MATH 262 - Homework 4.1

11. Find a basis for the linear space V below and determine its dimension.

    The space of all 2×2 matrices A that commute with

    $$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

    Note: Matrices A that commute with B means $AB = BA$.

    $$AB = \begin{bmatrix} a_{11} & (a_{11} + a_{12}) \\ a_{21} & (a_{21} + a_{22}) \end{bmatrix} \qquad BA = \begin{bmatrix} (a_{11} + a_{21}) & (a_{12} + a_{22}) \\ a_{21} & a_{22} \end{bmatrix}$$

    Let $AB = BA$:

    $$\begin{aligned} a_{11} &= a_{11} + a_{21} \\ 0 &= a_{21} \end{aligned} \qquad \begin{aligned} a_{11} + a_{12} &= a_{12} + a_{22} \\ a_{11} &= a_{22} \end{aligned} \qquad \begin{aligned} a_{21} + a_{22} &= a_{22} \\ 0 + a_{22} &= a_{22} \\ a_{22} &= a_{22} \end{aligned}$$

    Thus, A is of the form

    $$A = \begin{bmatrix} r & s \\ 0 & r \end{bmatrix}$$

    This can be expanded to

    $$\begin{aligned} A &= r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ A &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \end{aligned}$$

    Finally,

    $$\begin{aligned} \text{basis of V} &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \\ \dim(V) &= 2 \end{aligned}$$