Probability Theory — AI Bootcamp

Prob_theory_exercises

1.2

$P(A) = 0.1$, $P(B) = 0.2$

$P(A \cap B) = 0.05$

a) $P(A \cup B) = P(A) + P(B) - 0.05 = 0.25$

b) $P(A \cup B) - P(B) = 0.05$

c) $1 - P(A \cup B) = 0,75$

d) $P(A \cup B) - P(A \cap B) = 0.2$

1.4

~~let A be~~

A – at least 2 people share birthday

B – no two people share birthday

$P(A) = 1 - P(B)$

there are 365 possibilities for each person

$\Rightarrow$ 365^{23} overall possible assignments

if no two numbers are the same $\Rightarrow$

$\Rightarrow 365 \cdot (365-1) \cdot (365-2) \cdot \ldots \cdot (365-n+1)$

$n = 23$, $365 \cdot 364 \cdot \ldots \cdot 343$

$$P(B) = \frac{365 \cdot \ldots \cdot 343}{365^{23}} = 0.4927$$

$$P(A) = 1 - 0,4927 = 0.5073$$

2.3

$P(A) = 0.1 \quad P(B) = 0.05$

$P(A \cup B) = 0.8$

for independent:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A) \cdot P(B) = 0.005$$

Now,

$$P(A) + P(B) = P(A \cup D) + P(A \cap B)$$

$$0.15 \neq 0.8 + 0.005$$

$$\Downarrow$$

events are dependent

2.4

All probabilites are $\frac{1}{3}$ since the chance is 1 out of 3 for all of them.

$$a, b, c : \frac{1}{3}$$

2.6

a) A – 2 boys and 2 girls

B – first is a boy

$$P(A|B) \stackrel{*}{=} \frac{3/8 \cdot 1/2}{1/2} = 3/8$$

* $P(A) = 6/2^4 = \frac{6}{16} = \frac{3}{8}$

$P(B) = \frac{1}{2}$

A and B are independent $\Rightarrow$

$\Rightarrow P(A|B) = P(A) = 0.375$

b) A – 2 boys and 2 girls

B – at least one is a boy

B^c – none are boy

$P(B^c) = \frac{1}{16}$ $1 - P(B^c) = P(B) = \frac{15}{16}$

$P(A \cap B) = P(A) = \frac{3}{8}$

$$P(A|B) = \frac{3/8}{15/16} = \frac{6}{15} = 0.4$$

3.1

0.9 of ~~1~~ defective is classified

0.85 of correct is classified

10% are defective

A – unit is defective

B – it has been classified as defective

$P(A) = 0.1$

$P(B) = 0.1 \cdot 0.9 + 0.9 \cdot (1 - 0.85) =$

$= 0.09 + 0.9 \cdot 0.15 = 0.225$

$P(B|A) = 0.9$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{0.9 \cdot 0.1}{0.225} =$$

$$= \underline{\underline{0.4}}$$

3.3

0.99 of suffering get correct

0.95 of healthy get correct

6% get diagnosed

a)

$0.06 = x \cdot 0.99 + (1-x) \cdot (1 - 0.95)$

$0.06 = 0.99x + 0.05 - 0.05x$

$0.94x = 0.01$

$$x = \frac{0.01}{0.94} = \frac{1}{94}$$

b)

A – suffers from S

B – gets dignosed with S

$P(A) = \frac{1}{94}$ (as we found in (a))

$P(B) = 0.06$

$P(B|A) = 0.99$

$$P(A|B) = \frac{0.99 \cdot \frac{1}{94}}{0.06} = \frac{33}{188} \approx 0.1755$$

4.4 1% of drivers are intoxicated

a) prob. out of 50 at least one is intoxicated.

~~All at least one~~

This is Binomial distribution

$$1 - \binom{50}{0} \cdot 0.01^{0} \cdot (0.99)^{50} = 1 - (0.99)^{50} = $$
$$= 0.395$$

b) $E(x) = \frac{1}{p} = \frac{1}{0.01} = 100$

$Var(x) = \frac{1-p}{p} = \frac{1-0.01}{0.01} = 99$

$sd(x) = \sqrt{\frac{1-p}{p^2}} = 99.5$

5.1

a) $f(x) = \begin{cases} cx^2, & 0 < x < 6 \\ 0 & \text{othewise} \end{cases}$

$$\int_0^6 cx^2\,dx = 1$$

$$\int_0^6 cx^2\,dx = c \cdot \frac{x^3}{3}\Big|_0^6 = 72c$$

$$72c = 1$$

$$c = \frac{1}{72}$$

b)

$$F_x(x) = \int_0^x f(x) = \int_0^x \frac{x^2}{72}\,dx = \frac{x^3}{216}$$

$$F_x(x) = \begin{cases} \frac{x^3}{216}, & 0 < x < 6 \\ 0, & \text{otherwise} \end{cases}$$

c) $E(x) = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx =$

$$= \int_0^6 x \cdot \frac{x^2}{72}\,dx = \frac{x^4}{288}\Big|_0^6 = 4.5$$

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_0^6 x^2 \cdot \frac{x^2}{72}\,dx = \frac{x^5}{360}\Big|_0^6 = \frac{7776}{360} = 21.6$$

$$V(x) = 21.6 - 4.5^2 = 21.6 - 20.25 = 1.35$$

6.2

$X \in U(0,4) \qquad Y \in U(0,6)$

Unif Distribution $E(x) = \frac{1}{b-a}$

$$E(X) = \frac{1}{4-0} = \frac{1}{4}$$

$$E(X) = \frac{1}{6-0} = \frac{1}{6}$$

Prob of A waiting for B is

$$\frac{\frac{1}{6}}{\frac{1}{4}} = \frac{2}{3}$$

6.3

Exponential distribution

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = E(Y) = 2$$

$$E(\max(X,Y)) = \frac{1}{\lambda_x} + \frac{1}{\lambda_y} - \frac{1}{\lambda_x + \lambda_y}$$

$$E(\min(X,Y)) = \frac{1}{\lambda_x + \lambda_y}$$

$$E(\max(X,Y) - \min(X,Y)) = \frac{1}{\lambda_x} + \frac{1}{\lambda_y} - \frac{1}{\lambda_x + \lambda_y} -$$

$$- \frac{1}{\lambda_x - \lambda_y}$$

~~[illegible]~~

Since, $\frac{1}{\lambda_x} = \frac{1}{\lambda_y} = 2 \; | \Rightarrow$

$$\Rightarrow \lambda_x = \lambda_y = \frac{1}{2}$$

$$\frac{1}{\lambda_x + \lambda_y} = \frac{1}{\frac{1}{2} + \frac{1}{2}} = 1$$

$$E(\max(X, X) - \min(X, Y)) = E(U - V) =$$

$$= 2 + 2 - 1 - 1 = \underline{\underline{2}}$$

7.6 | <u>Geometric Distribution</u>

$$E(x) = \frac{1}{p}$$

for six toys $E(x) = \frac{6}{6}$

for five toy $E(x) = \frac{6}{5}$

and so on

getting every toy once will be

$$E(x) = \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} =$$

$$= 1 + 6 + \frac{462}{60} = \underline{\underline{14.7}}$$

8.4

Normal Distribution

max. 10 people or 800 kg

random person's weight is $N(70, 10)$ kg

Prob of 10 customers overloading?

10 people can have mean of up to $800/10 = 80$ kg

$P(\bar{x} > 80)$

$\mu = 70 \qquad \sigma_0 = 10$

$$\sigma = \frac{\sigma_0}{\sqrt{n}} = \frac{10}{\sqrt{10}} \approx 3.162$$

$$P(\bar{x} > 80) = P\left(\frac{x - \mu}{\sigma} > \frac{80-70}{\sqrt{10}}\right)$$

$$= P(Z > 3.162) = 0.000783$$