MATH 262 - Homework 7.1

5. Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$. Show that 2 and 4 are eigenvalues of A and find all corresponding eigenvectors. Find an eigenbasis of A and thus diagonalize A.

   First, show 2 and 4 are eigenvalues of A by showing $\det(A - \lambda I_2) = 0$ where λ is the eigenvalue being tested.

$$\begin{aligned} 0 &= \det(A - 2I_2) \\ &= \det\left(\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} \\ &= (0 \cdot 2) - (0 \cdot 3) \\ &= 0 \end{aligned} \qquad \begin{aligned} 0 &= \det(A - 4I_2) \\ &= \det\left(\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} -2 & 0 \\ 3 & 0 \end{bmatrix} \\ &= (-2 \cdot 0) - (0 \cdot 3) \\ &= 0 \end{aligned}$$

   To find all eigenvectors of A, find the kernel of $A - \lambda I_2$ for each eigenvalue.

$$\begin{aligned} \ker(A - 2I_2) &= \ker\begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} \\ &= \operatorname{span}\left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \right\} \end{aligned} \qquad \begin{aligned} \ker(A - 4I_2) &= \ker\begin{bmatrix} -2 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

   The vectors $\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form an eigenbasis for A, so that A is diagonalizable, with $S = \begin{bmatrix} -\frac{2}{3} & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.