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LINEAR ALGEBRA
Homework 5

Problem 1

1. If A is a solution to the equation, A must be invertible - A^{-1} exists, because $A^2 = AA = 1$. Hence, by definition, $A^{-1} = A$

2. Two possible solutions are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

3.

$$\begin{aligned} B^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} = I_2 \end{aligned}$$

4. B is the result of swapping the 2 rows of I_2

5. Let $C = SBS^{-1}$

$$\begin{aligned} C^2 &= SBS^{-1}SBS^{-1} \\ &= SB(S^{-1}S)BS^{-1} \\ &= SBI_2BS^{-1} \\ &= S(BI_2)BS^{-1} \\ &= SBBS^{-1} \\ &= S(BB)S^{-1} \\ &= SI_2S^{-1} \\ &= SS^{-1} \\ &= I_2 \quad \square \end{aligned}$$

6. Because there are infinite invertible 2×2 matrices, there are also infinite matrices $C = SBS^{-1}$. Thus, the equation has infinite solutions

Problem 2

1. If A was a non-square matrix, i.e. A was a $m \times n$ matrix where $m \neq n$, the product $A \cdot \vec{x}$ would produce a $m \times 1$ vector, which has a different number of rows compared to $\vec{x}$, which makes $\vec{x}$ and $A \cdot \vec{x}$ uncomparable.
2. $A \cdot \vec{0} = \vec{0}$ for any A, so $\vec{0}$ is a fixed point for any A
3. $A \cdot 2\vec{x} = 2A \cdot \vec{x} = 2(A \cdot \vec{x}) = 2\vec{x}$. Hence $2\vec{x}$ is also a fixed point of A
4. First, we show that if $\vec{x} \in \ker(A - I_n)$ then $\vec{x}$ is a fixed point in A. If that is the case, $\vec{x}$ is a solution to the equation $(A - I_n) \cdot \vec{x} = \vec{0}$

$$\begin{aligned}(A - I_n) \cdot \vec{x} &= \vec{0} \\ A \cdot \vec{x} - I_n \cdot \vec{x} &= \vec{x} - \vec{x} \\ A \cdot \vec{x} &= \vec{x} + (-\vec{x} + I_n \cdot \vec{x}) \\ A \cdot \vec{x} &= \vec{x} + \vec{0} = \vec{x}\end{aligned}$$

Therefore $\vec{x}$ is a fixed point in A $\quad\square$

Now we prove the other direction. Suppose $\vec{x}$ is a fixed point in A, then

$$\begin{aligned}A \cdot \vec{x} &= \vec{x} + \vec{0} \\ A \cdot \vec{x} &= \vec{x} + (-\vec{x} + I_n \cdot \vec{x}) \\ A \cdot \vec{x} - I_n \cdot \vec{x} &= \vec{x} - \vec{x} \\ (A - I_n) \cdot \vec{x} &= \vec{0}\end{aligned}$$

This implies $\vec{x}$ is a solution to the equation $(A - I_n) \cdot \vec{x} = \vec{0}$, or $\vec{x} \in \ker(A - I_n)$ $\quad\square$

5. We have showed that $\vec{x}$ is a fixed point in A iff $\vec{x} \in \ker(A - I_n)$. Thus, if a matrix were to have exactly 2 fixed points, there would be exactly 2 elements in $\ker(A - I_n)$, which would imply the equation $(A - I_n) \cdot \vec{x} = \vec{0}$ has exactly 2 solutions $\Rightarrow\!\Leftarrow$

So a matrix cannot have 2 fixed points

6. $\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$

$\ker\left(\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix} x | x \in \mathbb{R}\right\}$. These are also the fixed points of B

7. $\det(C - I_n) = \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right) = -2 \neq 0$

Thus C does not have non-zero fixed points