# Differential Geometry

Spring term 2022

## Exercise 3

1. Compute the evolute of the cycloid $\gamma(t) = (t - \sin t, 1 - \cos t)$. Show that the evolute is a congruent copy of the cycloid.

2. Show that the involute $\mathcal{I}_{\gamma,0}$ of the catenary $\gamma(t) = (t, \cosh t)$ is the tractrix.

3. Let $\gamma \colon [a, b] \to \mathbb{R}^2$ be a regular curve such that $\kappa(t) \neq 0$ and $\dot\kappa(t) \neq 0$ for all $t \in [a, b]$.
   a) Show that for any $a \leq t_1 < t_2 \leq b$ the length of the curve $\gamma|_{[t_1,t_2]}$ is equal to $\left|\frac{1}{\kappa(t_1)} - \frac{1}{\kappa(t_2)}\right|$.
   b) Prove the Tait-Kneser theorem: the osculating circles of $\gamma$ at $t_1$ and $t_2$ are nested.

4. Let $\gamma \colon I \to \mathbb{R}^2$ be a regular curve. A circle $\{p \in \mathbb{R}^2 \mid \|p - c\| = r\}$ is said to have *contact of order at least two* with $\gamma$ at $t_0 \in I$ if the function
$$f(t) := \|\gamma(t) - c\|^2 - r^2$$
   and its first and second derivative vanish at $t_0$.
   a) Show that if $\kappa(t_0) \neq 0$, then the osculating circle to $\gamma$ at $t_0$ has contact with $\gamma$ of order at least two at $t_0$.
   b) By studying the third derivative of $f$ show that if $\dot\kappa(t_0) \neq 0$, then the osculating circle at $t_0$ locally separates $\gamma$ around $\gamma(t_0)$.

5. * Let $\gamma \colon I \to \mathbb{R}^2$ be a regular space curve. The contact of order at least three with a sphere is defined similarly to the previous problem: the function $f(t) = \|\gamma(t) - c\|^2 - r^2$ must vanish at $t_0$ together with its derivatives up to order three. Find the center and the radius of the sphere which has contact with $\gamma$ of order at least three (the *osculating sphere*).

6. * Show that the osculating spheres of a space curve at sufficiently close points intersect each other.