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MAS 3105
Homework 7

Problem 1
1. F 2. F 3. T. 4. T 5. T

Problem 2

$A = \begin{bmatrix} 1 & 1 & 3 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 0 & c \end{bmatrix} = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_x} \end{bmatrix}$

$\vec{v_1}, \vec{v_2}$ are linearly independent, and $\vec{v_3} = \vec{v_1} + 2\vec{v_2}$. So for $\text{im}(A) = 2$, $\vec{v_x}$ must be a linear combination of $\vec{v_1}, \vec{v_2}$. Let $\vec{v_x} = \vec{v_1} + \vec{v_2}$, then $a = 2, b = 1, c = 0$

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $\ker(A) = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 - \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_4 \,|\, x_3, x_4 \in \mathbb{R} \right\}$ and $\dim(\ker(A) = 2$

Problem 3

$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are linearly independent so their span is $\mathbb{R}^2$. In other words, they form a basis of $\mathbb{R}^2$

$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent so they also form a basis of $\mathbb{R}^2$. $\vec{v_3} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \in \mathbb{R}^2$

Problem 4

$\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so $\ker(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 - \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_5 \,|\, x_2, x_5 \in \mathbb{R} \right\}$ (the span of these 2 vectors form the basis of $\ker(A)$) and $\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$ (this is also the basis of $\text{im}(A)$

Problem 5

1. Because $\vec{v_1}$ and $\vec{v_2}$ are linearly independent, they form a basis of $\mathbb{R}^2$. The transformation matrix with respect to that basis is $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

2. $\vec{v_B} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

3. The 90^o counter-clockwise rotation matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$T_B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}$

Problem 6

$$
\begin{aligned}
B(S^{-1}\cdot\vec{x}) &= S^{-1}AS((S^{-1}\cdot\vec{x})) \\
&= S^{-1}A(SS^{-1})\vec{x} \\
&= S^{-1}A\vec{x} \\
&= S^{-1}(A\vec{x}) \\
&= S^{-1}\vec{x}
\end{aligned}
$$

Hence $S^{-1}\vec{x}$ is a fixed point of B

From Homework 5, we have shown that a square matrix A has non-zero fixed points iff $\det(A - I_n) = 0$

$\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}, \det\left(\begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}\right) = 0$

$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}, \det\left(\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}\right) \neq 0$

Hence the matrices are not similar