MATH 262 - Homework 4.3

6.

Claim. Let $T : V \to W$ be a linear transformation of linear spaces V and W, suppose that a set of vectors $\{ \vec{v_1}, \vec{v_2}, \vec{v_3} \} \subset V$ has the property that $\{ T(\vec{v_1}), T(\vec{v_2}), T(\vec{v_3}) \} \subset W$ is linearly independent. Then, the set of vectors $\{ \vec{v_1}, \vec{v_2}, \vec{v_3} \}$ is linearly independent.

Proof. Suppose that $a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3} = \vec{0}$.
Then, applying T to both sides we get

$$T(a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3}) = T(\vec{0})$$

Since T is a linear transformation, we get

$$a_1T(\vec{v_1}) + a_2T(\vec{v_2}) + a_3T(\vec{v_3}) = T(\vec{0})$$

Since $\{ T(\vec{v_1}), T(\vec{v_2}), T(\vec{v_3}) \}$ are linearly independent we get that

$$a_1T(\vec{v_1}) + a_2T(\vec{v_2}) + a_3T(\vec{v_3}) = \vec{0}$$

This also means the only solution is the trivial $\vec{a} = \vec{0}$. Notice the supposition made at the beginning also uses $\vec{a}$. Thus, that supposition also only has $\vec{a} = \vec{0}$ as a solution and $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ are also linearly independent. $\square$