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Chapter 6 Exercises

Exercise 6.1

Observe that $-1, 1 \in \mathbb{Z}$ and $5(-1) + 7(1) = -5 + 7 = 2$ $\square$

Exercise 6.3

Existence: Fix $x \in \mathbb{R} \setminus \{0\}$. Let $y = \frac{8}{x}$. Then $xy = x \cdot \frac{8}{x} = 8$

Uniqueness: Let $x \in \mathbb{R} \setminus \{0\}$, $y \in \mathbb{R}$ be as above, and $z \in \mathbb{R} \ni xz = 8$. Then $xz = xy$, which leads to $xz \cdot \frac{1}{x} = xy \cdot \frac{1}{x}$, and so $z = y$

By the 2 parts, we have: $\forall x \in \mathbb{R} \setminus \{0\},\ \exists!\ y \in \mathbb{R} \ni xy = 8$ $\square$

Exercise 6.4

Let $c = \frac{a+b}{2}$. Because $a, b, \frac{1}{2} \in \mathbb{Q}$, $a + b \in \mathbb{Q}$ and $\frac{a+b}{2} \in \mathbb{Q}$ by Axiom of Closure. So $c \in \mathbb{Q}$

We have $c - a = \frac{a+b}{2} - \frac{2a}{2} = \frac{b-a}{2}$. Because $a < b$, $b - a \in \mathbb{R}^+$, which gives us $\frac{b-a}{2} \in \mathbb{R}^+$, which is equivalent to $c - a \in \mathbb{R}^+$. This tells us that $a < c$.

We also have $b - c = \frac{2b}{2} - \frac{a+b}{2} = \frac{b-a}{2}$. Similarly, $b - c \in \mathbb{R}^+$. This tells us that $c < b$.

Thus, we have shown that $a < c < b$ $\square$

Exercise 6.5

1. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$

Step 1. Let $n = 1$. We then need to show

$$\sum_{i=1}^{1} i^2 = \frac{(2 \cdot 1 + 1)(1+1)(1)}{6}$$

The left hand side simplifies to 1, and the right hand side simplifies to $\frac{(3)(2)(1)}{6} = 1$. So, we have $1 = 1$, and indeed,

$$\sum_{i=1}^{1} i^2 = \frac{(2 \cdot 1 + 1)(1+1)(1)}{6}$$

Step 2. We assume that when $n = k$,

$$\sum_{i=1}^{k} i^2 = \frac{(2k+1)(k+1)k}{6}$$

Step 3. We need to show

$$\sum_{i=1}^{k+1} i^2 = \frac{(2(k+1)+1)((k+1)+1)(k+1)}{6}$$

The left hand side can be expressed as

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\ &= \frac{(2k+1)(k+1)k}{6} + (k+1)^2 \\ &= \frac{(2k^2+k)(k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{(k+1)(2k^2+k+6(k+1))}{6} \\ &= \frac{(k+1)(2k^2+4k+3k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

The right hand side can be simplified to

$$\frac{(2(k+1)+1)((k+1)+1)(k+1)}{6} = \frac{(2k+3)(k+2)(k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

We observe that the final terms are equal. We then have

$$\sum_{i=1}^{k+1} i^2 = \frac{(2(k+1)+1)((k+1)+1)(k+1)}{6}$$

From our proof by induction, we have shown that

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}, \ \forall n \in \mathbb{N}$$

$\square$

2. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Step 1. Let $n = 1$. We then need to show

$$\sum_{i=1}^{1} i^3 = \frac{1^2(1+1)^2}{4}$$

The left hand side simplifies to 1, and the right hand side simplifies to $\frac{(1)(4)}{4} = 1$. So, we have $1 = 1$, and indeed,

$$\sum_{i=1}^{1} i^3 = \frac{1^2(1+1)^2}{4}$$

Step 2. We assume that when $n = k$,

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Step 3. We need to show

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

The left hand side can be expressed as

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2k^2}{4} + \frac{4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2+4(k+1))}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$

The right hand side can be simplified to

$$\frac{(k+1)^2((k+1)+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

We observe that the final terms are equal. We then have

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

From our proof by induction, we have shown that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4},\ \forall n \in \mathbb{N}$$

$\square$

3. $\displaystyle\sum_{i=1}^{n} r^{i-1} = \frac{r^n - 1}{r-1}$

Step 1. Let $n = 1$. We then need to show

$$\sum_{i=1}^{1} r^{i-1} = \frac{r^1 - 1}{r-1}$$

The left hand side simplifies to $r^0 = 1$, and the right hand side simplifies to $\frac{r-1}{r-1} = 1$. So, we have $1 = 1$, and indeed,

$$\sum_{i=1}^{1} r^{i-1} = \frac{r^1 - 1}{r-1}$$

Step 2. We assume that when $n = k$,

$$\sum_{i=1}^{k} r^{i-1} = \frac{r^k - 1}{r-1}$$

Step 3. We need to show

$$\sum_{i=1}^{k+1} r^{i-1} = \frac{r^{k+1} - 1}{r-1}$$

The left hand side can be expressed as

$$\begin{aligned}\sum_{i=1}^{k+1} r^{i-1} &= \sum_{i=1}^{k} r^{i-1} + r^{k+1-1} \\ &= \frac{r^k - 1}{r-1} + \frac{r^k \cdot r - r^k}{r-1} \\ &= \frac{r^{k+1}-1}{r-1}\end{aligned}$$

which is equivalent to the right hand side. We then have

$$\sum_{i=1}^{k+1} r^{i-1} = \frac{r^{k+1}-1}{r-1}$$

From our proof by induction, we have shown that

$$r \in \mathbb{R} \setminus \{1\},\ \sum_{i=1}^{n} r^{i-1} = \frac{r^n - 1}{r-1},\ \forall n \in \mathbb{N}$$

$\square$

Exercise 6.6
Step 1. Let $n = 1$. We then need to show

$$(2 \cdot 1 - 1) = 1^2$$

which is an obvious fact, as both sides are equal to 1.
Step 2. We assume that when $n = k$,

$$1 + 3 + 5 + ... + (2k - 1) = k^2$$

Step 3. We need to show

$$1 + 3 + 5 + ... + (2(k+1) - 1) = (k+1)^2$$

The left hand side can be rewritten as

$$\begin{aligned}1 + 3 + 5 + ... + (2k-1) + (2(k+1) - 1) &= k^2 + (2k + 2 - 1) \\ &= k^2 + 2k + 1\end{aligned}$$

The right hand side can be expressed as

$$(k+1)^2 = k^2 + k + k + 1 = k^2 + 2k + 1$$

We observe that the final terms are equal. We then have

$$1 + 3 + 5 + ... + (2(k+1) - 1) = (k+1)^2$$

From our proof by induction, we have shown that

$$1 + 3 + 5 + ... + (2n - 1) = n^2,\ \forall n \in \mathbb{N}$$

$\square$

Exercise 6.7
Step 1. Let $n = 9$. We then need to show

$$(9+1)^2 < 2^9$$

The left hand side simplifies to $10^2 = 100$, and the right hand side simplifies to 512. So, we have $100 < 512$, and indeed,

$$(9+1)^2 < 2^9$$

Step 2. We assume that when $n = k$,

$$(k+1)^2 < 2^k$$

Step 3. We need to show

$$((k+1)+1)^2 < 2^{k+1}$$

First, observe that $8 < k$ from our hypothesis. This implies $64 < k^2$. Combining this fact with $2 < 64$ gives us $2 < k^2$
This leads to

$$\begin{aligned} 2 &< k^2 \\ 2 + k^2 + 4k + 2 &< k^2 + k^2 + 4k + 2 \\ (k+2)^2 &< 2(k+1)^2 \\ ((k+1)+1)^2 &< 2(k+1)^2 \end{aligned}$$

Hence, we have

$$\begin{aligned} ((k+1)+1)^2 &< 2(k+1)^2 \\ ((k+1)+1)^2 &< 2 \cdot 2^k \\ ((k+1)+1)^2 &< 2^{k+1} \end{aligned}$$

which is what we need.
From our proof by induction, we have shown that

$$(n+1)^2 < 2^n, \ \forall n \in \mathbb{N} \ni n \geq 9$$

□

Exercise 6.13
Step 1. Let $n = 2$. We then need to show

$$\prod_{t=2}^{2} (1 - t^{-2}) = \frac{1}{2} + (2 \cdot 2)^{-1}$$

The left hand side simplifies to $1 - 2^{-2} = \frac{3}{4}$, and the right hand side simplifies to $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. So, we have $\frac{3}{4} = \frac{3}{4}$, and indeed,

$$\prod_{t=2}^{2} (1 - t^{-2}) = \frac{1}{2} + (2 \cdot 2)^{-1}$$

Step 2. We assume that when $n = k$,

$$\prod_{t=2}^{k} (1 - t^{-2}) = \frac{1}{2} + (2k)^{-1}$$

Step 3. We need to show

$$\prod_{t=2}^{k+1}(1-t^{-2})=\frac{1}{2}+(2(k+1))^{-1}$$

The left hand side can be expressed as

$$\begin{aligned}\prod_{t=2}^{k+1}(1-t^{-2})&=\prod_{t=2}^{k}(1-t^{-2})\cdot(1-(k+1)^{-2})\\&=\left(\frac{1}{2}+(2k)^{-1}\right)\left(1-\frac{1}{(k+1)^2}\right)\\&=\left(\frac{k}{2k}+\frac{1}{2k}\right)\left(\frac{k^2+2k+1}{(k+1)^2}-\frac{1}{(k+1)^2}\right)\\&=\frac{k+1}{2k}\cdot\frac{k^2+2k}{(k+1)^2}\\&=\frac{1\cdot(k+2)}{2\cdot(k+1)}\\&=\frac{k+2}{2(k+1)}\end{aligned}$$

The right hand side can be simplified to

$$\frac{1}{2}+(2(k+1))^{-1}=\frac{k+1}{2(k+1)}+\frac{1}{2(k+1)}=\frac{k+2}{2(k+1)}$$

We observe that the final terms are equal. We then have

$$\prod_{t=2}^{k+1}(1-t^{-2})=\frac{1}{2}+(2(k+1))^{-1}$$

From our proof by induction, we have shown that

$$\prod_{t=2}^{n}(1-t^{-2})=\frac{1}{2}+(2n)^{-1},\ \forall n\in\mathbb{N}\ni n\geq 2$$

$\square$

Exercise 6.15
Step 1. Let $n=1$. We then need to show

$$\prod_{i=1}^{1}(4i-2)=\frac{(2\cdot 1)!}{1!}$$

The left hand side simplifies to $\frac{4\cdot 1}{2}=2$, and the right hand side simplifies to $\frac{2!}{1!}=2$. So, we have $2=2$, and indeed,

$$\prod_{i=1}^{1}(4i-2)=\frac{(2\cdot 1)!}{1!}$$

Step 2. We assume that when $n = k$,

$$\prod_{i=1}^{k} (4i - 2) = \frac{(2k)!}{k!}$$

Step 3. We need to show

$$\prod_{i=1}^{k+1} (4i - 2) = \frac{(2(k+1))!}{(k+1)!}$$

The left hand side can be expressed as

$$\begin{aligned} \prod_{i=1}^{k+1} (4i - 2) &= \prod_{i=1}^{k} (4i - 2) \cdot (4(k+1) - 2) \\ &= \frac{(2k)!}{k!} \cdot (4k + 2) \end{aligned}$$

The right hand side can be rewritten as

$$\frac{(2(k+1))!}{(k+1)!} = \frac{(2k)!(2k+1)(2k+2)}{k!(k+1)} = \frac{(2k)!}{k!} \cdot \frac{(2k+1) \cdot 2(k+1)}{k+1} = \frac{(2k)!}{k!} \cdot 2(2k+1) = \frac{(2k)!}{k!} \cdot (4k+2)$$

We observe that the final terms are equal. We then have

$$\prod_{i=1}^{k+1} (4i - 2) = \frac{(2(k+1))!}{(k+1)!}$$

From our proof by induction, we have shown that

$$\prod_{i=1}^{n} (4i - 2) = \frac{(2n)!}{n!}, \ \forall n \in \mathbb{N}$$

$\square$