MATH 262 - Homework 4.1

5. Determine whether the following subset of P_2 is a subspace of P_2.

$$V = \left\{ p(t) : \int_0^1 p(t)dt = 0 \right\}$$

Claim. V is a linear subspace of P_2.

Proof. Take any $f(t)$ and $g(t)$ such that $f(t), g(t) \in V$. Therefore, the following is true

$$\int_0^1 f(t)dt = 0 \qquad \int_0^1 g(t)dt = 0$$

For V to be a linear subspace of P_2, it must be shown that $(f+kg)(t) \in V$. Let $k \in \mathbb{R}$ and consider $(f+kg)(t) = f(t) + kg(t)$, which is still a polynomial in P_2. Now consider

$$\begin{aligned} \int_0^1 (f+kg)(t)dt &= \int_0^1 f(t)dt + k\int_0^1 g(t)dt \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $(f+kg)(t) \in V$.

It must also be shown that the neutral element is in P_2. This is trivially true, as the definite integral of 0 is always 0:

$$\int_0^1 0dt = 0$$

It has been shown that V is closed under linear combinations and contains the neutral element in P_2. Therefore, the definition of a linear subspace is satisfied and the claim is true. $\square$

Note: P_n is the set consisting of the zero polynomial combined with the set of all polynomials of degree less than or equal to n.

Remember, to show that a subset IS a subspace you must show that for every f,g in the set and for every $k \in \mathbb{R}$ we have that $f + kg$ is also in the subset. To show that a subset is NOT a subspace you must either show that $\vec{0}$ is not in the subset or give specific f and g in the set and $k \in \mathbb{R}$ such that $f + kg$ is NOT in the subset.