MATH 262 - Homework 4.2b

2. Find the kernel and image of the linear transformation and state whether it is an isomorphism. Remember a linear transformation is an isomorphism if and only if the kernel is trivial (equals the zero element only) and the image is everything.

$$T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ from } \mathbb{R}^{2\times 2} \text{ to } \mathbb{R}^{2\times 2}$$

Let A $= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The kernel of $T(M)$ is defined as

$$\begin{aligned} ker(T) &= \left\{ M \in \mathbb{R}^{2\times 2} \;\middle|\; T(M) = 0 \right\} \\ &= \left\{ M \in \mathbb{R}^{2\times 2} \;\middle|\; M \cdot A = 0 \right\} \end{aligned}$$

$$\begin{aligned} M \cdot A &= 0 \\ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} &= 0 \\ \begin{bmatrix} m_{11} + 3m_{12} & 2m_{11} + 6m_{12} \\ m_{21} + 3m_{22} & 2m_{21} + 6m_{22} \end{bmatrix} &= 0 \end{aligned}$$

$$\begin{aligned} m_{11} + 3m_{12} = 0 &\longrightarrow m_{11} = -3m_{12} \\ m_{21} + 3m_{22} = 0 &\longrightarrow m_{21} = -3m_{22} \end{aligned}$$

$$\begin{aligned} \ker(T) &= \left\{ \begin{bmatrix} -3m_{12} & m_{12} \\ -3m_{22} & m_{22} \end{bmatrix} \;\middle|\; m_{12}, m_{22} \in \mathbb{R} \right\} \\ &= \left\{ m_{12} \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + m_{22} \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix} \;\middle|\; m_{12}, m_{22} \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix} \right\} \end{aligned}$$

Because nullity$(T) \neq 0$, $T(M)$ is not an isomorphism. Next, find the image anyway for the sake of exercise.

$$\begin{aligned}
\operatorname{im}(T) &= \left\{ M \cdot A \;\middle|\; M \in \mathbb{R}^{2\times 2} \right\} \\
&= \left\{ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} m_{11} + 3m_{12} & 2m_{11} + 6m_{12} \\ m_{21} + 3m_{22} & 2m_{21} + 6m_{22} \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} m_{11} + 3m_{12} & 2(m_{11} + 3m_{12}) \\ m_{21} + 3m_{22} & 2(m_{21} + 3m_{22}) \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{R} \right\} \\
&= \left\{ (m_{11} + 3m_{12}) \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + (m_{21} + 3m_{22}) \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\}
\end{aligned}$$