MATH 262 - Homework 7.3

2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

   Find all (real) eigenvalues for A. Then find a basis of each eigenspace, and diagonalize A, if you can. Do not use technology.

   Since A is a triangular matrix, its eigenvalues are its diagonal entries:

$$\lambda_1 = 1 \text{ with a multiplicity of } 1$$
$$\lambda_2 = 2 \text{ with a multiplicity of } 1$$
$$\lambda_3 = 3 \text{ with a multiplicity of } 1$$

   Now, use these eigenvalues to find the basis of each eigenspace.

$$E_1 = \ker(A - I_2) = \ker \begin{array}{c} \begin{matrix} 1 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \end{array} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$E_2 = \ker(A - 2I_2) = \ker \begin{array}{c} \begin{matrix} 1 & 1 & 0 \end{matrix} \\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_3 = \ker(A - 3I_2) = \ker \begin{array}{c} \begin{matrix} 1 & 2 & 1 \end{matrix} \\ \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\2\\1\end{bmatrix}$ form an eigenbasis for A, so that A is diagonalizable. $A = SBS^{-1}$ with $S = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$.