MATH 262 - Homework 4.2b

1. Find the kernel and image of the linear transformation and state whether it is an isomorphism. Remember a linear transformation is an isomorphism if and only if the kernel is trivial (equals the zero element only) and the image is everything.

$$T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M \text{ from } \mathbb{R}^{2\times 2} \text{ to } \mathbb{R}^{2\times 2}$$

Let A $= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The kernel of $T(M)$ is defined as

$$\begin{aligned} ker(T) &= \left\{ M \in \mathbb{R}^{2\times 2} \;\middle|\; T(M) = 0 \right\} \\ &= \left\{ M \in \mathbb{R}^{2\times 2} \;\middle|\; M \cdot A - A \cdot M = 0 \right\} \\ &= \left\{ M \in \mathbb{R}^{2\times 2} \;\middle|\; M \cdot A = A \cdot M \right\} \end{aligned}$$

Notice the kernel is the set of all $M \in \mathbb{R}^{2\times 2}$ that commute with A.

$$\begin{gathered} M \cdot A = A \cdot M \\ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \\ \begin{bmatrix} m_{11} & 2m_{11} + m_{12} \\ m_{21} & 2m_{21} + m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} + 2m_{21} & m_{12} + 2m_{22} \\ m_{21} & m_{22} \end{bmatrix} \end{gathered}$$

$$\begin{aligned} m_{11} &= m_{11} + 2m_{21} \longrightarrow m_{21} = 0 \\ 2m_{11} + m_{12} &= m_{12} + 2m_{22} \longrightarrow m_{11} = m_{22} \end{aligned}$$

$$\begin{gathered} M = \begin{bmatrix} r & s \\ 0 & r \end{bmatrix} = r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \ker(T) = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \end{gathered}$$

Because nullity$(T) \neq 0$, $T(M)$ is not an isomorphism. Next, find the image anyway for the sake of exercise.

$$
\begin{aligned}
\operatorname{im}(T) &= \left\{ \, M \cdot A - A \cdot M \;\middle|\; M \in \mathbb{R}^{2\times 2} \, \right\} \\
&= \left\{ \, \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{R} \, \right\} \\
&= \left\{ \, \begin{bmatrix} m_{11} & 2m_{11} + m_{12} \\ m_{21} & 2m_{21} + m_{22} \end{bmatrix} - \begin{bmatrix} m_{11} + 2m_{21} & m_{12} + 2m_{22} \\ m_{21} & m_{22} \end{bmatrix} \;\middle|\; m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{R} \, \right\} \\
&= \left\{ \, \begin{bmatrix} -2m_{21} & 2m_{11} - 2m_{22} \\ 0 & 2m_{21} \end{bmatrix} \;\middle|\; m_{11}, m_{21}, m_{22} \in \mathbb{R} \, \right\} \\
&= \left\{ \, 2m_{21} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + (2m_{11} - 2m_{22}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \;\middle|\; m_{11}, m_{21}, m_{22} \in \mathbb{R} \, \right\} \\
&= \operatorname{span} \left\{ \, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \, \right\}
\end{aligned}
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