Name: Dat Truong U78042625

Chapter 7 Exercises

Exercise 7.1

Starting on the left side, we want to prove $x \in \mathbb{Z}$ is odd if $\exists j \in \mathbb{Z} \ni x = 2j - 1$
Let $k = j + 1$. Because $j, 1 \in \mathbb{Z}$, $j + 1 \in \mathbb{Z}$ by Closure, which allows for $k \in \mathbb{Z}$
Also, $x = 2j - 1 = 2(k + 1) - 1 = 2k + 1$. Hence, by definition, x is odd. $\square$
Now we want to prove $\exists j \in \mathbb{Z} \ni x = 2j - 1$ if $x \in \mathbb{Z}$ is odd.
By definition, $\exists k \in \mathbb{Z} \ni x = 2k + 1$. Let $l = k - 1$. Because $k, -1 \in \mathbb{Z}$, $k - 1 \in \mathbb{Z}$ by Closure, which leads to $l \in \mathbb{Z}$
Also, $x = 2k + 1 = 2(l - 1) + 1 = 2l - 1$. Hence, we have shown that $\exists j \in \mathbb{Z} \ni x = 2j - 1$ $\square$
Therefore, $x \in \mathbb{Z}$ is odd iff $\exists j \in \mathbb{Z} \ni x = 2j - 1$

Exercise 7.2

1. We assume that $\exists n \ni (\exists a, b \in \mathbb{Z} \ni n = 2a$ and $n = 2b + 1)$
Then,

$$\begin{aligned} 2a &= 2b + 1 \\ 2a - 2b &= 1 \\ a - b &= \frac{1}{2} \end{aligned}$$

Because $a, -b \in \mathbb{Z}$, $a - b \in \mathbb{Z}$ by Closure. However, $\frac{1}{2} \notin \mathbb{Z}$ $\Rightarrow\!\Leftarrow$
Hence, there exists no integers that can be both even and odd. $\square$
2. Prove by induction
Base Case: Let $n = 1$. We need to show that n is either even or odd.
Obeserve that $0 \in \mathbb{Z}$ and $1 = 2 \cdot 0 + 1$, so 1 is odd; and indeed, 1 is either even or odd.
Induction Hypothesis: We assume that when $n = k$, k is either even or odd.
Induction Step: We need to show that $k + 1$ is either even or odd.
Case 1: k is even. So, $\exists a \in \mathbb{Z} \ni k = 2a$, which means that $k + 1 = 2a + 1$. By definition, $k + 1$ is odd. $\square$
Case 2: k is odd. So, $\exists b \in \mathbb{Z} \ni k = 2b + 1$, which means that $k + 1 = 2b + 2 = 2(b + 1)$. Because $b, 1 \in \mathbb{Z}$, $b + 1 \in \mathbb{Z}$ by Closure. Thus, by definition, $k + 1$ is even. $\square$
Therefore, $k + 1$ is either even or odd.
From our proof of induction, we have shown that n is either even or odd. $\square$
3. Let $m \in \mathbb{Z}^-$. Then $-m \in \mathbb{Z}^+$. We have shown that $-m$ is either even or odd.
Case 1: $-m$ is even. So, $\exists a \in \mathbb{Z} \ni -m = 2a$. This means that $m = -2a = 2(-a)$. Because $a, -1 \in \mathbb{Z}$, $-a \in \mathbb{Z}$ by Closure. Thus, by definition, m is also even. $\square$
Case 2: $-m$ is odd. So, $\exists b \in \mathbb{Z} \ni -m = 2b+1$. This means that $m = -(2b+1) = -2b-1 = 2(-b-1)+1$. Because $b, -1 \in \mathbb{Z}$, $-b \in \mathbb{Z}$ and $-b - 1 \in \mathbb{Z}$ by Closure. Thus, by definition, m is also odd. $\square$
Therefore, m is either even or odd. $\square$
4. Obeserve that $0 \in \mathbb{Z}$ and $0 = 2 \cdot 0$, so 0 is even. Furthermore, we have also shown that every number in $\mathbb{Z}^+$ and $\mathbb{Z}^-$ are either even or odd.
Because $\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$, we conclude that every number in $\mathbb{Z}$ are either even or odd. $\square$

Exercise 7.3

Because n is prime, $n \in \mathbb{N}$. Also, $n \in \mathbb{Z}$
Our hypothesis $n|(a + b)$ gives us $\exists i \in \mathbb{Z} \ni a + b = n \cdot i$. Based on another hypothesis $n|a$, we get $\exists j \in \mathbb{Z} \ni a = n \cdot i$. Then, $b = (a + b) - a = n \cdot i - n \cdot j = n(i - j)$
Because $i, -j \in \mathbb{Z}$, $i - j \in \mathbb{Z}$ by Closure. Hence, $n|b$ $\square$

Exercise 7.7
We are given that x is even. By definition, $\exists i \in \mathbb{Z} \ni x = 2i$
So, $x^2 = (2i)^2 = 4i^2 = 2(2i^2)$
Because $i, 2 \in \mathbb{Z}$, $2i^2 \in \mathbb{Z}$ by Closure. Hence, by definition, x^2 is even. $\square$

Exercise 7.9
We assume that there exists a largest real number. Let it be r.
Because $r, 1 \in \mathbb{R}$, $r+1 \in \mathbb{R}$ by Closure. Because of our assumption that r is the largest real number,

$$\begin{aligned} r+1 &\leq r \\ r-(r+1) \in \mathbb{R}^+ &\text{ or } r-(r+1) = 0 \\ -1 \in \mathbb{R}^+ &\text{ or } -1 = 0 \quad \Rightarrow\Leftarrow \end{aligned}$$

Hence, there exists no largest real number. $\square$

Exercise 7.10
We assume that there exists a smallest integer. Let it be z.
Because $z, 1 \in \mathbb{Z}$, $z-1 \in \mathbb{Z}$ by Closure. Because of our assumption that z is the smallest integer,

$$\begin{aligned} z &\leq z-1 \\ z-1-z \in \mathbb{R}^+ &\text{ or } z-1-z = 0 \\ -1 \in \mathbb{R}^+ &\text{ or } -1 = 0 \quad \Rightarrow\Leftarrow \end{aligned}$$

Hence, there exists no smallest integer. $\square$

Exercise 7.12
1. $\exists x, y \in \mathbb{Z} \ni 2x + 4y = 1$
We obtain $x + 2y = \frac{1}{2}$. Because $x, y, 2 \in \mathbb{Z}$, $x + 2y \in \mathbb{Z}$ by Closure. However, $\frac{1}{2} \notin \mathbb{Z}$ $\Rightarrow\Leftarrow$
Hence the initial statement is false. $\square$
2. $\exists x, y \in \mathbb{Z} \ni 3x + 5y = 2$
Observe that $-1, 1 \in \mathbb{Z}$ and $3 \cdot -1 + 5 \cdot 1 = 2$
Hence the initial statement is true. $\square$
3. $\forall x \in \mathbb{R}, \exists! y \in \mathbb{R} \ni xy = 4$
Observe that when $x = 0$, $xy = 0, \forall y \in \mathbb{R}$ $\Rightarrow\Leftarrow$
Hence the initial statement is false. $\square$
4. $\forall x \in \mathbb{N}, x^2 - x + 11$ is prime
Observe that when $x = 11$, $x^2 - x + 11 = 11^2 - 11 + 11 = 121$, and $11|121$ $\Rightarrow\Leftarrow$
Hence the initial statement is false. $\square$
5. $\exists x \in \mathbb{N} \ni x^x = 9x$
Observe that when $x = 3$, $x^x = 3^3 = 27$ and $9x = 9 \cdot 3 = 27$.
Hence the initial statement is true. $\square$
6. 231 can be written as the sum of 4 odd numbers.
We assume that this statement is true, so $\exists i, j, k, l \in \mathbb{Z} \ni 231 = (2i+1) + (2j+1) + (2k+1) + (2l+1)$
The terms can be rewritten as: $(2i+1)+(2j+1)+(2k+1)+(2l+1) = 2i+2j+2k+2l+4 = 2(i+j+k+l+2)$
Because $i + j + k + l + 2 \in \mathbb{Z}$, 231 is even by definition. But $231 = 2 \cdot 115 + 1$, so 231 is odd. $\Rightarrow\Leftarrow$
Hence the initial statement is false. $\square$
7. 132 can be written as the sum of 4 odd numbers.
Observe that $132 = 29 + 31 + 35 + 37$, and $29 = 2 \cdot 14 + 1,\ 31 = 2 \cdot 15 + 1,\ 35 = 2 \cdot 17 + 1,\ 37 = 2 \cdot 18 + 1$
Hence the initial statement is true. $\square$

8. 73 can be written as the sum of 1 odd and 2 even numbers.
Observe that $73 = 3 + 34 + 36$, and $3 = 2 \cdot 1 + 1,\ 34 = 2 \cdot 17,\ 36 = 2 \cdot 18$
Hence the initial statement is true. $\square$
9. 73 can be written as the sum of 1 even and 2 odd numbers.
We assume that this statement is true, so $\exists i, j, k \in \mathbb{Z} \ni 73 = 2i + (2j + 1) + (2k + 1)$
The terms can be rewritten as: $2i + (2j + 1) + (2k + 1) = 2i + 2j + 2k + 2 = 2(i + j + k + 1)$
Because $i + j + k + 1 \in \mathbb{Z}$, 73 is even by definition. But $73 = 2 \cdot 36 + 1$, so 73 is odd. $\Rightarrow\!\Leftarrow$
Hence the initial statement is false. $\square$

Exercise 7.14
We will prove the given statement by induction.
Base Case: Let $n = 1$. We need to show that $4|5^1 - 1$. We have $5^1 - 1 = 5 - 1 = 4$. So, $4|4$, and indeed, $4|5^1 - 1$.
Induction Hypothesis: We assume that when $n = k$, $4|5^k - 1$. This means that $\exists a \in \mathbb{N} \ni 5^k - 1 = 4a$
Induction Step: We want to show that $4|5^{k+1} - 1$.

$$\begin{aligned} 5^{k+1} - 1 &= 5 \cdot 5^k - (5 - 4) \\ &= 5 \cdot 5^k - 5 + 4 \\ &= 5(5^k - 1) + 4 \\ &= 5(4a) + 4 \\ &= 4(5a) + 4 \\ &= 4(5a + 1) \end{aligned}$$

Because $a, 5, 1 \in \mathbb{Z}$, $5a + 1 \in \mathbb{Z}$ by Closure. Thus, $4|5^{k+1} - 1$, which is what we need.
From our proof of induction, we have shown that $4|5^n - 1, \forall n \in \mathbb{N}$ $\square$
Exercise 7.19
Starting on the left side, we want to prove $3x - 5$ is even if x is odd
By definition, $\exists a \in \mathbb{Z} \ni x = 2a + 1$. So, $3x - 5 = 3(2a + 1) - 5 = 6a + 3 - 5 = 2(3a - 1)$.
Because $a, 3, -1 \in \mathbb{Z}$, $3a - 1 \in \mathbb{Z}$ by Closure. Hence, $3x - 5$ is even by definition. $\square$
Now we want to prove x is odd if $3x - 5$ is even. We assume that x is even.
By definition, $\exists b \in \mathbb{Z} \ni x = 2b$. So, $3x - 5 = 3(2b) - 5 = 6b - 5 = 6b - 6 + 1 = 2(3b - 3) + 1$.
Because $b, 3, -3 \in \mathbb{Z}$, $3b - 3 \in \mathbb{Z}$ by Closure. Hence, $3x - 5$ is odd by definition. Taking the contrapositive, we have if $3x - 5$ is even, then x is odd. $\square$

Exercise 7.20
We assume that x is odd.
By definition, $\exists a \in \mathbb{Z} \ni x = 2a + 1$. So, $x^2 - 1 = (2a + 1)^2 - 1 = 4a^2 + 4a + 1 - 1 = 4(a^2 + a)$
By Closure, $a^2 + a \in \mathbb{Z}$, which yields us $4|4(a^2 + a)$, or $4|x^2 - 1$
Therefore, if x is odd, then 4 divides $x^2 - 1$
Taking the contrapositive lets us conclude that if 4 does not divide $x^2 - 1$, then x is even. $\square$

Exercise 7.29
Theorem 7.10. We assume that $\sqrt{n} \in \mathbb{Q}$. In other words, $\exists a, b \in \mathbb{Z} \ni \sqrt{n} = \frac{a}{b}$ and a, b share no common factors other than 1.
We get $n = \frac{a^2}{b^2}$, which gives us $a^2 = b^2 n$. Because $b^2 \in \mathbb{Z}$ by Closure, $n|a^2$. Because n is prime, $n|a$ by Euclid's Lemma. So, $\exists i \in \mathbb{Z} \ni a = i \cdot n$ by definition.
Then, $(i \cdot n)^2 = b^2 n$, which takes us to $b^2 = i^2 n$. Because $i^2 \in \mathbb{Z}$ by Closure, $n|b^2$. Because n is prime, $n|b$ by Euclid's Lemma.
So, $n|a$ and $n|b$, which means a, b share n as a common factor. Also, $n > 1$ because it is prime. $\Rightarrow\!\Leftarrow$
Hence, the initial assumption is false, and $\sqrt{n} \in \mathbb{Q}^C$ $\square$

Corollary 7.11.

- $-\sqrt{n}$ is irrational
  We assume that $-\sqrt{n} \in \mathbb{Q}$. In other words, $\exists c, d \in \mathbb{Z} \ni -\sqrt{n} = \frac{c}{d}$. Then $\sqrt{n} = -\frac{c}{d} = \frac{-c}{d}$
  $-c \in \mathbb{Z}$ by Closure. Hence, by definition, $\frac{-c}{d} \in \mathbb{Q}$. But $\sqrt{n} \in \mathbb{Q}^C$ by Theorem 7.10. $\Rightarrow\Leftarrow$
  So the initial assumption is false, and $-\sqrt{n} \in \mathbb{Q}^C$ $\square$

- $\frac{1}{\sqrt{n}}$ is irrational
  We assume that $\frac{1}{\sqrt{n}} \in \mathbb{Q}$. In other words, $\exists e, f \in \mathbb{Z} \ni \frac{1}{\sqrt{n}} = \frac{e}{f}$. Then $\sqrt{n} = 1 \div \frac{e}{f} = \frac{f}{e}$
  By definition, $\frac{f}{e} \in \mathbb{Q}$. But $\sqrt{n} \in \mathbb{Q}^C$ by Theorem 7.10. $\Rightarrow\Leftarrow$
  So the initial assumption is false, and $\frac{1}{\sqrt{n}} \in \mathbb{Q}^C$ $\square$

Exercise 7.34
$x, y \in \mathbb{Q}$ and $w, z \in \mathbb{Q}^C$
1. $x + y \in \mathbb{Q}$
Because $x, y \in \mathbb{Q}$, $\exists a, b, c, d \in \mathbb{Z} \ni x = \frac{a}{b}, y = \frac{c}{d}$
$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$
$ad + bc \in \mathbb{Z}$ and $bd \in \mathbb{Z}$ by Closure. Therefore, by definition, $\frac{ad+bc}{bd} \in \mathbb{Q}$, or $x + y \in \mathbb{Q}$ $\square$
2. $w + z \in \mathbb{Q}^C$
Observe that $\sqrt{2}, -\sqrt{2} \in \mathbb{Q}^C$, and $\sqrt{2} + -\sqrt{2} = 0 \in \mathbb{Q}$
So the initial assumption is false. $\square$
3. $x + z \in \mathbb{Q}^C$
We assume that $x + z = \alpha \in \mathbb{Q}$. Then, $\exists a, b, e, f \in \mathbb{Z} \ni x = \frac{a}{b}, \alpha = \frac{e}{f}$
$z = \alpha - x = \frac{e}{f} - \frac{a}{b} = \frac{be-af}{bf}$
$be - af \in \mathbb{Z}$ and $bf \in \mathbb{Z}$ by Closure. Therefore, by definition, $\frac{be-af}{bf} \in \mathbb{Q}$, or $z \in \mathbb{Q}$ $\Rightarrow\Leftarrow$
Our initial assumption is false. Therefore, $x + z \in \mathbb{Q}^C$ $\square$
4. $xy \in \mathbb{Q}$
Because $x, y \in \mathbb{Q}$, $\exists a, b, c, d \in \mathbb{Z} \ni x = \frac{a}{b}, y = \frac{c}{d}$
$xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
$ac \in \mathbb{Z}$ and $bd \in \mathbb{Z}$ by Closure. Therefore, by definition, $\frac{ac}{bd} \in \mathbb{Q}$, or $xy \in \mathbb{Q}$ $\square$
5. $wz \in \mathbb{Q}^C$
Observe that $\sqrt{2}, -\sqrt{2} \in \mathbb{Q}^C$, and $\sqrt{2} \cdot -\sqrt{2} = -2 \in \mathbb{Q}$
So the initial assumption is false. $\square$
6. $xz \in \mathbb{Q}^C$
Observe that $0 \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{Q}^C$, but $0 \cdot \sqrt{2} = 0 \in \mathbb{Q}$
So the initial assumption is false. $\square$