Planar Graphs

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Günther Raidl

Algorithms and Complexity Group
Institute of Computer Graphics and Algorithms
TU Wien

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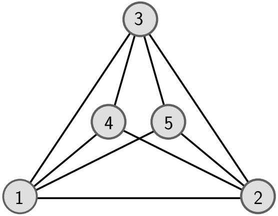
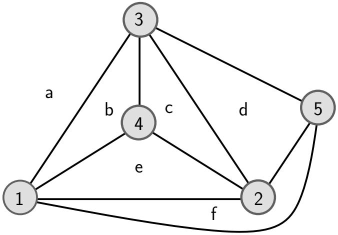
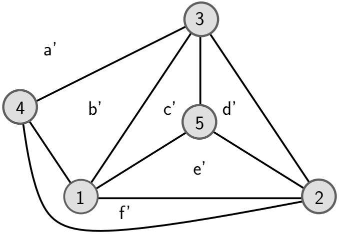
Planar Graphs

- A graph $G = (V, E)$ is called planar, if it can be drawn in the plane in a way such that no two edges intersect geometrically except at a vertex to which both are incident.
- Such a drawing is called planar embedding of G (or plane graph).
- A plane graph G divides the plane into connected regions called faces.
- The unbounded region is called the outer/external face of G.

Example: Graph G and Two Planar Embeddings of G







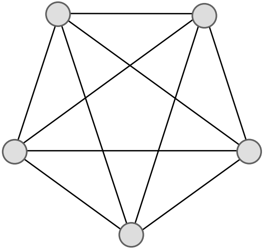


Figure: K_5

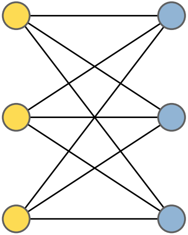


Figure: $K_{3,3}$

These graphs are often called Kuratowski graphs,
they cannot be drawn in the plane without crossing edges.

Theorem (Euler's formula)

Let $G = (V, E)$ be a connected, planar graph, and let f denote the number of faces. Then $|V| - |E| + f = 2$ holds for any planar embedding of G.

Proof e.g., by induction over the number of edges

Corollary

If G is a simple, planar graph with $|V| \geq 3$, then the number of edges is $|E| \leq 3|V| - 6$.

Corollary

For any simple, planar, bipartite graph with $|V| \geq 3$ vertices and $|E|$ edges, $|E| \leq 2|V| - 4$ holds.

Corollary

Each simple, planar graph contains a vertex v of degree $d(v) \leq 5$.

Definition (Subdivision)

Subdividing an edge $(u, v) \in E$ of a graph $G = (V, E)$ is the operation of deleting (u, v) and adding a path $P = (u, w_1, \ldots, w_k, v)$, $w_i \notin V$, $1 \leq i \leq k$ to G.

A graph G is called a subdivision of a graph G' if it is obtained by subdividing some of the edges of G'.

Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Definition (Graph Minor)

A graph H is a minor of G if H is obtained from G by deleting and contracting edges and/or deleting vertices.

Theorem (Wagner)

A graph is planar if and only if it has no minor isomorphic to K_5 *or* $K_{3,3}$.

Example: Peterson Graph

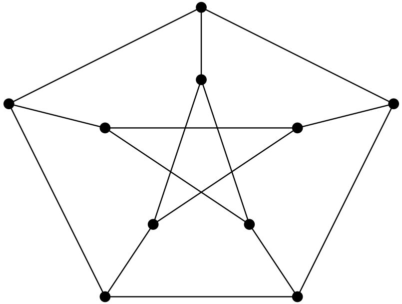


Figure: Peterson Graph

A graph which does not have K_5 or $K_{3,3}$ as a subgraph
but contains a subdivision of $K_{3,3}$ and has K_5 and $K_{3,3}$ as minors.

Non-planarity of the Peterson Graph

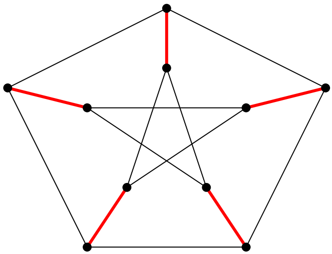


Figure: Contracting the red edges, we obtain the K_5

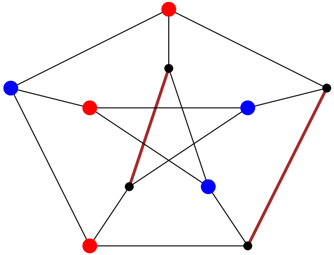


Figure: After deleting the brown edges, we obtain a subdivision of the $K_{3,3}$

Linear Time Planarity (Testing) Algorithms

Path-Addition Approach

- first linear-time planarity testing algorithm due to Hopcroft and Tarjan (1974)
- starts from a cycle and adds to it one path at a time
- Mehlhorn and Mutzel (1996) clarified how to construct the embedding

Vertex-Addition Approach

- first considered by Lempel, et al. (1967)
- adds vertices one-by-one following an st-numbering
- Booth and Lueker (1976) showed that it can be implemented in linear time
- Chiba et al. (1985) showed how to construct the embedding

Linear Time Planarity (Testing) Algorithms

Block Embedding Approach

- Shih and Hsu (1999), Boyer and Myrvold (1999)
- first construct a DFS-tree and then add back edges to iteratively construct a planar embedding (if it exists)
- Boyer and Myrvold (2004) proposed a new variant with a simpler implementation

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