MATH 262 - Homework 3.3

2. The redundant column is $\vec{v}_1 = \vec{0}$. The nonredundant column

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

forms a basis of the image of A. Thus $\dim(\text{im } A) = 1$.

Using the redundant vector $\vec{v}_1$, a vector in the kernel of A can be generated.

Redundant Vector	*Relation*	*Vector in Kernel of A*
$\vec{v}_1 = \vec{0}$	$\vec{v}_1 - 0\vec{v}_2 = \vec{0}$	$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Vector $\vec{w}_1$ constructed above forms a basis of the kernel of A. This can be verified by observing that $\vec{w}_1$ is trivially linearly independent and that $\dim(\ker A) = 1$.

Finally,

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$
$$\ker(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

6. The redundant column is $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$. The nonredundant columns

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a basis of the image of A. Thus $\dim(\text{im } A) = 2$.

Using the redundant vector $\vec{v}_3$, a vector in the kernel of A can be generated.

Redundant Vector	*Relation*	*Vector in Kernel of A*
$\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$	$-\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$	$\vec{w}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Vector $\vec{w}_3$ constructed above forms a basis of the kernel of A. This can be verified by observing that $\vec{w}_3$ is trivially linearly independent and that $\dim(\ker A) = 1$.

Finally,

$$\operatorname{im}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$
$$\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

8. The redundant column is $\vec{v}_1 = \vec{0}$. The nonredundant columns

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

form a basis of the image of A. Thus $\dim(\operatorname{im} A) = 2$.

Using the redundant vector $\vec{v}_1$, a vector in the kernel of A can be generated.

Redundant Vector	*Relation*	*Vector in Kernel of A*
$\vec{v}_1 = \vec{0}$	$\vec{v}_1 - 0\vec{v}_2 - 0\vec{v}_3 = \vec{0}$	$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Vector $\vec{w}_1$ constructed above forms a basis of the kernel of A. This can be verified by observing that $\vec{w}_1$ is trivially linearly independent and that $\dim(\ker A) = 1$.

Finally,

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

$$\ker(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

12. The redundant columns are $\vec{v}_1 = \vec{0}$ and $\vec{v}_3 = 2\vec{v}_2$. The nonredundant column

$$\vec{v}_2 = \begin{bmatrix} 1 \end{bmatrix}$$

forms a basis of the image of A. Thus $\dim(\text{im } A) = 1$.

Using the redundant vector $\vec{v}_1$, two vectors in the kernel of A can be generated.

Redundant Vector	*Relation*	*Vector in Kernel of A*
$\vec{v}_1 = \vec{0}$	$\vec{v}_1 - 0\vec{v}_2 - 0\vec{v}_3 = \vec{0}$	$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$\vec{v}_3 = 2\vec{v}_2$	$-0\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$	$\vec{w}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

Vectors $\vec{w}_1$ and $\vec{w}_3$ constructed above form a basis of the kernel of A. This can be verified by observing that they're linearly independent and that $\dim(\ker A) = 2$.

Finally,

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \end{bmatrix} \right\}$$

$$\ker(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

16. The redundant columns are $\vec{v}_2 = -2\vec{v}_1$ and $\vec{v}_4 = -\vec{v}_1 + 5\vec{v}_3$. The nonredundant columns

$$\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{v}_5 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

form a basis of the image of A. Thus $\dim(\text{im } A) = 3$.

Using the redundant vectors $\vec{v}_2$ and $\vec{v}_4$, two vectors in the kernel of A can be generated.

Redundant Vector	*Relation*	*Vector in Kernel of A*
$\vec{v}_2 = -2\vec{v}_1$	$2\vec{v}_1 + \vec{v}_2 - 0\vec{v}_3 - 0\vec{v}_4 - 0\vec{v}_5 = \vec{0}$	$\vec{w}_2 = \begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}$
$\vec{v}_4 = -\vec{v}_1 + 5\vec{v}_3$	$\vec{v}_1 - 0\vec{v}_2 - 5\vec{v}_3 + \vec{v}_4 - 0\vec{v}_5 = \vec{0}$	$\vec{w}_4 = \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}$

Vectors $\vec{w}_2$ and $\vec{w}_4$ constructed above form a basis of the kernel of A. This can be verified by observing that they're linearly independent and that $\dim(\ker A) = 2$.

Finally,

$$\text{im}(A) = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

$$\ker(A) = \text{span}\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}\right\}$$

18. The redundant column is $\vec{v}_3 = 3\vec{v}_1 + 2\vec{v}_2$. The nonredundant columns

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

form a basis of the image of A. Thus $\dim(\text{im } A) = 3$.

Using the redundant vector $\vec{v}_3$, a vector in the kernel of A can be generated.

Redundant Vector	*Relation*	*Vector in Kernel of A*
$\vec{v}_3 = 3\vec{v}_1 + 2\vec{v}_2$	$-3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 - 0\vec{v}_4 = \vec{0}$	$\vec{w}_3 = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Vector $\vec{w}_3$ constructed above forms a basis of the kernel of A. This can be verified by observing that $\vec{w}_3$ is trivially linearly independent and that $\dim(\ker A) = 1$.

Finally,

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

$$\ker(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

24. Solve the linear system $A\vec{x} = \vec{0}$ by Gaussian elimination.

$$B = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ can be solved by solving the simpler $B\vec{x} = \vec{0}$. The vectors in $\ker(B)$ are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2q & & & \\ q & & & \\ & r & & \\ & & s & \\ & & & t \end{bmatrix} = q \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}_1} + r \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}_2} + s \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_3} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_4}$$

where q, r, s, and t are arbitrary constants.

Note that $\ker(A) = \ker(B)$. The vectors $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$, and $\vec{w}_4$ form a basis of the kernel of A. The preceding equation, $\vec{x} = q\vec{w}_1 + r\vec{w}_2 + s\vec{w}_3 + t\vec{w}_4$, shows that these four vectors span the kernel. Furthermore, these vectors are linearly independent. Thus, the basis of the kernel of A is

$$\ker(A) = \operatorname{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and $\dim(\ker A) = 4$.

To construct the basis of the image of A, find the redundant columns of A. Using $B = rref(A)$, the redundant columns are the ones that lack a leading 1. The only redundant column is $\vec{b}_2 = 2\vec{b}_1$. Therefore, $\vec{a}_2$ in matrix A is also redundant via the same relation. Thus, a basis of the image of A is

$$\operatorname{im}(A) = \operatorname{span} \left\{ \begin{bmatrix} 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \right\}$$

and $\dim(\operatorname{im} A) = 1$.

30.

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; 2x_1 - x_2 + 2x_3 + 4x_4 = 0 \right\} \subseteq \mathbb{R}^4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 & & \\ 2x_1 + & 2x_3 + & 4x_4 \\ & x_3 & \\ & & x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

The basis for subspace S is

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$