MATH 262 - Homework 4.3

3. Find the matrix of the given linear transformation T with respect to the given basis. Determine whether T is an isomorphism. If T isn't an isomorphism, find bases of the kernel and image of T, and thus determine the rank of T.

$$T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M \text{ from } U^{2\times 2} \text{ to } U^{2\times 2}$$

For the space of $U^{2\times 2}$ of upper traingular 2×2 matrices, use the basis

$$\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Let A $= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The kernel of $T(M)$ is defined as

$$\begin{aligned} ker(T) &= \left\{ M \in U^{2\times 2} \mid T(M) = 0 \right\} \\ &= \left\{ M \in U^{2\times 2} \mid M \cdot A - A \cdot M = 0 \right\} \\ &= \left\{ M \in U^{2\times 2} \mid M \cdot A = A \cdot M \right\} \end{aligned}$$

Notice the kernel is the set of all $M \in U^{2\times 2}$ that commute with A.

$$M \cdot A = A \cdot M$$
$$\begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix}$$
$$\begin{bmatrix} m_{11} & 2m_{11} + m_{12} \\ 0 & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} + 2m_{22} \\ 0 & m_{22} \end{bmatrix}$$

$$2m_{11} + m_{12} = m_{12} + 2m_{22} \longrightarrow m_{11} = m_{22}$$

$$M = \begin{bmatrix} r & s \\ 0 & r \end{bmatrix} = r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\ker(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Because $\operatorname{nullity}(T) \neq 0$, $T(M)$ is not an isomorphism. Next, find the image.

$$\begin{aligned}
\operatorname{im}(T) &= \left\{ M \cdot A - A \cdot M \;\middle|\; M \in U^{2\times 2} \right\} \\
&= \left\{ \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{22} \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} m_{11} & 2m_{11} + m_{12} \\ 0 & m_{22} \end{bmatrix} - \begin{bmatrix} m_{11} & m_{12} + 2m_{22} \\ 0 & m_{22} \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{22} \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} 0 & 2m_{11} - 2m_{22} \\ 0 & 0 \end{bmatrix} \;\middle|\; m_{11}, m_{22} \in \mathbb{R} \right\} \\
&= \left\{ (2m_{11} - 2m_{22}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \;\middle|\; m_{11}, m_{22} \in \mathbb{R} \right\} \\
&= \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}
\end{aligned}$$

Thus, the rank of T is 1.