DRW New-Hire Learning Program

Module: Risk

Session D.3: Optimization

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Summer 2022

Outline

Mean-Variance

Excess Returns

Appendix

Mean-variance comparisons

We want to compare risk and return...

- Use mean return to score the portfolio's benefits.
- Use variance (or volatility) of return to score the portfolio's risk.

Consider the case of two assets:

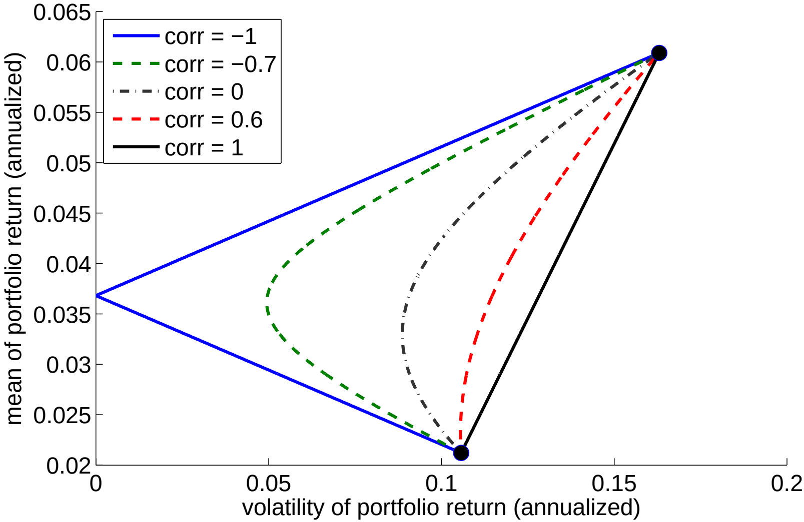


Figure: Example in mean-volatility space of diversification between two assets.

Diversification across n assets

With n securities, there is further potential for diversification.

- The set of all possible portfolios formed from this basis of assets forms a convex set in mean-variance space.
- The boundary of this set is known as the mean-variance frontier, and it forms a parabola.
- The boundary of the set in mean-volatility space forms a hyperbola.

We use **MV frontier** to refer to both the mean-variance and mean-volatility frontiers.

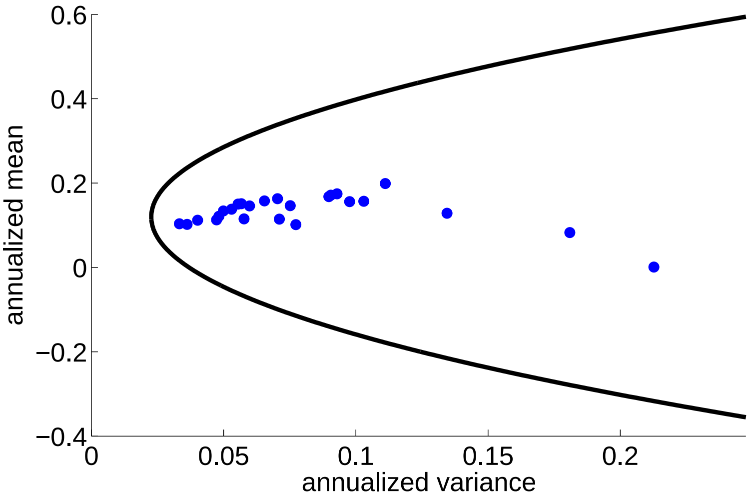


Figure: Mean-variance frontier formed by 25 U.S. equity portfolios, sorted by size and and book/market.

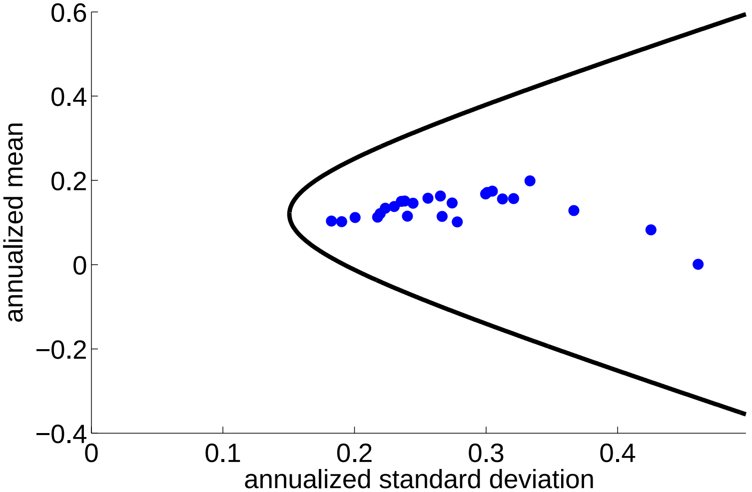


Figure: Mean-volatility frontier formed by 25 U.S. equity portfolios, sorted by size and and book/market.

Efficient portfolios

The top segment of the MV frontier is the set of efficient MV portfolios.

- These portfolios maximize mean return given the return variance.
- Contrast this with the lower segment of the MV frontier, the inefficient MV portfolios.
- The inefficient MV portfolios minimize mean return given the return variance.

Importance of MV analysis

- MV analysis is the most widely used tool in portfolio allocation.
- The model gives a tractable way to balance risk and return.
- Later in the course, we will a connection between MV analysis and beta-factor models.

Notation

Suppose there are n risky assets.

- $\boldsymbol{r}$ is an $n \times 1$ random vector. Each element is the return on one of the n assets.
- Let $\boldsymbol{\mu}$ denote the $n \times 1$ vector of mean returns. Let Σ denote the $n \times n$ covariance matrix of returns.

$$\begin{aligned} \boldsymbol{\mu} =& \mathbb{E}\left[\boldsymbol{r}\right] \\ \Sigma =& \mathbb{E}\left[\left(\boldsymbol{r}-\boldsymbol{\mu}\right)\left(\boldsymbol{r}-\boldsymbol{\mu}\right)'\right] \end{aligned}$$

- For now, we suppose no risk-free rate is available.
- Assume Σ is positive definite—no asset is a linear function of the others.

Portfolios

- An investor chooses a **portfolio**, defined as a $n \times 1$ vector of allocation weights, $\boldsymbol{\omega}$.

- These allocation weights must sum to unity:

$$\boldsymbol{\omega}'\mathbf{1} = 1$$

where $\mathbf{1}$ denotes a $n \times 1$ vector of ones.

- No shorting restriction here: elements of $\boldsymbol{\omega}$ can be negative.

Return moments

The portfolio return on some portfolio, $\boldsymbol{\omega}^p$, is

$$r^p = (\boldsymbol{\omega}^p)' \, \boldsymbol{r}.$$

The portfolio return moments are

$$\mu^p =: \mathbb{E}\left[r^p\right] = \; (\boldsymbol{\omega}^p)' \, \boldsymbol{\mu}$$

$$\sigma_p^2 =: \text{var}\left(r^p\right) = \; (\boldsymbol{\omega}^p)' \, \Sigma \boldsymbol{\omega}^p$$

$$\text{cov}\left(r^p, \boldsymbol{r}\right) = \Sigma \boldsymbol{\omega}^p$$

MV Portfolio

A Mean-Variance (MV) portfolio is a vector, $\boldsymbol{\omega}^*$, which solves the following optimization for some number μ^p:

$$\begin{aligned} \min_{\boldsymbol{\omega}} \quad & \boldsymbol{\omega}'\Sigma\boldsymbol{\omega} \\ \text{s.t.} \quad & \boldsymbol{\omega}'\boldsymbol{\mu} = \mu^p \\ & \boldsymbol{\omega}'\mathbf{1} = 1 \end{aligned}$$

- Note that the objective function is convex in w, given that Σ is positive definite.
- The constraint set is also convex.
- Thus, the solution, $\boldsymbol{\omega}^*$ is characterized by the first-order conditions.

MV solution

Thus, a portfolio $\boldsymbol{\omega}^*$ is MV iff exists $\delta \in (-\infty, \infty)$ such that

$$\boxed{\begin{gathered}\boldsymbol{\omega}^* = \delta \boldsymbol{\omega}^{\mathtt{t}} + (1-\delta)\boldsymbol{\omega}^{\mathtt{v}} \\ \boldsymbol{\omega}^{\mathtt{t}} \equiv \underbrace{\left(\frac{1}{\mathbf{1}'\Sigma^{-1}\boldsymbol{\mu}}\right)}_{\text{scaling}} \Sigma^{-1}\boldsymbol{\mu}, \qquad \boldsymbol{\omega}^{\mathtt{v}} \equiv \underbrace{\left(\frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right)}_{\text{scaling}} \Sigma^{-1}\mathbf{1}\end{gathered}}$$

$\boldsymbol{\omega}^{\mathtt{t}}$ and $\boldsymbol{\omega}^{\mathtt{v}}$ are themselves MV portfolios ($\delta = 0, 1$)

GMV and zero-tangency portfolios

$\boldsymbol{\omega}^{\mathrm{v}}$ is the Global Minimum Variance (GMV) portfolio. It solves,

$$\min_{\boldsymbol{\omega}} \quad \boldsymbol{\omega}'\Sigma\boldsymbol{\omega}$$
$$\text{s.t.} \quad \boldsymbol{\omega}'\mathbf{1} = 1$$

- This is the same as the MV problem, but dropping the first constraint, $(\boldsymbol{\omega}'\boldsymbol{\mu} = \mu^{p}.)$

$\boldsymbol{\omega}^{\mathrm{t}}$ is the portfolio tangent to the mean-volatility frontier and going through the origin. (See next slide.)

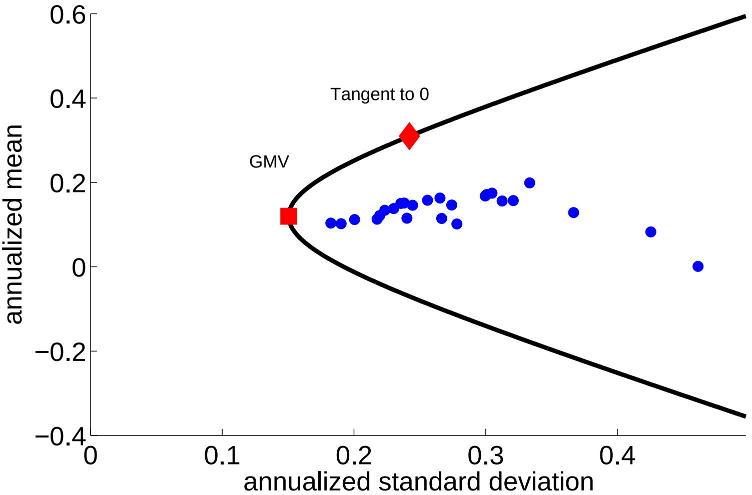


Figure: Illustration of two useful MV portfolios. The Global-Minimum-Variance portfolio as well as the zero-tangency portfolio.

MV investors

Consider MV investors, the investors for whom mean and variance of returns are sufficient statistics of the investment.

- Such investors will hold an MV portfolio, $\boldsymbol{\omega}^*$.
- Thus, these investors are holding linear combination of just two risky portfolios, $\boldsymbol{\omega}^{\mathtt{t}}$ and $\boldsymbol{\omega}^{\mathtt{v}}$.
- So if in real markets all investors were MV investors, everyone would simply invest in two funds.
- Those wanting higher mean returns would hold more in the high-return MV, $\boldsymbol{\omega}^{\mathtt{t}}$, while those wanting safer returns would hold more in the low-return MV, $\boldsymbol{\omega}^{\mathtt{v}}$.

Outline

With a riskless asset

Now consider the existence a risk-free asset with return, r^f.

- Suppose there are still n risky assets available, still notating the risky returns as $\boldsymbol{r}$
- Let $\boldsymbol{w}$ denote a $n \times 1$ vector of portfolio allocations to the n risky assets.
- Since the total portfolio allocations must add to one, we have

$$\text{allocation to the risk-free rate} = 1 - \boldsymbol{w}'\mathbf{1}$$

Mean excess returns

$\boldsymbol{\mu}$ denotes the vector of mean returns of risky assets, $\mathbb{E}\left[\boldsymbol{r}\right]$.

Let μ^p denote the mean return on a portfolio.

$$\mu^p = \left(1 - \boldsymbol{w}'\mathbf{1}\right) r^f + \boldsymbol{w}'\boldsymbol{\mu}$$

Use the following notation for excess returns:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \mathbf{1} r^f$$

Thus the mean return and mean excess return of the portfolio are

$$\begin{aligned}\mu^p =& r^f + \boldsymbol{w}'\tilde{\boldsymbol{\mu}}\\ \tilde{\mu}^p =& \boldsymbol{w}'\tilde{\boldsymbol{\mu}}\end{aligned}$$

Variance of returns

- The risk-free rate has zero variance and zero correlation with any security.
- Let Σ continue to denote the $n \times n$ covariance matrix of *risky* assets, (and is positive semi-definite.)
- The return variance of the portfolio, $\boldsymbol{w}^p$ is

$$\sigma_p^2 = \boldsymbol{w}'\Sigma\boldsymbol{w}$$

The $\tilde{\text{MV}}$ problem with a riskless asst

A Mean-Variance portfolio with risk-free asset ($\tilde{\text{MV}}$) is a vector, $\boldsymbol{w}^*$, which solves the following optimization for some mean excess return number $\tilde{\mu}^p$:

$$\min_{\boldsymbol{w}} \quad \boldsymbol{w}'\Sigma\boldsymbol{w}$$
$$\text{s.t.} \quad \boldsymbol{w}'\tilde{\boldsymbol{\mu}} = \tilde{\mu}^p$$

- In contrast to the MV problem, there is only one constraint.
- The allocation weight vector, $\boldsymbol{w}$ need not sum to one, as the remainder is invested in the risk-free rate.

Solving the $\tilde{\text{M}}\text{V}$ problem

Solving the problem is straitforward:

1. Set up the Lagrangian with just one constraint.
2. The FOC is sufficient given the convexity of the problem.
3. Finally, substitute the Lagrange multiplier using the constraint.

Refer to the solution as an $\tilde{\text{M}}\text{V}$ portfolio.

$\tilde{\text{MV}}$ solution

$$\boldsymbol{w}^* = \tilde{\delta}\, \boldsymbol{w}^{\mathtt{t}}$$

for the portfolio

$$\boldsymbol{w}^{\mathtt{t}} = \underbrace{\left(\frac{1}{\mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}\right)}_{\text{scaling}} \Sigma^{-1}\tilde{\boldsymbol{\mu}}$$

and allocation

$$\tilde{\delta} = \left(\frac{\mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{(\tilde{\boldsymbol{\mu}})'\,\Sigma^{-1}\tilde{\boldsymbol{\mu}}}\right)\tilde{\mu}^{p}$$

$\tilde{\text{MV}}$ portfolio variance formula

The return variance of an $\tilde{\text{MV}}$ portfolio is given by

$$\frac{\left(\tilde{\mu}^p\right)^2}{(\tilde{\boldsymbol{\mu}})' \Sigma^{-1} \tilde{\boldsymbol{\mu}}}$$

This implies that the return volatility (standard-deviation) is linear in the absolute value of the mean excess return:

$$\frac{\left|\tilde{\mu}^p\right|}{\sqrt{(\tilde{\boldsymbol{\mu}})' \Sigma^{-1} \tilde{\boldsymbol{\mu}}}}$$

Tangency portfolio

The result is that any $\tilde{\text{MV}}$ portfolio is a combination of the tangency portfolio, $\boldsymbol{w}^{\mathrm{t}}$, and a position in the riskless asset.

- The tangency portfolio, $\boldsymbol{w}^{\mathrm{t}}$ invests 100% in risky assets, $\mathbf{1}'\boldsymbol{w}^{\mathrm{t}} = 1$.
- $\boldsymbol{w}^{\mathrm{t}}$ is the unique portfolio which is on the risky MV frontier as well as the $\tilde{\text{MV}}$ frontier expanded by the risk-free asset.
- $\boldsymbol{w}^{\mathrm{t}}$ is the point on the risky MV frontier at which the tangency line goes through the risk-free rate. (See the figure below.)

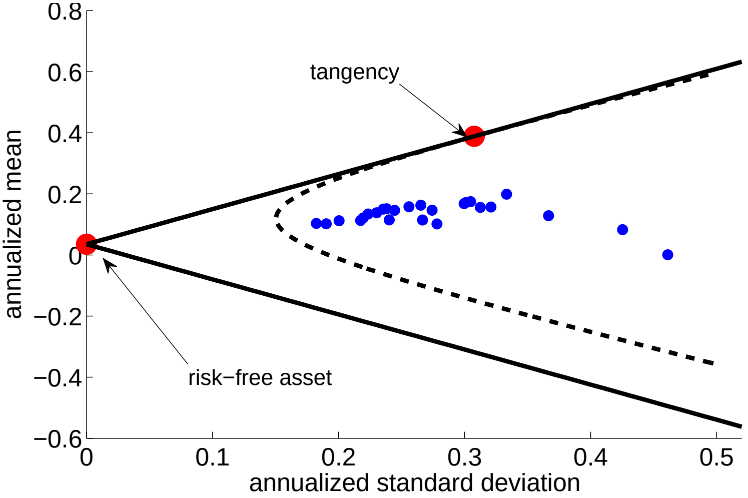


Figure: Illustration of the $\tilde{MV}$ frontier when a riskless asset is available. In this case, the $\tilde{MV}$ portfolio frontier consists of two straight lines. The curved frontier is the MV frontier when a riskless asset is unavailable.

Tangency portfolio and the Sharpe ratio

For an arbitrary portfolio, $\boldsymbol{w}^p$,

$$\mathrm{SR}\left(\boldsymbol{w}^p\right) = \frac{\mu^p - r^f}{\sigma^p} = \frac{\tilde{\mu}^p}{\sigma^p}$$

The tangency portfolio, $\boldsymbol{w}^{\mathrm{t}}$, is the portfolio on the risky MV frontier with maximum Sharpe ratio.

$$\mathrm{SR}\left(\boldsymbol{w}^*\right) = \pm\sqrt{\left(\tilde{\boldsymbol{\mu}}\right)' \Sigma^{-1} \tilde{\boldsymbol{\mu}}}$$

The SR magnitude is constant across all $\tilde{\text{MV}}$ portfolios.
(Sign depends on whether part of the efficient or inefficient frontier.)

Capital Market Line

The Capital Market Line (CML) is the efficient portion of the $\tilde{\text{MV}}$ frontier.

- The CML shows the risk-return tradeoff available to $\tilde{\text{MV}}$ investors.
- The slope of the CML is the maximum Sharpe ratio which can be achieved by any portfolio.
- The inefficient portion of the $\tilde{\text{MV}}$ frontier acheives the minimum (negative) Sharpe ratio by shorting the tangency portfolio.

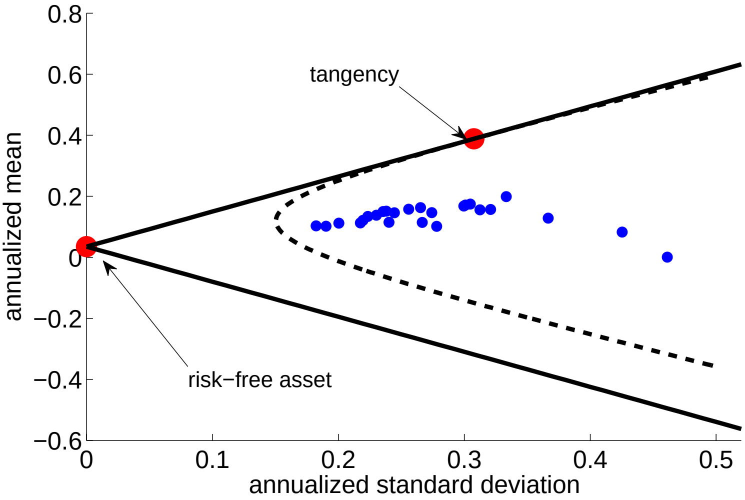


Figure: Illustration of the $\tilde{\text{M}}$V frontier when a riskless asset is available. In this case, the $\tilde{\text{M}}$V portfolio frontier consists of two straight lines. The curved frontier is the MV frontier when a riskless asset is unavailable.

Two-fund separation

Two-fund separation. Every $\tilde{\text{MV}}$ portfolio is the combination of the risky portfolio with maximal Sharpe Ratio and the risk-free rate.

Thus, for an $\tilde{\text{MV}}$ investor the asset allocation decision can be broken into two parts:

1. Find the tangency portfolio of risky assets, $\boldsymbol{w}^{\mathrm{t}}$.
2. Choose an allocation between the risk-free rate and the tangency portfolio.

Intuition of asset allocation

The two-fund separation says that

- Any investment in risky assets should be in the tangency portfolio since it offers the maximum Sharpe Ratio.
- One must decide the desired level of risk in the investment, which determines the split between the riskless asset and the tangency portfolio.

Conclusion

- Non-additivity of portfolio risk requires us to consider mathematics of diversification.
- Mean-variance optimization is the dominant approach in industry.
- But implementation will raise a number of challenges, related to computation and statistics.

References

- Back, Kerry. *Asset Pricing and Portfolio Choice Theory.* 2010. Chapter 5.

  Develops the mathematical formulas for optimization among n assets.

- Bodie, Kane, and Marcus. *Investments.* 2011. Chapter 7.

  Develops the intuition of mean-variance space and optimal portfolios.

Outline

Solving the MV problem: FOC

Solving with Lagrangian multipliers, (γ_1 and γ_2,) gives the unconstrained optimization:

$$\mathcal{L} = \frac{1}{2}\boldsymbol{\omega}'\Sigma\boldsymbol{\omega} - \gamma_1\left(\boldsymbol{\omega}'\boldsymbol{\mu} - \mu^p\right) - \gamma_2\left(\boldsymbol{\omega}'\mathbf{1} - 1\right)$$

The first derivative equations are (in matrix notation,)

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{\omega}'} = \Sigma\boldsymbol{\omega} - \gamma_1\boldsymbol{\mu} - \gamma_2\mathbf{1}$$

Get the first-order conditions of optimization by setting equal to zero and solve for $\boldsymbol{\omega}^*$:

$$\boldsymbol{\omega}^* = \Sigma^{-1}\begin{bmatrix}\boldsymbol{\mu} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix}$$

Solving the MV problem: portfolios $\boldsymbol{\omega}^{\mathtt{t}}$ and $\boldsymbol{\omega}^{\mathtt{v}}$

Rewrite this as

$$\boldsymbol{\omega}^* = \gamma_1 \Sigma^{-1} \boldsymbol{\mu} + \gamma_2 \Sigma^{-1} \mathbf{1}$$

which can be rewritten as the sum of two portfolios:

$$\boldsymbol{\omega}^* = \gamma_1 \left(\mathbf{1}' \Sigma^{-1} \boldsymbol{\mu} \right) \boldsymbol{\omega}^{\mathtt{t}} + \gamma_2 \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \boldsymbol{\omega}^{\mathtt{v}}$$

where

$$\boldsymbol{\omega}^{\mathtt{t}} \equiv \frac{1}{\mathbf{1}' \Sigma^{-1} \boldsymbol{\mu}} \Sigma^{-1} \boldsymbol{\mu}, \qquad \boldsymbol{\omega}^{\mathtt{v}} \equiv \frac{1}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \Sigma^{-1} \mathbf{1}$$

Solving the MV problem: eliminate γ_2

Note that $\boldsymbol{\omega}^{\mathtt{t}}$ and $\boldsymbol{\omega}^{\mathtt{v}}$ are proper portfolios:

$$\left(\boldsymbol{\omega}^{\mathtt{t}}\right)' \mathbf{1} = 1, \qquad \left(\boldsymbol{\omega}^{\mathtt{v}}\right)' \mathbf{1} = 1$$

Given that $\mathbf{1}'\boldsymbol{\omega}^* = \mathbf{1}'\boldsymbol{\omega}^{\mathtt{t}} = \mathbf{1}'\boldsymbol{\omega}^{\mathtt{v}} = 1$, the equation above implies

$$1 = \gamma_1 \left(\mathbf{1}'\Sigma^{-1}\boldsymbol{\mu}\right) + \gamma_2 \left(\mathbf{1}'\Sigma^{-1}\mathbf{1}\right)$$

Use this to rewrite the MV vector as

$$\boldsymbol{\omega}^* = \delta\boldsymbol{\omega}^{\mathtt{t}} + (1-\delta)\boldsymbol{\omega}^{\mathtt{v}}$$

where

$$\delta \equiv \gamma_1 \left(\mathbf{1}'\Sigma^{-1}\boldsymbol{\mu}\right)$$

MV formulas

For any MV portfolio $\boldsymbol{\omega}^*$, consider the mean, μ^p and variance σ_p^2,

Sub out γ_1 to get δ in terms of μ^p,

$$\delta = \frac{\mu^p - \boldsymbol{\mu}'\boldsymbol{\omega}^{\mathtt{v}}}{\boldsymbol{\mu}'\boldsymbol{\omega}^{\mathtt{t}} - \boldsymbol{\mu}'\boldsymbol{\omega}^{\mathtt{v}}}$$

The return variance, σ_p^2, is a quadratic function of μ^p,

$$\sigma_p^2 = \frac{1}{\phi_0\phi_2 - \phi_1^2}\left[\phi_0 - 2\phi_1\left(\mu^p\right) + \phi_2\left(\mu^p\right)^2\right]$$

where the coefficients, ϕ are characterized by

$$\phi_0 = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}, \qquad \phi_1 = \boldsymbol{\mu}'\Sigma^{-1}\mathbf{1}, \qquad \phi_2 = \mathbf{1}'\Sigma^{-1}\mathbf{1}$$

Two-fund separation

Consider any three MV portfolios, $\boldsymbol{\omega}_a$, $\boldsymbol{\omega}_b$, $\boldsymbol{\omega}^p$, which must satisfy the following for some $\delta_a, \delta_b, \delta_p$,

$$\begin{aligned}\boldsymbol{\omega}_a =& \delta_a \boldsymbol{\omega}^{\texttt{t}} + (1-\delta_a)\boldsymbol{\omega}^{\texttt{v}}\\ \boldsymbol{\omega}_b =& \delta_b \boldsymbol{\omega}^{\texttt{t}} + (1-\delta_b)\boldsymbol{\omega}^{\texttt{v}}\\ \boldsymbol{\omega}^p =& \delta_p \boldsymbol{\omega}^{\texttt{t}} + (1-\delta_p)\boldsymbol{\omega}^{\texttt{v}}\end{aligned}$$

- $\boldsymbol{\omega}^{\texttt{t}}$ and $\boldsymbol{\omega}^{\texttt{v}}$ are not unique in being able to decompose the MV portfolio, $\boldsymbol{\omega}^p$.
- Any MV portfolio can be written as a combo of $\boldsymbol{\omega}_a$ and $\boldsymbol{\omega}_b$.

$$\boldsymbol{\omega}^p = \vartheta \boldsymbol{\omega}_a + (1-\vartheta)\boldsymbol{\omega}_b, \qquad \vartheta \equiv \frac{\delta_p - \delta_b}{\delta_a - \delta_b}$$

Uncorrelated MV portfolios

Using 2-fund separation, convenient to decompose MV portfolios into two orthogonal portfolios.

- For any MV portfolio, $\boldsymbol{\omega}^p \neq \boldsymbol{\omega}^{\mathrm{v}}$, there exists another MV portfolio, $\boldsymbol{\omega}_o$ such that $\boldsymbol{\omega}_o$ orthogonal to $\boldsymbol{\omega}^p$.
- If $\boldsymbol{\omega}^p$ has mean return μ^p, then the orthogonal MV portfolio $\boldsymbol{\omega}_o$ has mean return, μ_o, where

$$\mu_o = \frac{\phi_1 \mu^p - \phi_0}{\phi_2 \mu^p - \phi_1}$$

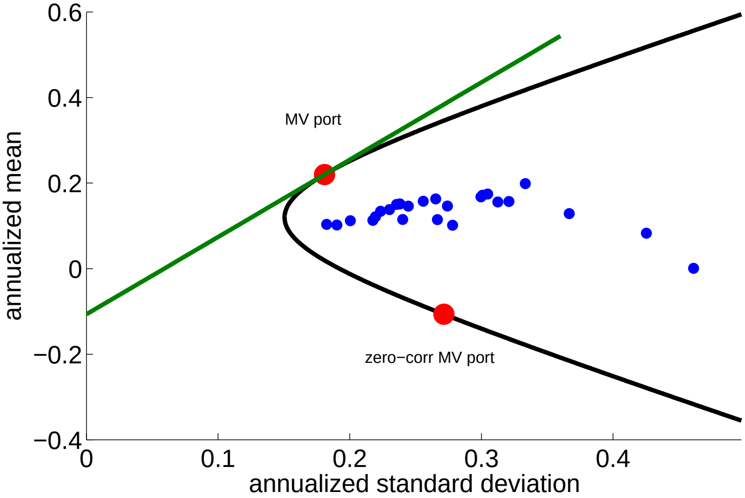
$$\phi_0 = \boldsymbol{\mu}' \Sigma^{-1} \boldsymbol{\mu}, \qquad \phi_1 = \boldsymbol{\mu}' \Sigma^{-1} \mathbf{1}, \qquad \phi_2 = \mathbf{1}' \Sigma^{-1} \mathbf{1}$$

Geometry of uncorrelated portfolios

In mean-volatility space, the orthogonal MV portfolio has a simple geometry.

- Draw the tangent line at the point of some MV portfolio.
- Find the value on this tangent line for volatility of zero, (where it hits the vertical axis.)
- The mean return at this point is, μ_o, the mean return of the orthogonal MV portfolio.

See the following figure.


0.6
0.4
0.2
0
−0.2
−0.4
annualized mean
MV port
zero−corr MV port
0
0.1
0.2
0.3
0.4
annualized standard deviation

Figure