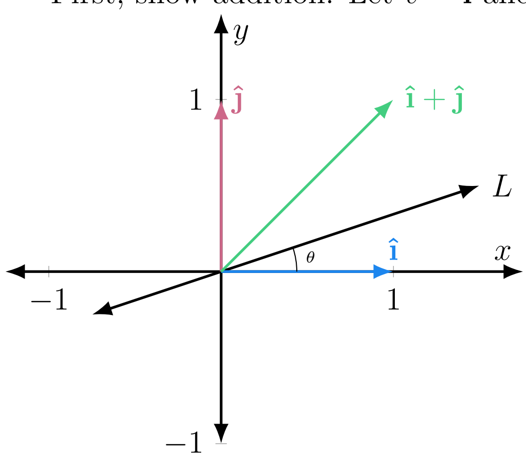
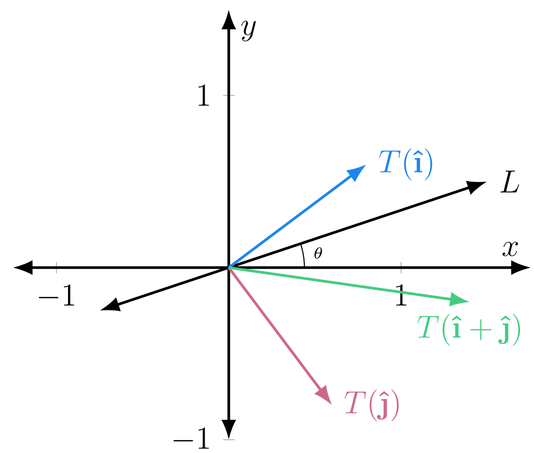
To illustrate that T is linear, let's show that it respects addition and scalar multiplication i.e. for two vectors $\vec{v}$ and $\vec{w}$ and a scalar k, the following holds true:

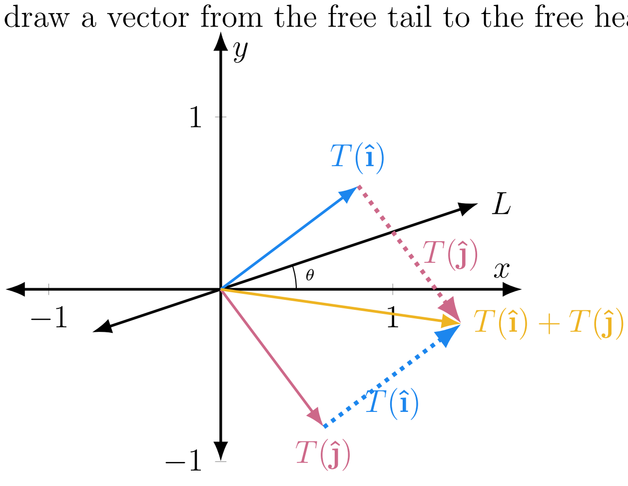
$$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$$
$$T(k\vec{v}) = kT(\vec{v})$$

First, show addition. Let $\vec{v} = \hat{\mathbf{\imath}}$ and $\vec{w} = \hat{\mathbf{\jmath}}$.



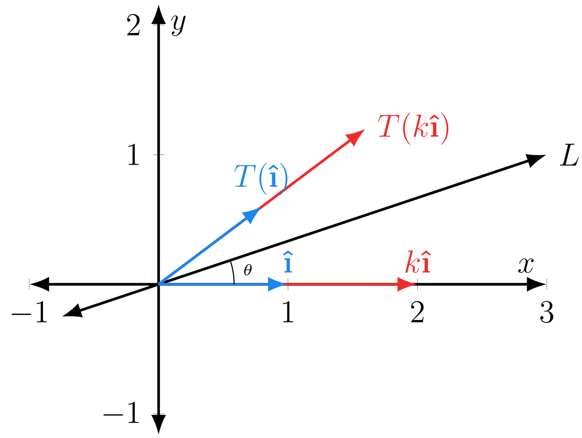


To plot the vector sum $T(\hat{\mathbf{\imath}}) + T(\hat{\mathbf{\jmath}})$, place the vectors head to tail and then draw a vector from the free tail to the free head.

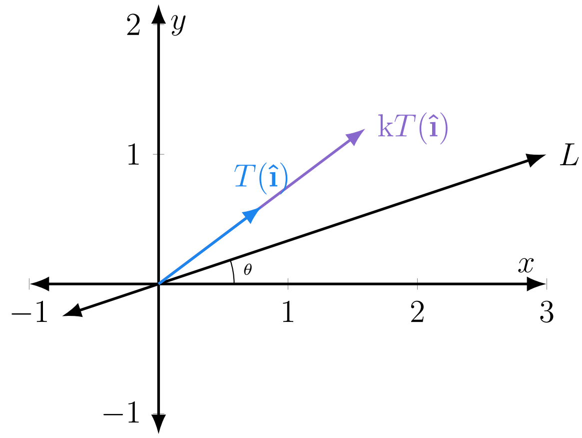


Compare the plots and observe that $T(\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}) = T(\hat{\mathbf{\imath}})+T(\hat{\mathbf{\jmath}})$. Thus, T respects addition.

Next show scalar multiplication. Let $k = 2$.



Let's simply multiply $T(\hat{\mathbf{i}})$ by k and plot that:



Compare the plots and observe that $T(k\hat{\mathbf{i}}) = kT(\hat{\mathbf{i}})$. Thus, T respects scalar multiplication.

The matrix T in terms of θ is

$$T_\theta(\vec{x}) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

It can be found by observing the results of applying the transformation to $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$:

$$T(\hat{\mathbf{i}}) = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$
$$T(\hat{\mathbf{j}}) = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}$$

Setting those results as columns of a 2×2 matrix would give a matrix that produces those results:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}$$