MATH 262 - Homework 4.2b

9.

Claim. If a, b, and c are distinct real numbers, then the polynomials $(x-b)(x-c)$, $(x-a)(x-c)$, and $(x-a)(x-b)$ must be linearly independent.

Proof. First, expand the polynomials.

$$
\begin{aligned}
(x-b)(x-c) &= bc - (b+c)x + x^2 \\
(x-a)(x-c) &= ac - (a+c)x + x^2 \\
(x-a)(x-b) &= ab - (a+b)x + x^2
\end{aligned}
$$

These polynomials are in $\mathcal{P}_2$, whose standard basis is $\mathfrak{B} = \{\, 1, x, x^2 \,\}$. Using $\mathfrak{B}$, represent the polynomials as a matrix A.

$$
A = \begin{bmatrix} bc & -(b+c) & 1 \\ ac & -(a+c) & 1 \\ ab & -(a+b) & 1 \end{bmatrix}
$$

To show the polynomials are linearly independent, show $\det(A) \neq 0$.

$$
\begin{aligned}
\det(A) = bc \begin{vmatrix} -(a+c) & 1 \\ -(a+b) & 1 \end{vmatrix} &- -(b+c) \begin{vmatrix} ac & 1 \\ ab & 1 \end{vmatrix} + 1 \begin{vmatrix} ac & -(a+c) \\ ab & -(a+b) \end{vmatrix} \\
&= bc[(a+b) - (a+c)] + (b+c)(ac-ab) + ab(a+c) - ac(a+b) \\
&= cb^2 - bc^2 + ac^2 - ab^2 + ba^2 - ca^2 \\
&= (a^2 - ab - ac + bc)(b-c) \\
&= (a-b)(a-c)(b-c)
\end{aligned}
$$

Notice that for $\det(A) = 0$, at least two values would have to equal 0. However, all three values are distinct, so this is impossible. Therefore, $\det(A) \neq 0$ and the polynomials are linearly independent. □