CLASS

Cosmological Linear Anisotropy Solving System

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Visit http://class-code.net/ for more info!

`class` in Les Karellis

What to expect in this *advanced* lecture:

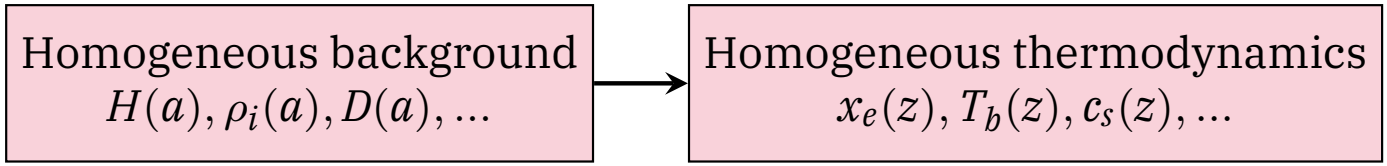
- Theory: What is `class` based upon?
- Coding: Structure of `class`
- Coding: Essential rules and conventions
- Coding: Implementing features (C and python)
- Coding: Using MontePython/Cobaya with `class`

We will learn the theory behind `class` and the fundamental rules of its code base.

class Theory

1. Fundamental layout of Einstein-Boltzmann solvers
2. Essential steps for each module
3. A few details for each of these steps

Fundamental layout of Einstein-Boltzmann solvers



Fundamental layout of Einstein-Boltzmann solvers

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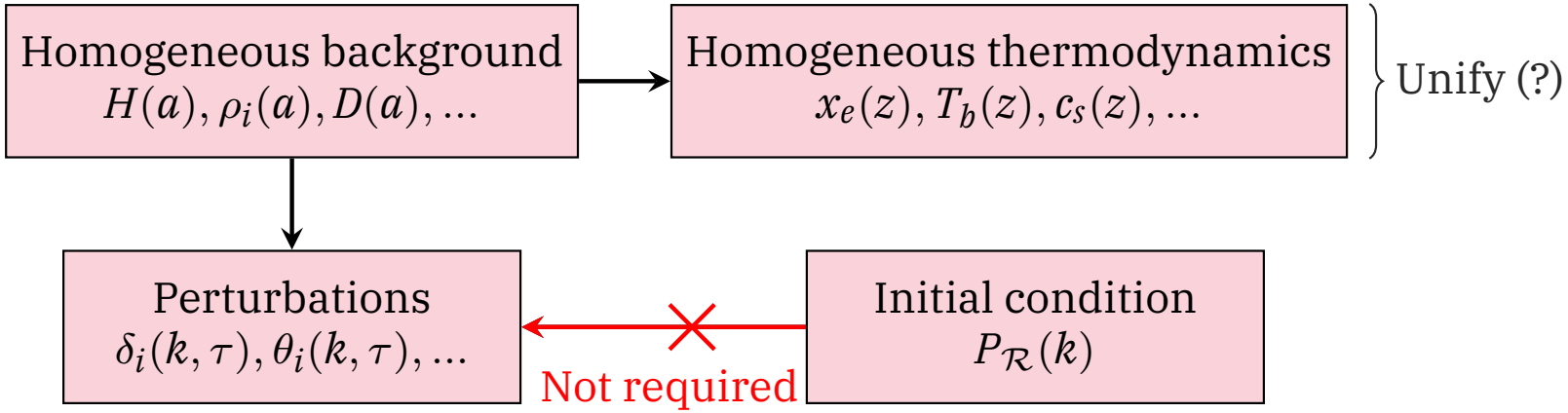

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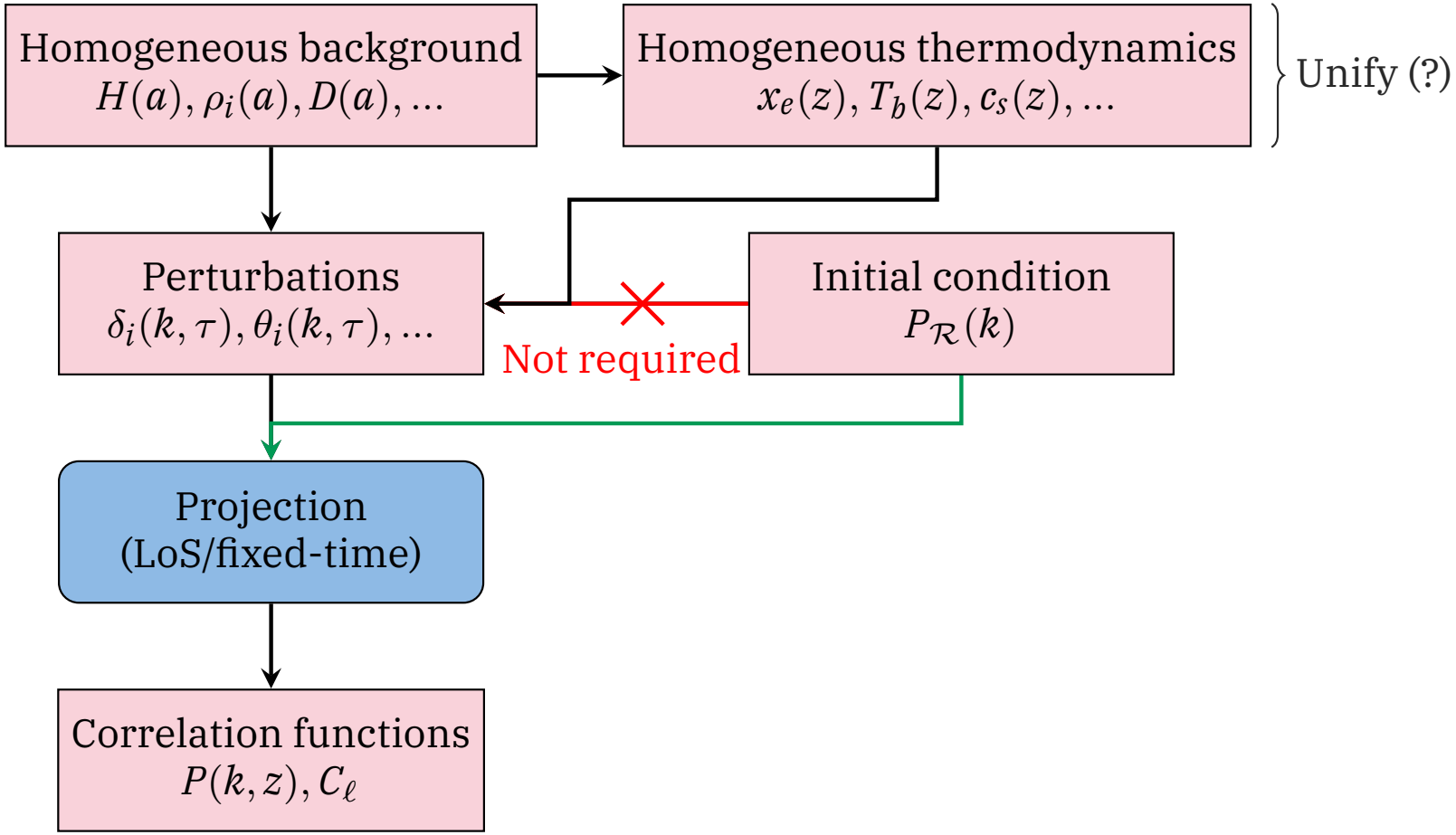

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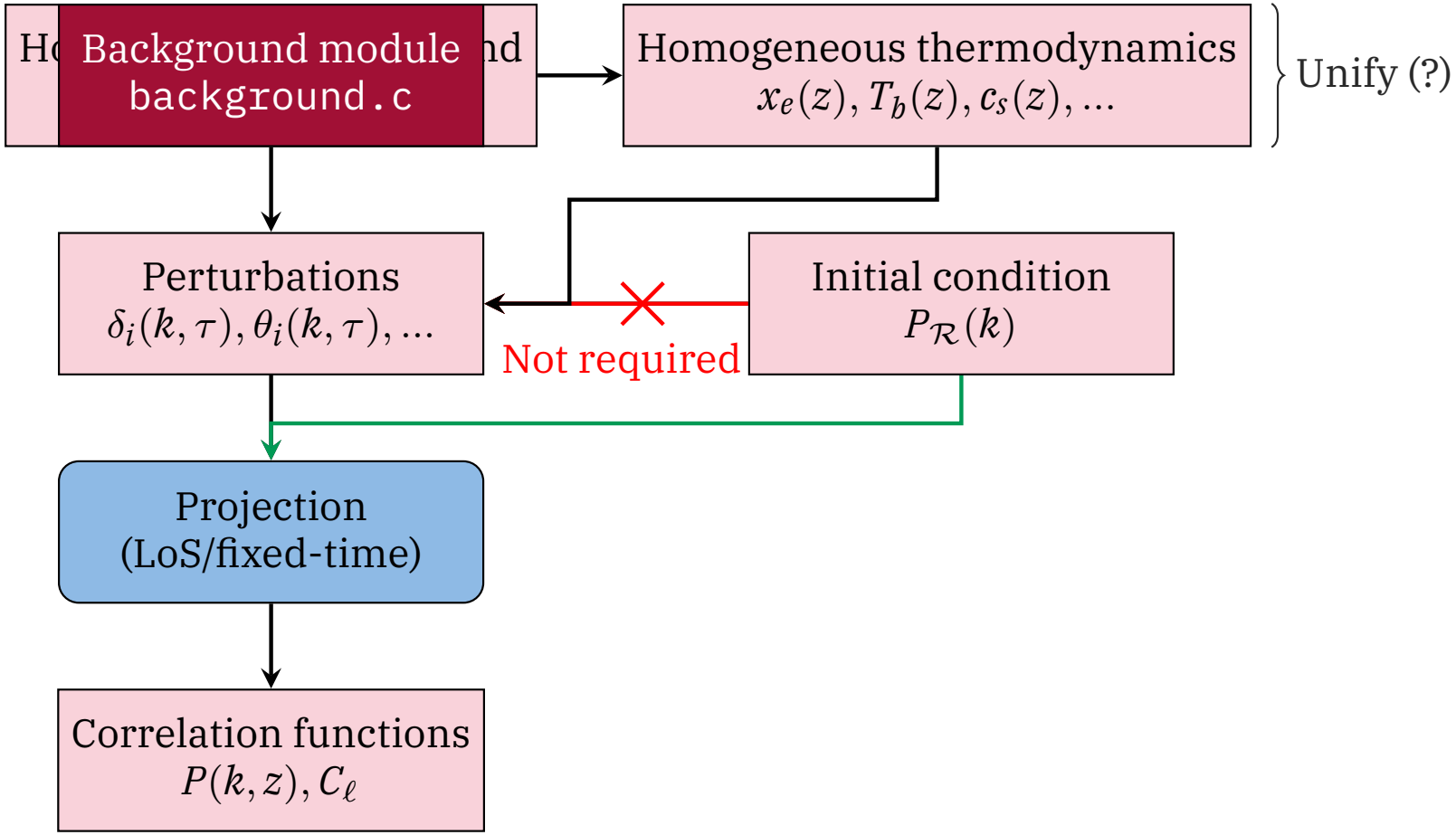

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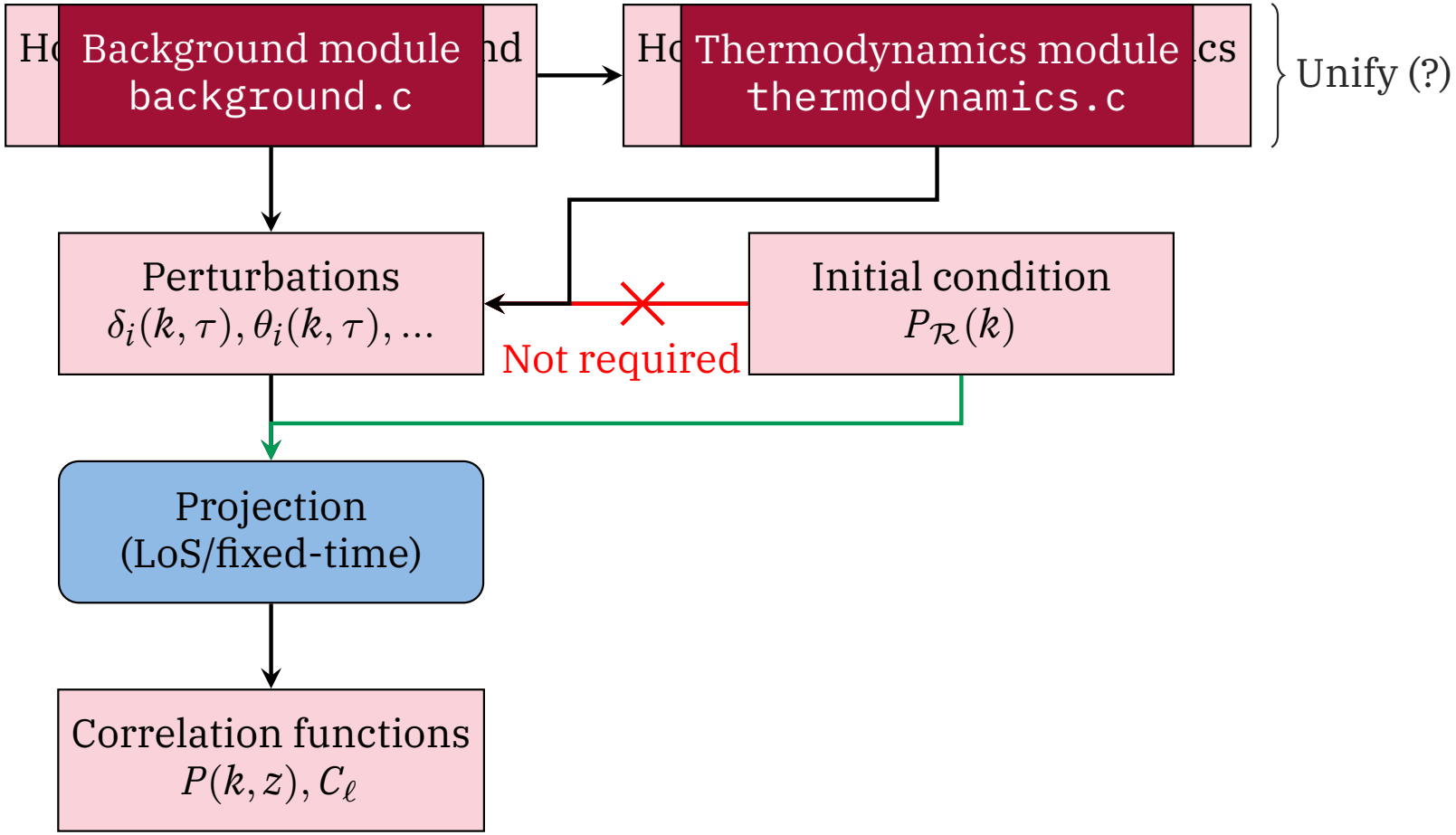

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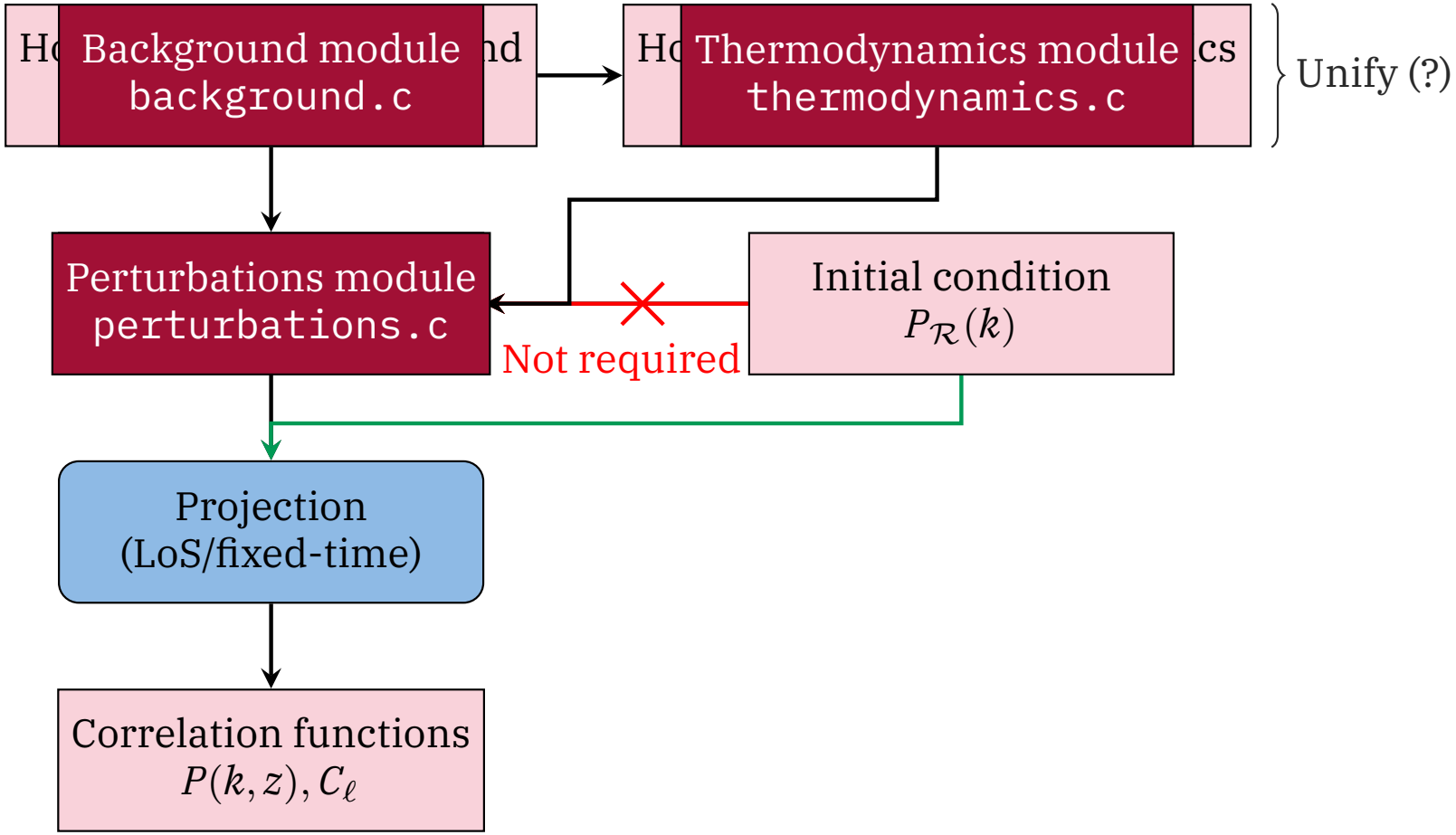

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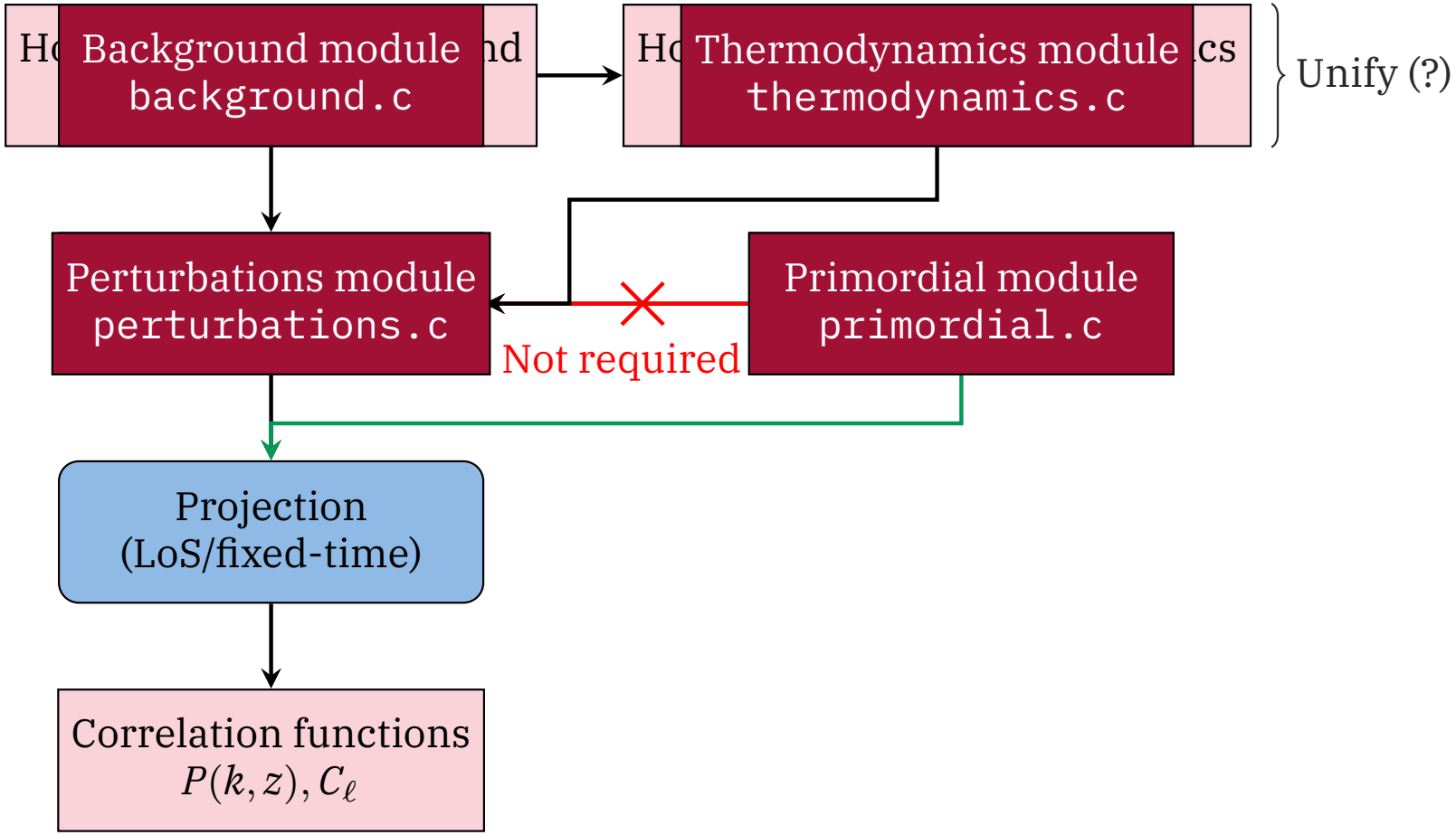

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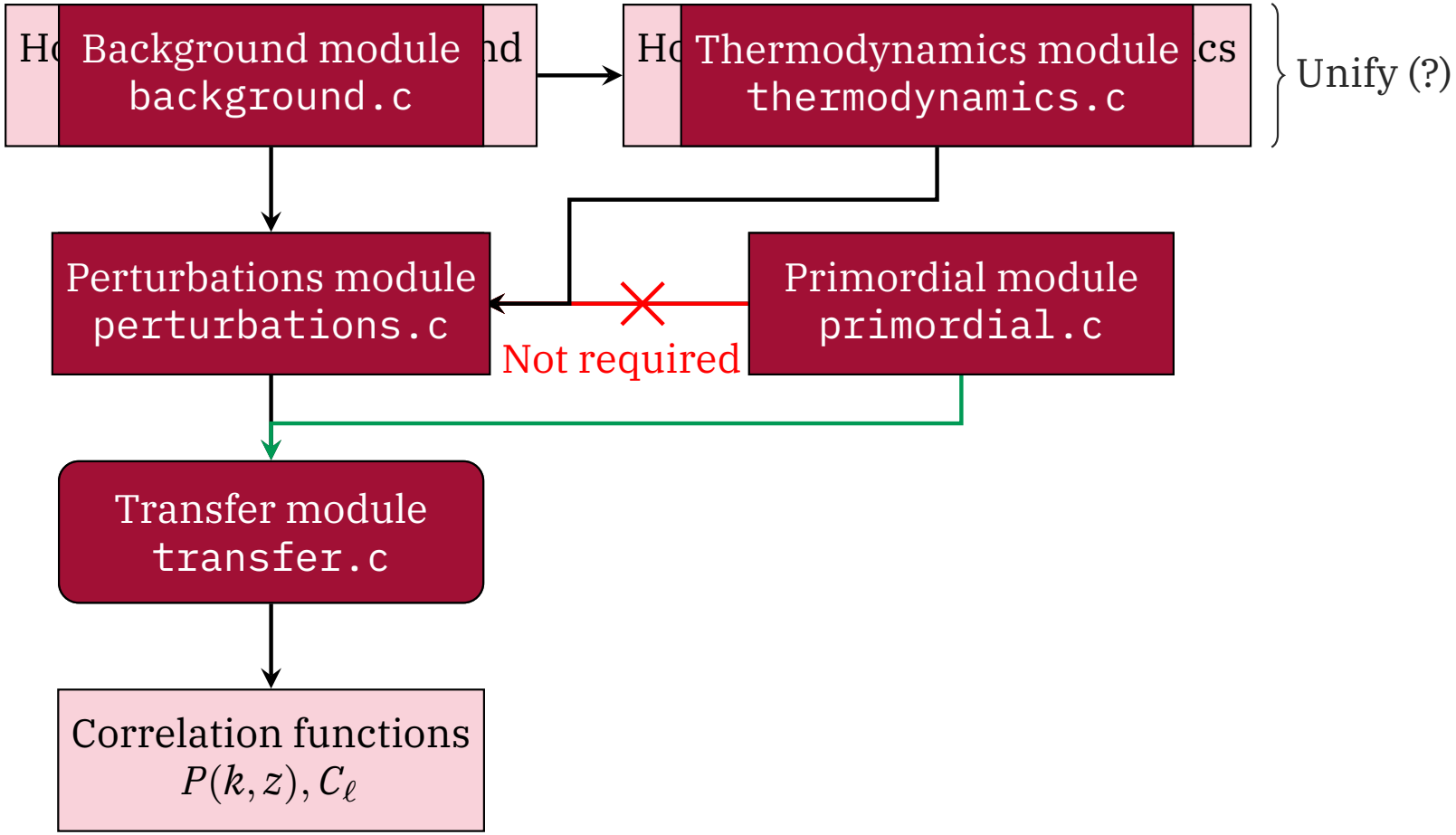

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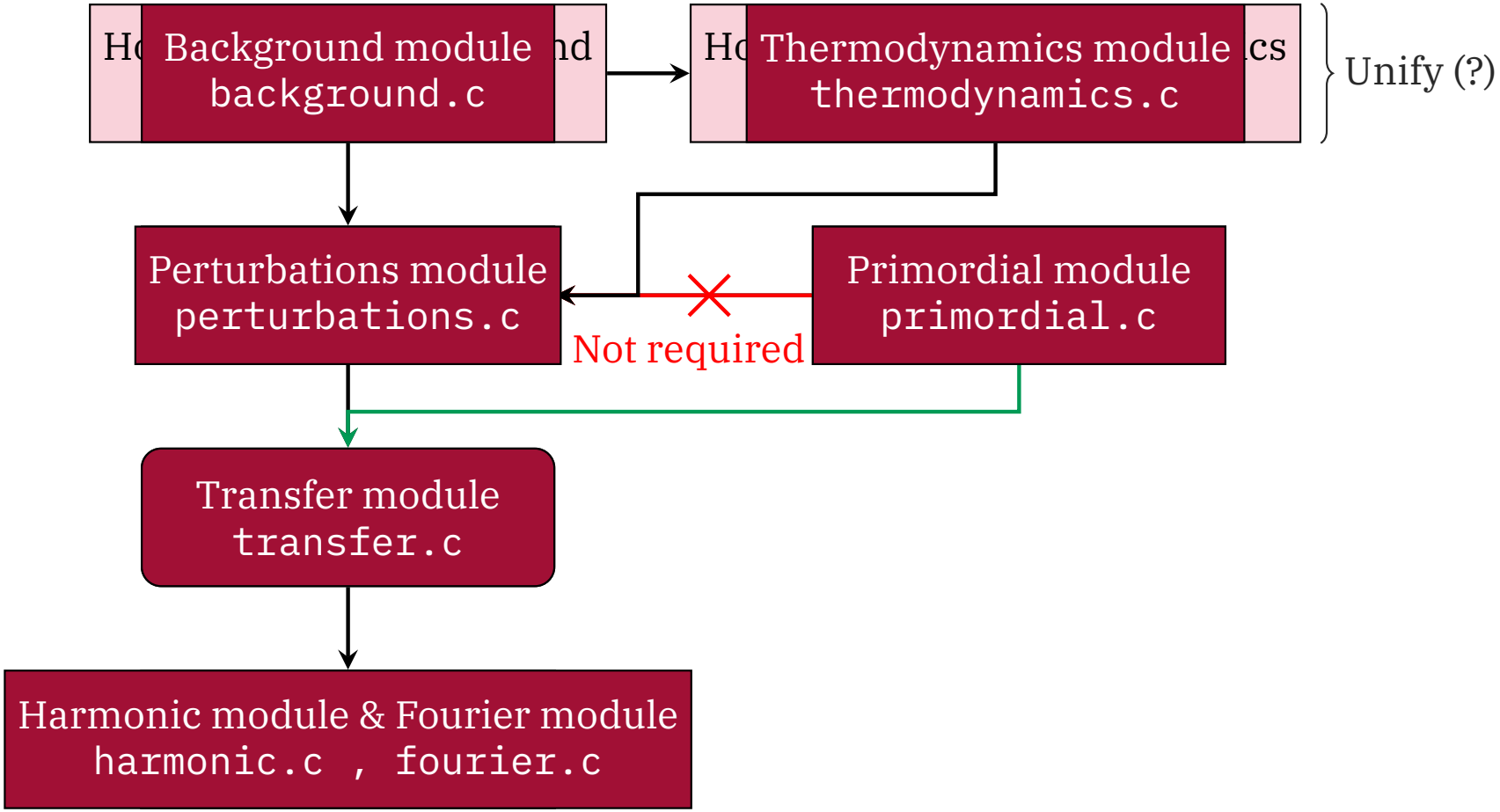

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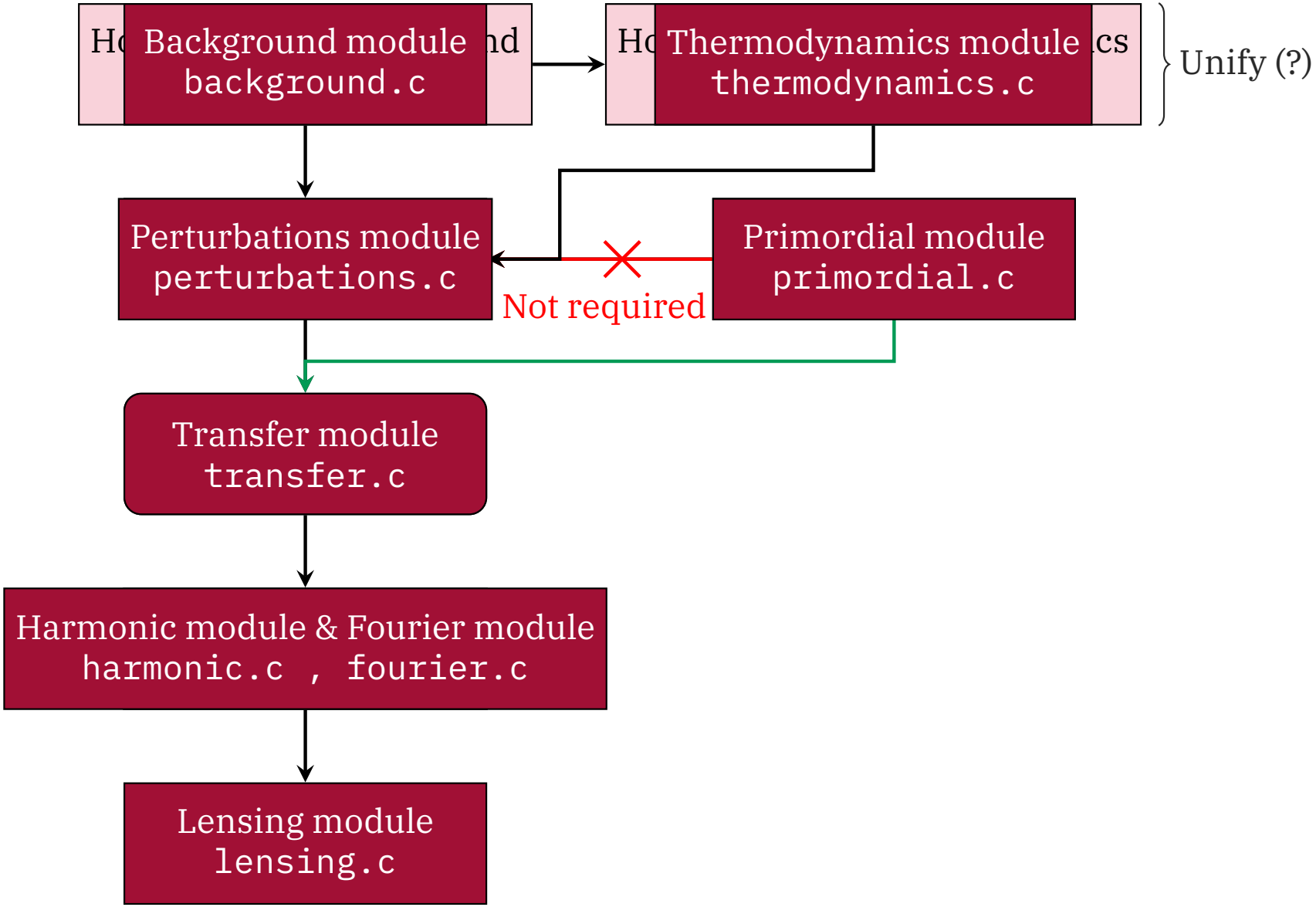

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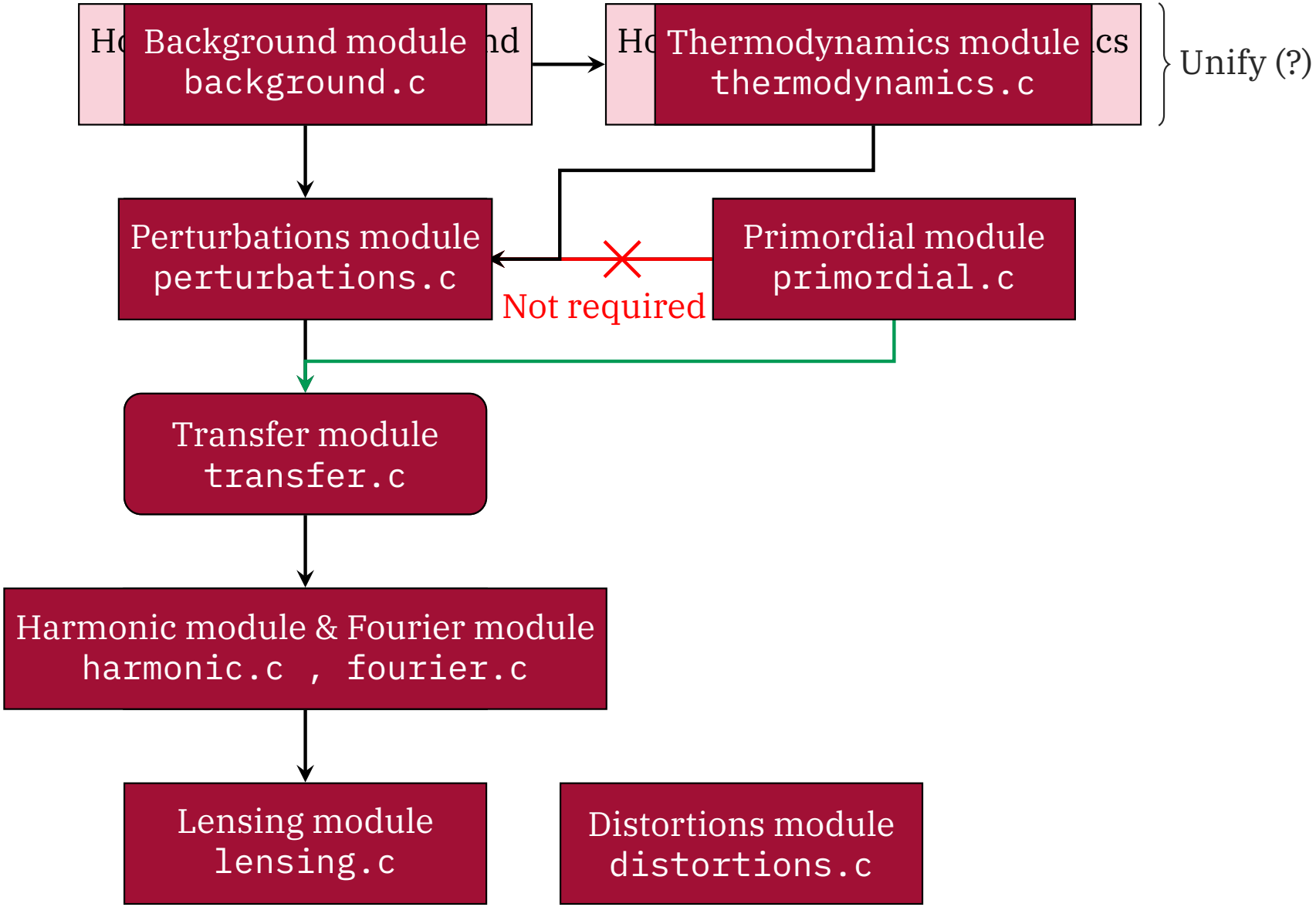

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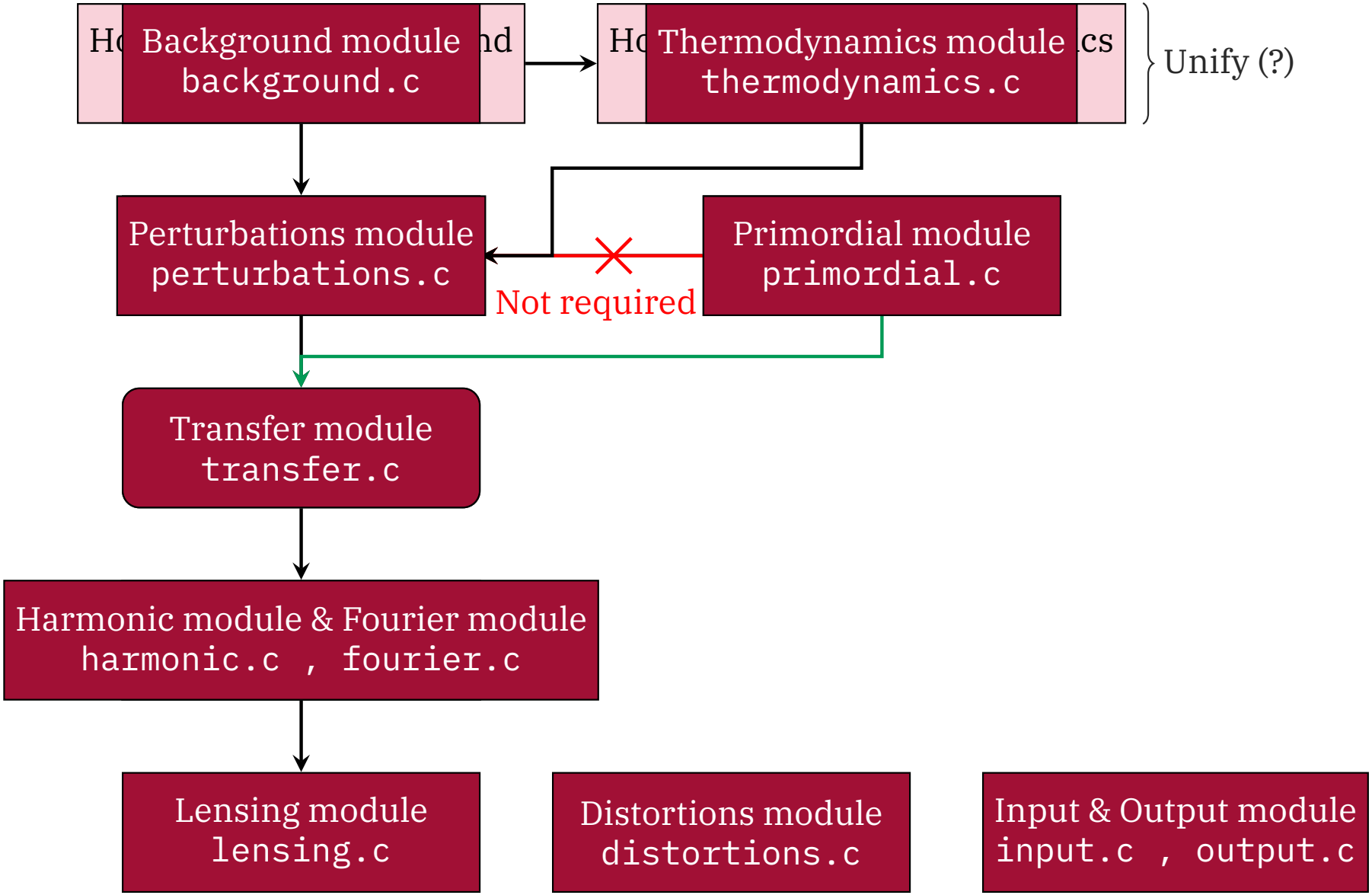

Essential steps in Einstein-Boltzmann solver

Let's make a journey through each module!

Essential steps in Einstein-Boltzmann solver

Module 1. Input

Read in input files, take care of *shooting*.

```
h = 0.7
#H0 = 70
Omega_m = 0.3
#omega_m = 0.14
sigma8=0.8
```

Special care for equivalent/unknown parameters

Input management in `class`

Terminal	Python wrapper
file `xxx.ini`	
↓	
`input_init(...)`	
(parser)	`.set(...)`
↓	↓
`struct file_content fc;`	(all parameter names/values stored as arrays of strings)
↓	
`input_read_from_file(...)`	
↓	
`input_read_parameters(...)`	
(assign all default values + interprete input + update some parameters)	
↓	
Only *relevant* parameters get stored in the structures of each module	

Input management in `class`

For indirect parameters, use shooting method

Repeated calls of `input_read_parameters(...)`, `class` executions, from `input_read_from_file(...)` until shooting target is met.

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Example:
How would you code the input parameter θ_s?
Use approximate formula $\rightarrow$ inflexible, inaccurate

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Example:
How would you code the input parameter θ_s?
Use approximate formula $\rightarrow$ inflexible, inaccurate

Try out a few values and narrow down (Example: User wants $100\theta_s = 1.04325$)

h	$100\theta_s$
0.7	1.0522492086422521
0.65	1.0270326366580724
0.68215616173	1.0437999980620178
0.68110138476	1.0432819283581667
0.6810337942	1.0432499363679562
0.6810365 0871	1.0432499921072458
0.68103652701	1.0432500079710365
...	...

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0.6810337942	1.0432499363679562
0.681036500871	1.0432499921072458
0.68103652701	1.0432500079710365
...	...

In practice, use more sophisticated Ridder's method / Newton's method

Input management in class

For shooting parameters, establish mapping between *target parameter*, *unknown parameter* and *level*. Currently:

target parameter	unknown parameter	level
$100 \times \theta_s$	h	thermodynamics
$\Omega_{\rm dcdm}$	$\rho^{\rm ini}_{\rm dcdm}$	background
~~σ_8~~	~~A_s~~	~~spectra~~
...	...	...

... plus a few others (alternative parametrizations of decaying CDM, quintessence parameters).

This is what is used e.g. in models of early dark energy!

If you need to add such parameters: see how it is done e.g. for `100*theta_s` and replicate the structure!

Special exception $\tau_{\rm reio} \leftrightarrow z_{\rm reio}$ only concerns reionization and is done independently in `thermodynamics.c`
New Special exception: $\sigma_8 \leftrightarrow A_s$ can be very simply analytically re-scaled (multiplicative property), therefore done independently in `input.c`

Input management in class

Budget equation:

$$\sum_X \Omega_X = 1 + \Omega_k$$

To avoid over-constraining the input, one of the last three (`Omega_Lambda`, `Omega_fld`, `Omega_scf`) must be left unspecified and `class` will assign it using budget equation.

Possibly more advanced in the future

- default: `Omega_Lambda` is automatically adjusted
- if you pass `Omega_Lambda`, `Omega_fld` is automatically adjusted
- if you pass `Omega_Lambda` and `Omega_fld`: `Omega_scf` is automatically adjusted (if you allow, by setting to -1)

This allows whatever combination.
E.g. to get Λ plus a DE fluid:
`Omega_Lambda=0.2, Omega_scf=0` or `Omega_fld=0.3, Omega_scf=0`

Helpful output by setting background verbose >= 2

Module 2. Background

Get all background quantities as function of a scale factor a.

This also gives mapping $a \leftrightarrow z \leftrightarrow t \leftrightarrow$ conf.time

The Background Module

Let's formalize problem!

Three types of parameters:

- $\{A\}$ are analytical functions of scale factor and $\{B\}$ quantities.
- $\{B\}$ need to be integrated over, and are used to compute $\{A\}$
- $\{C\}$ also need to be integrated over, but are not used to compute $\{A\}$.

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ΛCDM and many simple extensions:

- $\{A\} = \{\rho_i(a), p_i(a), H(a), ..., \}$ with e.g. $H(a) = \left(\sum_X \rho_x(a) - \frac{K}{a^2}\right)^{1/2}$
- $\{B\} = \{\}$ (eliminated since `v3.0`)
- $\{C\} = \{t, \tau, r_s, D, f\}$ with e.g. $\frac{dt}{da} = 1/H(a)$, $\frac{dr_s}{da} = c_s(a)/(a \cdot H(a))$

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Example of DE/DM/DR fluid:

- $\{A\} = \{\rho_i(a), p_i(a), H(a), ..., w_{\rm fld}(a)\}$
- $\{B\} = \{\rho_{\rm fld}\}$ with $\frac{d\rho_{\rm fld}}{da} = -3(1 + w_{\rm fld}(a))\rho_{\rm fld}$

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Exemple of extended cosmology with quintessence ϕ:

- $\{A\} = \{\rho_i, p_i, H, ..., V(\phi), \rho_\phi(\phi, \phi')\}$ with e.g. $\rho_\phi(\phi, \phi') = \frac{1}{2}(\phi')^2 + V(\phi)$
- $\{B\} = \{\phi, \phi'\}$ with $\frac{d\phi}{da} = \phi'/[aH(a)]$, $\frac{d\phi'}{da} = -2\phi' - aV(\phi)/H(a)$

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Also Cold Dark Matter decaying into Dark Radiation...

- $\{A\} = \{\rho_i, p_i, H, ...\}$
- $\{B\} = \{\rho_{\rm dcdm}, \rho_{\rm dr}\}$ with $\frac{d\rho_{\rm dcdm}}{da} = -3\rho_{\rm dcdm} - \Gamma(a)/H(a) \cdot \rho_{\rm dcdm}$

The Background Module

Small details:

- Quantities as $D_A(z), D_L(z), r_s, t_{\text{age}}$ can be derived after all A,B,C are computed
- Takes care of NCDM integration of phase-space distribution
- Useful checks & output
- $\rightarrow$ Budget equation output at verbosity level 2

Module 3. Thermodynamics

Get all thermodynamics quantities as function of a time variable (`class` $\rightarrow$ redshift z) after integrating differential equations like recombination equations:

$$\frac{dx_e}{dz}, \frac{dT_b}{dz} = \text{excitation, ionization, heating, ...}$$

Then $x_e(z) \rightarrow \kappa'(z)$ (Thomson scattering rate)
$\rightarrow \kappa(z)$ (Optical depth)
$\rightarrow \exp(-\kappa(z))$ (factor for Integrated Sachs-Wolfe effect)
$\rightarrow g(z)$ (visibility function for Sachs-Wolfe effect)
$\rightarrow g'(z)$ (factor for Doppler effect)

The Thermodynamics Module

Simplest model of recombination is the Saha equation.

It is well known that a non-relativistic ($T \ll m$) species in thermal equilibrium obeys

$$n(\mu, T) \approx g e^{\mu/T} \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \tag{1}$$

Thus we find using complete thermal equilibrium with $\mu_{\text{ionized}} + \mu_e = \mu_{\text{rec}}$ that

$$\frac{n_e n_{\text{ionized}}}{n_{\text{rec}}} \approx \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-E_{\text{bind}}/T} \times \underbrace{\left[e^{\mu_{\text{ionized}}+\mu_e-\mu_{\text{rec}}} \left(\frac{g_e g_{\text{ionized}}}{g_{\text{rec}}}\right) \left(\frac{m_{\text{ionized}}}{m_{\text{rec}}}\right)^{3/2}\right]}_{\approx 1}$$

This gives
$$\frac{x_e^2}{1-x_e} \approx \left(\frac{1.1 \cdot 10^{-10}}{n_{\text{H},0}/T_{\text{cmb},0}^3}\right) \left(\frac{\text{eV}}{T}\right)^{3/2} \exp(39.9 - 13.6 \frac{\text{eV}}{T}) \tag{2}$$

and thus recombination at $T \approx \frac{13.6\text{eV}}{39.9} \approx 0.34\text{eV} \to z \approx 1400$.

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recombination is a non-equilibrium process

The Thermodynamics Module

The effective multi-level atom is the basis for recombination codes.

1s 2s 2p 3s 3p 3d ... ionized $\rightarrow$ 1s 2s 2p ionized

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Reason: Intermediate transitions (4p$\rightarrow$3s) or (3s$\rightarrow$2p) are comparatively instant. Why? Direct transition $2s \rightarrow 1s$ is forbidden, and $2p \rightarrow 1s$ is immediately reversed by $1s \rightarrow 2p$. The medium is optically thick during recombination.

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Instead, focus on $2p \rightarrow 1s$ with subsequent redshifting of photon to escape reabsorption (slow) or $2s \rightarrow 1s$ with two-photon decay (slow).

Peeble's equation

$$\dot{x_e} \approx f_{\text{photo}-\text{ion}}(T) x_{\text{rec}} - f_{\text{rec}}(T) x_e x_{\text{ionized}} \tag{3}$$

Solved numerically, basis of recfast

recfast only resolves $2s \approx 2p$

The Thermodynamics Module

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Improvement: HyRec with EMLA resolves $2s$, $2p$. Even more, can do $2s$, $2p$, $3s$, ... with *effective* rates.

Fullest code to date: CosmoRec does full numerical computation (iteratively). Comparatively slow, but highest achievable accuracy

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Further complication: Helium (higher elements don't contribute)

The Thermodynamics Module

User can choose to model approximate recombination and get $x_e(z)$, $T_b(z)$ from:

- RECFAST (Wong, Moss & Scott 2008)
- HyRec-2 (Y. Ali-Haïmoud, N. Lee)
- Possibly soon? CosmoRec (J. Chluba)

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Recombination needs one more cosmological parameter: the primordial Helium fraction $Y_{\rm He}$.

- Fix it (`Y_He = 0.25`)
- Get it from BBN (`Y_He = BBN`). `class` has interpolation table pre-pcomputed with a BBN code (Parthenope), for each given value of $N_{\rm eff}$, ω_b (assumes $\mu_{\nu_e} = 0$, easy to generalize).
- BBN interpolation table located in separate directory (in `external/bbn/sBBN_2017.dat`, update inbound)

For reionization:

- tanh with complicated argument (like CAMB)
- multi-tanh
- half tanh
- from file (either linear or tanh)

The Thermodynamics Module

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Mini-shooting to find z_{reio} for given $\tau_{\text{reio}} = \kappa_{\text{reio}}$.
Optical depth $\kappa(z)$ = inverse number of expected interactions $\Rightarrow \kappa'(z) = a n_H x_e \sigma_T$

The Thermodynamics Module

We also include

- Energy injection (increases ionization, heats T_b)
  This can cause changes in scattering $\kappa(z)$ and thus be observable with CMB
- Time-dependent fundamental constants $\rightarrow$ Causes shift in recombination due to fundamental dependencies such as
  $E_{\text{binding}} = \frac{1}{2}\alpha^2 m_e = 13.6\text{eV}\,(137\alpha)^2\,\left(\frac{m_e}{511\text{keV}}\right)$
  We remind ourselves $1 + z_{\text{rec}} = T_{\text{rec}}/T_{\text{cmb}} \approx \frac{E_{\text{binding}}}{12.57\text{meV}}$
- Computation of useful quantities $z_{\text{rec}}, z_{\text{drag}}, z_*, D_A(z_{\text{rec}}), r_s(z_{\text{drag}}), \ldots$

Module 4. Perturbations

- Find all perturbations ($\delta_X(\tau, k)$, $\phi(\tau, k)$, ...) by integrating ODEs for each independent wavenumber k, each mode (scalar/tensor), each initial condition (adiabatic/isocurvature)
  Minimal set! (no massive neutrinos, or advanced DM/DE)

The Perturbations Module

- Find all perturbations ($\delta_X(\tau, k)$, $\phi(\tau, k)$, ...) by integrating ODEs for each independent wavenumber k, each mode (scalar/tensor), each initial condition (adiabatic/isocurvature):
  - Boltzmann
  - Continuity + Euler
  - linearized Einstein equations (one = ODE, others = constraint equations)

  Linear perturbations $\Rightarrow$ perturbations normalized to initial condition (`class` $\rightarrow$ curvature $\mathcal{R} = 1$ for scalar with adiabatic I.C.)

The Perturbations Module

Einstein Equations

$$k^2\phi + 3\mathcal{H}(\phi' + \mathcal{H}\psi) = -4\pi G a^2 \delta\rho \tag{4}$$

$$k^2(\phi' + \mathcal{H}\psi) = 4\pi G a^2 (\rho + P)\theta\phi'' + \mathcal{H} \tag{5}$$

$$(\psi' + \phi') + (2\mathcal{H}' + \mathcal{H}^2)\psi + \frac{1}{3}k^2(\phi - \psi) = 4\pi G a^2 \delta P \tag{6}$$

$$k^2(\phi - \psi) = 12\pi G a^2 (\rho + P)\sigma \tag{7}$$

and Boltzmann equations

$$\frac{\mathrm{d}F_0^{(\gamma)}}{\mathrm{d}\eta} + kF_1^{(\gamma)} = 4\phi' \tag{8}$$

$$\frac{\mathrm{d}F_1^{(\gamma)}}{\mathrm{d}\eta} - \frac{k}{3}\left[F_0^{(\gamma)} - 2F_2^{(\gamma)}\right] = \frac{4k}{3}\psi + \underbrace{\Gamma_{\gamma,b}}_{\text{from thermodynamics}}\left[F_1^{(b)} - F_1^{(\gamma)}\right] \tag{9}$$

$$\frac{\mathrm{d}F_2^{(\gamma)}}{\mathrm{d}\eta} - \frac{k}{5}\left[2F_1^{(\gamma)} - 3F_3^{(\gamma)}\right] = \underbrace{\Gamma_{\gamma,b}}_{\text{from thermodynamics}}\left[-F_2^{(\gamma)} + \Pi^{\mathrm{pol}}/10\right] \tag{10}$$

$$\frac{\mathrm{d}F_\ell^{(\gamma)}}{\mathrm{d}\eta} - \frac{k}{2\ell+1}\left[\ell F_{\ell-1}^{(\gamma)} - (\ell+1)F_{\ell+1}^{(\gamma)}\right] = 0 \qquad \text{(infinite hierarchy)} \tag{11}$$

The Perturbations Module

4 Einstein Equations (only one dynamical)
1 $\ell_{\rm max}^{(\gamma)}$ photon temperature hierarchy
1 $\ell_{\rm max}^{(\gamma)}$ photon polarization hierarchy (or 2 $\ell_{\rm max}^{(\gamma)}$)
2 baryon (density, velocity)
1/2 cdm equations (density?, velocity)
Either

a) 1 $\ell_{\rm max}^{(\rm dr)}$ massless neutrino hierarchy

b) 1 $\ell_{\rm max}^{(\rm dr)}$ massless neutrino hierarchy
+ $N_{\rm ncdm} \cdot \ell_{\rm max}^{(\rm ncdm)} \cdot N_q$ massive neutrino hierarchies

= Too many equations for simple solvers!
(also tight coupling == stiff equations)
(also sparse system)

The Perturbations Module

ODE Solver (customized for Einstein-Boltzmann equations)

- Stiff system require implicit method like backward Euler or more advanced:
  $\rightarrow$ find y_{n+1} as a solution of $\mathbf{y_{n+1}} = y_n + y'(\mathbf{y_{n+1}})\delta t$
- Still fast: Newton method with Jacobian recycling
- Robustness requires δt to be adaptive time step
- Source function required at predefined t_i: on-the-fly interpolation
- System is sparse: some algebra gives big speed up (sparse LU decomposition)

Everything gathered in `ndf15` by T. Tram (CLASS II 2011).
TCA could even be removed!

The Perturbations Module

ODE approximations (papers : CLASS II & CLASS IV 2011)
Idea of these approximations: Reduce number of evolved equations.

- Tight Coupling Approximation for baryons and γ at 2nd order $\rightarrow$ Suppresses shear & higher moments whenever $k\tau_{b\gamma} \ll 1$
- Ultrarelativistic Fluid Approximation (for massless ν, also one for massive ones): truncated Boltzmann, 3 equations $\rightarrow$ Suppresses higher order moments whenever $k\tau \gg 1$
- Radiation Streaming Approximation (for photons and massless ν): test particles, 0 equations $\rightarrow$ Only follow oscillation-averaged evolution when $k\tau \gg \ell$ (+ for photons $k\tau_{b\gamma} \gg 1$)

Source functions

- Keep memory not of everything, but anything useful for final calculation of observables:
  - raw transfer function ($\delta_m(\tau, k) \rightarrow P_m(k, z)$)
  - non-trivial combinations (photon, baryon, metric, thermodynamical functions $\rightarrow$ CMB source functions $S_{T_i}(k, \tau)$)

  All these are called *source functions* in `class`

Photon hierarchies

Two approaches to polarization in Boltzmann hierarchy:

- Ma & Bertschinger 1994 (optimal):
  $(F_\ell, G_\ell) \to (S_T, S_P) \to (\Delta_\ell^T, \Delta_\ell^E, \Delta_\ell^B)$: $2\ell_{\max}$ equations, only flat!
- Hu & White 1997 (TAM):
  $(\Theta_\ell, E_\ell, B_\ell) \to (S_T, S_E, S_B) \to (\Delta_\ell^T, \Delta_\ell^E, \Delta_\ell^B)$: $3\ell_{\max}$ equations!

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CMBFAST: first in flat space, second in curved space

CAMB: always second case

`class`: first case by default, thanks to new analytic results in curved space
(T. Tram & JL, JCAP 2013 [arXiv:1305.3261], approximation!)
User can select second case

Essentials 5: Primordial

Module 5. Primordial

Initial conditions for scalars (adiabatic, isocurvature) and tensors. Linear theory $\Leftrightarrow$ Gaussian independent Fourier modes $\Leftrightarrow$ only power spectrum required

- analytic: primordial power spectra as parametric functions (e.g. power-law)
- inflation mode: solve background+perturbation equation for single-field inflation and compute primordial scalar/tensor spectrum numerically

D. Primordial spectra : modes

P_k_ini type =	modes =	ic =
analytic_Pk	one or several of s,t	one or several of ad,bi,cdi,nid,niv
inflation_V	s,t	ad
inflation_H	s,t	ad
inflation_V_end	s,t	ad
external_Pk	one or several of s,t	ad

Module 6. Fourier space

- Linear matter power spectrum $P_m(k,z) \rightarrow$ integrated quantities $\sigma(R,z)$, $\sigma_8(z)$
- Linear baryon+CDM power spectrum $P_{cb}(k,z) \rightarrow$ integrated quantities $\sigma_{cb,8}(z)$
- Approximation for non-linear spectrum $P_m^{NL}(k,z)$ based on prescriptions like Halofit, HMcode...
- Keep also non-linear correction factor $\left(R^{NL}(k,z)\right)^2 = P_m^{NL}(k,z)/P_m(k,z)$ for e.g., CMB lensing, cosmic shear, number count C_ℓ's

How to emulate non-linear evolution with a halo model?

Halofit or HMcode require non-linearity scale $R_{\rm NL}(z)$ such that $\sigma(R_{\rm NL}(z), z) = 1$.
To get $P^{\rm NL}(k, z)$ ar higher z one should increase $k_{\rm max}$.

Halofit relies on simple similarity solution Ansatz:

$$\Delta^2_{1-\text{halo}}(k) = \underbrace{\frac{a_n y^{3f_2}}{(1 + b_n y^{f_2} + (f_3 c_n y)^{3-\gamma_n}}}_{\text{Original term, corrected with } f_2} \underbrace{\frac{1}{(1 + x_\mu y^{-1} + x_\nu y^{-2})(1 + 0.977 f_\nu)}}_{\text{Further corrections}}$$

with $y = k/k_{\text{nl}} = kR_{\text{NL}}(z)$.
Parameters calibrated to fit (early) simulations reasonably well.

$$\Delta^2_{\text{nl}} \approx \Delta^2_{2-\text{halo}} + \Delta^2_{1-\text{halo}} \text{ with } \Delta^2_{2-\text{halo}} \approx \frac{(1+\Delta^2_{\text{lin}})^\beta}{(1+\alpha\Delta^2_{\text{lin}})} \cdot \exp\left(-y/4 - y^2/8\right) \text{ with}$$

$$\Delta^2_{\text{lin}} = P_{\text{lin}}(k, z)\frac{k^3}{2\pi^2}$$

Summary: Simple analytical fitting formula

The Fourier Module

HMcode has a more complicated halo model:

- $\Delta^2_{\rm nl} \approx \left(\left(\Delta^2_{\rm 2-halo}\right)^{\alpha} + \left(\Delta^2_{\rm 1-halo}\right)^{\alpha}\right)^{1/\alpha}$
- $P_{\rm 2-halo} = \left[P_{\rm lin} + (f_{\rm dewiggle} - 1.)P_{\rm BAO-wiggle}\right] \times \{1 - f_{\rm damp} \cdot (k/k_{\rm damp})^{\alpha_{\rm damp}}/[1 + (k/k_{\rm damp})^{\alpha_{\rm damp}}]\}$
- $P_{\rm 1-halo} = \int n(\nu k^{\eta}) g(\nu) {\rm d}m$
- $\nu = \delta_c/\sigma(R(m))$
- $g(\nu) = A \cdot (1 + (q\nu^2)^{-p}) \exp(-q\nu^2/2)$ Sheth-Tormen HMF
- $n(x) =$ NFW halo profile

Summary: Physically motivated halo formula, reproduces well current simulations

Essentials 7+8: Transfer & Harmonic

Module 7: Transfer & Module 8: Harmonic

$\delta(k,\tau), \theta(k,\tau)$ (perturbations)

$\downarrow$

$S^{\mathrm{cmb}}(k,\tau), S^{m}(k,\tau)$ (source functions)

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$\Delta_\ell^x(k)$ (line-of-sight projection, `transfer.c`)

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$\int \Delta_\ell^x(k)\Delta_\ell^y(k)\mathrm{d}\ln k$

(correlation and summation, `harmonic.c`)

Transfer & Harmonic Modules

CMB spectrum depends on $\Delta_\ell^X(k) = \ell$-th multipole of anisotropy of photon temperature and polarisation ($X \in \{T, E, B\}$) today ($\tau = \tau_0$).

Since CMBFAST(Seljak & Zaldarriaga 1996): use "line-of-sight integral"

$$\Delta_\ell^X(k) = \int_\epsilon^{\tau_0} d\tau \; S^X(\tau, k) \, j_\ell(k(\tau_0 - \tau))$$

$S(\tau, k)$ = source function from above. Role of Bessel: projection from Fourier to harmonic space ($\theta \, d_a(z_{\rm rec}) = \frac{\lambda}{2}$ gives precisely $l = k(\tau_0 - \tau_{\rm rec})$):
Curved space: spherical bessel functions $\rightarrow$ modified Bessel functions (hypergeometric)

Transfer & Harmonic Modules

$$\Delta_\ell^X(k) = \int_\epsilon^{\tau_0} d\tau \; S^X(\tau, k) \; j_l(k(\tau_0 - \tau))$$

applies not just to CMB $X \in \{T, E, B\}$ but also all LSS C_ℓ's (one X per type of observable and redshift bin).

- CMB lensing + cosmic shear: ($S(\tau, k)$ involves broad window function)
- number count (galaxy clustering): $S(\tau, k)$ modeled fully relativistically (RSD, Doppler, lensing, other GR effects)
- may include non-linear correction factors $R^{NL}(k, z)$

Transfer & Harmonic Modules

Well known

$$\Delta_\ell(k) = \int_\epsilon^{\tau_0} d\tau \; S_T(\tau, k) \, j_\ell(k(\tau_0 - \tau))$$

$$\text{with} \quad S_T(\tau, k) \equiv \underbrace{g \, (\Theta_0 + \psi)}_{\text{SW}} + \underbrace{\left(g \, k^{-2} \theta_\text{b}\right)'}_{\text{Doppler}} + \underbrace{e^{-\kappa} (\phi' + \psi')}_{\text{ISW}} + \text{polarisation}$$

comes from integration by part of:

$$\begin{aligned}\Delta_l(k) = \int_{\tau_\text{ini}}^{\tau_0} d\tau \; \Big\{ & S_T^0(\tau, k) \, j_l(k(\tau_0 - \tau)) \\ & + S_T^1(\tau, k) \, \frac{dj_l}{dx}(k(\tau_0 - \tau)) \\ & + S_T^2(\tau, k) \, \frac{1}{2} \left[3 \frac{d^2 j_l}{dx^2}(k(\tau_0 - \tau)) + j_l(k(\tau_0 - \tau)) \right] \Big\}\end{aligned}$$

But $(S_T^1)'$, $(S_T^2)'$, $(S_T^2)''$ problematic! (Derivative of Einstein equation, massive neutrinos $\rightarrow$ finite differences...)

So we should rather stick to

$$\Delta_l(k) = \int_{\tau_{\rm ini}}^{\tau_0} d\tau \; \left\{ S_T^0(\tau,k)\, j_l(k(\tau_0-\tau)) \right.$$
$$+ S_T^1(\tau,k)\, \frac{dj_l}{dx}(k(\tau_0-\tau))$$
$$\left. + S_T^2(\tau,k)\, \frac{1}{2}\left[3\frac{d^2 j_l}{dx^2}(k(\tau_0-\tau)) + j_l(k(\tau_0-\tau)) \right] \right\}$$

CLASS v2.0 stores separately $S_T^0(\tau,k)$, $S_T^1(\tau,k)$, $S_T^2(\tau,k)$, and the transfer module will convolve them individually with respective bessel functions.

$$S_T^0 = g\left(\frac{\delta_g}{4} + \psi\right) + e^{-\kappa}(\phi' + \psi') \qquad S_T^1 = g\frac{\theta_b}{k} \qquad S_T^2 = \frac{g}{8}\left(G_0 + G_2 + F_2\right)$$

or

$$S_T^0 = g\left(\frac{\delta_g}{4} + \phi\right) + e^{-\kappa} 2\phi' + g'\theta_b + g\theta_b' \qquad S_T^1 = e^{-\kappa} k(\psi - \phi) \qquad S_T^2 = \frac{g}{8}\left(G_0 + G_2 + F_2\right)$$

Transfer & Harmonic Modules

Last step is (almost) trivial:

$$C_\ell^{XY} = \int \frac{dk}{k} \sum_{ij} \Delta^X_{\ell\, i}(k) \Delta^Y_{\ell\, j}(k) \mathcal{P}_{ij}(k)$$

with sum running over modes (scalar/tensor) and I.C. (adiabatic/isocurvature).

Essentials 9: Lensing

Module 9. Lensing

- metric fluctuations $(\phi, \psi) \rightarrow$ lensing potential source function $\rightarrow$ CMB lensing potential spectrum C_ℓ^{PP}
- several quadratic sums over $C_{\ell_1}^{XY} C_{\ell_2}^{PP} \rightarrow$ lensed CMB spectra $C_\ell^{TT,TE,EE,BB}$. Full-sky approach of Challinor & Lewis 2005.

Essentials 10: Spectral Distortions

Module 10. Spectral distortions

- Computations using `CosmoTherm` to derive thermalization Green's function
- Using Green's function to compute μ, y amplitudes

Simplified view:

$$a = \int \dot{Q} J_a(t) dt \tag{12}$$

with branching function $J_a(t)$.

Module 11. Output

Writes output files with correct headers and data

Implementing new features

If you want to implement:

- a new species
- a new approximation scheme to simplify some equations in some regime
- a new mathematical description of an existing species (switching on more precise corrections, etc.)
- a new observable or output (new source function, new transfer function, new spectrum...)

the logic is always the same:

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5. change `fld` into `earde`

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4. duplicate these occurences
5. change `fld` into `earde`
6. change some equations to describe the specific properties of your feature