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MAS 3105
Homework 13

Problem 1
1. T 2. T 3. F 4. T 5. T 6. T 7. T 8. F 9. T 10. F

Problem 2

$$\begin{aligned}
P_A(\lambda) &= \det(\lambda I_4 - A)\\
&= \det\left(\begin{bmatrix} \lambda-1 & -2 & -3 & -4\\ -5 & \lambda-6 & -7 & -8\\ 0 & 0 & \lambda-9 & -10\\ 0 & 0 & 0 & \lambda-11\end{bmatrix}\right)\\
&= (\lambda-1)\det\left(\begin{bmatrix} \lambda-6 & -7 & -8\\ 0 & \lambda-9 & -10\\ 0 & 0 & \lambda-11\end{bmatrix}\right) + 2\det\left(\begin{bmatrix} -5 & -7 & -8\\ 0 & \lambda-9 & -10\\ 0 & 0 & \lambda-11\end{bmatrix}\right)\\
&- 3\det\left(\begin{bmatrix} -5 & \lambda-6 & -8\\ 0 & 0 & -10\\ 0 & 0 & \lambda-11\end{bmatrix}\right) + 4\det\left(\begin{bmatrix} 5 & \lambda-6 & -7\\ 0 & 0 & \lambda-9\\ 0 & 0 & 0\end{bmatrix}\right)\\
&= (\lambda-1)(\lambda-6)(\lambda-9)(\lambda-11) - 10(\lambda-9)(\lambda-11)\\
&= (\lambda^2-7\lambda-4)(\lambda-9)(\lambda-11)
\end{aligned}$$

Problem 3

$$\begin{aligned}
\det(A-\lambda I_3) &= 0\\
\det\left(\begin{bmatrix} 1-\lambda & 2 & 3\\ 1 & 2-\lambda & 3\\ 1 & 2 & 3-\lambda\end{bmatrix}\right) &= 0\\
(1-\lambda)[(2-\lambda)(3-\lambda)-6] - 2(-\lambda) + 3\lambda &= 0\\
6\lambda^2 - \lambda^3 &= 0\\
\lambda = 0 \text{ or } \lambda &= 6
\end{aligned}$$

$\ker(A) = \left\{ \begin{bmatrix} -2\\ 1\\ 0\end{bmatrix} x_2 + \begin{bmatrix} -3\\ 0\\ 1\end{bmatrix} x_3 \,|\, x_2, x_3 \in \mathbb{R} \right\} \Rightarrow \begin{bmatrix} -2\\ 1\\ 0\end{bmatrix}$ and $\begin{bmatrix} -3\\ 0\\ 1\end{bmatrix}$ are the eigenvectors that correspond to $\lambda = 0$

$\ker(A-6I_3) = \ker\left(\begin{bmatrix} -5 & 2 & 3\\ 1 & -4 & 3\\ 1 & 2 & -3\end{bmatrix}\right) = \left\{ \begin{bmatrix} 1\\ 1\\ 1\end{bmatrix} x \,|\, x \in \mathbb{R} \right\} \Rightarrow \begin{bmatrix} 1\\ 1\\ 1\end{bmatrix}$ is the eigenvector that corresponds to $\lambda = 6$

Problem 4

$$\begin{aligned}
\det(A - \lambda I_2) &= 0 \\
\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) &= 0 \\
(a-\lambda)(d-\lambda) - bc &= 0 \\
(ad - bc) - (a+d)(\lambda) + \lambda^2 &= 0 \\
2 - 3\lambda + \lambda^2 &= 0 \\
\lambda = 1 \text{ or } \lambda &= 2
\end{aligned}$$

Problem 5

By properties of a diagonalizable matrix, we have the following: A is diagonalizable iff the eigenvectors of A form a basis of $\mathbb{R}^2$

We find the eigenvalues of A: $\det(A - \lambda I_2) = (1-\lambda)^2 - a = 0$

- $a < 0 \Rightarrow A$ doesn't have real eigenvalues
- $a = 0 \Rightarrow \lambda = 1$ with corresponding eigenspace $\ker\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. But $\dim\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq \mathbb{R}^2$
- $a > 0 \Rightarrow A$ has 2 distinct real eigenvalues, hence A has a diagonalization $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$