One of my fun projects in mathematics
Project Report

NIGHT & DAY

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1 Abstract

The objective of this project was to build a simple mathematical model that would calculate the length of daylight given a certain parallel and time of the year. By simple I mean a model that uses basic calculus and trigonometry and does not require advance knowledge in physics nor astronomy, just some basic facts and imagination.

2 Introduction

The idea is to build a deterministic model that calculates the length of daylight on a particular day. Also, my intention was to keep it simple whenever possible but not to over simplify.

If one decided to be very precise, a lot would have to be taken into consideration such as the elliptic trajectory of Earth around the sun, different speed along the trajectory and the fact that Earth moves along it's trajectory during a single day just to name a few. To avoid over complication but still get some good estimates of the daylight length I settled for the following assumptions:

- Earth is a perfect ball in R^3.
- Earth revolves around the sun along a circle with constant speed.
- Earth is tilted to one side at a constant angle.
- I do not combine the Earth's spin about it's north-south pole axis and it's revolution around the sun.

3 Project Idea

To model the length of daylight we place a sphere representing Earth in the xyz plane, so that the center of the sphere is at point $(0, 0, 0)$. Next, we draw a North-South axis through the sphere (in red in figure 1) and rotate the sphere by $23.5°$ about the y-axis. This is because Earth revolves around a tiled axis.

Imagine sun rays are parallel to the x-axis, shining from positive to negative x. In such a situation any point on the surface of the sphere for which $x > 0$ will be lighted. In other words all points with a positive signed distance to the plane given by normal vector $(1, 0, 0)$ are lighted. If we keep the normal direction of sunlight fixed and examine a parallel on the Northern hemisphere, as presented in figure 1 we can observe that the larger, black part of the parallel, is to the left of the plane $x = 0$, while as the orange part is smaller. This corresponds to the the night being longer than day. This situation represents the longest night and shortest day in the northern hemisphere and vice verse on the southern hemisphere which happens on the 22nd of December.

We can now start changing the direction of the sun rays, at constant rate in time (such that a full circle takes place during one year), in such a way that after some time Δt the lightened part of Earth will be given by all points on the positive side of the plane given by $cos(\Delta t)x + sin(\Delta t)y = 0$. Note the if $\Delta t = 0$ then we are back at $x = 0$. In other words, we model the route of Earth around the sun not by moving the Earth in the xyz coordinate system, as that would be rather complicated, Instead, we assume that the plane parallel to the z-axis and passing through the origin rotates steadily, covering a full 360° in one year.

Summarizing, what we need is, to take a sphere, fix a parallel, tilt the sphere by 23.5°, decide upon a time (day) of the year by choosing some $t \in [0, 2\pi]$ (remembering that $t = 0$ corresponds to the winter solstice) and check what part of the parallel is on the positive side of the plane given by a normal vector $(cos(t), sin(t), 0)$

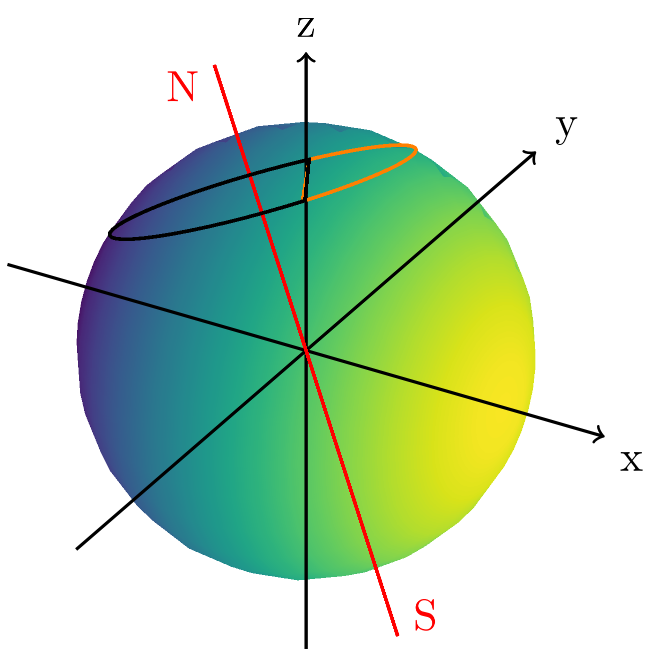

Figure 1: Model of Earth

There is one more tiny adjustment that we will make to the above described model. Although we assume for simplicity that Earth's orbit around the sun is a perfect circle, it turns out that the sun is not placed in the middle of this circle (see [1]). The sun is shifted to one side like in figure 2 (please note that the figure exaggerates grossly the size of the

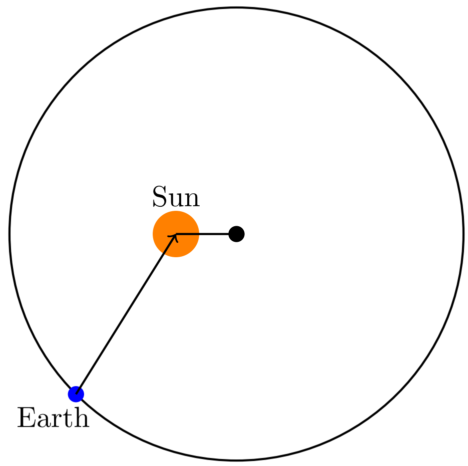

Figure 2: Sun shifted from the middle of the orbit

shift) On the 22^{nd} of December the sun is about 91 400 miles from Earth and on June 21^{st} it is 94 500 miles away. Taking this into account and rescaling yields a normal vector of the plane dividing night and day equal to $(cos(t) - 0.0166756, sin(t), 0)$.

4 Project Implementation

4.1 Polar coordinates

It will be convenient to assume that Earth's radius is equal to 1. This is just rescaling and will have no effect on daylight time calculations. We represent every point on Earth using so called polar coordinates. First we fix an angle $\alpha \in (0, 2\pi]$ which gives the direction on a xy plane. Next, we fix a latitude angle $\gamma \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ which informs how the point of consideration is directed above or below the xy plane. The third polar coordinate $\rho > 0$ would provide the distance of the point from the origin. Since we are considering only points on Earth's surface we will fix $\rho = 1$ and any point can be represented by $, \gamma$.

A parallel is a circle on the Earth's surface with the same latitude, i.e. a fixed value of γ. So any parallel may be parametrized by :

$$(0, 2\pi] \ni \alpha \to (\alpha, \gamma, 1)$$

However, we will eventually want to perform calculations in regular coordinates, so we need to know how to transform polar into regular coordinates. Using elementary trigonometry we can see that going from polar to regular coordinates is given by:

$$(\alpha, \gamma, 1) \to (cos(\alpha)cos(\gamma), sin(\alpha)cos(\gamma), sin(\gamma))$$

and a fixed parallel (γ is fixed) is consequently parametrized in xyz coordinates by:

$$(0, 2\pi] \ni \alpha \to (\alpha, \gamma, 1) \to (cos(\alpha)cos(\gamma), sin(\alpha)cos(\gamma), sin(\gamma))$$

4.2 Declination

Since Earth revolves about a tilted north pole - south pole axis, we need to tilt, or geometrically rotate the sphere (with our fixed parallel) by $\beta = 23.5°$ about the y axis. This can be accomplished by applying the following matrix transformation to any point (x, y, z) as so:

$$\begin{pmatrix} cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & cos(\beta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Applying the above rotation to a parallel we get

$$\begin{pmatrix} cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & cos(\beta) \end{pmatrix} \begin{pmatrix} cos(\alpha)cos(\gamma) \\ sin(\alpha)cos(\gamma) \\ sin(\gamma) \end{pmatrix} = \begin{pmatrix} cos(\alpha)cos(\gamma)cos(\beta) - sin(\beta)sin(\gamma) \\ sin(\alpha)cos(\gamma) \\ cos(\alpha)cos(\gamma)sin(\beta) + cos(\beta)sin(\gamma) \end{pmatrix} \quad (1)$$

4.3 Daylight calculation

As mentioned in the project idea section, starting from 22^{nd} od December which corresponds to time $t = 0$, the lighted part of Earth by the sun is given by points lying on the positive side of a plane given by the normal vector $(cos(t) - e, sin(t), 0)$ where t ranges from 0 to 2π. and $e = 0.0166756$ is the correction that we apply to compensate the fact that the sun is miscentered.

If we number days from 1 to 365, where 1 corresponds to the 22^{nd} od December and we choose a day $1 \leq n \leq 365$ for which we wish to calculate the length of daylight (given some parallel determined by γ), we really need to check the proportion of $[0, 2\pi)$ (remember that a parallel is parametrized by α ranging from 0 to 2π) that results in points with a positive signed distance from the plane given by normal vector $\vec{N_t} = (cos(\frac{2n\pi}{365}) - e, sin(\frac{2n\pi}{365}), 0)$.

Mathematically, to check if the singed distance of a point on the parallel - call it $\vec{p_\alpha}$ - is positive it is enough to check if the standard inner product is positive:

$$\vec{p_\alpha} \cdot \vec{N_t} > 0 \quad (2)$$

Now we need to plug in what we have in (1) for $\vec{p_\alpha}$ and $(cos(\frac{2n\pi}{365})-e, sin(\frac{2n\pi}{365}), 0)$ for $\vec{N_t}$. This yields:

$$\begin{pmatrix} cos(\alpha)cos(\gamma)cos(\beta) - sin(\beta)sin(\gamma) \\ sin(\alpha)cos(\gamma) \\ cos(\alpha)cos(\gamma)sin(\beta) + cos(\beta)sin(\gamma) \end{pmatrix} \left(cos(\tfrac{2n\pi}{365}) - e, sin(\tfrac{2n\pi}{365}), 0\right) > 0 \tag{3}$$

Multiplying out (3) and letting

$$\alpha = \frac{2n\pi}{365}$$

$$A = cos(\gamma)cos(\beta)(cos(\alpha) - e)$$

$$B = cos(\gamma)sin(\alpha)$$

$$C = sin(\beta)sin(\gamma)(cos(\alpha) - e)$$

we obtain a basic inequality

$$Acos(\alpha) + Bsin(\alpha) > C \tag{4}$$

Solving (4) for α, limited to $[0, 2\pi)$ yields an interval J.Denoting $m(J)$ as the length of interval J , we can calculate the proportion of daylight as $\frac{m(J)}{2\pi}$ or we can set the formula for daylight length in hours as

$$\text{Daylight} = 24 \cdot \frac{m(J)}{2\pi} \tag{5}$$

4.4 Solving inequality (4)

In order to calculate the length of daylight we need to solve (4). If exactly one out of A, B is equal zero, then we need to solve either:

- $sin(x) > C$ for $1 > C > 0$ has a solution limited to $[0, 2\pi)$ given by $x \in [\arcsin(C), \pi - \arcsin(C)]$. For $-1 < C < 0$ the solution interval is $[0, \pi] \cup [\pi, \pi - \arcsin(C)] \cup [2\pi + arcisn(C), 2\pi]$. In either case this yields an interval of solutions of length $\pi - 2 \cdot arcsin(C)$

- $cos(x) > C$ yields $x \in [0, \arccos(C)] \cup (2\pi - \arccos(C), 2\pi)$ for $|C| < 1$ which is as an interval with length equal to $2 \cdot \arccos(C)$.

It is important to note, that if $\phi \in [-\pi, \pi]$ then $cos(x - \phi) > C$ also yields a length of the interval of solutions equal to $2 \cdot \arccos(C)$. To see this we need to consider the set of solutions $[\phi - arccos(x), \phi + arccos(x)]$ and observe that when $\phi - arccos(x) < 0$ then we can wrap around 2π, i.e. $[2\pi + \phi - arccos(x), 2\pi]$ and $[0, \phi + arccos(x)]$ are valid intervals of solutions with a total length equal to $2arccos(\phi)$. A similar argument works for the case when $\phi + arccos(x) > 2\pi$

Now we tackle the general case of inequality (4) where $A \neq 0, B \neq 0$.

Draw a point (A, B) on the plane. Let ϕ be the angle in the range $[-\pi, \pi]$ representing the slope of the the vector (A, B). Then we have

$$cos(\phi) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$sin(\phi) = \frac{B}{\sqrt{A^2 + B^2}}$$

From this we can easily obtain:

$$\sqrt{A^2 + B^2}cos(\phi)cos(\alpha) = Acos(\alpha)$$

$$\sqrt{A^2 + B^2}sin(\phi)sin(\alpha) = Bsin(\alpha)$$

Add both sides together and use the fact that $\cos(\phi)cos(\alpha) + sin(\phi)sin(\alpha) = cos(\alpha - \phi)$:

$$Acos(\alpha) + Bsin(\alpha) = \sqrt{A^2 + B^2}cos(\alpha - \phi)$$

Hence, solving equation (4) comes dons to solving

$$cos(\alpha - \phi) > \frac{C}{\sqrt{A^2 + B^2}}$$

but we already showed that the length of the solution of the later equation is $2 \cdot \arccos(\frac{C}{\sqrt{A^2+B^2}})$.

5 Sanity Testing

We now have a mathematical (geometrical model) and will run a few of sanity tests, just to make sure it behaves as one would expect.

Sanity test cases:

1. The length of day at the equator should be constantly 12 hours.
2. The length of day at the north and south pole should be either 0 or 24 hours through out the year
3. Taking a parallel above the arctic circle (say latitude of 75 degrees) should result in fractions of the year with 24 hour daylight in summer and 0 daylight in the winter
4. If we take a parallel close to the equator (say 10 degrees) the oscillations around a 12 hour daylight should be relatively small

Having implemented a function that calculates daylight through out the year for given latitudes I generated plots for the the cases above and eyeballing them confirms that the model passes sanity tests.

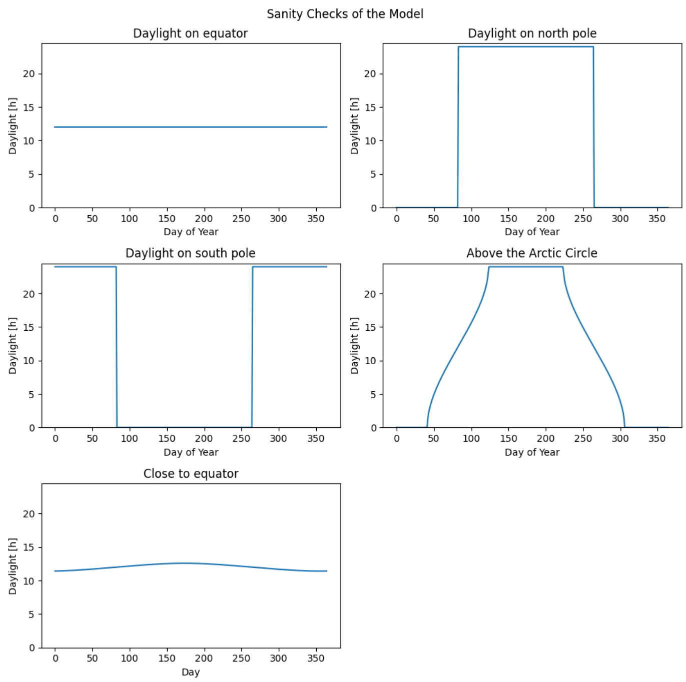

Figure 3: Sanity Tests of the Model

6 Data Analysis

Now we will compare the daylight returned by the model with some real data for two different cites - Warsaw lying on the northern and Sydney on the southern hemisphere. Please check figures 4 and 5.

In both cases the model is slightly off. Bigger errors occur in case of Warsaw with a maximum difference of daylight time reaching 28 minutes and a RMSE (root mean square error) of about 20 minutes.

The model does better on Sydney with a maximum difference of daylight time reaching 11.6 minutes and a RMSE of 7.5 minutes.

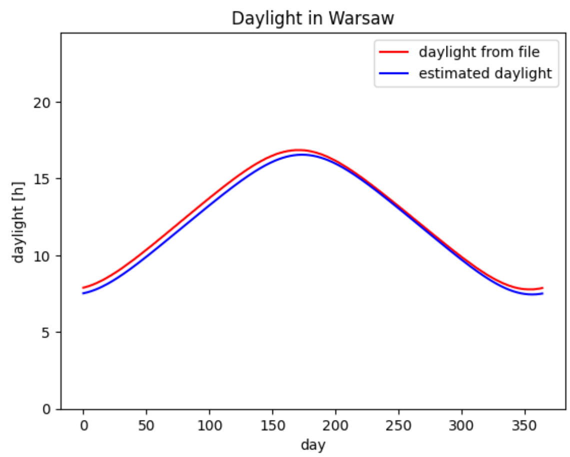

Figure 4: Modeled vs Internet found daylight in Warsaw

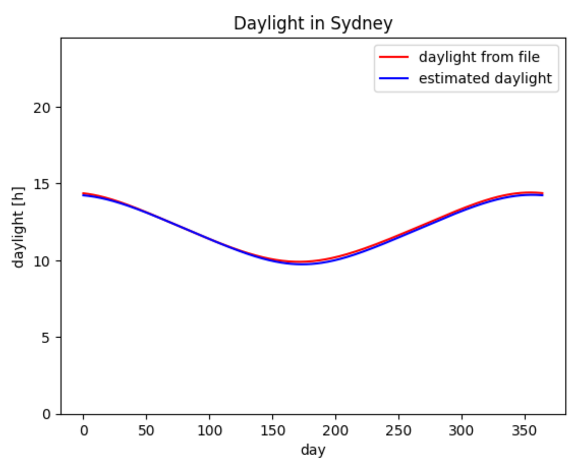

Figure 5: Modeled vs Internet found daylight in Sydney

7 Conclusion

Clearly the model is not perfect and there are much more accurate ways of calculating daylight. However, this model relies only on basic calculus and some analytical geometry typical for advanced high school students or first year STEM students. It doesn't exploit any physical or serious astronomical facts, that may not be easy to comprehend for those with little experience in those fields.

Despite it's shortcomings the model shows well, how daylight behaves in different latitudes including the equator and the poles. It also gives a good feeling how daylight is changes slowly near the winter and summer solstices and the rate of change becomes high near closer to the equinoxes.

8 Acknowledgment

I just want to thank myself for the stupor and determination to complete this kind of silly project.

References

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[2] J. Smith and J. Doe, "Title of the report," Department of Primary Industries and Regional Development, Western Australia, Tech. Rep. RMTR-1122, 2024, accessed: 2025-02-07. [Online]. Available: https://library.dpird.wa.gov.au/cgi/viewcontent.cgi?article=1122context=rmtr