Finding the first r value where bifurcations begin

logistic map:

$$x_{n+1} = r x_n (1 - x_n) = f(x_n) \quad (1) \quad (r > 0)$$

Locate fixed point x'

$$x' = r x' (1 - x')$$

we see $x' = 0$ is a fixed point

Considering $x' \neq 0$

$$\cancel{x'} = \cancel{x'} r (1 - x')$$

$$1 = r(1 - x')$$

$$\frac{1}{r} = 1 - x'$$

$$x' = 1 - \frac{1}{r} \quad \text{for} \quad r > 1$$

Now we consider checking for stability for r

we say that $x_n = x' + \delta_n$ (2)

s.t. $|\delta_n| << 1$

With this we consider the linearization of the map

$$f(x_n) = f(x' + \delta_n)$$

Using Taylor series approximation:

$$f(x_n) \simeq f(x') + \delta_n f'(x')$$

Since x' is a fixed point: $f(x') = x'$

and using ① we can rewrite as:

$$x_{n+1} = x' + \delta_n f'(x')$$

now using ② we say $x_{n+1} = x' + \delta_{n+1}$ so

$$\cancel{x'} + \delta_{n+1} = \cancel{x'} + \delta_n f'(x')$$

Cancelling x' gives

$$\delta_{n+1} = f'(x')\,\delta_n \quad ③$$

Recall $f(x) = rx(1-x)$

so $f'(x) = r(1-2x)$

we know that $x' \in \{0, 1-\frac{1}{r}\}$

Substituting $x'=0$ into ③ we obtain:

$$\delta_{n+1} = r(1-0)\,\delta_n$$

$$\delta_{n+1} = r\,\delta_n \quad ④$$

Substituting $x' = 1-\frac{1}{r}$ into ③ we obtain:

$$\delta_{n+1} = r\left(1-2+\frac{2}{r}\right)\delta_n$$

$$\Rightarrow \delta_{n+1} = (2-r)\,\delta_n \quad ☆☆$$

Now to define stability:

a linear recurrence relation U is stable if and only if the sequence converges to zero

i.e. $U_{n+1} = k U_n$

if $|k| \geq 1$ the sequence diverges hence it is not stable
($k=1 \to$ critical, $k>1 \to$ unstable)

if $|k|<1$ the sequence converges hence it is stable

Using this definition to find the range of r values for which the logistic map is stable

Firstly result obtained from ☆

$$\delta_{n+1} = r\,\delta_n$$

$$|r| < 1$$

$$r < -1 \qquad r < 1$$

$\Rightarrow$ Stable when $0 < r < 1$ (recall $r>0$)

Secondly using ☆☆

$$\delta_{n+1} = (2-r)\,\delta_n$$

$$|2-r| < 1$$

$2-r<-1$ $\qquad\qquad$ $2-r<1$

$\Rightarrow$ Stable when $1<r<3$

So we can see that at $r=1, r=3$ we have instability this corresponds to when bifurcations start.

So the bifurcations start at $r=1$, $r=3$