Name: Dat Truong U78042625

MAS 3105
Homework 8

Problem 1
1. T 2. T 3. T. 4. T 5. F

Problem 2
1. Not a linear transformation. $T(2I_2) \neq 2T(I_2)$
2. Is a linear transformation. Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

- Scalar-multiplicative: $T(kA) = M(kA) = k(MA) = kT(A)$
- Additive: $T(A+B) = M(A+B) = MA + MB = T(A) + T(B)$

3. Is a linear transformation

- Scalar-multiplicative: $T(kf) = (kf)'' - 2x(kf)' + kf = kf'' - 2kxf' + kf = kT(f)$
- Additive: $T(f+g) = (f+g)'' - 2x(f+g)' + (f+g) = (f'' - 2xf' + f) + (g'' - 2xg' + g) = T(f) + T(g)$

4. Is a linear transformation

- Scalar-multiplicative: $T(kf) = kf(0) = kT(f)$
- Additive: $T(f+g) = (f+g)(0) = f(0) + g(0) = T(f) + T(g)$

Problem 3
1. Are linearly independent
2. Are not linearly independent. $f_3 = \pi(f_1 + f_2)$
3. Are not linearly independent. $f_1 = 2f_2$

Problem 4
1. By Theorem 1, there exists a unique $f_1 \in \mathfrak{G}$ so that $f_1'(0) = 1$ and $f_1(0) = 0$, and a unique $f_2 \in \mathfrak{G}$ so that $f_2'(0) = 0$ and $f_2(0) = 1$
2. We assume they're linearly dependent, which means $\exists k, l \in \mathbb{R}\{0\} \ni kf_1 + lf_2 = 0$. However, $kf_1(0) + lf_2(0) = l \neq 0$, which is a contradiction. Similarly, their derivatives, f_1', f_2', are not linearly dependent. Hence they are linearly independent. $\square$
3. Define $g(x) = f'(0) \cdot f_1(x) + f(0) \cdot f_2(x) = af_1(x) + bf_2(x)$, $f \in \mathfrak{G}$

$$\begin{aligned} g'' - 3g' + 2g &= (af_1'' + bf_2'') - 3(af_1' + bf_2') + 2(af_1 + bf_2) \\ &= a(f_1'' - 3f_1' + 2f_1) + b(f_2'' - 3f_2' + 2f_2) = 0 \end{aligned}$$

$g(0) = f(0) = a, g'(0) = f'(0) = b$, hence $g \in \mathfrak{G}$, and $\mathfrak{G} = \text{Span}(f_1, f_2)$ $\square$
5. $\lambda^2 - 3\lambda + 2 = 0$, so $\lambda = 1$ or $\lambda = 2$
$g_1 = e^x, g_2 = e^{2x}$, and $\nexists k \in \mathbb{R} \ni g_1 = kg_2$, so they are linearly independent. We have established in part 3 that $\dim(\mathfrak{G}) = 2$, so g_1 and g_2 form a basis of $\mathfrak{G}$
$g_1(x) = f_1(x) + f_2(x), g_2(x) = 2f_1(x) + f_2(x)$
6. From the explicit formulas in part 5, g_1 has coordinates $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and g_2 has coordinates $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
7. f_1 has coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and f_2 has coordinates $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$