# Introduction To The Local Discontinuous Galerkin Method

Michael Harmon

March 8, 2017

We discuss the local discontinuous Galerkin (LDG) method to approximate solutions of the Poisson equation:

$$\begin{aligned} -\nabla \cdot (\ \nabla u\ ) &= f(\mathbf{x}) && \text{in } \Omega, \\ -\nabla u \cdot \mathbf{n} &= g_N(\mathbf{x}) && \text{on } \partial\Omega_N \\ u &= g_D(\mathbf{x}) && \text{on } \partial\Omega_D. \end{aligned} \tag{1}$$

We first introduce some notation. Let $\mathcal{T}_h = \mathcal{T}_h(\Omega) = \{\,\Omega_e\,\}_{e=1}^{N}$ be the general triangulation of a domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, into $N$ non-overlapping elements $\Omega_e$ of diameter $h_e$. The maximum size of the diameters of all the elements is $h = \max(\,h_e\,)$. We define $\mathcal{E}_h$ to be the set of all element faces and $\mathcal{E}_h^i$ to be the set of all interior faces of elements which do not intersect the total boundary $(\partial\Omega)$. We define $\mathcal{E}_D$ and $\mathcal{E}_N$ to be the sets of all element faces and on the Dirichlet and Neumann boundaries respectively. Let $\partial\Omega_e \in \mathcal{E}_h^i$ be a interior boundary face element, we define the unit normal vector to be,

$$\mathbf{n} \;=\; \text{unit normal vector to } \partial\Omega_e \text{ pointing from } \Omega_e^- \;\rightarrow\; \Omega_e^+. \tag{2}$$

We take the following definition on limits of functions on element faces,

$$w^-(\mathbf{x})\,|_{\partial\Omega_e} \;=\; \lim_{s\rightarrow 0^-} w(\mathbf{x} + s\mathbf{n}), \qquad\qquad w^+(\mathbf{x})\,|_{\partial\Omega_e} \;=\; \lim_{s\rightarrow 0^+} w(\mathbf{x} + s\mathbf{n}). \tag{3}$$

We define the average and jump of a function across an element face as,

$$\{f\} \;=\; \frac{1}{2}(f^- + f^+), \qquad \text{and} \qquad [\![ f ]\!] \;=\; f^+\mathbf{n}^+ + f^-\mathbf{n}^-, \tag{4}$$

and,

$$\{\mathbf{f}\} \;=\; \frac{1}{2}(\mathbf{f}^- + \mathbf{f}^+), \qquad \text{and} \qquad [\![ \mathbf{f} ]\!] \;=\; \mathbf{f}^+ \cdot \mathbf{n}^+ + \mathbf{f}^- \cdot \mathbf{n}^-, \tag{5}$$

where $f$ is a scalar function and $\mathbf{f}$ is vector-valued function. We note that for a faces that are on the boundary of the domain we have,

$$[\![ f ]\!] \;=\; f\,\mathbf{n} \qquad \text{and} \qquad [\![ \mathbf{f} ]\!] \;=\; \mathbf{f} \cdot \mathbf{n}. \tag{6}$$

We denote the volume integrals and surface integrals using the $L^2(\Omega)$ inner products by $(\,\cdot\,,\,\cdot\,)_\Omega$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ respectively.

As with the mixed finite element method, the LDG discretization requires the Poisson equations be written as a first-order system. We do this by introducing an auxiliary variable which we call the current flux variable $\mathbf{q}$:

$$\nabla\cdot\mathbf{q}_n \;=\; f(\mathbf{x}) \qquad \text{in } \Omega, \tag{7}$$

$$\mathbf{q}_n \;=\; -\nabla u \qquad \text{in } \Omega, \tag{8}$$

$$\mathbf{q}_n\cdot\mathbf{n} \;=\; g_N(\mathbf{x}) \qquad \text{on } \partial\Omega_N, \tag{9}$$

$$u = g_D(\mathbf{x}) \qquad \text{on } \partial\Omega_D. \tag{10}$$

In our numerical methods we will use approximations to scalar valued functions that reside in the finite-dimensional broken Sobolev spaces,

$$W_{h,k} \;=\; \left\{ w\in L^2(\Omega)\;:\; w|_{\Omega_e}\in\mathcal{Q}_{k,k}(\Omega_e), \quad \forall\,\Omega_e\in\mathcal{T}_h \right\}, \tag{11}$$

where $\mathcal{Q}_{k,k}(\Omega_e)$ denotes the tensor product of discontinuous polynomials of order $k$ on the element $\Omega_e$. We use approximations of vector valued functions that are as,

$$\mathbf{W}_{h,k} \;=\; \left\{ \mathbf{w}\in\left(L^2(\Omega)\right)^d \;:\; \mathbf{w}|_{\Omega_e}\in\left(\mathcal{Q}_{k,k}(\Omega_e)\right)^d, \quad \forall\,\Omega_e\in\mathcal{T}_h \right\} \tag{12}$$

We seek approximations for densities $u_h\in W_{h,k}$ and gradients $\mathbf{q}_h\in\mathbf{W}_{h,k}$. Multiplying (7) by $w\in W_{h,k}$ and (8) by $\mathbf{w}\in\mathbf{W}_{h,k}$ and integrating the divergence terms by parts over an element $\Omega_e\in\mathcal{T}_h$ we obtain,

$$\begin{aligned} -\left(\nabla w\,,\,\mathbf{q}_h\right)_{\Omega_e} + \langle w\,,\,\mathbf{q}_h\rangle_{\partial\Omega_e} \;&=\; (w,\,f)_{\Omega_e}\,, \\ \left(\mathbf{w}\,,\,\mathbf{q}_h\right)_{\Omega_e} - \left(\nabla\cdot\mathbf{w}\,,\,u_h\right)_{\Omega_e} + \langle\mathbf{w}\,,\,u_h\rangle_{\partial\Omega_e} \;&=\; 0, \end{aligned}$$

Summing over all the elements leads to the **weak formulation**:

Find $u_h\in W_{h,k}$ and $\mathbf{q}_h\in\mathbf{W}_{h,k}$ such that,

$$-\sum_e\left(\nabla w,\,\mathbf{q}_h\right)_{\Omega_e} + \langle[\![\, w\,]\!]\,,\,\widehat{\mathbf{q}_h}\rangle_{\mathcal{E}_h^i} + \langle[\![\, w\,]\!]\,,\,\widehat{\mathbf{q}_h}\rangle_{\mathcal{E}_D\cup\mathcal{E}_N} \;=\; (w,\,f)_\Omega \tag{13}$$

$$\sum_e\left(\mathbf{w}\,,\,\mathbf{q}_h\right)_{\Omega_e} - \sum_e\left(\nabla\cdot\mathbf{w},\,u_h\right)_{\Omega_e} + \langle\,[\![\,\mathbf{w}\,]\!]\,,\,\widehat{u_h}\rangle_{\mathcal{E}_h^i} + \langle[\![\,\mathbf{w}\,]\!]\,,\,\widehat{u_h}\rangle_{\mathcal{E}_D\cup\mathcal{E}_N} \;=\; 0 \tag{14}$$

for all $(w,\mathbf{w})\in W_{h,k}\times\mathbf{W}_{h,k}$.

The terms $\widehat{\mathbf{q}_h}$ and $\widehat{u_h}$ are the numerical fluxes. The numerical fluxes are introduced to ensure consistency, stability, and enforce the boundary conditions weakly. The flux $\widehat{u_h}$ is,

$$\widehat{u_h} \;=\; \begin{cases} \{u_h\} \;+\; \boldsymbol{\beta}\cdot[\![u_h]\!] & \text{in } \mathcal{E}_h^i \\ u_h & \text{in } \mathcal{E}_N \\ g_D(\mathbf{x}) & \text{in } \mathcal{E}_D \end{cases} \tag{15}$$

The flux $\widehat{\mathbf{q}_h}$ is,

$$\widehat{\mathbf{q}_h} \;=\; \begin{cases} \{\mathbf{q}_h\} \;-\; [\![\mathbf{q}_h]\!]\boldsymbol{\beta} \;+\; \sigma\,[\![\, u_h \,]\!] & \text{in } \mathcal{E}_h^i \\ g_N(\mathbf{x})\,\mathbf{n} & \text{in } \mathcal{E}_N \\ \mathbf{q}_h \;+\; \sigma\,(u_h - g_D(\mathbf{x}))\;\mathbf{n} & \text{in } \mathcal{E}_D \end{cases} \tag{16}$$

The term $\boldsymbol{\beta}$ is a constant vector which does not lie parallel to any element face in $\mathcal{E}_h^i$. For $\boldsymbol{\beta} = 0$, $\widehat{\mathbf{q}_h}$ and $\widehat{u_h}$ are called the central or Brezzi et. al. fluxes. For $\boldsymbol{\beta} \neq 0$, $\widehat{\mathbf{q}_h}$ and $\widehat{u_h}$ are called the LDG/alternating fluxes. The term $\sigma$ is the penalty parameter that is defined as,

$$\sigma \;=\; \begin{cases} \tilde{\sigma}\,\min\left(h_{e_1}^{-1}, h_{e_2}^{-1}\right) & \mathbf{x} \in \langle \Omega_{e_1}, \Omega_{e_2} \rangle \\ \tilde{\sigma}\,h_e^{-1} & \mathbf{x} \in \partial\Omega_e \cap \in \mathcal{E}_D \end{cases} \tag{17}$$

with $\tilde{\sigma}$ being a positive constant.

We can now substitute (15) and (16) into (13) and (14) to obtain the solution pair $(u_h, \mathbf{q}_h)$ to the semi-discrete LDG approximation to the drift-diffusion equation given by:

Find $u_h \in W_{h,k}$ and $\mathbf{q}_h \in \mathbf{W}_{h,k}$ such that,

$$\begin{aligned} a(\mathbf{w}, \mathbf{q}_h) \;+\; b^T(\mathbf{w}, u_h) \;&=\; G(\mathbf{w}) \\ b(w, \mathbf{q}_h) \;+\; c(w, u_h) \;&=\; F(w) \end{aligned} \tag{18}$$

for all $(w, \mathbf{w}) \in W_{h,k} \times \mathbf{W}_{h,k}$. This leads to the linear system,

$$\begin{bmatrix} A & -B^T \\ B & C \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{Q} \end{bmatrix} \;=\; \begin{bmatrix} \mathbf{G} \\ \mathbf{F} \end{bmatrix} \tag{19}$$

Where $\mathbf{U}$ and $\mathbf{Q}$ are the degrees of freedom vectors for $u_h$ and $\mathbf{q}_h$ respectively. The terms $\mathbf{G}$ and $\mathbf{F}$ are the corresponding vectors to $G(\mathbf{w})$ and $F(w)$ respectively. The matrix in for the LDG system is non-singular for any $\sigma > 0$.

The bilinear forms in (18) and right hand functions are defined as,

$$b(w, \mathbf{q}_h) \;=\; -\sum_e (\nabla w, \mathbf{q}_h)_{\Omega_e} + \langle [\![w]\!], \{\mathbf{q}_h\} - [\![\mathbf{q}_h]\!]\boldsymbol{\beta} \rangle_{\mathcal{E}_h^i} + \langle w, \mathbf{n} \cdot \mathbf{q}_h \rangle_{\mathcal{E}_D} \tag{20}$$

$$a(\mathbf{w}, \mathbf{q}_h) \;=\; \sum_e (\mathbf{w}, \mathbf{q}_h)_{\Omega_e} \tag{21}$$

$$-b^T(w, \mathbf{q}_h) \;=\; -\sum_e (\nabla \cdot \mathbf{w}, u_h)_{\Omega_e} + \langle [\![\mathbf{w}]\!], \{u_h\} + \boldsymbol{\beta} \cdot [\![u_h]\!] \rangle_{\mathcal{E}_h^i} + \langle w, u_h \rangle_{\mathcal{E}_N} \tag{22}$$

$$c(w, u_h) \;=\; \langle [\![w]\!], \sigma [\![u_h]\!] \rangle_{\mathcal{E}_h^i} + \langle w, \sigma u_h \rangle_{\mathcal{E}_D} \tag{23}$$

$$G(\mathbf{w}) \;=\; -\langle \mathbf{w}, g_D \rangle_{\mathcal{E}_D} \tag{24}$$

$$F(w) \;=\; (w, f) - \langle w, g_N \rangle_{\mathcal{E}_N} + \langle w, \sigma g_D \rangle_{\mathcal{E}_D} \tag{25}$$

For more information, see the deal.ii code-gallery website.