Differential Geometry

Spring term 2022

Exercise 1

1. A circle of radius 1 rolls on the x-axis. One point on the circle is marked, and at time $t = 0$ this point coincides with the coordinate origin $(0, 0)$ (thus the circle touches the x-axis at the origin at time $t = 0$). The trajectory of the marked point is called a *cycloid*.
   a) Find a parametrization of the cycloid corresponding to the rolling with a constant speed. Which points are regular and which are singular?
   b) Compute the curvature of the cycloid.

2. Let $f\colon I \to \mathbb{R}$ be a C^∞-function. The graph of f is the set
$$\Gamma(f) = \{(t, f(t)) \mid t \in I\} \subset \mathbb{R}^2.$$
   a) Find a regular parametrization of $\Gamma(f)$, that is a regular smooth curve $\gamma\colon J \to \mathbb{R}^2$ with $\gamma(J) = \Gamma(f)$ and γ injective.
   b) Let I be an open interval. Show directly that $\Gamma(f)$ is a 1-dimensional submanifold of $\mathbb{R}^2$.

3. For $a > 0$ consider the curve
$$\gamma\colon [-a, a] \to \mathbb{R}^2, \quad \gamma(t) = (t, \cosh t)$$
(the *catenary* or the chainette).
   a) Compute the length of γ and the unit-speed parametrization of γ.
   b) Compute the curvature of γ.

4. * Find a C^∞-parametrization of the set
$$L = \{(x, 0) \mid x \geq 0\} \cup \{(0, y) \mid y \geq 0\}$$
and show that this set has no regular parametrization.

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*Exercises marked with $*$ are bonus exercises, allowing you to receive additional points.

*Some of the results obtained here may be used in later exercises. It might be a good idea to keep the solutions for future reference.