DRW New-Hire Learning Program

Module: Risk

Session D.2: Nonlinear Risk

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Outline

Value at Risk

The τ-day, π% Value at Risk (VaR) of a portfolio is defined as $\text{VaR}_{\pi,\tau}$ such that

$$\Pr\left(L_{t,t+\tau} > \text{VaR}_{\pi,\tau}\right) = \pi$$

Thus,

- there is a π% chance
- that over a horizon of τ days
- the portfolio will lose an amount greater than VaR.

VaR as a distribution quantile

Let F^{ℓ}_{τ} denote the cdf of $L_{t,t+\tau}$.

- VaR can be written as the upper quantile of the cdf of losses.

$$\mathrm{VaR}_{\pi,\tau} = \inf\left\{L \mid F^{\ell}_{\tau}(L) \geq 1-\pi\right\}$$

- Assuming the the cdf is continuous and strictly increasing,

$$\begin{aligned}\mathrm{VaR}_{\pi,\tau} =& \left\{L \mid F^{\ell}_{\tau}(L) = 1-\pi\right\}\\ =& F^{\ell(-1)}_{\tau}(1-\pi)\end{aligned}$$

where $F^{\ell(-1)}_{\tau}$ denotes the inverse CDF.

Returns

Losses can be written as a linear function of returns:

$$L_{t,t+\tau} = -P_t\, r_{t,t+\tau} \tag{1}$$

- Low returns mean rate of return is negative. Thus losses, L, are positive.
- With this, VaR can be described in terms of the lower quantiles of the return distribution.

VaR through return distributions

Let F^r_τ denote the cdf of τ-horizon return rates, $r_{t,t+\tau}$.

- Assuming the the cdf is continuous and strictly increasing,

$$\begin{aligned}\text{VaR}_{\pi,\tau} &= \{-P_t\, r \mid F^r_\tau\,(\,r) = \pi\} \\ &= -\,P_t F^{r(-1)}_\tau(\pi)\end{aligned}$$

- $F^{r(-1)}_\tau$ denotes the inverse CDF.

Comparison of VaR and distributions

$$\text{VaR}_{\pi,\tau} = \left\{ L \mid F^{\ell}_{\tau}(L) = 1 - \pi \right\}$$

$$\text{VaR}_{\pi,\tau} = \{ -P_t\, r \mid F^{r}_{\tau}(r) = \pi \}$$

- It is common to see VaR discussed in terms of loss distribution or return distribution.
- Keep track which is being used: upper tail of losses or lower tail of returns!
- Regardless of the distribution, VaR expresses the dollar loss, L.

VaR expressed as return rate

Though VaR is quoted in terms of losses, often interested in the return **rate** associated with π-quantile of loss.

- The VaR expressed as a return rate is then,

$$r^{\text{VaR}}_{\pi,\tau} = F^{r\,(-1)}_{\tau}(\pi)$$

- And VaR, which is expressed as a dollar loss is

$$\text{VaR}_{\pi,\tau} = -\,P_t\, r^{\text{VaR}}_{\pi,\tau} \tag{2}$$

Problems of VaR

Many have argued that VaR is not only an insufficient measure of risk, but actually contributes to disaster risk.

- VaR only calculates the π-quantile, while saying nothing about more extreme outcomes.
- If a trader wants to maximize profits subject to meeting some VaR, then one might take extremely risky positions in the far tail.

Illustration of VaR

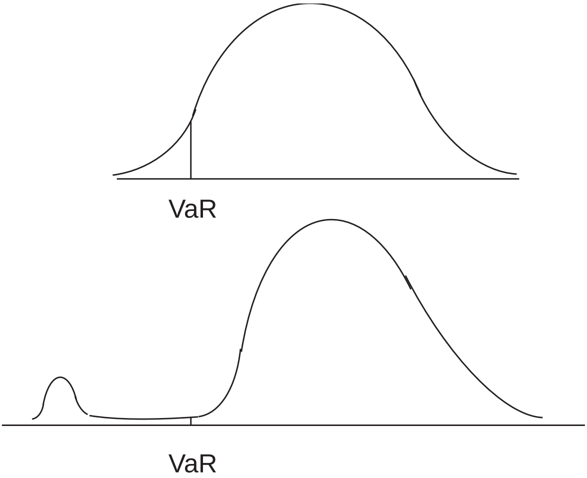

Figure: Illustration of identical VaR with differing distribution tails.
Source: Hull (2012)

Expected shortfall

Expected Shortfall (ES) refers to the expected loss conditional on a loss greater than $\text{VaR}_{\pi,\tau}$ occurring.

$$\text{ES}_{\pi,\tau} = \mathbb{E}\left[L_{t,t+\tau} \mid L_{t,t+\tau} > \text{VaR}_{\pi,\tau}\right]$$

- ES is the expected horizon-τ loss, conditional on a loss in the $(1-\pi)\,\%$ tail of the loss cdf occurring.
- While the VaR is identical in the two figures above, the Expected Shortfalls would be very different.

Expected shortfall with returns

Can write Expected Shortfall as a function of return rates.

$$\mathrm{ES}_{\pi,\tau} = -\,P_t\,\mathbb{E}\left[\,r_{t,t+\tau}\mid r_{t,t+\tau} < \; r^{\mathrm{VaR}}_{\pi,\tau}\right]$$

- May want return rate associated with a loss of ES,

$$r^{\mathrm{ES}}_{\pi,\tau} = \mathbb{E}\left[\,r_{t,t+\tau}\mid r_{t,t+\tau} < \; r^{\mathrm{VaR}}_{\pi,\tau}\right]$$

- Then ES expressed in losses is just a rescaling,

$$\mathrm{ES}_{\pi,\tau} = -\,P_t\,r^{\mathrm{ES}}_{\pi,\tau} \tag{3}$$

Outline

Normally distributed returns

Recall that future losses are linear in future returns.

$$L_{t,t+\tau} = -\, P_t \, r_{t,t+\tau}$$

So normally distributed **return rates** imply normally distributed losses.

$$r_{t,t+\tau} \sim \mathcal{N}\left(\mu_\tau, \sigma_\tau^2\right)$$

$$L_{t,t+\tau} \sim \mathcal{N}\left(-P_t\mu_\tau,\ P_t^2\sigma_\tau^2\right)$$

where the notation μ_τ and σ_τ simply reminds that the distribution parameters depend on the cumulation horizon, τ.

VaR for normal returns

In this case of normal returns, the VaR calculation is simple.

$$r^{\text{VaR}}_{\pi,\tau} = \mu_\tau + z_\pi \sigma_\tau$$

$$\text{VaR}_{\pi,\tau} = -\, P_t \, r^{\text{VaR}}_{\pi,\tau}$$

where z_π is the π% quantile of the standard normal cdf, Φ:

$$\Phi\left(z_\pi\right) = \pi$$

Table of standard normal quantiles

Table: Quantiles of the Standard Normal Distribution

CDF (%):	π	50	10	5	2.5	1	0.1	0.01
Quantile:	z_π	0	-1.28	-1.65	-1.96	-2.33	-3.09	-3.71
CDF (%):	π	50	90	95	97.5	99	99.9	99.99
Quantile:	z_π	0	1.28	1.65	1.96	2.33	3.09	3.71

- Obviously, you can get the normal cdf yourself.
- This table is just for reference.
- Note that in the table above, π is displayed as a percent (i.e. 50) though in the formulas it is a decimal. (i.e. 0.50.)

Conditional expectation of a normal

ES is calculated by taking the expectation conditional on passing some threshold.

- Consider a standard normal, z.
- Let $\phi(z)$ denote the pdf of a standard normal.
- It is well known that for any number, a,

$$\mathbb{E}\left[z \mid z < a\right] = -\frac{\phi(a)}{\Phi(a)}$$

Expected shortfall with normal distributions

In the case of normal returns, the ES is the expected value of a truncated normal.

$$r^{\text{ES}}_{\pi,\tau} = \mu_\tau - \frac{\phi\left(\mathsf{z}_\pi\right)}{\pi}\sigma_\tau$$

Expressed in terms of losses, scale by negative P,

$$\text{ES}_{\pi,\tau} = -P_t\, r^{\text{ES}}_{\pi,\tau}$$

VaR compared to ES with normality

Note the similarity between the equations for VaR and ES in the case of normality:

$$r_{\pi,\tau}^{\text{VaR}} = \mu_\tau + z_\pi \sigma_\tau \tag{4}$$

$$r_{\pi,\tau}^{\text{ES}} = \mu_\tau - \frac{\phi\left(z_\pi\right)}{\pi}\sigma_\tau \tag{5}$$

- The only difference is the scaling on the volatility parameter.
- With normality, every statistic can be reduced to a function of the mean and variance.
- Thus, the ES measure is more useful when we do not have normality.

Lognormal returns

Let $\mathtt{r}_{t,t+\tau}$ denote log returns, $\ln R_{t,t+\tau}$.

- Suppose that returns are lognormal.

$$\mathtt{r}_{t,t+1} \sim \mathcal{N}\left(\mu, \sigma^2\right)$$

- Also assume iid returns so

$$\mathtt{r}_{t,t+\tau} \sim \mathcal{N}\left(\tau\ \mu,\ \tau\ \sigma^2\right)$$

- Loss is exponential-affine in the log-return:

$$L_{t,t+\tau} = -P_t\left(\exp\left\{\mathtt{r}_{t,t+\tau}\right\} - 1\right)$$

VaR and ES for iid lognormal returns

For iid, lognormal returns, the VaR and ES calculations are,

$$\text{VaR}_{\pi,\tau} = -P_t\left(\exp\left\{\tau\mu + \text{z}_\pi\sqrt{\tau}\,\sigma\right\} - 1\right)$$

$$\text{ES}_{\pi,\tau} = -P_t\left(\exp\left\{\tau\mu - \frac{\phi\left(\text{z}_\pi\right)}{\pi}\sqrt{\tau}\,\sigma\right\} - 1\right)$$

Log-return risk stats

Express VaR and ES in log returns,

$$r^{\text{VaR}}_{\pi,\tau} = \tau\mu + z_\pi \sqrt{\tau}\,\sigma \tag{6}$$

$$r^{\text{ES}}_{\pi,\tau} = \tau\mu - \frac{\phi\left(z_\pi\right)}{\pi}\sqrt{\tau}\,\sigma \tag{7}$$

But loss is now only approximately a scaling of log-return risk stats:

$$\text{VaR}_{\pi,\tau} \approx -\,P_t\, r^{\text{VaR}}_{\pi,\tau}$$

$$\text{ES}_{\pi,\tau} \approx -\,P_t\, r^{\text{ES}}_{\pi,\tau}$$

Normal and lognormal results

- Normality
  VaR and ES (losses) are simply scalings of return-rate stats.

$$\mathrm{VaR}_{\pi,\tau} = -P_t \, r^{\mathrm{VaR}}_{\pi,\tau}, \qquad \mathrm{ES}_{\pi,\tau} = -P_t \, r^{\mathrm{ES}}_{\pi,\tau} \tag{8}$$

- Log-normality
  With $\mu = 0$, r^{VaR} and r^{ES} scale with horizon,

$$r^{\mathrm{VaR}}_{\pi,\tau} = \sqrt{\tau} \;\; r^{\mathrm{VaR}}_{\pi,1}, \qquad r^{\mathrm{ES}}_{\pi,\tau} = \sqrt{\tau} \;\; r^{\mathrm{ES}}_{\pi,1} \tag{9}$$

- Both VaR and ES are linear in mean and volatility,

$$r^{\mathrm{VaR}}_{\pi,1} = \mu + \mathrm{z}_\pi \sigma, \qquad r^{\mathrm{ES}}_{\pi,1} = \; \mu - \frac{\phi\left(\mathrm{z}_\pi\right)}{\pi}\sigma \tag{10}$$

Approximations

In practice, whether normal or log-normal, approximate VaR with

$$r^{\text{VaR}}_{\pi,1} = \mu + z_\pi \sigma$$

$$\text{VaR}_{\pi,1} \approx -P_t \, r^{\text{VaR}}_{\pi,1}$$

$$\text{VaR}_{\pi,\tau} \approx \sqrt{\tau}\ \text{VaR}_{\pi,1}$$

Approximate ES with,

$$r^{\text{ES}}_{\pi,1} = \mu - \frac{\phi\left(z_\pi\right)}{\pi}\sigma$$

$$\text{ES}_{\pi,1} \approx -P_t \, r^{\text{ES}}_{\pi,1}$$

$$\text{ES}_{\pi,\tau} \approx \sqrt{\tau}\ \text{ES}_{\pi,1}$$

Where μ and σ can be stats of return rate or log return.

Semi-parametric VaR

Semi-parametric models of VaR use some type of statistical modeling for the tails of the loss distribution, without giving a full, dynamic probability model of losses.

- Extreme value theory is widely used in this way. A comprehensive study is beyond the scope of this course.
- Quantile regression is another methodology which can be applied, but we do not cover here.
- Various semiparametric implementations of GARCH.

Outline

Value-at-Risk

Normal Distributions

Simulation

Portfolio VaR

Historical simulation

Historical simulation is one of the simplest and most widely used approaches to calculating VaR.

- The historical simulation does not assume normally distributed losses.
- However, it is an unconditional VaR, in that it assumes independent observations.
- It takes the cdf of losses from the histogram of past losses.

VaR from historical simulation

To calculate $\text{VaR}_{\pi,1}$ in this way,

1. Take a subsample of the historical returns of size N: $\{\, r_{t-N+1}, \ldots\ r_t \}$.
2. Sort this subsample into ascending order. Define the sorted data as $\left\{\, r_{(1)}, \ldots, \, r_{(N)} \right\}$.
3. Then $\text{VaR}_{\pi,1}$ is the $\pi\%$ order statistic: the ranked observation $r_{(\pi N)}$ where

Working with the discrete cdf

With sample size of N, may be that πN is not an integer.

- Natural to just take a linear interpolation.
- For example, just average the sample points just above and below the π percentile.

$$r_{\pi,1}^{\text{VaR}} = \frac{r_{(\pi N-)} + r_{(\pi N+)}}{2}$$

where $\pi N-$ denotes the closest integer below πN, and $\pi N+$ is the closest integer above πN.

Historic estimation for ES

Obtain the ES again using the order statistics to build the cdf:

$$r^{\text{ES}}_{\pi,1} = \frac{1}{\pi N} \sum_{i=1}^{\pi N} r_{(i)}$$

This is just the sample mean of the left tail of the sample cdf.

Advantages of historical simulation

This approach estimates a cdf of losses nonparametrically by assuming the subsample frequency reflects the actual probabilities.

- Thus, the estimated cdf can have any shape, based on the historic observations.
- The two main advantages here are the ease of implementation and the flexibility in not assuming a probability distribution, (such as the normal,) ex-ante.

Disadvantage 1 of historical simulation

First, the VaR depends on having a good estimate of the $\pi\%$ tail of the distribution.

- If the sample size, N, is small, then there will be large standard errors on the $\pi\%$ order statistic. (The standard errors shrink by the square root of the sample size.)
- If stress-testing extreme market conditions, need a sample of 10,000 just to get 10 observations in the 0.1% tail of the distribution.

Disadvantage 2 of historical distribution

Approach assumes returns are iid, which they are not.

- Take too big of a sample, and may be including irrelevant data, data which came from different distribution compared to data going forward.
- But too small a sample, and no precision.

Monte Carlo

Monte Carlo simulation generates data according to some statistical model.

1. Use each simulated observation to construct the corresponding portfolio loss.
2. Build an empirical cdf (histogram) from these simulated losses, and select the appropriate quantile.

MC in general

MC has many applications:

- Historical approach simply skips the first step by taking observed past as the generated data.
- By using simulated cdf no need to worry about keeping the cdf tractable. Important for complicated dynamics or nonlinear valuation.

MC for nonlinear analysis

Monte Carlo simulation is usefully applied to cases where the portfolio value is nonlinear in the simulated factors.

- Consider simulating stock prices and then plugging the simulated data into Black-Scholes to obtain simulated losses on options.
- Simulate interest rates and then calculate bond portfolio losses as nonlinear function of these simulated data.

Evaluating VaR

The Hit Test is a common way of backtesting a VaR methodology.

- It checks historically what the daily VaR would have been, given the information known at that time. It compares this to the actual performance for the day.
- If the VaR is working well, then the day-t loss should only exceed the day-t $\text{VaR}_{\pi,1}$ on about $\pi\%$ of the days.
- But what if future market environment is very different than past environment with which fit test was run? Many VaR models looked okay before the 2007-2008 crisis hit!

Outline

Portfolio return stats

Portfolio positions are frequently changing, so do not have long stable sample of portfolio return.

- Must use stats of underlying securities to get return stats for the current portfolio position.
- Requires estimation of many parameters given number of underlying securities.
- Using longer historical sample allows for better estimation, but also risks including stale data.
- Common to use sample of 500 to 1000 days of data for estimating security return stats.

VaR of portfolio

Suppose there are k securities,

- $\boldsymbol{r}_{t,t+\tau}$ is a $k \times 1$ vector of time t return **rates**.
- $\boldsymbol{\mu} = \mathbb{E}\left[\boldsymbol{r}_{t,t+\tau}\right]$ is a $k \times 1$ vector of mean returns.
- $\Sigma = \text{cov}\left[\boldsymbol{r}_{t,t+\tau}\right]$ is a $k \times k$ covariance matrix of returns.
- $\boldsymbol{\omega}$ is a $k \times 1$ vector of portfolio weights, satisfying $\boldsymbol{\omega}'\mathbf{1} = 1$.

If $\boldsymbol{r}_t$ is normally distributed, then the formulas above apply, where the portfolio return rate, r^p, has mean and variance,

$$\mu_\tau = (\boldsymbol{\omega})'\boldsymbol{\mu}, \qquad \sigma_\tau^2 = (\boldsymbol{\omega})'\Sigma\boldsymbol{\omega}$$

Portfolio estimation challenge

In application, financial firms may hold portfolios with thousands of assets.

- With k assets, estimating $\boldsymbol{\mu}$ and Σ requires $k + \frac{k(k+1)}{2}$ parameters. For just $k = 10$, this is 65 parameters!
- Factor decompositions (LFD) reduce the number of assets and parameters needed to understand the portfolio value.
- For 3 factors and $k = 10$ securities, this factor decomposition reduces 65 necessary parameters to 9.

Factors for portfolio

Decompose portfolio return rate, r^p, in terms of n factors:

$$r_t^p = \alpha + \beta_1 x_t^1 + \beta_2 x_t^2 + \ldots + \beta_n x_t^n + \epsilon_t$$

- Use small number, n, of factors.
- Note these should form a strong Linear Factor Decomposition (high R-squared,) not factors from a Linear Factor Pricing model, (small alpha.)
- Small number, n, of principal components may work well, and will be uncorrelated from each other.

VaR with factors

- With LFD, estimate distribution of portfolio return just using statistics of the n factors instead of $k > n$ securities.

$$r_t^p = \alpha + \beta_1 x_t^1 + \beta_2 x_t^2 + \ldots + \beta_n x_t^n + \epsilon_t$$

- Let $\boldsymbol{\mu}_x$ and Σ_x are the n-dimensional mean and covariance of factor returns.

$$\mu_\tau = \alpha + \boldsymbol{\beta}' \boldsymbol{\mu}_x, \qquad \sigma_\tau^2 = \boldsymbol{\beta}' \Sigma_x \boldsymbol{\beta}$$

- If returns are normally distributed, these portfolio stats are enough to calculate VaR and ES using (4) and (5).

Factors other than PC

Principal components will give the best fit in-sample, but may be useful to express portfolio in terms of well-known securities.

- Consider a portfolio of stock options, $R^{\text{opt}} = f\left(R^s\right)$.
- Choose factors linear in the stock and quadratic in the stock.
- Now write the option as linear function of these two factors—so-called "delta" and "gamma"

$$r_t^{\text{opt}} = \alpha + \beta_1 r_t^s + \beta_2 \left(r_t^s\right)^2 + \epsilon_t$$

Portfolio VaR to individual VaR

VaR of a portfolio can be broken into the VaR of the underlying securities,

$$\left(\mathrm{VaR}^{p}_{\pi,\tau}\right)^2 = \sum_{j=1}^{k} \omega_j^2 \left(\mathrm{VaR}^{j}_{\pi,\tau}\right)^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k} \omega_i \omega_j \rho_{i,j} \mathrm{VaR}^{i}_{\pi,\tau}\mathrm{VaR}^{j}_{\pi,\tau}$$

where ω is the weight vector of the portfolio, and ω_i denotes the weight in security i.

Marginal and component VaR

- Often want to know impact of security holding on portfolio VaR.
- Marginal VaR (MVaR) measures marginal impact of security i on the portfolio VaR.
- Assume normality, then Marginal VaR for portfolio p at threshold, π, horizon τ, with respect to security i is

$$\frac{\partial\, r^{\text{VaR}}_{\pi,1}}{\partial \boldsymbol{\omega}_i} = R^{\text{MVaR}}_{\pi,1,i} = \mu_i + \mathsf{z}_\pi \frac{1}{\sigma_p} \sum_{j=1}^{k} \boldsymbol{\omega}_j \sigma^{i,j}$$

$$\text{MVaR}_{\pi,1,i} = -\, P_t\, R^{\text{MVaR}}_{\pi,1,i}$$