Machine Learning

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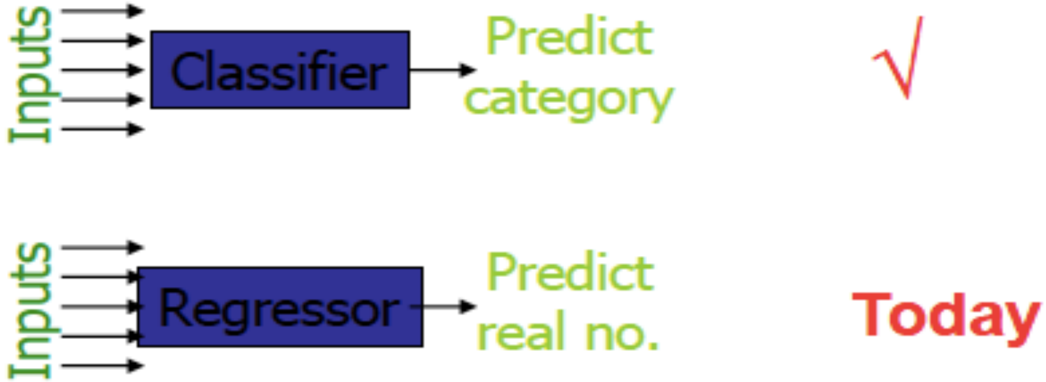
Where we are



Inputs
Classifier
Predict
category
√
Inputs
Regressor
Predict
real no.
Today

Choosing a restaurant

- In everyday life we need to make decisions by taking into account lots of factors
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Reviews (out of 5 stars)	$	Distance	Cuisine (out of 10)
4	30	21	7
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- The question is what weight we put on each of these factors (how important are they with respect to the others).
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- If we have many observations we may be able to recover the weights

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?

Linear regression

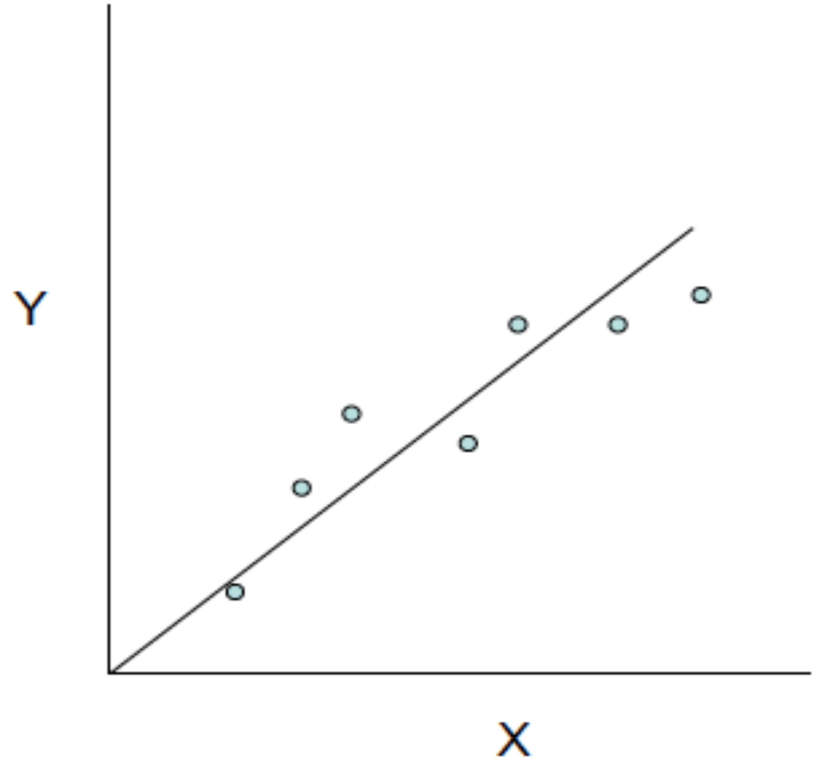
Linear regression

- Given an input x we would like to compute an output y
- For example:

  - Predict height from age

  - Predict Google's price from Yahoo's price

  - Predict distance from wall using sensor readings



Note that now Y can be **continuous**

Linear regression

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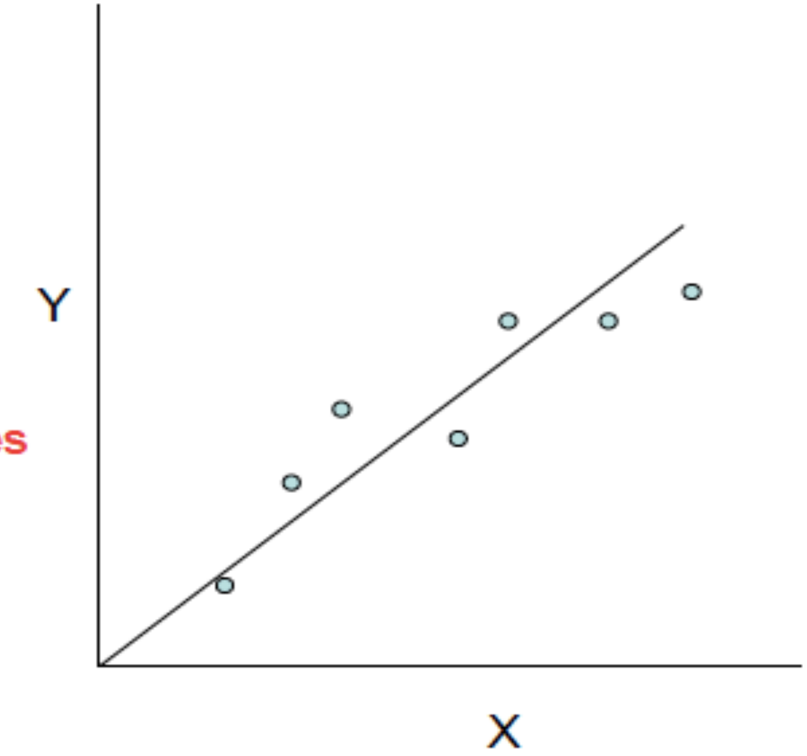
- In linear regression we assume that y and x are related with the following equation:

What we are trying to predict

Observed values

$$y = wx+\varepsilon$$



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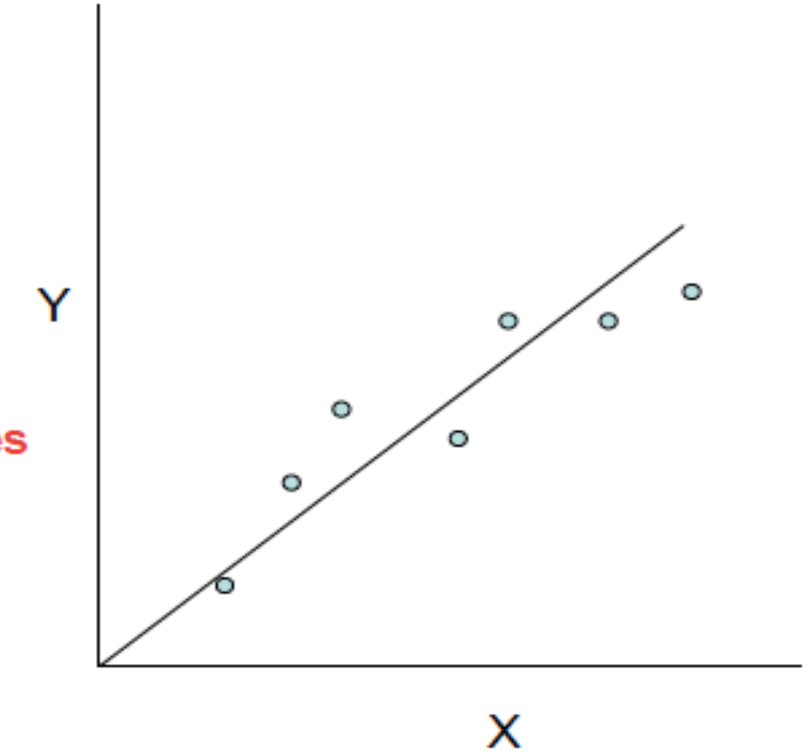
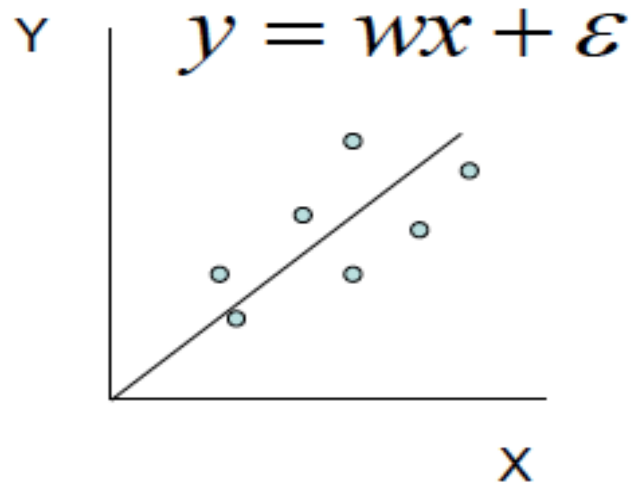
where w is a parameter and ε represents measurement or other noise



Linear regression

• Our goal is to estimate *w* from a training data of $<x_i,y_i>$ pairs



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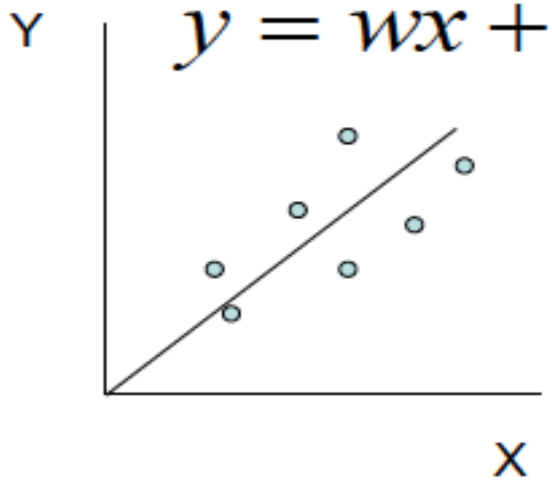

- Our goal is to estimate *w* from a training data of $<x_i,y_i>$ pairs
- One way to find such relationship is to minimize the a least squares error:

$$\arg\min_w \sum_i (y_i - wx_i)^2$$

- Several other approaches can be used as well
- So why least squares?
  - minimizes squared distance between measurements and predicted line
  - has a nice probabilistic interpretation
  - easy to compute

If the noise is Gaussian with mean 0 then least squares is also the maximum likelihood estimate of w

Solving linear regression using least squares minimization

- We just take the derivative w.r.t. to w and set to 0:

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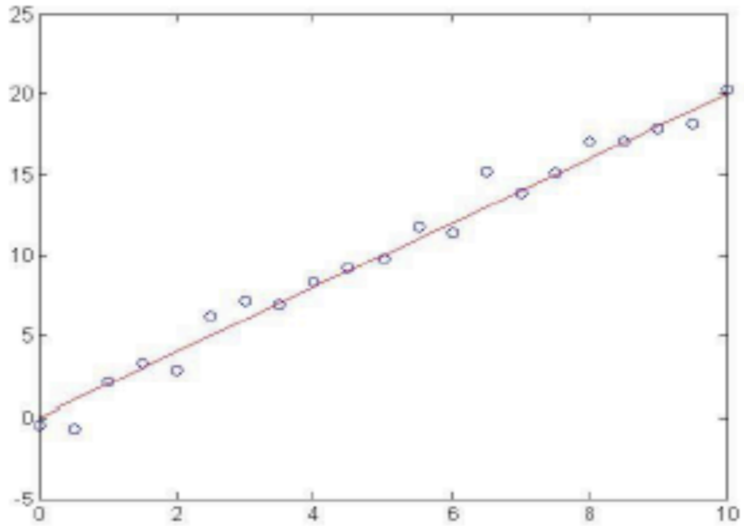
$$\sum_i x_i y_i = \sum_i wx_i^2 \Rightarrow$$

$$w = \frac{\sum_i x_i y_i}{\sum_i x_i^2}$$

Regression example

- Generated: w=2
- Recovered: w=2.03
- Noise: std=1



Regression example

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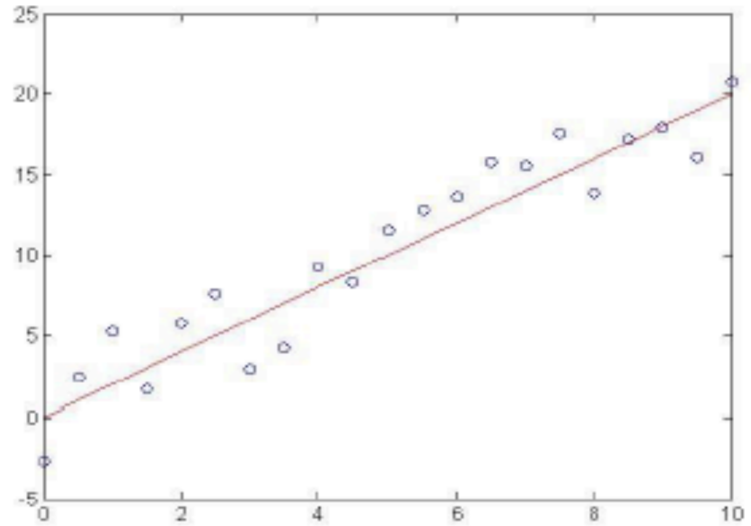
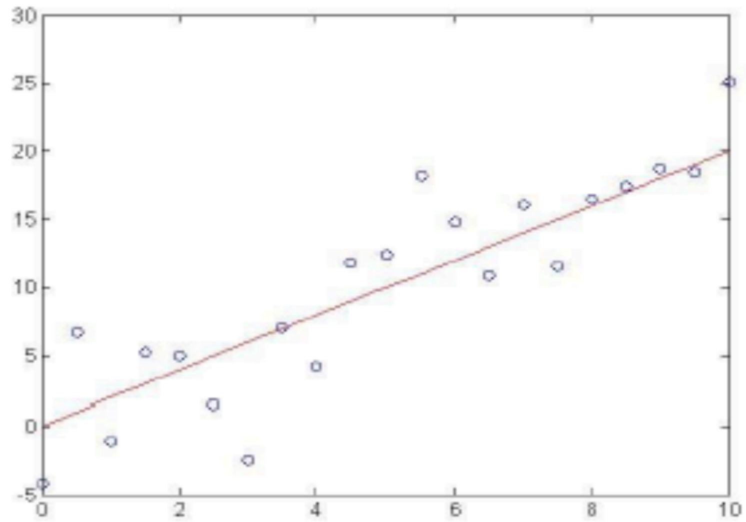
- Noise: std=2



Regression example

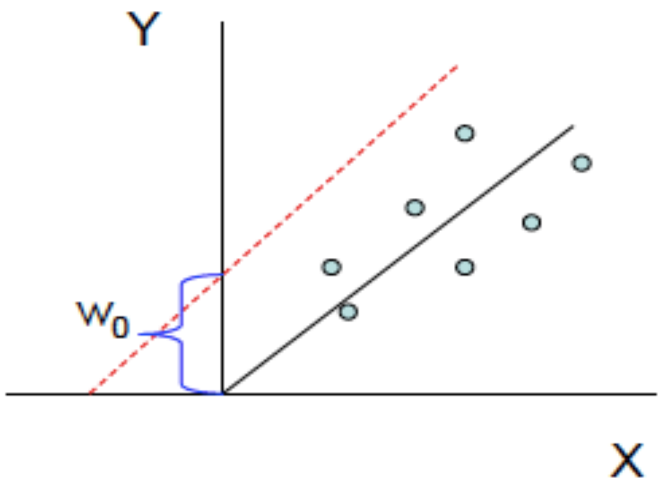
- Generated: w=2
- Recovered: w=2.08
- Noise: std=4



Bias term

- So far we assumed that the line passes through the origin
- What if the line does not?
- No problem, simply change the model to

$$y = w_0 + w_1 x + \varepsilon$$



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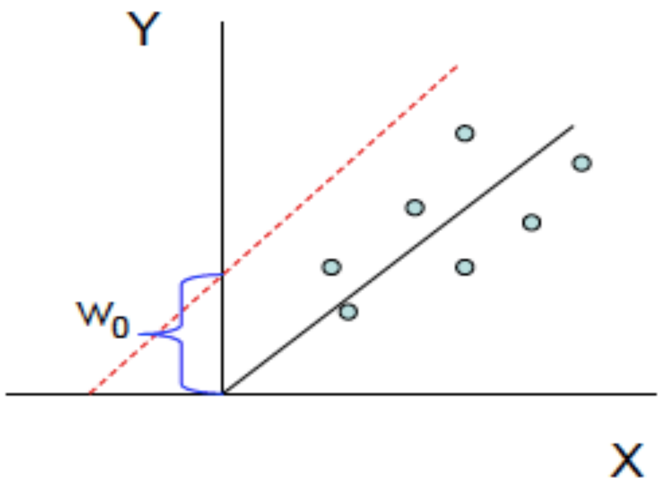
$$w_0 = \frac{\sum_i y_i - w_1 x_i}{n}$$



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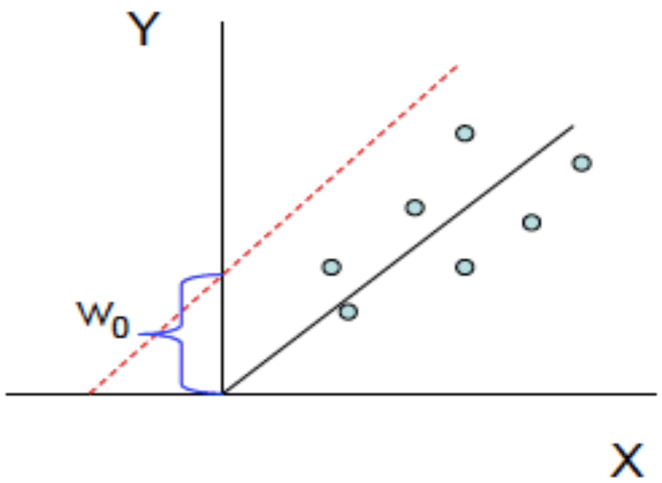

$$w_0 = \frac{\sum_i y_i - w_1 x_i}{n} \qquad w_1 = \frac{\sum_i x_i (y_i - w_0)}{\sum_i x_i^2}$$

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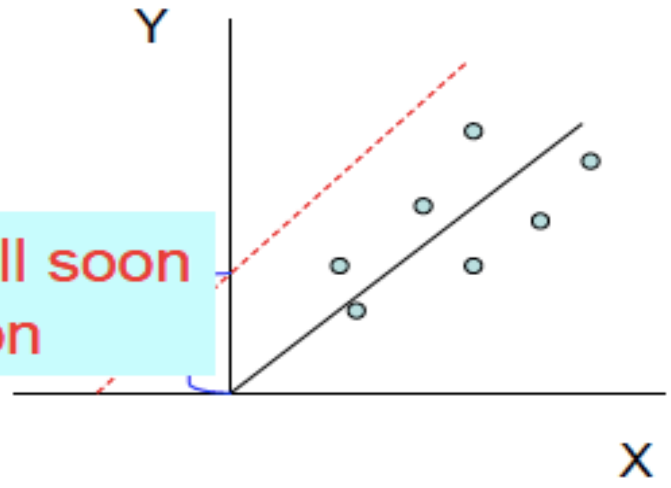
y = w

Just a second, we will soon give a simpler solution



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Multivariate regression

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Google's stock price

Microsoft's stock price

Yahoo's stock price

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Not all functions can be approximated using the input values directly

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Yes. As long as the coefficients are linear the equation is still a linear regression problem!

Five mins break!

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- A few common examples:

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- Gaussian: $$\phi_j(x) = \frac{(x-\mu_j)}{2\sigma_j^2}$$

- Sigmoid: $$\phi_j(x) = \frac{1}{1+\exp(-s_j x)}$$

Any function of the input values can be used. The solution for the parameters of the regression remains the same.

General linear regression problem

- Using our new notations for the basis function linear regression can be written as

$$y = \sum_{j=0}^{n} w_j \phi_j(x)$$

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- Once again we can use ‘least squares’ to find the optimal solution.

LMS for the general linear regression problem

$$y = \sum_{j=0}^{n} w_j \phi_j(x)$$

Our goal is to minimize the following loss function:

$$J(\mathrm{w}) = \sum_i (y^i - \sum_j w_j \phi_j(x^i))^2$$

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w – vector of dimension k+1

$\phi(x^i)$ – vector of dimension k+1

y^i – a scaler

Moving to vector notations we get:

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Define:

$$\Phi = \begin{pmatrix} \phi_0(x^1) & \phi_1(x^1) & \cdots & \phi_m(x^1) \\ \phi_0(x^2) & \phi_1(x^2) & \cdots & \phi_m(x^2) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_0(x^n) & \phi_1(x^n) & \cdots & \phi_m(x^n) \end{pmatrix}$$

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Then deriving w we get:

$$\mathbf{w} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$$

LMS for general linear regression problem

$$J(\mathbf{w}) = \sum_i (y^i - \mathbf{w}^T \phi(x^i))^2$$

Deriving w we get: $\mathbf{w} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$

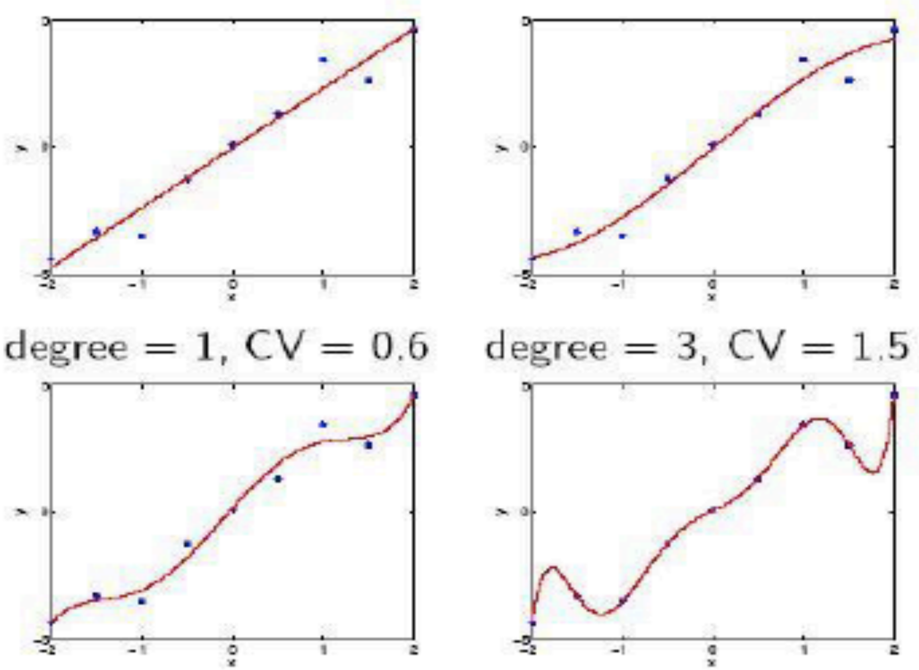
k+1 entries vector

n by k+1 matrix

n entries vector

This solution is
also known as
'psuedo inverse'

Example: Polynomial regression



degree = 1, CV = 0.6 degree = 3, CV = 1.5

degree = 5, CV = 6.0 degree = 7, CV = 15.6

A probabilistic interpretation

Our least squares minimization solution can also be motivated by a probabilistic in interpretation of the regression problem: $y = \mathbf{w}^{\mathrm{T}} \phi(x) + \varepsilon$

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The MLE for w in this model is the same as the solution we derived for least squares criteria:

$$\mathbf{w} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$$

Other types of linear regression

- Linear regression is a useful model for many problems
- However, the parameters we learn for this model are **global**; they are the same regardless of the value of the input x
- Extension to linear regression adjust their parameters based on the region of the input we are dealing with

Five mins break!

Back to classification

1. Instance based classifiers
   - Use observation directly (no models)
   - e.g. K nearest neighbors

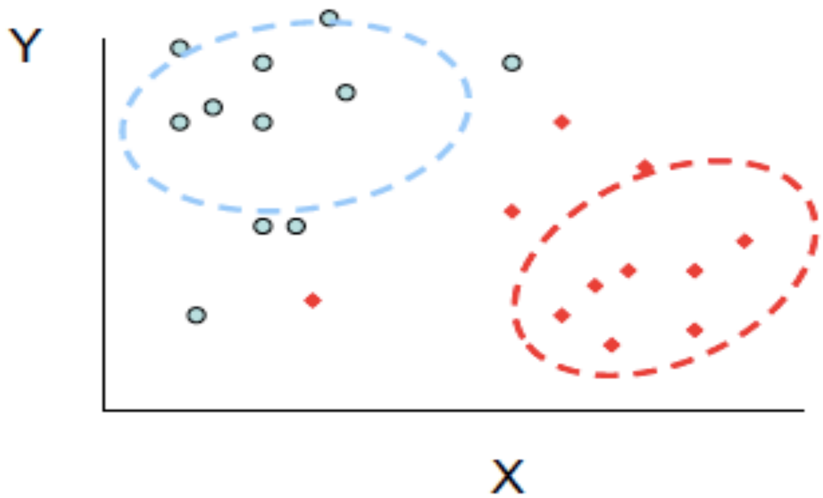
2. Generative:
   - build a generative statistical model
   - e.g., Bayesian networks

3. Discriminative
   - directly estimate a decision rule/boundary
   - e.g., decision tree

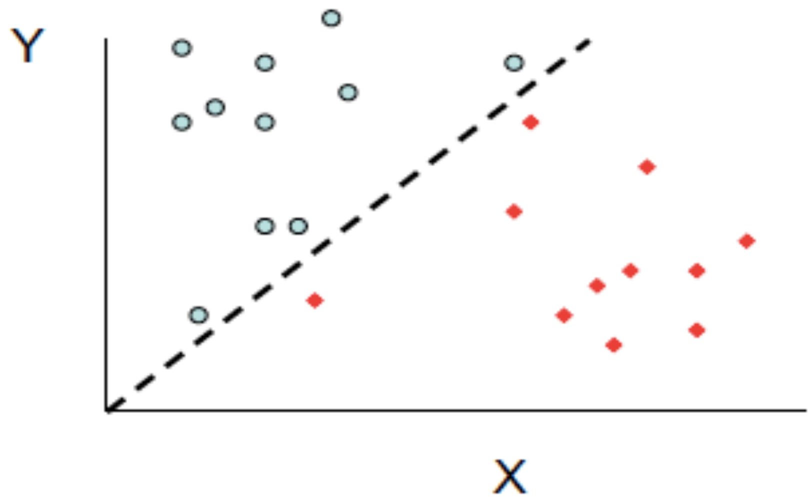
Generative vs. discriminative classifiers

- When using generative classifiers we relied on all points to learn the generative model
- When using discriminative classifiers we mainly care about the boundary

Generative model



Discriminative model



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- However, since the output is usually binary or discrete there are more efficient regression methods
- Recall that for classification we are interested in the conditional probability $p(y \mid X ; \theta)$ where θ are the parameters of our model
- When using regression θ represents the values of our regression coefficients (w).

Regression for classification

- Assume we would like to use linear regression to learn the parameters for $p(y \mid X ; \theta)$
- Problems?

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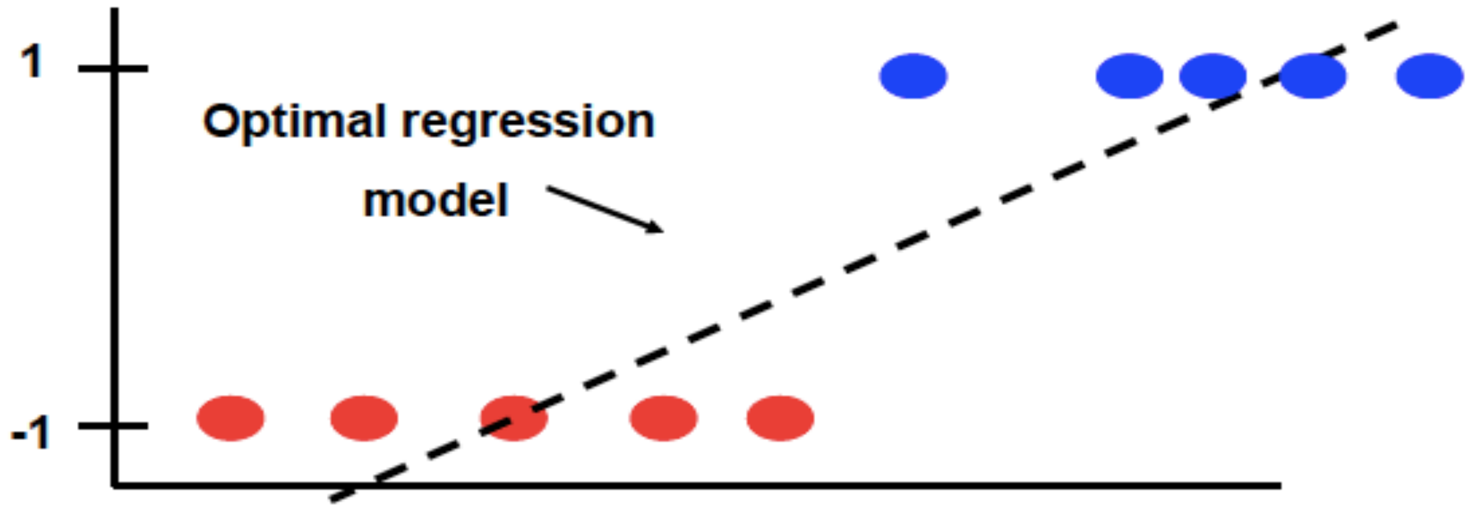
$w^T X \geq 0 \Rightarrow$ classify as 1

$w^T X < 0 \Rightarrow$ classify as -1



The sigmoid function

$$p(y \mid X; \theta)$$

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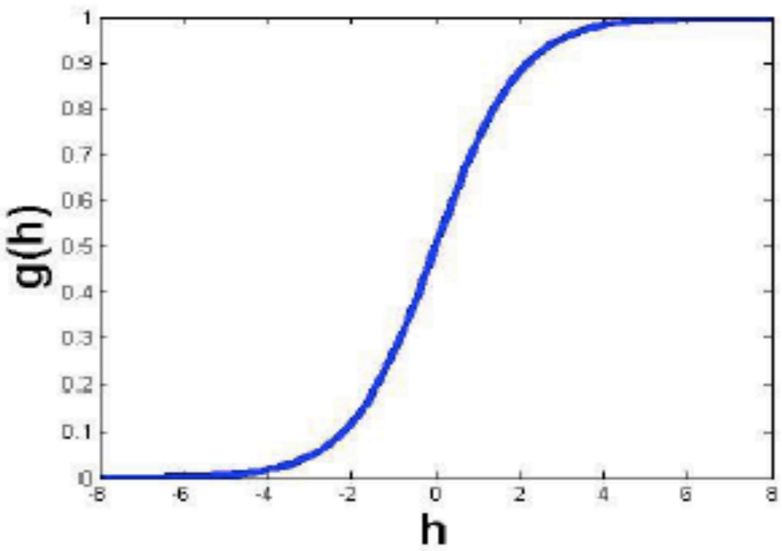

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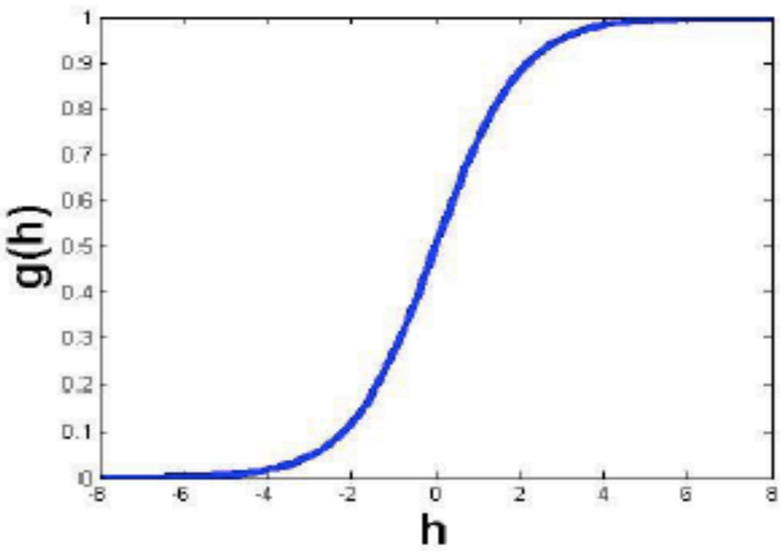

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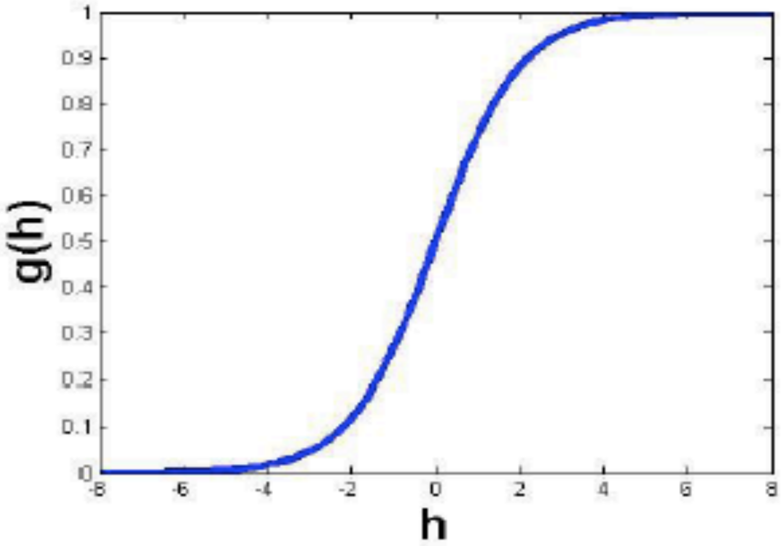

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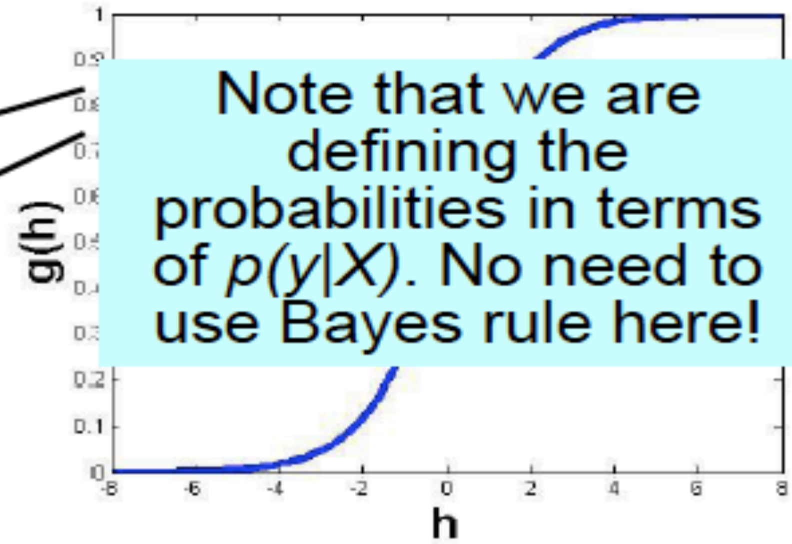

Note that we are defining the probabilities in terms of *p(y|X)*. No need to use Bayes rule here!

Logistic regression vs. Linear regression

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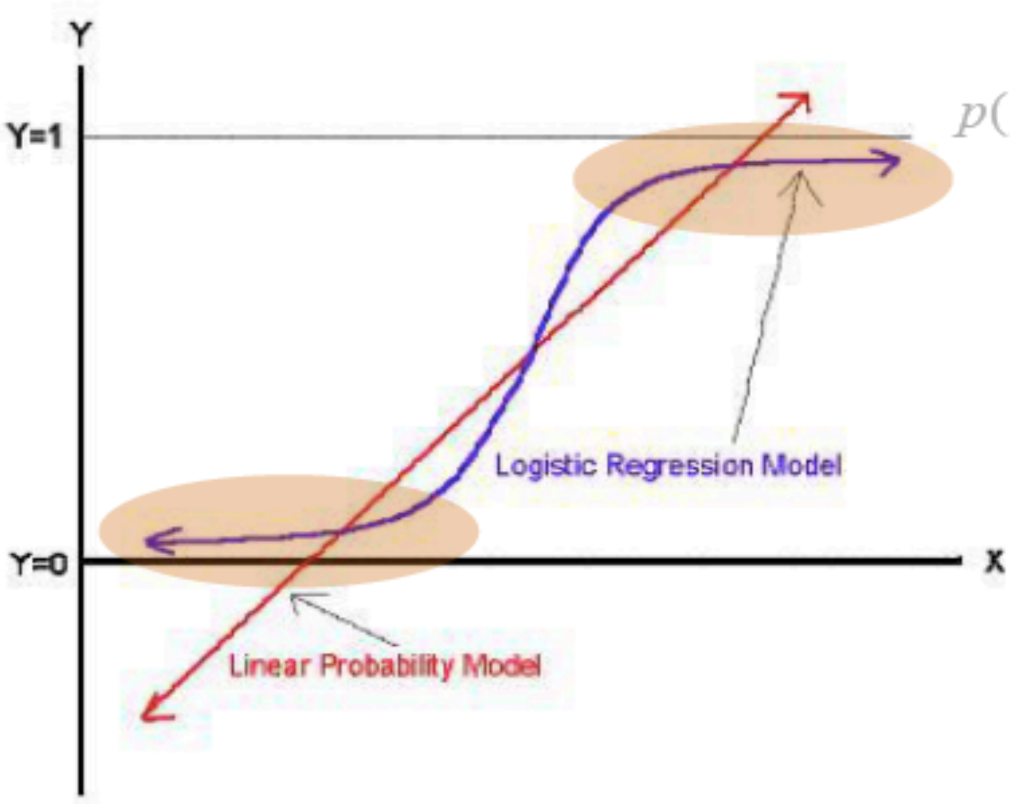

Determining parameters for logistic regression problems

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$$\begin{aligned}
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&= \sum_{i=1}^{N} y_i \mathrm{w}^{\mathrm{T}} X_i - \ln(1+e^{w^T X_i})
\end{aligned}$$

Maximum likelihood estimation

$$\frac{\partial}{\partial w^j} l(w) = \frac{\partial}{\partial w^j} \sum_{i=1}^{N} \{y_i \mathrm{w}^{\mathrm{T}} X_i - \ln(1 + e^{\mathrm{w}^{\mathrm{T}} X_i})\}$$

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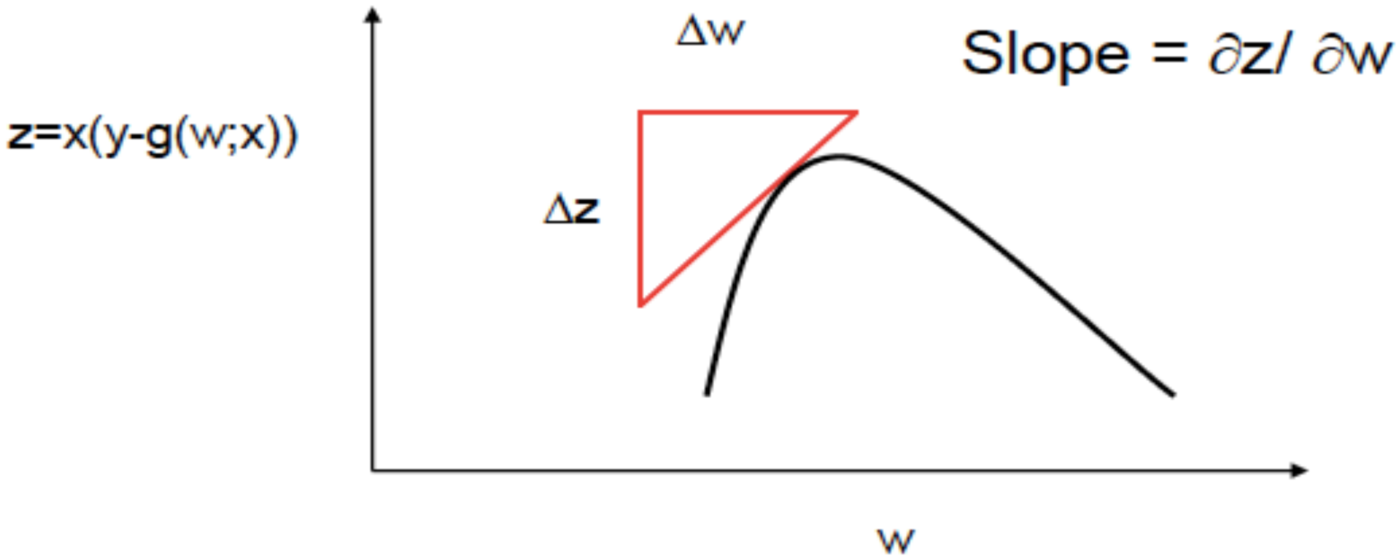
Taking the partial derivative w.r.t. each component of the **w** vector

Bad news: No close form solution!

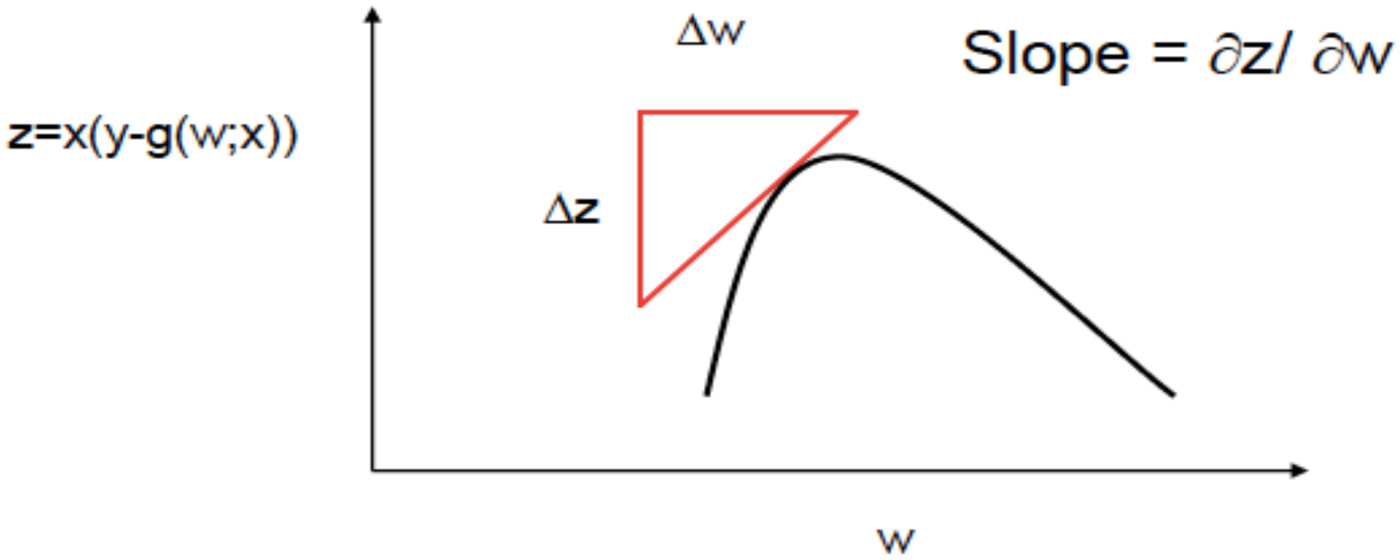
Good news: Concave function

Five mins break!

Gradient ascent

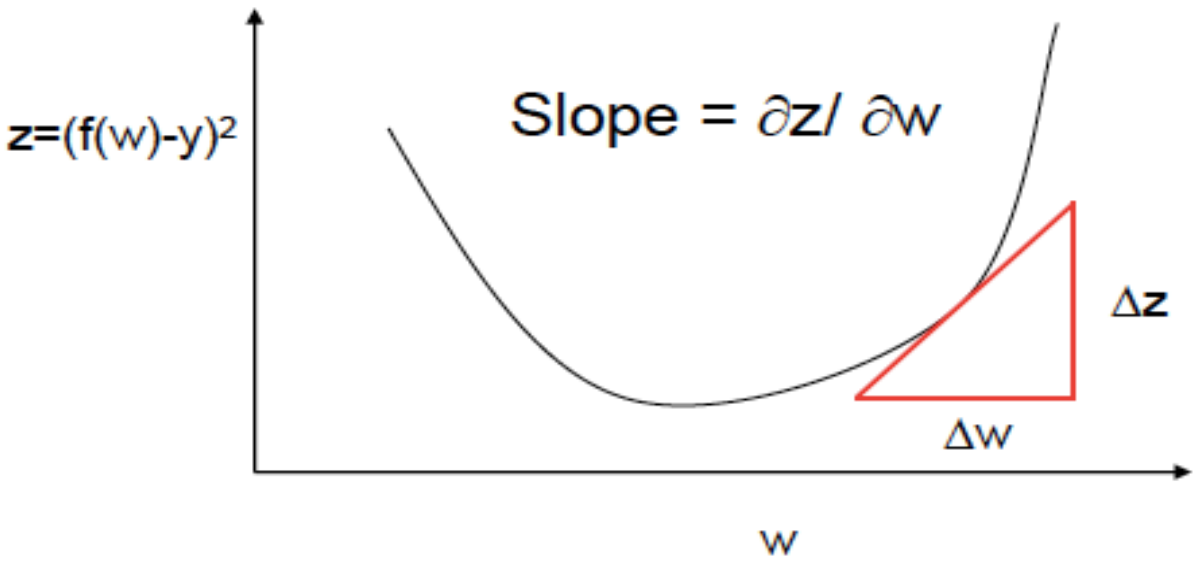


Gradient ascent



- Going in the direction to the slope will lead to a larger z
- But not too much, otherwise we would go beyond the optimal w

Gradient descent



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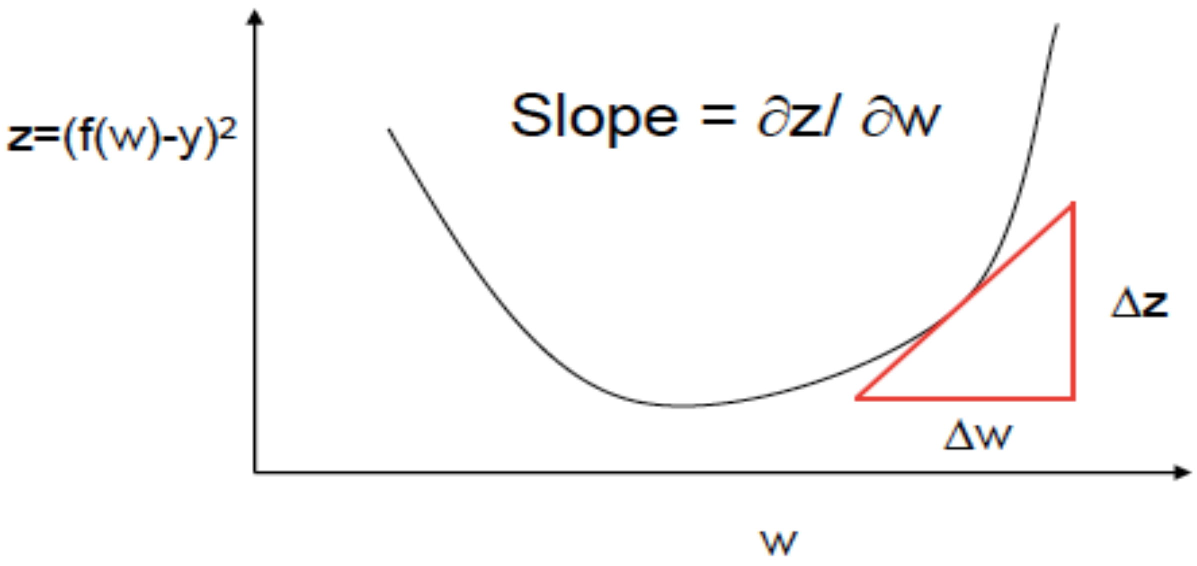

- Going in the *opposite* direction to the slope will lead to a smaller z
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Gradient ascent for logistic regression

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We use the gradient to adjust the value of w:

$$w^j \leftarrow w^j + \varepsilon \sum_{i=1}^{N} X_i^j \{y_i - (1 - g(X_i; w))\}$$

Where ε is a (small) constant

Algorithm for logistic regression

1. Chose λ
2. Start with a guess for **w**
3. For all j set
$$w^j \leftarrow w^j + \varepsilon \sum_{i=1}^{N} X_i^j \{y_i - (1 - g(X_i; w))\}$$
4. If no improvement for
$$LL(y \mid X; w) = \sum_{i=1}^{N} y_i \ln(1 - g(X_i; w)) + (1 - y_i) \ln g(X_i; w)$$

   stop. Otherwise go to step 3

Example

Regularization

- Similar to other data estimation problems, we may not have enough samples to learn good models for logistic regression classification
- One way to overcome this is to ‘regularize’ the model, impose additional constraints on the parameters we are fitting.

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Regularization

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- One way to overcome this is to ‘regularize’ the model, impose additional constraints on the parameters we are fitting.
- For example, lets assume that w^j comes from a Gaussian distribution with mean 0 and variance σ^2 (where σ^2 is a user defined parameter): $w^j \sim N(0, \sigma^2)$
- In that case we have **a prior** on the parameters and so:

$$p(y=1,\theta \mid X) \propto p(y=1 \mid X;\theta)p(\theta)$$

Regularization

- If we regularize the parameters we need to take the prior into account when computing the posterior for our parameters

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- Thus, the log likelihood changes to :

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- And the new update rule (after taking the derivative w.r.t. w^j) is:

$$w^j \leftarrow w^j + \varepsilon \sum_{i=1}^{N} X_i^j \{y_i - (1 - g(X_i;w))\} - \varepsilon \frac{w^j}{\sigma^2}$$

Also known as the MAP estimate

The variance of our prior model

Regularization

- There are many other ways to regularize logistic regression
- The Gaussian model leads to an L2 regularization (we are trying to minimize the square value of w)
- Another popular regularization is an L1 which tries to minimize $|w|$

Important points

- Advantage of logistic regression over linear regression for classification
- Sigmoid function
- Gradient ascent / descent
- Regularization
- Logistic regression for multiple classes

Logistic regression

- The name comes from the **logit** transformation:

$$\log \frac{p(y=i \mid X;\theta)}{p(y=k \mid X;\theta)} = \log \frac{g(z_i)}{g(z_k)} = w_i^0 + w_i^1 x^1 + \ldots + w_i^d x^d$$

That’s all!