MATH 262 - Homework 7.2

5. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and a, b, and c are nonzero constants For which values of a, b, and c does A have two distinct eigenvalues?

   First, calculate the characteristic polynomial and set it equal to 0.

$$\begin{aligned} 0 &= \det(A - \lambda I_2) \\ &= \det\left(\begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}\right) \\ &= (a-\lambda)(c-\lambda) - b^2 \\ &= \lambda^2 - (a+c)\lambda + ac - b^2 \end{aligned}$$

   Notice that the result is a quadratic equation. Thus, the quadratic formula can be used. For the quadratic formula to yield two real and distinct solutions (i.e. two distinct eigenvalues), the discriminant must be positive.

$$\begin{aligned} (a+c)^2 - 4(ac - b^2) &> 0 \\ a^2 + 2ac + c^2 - 4ac + 4b^2 &> 0 \\ a^2 - 2ac + c^2 + 4b^2 &> 0 \\ (c-a)^2 + 4b^2 &> 0 \end{aligned}$$

   Notice this inequality holds true for $b \neq 0$, $a, c \in \mathbb{R}$ and for $b = 0$, $c \neq a$. Since all constants are nonzero, the latter can be ignored. Thus, any real values for a, b, and c will yield two distinct eigenvalues.