. Observe that $(i) + (ii')$ implies $\mathcal{A}_0 \supseteq \mathcal{A}_0'$. Similarly, $(i') + (ii)$ implies $\mathcal{A}_0' \supseteq \mathcal{A}_0$. So, $\mathcal{A}_0 = \mathcal{A}_0'$, as required.

□

Definition 7.5. Take $\mathcal{C} \subseteq 2^{\Omega}$. The σ-algebra found in Proposition 7.4 is called the *σ-algebra generated by* $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$.

An important example discussed in class found in Homework Assignment 5 is when (Ω, d) is a metric space and we consider its *Borel* σ-algebra, $\mathcal{B} := \sigma(\mathcal{T})$, where $\mathcal{T}$ is the topology of Ω.

8. Finite Positive Measure

8.1. June 7, Lecture 15.

Definition 8.1. A *finite positive measure* on a σ-algebra $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty)$ which satisfies the following:

Whenever $A_1, A_2, \cdots \in \mathcal{A}$ such that $A_i \cap A_j =, \forall i \neq j$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu\left(A_n\right) \tag{σ-Add}$$

Definition 8.2. Let $\mu : \mathcal{A} \to [0, \infty)$ be a finite positive measure. Then, the triple $(\Omega, \mathcal{A}, \mu)$ is called a *finite measure space.*

Remark 8.3. *Properties of a finite positive measure.*

(1) $\mu(\emptyset) = 0$: Denote $\mu(\emptyset) = \alpha \in [0, \infty)$, and let $A_1 = A_2 = \cdots = \emptyset$. Then by σ-Add, $\mu(\emptyset) = \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(A_k) = \lim_{k\to\infty} k\alpha$. Notice $\alpha = k\alpha$ only when $\alpha = 0$.

(2) (Finite Add): ...

(3) Increasing Property: ...

(4) Compliment formula: ...

Proposition 8.4. *Continuity along Chains - Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space.*

(1) Suppose $(B_n)_{n=1}^{\infty}$ is a family of sets from $\mathcal{A}$ such that $B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots$. Then,

$$\mu(\cup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n)$$

(2) Symmetrically, suppose $(C_n)_{n=1}^{\infty} \in \mathcal{A}$ such that $C_1 \supseteq \cdots \supseteq C_n \supseteq \cdots$. Then,

$$\mu(\cap_{n=1}^{\infty} C_n) = \lim_{n\to\infty} \mu(C_n)$$

8.2. **June 10, Lecture 16.**

Proof of (1) Put $A_1 = B_1, A_2 = B_2 \setminus B_1, \cdots, A_n = B_n \setminus B_{n-1}$. Observe $A_i \in \mathcal{A}, \forall i \in \mathbb{N}$, $A_i \cap A_j = \emptyset, \forall i \neq j$, and $A_1 \cup \cdots \cup A_n = B_n, \forall n \in \mathbb{N}$. Then,

$$B_1 \cup B_2 \cup \cdots = A_1 \cup (A_1 \cup A_2) \cup \cdots = A_1 \cup A_2 \cup \cdots$$

. Hence,

$$\begin{aligned}\mu\left(\cup_{n=1}^{\infty} B_n\right) = \mu\left(\cup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \mu(A_n) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(A_n) \\ &= \lim_{k\to\infty} \left(\mu(A_1 \cup \cdots \cup A_k)\right) = \lim_{k\to\infty} \mu(B_k)\end{aligned}$$

□

Proof of (2) Let $B_n = \Omega \setminus C_n, \forall n \in \mathbb{N}$. Then, $B_n \in \mathcal{A}$, and we have $B_1 \subseteq B_2 \subseteq, \cdots$. Moreover, $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty}(\Omega \setminus C_n)$. Hence, $\cap_{n=1}^{\infty} C_n = \Omega \setminus (\cup_{n=1}^{\infty} B_n)$. We have Remark 8.3.4 which says

$$\begin{aligned}\mu\left(\cap_{n=1}^{\infty} C_n\right) = \mu(\Omega) - \mu\left(\cup_{n=1}^{\infty} B_n\right) &= \mu - \lim_{n\to\infty} \mu(B_n) \\ &= \lim_{n\to\infty} \left[\mu(\Omega) - \mu(B_n)\right] \lim_{n\to\infty} \mu(C_n)\end{aligned}$$

□

Proposition 8.5. *Subadditivity properties of μ*

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space.

(1) $\forall n \in \mathbb{N}$, and any $E_1, \cdots, E_n \in \mathcal{A}$, we have that

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu(E_n) \tag{Finite-SubAdd}$$

(2) $\forall E_1, E_2, \cdots \in \mathcal{A}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n) \in [0, \infty] \tag{σ-SubAdd}$$

Proof. ... □

9. Regularity Properties: *For a finite positive measure on a Borel σ-algebra*

9.1. **June 12, Lecture 17.**

Definition 9.1. Let (Ω, d) be a metric space, take $\mathcal{T}$ to be its topology. We define $\mathcal{B} = \sigma(\mathcal{T})$ to be the *Borel σ-algebra*, which is the smallest σ-algebra of Ω which contains all open sets.

$\mathcal{B}$ also contains all the closed sets, and all sets of type G_δ and F_σ.

Definition 9.2. Let (Ω, d) be a metric space, and let $\mathcal{B} \subseteq 2^\Omega$ be its Borel σ-algebra. Let $\mu : \mathcal{B} \to [0, \infty)$ be a finite positive measure. We say μ is *regular* to mean that

$$\begin{gathered}\forall B \in \mathcal{B}, \epsilon > 0, \exists K, G \subseteq \Omega \text{ such that } K \text{ is compact}, G \text{ is open}, \\ K \subseteq B \subseteq G, \text{ and } \mu(G) - \mu(K) < \epsilon\end{gathered} \tag{Reg}$$ [3]

We say that μ is *closed regular* to mean

$$\begin{gathered}\forall B \in \mathcal{B}, \epsilon > 0, \exists F, G \subseteq \Omega \text{ such that } F \text{ is closed}, G \text{ is open}, \\ F \subseteq B \subseteq G, \text{ and } \mu(G) - \mu(F) < \epsilon\end{gathered} \tag{Closed-Reg}$$

It's obvious that Reg $\Rightarrow$ Closed-Reg, and they coincide when (Ω, d) is compact.

Proposition 9.3. *Rephrasing of Closed Regularity*

Let (Ω, d) be a metric space, $\mathcal{B} \subseteq 2^\Omega$ be the Borel σ-algebra, and $\mu : \mathcal{B} \to [0, \infty)$ be a finite positive measure. Then,

$$\mu \textit{ is closed regular} \Leftrightarrow \forall B \in \mathcal{B}, \mu(B) = \inf\{\mu(G) \mid G \textit{ open}, G \supseteq B\} = \sup\{\mu(F) \mid F \textit{ closed}, F \subseteq B\}$$

Proof. ... □

Corollary 9.4. *Let $\mu, \nu : \mathcal{B} \to [0, \infty)$ be finite positive measures which are closed regular and agree on open sets*[4]*. Then, $\mu = \nu$. That is, $\mu(B) = \nu(B), \forall B \in \mathcal{B}$.*

---

[3] Note that $\mu(G) - \mu(K) = \mu(G \setminus K)$, so we could have $\mu(G \setminus K) < \epsilon$.

[4] That is, $\mu(G) = \nu(G), \forall G \in \mathcal{T}$.

Proof.

$$\begin{aligned}\forall B \in \mathcal{B}, \mu(B) &= \inf\{\mu(G) \mid G \text{ open}, G \supseteq B\} \\ &= \inf\{\nu(G) \mid G \text{ open}, G \supseteq B\} = \nu(B)\end{aligned}$$

□

9.2. **June 14, Lecture 18.**

Theorem 9.5. *Let (Ω, d) be a metric space, $\mathcal{B}$ be the Borel σ-algebra, and let $\mu : \mathcal{B} \to [0, \infty)$ be a finite positive measure. Then, μ is closed regular.*

Proof We use "the method of the friendly sets". Let

$$\mathcal{F} = \{B \in \mathcal{B} \,|\, \forall \epsilon > 0, \exists F \text{ closed}, G \text{ open} \subseteq \Omega, F \subseteq B \subseteq G, \mu(G \setminus F) < \epsilon\}$$

We have $\mathcal{F} \subseteq \mathcal{B}$ by definition, and our goal is to show $\mathcal{F} = \mathcal{B}$. We verify two facts.

Fact 1: "Every open set is friendly". That is,

$$\mathcal{F} \supseteq \mathcal{T}, \text{ where } \mathcal{T} := \{G \subseteq \Omega \,|\, G \text{ is open}\}$$

Fact 2: $\mathcal{F}$ is a σ algebra of subsets of Ω.

Once we prove these facts, we will be done because $\mathcal{T} \supseteq \mathcal{T}$ being a σ-algebra implies $\mathcal{F} \supseteq \sigma(\mathcal{T}) =: \mathcal{B}$. Hence, $\mathcal{F} \supseteq \mathcal{B}$, and $\mathcal{F} \subseteq \mathcal{B}$ gives us our desired consclusion.

Verification of Fact 1: We must show that $G \subseteq \Omega$ open implies $G \in \mathcal{F}$. So, pick $G \subseteq \Omega$ open, $\epsilon > 0$. We must find F' closed, G' open $\subseteq \Omega$ such that $F' \subseteq G \subseteq G'$ and $\mu(G') - \mu(F') < \epsilon$. Recall that G is a set of type F_σ, and therefore we can find an increasing chain of closed sets, $F_1 \subseteq F_2 \subseteq \cdots$ such that $\cup_{n=1}^{\infty} F_n = G$. Continuity of μ along increasing chains says that $\lim_{n\to\infty} \mu(F_n) = \mu(G)$. So, we can find an $n_0 \in \mathbb{N}$ such that $\mu(F_{n_0}) > \mu(G) - \epsilon$. Let $F' = F_{n_0}, G' = G$, which are closed and open respectively, then we have $F' \subseteq G \subseteq G'$, and

$$\mu(G') - \mu(F') = \mu(G) - \mu(F') < \epsilon$$

Verification of Fact 2: We must show $\mathcal{F}$ satisfies the 3 facts of a σ-algebra.

- $\emptyset \in \mathcal{F}$ follows from fact 1, since $\emptyset$ is open. So, given $\epsilon > 0$, let $F = G = \emptyset$. Thus, all needed things are satisfied.
- Let $B \in \mathcal{F}$, and let us look at $\Omega \setminus B$. Given $\epsilon > 0$, we want to find $F, G \subseteq \Omega$ such that $F \subseteq \Omega \subseteq G$, where F closed, G open, and $\mu(G) - \mu(F) < \epsilon$. Since $B \in \mathcal{F}$, we can find $F', G' \subseteq \Omega$ such that $F' \subseteq B \subseteq G'$ where F' closed, G' open, and $\mu(G') - \mu(F') < \epsilon$. Let $G = \Omega \setminus F', F = \Omega \setminus G'$, and observe

  $$F' \subseteq B \subseteq G' \Rightarrow \Omega \setminus F' \supseteq \Omega \setminus B \supseteq \Omega \setminus G' \Rightarrow F \subseteq \Omega \setminus B \subseteq G$$

  Moreover,

  $$\mu(G) - \mu(F) = \big[\mu(\Omega) - \mu(F')\big] - \big[\mu(\Omega) - \mu(G')\big] = \mu(G') - \mu(F') < \epsilon$$

- First, we show by induction $n \in \mathbb{N}$ and $B_1, \cdots, _n \in \mathcal{F} \Rightarrow \cap_{i=1}^{n} B_i \in \mathcal{F}$. Starting with the base case, $n = 2$, take $B_1, B_2 \in \mathcal{F}$, and $\epsilon > 0$. Then, $B_1 \in \mathcal{F} \Rightarrow \exists F_1, G_1 \subseteq \Omega$ such that F_1 closed, G_1 open, $F_1 \subseteq B_1 \subseteq G_1$, and $\mu(G_1) - \mu(F_1) < \epsilon/2$ Likewise, find F_2, G_2 for B_2. Let $F = F_1 \cap F_2$ which is closed, and $G = G_1 \cap G_2$ which is open, and observe

$$\left.\begin{array}{l} F_1 \subseteq B_1 \subseteq G_1 \\ F_2 \subseteq B_2 \subseteq G_2 \end{array}\right\} \Rightarrow \underbrace{F_1 \cap F_2}_{=F} \subseteq B_1 \cap B_2 \subseteq \underbrace{G_1 \cap G_2}_{=G}$$

Also observe $G \setminus F = (G_1 \cap G_2) \setminus (F_1 \cap F_2) \subseteq (G_1 \setminus F_1) \cup (G_2 \setminus F_2)$. Indeed, for $x \in (G_1 \cap G_2) \setminus (F_1 \cap F_2)$, $x \in G_1, x \in G_2$ and either missing from F_1 or F_2. If $x \notin F_1$, then $x \in G_1 \setminus F_1$. If $x \notin F_2$, then $x \in G_2 \setminus F_2$. So,

$$\begin{aligned} \mu(G) - \mu(F) = \mu(G \setminus F) &\leq \mu((G_1 \setminus F_1) \cup (G_2 \setminus F_2)) \\ &\leq \mu(G_1 \setminus F_1) + \mu(G_2 \setminus F_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

It's easy to see by induction on n that $B_1, \cdots, B_n \in \mathcal{F} \Rightarrow \cap_{i=1}^{n} B_n \in \mathcal{F}$. Then, by Homework Assignment 7A, $C_1 \supseteq C_2 \supseteq \cdots \in \mathcal{F} \Rightarrow C := \cap_{n=1}^{\infty} \in \mathcal{F}$.

$\square$

9.3. **June 17, Lecture 19.** We start by completing the proof of Theorem 9.5. ...

Corollary 9.6. *Restatement of 9.4 Let (Ω, d) be a metric space, $\mathcal{B}$ be the Borel σ-algebra, and let $\mu, \nu : \mathcal{B} \to [0, \infty)$ be finite positive measures. If $\mu(G) = \nu(G), \forall G \in \mathcal{T}$, then $\mu = \nu$.*

10. The Lebesgue Measure: *On an interval* $[a, b] \subseteq \mathbb{R}$

10.1. **Lecture 20, June 19.**

Definition 10.1. Let $a < b \in \mathbb{R}$, and put $\Omega = [a, b]$. Consider its usual distance, let $\mathcal{B}$ be the Borel σ-algebra of (Ω, d), and let $\mu : \mathcal{B} \to [0, \infty)$ be a finite positive measure. We say μ is *length-fitting* when it measures the open balls of Ω according to there length. That is, $\mu((\alpha, \beta)) = \beta - \alpha, \forall a \leq \alpha < \beta \leq b$.

Equivalently, we can characterize these intervals using the collection of open balls,

$$\mathcal{U} := \{U \subseteq \Omega \mid \exists x \in \Omega, r > 0 \text{ s.t } U = B(x; r)\}$$

Hence, we get that μ is length fitting when $\mu(U) = \sup(U) - \inf(U), \forall u \in \mathcal{U}$.

Definition 10.2. Using the notation as above, there exists a unique finite positive measure, $\mu : \mathcal{B} \to [0, \infty)$, such that μ is length-fitting. This μ is called the *Lebesgue measure* on the interval $[a, b]$.

Remark 10.3. Uniqueness of the Lebesgue measure on $[a, b]$ comes as a consequence of *closed regularity*, and the *Structure Theorem* for the open subsets of this particular metric space, (Ω, d). More precisely, every not-empty open set $G \subseteq \Omega$ can be written as the countable union of disjoint open balls. So for finite positive measures $\mu, \nu :\to [0, \infty)$ such that $\mu(U) = \nu(U), \forall U \in \mathcal{U}$, it follows by σ-Add that $\mu(G) = \nu(G), \forall G \in \mathcal{T}$. From there, Corollary 9.4 (a concequence of closed regularity) tells us that $\mu = \nu$.

Existence of the Lebesgue measure on $[a, b]$ - we discuss the "pedestrian approach". The idea is we want the notion of length for "nice" subsets of Ω.

Notation 10.4. Let $\mathcal{U} = \{U \subseteq \Omega \mid \exists x \in \Omega, r > 0 \text{ s.t. } U = B(x;r)\}$ be the collection of open balls in Ω, and define $\lambda_{ball} : \mathcal{U} \to [0,\infty)$, where $\lambda_{ball}(U) := \sup(U) - \inf(U), \forall U \in \mathcal{U}$.

Notation 10.5. Let $\mathcal{T}$ be the topology of Ω, and define $\lambda_{open} : \mathcal{T} \to [0,\infty)$ as follows. First, $\lambda_{open}(\emptyset) = 0$. Then, for non-empty $G \in \mathcal{T}$, put

$$\lambda_{open}(G) := \sup\left\{\sum_{i=1}^{n} \lambda_{ball}(U_i) \mid n \in \mathbb{N}, U_i \in \mathcal{U}, U_i \subseteq G, \forall 1 \leq i \leq n, U_i \cap U_j = \emptyset, \forall i \neq j\right\}$$

Remark 10.6. Suppose $U_1, \cdots, U_n \in \mathcal{U}$ such that $U_i \cap U_j = \emptyset, \forall i \neq j$. Then, we can relabel $U_1, \cdots, U_n$ such that

$$\sup(U_1) \subseteq \inf(U_2), \sup(U_2) \subseteq \inf(U_3), \cdots, \sup(U_{n-1}) \subseteq \inf(U_n)^5$$

10.2. **Lecture 21, June 21.**

Remark 10.7.
(1) It $\emptyset \neq G \in \mathcal{T}$, then $\lambda_{open}(G) > 0$. To see this, take $U_1 = B$. Then, $\lambda_{open}(G) \geq \sum \lambda_{ball}(U_1) = \sup(U_1) - \inf(U_1) > 0$
(2) If $G_1 \subseteq G_2 \in \mathcal{T}$, then $\lambda_{open}(G_1) \leq \lambda_{open}(G_2)$. *Verification...*
(3) If $U \in \mathcal{U} \subseteq \mathcal{T}$, then $\lambda_{open}(U) = \lambda_{ball}(U)$. *Why?* In the sup which defines $\lambda_{open}(U)$ we can make $n = 1$, and $U_1 = U$. This gives $\lambda_{open}(U) \geq \lambda_{open}(U_1)$. To show $\leq$, we must prove for $n \in \mathbb{N}, U_1, \cdots, U_n \in \mathcal{U}$ with $U_1, \cdots, U_n \subseteq U$, and $U_i \cap U_j = \emptyset$ when $i \neq j$.

    Verification...
(4) $\lambda_{open}(G) \leq b - a, \forall G \in \mathcal{T}$. Observe that $G \subseteq \Omega \Rightarrow \lambda_{open}(G) \leq \lambda_{open}(\Omega)$. By 2, $\lambda_{ball}(\Omega) = \sup(\Omega) - \inf(\Omega) = b - a$.

10.3. **Lecture 22, June 24.**

Proposition 10.8. *Let $G_1, G_2 \in \mathcal{T}$ be such that $G_1 \cap G_2 = \emptyset$. Then,*

$$\lambda_{open}(G_1 \cup G_2) = \lambda_{open}(G_1) + \lambda_{open}(G_2)$$

Proof. ... □

11. The Lebesgue Measure: *Length for compact subsets*

11.1. **Lecture 23, June 26.**

Remark and Notation 11.1. Let $\mathcal{K} := \{K \subseteq \Omega \mid K \text{compact}\} = \{\Omega \setminus G \mid G \in \mathcal{T}\}$. Define $\lambda_{cpct}(K) : \mathcal{K} \to \mathbb{R}$ such that for $K \in \mathcal{K}$, $G = \Omega \setminus K$, and put

$$\lambda_{cpct}(K) = \lambda_{open}(\Omega) - \lambda_{open}(G) = (b - a) - \lambda_{open}(G)$$

[5] This is called the Canonical ordering of $U_1, \cdots, U_n$

Remark 11.2. *Can λ_{open} and λ_{cpct} disagree?*
Since $\Omega = [a, b]$ is connected, we have $\mathcal{K} \cap \mathcal{T} = \{\emptyset, \Omega\}$. We know $0 \leq \lambda_{open}(G) \leq b - a$ implies $0 \leq \lambda_{cpct}(K) \leq b - a$ as well. We also have $\lambda_{open}(\emptyset) = 0$, and

$$\lambda_{open}(\Omega) = \lambda_{ball}(\Omega) = \sup(\Omega) - \inf(\Omega) = b - a$$

This implies $\lambda_{cpct}(\emptyset) = (b - a) - \lambda_{open}(\Omega \setminus \emptyset) = (b - a) - (b - a) = 0$, and

$$\lambda_{cpct}(\Omega) = (b - a) - \lambda_{open}(\Omega \setminus \Omega) = (b - a) - 0 = (b - a)$$

Definition 11.3. A set $G \in \mathcal{T}$ is *elementary* when it can be written as the finite union of disjoint open balls.

$$U_1 \cup \cdots \cup U_n = G, \text{such that } U_i \cap U_j = \emptyset, U_i \in \mathcal{U}, \forall i \neq j$$

Remark 11.4. If $G = U_1 \cap \cdots \cap U_n$, then

$$\lambda_{open}(G) = \sum_{i=1}^{n} \lambda_{open}(U_i) = \sum_{i=1}^{n} \lambda_{ball}(U_i) = \sum_{i=1}^{n} \sup(U_i) - \inf(U_i)$$

11.2. **Lecture 24, June 28.**

Lemma 11.5. *Suppose $K \in \mathcal{K}, G \in \mathcal{T}$ such that $K \subseteq G$. We can decompose $G = G' \cup G''$ with $G', G'' \in \mathcal{T}$, such that*

(1) $G' \cap G'' = \emptyset$
(2) G' elementary
(3) $K \subseteq G'$

Proof. If G is elementary, take $G' = G, G'' = \emptyset$. So, suppose G isn't elementary. Then, "Fact S" says that $G = \cup_{n=1}^{\infty}, U_i \in \mathcal{U}, \forall i \in \mathbb{N}$, with $U_i \cap U_j = \emptyset$. The open cover $K \subseteq \cup_{i=1}^{\infty} U_i$ must contain a finite cover since K is compact. Hence, $\exists n \in bN$ sich that $K \subseteq \cup_{i=1}^{n} U_i$.

Let $G' = \cup_{i=1}^{n} U_i, G'' = \cup_{i=1}^{\infty} U_i$. Then, G', G''have the required properties. □

Lemma 11.6. *Let $K \in \mathcal{K}, G \in \mathcal{T}$ be such that G is elementary and $K \subseteq G$. Let $D = G \setminus K$. Then,*

$$(\blacklozenge) \qquad \lambda_{open}(D) + \lambda_{cpct}(K) = \lambda_{open}(G)$$

Proof. Homework Assignment 7B, Question 2 □

Proposition 11.7. *Let $K \in \mathcal{K}, G \in \mathcal{T}$ such that $K \subseteq G$. Let $D = G \setminus K \in \mathcal{T}$. Then,*

$$(\blacklozenge\blacklozenge) \qquad \lambda_{open}(D) + \lambda_{cpct}(K) = \lambda_{open}(G)$$

Proof Decompose $G = G' \cup G''$ as in Lemma 11.5. So, G' is elementary, and $K \subseteq G'$. Let $D' = G' \setminus K \in \mathcal{T}$. From Lemma 11.6, we get

$$\text{(I)} \qquad \lambda_{open}(D') + \lambda_{cpct}(K) = \lambda_{open}(G')$$

Observe $D = G \setminus K = (G' \setminus K) \cup G'' = D' \cup G''$ with $D' \cap G'' = \emptyset$. By Proposition 10.8,

$$\text{(II)} \qquad \lambda_{open}(D) = \lambda_{open}(D') + \lambda_{open}(G'')$$

Adding (I) + (II).

$$\left.\begin{aligned}\lambda_{open}(D') + \lambda_{cpct}(K) = \qquad\qquad\qquad & \lambda_{open}(G') \\ + \lambda_{open}(D) \qquad + \lambda_{open}(D') + & \lambda_{open}(G'')\end{aligned}\right\} = \lambda_{open}(G)$$

□

Corollary 11.8.

$$K \in \mathcal{K}, G \in \mathcal{T}, \text{ such that } K \subseteq G \Rightarrow \lambda_{cpct}(K) \leq \lambda_{open}(G)$$

Proof. (◆◆) gives us $\lambda_{open}(G) = \lambda_{cpct}(K) + \underbrace{\lambda_{open}(D)}_{\geq 0} \geq \lambda_{cpct}(K)$ □

Definition 11.9. Let $\mathcal{B} := \sigma(\mathcal{T}) \subseteq 2^{\Omega}$.

(1) $B \in \mathcal{B}$ is *friendy* when

$$\forall \epsilon > 0, \exists K \in \mathcal{K}, G \in \mathcal{T}, \text{ such that } K \subseteq B \subseteq G, \text{ and } (0 \leq)\lambda_{open}(G) - \lambda_{cpct}(K) < \epsilon$$

(2) Denote $\mathcal{F} := \{B \in \mathcal{B} \mid B \text{ is friendly}\}$.

Remark 11.10. One can show $\mathcal{F} = \mathcal{B}$. Thus, every Borel set is friendly.

Remark 11.11. Another way to obtain the "friendly" machinery is found in chapter 2 if Rudin's *Real and Complex Analysis* [4].

12. Measurable Functions: *On the space Bor*$(\Omega, \mathbb{R})$

12.1. Lecture 25, July 3.

Definition 12.1. A *measurable space* is a pair $(\Omega, \mathcal{A})$ where Ω is non-empty, and $\mathcal{A} \subseteq 2^{\Omega}$ is a σ-albegra.

We focus on a special case of interest: the measurable space $(\Omega, \mathcal{B})$ where (Ω, d) is a metruc space, and $\mathcal{B} \subseteq 2^{\Omega}$ is the Borel σ-algebra of $(\Omega, \mathcal{B})$.

Definition 12.2. Let $(\Omega, \mathcal{A}), (\Gamma, \mathcal{M})$ be measurable spaces, and let $f : \Omega \to \Gamma$ be a function. We say f is $\mathcal{A}/\mathcal{M}$*-measurable* to mean

$$f^{-1}(M) := \{x \in \Omega \mid f(x) \in \mathcal{M}\} \in \mathcal{A}, \forall M \in \mathcal{M}$$

Two measurability lemmas

Lemma 12.3. ***Composites Lemma***
Let $(\Omega, \mathcal{A}), (\Gamma, \mathcal{M}), (\Lambda, \mathcal{N})$ be measurable spaces. Suppose we have the functions

$$f : \Omega \to \Gamma \text{ which is } \mathcal{A}/\mathcal{M} - \text{measurable}$$
$$g : \Gamma \to \Lambda \text{ which is } \mathcal{M}/\mathcal{N} - \text{measurable}$$

Let $h = g \circ f : \Omega \to \Lambda$. Then, h is $\mathcal{A}/\mathcal{N}$-measurable.

Proof. ... □

Lemma 12.4. ***"Give me a break" Lemma***
Let $(\Omega, \mathcal{A}), (\Gamma, \mathcal{M})$ be measurable spaces, where we're also given a collection of sets, $\mathcal{C} \subseteq \mathcal{M}$, such that $\sigma(\mathcal{C}) = \mathcal{M}$. Let $f : \Omega \to \Gamma$ be a function, and suppose $f^{-1}(C) \in \mathcal{A}, \forall C \in \mathcal{C}$. Then, f is $\mathcal{A}/\mathcal{M}$-measurable.

Proof. ... □

12.2. **Lecture 26, July 5.**

Proposition 12.5. *Continuous implies Measurable*
Let $(\Omega, d), (\Gamma, d')$ be metric spaces and let $\mathcal{B}_\Omega \subseteq 2^\Omega, \mathcal{B}_\Gamma \subseteq 2^\Gamma$ be their corresponding σ-algebras. Let $f : \Omega \to \Gamma$ be a function. If f is continuous, then f is $\mathcal{B}_\Omega/\mathcal{B}_\Gamma$-measurable.

Proof. Let $\mathcal{T}_\Omega, \mathcal{T}_\Gamma$ be the topologies corresponding to Ω and Γ. We know $\mathcal{B}_\Gamma = \sigma(\mathcal{T}_\Gamma)$. Moreover, observe that $G \in \mathcal{T}_\Gamma \Rightarrow f^{-1}(G) \in \mathcal{T}_\Omega$, since f is continuous. This implies $f^{-1}(G) \in \mathcal{B}_\Omega$ since $\mathcal{T}_\Omega \subseteq \mathcal{B}_\Omega$. So, $f^{-1}(G') \in \mathcal{B}_\Omega, \forall G \in \mathcal{T}$. Lemma 12.4 implies that $f^{-1}(B) \in \mathcal{B}_\Omega, \forall B \in \mathcal{B}_\Gamma$, that is, f is $\mathcal{B}_\Omega/\mathcal{B}_\Gamma$-measurable, as required. □

Notation 12.6. Let $(\Omega, \mathcal{A})$ be a measurable space. For a second measurable space, $(\Gamma, \mathcal{M})$, pick $\Gamma = \mathbb{R}$, and $\mathcal{M} = \mathcal{B}_\mathbb{R}$. We denote the space of Borel functions,

$$\mathrm{Bor}(\Omega, \mathbb{R}) := \{f : \Omega \to \mathbb{R} \mid f \text{ is } \mathcal{A}/\mathcal{B}_\mathbb{R} - \text{measurable}\}$$

Corollary 12.7. *Let (Ω, d) be a metric space. We have $C(\Omega, \mathbb{R}) \subseteq Bor(\Omega, \mathbb{R})$.*

Proof. This is a special case of Proposition 12.5. □

Lemma 12.8. *Let $(\Omega, \mathcal{A})$ be a measurable space, $k \in \mathbb{N}$ be a dimension, and let $f : \Omega \to \mathbb{R}^k$ be a function. $\forall x \in \Omega$, write $f(x) = (f_1(x), \cdots, f_k(x)) \in \mathbb{R}^k$. This gives us functions $f_1, \cdots, f_k : \Omega \to \mathbb{R}$ called the components of f. Then,*

$$f \textit{ is } \mathcal{A}/\mathcal{B}_{\mathbb{R}^k} - \textit{measurable} \Leftrightarrow \textit{Each of } f_1, \cdots, f_k \in Bor(\Omega, \mathcal{B}).$$

Proof. ... □

12.3. **Lecture 27, July 8.** With Lemma 12.8, we now prove the stability of $\mathrm{Bor}(\Omega, \mathbb{R})$ under operations.

Proposition 12.9. *Let $(\Omega, \mathcal{A})$ be a measurable space, let $f, g : \Omega \to \mathbb{R}$, for which we form:*
Sum:

$$(f + g)(x) = f(x) + g(x)$$

Product:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Minimum:

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

Minimum:

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

$$\textit{If } f, g \in Bor(\Omega, \mathbb{R}), \textit{ then } f + g, f \cdot g, f \wedge g, f \vee g \in Bor(\Omega, \mathbb{R})$$

Proof. ... □

13. Convergence Properties of Bor$(\Omega,\mathbb{R})$

13.1. **Lecture 28, July 10.**

Proposition 13.1. *Let $f:\Omega\to\mathbb{R}$, be a function and suppose we found a sequence, $(f_n)_{n=1}^{\infty}\in Bor(\Omega,\mathbb{R})$ such that*

$$\lim_{n\to\infty} f_n(x)=f(x),\forall x\in\Omega$$

Then, $f\in Bor(\Omega,\mathbb{R})$.

Next, we state a stronger argument.

Proposition 13.2.
1. *Let $f:\Omega\to\mathbb{R}$ be a function and suppose we found a sequence, $(f_n)_{n=1}^{\infty}\in Bor(\Omega,\mathbb{R})$ such that*
$$f(x)=\limsup_{n\to\infty} f_n(x),\forall x\in\Omega$$
Then, $f\in Bor(\Omega,\mathbb{R})$.
2. *Let $g:\Omega\to\mathbb{R}$ be a function and suppose we found a sequence, $(g_n)_{n=1}^{\infty}\in Bor(\Omega,\mathbb{R})$ such that*
$$g(x)=\liminf_{n\to\infty} g_n(x),\forall x\in\Omega$$
Then, $g\in Bor(\Omega,\mathbb{R})$.

Remark 13.3. Pop-Quiz
Let $(\Omega,\mathcal{A})$ be a measurable space, and let $S\subseteq\Omega$, and consider the *indicator function*, $\chi_S:\Omega\to\mathbb{R}$,

$$\chi_S(x)=\begin{cases}1 & \text{If } x\in S\\ 0 & \text{If } x\notin S\end{cases}$$

(1) *Prove $S\in\mathcal{A}\Rightarrow\chi_S\in Bor(\Omega,\mathbb{R})$*
(2) *Is the converse true?* - The answer is yes.

Lemma 13.4.
1. *Let $f:\Omega\to\mathbb{R}$, and suppose $(f_n)_{n=1}^{\infty}\in Bor(\Omega,\mathbb{R})$ such that $f(x)=\sup\{f_n(x)\mid n\in\mathbb{N}\},\forall x\in\Omega$. Then, $f\in Bor(\Omega,\mathbb{R})$.*
2. *Let $g:\Omega\to\mathbb{R}$, and suppose $(g_n)_{n=1}^{\infty}\in Bor(\Omega,\mathbb{R})$ such that $g(x)=\inf\{g_n(x)\mid n\in\mathbb{N}\},\forall x\in\Omega$. Then, $g\in Bor(\Omega,\mathbb{R})$.*

Proof of (1) We use Homework Assignment 8, Q3 which says it suffices to check

$$(✺)\qquad \forall t\in\mathbb{R},\{x\in\Omega\mid f(x)\le t\}\in\mathcal{A}\Rightarrow f^{-1}((-\infty,t])\in\mathcal{A}$$

So, fix a $t\in\mathbb{R}$ for which we verify (✺) holds. Observe for $x\in\Omega$, we have

$$f(x)\le t\Leftrightarrow\sup\{f_n(x)\mid n\in\mathbb{N}\}\le t\Leftrightarrow f_n(x)\le t,\forall n\in\mathbb{N}$$

Hence,

$$\{x\in\Omega\mid f(x)\le t\}=\cap_{n=1}^{\infty}\{x\in\Omega\mid f_n(x)\le t\}=\cap_{n=1}^{\infty}\underbrace{f_n^{-1}((-\infty,t])}_{\in\mathcal{A}}$$

since $f_n\in\mathrm{Bor}(\Omega,\mathbb{R})$, and $(-\infty,t]\in\mathcal{B}_{\mathbb{R}}$. Since $\mathcal{A}$ is stable under countable intersections, it follows that $\{x\in\Omega\mid f(x)\le t\}\in\mathcal{A}$. □

Proof of (2) Let $f := -g : \Omega \to \mathbb{R}$, and let $f_n = -g_n \in \mathrm{Bor}(\Omega, \mathbb{R}), \forall n \in \mathbb{N}$. Observe that

$$f(x) = -g(x) = -\inf\{g_n(x) \mid n \in \mathbb{N}\} = \sup\{-g_n(x) \,:\, n \in \mathbb{N}\} = \sup\{f_n(x) \,:\, n \in \mathbb{N}\}$$

So part (1) applies to give us $f \in \mathrm{Bor}(\Omega, \mathbb{R})$. Finally, $g = -f \in \mathrm{Bor}(\Omega, \mathbb{R})$, as well. $\square$

14. Integration of Functions in $\mathrm{Bor}^+(\Omega, \mathbb{R})$, I

14.1. **Lecture 29, July 12.** For the rest of the chapter, fix the finite measure space, $(\Omega, \mathcal{A}, \mu)$, and consider the space of of Borel functions,

$$\mathrm{Bor}(\Omega, \mathbb{R}) := \{f : \Omega \to \mathbb{R} \mid f^{-1}(B) \in \mathcal{A}, \forall B \in \mathcal{B}_{\mathbb{R}}\}$$

Remark and Notation 14.1. Denote the space of positive Borel functions,

$$\mathrm{Bor}^+(\Omega, \mathbb{R}) := \{f \in \mathrm{Bor}(\Omega, \mathbb{R}) \mid f(x) \geq 0, \forall x \in \Omega\}$$

- Observe that $\mathrm{Bor}^+(\Omega, \mathbb{R})$ has good properties with respect to operations with functions. For instance,
$$f, g \in \mathrm{Bor}^+(\Omega, \mathbb{R}) \Rightarrow f \wedge g \in \mathrm{Bor}^+(\Omega, \mathbb{R})$$
Indeed, we saw $f, g \in \mathrm{Bor}(\Omega, \mathbb{R}) \Rightarrow f \wedge g \in \mathrm{Bor}^+(\Omega, \mathbb{R})$ in chapter 12, and we also have $(f \wedge g)(x) = \min\{f(x), g(x)\} \geq 0, \forall x \in \Omega$.
- For $f, g \in \mathrm{Bor}^+(\Omega, \mathbb{R})$, we write "$f \leq g$" to mean that $f(x) \leq g(x), \forall x \in \Omega$.

Definition 14.2. Let us use the name *"measurable scale"* for a system

$$\sigma = \begin{pmatrix} A_1, \cdots, A_n \\ \alpha_1, \cdots, \alpha_n \end{pmatrix} \quad \text{with } n \in \mathbb{N}, \quad \begin{array}{c} \alpha_1, \cdots \alpha_n \in [0, \infty), \\ A_1, \cdots, A_n \in \mathcal{A}, \\ A_i \cap A_j = \emptyset, \forall i \neq j \end{array}$$

For such σ we define the *weight of* σ to be $W(\sigma) := \alpha_1 \mu(A_1) + \cdots + \alpha_n \mu(A_n) \in [0, \infty)$.

Definition 14.3. Let $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$, and let $\sigma = \begin{pmatrix} A_1, \cdots, A_n \\ \alpha_1, \cdots, \alpha_n \end{pmatrix}$ be a measurable scale. We write $\sigma \prec f$ to mean that

$$\begin{array}{ccc} x \in A_1 & \Rightarrow & f(x) \geq \alpha_1 \\ \vdots & & \vdots \\ x \in A_n & \Rightarrow & f(x) \geq \alpha_n \end{array}$$

Definition 14.4. Let $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$ and consider various measurable scales σ such that $\sigma \prec f$. [6] Let us denote

$$L^+(f) := \sup\{W(\sigma) \mid \sigma \text{ is a measurable scale such that } \sigma \prec f\}$$

The convention if $\{W(\sigma) \mid \sigma \prec f\}$ is an unbounded subset of $[0, \infty)$, then declare it's sup to be "$+\infty$". Otherwise, if $\{W(\sigma) \mid \sigma \prec f\}$ is bounded from above, we have the usual notion of sup.

---

[6] Note that such σ's always exist, for example, $\sigma_0 = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} = \begin{pmatrix} A_1, \cdots, A_n \\ 0, \cdots, 0 \end{pmatrix}$ with $n = 1, A_1 = \Omega, \sigma_1 = 0$. Then, $\sigma_0 \prec f$ holds simply because $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$.

Remark 14.5. From Definition 14.4, we get a map $L^+ : \mathrm{Bor}^+(\Omega, \mathbb{R}) \to [0, \infty]$. This is one of the incarnations of the Lebesgue measure.

Remark 14.6. It's immediate $f, g \in \mathrm{Bor}^+(\Omega, \mathbb{R})$ such that $f \leq g$ implies $L^+(f) \leq L^+(g)$. This is because for measurable scale σ, we have $\sigma \prec f \Rightarrow \sigma \prec g$. Hence $L^+(g)$ is defined via a richer sup than the one defining $L^+(f)$.

Remark 14.7. Another immediate property is $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$ with $\alpha \in (0, \infty)$ implies $\alpha \cdot f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$, and $L^+(\alpha f) = \alpha L^+(f)$. [7] The observation to be made here is if $\sigma = \begin{pmatrix} A_1, \cdots, A_n \\ \alpha_1, \cdots, \alpha_n \end{pmatrix} \prec f$, then $\tau = \begin{pmatrix} A_1, \cdots, A_n \\ \alpha \cdot \alpha_1, \cdots, \alpha \cdot \alpha_n \end{pmatrix} \prec \alpha \cdot f$ has $W(\sigma) = \alpha \cdot W(\tau)$.

Note we can use $\alpha = 0 \Rightarrow \alpha \cdot f = \underline{0}$, where $\underline{0} : \Omega \to \mathbb{R}$ such that $\underline{0}(x) = 0, \forall x \in \Omega$. Hence, $L^+(\alpha f) = L^+(\underline{0}) = 0$ because the only measurable scale with $\sigma \prec \underline{0}$ is σ_0, as discussed in Definition 14.4. Hence, the equality $L^+(\alpha f) = \alpha L^+(f)$, setting $\alpha = 0$ makes LHS 0.

The RHS is $0 \cdot L^+(f)$, which is 0 as well, with the convention that $0 \cdot \infty = 0$.

14.2. **Lecture 30, July 15.**

Proposition 14.8. *Let $f, g \in Bor^+(\Omega, \mathbb{R})$. One has*

$$L^+(f + g) \geq L^+(f) + L^+(g) \tag{❏}$$

Proof. ...

Conclusion of the proof :

$$\begin{aligned} L^+(f + g) \geq W(\rho) = W(\sigma) + W(\tau) &> \Big(L^+(f) - \frac{\epsilon}{2}\Big) + \Big(L^+(g) - \frac{\epsilon}{2}\Big) \\ &= L^+(f) + L^+(g) - \epsilon. \end{aligned}$$

□

Remark 14.9. We would like a stronger claim than (❏).

$$L^+(f + g) = L^+(f) + L^+(g) \tag{❏❏}$$

But the method of proving Proposition 14.8 doesn't work to get "$\leq$". We instead use a different approach, based on approximating with simple functions.

15. Integration of $\mathrm{Bor}^+(\Omega, \mathbb{R})$, II: *Approximation by simple functions*

15.1. **Lecture 31, July 17.**

Definition 15.1. A function, $f : \Omega \to \mathbb{R}$, is *simple* when its image, $f(\Omega) := \{f(x) \mid x \in \Omega\}$, is finite. We denote $\mathrm{Bor}_s^+(\Omega, \mathbb{R})$ to be the space of all non-negative Borel simple functions.

Remark 15.2. Suppose $f \in \mathrm{Bor}_s^+(\Omega, \mathbb{R})$. Then, $\exists M_1, \cdots ,_k \in \mathcal{A}$ with $\emptyset \neq M_i$'s disjoint, and $\gamma_1 > \cdots > \gamma_k > 0$ such that

$$f = \sum_{i=1}^{k} \gamma_i \chi_{M_i} \tag{$\boxed{\text{S}}$}$$

[7] With $\alpha \cdot \infty = \infty$.

Fact List 15.3. (1) Let $f \in \text{Bor}_s^+(\Omega, \mathbb{R})$ and write $f = \sum_{i=1}^k \gamma_i \chi_{M_i}$ as in ($\boxed{\text{s}}$). Then, $L^+(f) = \sum_{i=1}^k \gamma_i \mu(M_i)$. Moreover, if $f = \sum_{j=1}^\ell \beta_i \chi_{N_i}$ such that $\beta_j \in [0, \infty)$, and N_i's disjoint, then

$$L^+(f) = \sum_{j=1}^{\ell} \beta_j \mu(N_j)$$

(2) If $f, g \in \text{Bor}_s^+(\Omega, \mathbb{R})$, then $L^+(f+g) = L^+(f) + L^+(g)$ $\boxed{\square\square}$

(3) Let $f \in \text{Bor}^+(\Omega, \mathbb{R})$. Then, $\exists (f_n)_{n=1}^\infty \in \text{Bor}_s^+(\Omega, \mathbb{R})$ such that
 (a) $f_1 \leq f_2 \leq \cdots$
 (b) $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in \Omega$

(4) Let $f \in \text{Bor}^+(\Omega, \mathbb{R})$, by (3), $\exists (f_n)_{n=1}^\infty$ as in (3).Then,
 (a) $L^+(f_1) \leq L^+(f_2) \leq \cdots$
 (b) $\lim_{n\to\infty} L^+(f_n) = L^+(f)$

Proof of fact 2. ... □

Proposition 15.4. *Let $f, g \in Bor^+(\Omega, \mathbb{R})$. Then,*

$$L^+(f+g) = L^+(f) + L^+(g)$$

Proof. First, we know "$\geq$", so "$\leq$" is left.

Let $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty \in \text{Bor}_s^+(\Omega, \mathbb{R})$ be as in Fact (3). Let $h_n := f_n + g_n$. Clearly, $h_1 \leq h_2 \leq \cdots$, with $h_n \in \text{Bor}_s^+(\Omega, \mathbb{R})$, and $h_n \to f + g$ converging pointwise. By Fact (4), $L^+(h_n) \to L^+(f) + L^+(g)$, $L^+(f_n) \to L^+(f)$, and $L^+(g_n) \to L^+(g)$. So,

$$L^+(f_n + g_n) = L^+(f_n) + L^+(g_n) \to L^+(f+g) = L^+(f) + L^+(g)$$

□

16. The $\mathcal{L}^1(\mu)$ Space

Fix for the chapter a finite measure space, $(\Omega, \mathcal{A}, \mu)$. The goal of this chapter is to find a second incarnation of the Lebesgue measure defined for f is a suitable subspace, $\mathcal{L}^1(\mu) \subseteq \text{Bor}(\Omega, \mathbb{R})$.

16.1. **Lecture 32, July 19.**

Definition 16.1. For $f \in \text{Bor}(\Omega, \mathbb{R})$, we define its positive and negative parts, $f_+, f_- : \Omega \to \mathbb{R}$, as follows.

($\blacklozenge$)
$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

Remark 16.2. (1) If $f \in \mathrm{Bor}(\Omega, \mathbb{R})$, then $f_+ \in \mathrm{Bor}^+(\Omega, \mathbb{R})$. This is because $f_+ = f \vee \underline{0}$, and invoke Proposition 12.9.

(2) If $f \in \mathrm{Bor}(\Omega, \mathbb{R})$, then $f_- \in \mathrm{Bor}^+(\Omega, \mathbb{R})$. Observe $f_- = (-f)_+$, and apply (1) to $-f$

(3) If $f \in \mathrm{Bor}(\Omega, \mathbb{R})$, then

$$(\blacklozenge\blacklozenge) \qquad \begin{cases} f_+(x) + f_-(x) = & |f(x)| \\ f_+(x) - f_-(x) = & f(x) \end{cases}$$

Proposition and Definition 16.3. *For $f \in Bor(\Omega, \mathbb{R})$, we have*

$$(\heartsuit) \qquad L^+(|f|) < \infty \Leftrightarrow L^+(f_+) < \infty \text{ and } L^+(f_-) < \infty$$

If the following equivalence holds, then we say f is integrable wrt μ, and define its integral

$$(\heartsuit\heartsuit) \qquad \int_\Omega f d\mu := L^+(f_+) - L^+(f_-) \in \mathbb{R}$$

Proof of equivalence in ($\heartsuit$). $\Rightarrow$**:** We have $f_+, f_- \leq |f|$. Hence, $L^+(f_+) \leq L^+(|f|) < \infty$, and $L^+(f_-) \leq L^+(|f|) < \infty$.

$\Leftarrow$**:** We have $|f| = f_+ + f_-$, hence $L^+(|f|) = L^+(f_+ + f_-) = L^+(f_+) + L^+(f_-)$.

□

Notation 16.4.

$$\mathcal{L}^1(\mu) := \{f \in \mathrm{Bor}(\Omega, \mathbb{R}) \mid f \text{ is integrable}\}$$

Proposition 16.5. *$\mathcal{L}^1(\mu)$ is a linear subspace of $Bor(\Omega, \mathbb{R})$.*

Proof. **Claim 1:** Let $f, g \in \mathcal{L}^1(\mu)$. Then, $f + g \in \mathcal{L}^1(\mu)$
We have $|f+g| \leq |f|+|g|$ implies $L^+(|f+g|) \leq L^+(|f|+|g|) = L^+(|f|)+L^+(|g|) < \infty$, because $f, g \in \mathcal{L}^1(\mu)$. Hence, $L^+(|f+g|) < \infty$, and $f + g \in \mathcal{L}^1(\mu)$.

Claim 2: Let $f \in \mathcal{L}^1(\mu)$ and $\alpha \in \mathbb{R}$. Then, $\alpha f \in \mathcal{L}^1(\mu)$.
We have $|\alpha f| = |\alpha| \cdot |f|$. Hence, $L^+(|\alpha f|) = L^+(|\alpha| \cdot |f|) = |\alpha| \cdot L^+(|f|) < \infty$. Hence, $\alpha f \in \mathcal{L}^1(\mu)$.

□

Next, we state a lemma to show the map, $\left\{ \begin{array}{l} I : \mathcal{L}^1(\mu) \to \mathbb{R} \\ I(f) = \int_\Omega f d\mu \end{array} \right\}$, is linear.

Lemma 16.6. *Let $f \in \mathcal{L}^1(\mu)$, and suppose $f = h_1 - h_2$, with $h_1, h_2 \in Bor^+(\Omega, \mathbb{R})$, such that $L^+(h_1) < \infty, L^+(h_2) < \infty$. Then,*

$$\int_\Omega f d\mu = L^+(h_1) - L^+(h_2)$$

Proof. We have $f = f_+ - f_- = h_1 - h_2$, which implies $f_+ + h_2 = f_- + h_1$. So, we get

$$L^+(f_+ + h_2) = L^+(f_- + h_2)$$

$$(\blacklozenge) \qquad \Rightarrow L^+(f_+) + L^+(h_2) = L^+(f_-) + L^+(h_1)$$

All quantities in (✦) are finite. So, we can do algebra with them to get

$$L^+(h_1) - L^+(h_2) = L^+(f_+) - L^+(f_-) = \int_\Omega f d\mu$$

□

Proposition 16.7. *(1) If $f, g \in \mathcal{L}^1(\mu)$, then $\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$.*

(2) If $f \in \mathcal{L}^1(\mu), \alpha \in \mathbb{R}$, then $\int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$.

Proof of (1). Let $f = f_+ - f_-, g = g_+ - g_-$, which gives

$$f + g = (f_+ - f_-) + (g_+ - g_-) = (f_+ + g_+) - (f_- + g_-) = h_1 - h_2$$

Where $h_1 = f_+ + g_+, h_2 = f_- + g_-$. Observe $h_1, h_2 \in \mathrm{Bor}^+(\Omega, \mathbb{R})$, with

$$L^+(h_1) = L^+(f_+ + g_+) = L^+(f_+) + L^+(g_+) < \infty$$
$$L^+(h_2) = L^+(f_- + g_-) = L^+(f_-) + L^+(g_-) < \infty$$

So, by Lemma 16.6 we get

$$\begin{aligned}\int_\Omega f d\mu = L^+(h_1) - L^+(h_2) &= \big(L^+(f_+) + L^+(g_+)\big) - \big(L^+(f_-) - L^+(g_-)\big) \\ &= \big(L^+(f_+) - L^+(f_-)\big) + \big(L^+(g_+) - L^+(g_-)\big) \\ &= \int_\Omega f d\mu + \int_\Omega g d\mu\end{aligned}$$

□

Remark 16.8. For $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$, we have $f \in \mathcal{L}^1(\mu) \Leftrightarrow L^+(f) < \infty$, and if these two facts hold, then we have

$$\int_\Omega f d\mu = L^+(f) \in [0, \infty)$$

Indeed, for $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$, we have $f_+ = f, f_- = \underline{0}$. Hence,

$$f \in \mathcal{L}^1(\mu) \Leftrightarrow \begin{array}{l} L^+(f_+) < \infty, \\ L^+(f_-) < \infty \end{array} \Leftrightarrow \begin{array}{l} L^+(f) < \infty, \\ L^+(\underline{0}) < \infty \end{array} \Leftrightarrow L^+(f) < \infty$$

Moreover, if it foes hold that $f \in \mathcal{L}^+(f)$, then we have

$$\int_\Omega df\mu = L^+(f_+) - L^+(f_-) = L^+(f) - L^+(\underline{0}) = l^+(f)$$

Hence, we have 2 incarnations of the Lebesgue measure:

"L$^+$": $\int_\Omega f d\mu = L^+(f) \in [0, \infty]$ for $f \in \mathrm{Bor}^+(\Omega, \mathbb{R})$. Note $\int_\mu$ is always defined, but may be equal to ∞.

"$\mathcal{L}^1$": $\int_\Omega f d\mu \in \mathbb{R}$, for $f \in \mathcal{L}^1(\mu)$ Note this is always a finite quantity, but only defined when

$$\int_\Omega |f| d\mu = L^+(|f|) < \infty$$

Remark 16.9. (1)

$$f, g \in \mathcal{L}^1(\mu), f \leq g \Rightarrow \int_\Omega f d\mu \leq \int_\Omega g d\mu$$

(2)

$$f \in \mathcal{L}^1(\mu) \Rightarrow \left| \int_\Omega f d\mu \right| \leq \int_\Omega |f| d\mu = L^+(|f|) < \infty$$

Why is this?

(1) ...

(2) ...

17. The "$\mathcal{L}^2(\mu)$" Space: *And its variation, $L^2(\mu)$*

17.1. Lecture 33, July 22.

Notation 17.1.

$$\mathcal{L}^2(\mu) := \left\{ f \in \mathrm{Bor}(\Omega, \mathbb{R}) \mid \int_\Omega f^2 d\mu < \infty \right\}$$
$$= \left\{ f \in \mathrm{Bor}(\Omega, \mathbb{R}) \mid f^2 \in \mathcal{L}^1(\mu) \right\}$$

Note that $\int_\Omega f^2 d\mu < \infty$ is the same as $L^+(f^2) < \infty$.

Proposition 17.2. *$\mathcal{L}^2(\mu)$ is a linear subspace of Bor$(\Omega, \mathbb{R})$.*

Proof. ... □

Proposition and Definition 17.3. *If $f, g \in \mathcal{L}^2(\mu)$, then $f \cdot \in \mathcal{L}^1(\mu)$, and hence it makes sense to define*

(❖) $$\langle f, g \rangle = \int_\Omega f \cdot g d\mu \in \mathbb{R}$$

Proof. We have $|f \cdot g| \leq \frac{1}{2}\left(f^2 + g^2\right)$, since

$$\frac{1}{2}\left(|f| - |g|\right)^2 \geq 0 \Rightarrow \frac{1}{2}\left(|f|^2 - 2|f| \cdot |g| + |g|^2\right) \geq 2$$

Hence,

$$\int_\Omega |f \cdot g| d\mu \leq \frac{1}{2} \int_\Omega f^2 d\mu + \frac{1}{2} \int_\Omega g^2 d\mu < \infty$$

Thus, $f \cdot g \in \mathcal{L}^1(\mu)$. □

Remark 17.4. A consequence of Proposition 17.3 is that $cL^2(\mu) \subseteq cL^1(\mu)$. This is because we can use $g := \underline{1}$, where $\underline{1}(x) = 1, \forall x \in \Omega$. So we have

$$\int_\Omega \underline{1}^2 d\mu = \int_\Omega \underline{1} d\mu = \mu(\Omega) < \infty$$

Observe then that $f \in \mathcal{L}^2(\mu)$ implies $f = f \cdot \underline{1} \in \mathcal{L}^1(\mu)$.

Proposition 17.5. *The rule indicated in (❖) has the following properties:*
Bilinear:

$$\begin{cases} \langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle & = \alpha_1\langle f_1, g\rangle + \alpha_2\langle f_2, g\rangle, \quad \forall f_1, f_2, g \in \mathcal{L}^2(\mu), \alpha_1, \alpha_2 \in \mathbb{R} \\ \langle f, \beta_1 g_1 + \beta_2 g_2\rangle & = \beta_1\langle f, g_1\rangle + \beta_2\langle f, g_2\rangle, \quad \forall f, g_1, g_2 \in \mathcal{L}^2(\mu), \beta_1, \beta_2 \in \mathbb{R} \end{cases}$$

Symmetric: $\langle f, g\rangle = \langle g, f\rangle, \forall f, g \in \mathcal{L}^2(\mu)$

Non-Negative Definite: $\langle f, f\rangle \geq 0, \forall f \in \mathcal{L}^2(\mu)$

NOTE: We don't have $\langle f, f\rangle = 0 \Leftrightarrow f = \underline{0}$. Because of this, $\big(\mathcal{L}^2(\mu), \langle\cdot,\cdot\rangle\big)$ is not exactly an inner product space in the sense we disscused in Chapter 1.

Notation 17.6. For $f \in \mathcal{L}^2(\mu)$, we denote

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = \sqrt{\int_\Omega f^2 d\mu} \in [0, \infty)]$$

Proposition 17.7. *(1)*

((C-S)) $$|\langle f, g\rangle| \leq \|f\|_2 \cdot \|g\|_2, \forall f, g \in \mathcal{L}^2(\mu)$$

(2) The map $\begin{pmatrix} \mathcal{L}^2(\mu) \to \mathbb{R} \\ f \to \|f\|_2 \end{pmatrix}$ *is a semi-norm, which means it has the following properties.*
- $\|f\|_2 \geq 0$
- $\|\alpha f\|_2 = |\alpha| \cdot \|f\|_2$
- $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$

17.2. **Lecture 34, July 24 - *How to Handle a Semi-Norm.*** For the lecture, suppose we have a vector space, V, over $\mathbb{R}$, and a semi-norm, $\|\cdot\|$, on V.

Lemma 17.8. *Null-Space*
Let $\mathcal{N} = \{v \in V \mid \|v\| = 0\}$. Then, $\mathcal{N}$ is a linear subspace of V.

Proof. ... □

Notation 17.9. Let $\mathcal{N} \subseteq V$ be as in Lemma 17.8, and let us denote $\mathcal{Q} = V/\mathcal{N}$ to be the *quotient vector space.*

Our way to think about $\mathcal{Q}$ is that it's a vector space over $\mathbb{R}$, and one has a linear surjective map,

$$\begin{cases} V \to \mathcal{Q} \\ v \to \hat{v} \end{cases} \text{ such that for } v_1, v_2 \in V, \hat{v}_1 = \hat{v}_2 \Leftrightarrow v_1 - v_2 \in \mathcal{N}$$

We call this the *"hat placing map"*, and it is surjective to mean every $\xi \in \mathcal{Q}$ is of the form $\hat{v}$ for some $v \in V$. That is, ξ is not unique to v.

Proposition and Definition 17.10. *Take $\mathcal{N} \subseteq V$ and $\mathcal{Q} = V/\mathcal{N}$ as above. Define a map,*

$$(✾) \qquad \begin{pmatrix} \mathcal{Q} \to \mathbb{R} \\ \xi \to \|\xi\|_{\mathcal{Q}} \end{pmatrix} \text{ as follows:}$$

Given $\xi \in \mathcal{Q}$, pick $v \in V$ such that $\xi = \hat{v}$, and put $\|xi\|_{\mathcal{Q}} = \|v\|$. Then, the map $\|\cdot\|$ from (✾) is well defined, and $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is a normed vector space.

Proof. *Why is $\|\xi\|_{\mathcal{Q}}$ well defined?*

Suppose $\xi \in \mathcal{Q}$ is written as $\xi = \hat{v}_1$ and $\xi = \hat{v}_2$, with $v_1, v_2 \in V$. So then $\hat{v}_1 = \hat{v}_2\,(=\xi)$, hence $\|v_1 - v_2\| = 0$. But, then we write

$$\|v_1\| = \|(v_1 - v_2) + v_2\| \leq \underbrace{\|v_1 - v_2\|}_{=0} + \|v_2\|$$

Hence, $\|v_1\| \leq \|v_2\|$. Likewise, we find that $\|v_2\| = \|(v_2 - v_1) + v_1\| \leq \|v_1\|$. Thus, we get $\|v_1\| = \|v_2\|$, confirming that $\|\xi\|_{\mathcal{Q}}$ is well defined.

Why is $\|\cdot\|_{\mathcal{Q}}$ a norm on $\mathcal{Q}$?

We have

$$\begin{array}{ll} ✷\|\alpha\xi\|_{\mathcal{Q}} = |\alpha| \cdot \|\xi\|_{\mathcal{Q}} & ✷\|\xi_1 + \xi_2\|_{\mathcal{Q}} \leq \|\xi_1\|_{\mathcal{Q}} + \|\xi_2\|_{\mathcal{Q}} \\ ✷\|\alpha\xi\|_{\mathcal{Q}} \geq 0 & ✷ \text{ and } \|\alpha\xi\|_{\mathcal{Q}} = 0 \Leftrightarrow \xi = 0_{\mathcal{Q}} \end{array}$$

That is, ${}_{\mathcal{Q}} = \hat{0}_V$.

"$\Rightarrow$" Take $\xi \in \mathcal{Q}$ such that $\|\xi\|_{\mathcal{Q}} = 0$. Write $\xi = \hat{v}$ with $v \in V$, and observe that $0 = \|\xi\|_{\mathcal{Q}} = \|v\|$ which implies $v \in \mathcal{N}$, and for $\hat{v} = \hat{0}_V = 0_{\mathcal{N}}$, since $v = 0_V \in \mathcal{N}$. Hence, $\xi = 0_{\mathcal{Q}}$, as required. □

17.3. Lecture 35, July 26 - *The "a.e wrt μ" annoyance and the space $L^2(\mu)$.*

Notation 17.11. Let $\mathcal{N} + \{f \in \mathcal{L}^2(\mu) \mid \|f\|_2 = 0\}$. We denote

$$L^2(\mu) := \mathcal{L}^2(\mu)\big/\mathcal{N} = \{\hat{f} \mid f \in \mathcal{L}^2(\mu)\}$$
$$\text{with } \hat{f} = \hat{g} \Leftrightarrow f - g \in \mathcal{N}$$

On $L^2(\mu)$, we consider the norm $\|\cdot\|_2$ defined by

$$\|\hat{f}\|_2 = \|f\|_2 = \sqrt{]int_\Omega f^2 d\mu}, f \in \mathcal{L}^2(\mu)$$

Thus, we get a normed vector space, $\big(L^2(\mu), \|\cdot\|_2\big)$.

Definition and Remark 17.12. Take $f, g \in \mathrm{Bor}(\Omega, \mathbb{R})$. We say that f and g are equal *almost everywheyre wrt μ* [8] to mean $\exists S \in \mathcal{A}$ with $\mu(S) = 0$ such that $f(x) = g(x), \forall x \in \Omega \backslash S$.

Proposition 17.13. *Let $\mathcal{N} - \{f \in \mathcal{L}^2(\mu) \mid \|f\|_2 = 0\} = \{f \in \mathcal{L}^2(\mu) \mid \int_\Omega f^2 d\mu < \infty\}$, as in Notation 17.11. Then,*

$$\mathcal{N} = \{f \in Bor(\Omega, \mathbb{R}) \mid f = \underline{0} \ a.e\text{-}\mu\}$$

[8] This is shortened to *a.e-μ*. In probability, we instead use *almost surely (a.s)*.

Remark 17.14. For $f, g \in \mathcal{L}^2(\mu)$, we now see that $\hat{f} = \hat{g} \Leftrightarrow f = g$ a.s-μ. Indeed,

$$\begin{aligned} \hat{f} = \hat{g} &\Leftrightarrow f - g \in \mathcal{N} \\ \text{(by Prop 17.13)} \qquad &\Leftrightarrow f - g = \underline{0} \text{ a.s-}\mu \\ &\Leftrightarrow f = g \text{ a.s-}\mu \end{aligned}$$

Part 3. Fourier Series

18. Back to Hilbert Spaces: *The case of* $L^2[-\pi, \pi]$

18.1. **Lecture 36, July 29.** We now look at the measure space $(\Omega, \mathcal{A}, \mu)$, with $\Omega = [-\pi, \pi]$, $\mathcal{A} = \mathcal{B}_{[-\pi,\pi]}$ is its Borel σ-algebra, and $\mu = \mu_{[-\pi,\pi]}$ is the Lebesgue measure on $[-\pi, \pi]$. For this measure space, we consider the space $\mathcal{L}^2(\mu)$ and its version $L^2(\mu)$, which is usually denoted as $L^2[-\pi, \pi]$. So, $L^2[-\pi, \pi] = \{\hat{f} \mid f \in \mathcal{L}^2(\mu)\}$, which means $f : [-\pi, \pi] \to \mathbb{R}$, is ${}^{\mathcal{B}_{[-\pi,\pi]}}/_{\mathcal{B}_{\mathbb{R}}}$-measurable, and $\int_{[-\pi,\pi]} f^2 d\mu < \infty$. Thus, we get the normed vector space, $\big(L^2[-\pi, \pi], \| \cdot \|_2\big)$. Comparing this to $(C([-\pi, \pi], \mathbb{R}), \| \cdot \|_\infty)$, we get the following result.

Remark 18.1. If $f \in C([-\pi, \pi], \mathbb{R})$, then $f \in \mathcal{L}^2[-\pi, \pi]$, and $\|f\|_2 \leq \sqrt{2\pi}\|f\|_\infty$. This is because f being continuous implies it's ${}^{\mathcal{B}_{[-\pi,\pi]}}/_{\mathcal{B}_{\mathbb{R}}}$-measurable. Denote $\|f\|_\infty =: \alpha \in [0\infty)$. Then,

$$f^2 \leq \alpha \cdot \underline{1} \Rightarrow \int_{[-\pi,\pi]} f^2 d\mu \leq \int_{[-\pi,\pi]} \alpha^2 \underline{1} d\mu = \alpha^2 \mu([-\pi, \pi])$$

Proposition 18.2. *Let* $\Phi : C[-\pi, \pi] \to L^2[-\pi, \pi], \Phi(f) = \hat{f}$. *Then,*

(1) Φ *is linear and continuous (infact Lipschitz).*
(2) Φ *is injective.*
(3) Φ *has dense range. i.e,* $Ran(\Phi) = \{\hat{f} | f \in C[-\pi, \pi]\}$ *is a dense subspace of* $L^2[-\pi, \pi]$.

Proof. ... □

Theorem 18.3. *On* $L^2[-\pi, \pi]$, *it makes sense to define an inner product by putting*

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle = \int_{[-\pi,\pi]} f \cdot g d\mu$$

This gives us a Hilbert space.

Remark 18.4. In Question 2 of Homework Assignment 4, we had an inner product on $C[-\pi, \pi]$ where

$$\text{(Riemann Intagral)} \qquad \langle f, g \rangle = \int_{pi}^{\pi} f(x)g(x)dx$$

Recall that $(C[-\pi, \pi], \langle \cdot, \cdot \rangle)$ is an inner product space, but nor complete. However, Proposition 18.2 and Theorem 18.3 say that $L^2[-\pi, \pi]$ is the completion of that inner product space.

Remark 18.5. Question 2 of Homework Assignment 4 also introduced an orthonormal system $(\xi_n)_{n=1}^{\infty}$ on $[-\pi, \pi]$ where $\xi_1(x) = 1/\sqrt{2}\pi$, and

$$\xi_{2k} = \frac{1}{\sqrt{\pi}} \sin(xk), \xi_{2k+1} = \frac{1}{\sqrt{\pi}} \cos(xk), \forall k \in \mathbb{N}, -\pi \leq x \leq \pi$$

It can be shown the vectors $(\hat{\xi}_n)_{n=1}^{\infty}$ form an orthonormal basis on $L^2[-\pi, \pi]$.

Definition 18.6. For every $f \in \mathcal{L}^2[-\pi, \pi]$, the sequence of real numbers $(c_n)_{n=1}^{\infty}$ with $c_n = \langle f, \xi_n \rangle, n \in \mathbb{N}$, are called *fourier series coefficients.* [9]

Remark 18.7. Results we found in Chapter 5 still apply! Namely, Rieze-Fisher (Theorem 5.4), and Corollary 5.6.

(Parseval)
$$\|f\|_2^2 = \sum_{n=1}^{\infty} c_n^2$$

Example 18.8. Let $f : [-\pi, \pi] \to \mathbb{R}$, such that $f(t) = t$. Compute the fourier coefficients,

$$c_1 = 1/\sqrt{2}\pi,\ c_{2k+1} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \underbrace{cos(kt)}_{\text{odd}} \cdot tdt = 0$$

$$c_{2k} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin(kt) \cdot tdt = \frac{(-1)^{k+1} 2 \cdot \sqrt{\pi}}{k}$$

By Parseval,

$$\|f\|_2^2 = \sum_{k=1}^{\infty} \frac{4\pi}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Since

$$\|f\|_2^2 = \int_{\pi}^{\pi} f^2 dt = \frac{2\pi^2}{3}$$

[9]These are precicely the $\hat{\xi}_n$-coefficients of $\hat{f} \in L^2[-\pi, \pi]$ we saw in Chapter 5.

References

[1] Andrew M. Bruckner, Judith B. Bruckner, Brian S. Thomson. *Real Analysis*, Prentice-Hall Publishers, 1997.

[2] A. Nica. Lecture summaries for PMath 450 lectures in Summer Term 2024, available on the Learn web-site of the course.

[3] H.L. Royden. *Real Analysis, 3rd edition*, Prentice-Hall Publishers, 1988.

[4] Walter Rudin. *Real And Complex Analysis*, McGraw-Hill, 1986.

Faculty of Mathematics, University of Waterloo
Email address: magagri@uwaterloo.ca