The Error Function

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Abstract

In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\,dt$. In statistics, for nonnegative values of x, the error function has the following interpretation: for a random variable Y that is normally distributed with mean 0 and variance 1/2, erf(x) describes the probability of Y falling in the range [-x, x].

1 The name "error function"

The name and abbreviation for the error function (and the error function complement) were developed by J. W. L. Glaisher in 1871 on account of its connection with "the theory of Probability, and notably the theory of Errors."

2 Properties

The property $\mathrm{erf}(-z) = \mathrm{erf}(z)$ means that the error function is an odd function. This directly results from the fact that the integrand e^{-t^2} is an even function.

For any complex number z:

$$\mathrm{erf}(\bar{z}) = \overline{\mathrm{erf}}(z) \tag{1}$$

where $\bar{z}$ is the complex conjugate of z.

The error function at $+\infty$ is exactly 1 (see Gaussian integral). At the real axis, $\mathrm{erf}(z)$ approaches unity at $z \to \infty$ and -1 at $z \to -\infty$. At the imaginary axis, it tends to $\pm i\infty$.

The error function is an entire function; it has no singularities (except that at infinity) and its Taylor expansion always converges.

The shape of the error function can be seen in Figure 1.

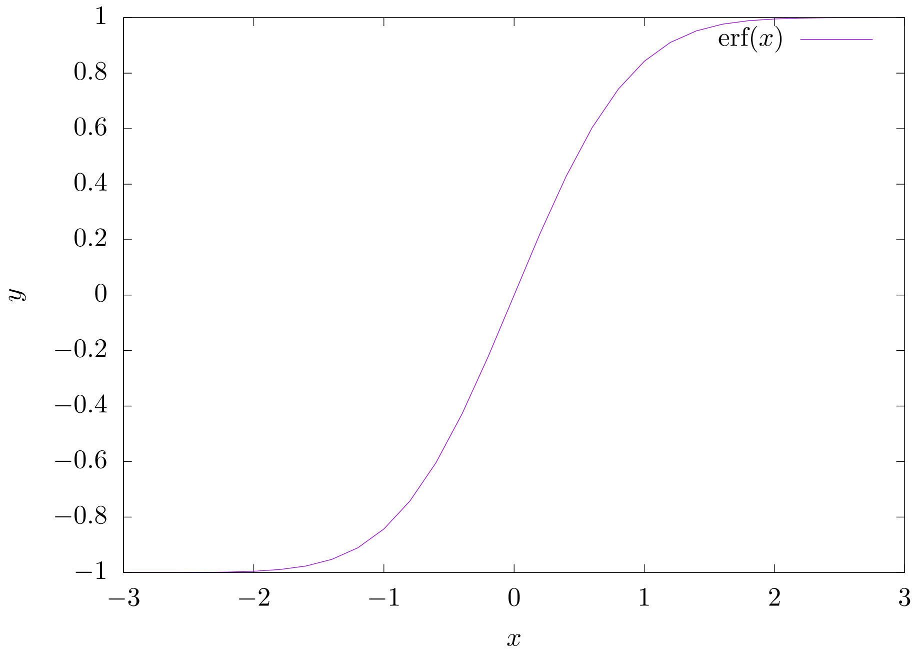

Figure 1: A plot of the error function calculated from the differential equation.