AIML

S.V.D

Q1 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$\therefore A' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

$A \cdot A' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

$A A' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$A' \cdot A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$A' \cdot A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Finding Eigenvalues and Eigenvectors of $A \cdot A'$

$|AA' - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

By using $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

where,

S_1 = sum of diagonal elements

$S_1 = 1+2+1 = 4$

S_2 = sum of minors of diagonal element

$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$

$S_2 = (2-1) + (1-0) + (2-1)$

$S_2 = 1+1+1 = 3$

$|A| = 0$

i.e $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

Solving, it we get,

$\lambda = 3, 1, 0$

Eigenvector of $\lambda = 3$

$[A \cdot A' - \lambda I] X = 0$

$$\begin{bmatrix} 1-3 & 1 & 0 \\ 1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations formed are

$-2x_1 + x_2 + 0x_3 = 0 \quad ; \quad x_1 - x_2 + x_3 = 0$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{-x_2}{-2} = \frac{x_3}{1}$$

Let $\dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{1} = t$

$\therefore x_1 = t \quad , \quad x_2 = 2t \quad , \quad x_3 = t$

$$\therefore X = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vectors at $\lambda = 3$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$$\|X\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$\therefore$ Normalized eigen vector u_1 = $\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

Eigenvector of $\lambda = 1$

$$[A \cdot A' - \lambda I] X = 0$$

$$\begin{bmatrix} 1-1 & 1 & 0 \\ 1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations formed are,

$0x_1 + x_2 + 0x_3 = 0 \quad ; \quad x_1 + x_2 + x_3 = 0$

By Cramers Rule,

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{-x_2}{0} = \frac{x_3}{-1}$$

Let $\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = t$

$\therefore x_1 = t, \; x_2 = 0, \; x_3 = -t$

$\therefore X = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\|X\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

Normalized Eigen vector $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

Eigenvector at $\lambda = 0$:

$[A \cdot A' - \lambda I]X = 0$

$$\begin{bmatrix} 1-0 & 1 & 0 \\ 1 & 2-0 & 1 \\ 0 & 1 & 1-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations formed are

$x_1 + x_2 + 0x_3 = 0 \; ; \; x_1 + 2x_2 + x_3 = 0$

By Cramer's Rule,

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{-x_2}{1} = \frac{x_3}{1}$$

Let $\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = t$

$\therefore \; x_1 = t, \; x_2 = -t, \; x_3 = t$

$$x = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\|x\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

Normalized eigen vector $u_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$\therefore \; U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

Finding Eigenvalues and Eigenvectors of $A' \cdot A$

$$|A' \cdot A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

Solving, we get

$$\lambda = 3, 1$$

Eigenvector at $\lambda = 3$,

$$[A' \cdot A - \lambda I]\, x = 0$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equations formed are.

$-x_1 + x_2 = 0 \; ; \quad x_1 - x_2 = 0$

By Cramer's Rule.

or

$R_2 \rightarrow R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equation formed is

$-x_1 + x_2 = 0$

No. of independent variables = No. of variables - Rank

$= 2 - 1 = 1$

So, considering 1 variable as t.

$\therefore \quad x_1 = t$

$\therefore \quad -t + x_2 = 0$

$\therefore \quad x_2 = t$

$$\therefore \quad X = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\|x\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\therefore \text{Normalized eigen vector } v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For eigenvector of $\lambda = 1$

$[A'A - \lambda I] X = 0$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

~~Equations formed are,~~

~~$-x_1 + x_2 = 0$; $x_1 - x_2 = 0$~~

~~By Cramer's Rule,~~

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equation formed is

$-x_1 + x_2 = 0$

No. of independent variables = No. of variables - Rank

$= 2 - 1 = 1$

So, considering 1 variable as t

$\therefore x_1 = t$

$\therefore -t + x_2 = 0$

$\therefore x_2 = t$

$$\therefore X = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\|X\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\therefore \text{Normalized eigen vector } v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For eigenvector of $\lambda = 1$

$[A'A - \lambda I]\, X = 0$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equations formed are

$x_1 + x_2 = 0$

No. of independent variables = No. of variables - Rank

$= 2 - 1 = 1$

So, let one variable $x_1 = t$

$\therefore t + x_2 = 0$

$\therefore x_2 = -t$

$$\therefore X = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\|X\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\therefore$ Normalized eigen vector $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

To find singular values, $Av_i = \sigma_i u_i$

$i = 1, 2$

$\therefore Av_1 = \sigma_1 u_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sigma_1 \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$\sigma_1 = \sqrt{3}$

$\therefore Av_2 = \sigma_2 u_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \sigma_2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$\sigma_2 = 1$

$$\therefore \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

S.V.D of A is

$$U \Sigma V' = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Q.2

	Travelled	Not Travelled	Total
Literate	120 ~~300~~	80	200
Illiterate	180	420	600
Total	300	500	800

Step 1: Null Hypothesis H_0: There is no relation between literacy and travelling.

Alternate Hypothesis H_1: There is ~~no~~ a such relation.

Step 2: Calculation of test static

Expected frequencies are

	Travelled	Not Travelled	Total
Literate	75	125	200
Illiterate	225	375	600
Total	300	500	800

Expected frequency in 1st cell = $\frac{200 \times 300}{800} = 75$

" " in 2nd cell = 200 - 75 = 125

" " in 3rd cell = 300 - 75 = 225

" " in 4th cell = 500 - 125 = 375

$\chi^2 =$

O	E	O - E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
120	75	45	2025	27
80	125	-45	2025	16.2
180	225	-45	2025	9
420	375	45	2025	5.4
				$\Sigma = 57.6$

$$\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right] = 57.6$$

Step 3 :-

Level of significance $= 5\% = \frac{5}{100} = 0.05$

Degree of freedom $= (r-1)(c-1) = (2-1)(2-1) = 1$

Step 4 :-

The value of χ^2_α at d.o.f 1 and 0.05 level of significance is

$$\chi^2_\alpha = 3.84$$

Step 5 :-

As the value of χ^2_α is less than χ^2, so we accept the alternate hypothesis and reject the null hypothesis.

$\therefore$ There is ~~no~~ a relation between literacy and travelling.

Q 3

x	8	12	16	20	24
$P(X=x)$	$1/8$	m	n	$1/4$	$1/12$

$E(x) = 16$

Sol^n:- For random variable X,

$\sum P(X=x) = 1$

$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1$

$m+n = 1 - \frac{11}{24}$

$m+n = \frac{13}{24}$ — ①

Also, expectation of x is given by

$E(x) = \sum(x \cdot P(x))$

$16 = 8 \times \frac{1}{8} + 12 \times m + 16 \times n + 20 \times \frac{1}{4} + 24 \times \frac{1}{12}$

$16 = 1 + 12m + 16n + 5 + 2$

$16 = 12m + 16n + 8$

i.e $12m + 16n = 8$ — ②

Solving equations ① and ②, we get

$m = \frac{1}{6}$, $n = \frac{3}{8}$

For variance of x

$E(x^2) = \sum[x^2 \cdot P(X=x)]$

$= 64 \times \frac{1}{8} + 144 \times \frac{1}{6} + 256 \times \frac{3}{8} + 400 \times \frac{1}{4} + 576 \times \frac{1}{12}$

$= 8 + 24 + 96 + 100 + 48$

$E(x^2) = 276$//

Q. 4 Solⁿ :-

$N = 12$

$$\text{Mean } (\bar{x}) = \frac{\Sigma X}{N}$$

$$= \frac{5+2+8-1+3+0+6-2+1+5+0+4}{12}$$

$$\bar{X} = \frac{31}{12} = 2.583$$

Let the assumed mean be 3

x_i	5	2	8	-1	3	0	6	-2	1	5	0	4	Total
$d_i = x_i - 3$	2	-1	5	-4	0	-3	3	-5	-2	2	-3	1	-5
d_i^2	4	1	25	16	0	9	9	25	4	4	9	1	107

$$\Sigma(x_i - X)^2 = \Sigma d_i^2 - \frac{\Sigma(d_i)^2}{n}$$

$$= 107 - \frac{(-5)^2}{12}$$

$$= 107 - \frac{25}{12}$$

$$\Sigma(x_i - X)^2 = 104.9167$$

$$\therefore \quad S = \sqrt{\frac{\Sigma(x_i - X)^2}{n}}$$

$$= \sqrt{\frac{104.9167}{12}}$$

$$S = 2.9568$$

Step 1 :- Null Hypothesis $H_0 : \mu = 0$

Alternate Hypothesis $H_1 : \mu > 0$ (1-tailed test)

Step 2 :- Calculation of test static

As the sample is small, we use t - distribution,

$$|t| = \left| \frac{\bar{x} - \mu}{s / \sqrt{n-1}} \right|$$

$$|t| = \left| \frac{2.583 - 0}{2.9568 / \sqrt{12-1}} \right|$$

$$|t| = \left| \frac{2.583}{2.9568 / \sqrt{11}} \right|$$

$$|t| = 2.897$$

Step 3 :- Level of Significance

As the level of significance is $5\% = \frac{5}{100} = 0.05$

Step 4 :- Critical Value

So, the value of t_α for 0.05 level of significance and degree of freedom $= n-1 = 12-1 = 11$ is $t_\alpha = 1.796$

Step 5 :- Conclusion

As the value of t_α is less than $|t|$, so we accept the alternate hypothesis and reject the null hypothesis.

$\therefore$ The injection led to an increase in B.P.

6. $n = 10$

$p = \frac{1}{500}$

$\therefore m = np = 10 \times \frac{1}{500} = 0.02$

By Poisson Distribution,

$$P(X=x) = \frac{e^{-m} m^{x}}{x!}$$

Q.5 1. → 3 green, 2 white
2. 5 green, 6 white
3. 2 green, 4 white

prob of white from 1.

$P(\text{white}, 1) = \frac{2}{5}$ $\quad \therefore E(x_1) = \frac{2}{5}$

$P(\text{white } 2) = \frac{6}{11}$ $\quad \therefore E(x_2) = \frac{6}{11}$

$P(\text{white}, 3) = \frac{4}{6} = \frac{2}{3}$ $\quad E(x_3) = \frac{2}{3}$

$\therefore$ Total no of white balls drawn

$$\therefore E(x) = \frac{2}{5} + \frac{6}{11} + \frac{2}{3}$$

$$= 1.61$$

$$\approx 2 \text{ balls}$$

a] No defective ($x=0$)

$\therefore P(X=0) = \dfrac{e^{-0.02} \cdot (0.02)^0}{0!}$

$P(x=0) = 0.9802$

$\therefore$ No. of blades = $10000 \times 0.9802 = 9802$ blades

b] One defective ($x=1$)

$\therefore P(X=1) = \dfrac{e^{-0.02} \cdot (0.02)^1}{1!}$

$P(X=1) = 0.01960$

$\therefore$ No. of blades = $10000 \times 0.01960 = 196$ blades

c] Two defective ($x=2$)

$\therefore P(X=2) = \dfrac{e^{-0.02} (0.02)^2}{2!}$

$P(x=2) = 1.960 \times 10^{-4}$

$\therefore$ No. of blades = $10000 \times 1.96 \times 10^{-4} = 10^4 \times 1.96 \times 10^{-4}$

$= 1.96 \approx 2$ blades

Q.7

		Fathers		
		Educated	Uneducated	Total
Son	Intelligent	40	35	75
	Unintelligent	40	85	125
	Total	80	120	200

Sol^n:- Step 1 : Null Hypothesis H_0 : There is no association between educated father and intelligent sons.

Alternate Hypothesis H_1 : There is an association

Step 2 : Calculation of test static

Expected frequency will be,

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Solving this, we get

$\lambda = 1, 3, 2$

For $\lambda = 1$

$[A - \lambda I]\, x = 0$

$$\begin{bmatrix} 8-1 & -8 & -2 \\ 4 & -3-\lambda 1 & -2 \\ 3 & -4 & 1-\lambda 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~By Cramers Rule,~~

Equations formed are

$7x_1 - 8x_2 - 2x_3 = 0, \quad 4x_1 - 4x_2 - 2x_3 = 0$

By Cramers Rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2}$$

Let $\dfrac{x_1}{4} = \dfrac{x_2}{3} = \dfrac{x_3}{2} = t$

$\therefore x_1 = 4t, \quad x_2 = 3t, \quad x_3 = 2t$

	Educated father	Uneducated father	Total
Intelligent son	30	45	75
Unintelligent son	50	75	125
Total	80	120	200

Expected frequency of 1^{st} cell $= \frac{75 \times 80}{200} = 30$

" " of 2^{nd} cell $= \cancel{8}\ 75 - 30 = 45$

" " of 3^{rd} cell $= 80 - 30 = 50$

" " of 4^{th} cell $= 125 - 50 = 75$

O	E	O-E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
40	30	10	100	3.33
35	45	-10	100	2.22
40	50	-10	100	2
85	75	10	100	1.33
				$\Sigma = 8.88$

$$\chi^2 = \Sigma \left[\frac{(O-E)^2}{E} \right]$$

$$\chi^2 = 8.88$$

Step 3 :- Level of Significance

As the level of significance $= 5\% = \frac{5}{100} = 0.05$

Degree of Freedom $= (r-1)(c-1) = 1$

Step 4 :- Critical Value

The value of χ^2_α at 0.05 level of significance and 1 d.o.f is given as $\chi^2_\alpha = 3.84$

Step 5 :- Conclusion

As the value of $\chi^2_\alpha = 3.84$ is less than $\chi^2 = 8.88$, the alternate hypothesis is accepted and null hypothesis is rejected.

$\therefore$ There is an association of intelligent sons and educated fathers.

8

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

The characteristic equation is given by,

$$A - \lambda I = 0$$

$$\begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix} = 0$$

The characteristic polynomial is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

It can be written as,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1 = sum of diagonal elements

$S_1 = 8-3+1 = 6$

S_2 = sum of minors of diagonal elements

$= (-3-8) + (8+6) + (-24+32)$

$S_2 = (-11 + 14 + 8 = 11$

$|A| = 6$

$\therefore\ \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Solving this, we get

$\lambda = 1, 3, 2$

For $\lambda = 1$

$[A - \lambda I]\, X = 0$

$$\begin{bmatrix} 8-1 & -8 & -2 \\ 4 & -3-1 & -2 \\ 3 & -4 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~By Cramers Rule,~~

Equations formed are

$7x_1 - 8x_2 - 2x_3 = 0, \quad 4x_1 - 4x_2 - 2x_3 = 0$

By Cramers Rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2}$$

Let $\dfrac{x_1}{4} = \dfrac{x_2}{3} = \dfrac{x_3}{2} = t$

$\therefore\ x_1 = 4t, \quad x_2 = 3t, \quad x_3 = 2t$

$\therefore \quad X = t \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$\therefore$ Eigen vector at $\lambda = 1$ is $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

For $\lambda = 3$

$[A - \lambda I] X = 0$

$$\begin{bmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations formed are,

$5x_1 - 8x_2 - 2x_3 = 0 \quad ; \quad 4x_1 - 6x_2 - 2x_3 = 0$

By Cramers Rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

Let $\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1} = t$

$\therefore x_1 = 2t, \; x_2 = t, \; x_3 = t$

$$\therefore X = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vector at $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$,

$[A - \lambda I] X = 0$

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations formed are,

$6x_1 - 8x_2 - 2x_3 = 0 \quad ; \quad 4x_1 - 5x_2 - 2x_3 = 0$

By Cramer's Rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

Let $\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = t$

$\therefore x_1 = 3t,\ x_2 = 2t,\ x_3 = t$

$\therefore X = t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector at $\lambda = 2$ is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Q. 9 $\quad A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

Solⁿ: Reducing A to Echelon form

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The first and second columns are pivot columns, so they are the basis of column space of A.

Basis :- $Col(A) = Span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right)$

$\therefore Dim(Col(A)) = 2$

The non-zero rows of row echelon form is the basis for row space.

$Row(A) = span\{[1, 2, 0, 1], [0\ 1\ 1\ 0]\}$

$\therefore$ Basis $= \{[1\ 2\ 0\ 1], [0\ 1\ 1\ 0]\}$

$Dim(Row(A)) = 2$

For null space of A,

$AX = 0$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 + x_4 = 0$

$x_2 + x_3 = 0$

No. of independent variables = No. of variables − Rank

$= 4 - 2$

$= 2$

Considering $x_3 = t$, $x_4 = s$

$\therefore x_2 + x_3 = 0$

$x_2 + t = 0$

$x_2 = -t$

$\therefore x_1 + 2x_2 + x_4 = 0$

$x_1 + 2(-t) + s = 0$

$x_1 - 2t + s = 0$

$x_1 = 2t - s$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t-s \\ -t \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for nul (A) = $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Dimension of nul(A) = 2

For left null space of A,

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Reduce to Row-Echelon form. $R_4 \to R_4 - R_1$

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A'y = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 + y_3 = 0$

$y_2 = 0$

Let $y_3 = -t$

$\therefore y_1 = t$

$y_2 = 0$

$y_3 = -t$

$$\therefore y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore$ Basis for left null of A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$\therefore$ Dimension $(Nul(A^T)) = 1$

Shortcut for Dimension

$Col(A) = r$

$Row(A) = r$

$Nul(A) = n - r$

$Nul(A^T) = m - r$

~~Q-16~~

Q.10 $\begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix}$

~~UAV~~

$U \Sigma V^T$

~~$A \cdot A^T$~~

~~$= \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix} \begin{bmatrix} 4 & 4 \\ 3 & \end{bmatrix}$~~

For U

$A \cdot A^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

diagonal matrix

$\therefore d_1 = 32, \ d_2 = 18$

~~$\therefore V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad V_2 =$~~

$\therefore x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\sqrt{1^2 + 0^2} = 1 \qquad \sqrt{0^2 + 1^2} = 1$

~~$V = A^T A$~~

~~$= \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$~~

~~$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$~~

$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

for U

$$A^T \cdot A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

$$\therefore |A^T A - \lambda I| = \begin{vmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{vmatrix}$$

$\therefore \lambda^2 - \lambda(\text{sum of dia}) + |A^T A| = 0$

$\lambda^2 - 50\lambda + 576 = 0$

$\lambda_1 = 32, \lambda_2 = 18$

$\lambda_1 = 32$

$$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 7x_2 = 0$

$7x_1 - 7x_1 = 0$

$x_1 = 1, x_2 = 1$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \sqrt{2} \quad L(x_1) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\lambda_2 = 18$

$$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$7x_1 + 7x_2 = 0$

$7x_1 + 7x_2 = 0$

$x_1 = 1, x_2 = -1$

$$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \sqrt{2}$$

$$L(x_2) = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \cancel{\begin{matrix} 0 \\ 0 \end{matrix}}$$

$$\cancel{\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 & 0 \\ 0 & 18 & 0 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Sample	population
Q.11 $n = 26$	$\mu = 140$
$\bar{x} = 147$ (sample mean)
$s = 16$
~~$\mu = 140$~~

Step 1: Null hypothesis : $\mu = 140$
Alternate hypothesis : $\mu \neq 140$

Step 2: Test statistics

$$t = \left| \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \right|$$

$$= \frac{147 - 140}{16 \sqrt{26-1}}$$

$$\therefore \boxed{t_{cal} = 2.16}$$

Step 3: Level of significance (default $\alpha = 5\%$)

$$\alpha = 5\%$$

Step 4: degree of freedom

$$\nu = n-1$$
$$= 25$$

Step 5: Critical value

for $\alpha = 5\%$, $\nu = 25$

$\therefore \boxed{t_{tab} = 2.06}$ → from t-table

Step 6: Decision

$t_{cal} = 2.16$ $\quad t_{tab} = 2.06$

$t_{cal} > t_{tab}$

∴ Null hypothesis is rejected.

~~∴ Yes, Null hypothesis is rejected.~~

∴ Yes, Advertisement is effective at $\alpha = 5\%$

∴ There is relation.

Q.72	Dark	Not Dark	Total
Dark	48	90	138
Not dark	80	782	862
Total	128	872	1000

son (rows), father (columns)

Step 1: Null hypo: There is no relation
Alter hypo: There is relation

S2: Test statistic

expected table

	Dark	Not dark	Total
Dark	S1 = 18	S2 = 120	~~128~~ 138
Not dark	S3 110	S4 = 752	862
total	128	872	1000

$$S1 = \frac{128 \times 138}{1000} = 17.6 \approx 18$$

O	E	$(O-E)^2$	$(O-E)^2/E$
48	18	900	50
90	120	900	7.5
80	110	900	8.18
782	752	900	1.19
			$\Sigma = 66.87$

$\therefore \chi^2_{test\,cal} = 66.87$

S3: $\alpha = 5\%$

S4: $v = (2-1)(2-1) = 1$

S5: critical value

for $\alpha = 5\%$, $v = 1$

$\chi^2_{tab} = 3.84$

$\therefore$

S6: $\chi^2_{tab} < \chi^2_{cal}$

$\therefore$ null hypo is false

$\therefore$ There is relation.

$(a,b) : a, b \in R$

Q.13 $V = \{(x,y) : x, y \in R\}$

$(a,b) + (c,d) = (a+c, b+d)$

$a(a,b) = (a, ab)$

~~is V(+, . is V(~~

is $(V, +, \cdot)$ ~~VS~~ VS?

→ Let $a = 2$, $b = 3$, $v = (5,4)$

by 8th property $(a+b) V = \cancel{ab} + a \cdot V + b \cdot V$

$\therefore$ L.H.S. $= (2+3) \cdot (5,4)$

$= 5(5,4)$

due to question → $a(a,b) = (a, ab)$

$= (5, 20)$

R.H.S. $= 2(5,4) + 3(5,4)$

$= (5,8) + (5,12)$

due to question,

$= (5+5, 8+12)$

$= (10, 20)$

L.H.S $\neq$ R.H.S.

$\therefore$ It is not a vector space

Q.14 $V = \{(x, y) : x, y \in R\}$

$(a, b) + (c, d) = (a+c, b+d)$

$a(a, b) = (aa, a^2 b)$

Show that $(V, +, \cdot)$ is not VS

$\Rightarrow$ 8^{th} property, $(a+b)V = a \cdot V + b \cdot V$

Let $a = 2$, $b = 3$, $V = (5, 4)$

$$
\begin{aligned}
L.H.S. &= (2+3)(5, 4) \\
&= 5(5, 4) \\
&= (25, 25 \times 4) \\
&= (25, 100)
\end{aligned}
$$

$$
\begin{aligned}
R.H.S. &= a \cdot V + b \cdot V \\
&= 2(5, 4) + 3(5, 4) \\
&= (10, 16) + (15, 36) \\
&= (25, 52)
\end{aligned}
$$

$L.H.S. \neq R.H.S.$

$\therefore$ it is not V.S.

15. $V = \{(x,y,z) ; x,y,z \in \mathbb{R}\}$ $\therefore V = \mathbb{R}^3$

$W = \{(x,y,z) : x - 3y + 4z = 0\}$

Let $(x,y,z) = \cancel{(\ )}, (-5,1,2)$

$\therefore -5 - 3 + 4(2)$
$= -8 + 8$
$= 0$

$\therefore (-5,1,2) \in W$

$\therefore v_1 = (-5,1,2)$
Let $v_2 = (2,2,1)$
$v_3 = (-1,1,1)$ } from W

Let $\alpha = 2$
$\beta = 0$
$\gamma = 1$ } from V

$\therefore \alpha v_1 + \beta v_2 + \gamma v_3$
$= 2(-5,1,2) + 0(2,2,1) + 1(-1,1,1)$
$= (-10,2,4) + (-1,1,1)$
$= (-11,3,5)$ → should belong in W

$x - 3y + 4z = 0$
$L.H.S = -11 - 3(3) + 4(5)$
$= -11 - 9 + 20$
$= 0$
$= R.H.S.$

{check for 2-3 more α, β, γ}

$\therefore$ for any values of α, β, γ
$\cancel{\alpha v_1 + \alpha_2\beta + \alpha_3}$ $\alpha v_1 + \beta v_2 + \gamma v_3 \in W$
$\therefore W$ is subspace of V