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Computation of Nash equilibria in bimatrix games using homotopy methods

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Contents

1 Introduction

This seminar paper mainly deals with the homotopy methods described in Herings and Peeters(2010) which are used to solve for Nash equilibria in non-degenerate bimatrix games. These methods are discussed in section (2). The homotopy method determines a path that eventually leads to a Nash equilibrium in a given game. This so called homotopy path coincides with the path generated by the Lemke-Howson algorithm.[1] Therefore the algorithm is briefly explained in section (3) before it is used to graphically illustrate the homotopy path for two given examples. In section (4) the implementation of the homotopy method into MATLAB is discussed. During the attempt to calculate Nash equilibria using MATLAB several issues occurred, which will be summarized in the last section (5).

1.1 Notation

In the following a normal form game $\Gamma = [I, \{S_i\}, \{u_i\}]$ with $u_i = S_1 \times S_2 \longrightarrow \mathbb{R}$ and $I = \{1, 2\}$ is considered, where S_1 and S_2 define the action sets of player 1 and 2 respectively.

Let A and B be the $m \times n$ payoff matrices of player 1 and 2. Player 1 has $m \in \mathbb{N}$ pure strategies to choose of, player 2 has $n \in \mathbb{N}$. To simplify the notation in the following sections, assume that m and n are disjoint in the sense that a pure strategy of player 1 is denoted by $s_i \in S_1$ with $i = 1, ..., m$, whereas a pure strategy for player 2 is denoted by $s_k \in S_2$ with $k = m+1, ..., n$. A mixed strategy of player 1, $\sigma_1 \in \mathbb{R}^m$, is a probability distribution over player 1's pure strategies written as a $m \times 1$ column vector. Analogously a mixed strategy of player 2 is denoted by $\sigma_2 \in \mathbb{R}^n$. Let σ be the mixed strategy profile $\sigma = (\sigma_1, \sigma_2)$.

Let a_i be the i-th row of A and b_k be the k-th row of B^T such that $a_i \sigma_2 \in \mathbb{R}$ is player 1's expected payoff when playing the pure strategy $s_i \in S_1$ against player 2's mixed strategy σ_2. Analogously $b_k \sigma_1$ defines player 2's expected payoff when playing $s_k \in S_2$ against σ_1.

Assume that the player's preferences satisfy the assumptions made by von Neumann and Morgenstern's expected utility theorem and further that they maximize their expected utility. Assume with out loss of generality, that the expected utility equals the expected payoff. A best response[2] of player 1 against a given mixed strategy σ_2 of player 2 is a mixed strategy σ_1 that maximizes his expected payoff $\sigma_1^T A \sigma_2$.[3] Similarly player 2's best response to σ_1 is a strategy σ_2 that maximizes $\sigma_1^T B \sigma_2$. A Nash Equilibrium[4] is a strategy profile σ where σ_1 is a best response to σ_2 and vice verse. Following Nash(1951) one can

[1] See Lemke and Howson(1964).
[2] See Definition (6.3).
[3] Note that mixed strategies in this context do not exclude playing one strategy with probability one.
[4] See Definition (6.4).

show that σ is a Nash equilibrium of Γ if and only if every pure strategy assigned with a positive probability is a best response.[5]

In the following section a homotopy that can be used to solve bimatrix games for Nash equilibria is described.

2 The Homotopy Method

Two continuous functions between two topological spaces are called homotopic if one function can be continuously deformed into the other. Let X,Y be two topological spaces and let $f,g : X \longrightarrow Y$ be homotobic mappings. The deformation between these two continuous functions is called homotopy denoted by H. The homotopy H is defined to be a continuous function $H : [0,1] \times X \longrightarrow Y$ with $H(0,x) = f(x)$ for all $x \in X$ and $H(1,x) = g(x)$ for all $x \in X$. Let $t \in [0,1]$ be the homotopy parameter such that $H_t = H(t,\cdot) \longrightarrow Y$. If the homotopy is well defined and the fixed points of f are known, gradually increasing t and solving for the fixed points of the corresponding nearby functions leads eventually to fixed points of g. This process is called homotopy method. The authors claim that by following Browder(1960) and Mas-Colell(1974) one can show that if H is an upper hemi-continuous correspondence[6] that is non-empty and convex-valued and given a unique fixed point in $H(0,x)$, there exists a unique path connecting the fixed point in $H(0,x)$ with a fixed point in $H(1,x)$. This path is called homotopy path.

In the following it is described how homotopy methods can be used to solve Γ for Nash equilibria.[7] First define the set of feasible mixed strategies Σ_1 and Σ_2 for player 1 and 2 respectively. Note that the probabilities a player assigns to his pure strategies must sum up to one. This yields a linear constraint we define as follows: Let $e \in \mathbb{R}^m$ be a $m \times 1$ column vector with $e_i = 1$ for all $i = 1,...,m$, similarly, let $q \in \mathbb{R}^n$ be a $n \times 1$ column vector with $q_k = 1$ or all $k = m+1,...,n$, such that

$$\Sigma_1 = \{\sigma_1 \in \mathbb{R}^m \mid e^T \sigma_1 = 1, \sigma_1 \geq \underline{0}\}$$
$$\Sigma_2 = \{\sigma_2 \in \mathbb{R}^n \mid q^T \sigma_2 = 1, \sigma_2 \geq \underline{0}\}$$

Note that in both sets $\underline{0}$ is the zero vector with length following the context, which ensures that probabilities are at least zero. Define $\Sigma = \Sigma_1 \times \Sigma_2$ such that $\sigma \in \Sigma$.

Let $H : [0,1] \times \Sigma \longrightarrow \Sigma$ be a homotopy and $\Gamma(t = 1)$ be the game that is supposed to be solved. In order to apply the homotopy method, H must have certain properties, first

[5]See Theorem (6.7).

[6]See Definition (6.6).

[7]Note that homotopy methods can also be used to solve for Nash equilibrium in games with more than two players.

define f and g such that every fixed point yields a Nash equilibrium. Therefore let β^i be a mapping $\beta^i : \Sigma \longrightarrow \Sigma$ such that given σ_{-i}, β^i yields the best response of player i:

$$\beta^i(\sigma) = \underset{\tilde{\sigma}_i \in \Sigma_i}{argmax} \quad u_i(\tilde{\sigma}_i, \sigma_{-i})$$

A fixed point of β^i ensures that player i has no incentive to deviate from his strategy σ_i. Let $H(1,\sigma) = \prod_{i \in I} \beta^i(\sigma)$ such that each fixed point of $H(1,\sigma)$ defines a Nash equilibrium of Γ.

Further there shall exist an initial game $\Gamma(0)$ with an easy to calculate unique Nash equilibrium given by the unique fixed point of $H(0,\sigma)$. Moreover, starting at $H(0,\sigma)$ and increasing t must lead to $H(1,\sigma)$. To ensure these properties let $\Gamma(t) = [I, \{S_i\}, \{v_i(\sigma,t)\}]$ with $v_{-i}(t,\sigma) = u_{-i}(\sigma)$ and $v_i(t,\sigma) = u_i(\sigma)$ for all $\sigma_k \neq \sigma_j$. Let the remaining strategy σ_j be a pure strategy $s_j^* \in S_i$. Player i's utility does only depend on t if he plays s_j^* with $v_i(s_j^*,t) = u_i(s_j^*) + (1-t)b$. That is, in the initial game $\Gamma(0)$ one pure strategy for a given player i, $s_j^* \in S_i$, gets a bonus b. If for example $s_j^* \in S_1$, add b to the j-th row of A and leave all other rows as well as matrix B unchanged. The second desired property is $\lim_{t \to 1} H(t,\sigma) = H(1,\sigma)$ which follows immediately. Let b be sufficiently large such that s_j^* is the only strategy of player i that survives iterated elimination of strictly dominated strategies in game $\Gamma(0)$. Remember that Γ is non-degenerate such that the best response of player $-i$ against s_j^* is a pure strategy, $s_k \in S_{-i}$. The unique Nash equilibrium of game $\Gamma(0)$ is (s_j^*, s_k) which is easy to calculate. Therefore $\Gamma(0)$ also comprehends the first desired property.

In addition one can show that $H : [0,1] \times \Sigma \longrightarrow \Sigma$ satisfies all remaining requirements of Theorem (6.11) such that there exists a connected set. The connection between the unique Nash equilibrium of $\Gamma(0)$ and a Nash equilibrium of $\Gamma(1)$ determines the unique homotopy path.[8]

The homotopy method solves for Nash equilibria in $\Gamma(1)$ by following the homotopy path starting at the unique Nash equilibrium of the initially constructed game $\Gamma(0)$. According to Herings and Peeters(2010) the homotopy path described above precisely coincides with the strategy profiles generated by the Lemke-Howson algorithm. Therefore section (3) sheds light on the Lemke-Howson algorithm and provides a graphically illustration of the homotopy path. The numerical method that follows the homotopy path is called complementary pivoting. This method will be discussed in section (4).

[8]Note that the homotopy path is unique for a given $\Gamma(0)$ but might change if the initially constructed game $\Gamma(0)$ changes.

3 The Lemke-Howson-Algorithm

Define $S = S_1 \times S_2$. Following Shapley(1974) assign to any mixed strategy $\sigma_i \in \Sigma_i$ a set of labels which are elements of S:

$$L(\sigma_1) = \{s_i \in S_1 \mid \sigma_1^i = 0\} \cup \{s_j \in S_2 \mid b_j\sigma_1 \geq b_k\sigma_1 \quad \forall k = m+1,...,n\}$$
$$L(\sigma_2) = \{s_j \in S_2 \mid \sigma_2^j = 0\} \cup \{s_i \in S_1 \mid a_i\sigma_2 \geq a_k\sigma_2 \quad \forall k = 1,...,m\}$$

That is, a label for σ_1 is the union of player 1's strategies played with zero probability and the pure strategies of player 2 that are best responses to σ_1. Similar for $L(\sigma_2)$. The mixed strategy profile σ is labeled by

$$L(\sigma) = L(\sigma_1) \cup L(\sigma_2)$$

The mixed strategy profile σ is called completely labeled if $L(\sigma) = S$ and it is called s_i-almost completely labeled if $L(\sigma) \cup \{s_i\} = S$ with $i = 1,...,m,m+1,...,n$. One can show that that σ constitutes a Nash equilibrium of Γ if and only if $L(\sigma) = S$.[9] In non-degenerate games the size of $L(\sigma_1)$ does not exceed m and the size of $L(\sigma_2)$ does not exceed n. Further in Von Stengel(2002) it is shown that in Γ the amount of mixed strategies $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ with exactly m and n labeles respectively, is finite.[10] This Theorem has an important implication for the graphical illustration as well as the computational problem. The graphical implication is that there exist two graphs G_1 and G_2 whose vertices are the points $L(\sigma_1)$ with exactly m labels and $L(\sigma_2)$ with exactly n labels. Further it follows immediately that there exist only finitely many pairs (σ_1, σ_2) for which $L(\sigma)$ is completely labeled, i.e. the game has finitely many Nash equilibria. This will have an impact on the computational problem which will be discussed in section (4).

3.1 Graphical Illustration

In order to apply the Lemke-Howson algorithm an additional vertex $\underline{0}_i$ with $i = 1,2$ must be added to G_1 and G_2. The vertex $\underline{0}_i$ has all pure strategies of player i as label, that is $L(\underline{0}_i) = S_i$. It immediately follows that $(\underline{0}_1, \underline{0}_2)$ is completely labeled. Any two vertices are joined by an edge if they only differ in exactly one label.

Let $G = G_1 \times G_2$ be the product graph with vertex (σ_1, σ_2) if σ_1 is a vertex of G_1 and σ_2 a vertex of G_2. In line with the definition from the previous chapter a vertex pair (σ_1, σ_2) of G is called s_i-almost completely label if $L(\sigma) \cup \{s_i\} = S$. Two vertex pairs (σ_1, σ_2) and $(\tilde{\sigma}_1, \sigma_2)$ are called adjacent if they are connected by an edge, i.e. if σ_1 and $\tilde{\sigma}_1$ only differ in one label s_i.

[9] See Theorem (6.8).
[10] See Theorem (6.10).

The Lemke-Howson algorithm starts at the initial completely labeled vertex $(\underline{0}_1, \underline{0}_2)$ and chooses one strategy $s_i \in S$ randomly. Player i, with $s_i \in S_i$, plays s_i with probability one such that $(\underline{0}_1, \underline{0}_2)$ is adjacent to a s_i-almost completely labeled vertex $(s_i, \underline{0}_{-i})$. The set of s_i-almost completely labeled vertices consist of disjoint paths and cycles. But there exists one unique path of s_i-almost completely labeled vertices in G that connects the initial completely labeled vertex $(\underline{0}_1, \underline{0}_2)$ with an other completely labeled vertex (σ_1, σ_2), i.e. with a Nash equilibrium.[11] Equilibrium points can graphically be distinguished from non-equilibrium points by the fact that only equilibrium points have one instead of two adjacent vertices.

In the following section (3.2) the s_i-almost completely labeled path is illustrated for several examples. According to section (2) this path coincides with the homotopy path. Note that the homotopy path does not start at $(\underline{0}_1, \underline{0}_2)$ but at the first vertex pair which does neither include $\underline{0}_1$ nor $\underline{0}_2$.

3.2 Examples

Consider the two bimatrix and non-degenerate games **Game 1**[12] and **Game 2** with $S_1 = \{S^1_A, S^1_B\}$ and $S_2 = \{S^2_A, S^2_B\}$. The payoffs for each player in each game are presented in the following tables:

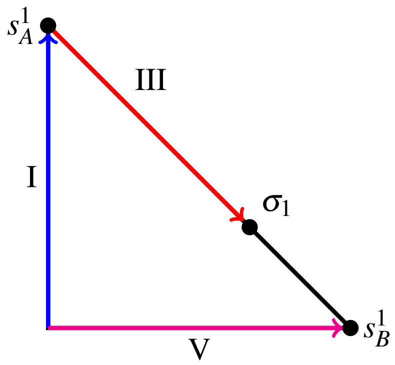
Game 1

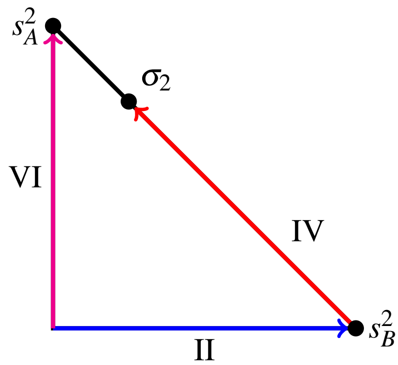
	s^2_A	s^2_B
s^1_A	2,2	1,4
s^1_B	1,4	4,0

Game 2

	s^2_A	s^2_B
s^1_A	5,1	0,0
s^1_B	4,4	1,5

Further in order to draw the homotopy path consider G_1 and G_2 separately instead of G. The algorithm describes movement along the edges in turns, i.e. if the first move takes place in G_1, the second will take place in G_2 and so on. Moreover, as probabilities over strategies must sum up to one, movement is restricted to the edges. Consider G_1 and G_2 respectively:





[11] See Theorem (6.9).
[12] See Herings and Peeters(2010) p. 129.

Game 1 has a unique mixed strategy Nash equilibrium $\sigma^{G1} = ((\frac{2}{3},\frac{1}{3}),(\frac{3}{4},\frac{1}{4}))$. Game 2 has three Nash equilibria, two in pure strategies $\sigma_1^{G2} = ((1,0),(1,0))$, $\sigma_2^{G2} = ((0,1),(0,1))$ and one in mixed strategies $\sigma_3^{G2} = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$.

In section (1) it was assumed that m and n are disjoint, this is used now in order to simplify the following notation. Each $s_i \in S$ can be assigned to exactly one natural number. Let $s_A^1 = 1$, $s_B^1 = 2$, $s_A^2 = 3$ and $s_B^2 = 4$ such that for example in Game 1, $L(s_A^1, s_A^2) = ((2,4),(1,4))$.

First Consider Game 1. Let s_A^1 be the randomly chosen strategy such that a path of s_A^1-almost completely labeled vertices connects $(\underline{0}_1, \underline{0}_2)$ with the unique Nash equilibrium σ^{G1}. Choosing $s_A^1 \in S_1$ is graphically illustrated by the blue arrow I, i.e. a shift upward from $\underline{0}_1$ to s_A^1. This vertex can be understood as the strategy profile where player 1 assigns probability one to s_A^1. Note that the line between s_A^1 and s_B^1 equals Σ_1.

Before the next move, which will take place in G_2, is described, one might want to check once that $(\underline{0}_1, \underline{0}_2)$ and $(s_A^1, \underline{0}_2)$ are indeed connected by an edge in G, i.e. only differ in one label:

$$L(\underline{0}_1, \underline{0}_2) = ((1,2),(3,4))$$
$$L(s_A^1, \underline{0}_2) = ((4,2),(3,4))$$

Thus $(s_A^1, \underline{0}_2)$ is indeed s_A^1-almost completely labeled and only differs from the initial vertex in $s_A^1 = 1$ such that $(s_A^1, \underline{0}_2)$ lies on the path connecting the initial equilibrium with the unique Nash equilibrium. Even though it is not done for the following steps, one can show by a similar argument that each of the vertices below are connected with each other in G.

Now consider G_2, from $4 \in L(s_A^1)$ the best response of player 2 is known, indicated by the blue arrow II from $\underline{0}_2$ to S_B^2. Analogously this means player 2 plays s_B^2 with probability one. The new label is $L(s_A^1, s_B^2) = ((4,2),(3,2))$ such that (s_A^1, s_B^2) is a s_A^1-almost completely labeled vertex of G. Clearly this implies that (s_A^1, s_B^2) is not completely labeled which means it is no endpoint and therefore has two adjacent vertices in G. Consider G_1 again in order to find the second one.

Not that $2 \in L(s_B^2)$ implies that s_B^1 is a best response of player 1, as he already assigns positive probability on s_A^1 it follows by theorem (6.3) that both s_A^1 and s_B^1 have to be best responses. As non-degenerate games are considered it already follows that player 2 randomizes as well such that $\sigma_1 = (\frac{2}{3},\frac{1}{3})$ in G_1 can be calculated.[13] Player 1's shift from a pure to a mixed strategy is illustrated by the red arrow III. The new vertex (σ_1, s_B^2) is not completely labeled with $L(\sigma_1, s_B^2) = ((4,3),(3,2))$, thus G_2 needs to be considered again.

From the conclusion above it is already known that player 2 randomizes as well. Player

[13]The calculation is discussed in section (4).

2's shift towards a mixed strategy σ_2 is indicated by the red arrow IV in G_2[14]. The strategy profile (σ_1, σ_2) is labeled $L(\sigma_1, \sigma_2) = ((4,3),(1,2))$ i.e. is completely labeled. This means the algorithm stops at this point as (σ_1, σ_2) has no more adjacent vertices in G i.e. is an endpoint and therefore a Nash equilibrium for Game 1.

Game 1 has a unique Nash equilibrium. Therefore it does not matter which strategy is randomly chosen in the beginning, each corresponding path leads to σ^{G1}. The same is not true for Game 2 which will be discussed in the following.

Assume that s_A^1 is again chosen randomly to begin with. Analogously to the argumentation above one can show that the blue arrow I and the magenta arrow IV lead to the completely labeled vertex (s_A^1, s_A^2) of G and thus to a Nash equilibrium of Game 2. If s_A^2 is chosen in the beginning the order is altered such that first the magenta arrow IV and then the blue arrow I must be considered but note the resulting Nash equilibrium is the same. If however s_B^1 (s_B^2) is chosen first, the magenta arrow V (blue arrow II) and the blue arrow II (magenta arrow V) lead to σ_2^{G2}.

In conclusion it holds for Game 2 that the Nash equilibrium reached by the Lemke-Howson algorithm depends on the initially chosen strategy. Moreover as all possible starting points were considered, it holds that the Lemke-Howson algorithm never reaches the mixed strategy equilibrium σ_3^{G2}. This already indicates that more flexible starting points are desirable if one is interested in finding several Nash equilibria for a given game. This issue will be discussed in section (5).

4 Computing

In Von Stengel(2002) it is shown that the vertices and edges of G can be represented by certain polyhedra.[15] As discussed in the previous example and even in more complex games, the Nash equilibrium is reached by traversing the polyhedron along its edges. These movements along the edges and thus also the homotopy path, can algebraically be implemented by pivoting, which is the change of basis for a given linear system of equations. In linear programming (LP) the variable that enters the basis is determined by the improvement of the objective function. However in section (3) it was implicitly claimed that a consideration of complementary conditions[16] determines the change of basis rather than an improvement of the objective function. Therefore the Lemke-Howson algorithm yields a solution to a so called linear complementarity problem (LCP). In the following it is shown that a Nash equilibrium can be represented as a LCP.

[14] As before σ_2 can be calculated which will be discussed in section (4).
[15] See Definition (6.5).
[16] See section (4.1).

4.1 Linear Complementary Problem

For player 1 a best response to a given mixed strategy σ_2 is a mixed strategy $\sigma_1 \in \Sigma_1$ that maximizes $\sigma_1^T A \sigma_2$. This means that a best response is the solution to the LP

$$\max_{\sigma_1} \quad \sigma_1^T A \sigma_2 \qquad \text{subject to} \qquad e^T \sigma_1 = 1,\ \sigma_1 \geq \underline{0}.$$

Following the exposition of Von Stengel(2002) regarding the strong duality theorem of linear programming, σ_1 is a optimal solution if and only if there exists a $u \in \mathbb{R}$ satisfying

$$\min_{u} \quad u \qquad \text{subject to} \qquad e \cdot u \geq A \cdot \sigma_2. \tag{1}$$

with utility of player 1

$$\begin{aligned} u &= \sigma_1^T A \sigma_2 \\ &\Leftrightarrow \sigma_1^T (eu) = \sigma_1^T (A\sigma_2) \\ &\Leftrightarrow \sigma_1^T (eu - A\sigma_2) = 0. \end{aligned} \tag{2}$$

Note that by assumption $\sigma_1 \geq 0$ and by (1) $eu - A\sigma_2 \geq 0$, i.e. both expressions are non-negative. This implies for (2) to be true that if strategy $s_i \in S_1$ is played with positive probability it must hold that $u - a_i \sigma_2 = 0$. In that sense σ_1 and $(eu - A\sigma_2)$ have to be complementary as they can not have positive values in the same row. Note that (2) is equivalent to the first equation in Theorem (6.7).

For Player 2 the same argument is true such that σ_2 is a optimal solution if and only if there exists a $v \in \mathbb{R}$ satisfying

$$\min_{v} \quad v \qquad \text{subject to} \qquad q \cdot v \geq B^T \cdot \sigma_1$$

with

$$\sigma_2^T (qv - B^T \sigma_1) = 0 \tag{3}$$

being equivalent to equation 2 in Theorem (6.7). It follows that σ is a Nash equilibrium of Γ if and only if

$$e^T \cdot \sigma_1 = 1, \tag{4}$$

$$q^T \cdot \sigma_2 = 1, \tag{5}$$

$$\begin{aligned} \sigma_1^T (e \cdot u - A\sigma_2) &= 0, \\ \sigma_2^T (q \cdot v - B^T \sigma_1) &= 0, \end{aligned} \tag{6}$$

$$\sigma_1, \sigma_2 \geq 0, \tag{7}$$

$$e \cdot u - A\sigma_2 \geq 0, \tag{8}$$

$$q \cdot v - B^T \sigma_1 \geq 0. \tag{9}$$

Note that by assuming A and B^T are non-negative and have no zero column, which can be done without loss of generality, the inequalities (8) and (9) are always true. Moreover the complementary conditions (2) and (3) where used in section (3.2) to calculate the mixed strategy equilibria. These conditions define the desired LCP.

4.2 Implementing

Consider s_j^* as defined in section (1), for simplicity assume that $s_j^* \in S_1$. The homotopy method starts at (s_j^*, s_k), the starting point is certainly s_j^*-almost completely labeled but not necessarily completely labeled.

By construction it holds that $(s_j^*, s_k) \in H(0, (s_j^*, s_k))$. If $(s_j^*, s_k) \in H(t, (s_j^*, s_k))$ for all $t \in [0,1]$ then it holds that the initially chosen strategy profile (s_j^*, s_k) already defines a Nash equilibrium of $\Gamma(1)$. An example for the corresponding homotopy path would be one of the paths considered for Game 2 in section (3.2). However if it is not true for all t, then there exists a $t' \in [0,1]$ for which a second pure strategy of player 1, $s'_j \in S_1$, becomes a best response. Let σ_1 be the mixed strategy assigning positive probability on s_j^* and s'_j. Further let the best responses of player 2 against σ_1 be the strategy σ_2. If $\sigma \in H(t', \sigma)$ then t is increased beyond t'. After a finite amount of these steps a Nash equilibrium of $\Gamma(1)$ is reached. In the following the code for computing Nash equilibria using the homotopy method is discussed.[17]

When implementing the described homotopy method into MATLAB one needs to consider some crucial parts. First and for most the theory is based on a continuous t which is not possible to cover in MATLAB. This yields a problem as player 1 is willing to randomize between two strategies if and only if they offer the exact same payoff against a given strategy of player 2.[18] It is impossible to ensure that this exact point is reached even if the step size with which t converges to 1 is chosen to be very small. Looking at the code, this issue is tackled in line 71 in which the strategies assigned with positive probability by player 1 are determined. Instead of considering the exact payoffs player 1 obtains given σ_2 and t, the rounded values are considered.[19] This makes mixed strategy equilibria more likely to occur. However in all 5 Games[20] that were considered the impact turned out to be rather small. Only in Game 3 starting with strategy s_A^1 and Game 5 starting with strategy s_B^1, the Homotopy function leads to an other Nash equilibrium than the Lemke-Howson

[17] The function I constructed in MATLAB is called "Homotopy".

[18] See Definition (6.3).

[19] Note that depending on the matrices A and B one might **not** want to round to the nearest integer.

[20] At the end of the code all tested Games are displayed.

algorithm.

Secondly one has to distinguish whether player 1 plays pure or mixed strategies. If player 1 plays a pure strategy, the complementary condition (2) is already fullfilled if player 2 plays his best response in pure strategies. However if player 1 assigns positive probability to more than one strategy, the complementary conditions turn out to be more complex. Looking at the code line 81 determines which type of strategy player 1 plays. If it is a pure strategy the best response of player 2 is defined in lines 157-158.

If not, the complementary conditions as for example displayed in lines 84-95 have to be considered. Let t' be the value of t for which player 1 wants to randomize. The strategies to which he assigns positive probability are already known. Let j be the rows of σ_1 with entries greater zero and a_j be the corresponding rows of A. It is enough to make sure that $u - a_j\sigma_2 = 0$ for all j, which yields a system of linear equations. Consider the following example: During the calculation of Game 1's Nash equilibrium starting with the strategy s_A^1 the following system of linear equations needs to be solved:

$$\underbrace{\begin{pmatrix} 1 & -2 & -1 \\ 1 & -1 & -4 \\ 0 & 1 & 1 \end{pmatrix}}_{AA} \sigma' = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{d}$$

Note that u is equal for each a_j such that it can be replaced by ones in AA. Additionally the last row of AA is needed to ensure that condition (5) is met. When solving the system of linear equations the MATLAB-Function *"lsqnonneg"* is used to ensure that condition (7) is met as well. Note that by replacing the u's with ones in AA the first entry of σ' will yield the actual value of u. The value of u is not needed for further calculations such that the first row of σ' must be deleted in order to get $\sigma' = \sigma_2$. This last step is done in line 111.

Not changing t' and given the just calculated σ_2 it is left to check whether player 1's pure best responses stay unchanged. This is done in lines 117-120. If they have changed it is necessary to calculate σ_2 and σ_1 again before increasing t any further.

5 Conclusion

In section (3.2) it is shown that in Game 2 the mixed strategy equilibrium σ_3^{G2} is never reached by the homotopy path. This claim is supported by the Homotopy function even though the function is more likely to find mixed strategy equilibrium in comparison to the theory as described above. Independent of the initially chosen strategy the computational process never reaches the mixed strategy equilibrium.

This might be an issue if one is interested in finding several Nash equilibria for a given game. In Herings and Peeters(2010) it is shown that a homotopy method based on the van den Elzen-Talmann algorithm[21] allows for arbitrary starting points. This feature is desirable as one might have a guess which probability distribution over S_1 and S_2 leads to a mixed strategy equilibrium. For example in Game 2 using van den Elzen-Talmann algorithm with starting point $((\frac{1}{2}),(\frac{1}{2}))$ might lead to the mixed strategy Nash equilibrium.

Lastly the assumption made in the beginning that the game has to be non-degenerate is rather strong. In addition it is involved to check for non-degeneracy in large games. But as claimed by Von Stengel(2002) using a so called lexicographic pivoting technique rather than complementary pivoting, allows for extending the Lemke-Howson algorithm to degenerate games. This implies that the Homotopy function has to be changed as well but nonetheless could be used to solve for Nash equilibria in degenerate games.

[21] See Van den Elzen and Talman(1991).

6 Appendix

Definition 6.1 ***Support***
The support of a mixed strategy $\sigma_i \in \Sigma$ is the set of pure strategies that are played with positive probability.

Definition 6.2 ***Non-degenerate***
A bimatrix game is called non-degenerate if the number of pure best responses to a given mixed strategy never exceeds the size of its support.

Definition 6.3 ***Best response***
In a normal form game $\Gamma = [I, \{S_i\}, \{u_i\}]$, strategy σ_i is a best response for player i against σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\tilde{\sigma}_i, \sigma_{-i}) \qquad \forall\, \tilde{\sigma}_i \in \Sigma_i.$$

Definition 6.4 ***Nash Equilibrium***
A strategy profile $(\sigma_1, ..., \sigma_I)$ is a mixed strategy Nash Equilibrium of the game $\Gamma' = [I, \{\Sigma_i\}, \{u_i\}]$ if for every $i = 1, ..., I$ and given σ_{-i}

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\tilde{\sigma}_i, \sigma_{-i}) \qquad \forall\, \tilde{\sigma}_i \in \Sigma_i.$$

Definition 6.5 ***Polyhedron***
A polyhedron $\mathbb{P} \subset \mathbb{R}^d$ is a set $\{z \in \ \mid Cz \leq q\}$ where z and q are vectors whereas C is some matrix. If $\mathbb{P}$ has dimension d it is called full-dimensional. If $\mathbb{P}$ is bounded it is called polytope.

- *A **face** of $\mathbb{P}$ is a set $\{z \in \mathbb{P} \mid c^T z = q_0\}$ for some $c \in \mathbb{R}^d$ and $q_0 \in \mathbb{R}$ for which holds that $c^T z \leq q_0 \quad \forall\, z \in \mathbb{P}$.*
- *An **edge** of $\mathbb{P}$ is a one dimensional face.*
- *A **vertex** of $\mathbb{P}$ is the unique element of a 0-dimensional face.*
- *A **facet** is a face of dimension $d-1$.*

*Note that one can show that any nonempty face of $\mathbb{P}$ can be obtained by turning some of the inequalities defining $\mathbb{P}$ into equalities. A facet is defined by a single binding inequality. The polyhedron $\mathbb{P}$ is called **simple** if no $z \in \mathbb{P}$ belongs to more than d facets, i.e. if there exists no dependency between facet defining inequalities.*

Definition 6.6 ***Upper hemi-continuous correspondence***
A correspondence $g : X \longrightarrow Y$ is a mapping which associates each $x \in X$ with a subset

$g(x)$ of Y. Note that $g(x)$ might consist of more than one element. The correspondence is upper hemi-continuous if it has a closed graph and if the image of g is compact.

Theorem 6.7 ***Nash(1951) p. 287***
The mixed strategy profile σ is a Nash equilibrium of Γ if and only if

$$\sigma_1^i > 0 \Rightarrow a_i\sigma_2 = \max_{k\in M} a_k\sigma_2 \quad \forall\, i = 1,...,m\ ,$$
$$\sigma_2^j > 0 \Rightarrow b_j\sigma_1 = \max_{k\in N} b_k\sigma_1 \quad \forall\, j = m+1,...,n.$$

Note that σ_1^i is the i-th row of σ_1 and therefore the probability assigned to player 1's i-th strategy. Similar for σ_2^j.

Theorem 6.8 ***Shapley(1974) p. 178***
A mixed strategy pair (σ_1, σ_2) constitutes a Nash equilibrium of the bimatrix game Γ if and only if it is completely labeled.

Theorem 6.9 ***Lemke and Howson(1964) p. 420***
Let Γ be a non-degenerated bimatrix game and s_i a label in S. Then the set of s_i-almost completely labeled vertices and edges in G consists of disjoint arcs and loops. The end-points of the arcs are the completely labeled vertices i.e. Nash equilibria, as well as the completely labeled additional vertex $(\underline{0}, \underline{0})$. The number of Nash equilibria of the game is odd.

Theorem 6.10 ***Von Stengel(2002) p. 1732***
In a non-degenerate $m \times n$ bimatrix game Γ, only finitely many points $\sigma_1 \in \Sigma_i$ have m labels and only finitely many points in Σ_2 have n labels.

Theorem 6.11 ***Mas-Colell(1974) p. 230***
Let Σ be a non-empty, compact, convex subset of $\mathbb{R}^d$ and let $H : [0,1] \times \Sigma \longrightarrow \Sigma$ be an upper hemi-continuous correspondence[22] that is non-empty and convex-valued. Then the set of fixed points, $F_H = \{(t, \sigma) \in [0,1] \times \Sigma \mid \sigma \in H(t, \sigma)\}$ contains a connected set, F_H^c, such that $(\{0\} \times \Sigma) \cap F_H^c \neq \emptyset$ and $(\{1\} \times \Sigma) \cap F_H^c \neq \emptyset$.

[22] See Definition (6.6).

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