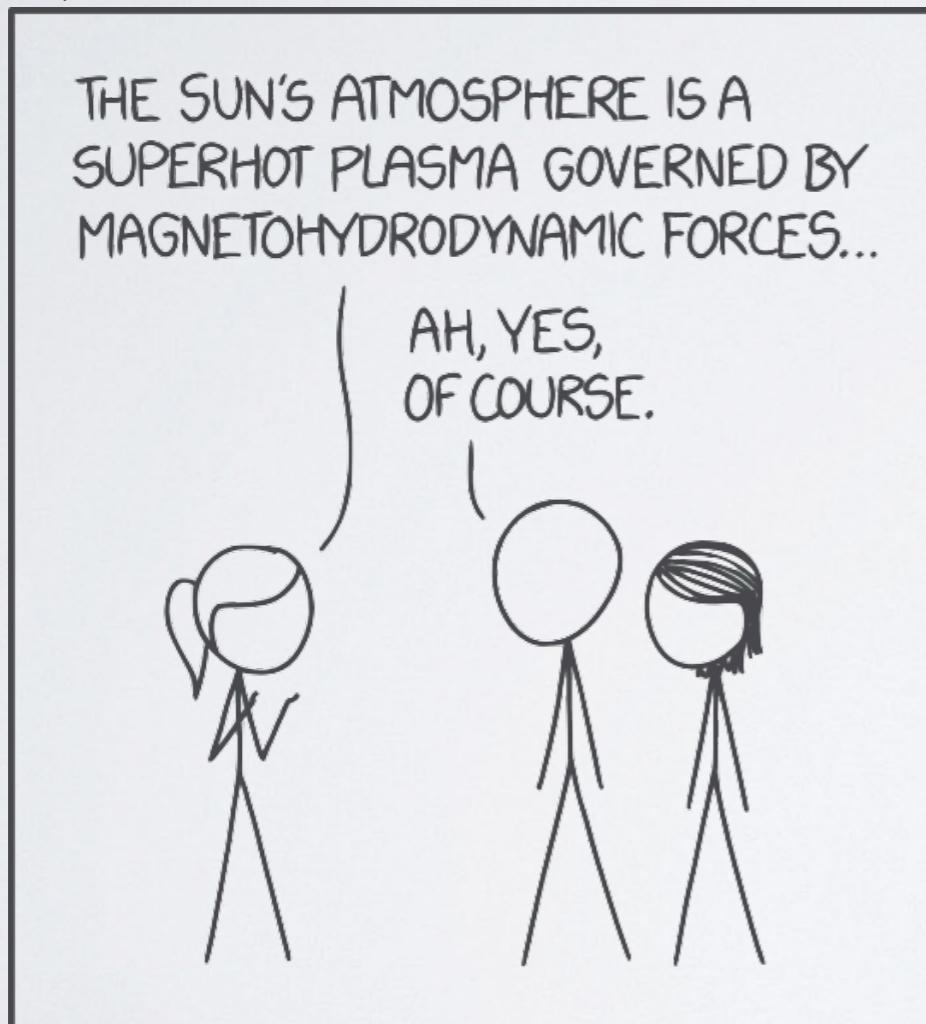


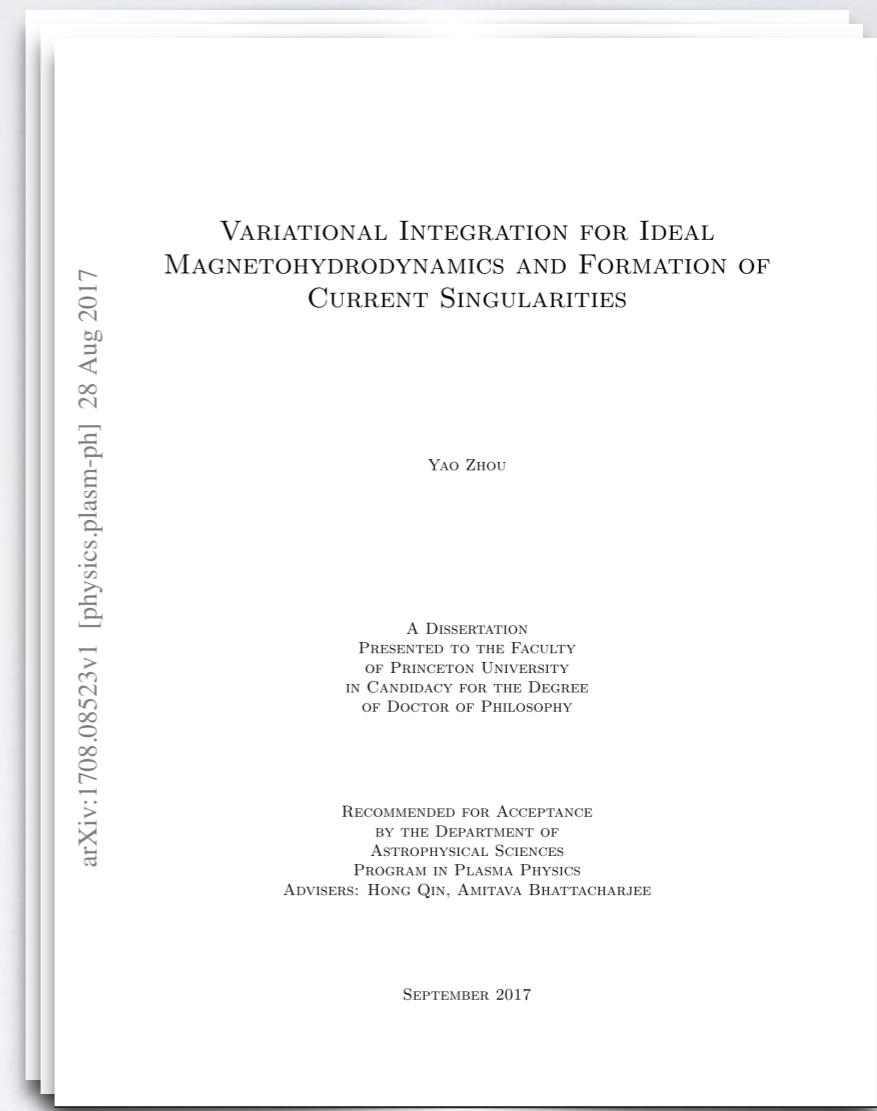
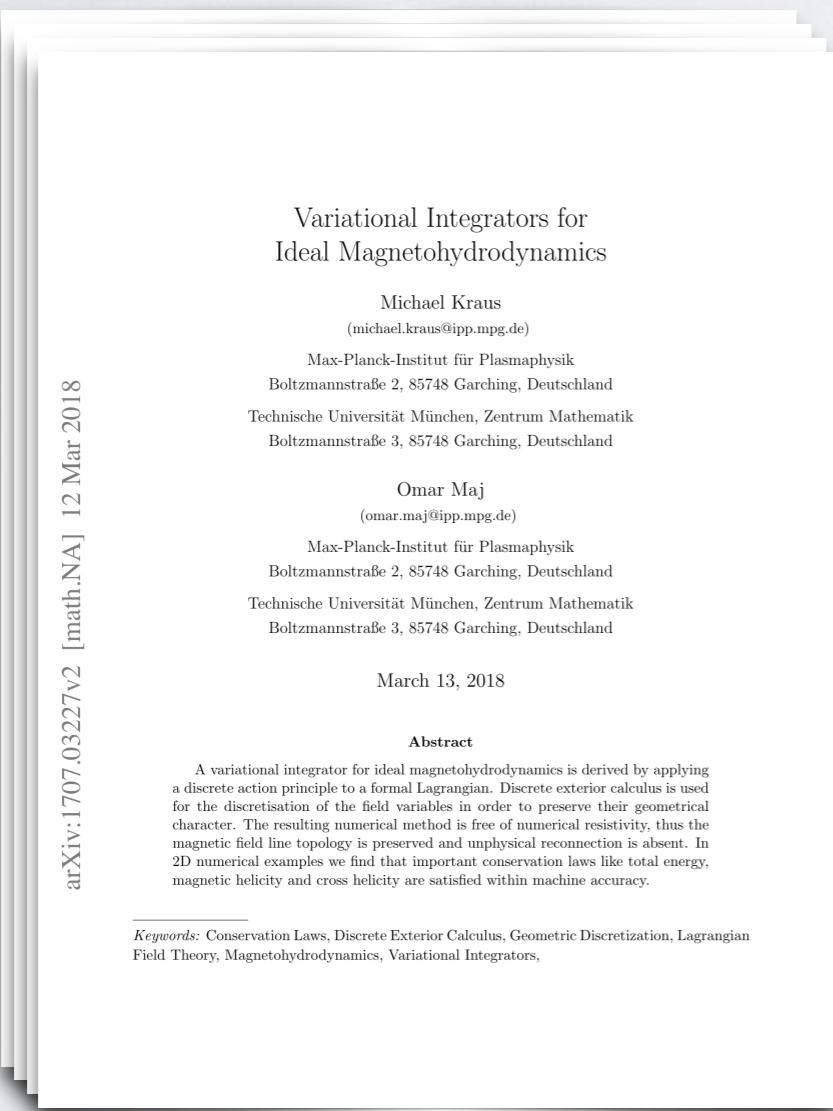
Magnetohydrodynamics and the Geometry of Conservation Laws in Physics

<https://xkcd.com/1851/>



Mark Gillespie

Why?



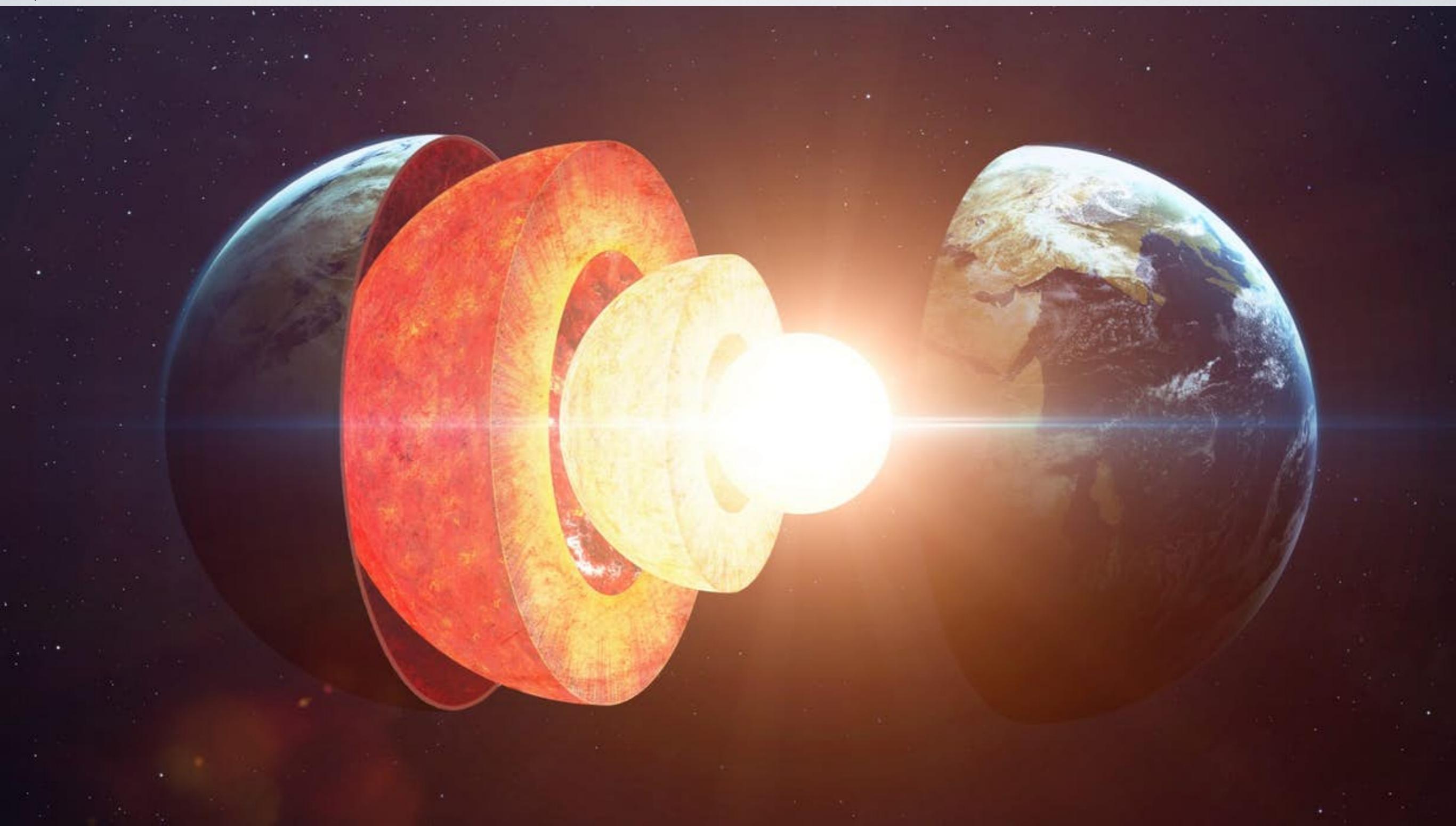
Why?

https://upload.wikimedia.org/wikipedia/commons/e/e3/Magnificent_CME_Erupts_on_the_Sun_-_August_31.jpg



Why?

<https://newatlas.com/earth-inner-core-solid-soft/56882/>



Structure-Preserving Integrators

Eurographics/ACM SIGGRAPH Symposium on Computer Animation (2006)
M.-P. Cani, J. O'Brien (Editors)

Geometric, Variational Integrators for Computer Animation

L. Kharevych Weiwei Y. Tong E. Kanso[†] J. E. Marsden P. Schröder M. Desbrun

Abstract
We present a new class of geometric, variational integrators for particular time-stepping methods. These methods are able to preserve properties such as linear behavior, even when the system is non-linear. They are also simply; finally they are able to preserve properties set by the user. An update step is added to the implementation of two integrators. This implementation is called the “geometric variational integrator” and it is applicable to the application.

1. Introduction

Mathematical models of physical systems (whether in computer animation or in other fields) generally involve differential equations. Solving a physical system's equations involves integrating the system forward in time. This is often done by allowing the computer to move a ball (*i.e.*, its position) through space over time. Although this is a simple, direct solution, it can be inaccurate. In some cases, a system is so complex that it is necessary to resort to numerical methods. A numerical description of a system is a mathematical model that describes how the system changes over time. A significant amount of work has been done on how to deal with some types of systems, leading to a wide variety of numerical methods. One of the most common approaches is to use finite difference methods. These methods are based on the idea of approximating derivatives by differences between values at different points in time. Another approach is to use finite element methods. These methods are based on the idea of approximating derivatives by differences between values at different points in space.

Keywords: discrete differential geometry, fluid simulation, Schrödinger operator
Concepts: • Mathematics of computing → Partial differential equations; • Computing methodologies → Physical simulation; • Applied computing → Physics;

1 Introduction

We introduce *incompressible Schrödinger flow* (ISF), a new method to simulate incompressible fluids. In it, the fluid state is represented by a C^2 -valued wave function evolving under the Schrödinger equation subject to incompressibility constraints. The underlying dynamical system is Hamiltonian and governed by the kinetic energy of the fluid together with an energy of Landau-Lifshitz type. The latter ensures that dynamics due to thin vortical structures, all important for visual simulation, are faithfully reproduced. This enables robust simulation of intricate phenomena such as vortical wakes and interacting vortex filaments, even on modestly sized grids. Our implementation uses a simple splitting method for time integration, employing the FFT for Schrödinger evolution as well as constraint projection. Using a standard penalty method we also allow arbitrary obstacles. The resulting algorithm is simple, unconditionally stable, and efficient. In particular it does not require any Lagrangian techniques for advection or to counteract the loss of vorticity. We demonstrate its use in a variety of scenarios, compare it with experiments, and evaluate it against benchmark tests. A full implementation is included in the ancillary materials.

Keywords: discrete differential geometry, fluid simulation, Schrödinger operator

Concepts: • Mathematics of computing → Partial differential equations; • Computing methodologies → Physical simulation; • Applied computing → Physics;

1 Introduction

We introduce *incompressible Schrödinger flow* (ISF), a new method to simulate incompressible fluids (Fig. 1, middle). Instead of describing the fluid evolution in terms of the velocity or vorticity field, ISF evolves a two-component wave function $\psi = (\psi_1, \psi_2)^T: M \rightarrow \mathbb{C}^2$, which encodes the fluid state on a 3D domain M . The classical fluid density ρ and fluid velocity $v = (v_1, v_2, v_3)^T$ are extracted from ψ as

$$\rho|\psi|^2 = \langle\psi, \psi\rangle_R \quad \text{and} \quad \rho v_\alpha = \hbar \left(\frac{\partial \psi}{\partial x_\alpha}, i\psi \right)_R \quad \alpha = 1, 2, 3 \quad (1)$$

where $\langle\phi, \psi\rangle_R = \operatorname{Re}(\langle\phi, \psi\rangle_C) = \operatorname{Re}(\bar{\phi}_1\psi_1 + \bar{\phi}_2\psi_2)$. The time evolution of these wave functions is governed by the Schrödinger equation

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2}\Delta\psi + p\psi - \frac{\varepsilon\psi}{\hbar^2}|_{\partial M} = 0 \quad (2)$$

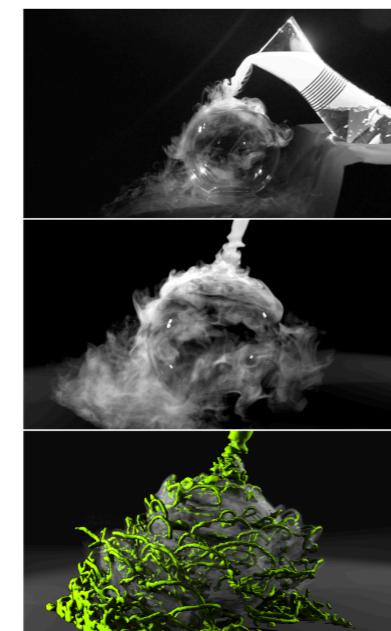


Figure 1: Comparing experiment (dry ice vapor, top) with ISF simulation (middle), followed by a visualization of the underlying wave function ψ . Vorticity is concentrated within the green region.

subject to the constraints

$$\langle\Delta\psi, i\psi\rangle_R = 0 \quad \text{and} \quad |\psi|^2 = 1, \quad (2)$$

which correspond to $\operatorname{div}(v) = 0$ and $\rho = 1$ in the classical variables (Sec. 4.1). The scalar potential $p: M \rightarrow \mathbb{R}$ in Eq. (1) is the Lagrange multiplier for the divergence constraint (App. A), and we will refer to it as *pressure* in analogy to the Euler equation. The reduced Planck constant \hbar of quantum Physics becomes the *only parameter* for our fluid and controls the quantization of vorticity. For a large range of initial conditions ISF tends to concentrate vorticity in filaments of strength $2\pi\hbar$ (Fig. 1, bottom).

We call Eqs. (1) and (2) the *incompressible Schrödinger equations* and the corresponding flow the *incompressible Schrödinger flow*.

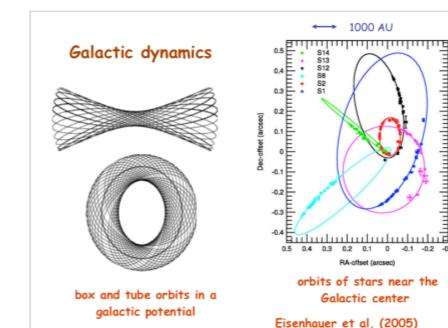
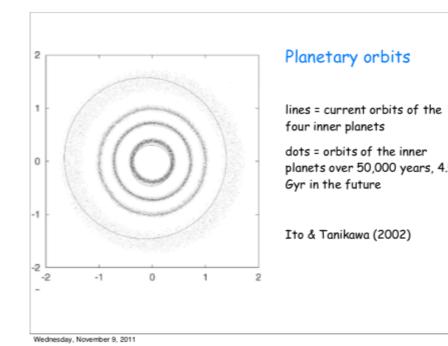
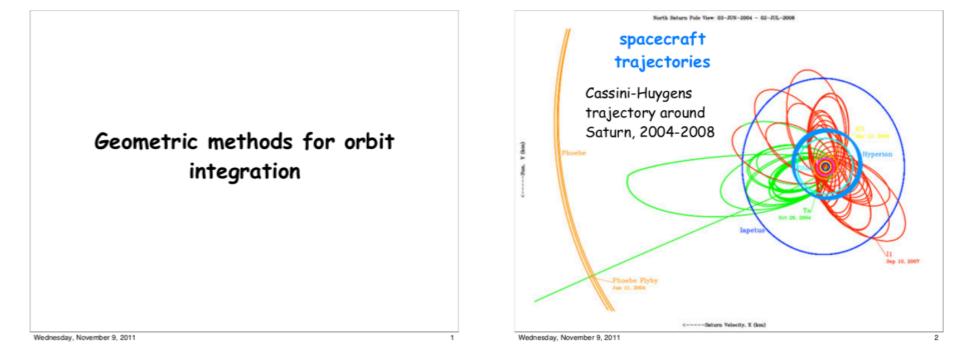
Energy-Preserving Integrators for Fluid Animation

Patrick Mullen Keenan Crane Dmitry Pavlov Yiying Tong Mathieu Desbrun
Caltech Caltech Caltech MSU Caltech

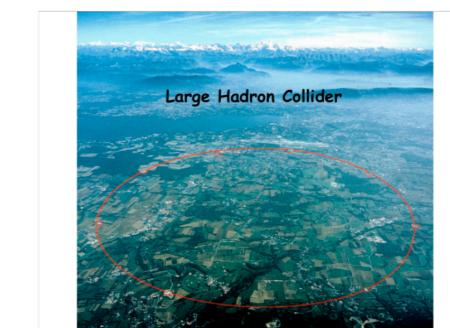


Figure 1: By applying viscosity in fluid simulation, the results become more realistic.

Geometric methods for orbit integration

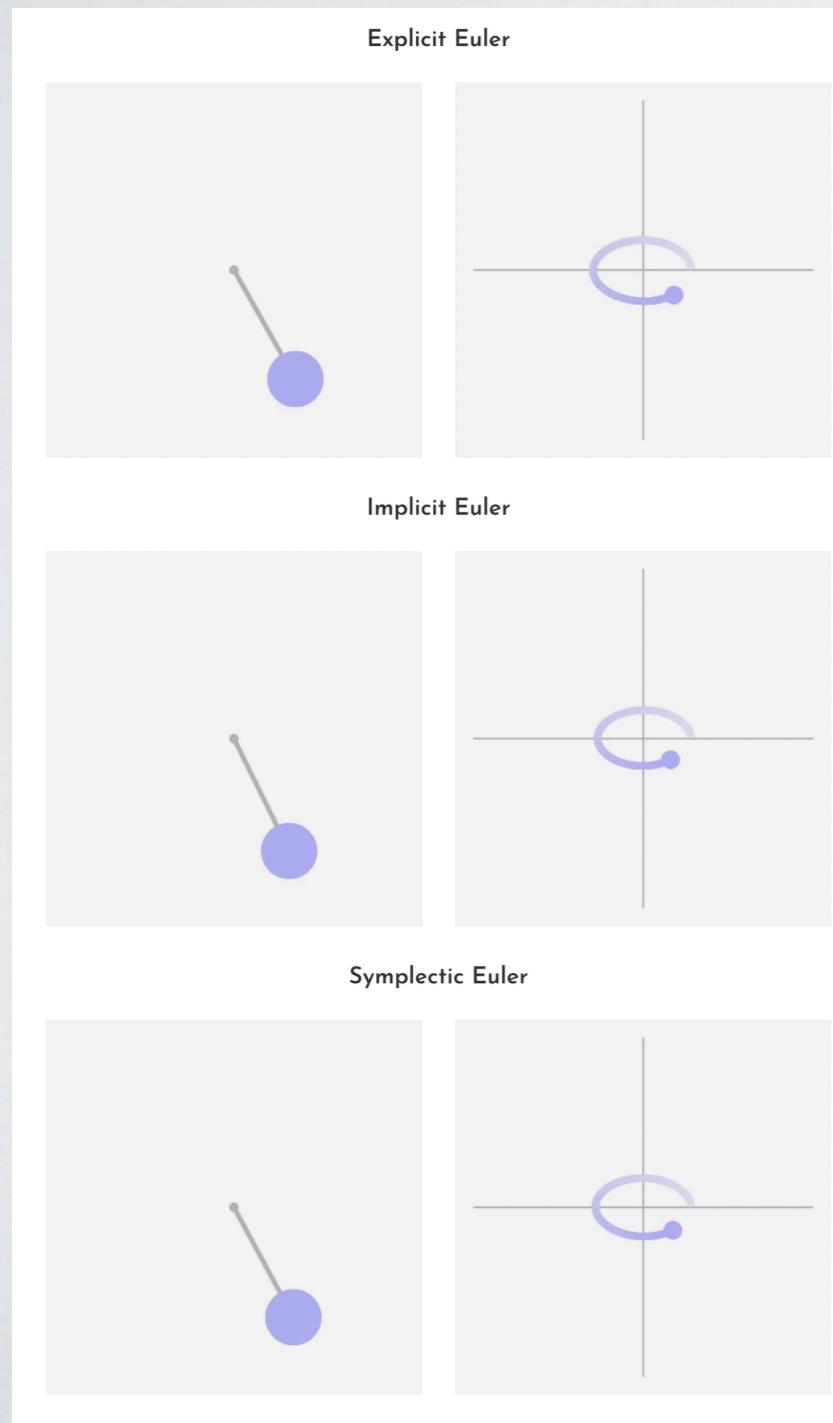


Wednesday, November 9, 2011



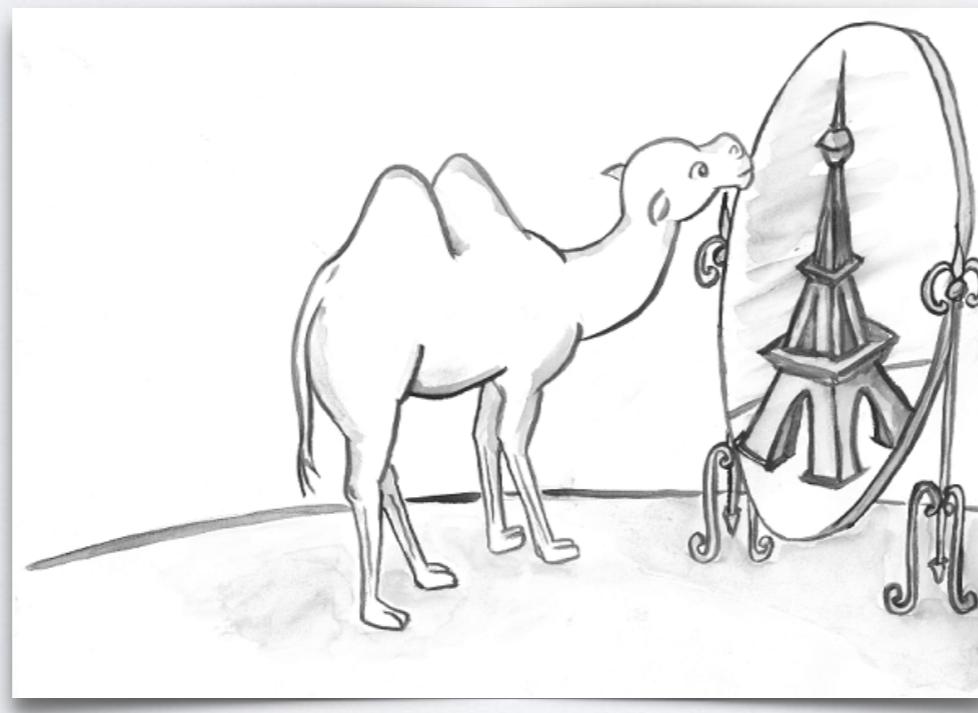
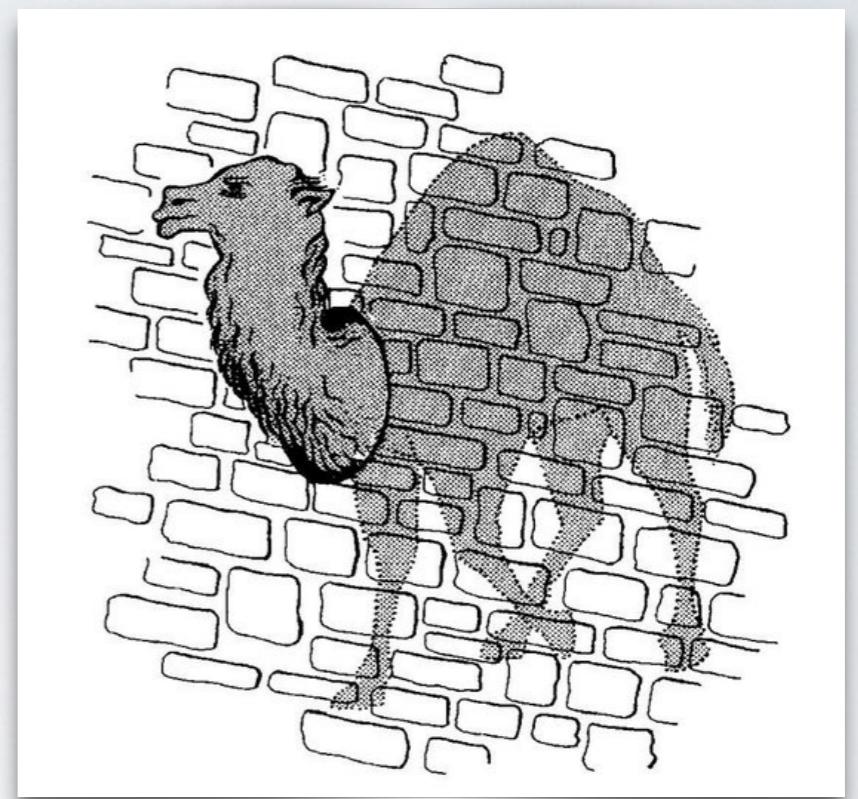
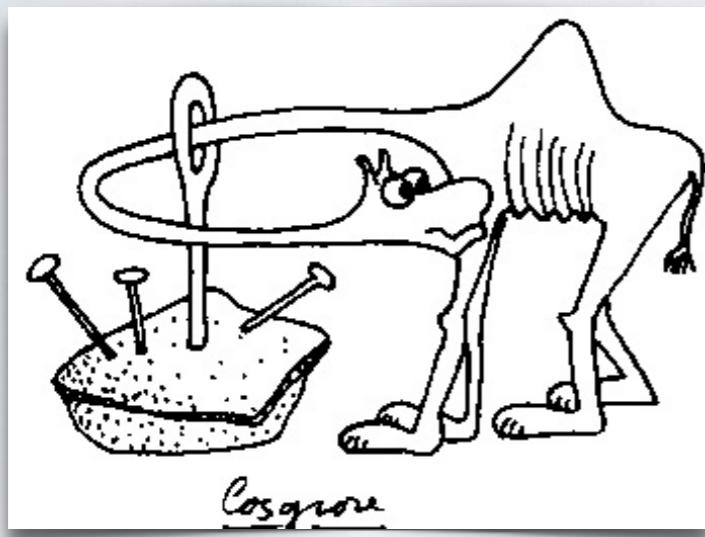
Wednesday, November 9, 2011

Symplectic Integrators

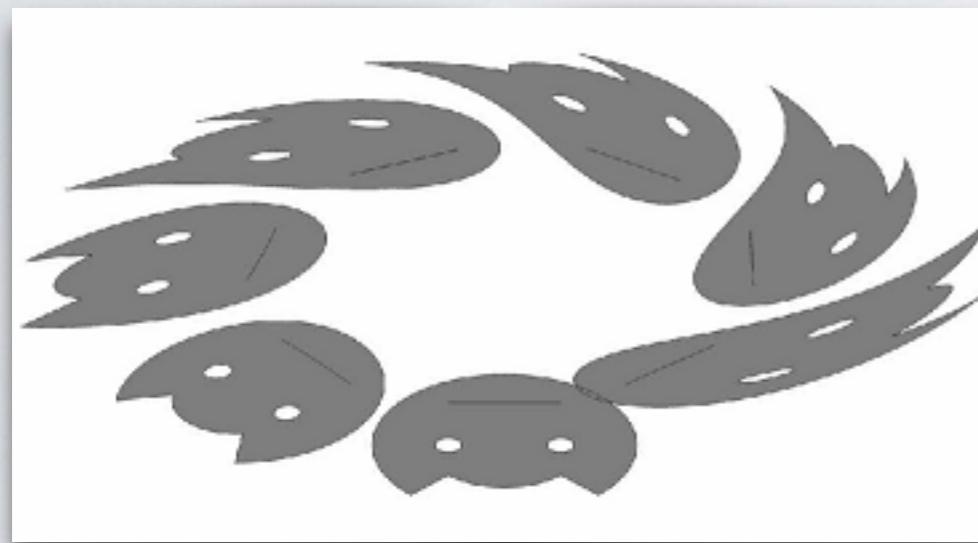


- How you update velocity makes a big difference
- One option makes your simulation *symplectic*
- It captures important features

Symplectic?



Symplectic?



<http://people.bath.ac.uk/tjs42/BNA/bna-res.html>

- In 2D, *symplectic* just means *area-preserving*
- Really, symplectic maps generalize area-preserving maps

Symplectic Euler is Symplectic

- Luckily for us, the pendulum's phase space is 2D.
- How do we measure areas in 2D?
 - Determinants
- Useful fact:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{pmatrix} a \\ c \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$$

Symplectic Euler is Symplectic

- So the matrix $\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ measures (infinitesimal) areas
- Our simulation preserves area (and is thus symplectic) if it “preserves” this matrix

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

Symplectic Euler is Symplectic

- Given a force function $F(q)$, and a time step h , symplectic Euler updates positions (q) and momenta (p) by

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} q_n + hp_n + h^2F(q_n) \\ p_n + hF(q_n) \end{pmatrix} =: T \begin{pmatrix} q_n \\ p_n \end{pmatrix}$$

- What does this do to Ω ?

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{"Area matrix"}$$

T

Update rule

Symplectic Euler is Symplectic

- How is the area of a parallelogram related to the area of $T(\text{parallelogram})$?
- If the parallelogram is tiny, T looks like a linear map, its *linearization* (or *Jacobian*) $L = dT$
- We only need to check that L preserves area

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{"Area matrix"}$$

T Update rule

$L = dT$ Linearization of T

Symplectic Euler is Symplectic

- Note that $\text{Area}(Lv, Lw) = (Lv)^T \Omega(Lw)$
 $= v^T L^T \Omega L w$
 $= v^T [L^T \Omega L] w$
- So our transformation preserves area (and is symplectic) as long as
$$\Omega = L^T \Omega L$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{"Area matrix"}$$

T Update rule

$L = dT$ Linearization of T

Symplectic Euler is Symplectic

- Now, we can just do this computation

$$T \begin{pmatrix} (q_n) \\ (p_n) \end{pmatrix} = \begin{pmatrix} q_n + hp_n + h^2 F(q_n) \\ p_n + hF(q_n) \end{pmatrix} \quad L = \begin{pmatrix} 1 + h^2 \partial_q F & h \\ h \partial_q F & 1 \end{pmatrix}$$

...

$$\Omega = L^T \Omega L$$

(You can also observe that $\det(L) = 1$)

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{"Area matrix"}$$

T Update rule

$L = dT$ Linearization of T

Symplectic?

- In n dimensions, we can define* $\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$
- Symplectic maps are still the maps which satisfy

$$\Omega = L^T \Omega L$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{"Area matrix"}$$

T Update rule

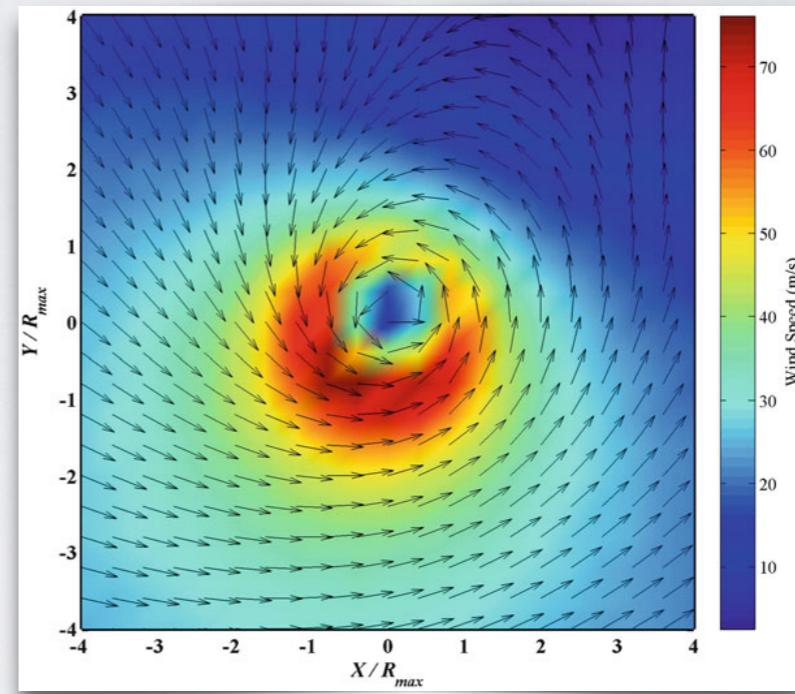
$L = dT$ Linearization of T

How does this relate to camels?

- Symplectic geometry was confusing, even to mathematicians at first
- It's "clear" that symplectic maps preserve volume
- Gromov's non-squeezing theorem

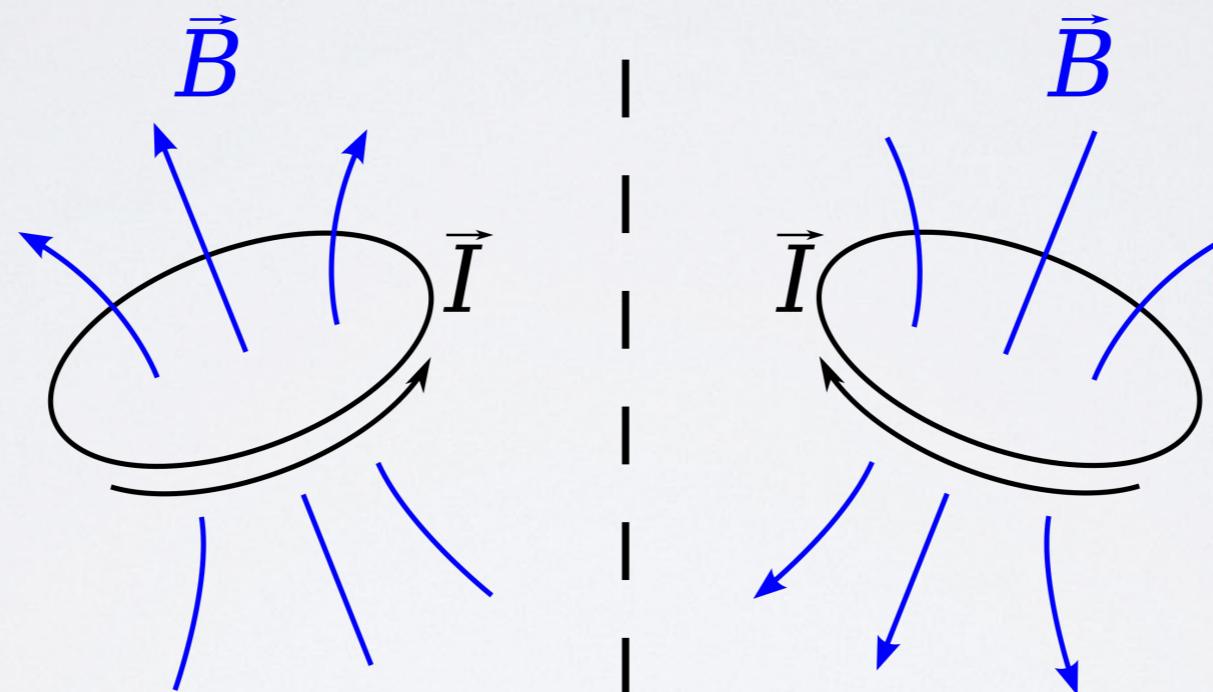
Magnetohydrodynamics

- Physics of conducting fluids
- Key ingredients
 - Velocity field (1 -form) η
 - Magnetic field (2 -form) β



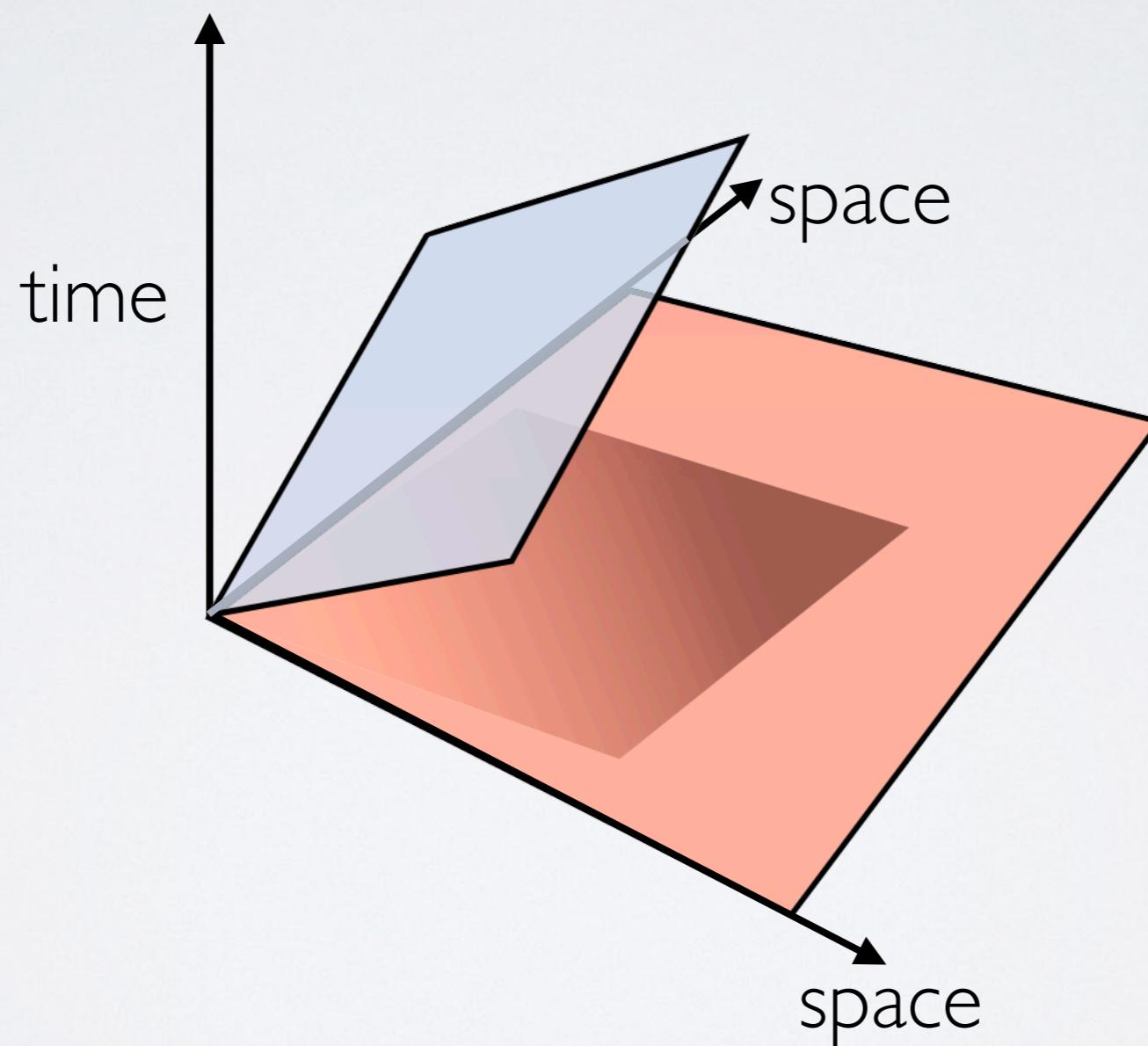
Aside: The Magnetic Field is not a Vector Field

- Wrong symmetries under reflection



- More like an oriented plane than a vector
 - 2D!

Even More Aside: Faraday 2-Form



Fluids

- Fluids are difficult
- Simplifying assumptions:
 - Incompressible
 - No viscosity
 - Unit density

Fluids

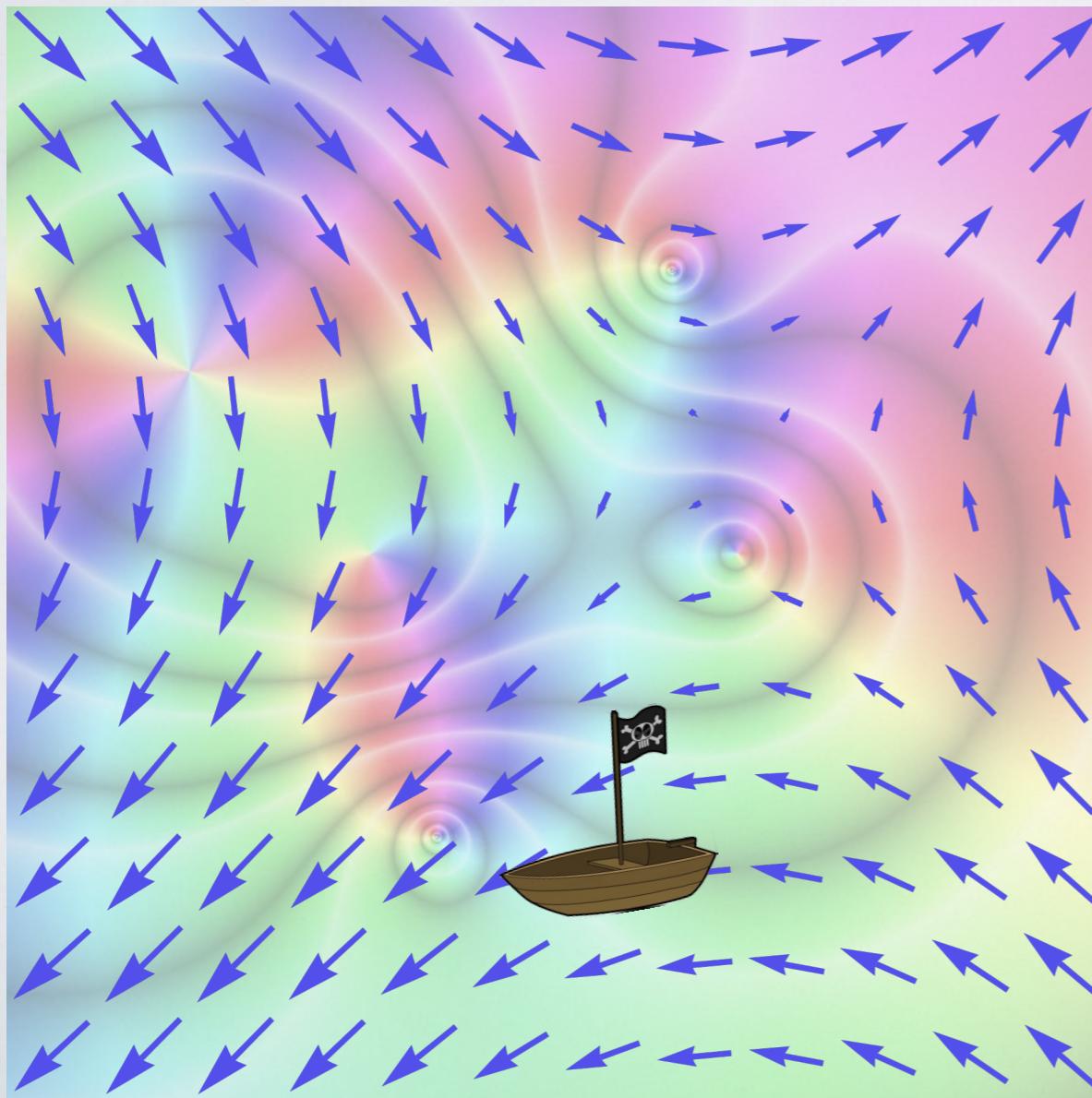
- How does a fluid's velocity change over time?
- Fluid is made up of small particles which each have a velocity
- The fluid drags the velocity field along

$$\frac{d\eta}{dt} = - \mathfrak{L}_{\eta^\#}\eta$$

Definitions
 η Velocity

Aside: Lie Derivative

- What is that weird \mathfrak{L} ?

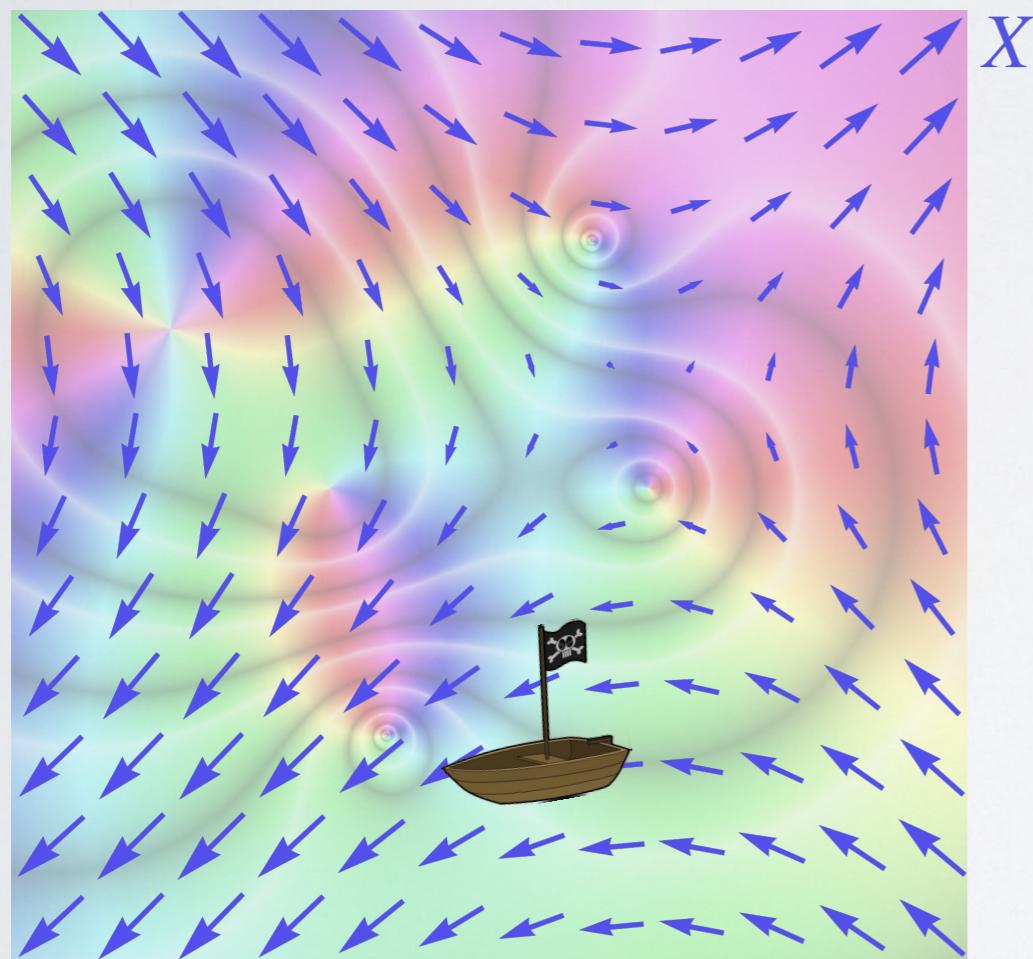


X

$\mathfrak{L}_X Y$

“How much does
 Y change as I flow
along X ? ”

Aside: Lie Derivative



φ_t Where do I wind up after time t ?

$\varphi_t^* \omega$ What does ω look like at my position after time t ?

$$\mathfrak{L}_X \omega = \frac{d}{dt} \Big|_{t=0} \varphi_t^* \omega$$

Fluids

- Not incompressible yet
- Incompressible \Leftrightarrow Divergence free

$$\frac{d\eta}{dt} = - \mathfrak{L}_{\eta^\#}\eta + dp$$

$$\delta\eta = 0$$

“Euler equation”

Definitions

η Velocity

p Pressure

dp Gradient of pressure

$\delta\eta$ Divergence of velocity

Magnetism

- Lorentz force law

$$F = J \times B$$

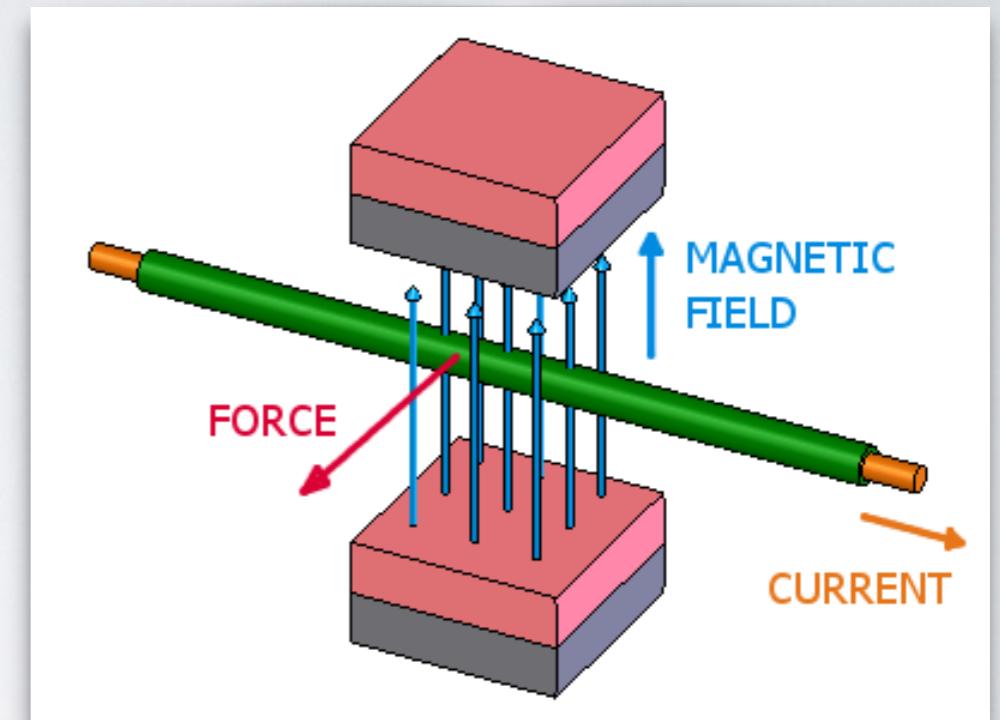
- Maxwell's fourth equation*

$$J = \nabla \times B$$

- Combining,

$$F = (\nabla \times B) \times B$$

$$= -\iota_{(\delta\beta)^{\#}}\beta$$



<https://www.kjmagnetics.com/images/blog/forcediagram1.png>

Definitions

β Magnetic field

$\delta\beta$ “Curl of β ”

$-\iota_{(\delta\beta)^{\#}}\beta$ Magnetic force from induced current

* in the case of low E-field oscillation, and with some constants dropped

Magnetism

- Magnetic field caused by little ions
- Carried by flow

$$\frac{d\beta}{dt} = - \mathfrak{L}_{\eta^\#}\beta$$

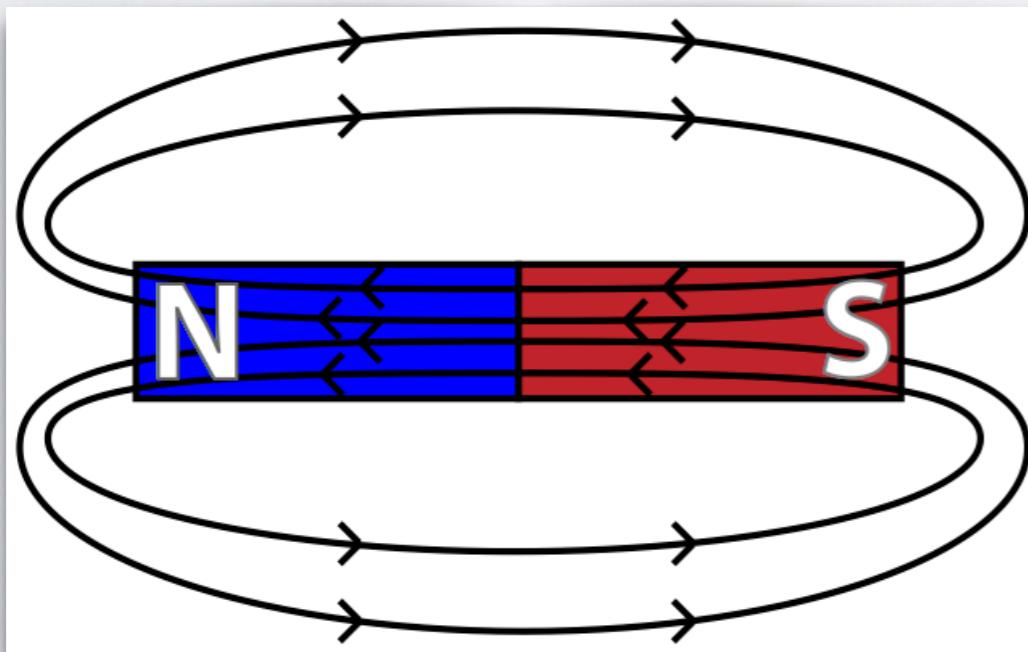
Definitions

β	Magnetic field
η	Velocity
$\mathfrak{L}_{\eta^\#}$	Lie derivative along velocity field

Magnetism

- One last constraint: no magnetic monopoles

$$d\beta = 0$$



Definitions

β

Magnetic field

$d\beta$

Divergence of
magnetic field

MHD Equations

$$\frac{d\eta}{dt} = - \mathfrak{L}_{\eta^\#}\eta + dp - \iota_{(\delta\beta)^\#}\beta$$

$$\frac{d\beta}{dt} = - \mathfrak{L}_{\eta^\#}\beta$$

$$\delta\eta = 0$$

$$d\beta = 0$$

Advection along velocity field

Force from pressure

Force from magnetic field

Constraints

Definitions

η Velocity

β Magnetic field

p Pressure

dp Gradient of pressure

$\delta\eta$ Divergence of η

$d\beta$ Divergence of β

$-\iota_{(\delta\beta)^\#}\beta$ Magnetic force from induced current

$\mathfrak{L}_{\eta^\#}$ Lie derivative along velocity field

MHD Equations

$$\frac{d\eta}{dt} = - \mathfrak{L}_{\eta^\#}\eta + dp - \iota_{(\delta\beta)^\#}\beta$$

$$\frac{d\beta}{dt} = - \mathfrak{L}_{\eta^\#}\beta$$

$$\delta\eta = 0$$

~~$$d\beta = 0$$~~

Advection along velocity field

Force from pressure

Force from magnetic field

Constraints

Definitions

η Velocity

β Magnetic field

p Pressure

dp Gradient of pressure

$\delta\eta$ Divergence of η

$d\beta$ Divergence of β

$-\iota_{(\delta\beta)^\#}\beta$ Magnetic force from induced current

$\mathfrak{L}_{\eta^\#}$ Lie derivative along velocity field

Conservation Laws: Energy

$$E = \frac{1}{2} \int (\|\eta\|^2 + \|\beta\|^2) dV$$

Kinetic Energy

Potential Energy (Magnetic field strength)

Definitions

η Velocity

β Magnetic field

Conservation Laws: Magnetic Helicity

- Suppose we have a vector-potential such that

$$\beta = d\alpha$$

- Then we define

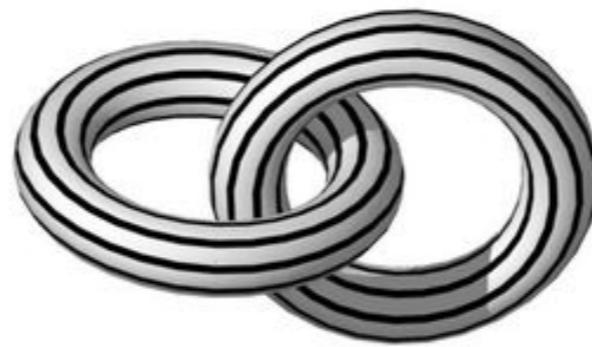
$$H_M := \int \alpha \wedge \beta$$



$H=0$



$H=T\Phi^2$



$H=\pm 2\Phi_1\Phi_2$

Definitions

α Vector potential

β Magnetic field

Conservation Laws: Cross Helicity

- Similarly

$$H_\chi := \int \eta \wedge \beta$$

“How linked are the velocity and magnetic fields?”

Definitions

η Velocity field

β Magnetic field

Discretization

- Standard Discrete Exterior Calculus gives us d, δ
- Now, we just need ι, \mathfrak{L}
- In fact, thanks to
Cartan's Magic Formula,
we only need ι

$$\mathfrak{L}_X\omega = \iota_X d\omega + d\iota_X\omega$$

MHD Equations

$$\begin{aligned}\frac{d\eta}{dt} &= -\mathfrak{L}_{\eta^\sharp}\eta + dp - \iota_{(\delta\beta)^\sharp}\beta \\ \frac{d\beta}{dt} &= -\mathfrak{L}_{\eta^\sharp}\beta \\ \delta\eta &= 0\end{aligned}$$

Discretization

- The proofs of conservation of energy and cross helicity rely on the fact that ι and Λ are adjoint
- We can use the standard discrete wedge product
- Also gives us behavior on boundaries

MHD Equations

$$\begin{aligned}\frac{d\eta}{dt} &= - \iota_{\eta^\#} d\eta + dp - \iota_{(\delta\beta)^\#} \beta \\ \frac{d\beta}{dt} &= - d\iota_{\eta^\#} \beta \\ \delta\eta &= 0\end{aligned}$$

Discretization

- This conserves energy and cross helicity for free!
- Magnetic helicity doesn't work out so well

MHD Equations

$$\frac{d\eta}{dt} = - \iota_{\eta^\sharp} d\eta + dp - \iota_{(\delta\beta)^\sharp} \beta$$

$$\frac{d\beta}{dt} = - d\iota_{\eta^\sharp} \beta$$

$$\delta\eta = 0$$

Complication: 2D MHD

- Limit of MHD equations to a thin layer of fluid in strong background magnetic field
- Equations look a bit different

2D MHD Equations

$$\frac{d\eta}{dt} = - \iota_{\eta^\sharp} d\eta + dp + \iota_{b^\sharp} db$$

$$\frac{db}{dt} = - \delta(b \wedge \eta)$$

$$\delta\eta = 0$$

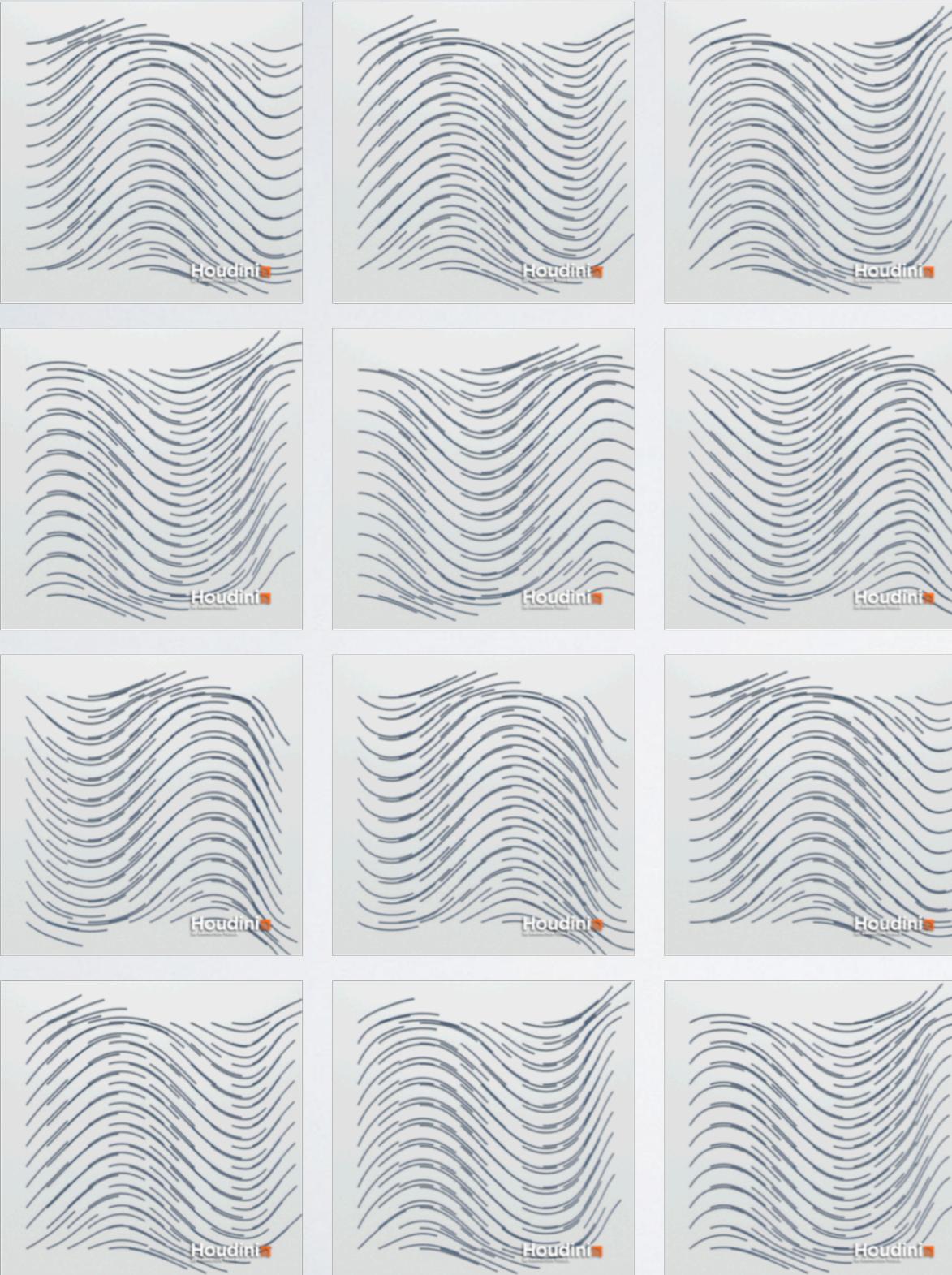
3D MHD Equations

$$\frac{d\eta}{dt} = - \iota_{\eta^\sharp} d\eta + dp - \iota_{(\delta\beta)^\sharp} \beta$$

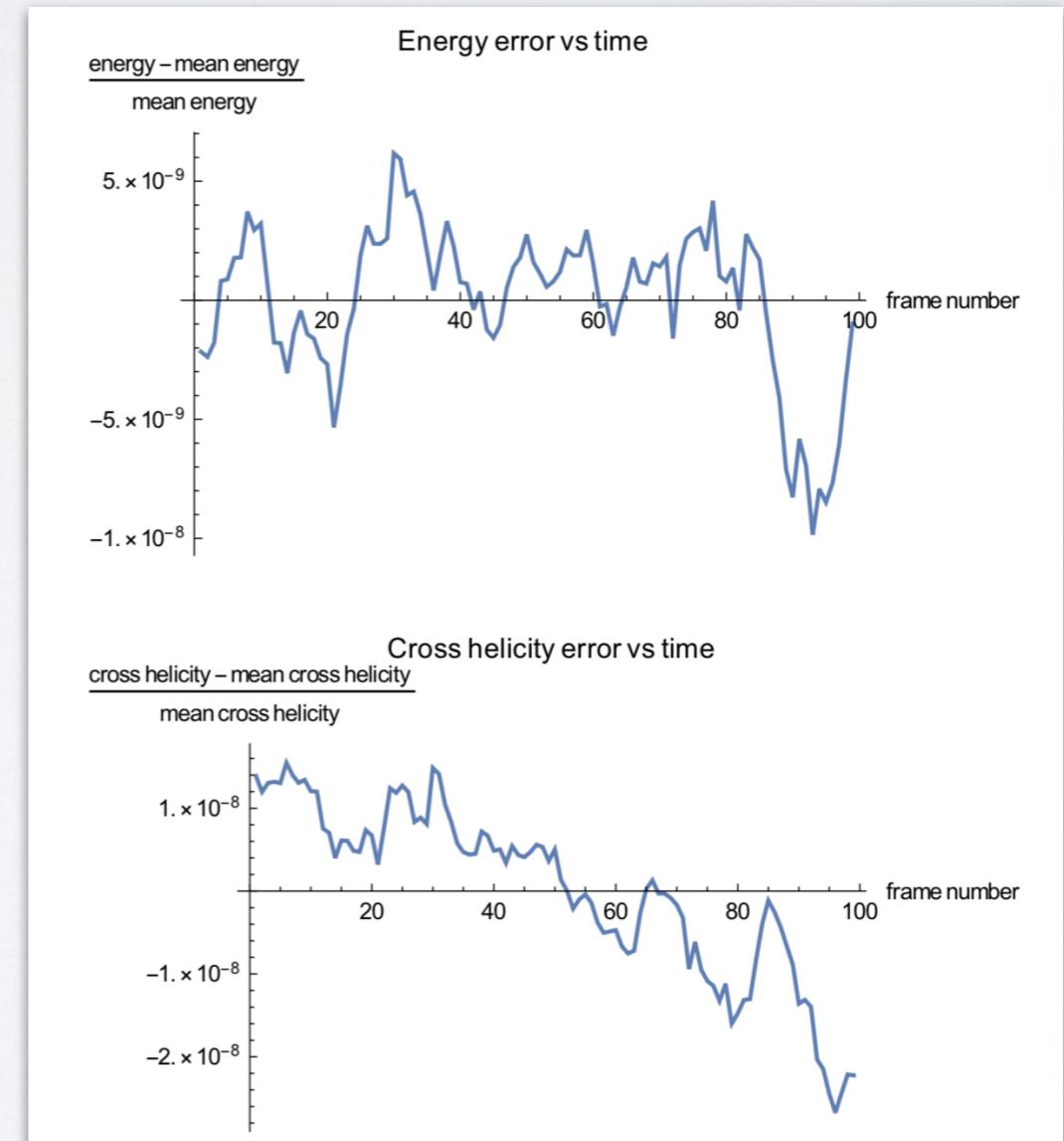
$$\frac{d\beta}{dt} = - d\iota_{\eta^\sharp} \beta$$

$$\delta\eta = 0$$

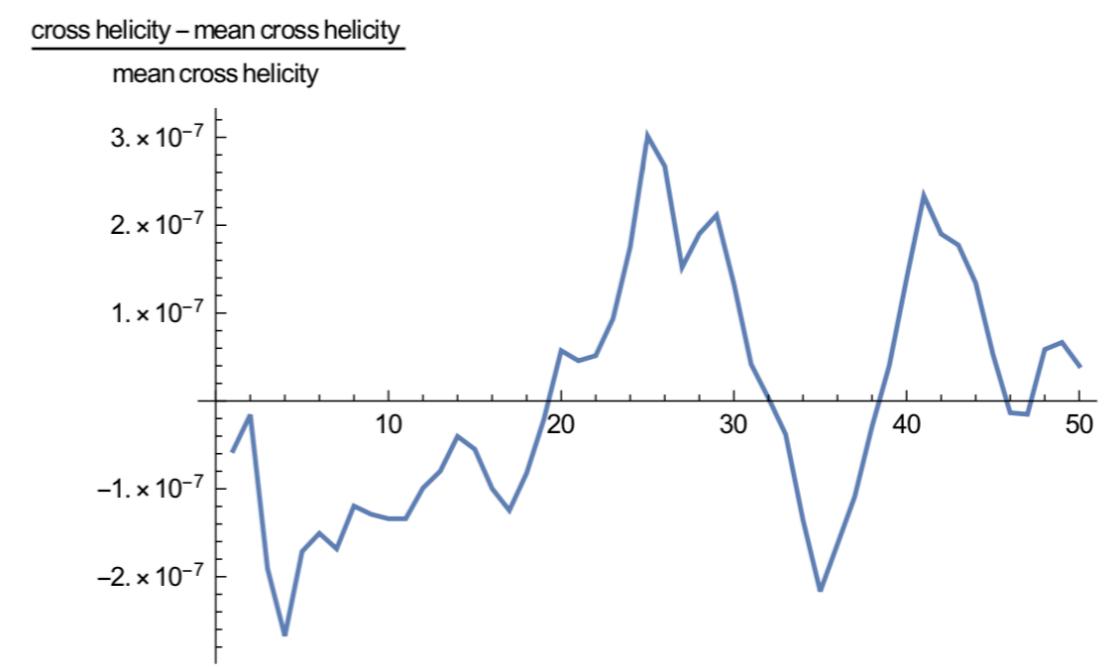
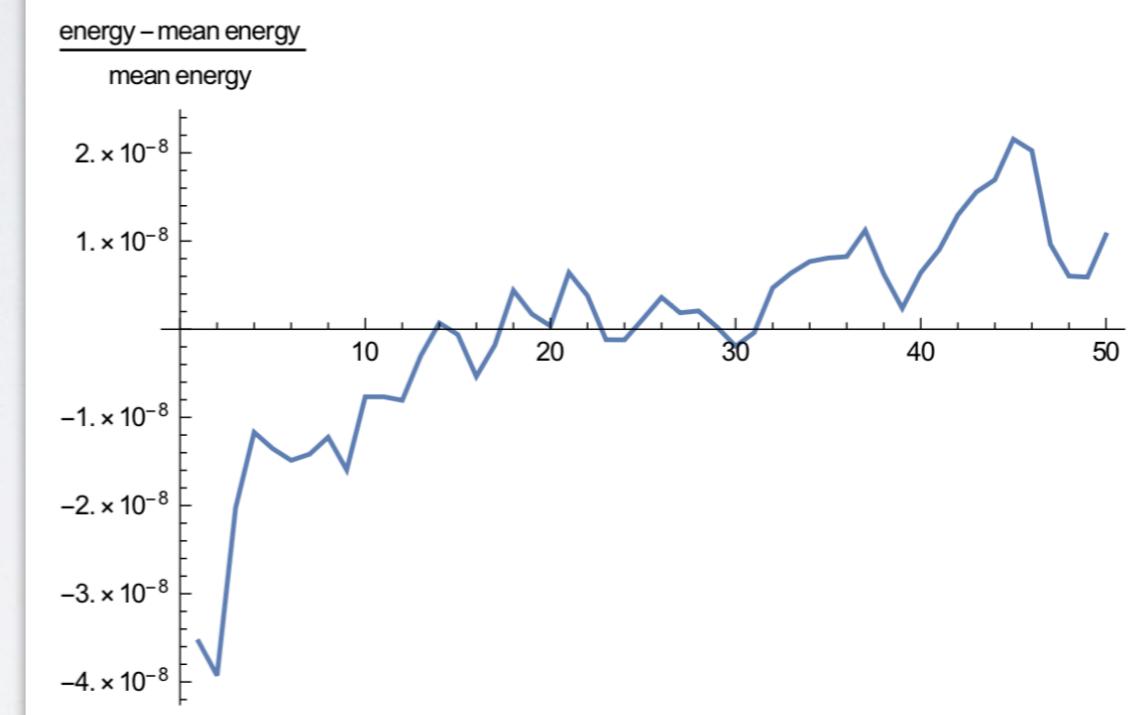
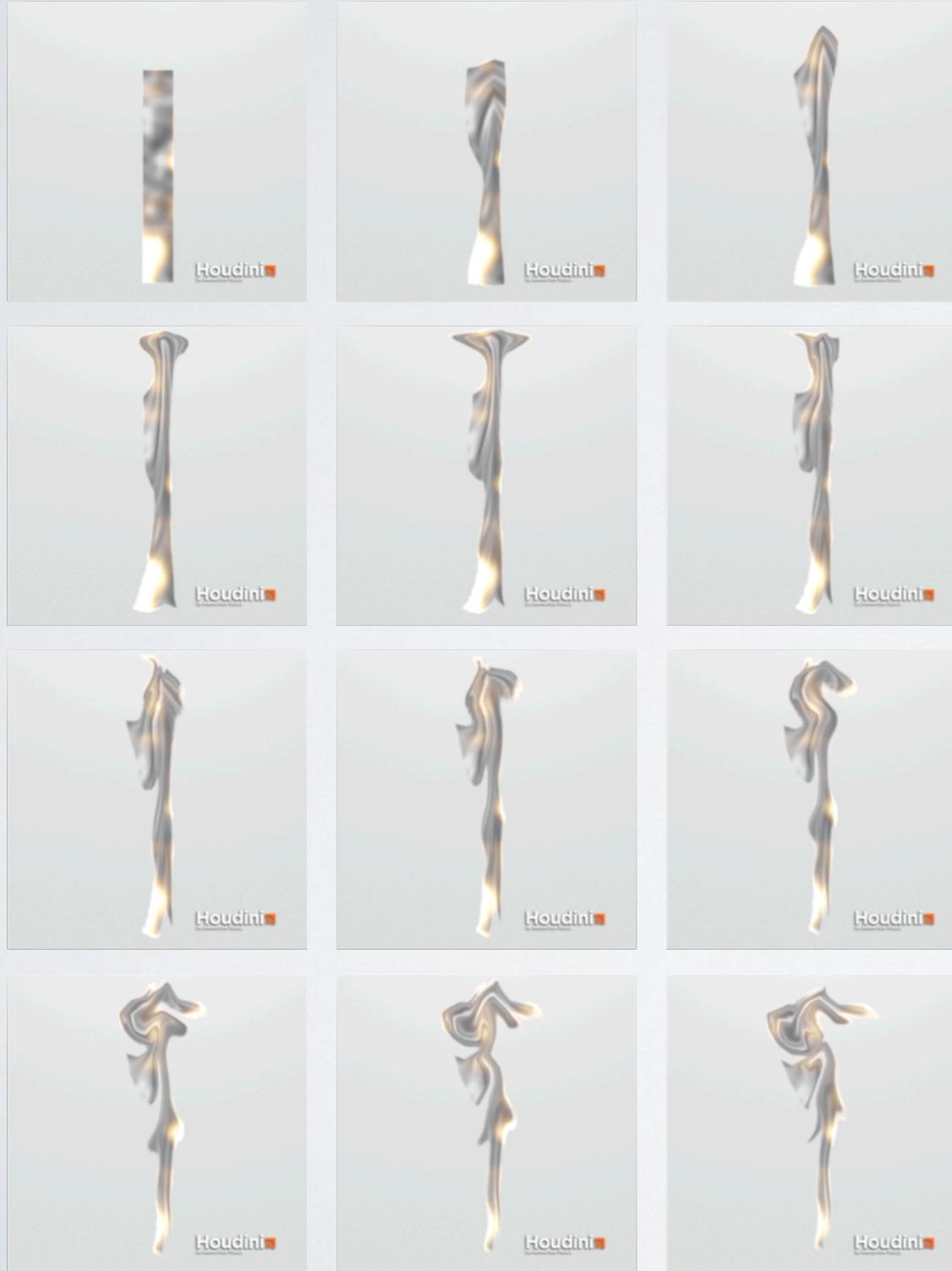
Test Case: Alfvén Wave



- Waves in the magnetic field lines



Test Case: Plume in a Box



Further questions?

- Is this integrator (multi)symplectic?
- Topological properties of magnetic field
- 3D simulation
- Boundaries between fluids

Quick Aside: Hamiltonian Mechanics is Symplectic

- Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$X := \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Omega \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ Symplectic Form}$$

Quick Aside: Hamiltonian Mechanics is Symplectic

$$X := \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Omega \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

$$\begin{aligned} X^T \Omega Y &= \left(\frac{\partial H}{\partial q} \quad \frac{\partial H}{\partial p} \right) \Omega^T \Omega Y \\ &= \left(\frac{\partial H}{\partial q} \quad \frac{\partial H}{\partial p} \right) Y \\ &= dH(Y) \end{aligned}$$

$$dH = X^T \Omega =: \iota_X \Omega$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ Symplectic Form}$$

Quick Aside: Hamiltonian Mechanics is Symplectic

- Change in Ω along time evolution: $\mathfrak{L}_X\Omega$
- Cartan: $\mathfrak{L}_X\Omega = d\iota_X\Omega + \iota_X d\Omega$ 

$$\mathfrak{L}_X\Omega = d(d\Omega) = 0$$

Definitions / Facts

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ Symplectic Form}$$

$$dH = \iota_X\Omega$$

Thanks!

