DRW New-Hire Learning Program

Module: Risk

Session D.1: Measuring Risk and Return

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Outline

Key results

- Portfolio risk is a nonlinear function of security risk.
- If we assume frictionless markets, then we can analytically solve for “optimal” return-risk allocations.
- The optimal formula penalizes securities for marginal risk (covariance), not total risk (volatility.)
- The result is analytical, efficiently implemented, and maximizes portfolio Sharpe Ratio.

Key questions

- What do we mean by "optimal"?
- What is the right measure of risk?
- How do we forecast returns?
- How well does this work in practice?

Portfolio Management Procedure

1. Define the security universe
2. Model security risk and performance
3. Forecast returns
4. Define the portfolio's objective
5. Define portfolio's constraints
6. Simulate the candidate portfolios
7. Optimize among the portfolios
8. Assess the constructed portfolio's performance

Getting Started

To begin,

- we need to understand diversification and non-linearity of risk.
- we will examine the full portfolio optimization process.

During the course, we will dig deeper into each of the steps of the process.

Outline

The Big Picture

Diversification

Measuring Performance

Evaluating Performance

Hedging and Tracking

Return notation: one-period

notation	description	formula	example
r^i	return rate of asset i		
r^f	risk-free return rate		
$\tilde{r}^i$	excess return rate of asset i	$r^i - r^f$	

Two investments: bonds and stocks

Consider the following portfolio example

Table: Portfolio example

	return	allocation weight
bonds	r^b	w
stocks	r^s	$1-w$

Table: Return statistics notation

mean	variance	correlation
μ	σ^2	ρ

Portfolio return stats

Investment portfolio return r^p has mean and variance of

$$\mu^p = w\mu^b + (1-w)\mu^s$$

$$\sigma_p^2 = w^2\sigma_b^2 + (1-w)^2\sigma_s^2 + 2w(1-w)\rho\sigma_s\sigma_b$$

Perfect correlation

Suppose that $\rho = 1$.

- Then the volatility (standard deviation) of the portfolio is proportional to the asset allocation weights:

$$\sigma_p = w\sigma_b + (1 - w)\sigma_s$$

- Thus, both mean and volatility are linear in the allocations.

Imperfect correlation

Suppose that $\rho < 1$.

- The volatility function is convex,

$$\sigma_p < w\sigma_b + (1-w)\sigma_s$$

- Yet the mean return is still linear in the portfolio allocation:

$$\mu^p = w\mu^b + (1-w)\mu^s$$

Diversification

Portfolio diversification refers to this case where

- mean returns are linear in allocations
- while volatility of returns is less than linear in allocation.

This only required $\rho < 1$.

A perfect hedge

For $\rho = -1$,

- The portfolio variance can be as small as desired, by choosing the appropriate allocation, w.
- In fact, $\sigma_p = 0$ if

$$w = \frac{\sigma_s}{\sigma_b + \sigma_s}$$

- Thus, a riskless portfolio can be formed from the two risky assets.

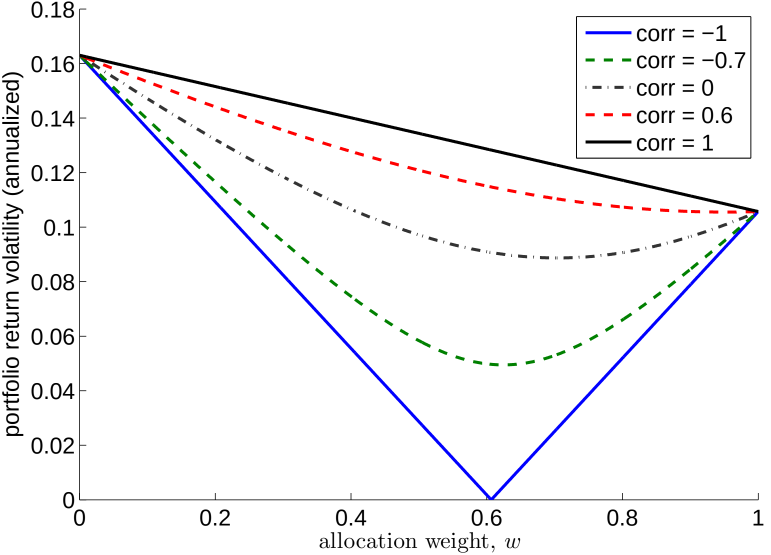

Figure: Diversification of investment portfolio between two risky assets.

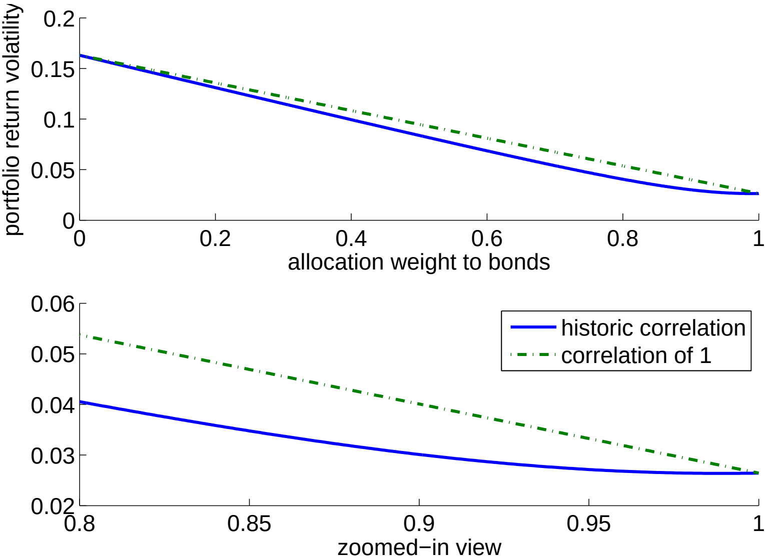

Figure: Diversification over investment in U.S. market index and 10-year T-note. Source: CRSP and N.Y. Fed. July 1971 to June 2012.

Allocation among n assets

Consider the following portfolio allocation problem:

- n risky securities,
- return volatility (std.dev.) denoted σ_i
- return covariance between security i and j denoted by $\sigma_{i,j}$.
- w^i denotes the fraction of the portfolio allocated to asset i, with $\sum_{i=1}^n w^i = 1$.

Then

$$\sigma_p^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} w^i w^j \sigma_{i,j}$$

Variance of the equally weighted portfolio

Consider an equally-weighted portfolio, with $w^i = 1/n$ for each asset. Then

$$\sigma_p^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 + \frac{1}{n^2}\sum_{j\neq i}\sum_{i=1}^{n}\sigma_{i,j}$$

In the earlier example with bonds and stocks, $n = 2$,

$$\sigma_p^2 = \frac{1}{4}\sigma_b^2 + \frac{1}{4}\sigma_s^2 + \frac{1}{2}\underbrace{\sigma_{b,s}}_{\rho\sigma_b\sigma_s}$$

Portfolio variance as average covariances

Use the following notation for averaging the variances and covariances across the n assets:

$$\text{avg}\left[\sigma_i^2\right] \equiv \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$$

$$\text{avg}\left[\sigma_{i,j}\right] \equiv \frac{1}{n(n-1)}\sum_{j\neq i}\sum_{i=1}^{n}\sigma_{i,j}$$

So the portfolio variance can be written as

$$\sigma_p^2 = \frac{1}{n}\text{avg}\left[\sigma_i^2\right] + \frac{n-1}{n}\text{avg}\left[\sigma_{i,j}\right]$$

Portfolio irrelevance of individual security variance

As number of securities in portfolio, n, gets large,

$$\lim_{n \to \infty} \sigma_p^2 = \text{avg}\left[\sigma_{i,j}\right]$$

- Individual security variance is unimportant!
- Overall portfolio variance is average of individual security covariance.

Diversified portfolio

Obtained this result using equally-weighted portfolio, $w^i = 1/n$.

- Don't need equal weighting, just that

$$\lim_{n\to\infty} w^i = 0$$

- That is, as n gets large the portfolio must have trivial exposure to security i.
- This is the sense in which portfolio must be diversified for individual variances to become unimportant.

Portfolio variance decomposition

Above we saw the equally-weighted portfolio variance:

$$\sigma_p^2 = \frac{1}{n}\text{avg}\left[\sigma_i^2\right] + \frac{n-1}{n}\text{avg}\left[\sigma_{i,j}\right]$$

Variance has a term which can be diversified to zero, and another term that remains.

Suppose that asset returns have

- identical volatilities, $\sigma_i = \sigma$
- identical correlations, $\rho_{i,j} = \rho$

Systematic risk

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

$$\lim_{n\to\infty} \sigma_p^2 \to \underbrace{\rho\sigma^2}_{\text{systematic}}$$

- A fraction, ρ, of the variance is systematic.
- No amount of diversification[1] can get portfolio variance lower:

$$\sigma_p^2 \geq \rho\sigma^2$$

[1] Inequality holds for any n and any set of allocations $\{w^i\}$.

Idiosyncratic risk

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

- Idiosyncratic risk refers to the diversifiable part of σ_p^2.
- An equally-weighted portfolio [2] has idiosyncratic risk equal to $\frac{1}{n}\sigma^2$.

[2]For general weights, w^i, remaining idiosyncratic risk is bounded by $\max_i w^i\sigma^2$.

Correlation and diversified portfolios

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

- For $\rho = 1$, there is no possible diversification, regardless of n.

$$\sigma_p^2 = \sigma^2$$

- For $\rho = 0$, there is no systematic risk, only variance is remaining idiosyncratic:

$$\sigma_p^2 = \frac{1}{n}\sigma^2$$

  And as n gets large the portfolio is riskless,

$$\lim_{n\to\infty} \sigma_p^2 = 0$$

Riskless portfolios

- Above, we found that a riskless portfolio could be created if $\rho = -1$.
- Here, we found that a riskless portfolio can be created if $\rho = 0$.

Question:
How did the assumptions behind these conclusions differ?

Answer:

- In the *case of just two underlying assets*, complete diversification is achieved with $\rho = -1$.
- In the case of many assets, complete diversification is achieved when all assets are uncorrelated, *and the number of assets in the portfolio goes to infinity.*

Outline

Log returns

Let r denote the log-return:

$$\mathrm{r}_t \equiv \log(1 + r_t)$$

Log returns are particularly useful when dealing with compounding returns across different time horizons.

$$\mathrm{r}_{t,t+h} \equiv \log(1 + r_{t,t+h}) = \sum_{i=1}^{h} \mathrm{r}_{t+i}$$

Thus, the cumulative return from t to $t + h$ is just the sum of one-period returns.

Annualizing Returns

- Suppose that log returns are iid (independent, and identically distributed).
- Then the mean and variance are linear in the cumulative return horizon, h.

$$\mathbb{E}\left[\mathrm{r}_{t,t+h}\right] = h\left(\mathbb{E}\left[\mathrm{r}_t\right]\right)$$
$$\mathrm{var}\left[\mathrm{r}_{t,t+h}\right] = h\left(\mathrm{var}\left[\mathrm{r}_t\right]\right)$$

Example: Suppose the *monthly* return of a security has unconditional mean, μ. Then the *annual* return of the security is 12μ.

Interview question

Question

- Daily excess returns on the U.S. stock index from July 1963 to June 2012 have a volatility of 0.0010 (0.1%).
- Assume there are 21 business days per month.

What is the volatility of monthly returns on the U.S. stock index over this time?

Interview question

Answer

- Calculate $\sqrt{21} = 4.58$.
- According to iid cumulation formula, monthly volatility would be 0.458%.
- Empirically, monthly excess return volatility is 4.55 times as large as the daily volatility.

Source: CRSP index. July 1963 - June 2012.

The iid assumption

These scalings are widely used in quoting stats.

- Theoretically requires iid assumption, but this is often ignored.
- Return series have low autocorrelation, so assumption of independence may not be too bad.
- However, returns are often distributed in clearly non-identical ways, as illustrated by the following figure.

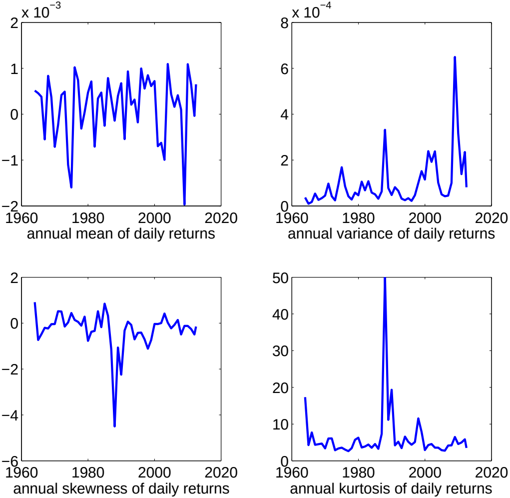

Figure: Sub-sample estimates of first four moments of excess log-returns on equity index. Source: CRSP U.S. stock index. July 1963 to June 2012.

Higher moments

Consider a random variable x, with

$$\mu = \mathbb{E}[x]$$
$$\sigma^2 = \mathbb{E}\left[(x-\mu)^2\right]$$

We are not assuming x is normally distributed.

Skewness is defined as the (scaled) third centralized moment of the distribution:

$$\varsigma = \frac{\mathbb{E}\left[(x-\mu)^3\right]}{\sigma^3}$$

Kurtosis is defined as

$$\kappa = \frac{\mathbb{E}\left[(x-\mu)^4\right]}{\sigma^4}$$

A normal distribution has kurtosis equal to 3, so *excess kurtosis* refers to $\kappa - 3$.

Scaling higher moments

Unlike quotes of return mean and volatility,

- Skewness and kurtosis of returns are not typically scaled with cumulative horizon.
- The formulas for how skewness and kurtosis compound for $\mathrm{r}_{t,t+h}$ are messier, especially without the iid assumption.

Example: higher moment market returns

Consider excess returns from the U.S. equity market.

- Skewness of **daily** returns: -0.8150.
  Skewness of **monthly** returns: -0.8023.
- Kurtosis of **daily** returns: 22.1085.
  Kurtosis of **monthly** returns: 5.8157.

Source: CRSP Index . July 1963 to June 2012

Estimating higher moments

Given samples of size T, the sample mean, variance, skewness, and kurtosis are given by

$$\bar{\mu}_x = \frac{1}{T}\sum_{i=1}^{T} x_i$$

$$\bar{\sigma}_x^2 = \frac{1}{T-1}\sum_{i=1}^{T}(x_i - \bar{\mu}_x)^2$$

$$\bar{\varsigma}_x = \left(\frac{1}{\bar{\sigma}_x^3}\right)\frac{1}{T-1}\sum_{i=1}^{T}(x_i - \bar{\mu}_x)^3$$

$$\bar{\kappa}_x = \left(\frac{1}{\bar{\sigma}_x^4}\right)\frac{1}{T-1}\sum_{i=1}^{T}(x_i - \bar{\mu}_x)^4$$

Value at Risk

The τ-day, π% Value at Risk (VaR) of a portfolio is defined as $\text{VaR}_{\pi,\tau}$ such that

- there is a π% chance
- that over a horizon of τ days
- the portfolio will lose an amount greater than VaR.

VaR as a Return

The VaR expressed as a return rate is then,

$$r^{\text{VaR}}_{\pi,\tau} = F^{r\,(-1)}_{\tau}(\pi)$$

where F^{r}_{τ} is the cdf of the return distribution.

The VaR in terms of returns is simply the π quantile of the observed returns.

Expected Shortfall

Expected Shortfall (ES) refers to the expected loss conditional on a loss greater than $\text{VaR}_{\pi,\tau}$ occurring.

- ES is the expected horizon-τ loss, conditional on a loss in the $(1-\pi)\,\%$ tail of the loss cdf occurring.
- That is, ES looks at expected loss given that a loss of at least $\text{VaR}_{\pi,\tau}$ has occurred.

Thus, ES is simply the mean of all returns smaller than the π quantile.

ES is also sometimes known as Conditional Value-at-Risk (CVaR).

Maximum Drawdown

The maximum drawdown (MDD) of a return series is the maximum **cumulative** loss suffered during the time period.

- Visually, this is the largest peak-to-trough during the sample.
- This is a path-dependent statistic, and it is much less precise to estimate for the future.
- It is widely cited in performance evaluation to understand how badly the investment might perform.

Outline

Risk-adjusted performance

- Asset may have impressive $\mathbb{E}[r]$, but we want to know how this compares to risk.
- Otherwise, a fund manager might obtain high returns by loading excessively on risk.

Mean-Variance

Mean-variance analysis is adjusting for one type of risk: variance.

- Optimize mean per variance (or equivalently, volatility.)
- The Sharpe Ratio is a measure of this risk and return tradeoff.
- Did not consider any other measure of return risk.

Factor decomposition of return variation

A Linear Factor Decomposition (LFD) of $\tilde{r}^i$ onto the factor $\mathbf{x}_t$ is given by the regression,

$$\tilde{r}^i_t = \alpha + \boldsymbol{\beta}^{i,\mathbf{x}}\mathbf{x}_t + \epsilon_t$$

- The variation in returns is decomposed into the variation explained by the benchmark, x_t and by the residual, ϵ_t.
- These factors, $\mathbf{x}$, in the LFD should give a high R-squared in the regression if they really explain the variation of returns well.

Elements of the regression

Consider the following elements of the regression:

- Alpha. Expected return beyond what can be explained by the factor.
- Beta. Risk related to the factor. If x moves, how much will our return move?
- Residual. The risk of the return uncorrelated to the factor.

Interpreting alpha

Using alpha as a measure of performance is sensitive to which factors are used in the regression.

- High α will always lead to the question of whether the performance is good, or whether we used a bad model.
- Perhaps alpha is really just some missing beta from the model.
- Still, if this missing beta is not widely known or understood, it may make sense for investors to pay fees to get access to this beta knowledge, even though the fund is passively tracking a model.

Luck or skill

Estimating alpha is statistically imprecise.

- It would be easy to get a large α due to in-sample luck.
- So when faced with a large α, it may be a sign of high performance, or it may just be luck in that sample.

Treynor's Ratio

Treynor's measure is an alternative measure of the risk-reward tradeoff. For the return of asset, i,

$$\text{Treynor Ratio} = \frac{\mathbb{E}\left[\tilde{r}^i\right]}{\beta^{i,m}}$$

Information ratio

The **information ratio** refers to the Sharpe Ratio of the non-factor component of the return: $\alpha + \epsilon_t$.

$$\mathrm{IR} = \frac{\alpha}{\sigma_\epsilon}$$

where σ_ϵ and α come from

$$\tilde{r}^i_t = \alpha + \beta^{i,j}\tilde{r}^j_t + \epsilon_t$$

- α measures the excess return beyond what is explained by the factor, j.
- σ_ϵ measures the non-factor volatility.

Manipulating the benchmark

Investment managers (of hedge funds, mutual funds, etc.) might prefer to be evaluated against a benchmark which does not capture all risks.

- Hedge funds may want to be evaluated against a simple equity benchmark.
- But these funds are likely achieving excess returns by taking on other forms of risk which do not show up in the regression.

Outline

The Big Picture

Diversification

Measuring Performance

Evaluating Performance

Hedging and Tracking

Net exposure

Investor is long \$1 of i and hedges by selling h\$ of j.

- Time t net exposure is,

$$\epsilon_t = r_t^i - h\, r_t^j$$

- This net exposure after hedging is known as **basis**.
- A position in i is perfectly hedged over horizon t if $\epsilon_t = 0$ with probability one.

Why basis?

Why cross-hedging rather than perfectly hedging with i through futures, options, etc.

- Maybe asset i is a non-tradable exposure.
- Maybe asset i is a traded security, but it has no market in futures and shorting i is costly.

Instead, the investor must hedge using asset j.

Basis risk

$$\epsilon_t = r_t^i - h\, r_t^j$$

Basis risk refers to volatility in ϵ_t, denoted as σ_ϵ.

$$\sigma_\epsilon^2 = \sigma_i^2 + h^2\sigma_j^2 - 2h\,\sigma_i\sigma_j\,\rho_{i,j}$$

- Denoted as σ_ϵ^2 because basis is the error in the hedge.
- For $\rho_{i,j} = \pm 1$, the basis risk can be eliminated.

Optimal hedge ratio

The **optimal hedge ratio**, h^*, minimizes basis risk.

$$\begin{aligned} h^* &= \arg\min_h \sigma_\epsilon^2 \\ &= \arg\min_h \left\{\sigma_i^2 + h^2\sigma_j^2 - 2h\,\sigma_i\sigma_j\,\rho_{i,j}\right\} \end{aligned}$$

Solve by taking derivative,

$$h^* = \frac{\sigma_i}{\sigma_j}\,\rho_{i,j}$$

- Higher correlation implies larger hedge ratio, h.
- High relative volatility of i implies larger hedge ratio.
- With negative correlation, must go long the hedging security.

Basis as a regression residual

From the previous slide, we can write

$$r_t^i = \beta^{i,j} r_t^j + \epsilon_t$$

where

$$\beta^{i,j} = h^*$$

- The optimal hedge ratio, h^*, is simply a regression beta!
- Optimized basis risk is simply the regression residual variance.
- (Thus the notation of using ϵ to denote basis.)

Hedging returns

These results also apply to hedging with multiple assets:

$$r_t^i = \beta^{i,1} r_t^1 + \beta^{i,2} r_t^2 + \ldots + \beta^{i,k} r_t^k + \epsilon_t$$

- Basis is then the net return exposure.
- Optimal hedge ratios are given by the betas in the return regression above.

Hedging excess returns

These results also apply to hedging excess returns.

$$\tilde{r}_t^i = \beta^{i,1}\tilde{r}_t^1 + \beta^{i,2}\tilde{r}_t^2 + \ldots + \beta^{i,k}\tilde{r}_t^k + \epsilon_t$$

- Optimal hedge ratios are given by the regression beta.
- Intercept is portion of mean returns which can not be replicated by the hedge strategy.

Include an intercept?

In regression for optimal hedge ratio, should we include a constant, (alpha?) Depends on our purpose...

- Do we want to explain the total return (including the mean) or simply the excess-mean return?
- In short samples, mean returns may be estimated inaccurately, (whether in r^i or $\tilde{r}^i$,) so we may want to include α (eliminate means) to focus on explaining variation.

Investment with hedging a factor

- Suppose a hedge fund wants to trade on information regarding a certain asset return, r^i.
- But does not want the trade to be subject to the overall market factor, r^m.
- More generally, imagine anyone that wants to trade on the performance of return r^i **relative** to another factor r^j.

This idea of trading on specific information while hedging out broader market movements is the origination of the term, hedge funds.

Building the market-hedged position

A hedge fund would first run the regression

$$\tilde{r}_t^i = \alpha + \beta^{i,m}\tilde{r}_t^m + \epsilon_t$$

- Then the hedge-fund can go long $\tilde{r}^i$, while shorting $\beta^{i,m}$ times the overall market.
- The fund is then holding

$$\tilde{r}_t^i - \beta^{i,m}\tilde{r}_t^m = \alpha + \epsilon_t$$

Properties of the market-hedged position

$$\tilde{r}^i_t - \beta^{i,m} \tilde{r}^m_t = \alpha + \epsilon_t$$

- This hedged position has mean excess return α; volatility σ_ϵ.
- Compared to simply going long $\tilde{r}^i$, the strategy is no longer subject to the volatility coming from $\beta^{i,m} \tilde{r}^m_t$.
- This allows the hedge fund to minimize the variance of the hedged position.

Hedging vs Tracking

- We have considered the case where an investor wants to completely hedge out some factor, r^j.
- This optimal hedging allows the investor to just trade on the portion of r^i uncorrelated with r^j: $\alpha + \epsilon$.
- Now consider a **tracking portfolio**, r^i, which tracks a factor, r^j, rather than hedging it out.

Tracking portfolios

Regress

$$\tilde{r}_t^i = \alpha + \beta \tilde{r}_t^j + \epsilon_t$$

- ϵ is known as the **tracking error** of $\tilde{r}^i$ relative to $\tilde{r}^j$.
- R-squared measures how well j tracks i.
- The Information Ratio, α/σ_ϵ, measures the tradeoff between obtaining extra mean return α at the cost of taking on tracking error ϵ from the target portfolio.

Of course, this is just another way of looking at the hedging problem.

Tracking funds and hedged funds

For broad market factors, mutual funds are often tracking some factor while hedge funds are trying to hedge it out.

$$\tilde{r}^i_t = \underbrace{\beta \tilde{r}^j_t}_{\text{mutual fund position}} + \underbrace{\alpha + \epsilon_t}_{\text{hedge fund position}}$$

for factors such as the overall market index, industry indexes, value/growth indexes, etc.

- This is not exactly true.
- Hedge funds retain some factor exposure, β, while mutual funds deviate from their benchmark.

Mutual funds benchmarks

Most mutual funds explicitly state that they track some type of benchmark. r^j may equal...

- Market index
- Value index
- Growth index
- Small stock index
- Large stock index
- Foreign equities
- AAA Corporate bonds

Along with many others...