MATH 262 - Homework 4.1

12. Find a basis for the following linear space, and thus determine the dimension.

$$V = \{\, f \in P_4 : f \text{ is even} \,\}$$

Note: A function is even if $f(-x) = f(x)$ for all x.
P_n is the set consisting of the zero polynomial combined with the set of all polynomials of degree less than or equal to n.

$$\begin{aligned}
f(x) &= f(-x) \\
a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 &= a_4(-x)^4 + a_3(-x)^3 + a_2(-x)^2 + a_1(-x) + a_0 \\
\cancel{a_4x^4} + a_3x^3 + \cancel{a_2x^2} + a_1x + \cancel{a_0} &= \cancel{a_4x^4} - a_3x^3 + \cancel{a_2x^2} - a_1x + \cancel{a_0} \\
a_3x^3 + a_1x &= -a_3x^3 - a_1x \\
2a_3x^3 + 2a_1x &= 0 \\
a_3x^3 + a_1x &= 0 \\
a_1x &= -a_3x^3
\end{aligned}$$

Any polynomial in V can be written as

$$\begin{aligned}
f(x) &= a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\
&= a_4x^4 + \cancel{a_3x^3} + a_2x^2 - \cancel{a_3x^3} + a_0 \\
&= a_4x^4 + a_2x^2 + a_0
\end{aligned}$$

Notice that since the first and third degrees must sum to 0, they can just be excluded. Finally,

$$\begin{aligned}
\text{basis of } V &= \{\, x^4, x^2, 1 \,\} \\
\dim(V) &= 3
\end{aligned}$$