Understanding Bonds Pricing and the Risks

"My name is Bond, James Bond"

Saïd Business School, Oxford University

Nicholas Burgess

nburgessx@gmail.com

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Abstract

In this article we illustrate how to price bonds and calculate the accrued interest, clean- and dirty price, from which we can compute a bond invoice i.e., the present value for a given cash investment in the bond. We present the classical bond pricing formulae and show how to modify this formula to account for accrued interest.

To be able to evaluate bond cash flows and compute their present value we must compute the yield to maturity (YTM) or the internal rate of return (IRR) of the bond. There is no closed-form analytical solution. Therefore, it must be solved for using an optimization algorithm such as Newton-Raphson. This requires an initial guess for the YTM. In this article, we present a highly accurate approximation formula for the YTM, which can be used as a good initial guess for low latency calculations.

To conclude we demonstrate bond pricing with a live market example and compute the accrued interest, clean- and dirty price for a US treasury bond. We show how to solve for its yield to maturity and compute the present value of the bond cash flows. With this information we show how to compute the present value of any cash investment in this underlying bond and assess the DV01 risk, that is the change in value of the investment for a one basis point fall in the bond yield. With this article we provide an Excel workbook to demonstrate all calculations.

Keywords: Bond, dirty price, clean price, accrued interest, yield to maturity, internal rate of return, cash flows, optimization, Newton-Raphson, low latency, DV01

1. Introduction to Bond Pricing

Bonds prices are quoted in percent terms and are quoted like a discount or growth factor. Equivalently, they can also be quoted as a yield to maturity, which is the internal rate of return (IRR). Using the bond price, we can compute the present value of a bond investment; simply multiply the investment notional by the price in percent to get the present value.

$$Present\ Value = Investment\ \times Bond\ Price \qquad (1)$$

Consider a bond quoted in with a price of 140.958963%. If we make an investment of USD 1,000,000 in this bond then the present value of our invested funds plus interest earned will be worth USD 1,000,000 x 140.958963% i.e., USD 1,409,589.63.

2. Calculation Types

Bonds have many bespoke features and idiosyncrasies that must be accounted for when pricing and trading bonds. Such pricing features may modify the bond coupon schedule, discount factor calculations, tax considerations and many other parts of the bond pricing calculation. To simplify pricing considerations and mitigate legal recourse, bonds are grouped into calculation types by pricing feature. There are over 5,000 bond types of which around 50 are liquid and actively trade. US Treasuries have bond calculation type 1, this bond calculation type is sometimes referred to as the "US Treasury Street" method. On the Bloomberg terminal the calculation type can be found on the bond description page. Bloomberg also provides a full list of Bond calculation types and gives a brief description of the calculation methodology, (Bloomberg, 2016)

3. Bond Present Value & Pricing Formulae

To compute the present value (PV) of a bond given the yield to maturity the following formula is used,

$$PV = \frac{C/t}{(1+y/t)^1} + \frac{C/t}{(1+y/t)^2} + \frac{C/t}{(1+y/t)^3} + \cdots + \frac{Principal + C/t}{(1+y/t)^{n\times t}} \qquad (2a)$$

Equivalently we can write this as,

$$PV = \sum_{i=1}^{n} C/t \cdot DF(i) + Principal \cdot DF(n) \qquad (2b)$$

with $DF(i) = 1/(1 + y/t)^i$ and where C is the annual coupon rate, t is the number of coupons per year, y is the yield to maturity (or internal rate of return, IRR) and n is the number of years to maturity.

We can compute the bond price in percent from the bond present value as follows,

$$Price = \left({}^{PV}/_{Denomination}\right) \qquad (3)$$

To illustrate bond pricing, let us consider the following bond,

Name:	Coca-Cola 7.5% maturing 27-Sep-2022
Type:	Eurobond, Annual Coupons
Settlement:	27-Sep-2022
Denomination:	USD 1,000
Coupon:	7.50%
Bond Yield or IRR:	8.00%

As illustrated in figure (1), we calculate the bond present value using equation (2b), from which we can compute the bond price using equation (3).

Firstly, we compute the discount factors at each time period. Secondly, we scale each cash flow by the corresponding discount factor to compute each cash flow's present value. Thirdly, we sum cash flow PV's giving a total bond PV of USD 987.11. Finally, we divide the bond PV by the bond denomination USD 1,000, which gives the bond price of 98.711%.

Figure 1: Illustration of Coca-Cola Bond Cash Flows and Price

When pricing bonds part-way through a coupon period the bond pricing formula is modified as follows,

$$PV = \frac{C/t}{(1+y/t)^{a}} + \frac{C/t}{(1+y/t)^{a+1}} + \frac{C/t}{(1+y/t)^{a+2}} + \cdots + \frac{Principal + C/t}{(1+y/t)^{a+m}} \quad (4a)$$

Equivalently we can write this as,

$$PV = \sum_{i=1}^{n} C/t \cdot DF(i) + Principal \cdot DF(n) \quad (4b)$$

with $DF(i) = 1/(1+y/t)^{a+i-1}$ and where a is the fraction of the current coupon remaining i.e., the number of days to the next coupon / the total number of days in the current coupon period and m is the number of whole coupons to maturity.

If we price the same Coca-Cola bond but this time with a mid-coupon settlement date that is 133 days after the coupon start date, then using equations (3) and (4b) we arrive at the price illustrated in figure (2) following the same steps as in figure (1).

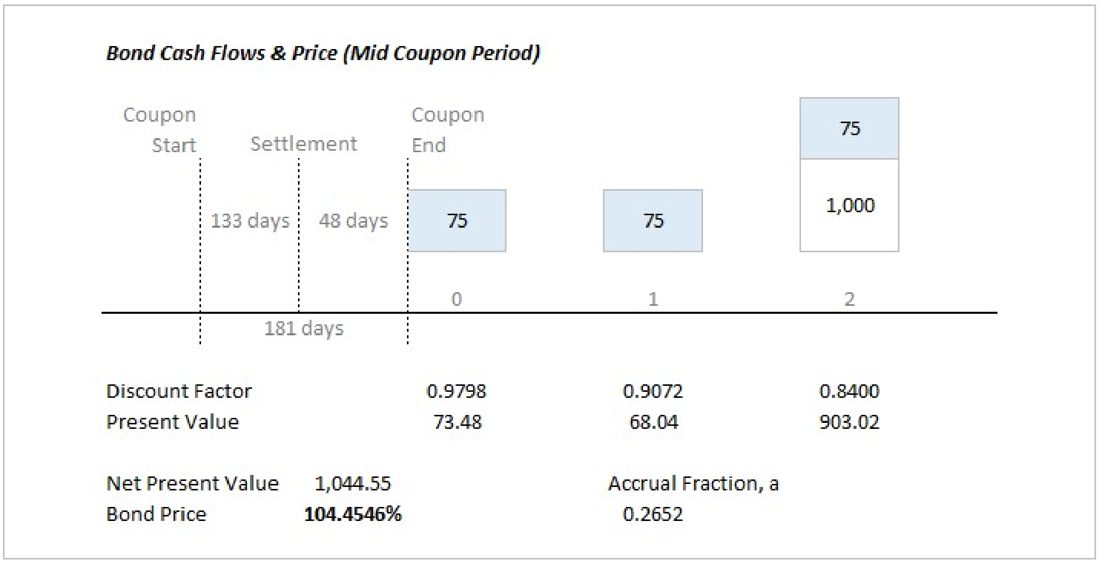


Figure 2: Illustration of Coca-Cola Bond Price when Pricing Mid-Coupon

4. Accrued Interest, Clean and Dirty Prices

When trading bonds mid-coupon, the partial interest accrued from the coupon start date to the transaction settlement date is called the **accrued interest**.

So far, all the prices we have looked at have been **dirty prices**, which is the present value of all bond coupons to be received. However, any accrued interest belongs to the seller and must be paid to the seller on the settlement date. The present value of remaining coupons, when normalized into percent, is called the **clean price**. It is calculated by subtracting the accrued interest in percent from the dirty price.

$$Clean\ Price = Dirty\ Price - Accrued\ Interest \tag{5}$$

The accrued interest in cash terms and as a percent is calculated as follows,

$$Accrued\ Interest = \left(\frac{No.\,days\ Accrued}{Total\ Days\ in\ Coupon\ Period}\right) \times Full\ Coupon \tag{6a}$$

$$Accrued\ Interest\ in\ \% = \left({}^{Accrued\ Interest}\!/_{Denomination}\right) \tag{6b}$$

For the Coca-Cola bond the accrued interest is calculated as,

$$Accrued\ Interest = \left(\frac{133}{181}\right) \times 75 = 55.11$$

$$Accrued\ Interest\ in\ \% = \left(\frac{55.11}{1{,}000}\right) = 5.5110\%$$

Having calculated the dirty price as 104.4546% we can subtract the accrued interest of 5.5110% to obtain a clean price of 98.943% as follows,

Bond Prices		PV		
Dirty Price	104.4546%	1,044.55		Accrual Fraction
Accrued Interest	5.5110%	55.11		0.7348
Clean Price	98.9435%	989.44		

Figure 3: Illustration of Coca-Cola Bond Dirty and Clean Prices

5. Yield to Maturity (YTM)

Bond prices can also be quoted as a yield to maturity (YTM), which is the bond's internal rate of return (IRR). Given a bond price we can compute the YTM by reverse engineering equation (4a) for the yield to maturity i.e., the y term,

$$PV = \frac{C/t}{(1+y/t)^a} + \frac{C/t}{(1+y/t)^{a+1}} + \frac{C/t}{(1+y/t)^{a+2}} + \cdots + \frac{Principal + C/t}{(1+y/t)^{a+m}}$$

There is no analytical solution for the yield to maturity, it must be solved for using the Newton-Raphson technique or a similar search method. The Newton-Raphson method requires an initial guess. The better the guess the faster we can solve for a solution with fewer solver iteration steps.

A reasonable guess for the yield to maturity might be the bond coupon rate. This works well for bonds trading close to par or 100, however bond coupons can be zero and many bonds trade far from par. A better, low latency initial guess that is highly accurate and reduces the number of search iterations is specified below,

$$Initial\ \ Guess = \left(Coupon + \left(\frac{Par - Clean\ Price}{Time\ to\ Maturity} \right) \right) \Big/ \left(\frac{Par + Clean\ Price}{2} \right) \quad (7)$$

The intuition behind this initial guess is to consider that each year we earn coupon interest plus we gain/lose on the bond price if not at par, we accrue this on average between the initial bond price (par) and the maturity price.

Using the initial guess from equation (7) we can quickly find the yield to maturity in approximately three solver iterations.

6. DV01

The DV01 is a measure of bond risk, it captures the change in present value of a bond investment for a down-shift of one basis point (0.01%) in the bond yield to maturity. It is simple to calculate numerically, we reprice the bond with the yield shifted down by one basis point (1bps) compute the change in value.

$$DV01 = Bond\ PV(YTM - 0.01\%) - Bond\ PV(YTM) \quad (8)$$

7. US Treasury Bond Example

To illustrate how to use all the different formulae presented, let us consider the following US treasury bond. It pays an annual coupon of 6.0% in two semi-annual payments of 3.0% and matures on 15th Feb 2025.

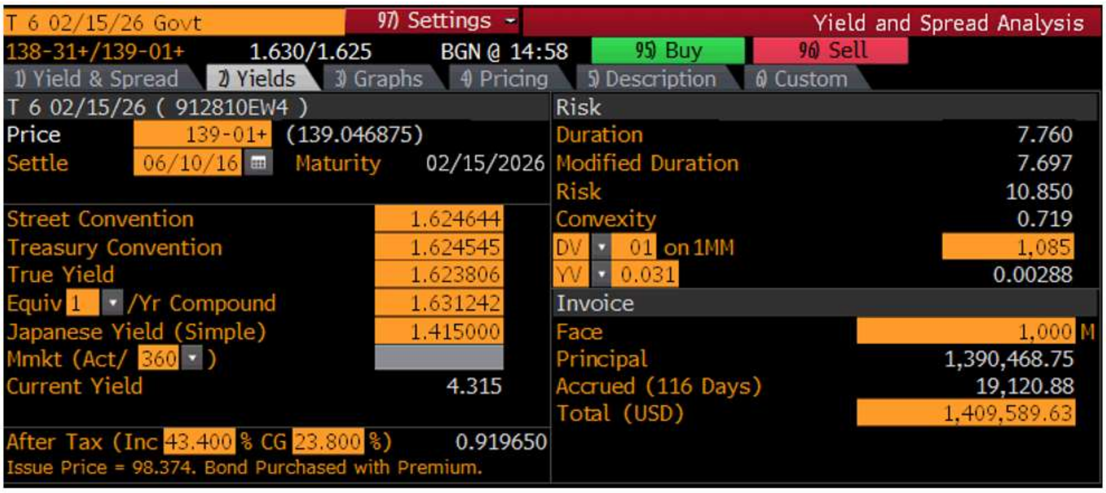


Figure 4: US Treasury Bond, T 6.0% 15-Feb-2026

(Source: Bloomberg)

Given a settlement or valuation date of 10th October 2016 there are 116 days of accrued interest owed to the seller out of a total of 182 accrual days within the current coupon period.

Current Coupon

Accrued Days	116
Remaining	66
Total	182
Accrued Fraction	0.63736
Coupon Fraction	0.36264

Figure 5: Accrued Interest Days

The bond cash flows are based on the bond denomination of USD 1,000 as illustrated in figure (6). The present value of these bond coupons is evaluated using equation (4b) requoted below,

$$PV = \sum_{i=1}^{n} C/t \cdot DF(i) + Principal \cdot DF(n)$$

We pay special attention that the discount factor for the ith bond coupon is calculated as follows to allow for accrued interest,

$$DF(i) = \frac{1}{(1 + y/t)^{a+i-1}} \tag{8}$$

where y denotes the yield to maturity in percent, t the number of coupons paid per year and ‘a’ denotes the coupon fraction i.e., the fraction of the current coupon yet to accrue and be paid.

Bond Cash Flows

Coupon	Start	End	Days	Coupon	DiscFact	PV
1	15/02/2016	15/08/2016	182	30.00	0.99707	29.91
2	15/08/2016	15/02/2017	184	30.00	0.98904	29.67
3	15/02/2017	15/08/2017	181	30.00	0.98107	29.43
4	15/08/2017	15/02/2018	184	30.00	0.97316	29.19
5	15/02/2018	15/08/2018	181	30.00	0.96532	28.96
6	15/08/2018	15/02/2019	184	30.00	0.95754	28.73
7	15/02/2019	15/08/2019	181	30.00	0.94983	28.49
8	15/08/2019	15/02/2020	184	30.00	0.94217	28.27
9	15/02/2020	15/08/2020	182	30.00	0.93458	28.04
10	15/08/2020	15/02/2021	184	30.00	0.92705	27.81
11	15/02/2021	15/08/2021	181	30.00	0.91958	27.59
12	15/08/2021	15/02/2022	184	30.00	0.91217	27.37
13	15/02/2022	15/08/2022	181	30.00	0.90482	27.14
14	15/08/2022	15/02/2023	184	30.00	0.89753	26.93
15	15/02/2023	15/08/2023	181	30.00	0.89030	26.71
16	15/08/2023	15/02/2024	184	30.00	0.88312	26.49
17	15/02/2024	15/08/2024	182	30.00	0.87601	26.28
18	15/08/2024	15/02/2025	184	30.00	0.86895	26.07
19	15/02/2025	15/08/2025	181	30.00	0.86195	25.86
20	15/08/2025	15/02/2026	184	1,030.00	0.85500	880.65

Bond Price

Clean Price	139.0469%
Accrued	1.9121%
Dirty Price	140.9590%

Total Coupons

1,600.00

Based on Denomination

Total PV

1,409.59

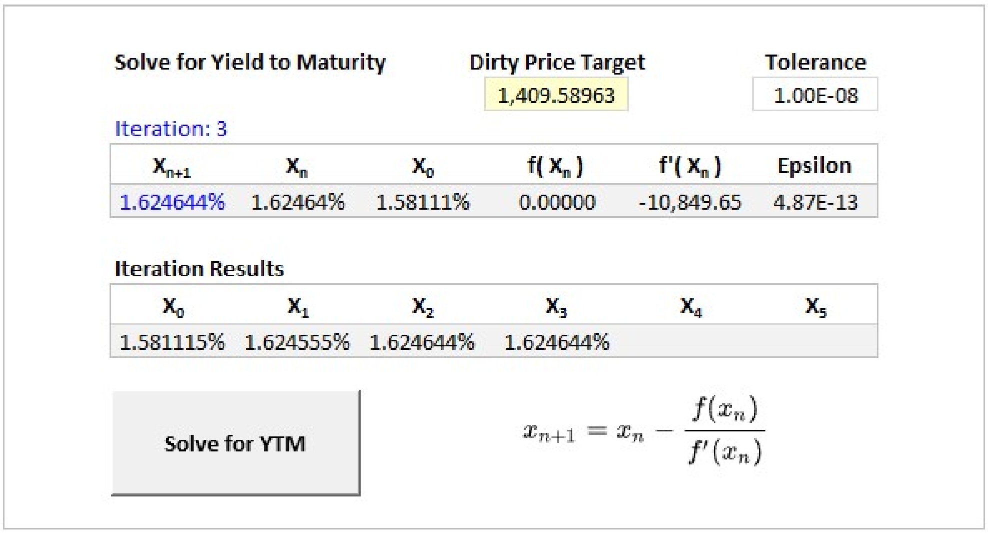
(Dirty)

Figure 6: Bond Cash Flows

To value the bond cash flows, we need to compute the yield to maturity given the target dirty bond price. We compute the yield to maturity using the Nelson-Raphson method and a highly accurate initial guess calculated using equation (7) requoted below,

$$Initial\ Guess = \left(Coupon + \left(\frac{Par\ - Clean\ Price}{Time\ to\ Maturity} \right) \right) \Big/ \left(\frac{Par + Clean\ Price}{2} \right)$$

Iteratively pricing the cash flows using the solver's iterative guess for the yield to maturity leads to an exact solution for the yield to maturity in three solver iterations, see figure (8).



Solve for Yield to Maturity | Dirty Price Target: 1,409.58963 | Tolerance: 1.00E-08

Iteration: 3

X_{n+1}	X_n	X_0	$f(X_n)$	$f'(X_n)$	Epsilon
1.624644%	1.62464%	1.58111%	0.00000	-10,849.65	4.87E-13

Iteration Results

X_0	X_1	X_2	X_3	X_4	X_5
1.581115%	1.624555%	1.624644%	1.624644%		

Solve for YTM

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Figure 7: Solving for Yield to Maturity using Newton-Raphson Technique

Having discovered the bond yield of 1.624644% we compute the total present value of the bond cash flows in figure (6). We compute the dirty bond price using equation (3), the accrued interest using equation (6) and the clean price using equation (5).

$$Price = \left({Present\ Value}\big/{Denomination} \right)$$

Clean Price	139.0469%
Accrued	1.9121%
Dirty Price	140.9590%

Figure 8: US Treasury Bond Dirty and Clean Price

If we invest USD 1,000,000 in this US treasury bond, then we can compute the invoice or present value of this investment with coupons accruing interest of 6.00% by scaling by the appropriate bond price by the bond price using equation (1) as shown in figure (10). The total PV corresponds to the dirty price and the total coupons received. However, we must pay accrued interest to the bond seller. The net present value less accrued interest is denoted principal in figure (10) and corresponds to the clean price.

Invoice

Face	1,000,000.00
Coupons Due	1,600,000.00
Principal	1,390,468.75
Accrued	19,120.88
Total PV	1,409,589.63
DV01	1,085.48

Figure 9: Invoice Calculation for a USD 1,000,000 Investment in the US Treasury

The DV01 risk reflects the bond risk and is calculated numerically using equation (7). Should the yield to maturity fall (rise) by 1 basis point we will make a profit (loss) of USD 1,085.48,

Appendix

The bond Excel workbook provided with this article looks as follows, kindly email the author should there be any issues accessing or obtaining the workbook.

For a given US treasury bond we are given that the clean price is 139-01+. As US treasury prices are quoted in denominations of 1/32 this means the clean price is 139 1/32. The plus after the price indicates we must also add an additional 1/64 of a percentage point to the price.

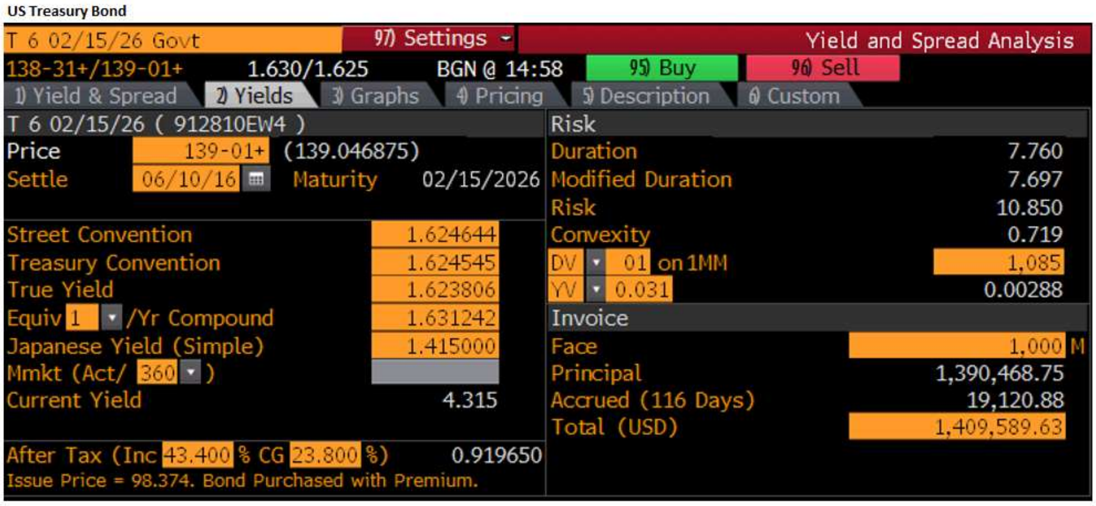


Figure 10: US Treasury Bond Quote

The yield to maturity is calculated using the Newton-Raphson algorithm, see (Burgess 2021), which solves for the yield that matches the price of the bond.

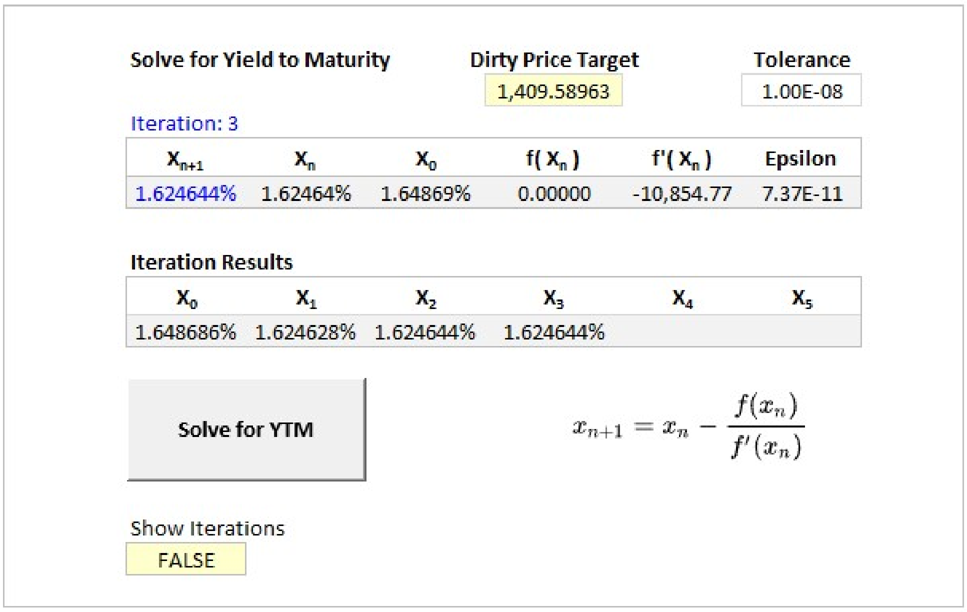


Figure 11: Bond Pricing, Cash Flow Analysis & Risk

Given the US treasury bond inputs in yellow we calculate the bond coupons for the given bond denomination or face value. We compute the present value of these bond coupons from which we can calculate the accrued interest, dirty and clean prices of the bond. With this information, we can compute the present value for any invoice or cash investment in this US treasury bond. For completeness, we also compute the DV01 risk by numerically shifting the yield downwards by one basis point.

Bond Details

Settlement	10-Jun-16	
Denomination	1,000	
Par	100.0000%	
Clean Price[1]	139.0469%	
Coupon	6.0000%	
Frequency	2	
Maturity	15-Feb-26	9.69

Current Coupon

Accrued Days	116
Remaining	66
Total	182
Accrued Fraction	0.63736
Coupon Fraction	0.36264

Yield to Maturity (YTM)

Initial guess[2]	1.648686%
YTM Used	1.624644%
Shift Size	-0.0100%

Bond Cash Flows

Coupon	Start	End	Days	Coupon	DiscFact	PV	Shifted DF	Shifted PV
1	15/02/2016	15/08/2016	182	30.00	0.99707	29.91	0.99709	29.91
2	15/08/2016	15/02/2017	184	30.00	0.98904	29.67	0.98910	29.67
3	15/02/2017	15/08/2017	181	30.00	0.98107	29.43	0.98118	29.44
4	15/08/2017	15/02/2018	184	30.00	0.97316	29.19	0.97332	29.20
5	15/02/2018	15/08/2018	181	30.00	0.96532	28.96	0.96553	28.97
6	15/08/2018	15/02/2019	184	30.00	0.95754	28.73	0.95780	28.73
7	15/02/2019	15/08/2019	181	30.00	0.94983	28.49	0.95013	28.50
8	15/08/2019	15/02/2020	184	30.00	0.94217	28.27	0.94252	28.28
9	15/02/2020	15/08/2020	182	30.00	0.93458	28.04	0.93497	28.05
10	15/08/2020	15/02/2021	184	30.00	0.92705	27.81	0.92748	27.82
11	15/02/2021	15/08/2021	181	30.00	0.91958	27.59	0.92005	27.60
12	15/08/2021	15/02/2022	184	30.00	0.91217	27.37	0.91268	27.38
13	15/02/2022	15/08/2022	181	30.00	0.90482	27.14	0.90538	27.16
14	15/08/2022	15/02/2023	184	30.00	0.89753	26.93	0.89812	26.94
15	15/02/2023	15/08/2023	181	30.00	0.89030	26.71	0.89093	26.73
16	15/08/2023	15/02/2024	184	30.00	0.88312	26.49	0.88380	26.51
17	15/02/2024	15/08/2024	182	30.00	0.87601	26.28	0.87672	26.30
18	15/08/2024	15/02/2025	184	30.00	0.86895	26.07	0.86970	26.09
19	15/02/2025	15/08/2025	181	30.00	0.86195	25.86	0.86273	25.88
20	15/08/2025	15/02/2026	184	1,030.00	0.85500	880.65	0.85582	881.50

Invoice[3]

Face	1,000,000.00
Coupons Due	1,600,000.00
Principal	1,390,468.75
Accrued	19,120.88
Total PV	1,409,589.63
DV01[4]	1,085.48

Bond Price

Clean Price	139.0469%
Accrued	1.9121%
Dirty Price	140.9590%

Total Coupons

1,600.00

Based on Denomination

Total PV

1,409.59

(Dirty)

Shifted Total PV

1,410.68

(Dirty)

Numerical DV01[4]

1.08548

Based on Denomination

Figure 12: Bond Pricing, Cash Flow Analysis & Risk

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