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# Mark Philip Roach - Thesis Defence

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Code/slides available at <https://github.com/MarkPhilipRoach>



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# Near-field Ptychography

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- Ptychography is a particular form of masked phase retrieval in which the collections of masks are generated by taking one physical mask and shifting it in space.
- Near-field Ptychography is when the distance between the lens, object, and detector are small (microscopic imaging).

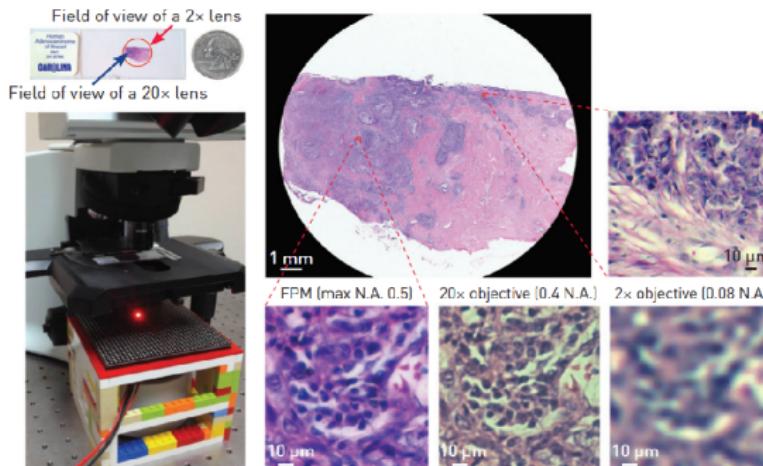


Figure: "Wide-field, high-resolution Fourier ptychographic microscopy" - G. Zheng et al., 2013



# Near-field Ptychographic Measurements

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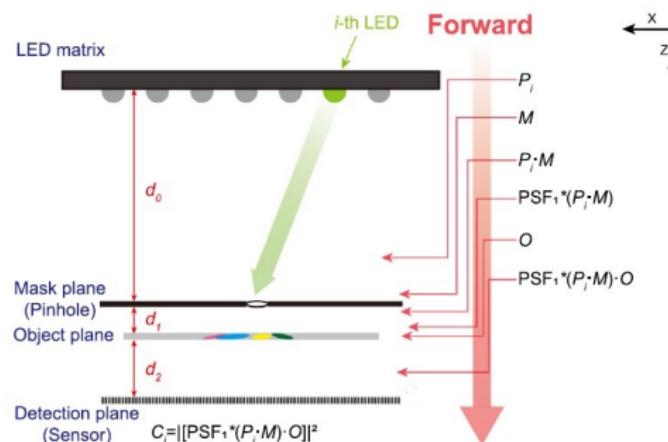
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- Let  $\mathbf{x} \in \mathbb{C}^d$  denote the unknown object.
- Let  $\mathbf{m}$  denote the known mask,  $\mathbf{p}$  the known point spread function (PSF).
- Measurements:  $Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2 + N_{k,\ell}, (k, \ell) \in \mathcal{K} \times \mathcal{L} \subseteq [d] \times [d]$ .
- Global phase ambiguity:**  $\mathbf{x}$  solution  $\rightarrow e^{i\phi} \mathbf{x}$  solution,  $\forall \phi \in [0, 2\pi)$ .
- $\mathcal{K}$ : set of shifts,  $\mathcal{L}$ : set of frequencies.
- Circular shift:  $(S_k \mathbf{m})_n = m_{n+k \bmod d}$ .
- Discrete convolution:  $(\mathbf{u} * \mathbf{v})_n = \sum_{k=0}^{d-1} u_k v_{n-k \bmod d}$ .
- Pointwise (Hadamard) product:  $(\mathbf{u} \circ \mathbf{v})_n = u_n v_n$ .





# Reducing To Inner Product Form

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- Consider noiseless measurements  $Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2$ ,  $(k, \ell) \in [d] \times [2\delta - 1]$ .
- By letting  $Y_{k,\ell} = Y_{-k,\ell+k}$ ,  $\check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}$ ,  $(\tilde{\mathbf{p}})_n = p_{-n}$ , one can show that

$$Y_{k,\ell} = |\langle \check{\mathbf{m}}_\ell^{(p,m)}, S_k \mathbf{x} \rangle|^2, \quad (k, \ell) \in [d] \times [2\delta - 1].$$

- Then  $\text{vec}(\mathbf{Y}) = \check{\mathbf{M}}\mathbf{z}$  for  $\mathbf{z} \in \mathbb{C}^d$  being a portion of  $\text{vec}(\mathbf{x}\mathbf{x}^*)$ , for block circulant matrix  $\check{\mathbf{M}}$  and  $\delta \ll d$  supported masks.
- Let  $D = d(2\delta - 1)$ . We then define the block circulant matrix  $\check{\mathbf{M}} \in \mathbb{C}^{D \times D}$  via

$$\check{\mathbf{M}} := \begin{pmatrix} \check{\mathbf{M}}_0 & \check{\mathbf{M}}_1 & \dots & \check{\mathbf{M}}_{\delta-1} & 0 & 0 & \dots & 0 \\ 0 & \check{\mathbf{M}}_0 & \check{\mathbf{M}}_1 & \dots & \check{\mathbf{M}}_{\delta-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \check{\mathbf{M}}_1 & \dots & \check{\mathbf{M}}_{\delta-1} & 0 & 0 & 0 & \dots & \check{\mathbf{M}}_0 \end{pmatrix},$$

where the matrices  $\check{\mathbf{M}}_k \in \mathbb{C}^{(2\delta-1) \times (2\delta-1)}$  are defined entry-wise by

$$(\check{\mathbf{M}}_k)_{ij} := \begin{cases} (\check{\mathbf{m}}_i)_k \overline{(\check{\mathbf{m}}_i)_{j+k}}, & 0 \leq j \leq \delta - k \\ (\check{\mathbf{m}}_i)_k (\check{\mathbf{m}}_i)_{j+k-2\delta}, & 2\delta - 1 + k \leq j \leq 2\delta - 2 \wedge k < \delta. \\ 0, & \text{otherwise} \end{cases}$$



# Condition Number Bound

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- Then  $\mathbf{z} = \check{\mathbf{M}}^{-1} \text{vec}(\mathbf{Y})$  and we reshape  $\mathbf{z}$  to recover  $\widehat{\mathbf{X}}$  whose non-zero entries are estimates of the  $\mathbf{x}\mathbf{x}^*$ .
- Angular synchronization is performed on  $\widehat{\mathbf{X}}$  to recover  $\mathbf{x}_{\text{est}}$ .
- Iwen et al. demonstrated that exponential masks  $\check{\mathbf{m}}_\ell^{(fpr)}$  defined by

$$(\check{\mathbf{m}}_\ell^{(fpr)})_n = \begin{cases} \frac{e^{-(n+1)/a}}{\sqrt[4]{2\delta-1}} \cdot e^{\frac{2\pi i n \ell}{2\delta-1}}, & n \in [\delta] \\ 0, & \text{otherwise} \end{cases}, \quad a := \max \left\{ 4, \frac{\delta-1}{2} \right\},$$

lead to lifted linear systems that are well-conditioned and allow for recovery guarantees.

Theorem (Theorem 4 - M. Iwen., et. al.)

For collection of masks  $\check{\mathbf{m}}_\ell^{(fpr)}$ , the condition number has the bound

$$\kappa(\check{\mathbf{M}}) = \frac{\sigma_{\max}}{\sigma_{\min}} \leq C \frac{\delta}{\frac{1}{\delta}} = C\delta^2, \quad C \in \mathbb{R}^+,$$

Furthermore,  $\check{\mathbf{M}}$  can be inverted in  $O(\delta \cdot d \log d)$ -time.



# Angular Synchronization

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## Example (Angular Synchronization)

Let  $d = 4, \delta = 2$ . Then

$$\widehat{\mathbf{X}} = \begin{pmatrix} |(\mathbf{x}_{\text{est}})_0|^2 & (\mathbf{x}_{\text{est}})_0 \overline{(\mathbf{x}_{\text{est}})_1} & 0 & (\mathbf{x}_{\text{est}})_0 \overline{(\mathbf{x}_{\text{est}})_3} \\ (\mathbf{x}_{\text{est}})_1 \overline{(\mathbf{x}_{\text{est}})_0} & |(\mathbf{x}_{\text{est}})_1|^2 & (\mathbf{x}_{\text{est}})_1 \overline{(\mathbf{x}_{\text{est}})_2} & 0 \\ 0 & (\mathbf{x}_{\text{est}})_2 \overline{(\mathbf{x}_{\text{est}})_1} & |(\mathbf{x}_{\text{est}})_2|^2 & (\mathbf{x}_{\text{est}})_2 \overline{(\mathbf{x}_{\text{est}})_3} \\ (\mathbf{x}_{\text{est}})_3 \overline{(\mathbf{x}_{\text{est}})_0} & 0 & (\mathbf{x}_{\text{est}})_3 \overline{(\mathbf{x}_{\text{est}})_2} & |(\mathbf{x}_{\text{est}})_3|^2 \end{pmatrix}.$$

Lead eigenvector is  $\mathbf{u} = (e^{i\theta_0} \ e^{i\theta_1} \ e^{i\theta_2} \ e^{i\theta_3})^T$  and therefore

$$\mathbf{x}_{\text{est}} = \sqrt{\text{diag}(\widehat{\mathbf{X}})} \circ \mathbf{u} = (|(\mathbf{x}_{\text{est}})_0|e^{i\theta_0} \ |(\mathbf{x}_{\text{est}})_1|e^{i\theta_1} \ |(\mathbf{x}_{\text{est}})_2|e^{i\theta_2} \ |(\mathbf{x}_{\text{est}})_3|e^{i\theta_3})^T.$$

## Example

$$\text{Let } d = 4, \delta = 2. \text{ Then } \mathbf{X} = \begin{pmatrix} |(\mathbf{x})_0|^2 & (\mathbf{x})_0 \overline{(\mathbf{x})_1} & 0 & (\mathbf{x})_0 \overline{(\mathbf{x})_3} \\ (\mathbf{x})_1 \overline{(\mathbf{x})_0} & |(\mathbf{x})_1|^2 & (\mathbf{x})_1 \overline{(\mathbf{x})_2} & 0 \\ 0 & (\mathbf{x})_2 \overline{(\mathbf{x})_1} & |(\mathbf{x})_2|^2 & (\mathbf{x})_2 \overline{(\mathbf{x})_3} \\ (\mathbf{x})_3 \overline{(\mathbf{x})_0} & 0 & (\mathbf{x})_3 \overline{(\mathbf{x})_2} & |(\mathbf{x})_3|^2 \end{pmatrix}$$



# Admissible Selection of PSF and Mask

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- We would like to show there are realistic  $\mathbf{p}$  and  $\mathbf{m}$  with  $\check{\mathbf{m}}_\ell^{(fpr)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}$ .
- One can show that there does not exist  $\mathbf{p}$  and  $\mathbf{m}$  that are independent of  $\ell$  that accomplishes this, however we can approximate it up to a phase factor.

## Lemma (Admissible Selection of PSF and Mask)

Let  $\mathbf{p}, \mathbf{m} \in \mathbb{C}^d$  have entries given by

$$p_n := e^{-\frac{2\pi i n^2}{2\delta-1}}, \quad \text{and} \quad m_n := \begin{cases} \frac{e^{-n+1}/a}{\sqrt[4]{2\delta-1}} \cdot e^{\frac{2\pi i n^2}{2\delta-1}}, & n \in [\delta]_0 \\ 0, & \text{otherwise} \end{cases},$$

where  $a := \max \left\{ 4, \frac{\delta-1}{2} \right\}$ . Then for all  $\ell \in [2\delta-1]_0$ ,  $\check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}$  satisfies

$$\check{\mathbf{m}}_\ell^{(p,m)} = e^{\frac{2\pi i \ell^2}{2\delta-1}} \cdot \check{\mathbf{m}}_{2\ell \bmod 2\delta-1}^{(fpr)}.$$

- Let  $\tilde{\mathbf{M}}^{(fpr)}$  and  $\tilde{\mathbf{M}}^{(p,m)}$  be our lifted linear measurement matrices obtained by  $\check{\mathbf{m}}_\ell^{(fpr)}$  and  $\check{\mathbf{m}}_\ell^{(p,m)}$ , respectively.
- Then  $\tilde{\mathbf{M}}^{(p,m)} = \tilde{\mathbf{P}} \tilde{\mathbf{M}}^{(fpr)}$ , where  $\mathbf{P}$  is a  $D \times D$  block diagonal permutation matrix.
- Thus  $\tilde{\mathbf{M}}^{(p,m)}$  and  $\tilde{\mathbf{M}}^{(fpr)}$  have the same singular values and

$$\kappa\left(\tilde{\mathbf{M}}^{(p,m)}\right) = \kappa\left(\tilde{\mathbf{M}}^{(fpr)}\right) \leq C\delta^2.$$



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# Magnitude Error Bound

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- For  $\mathbf{x} \in \mathbb{C}^d$ , we write its  $n^{\text{th}}$  entry as  $x_n =: |x_n|e^{i\theta_n}$ .
- Let  $\mathbf{x}^{(\text{mag})} := (|x_0|, \dots, |x_{d-1}|)^T$ ,  $\mathbf{x}^{(\theta)} := (e^{i\theta_0}, e^{i\theta_1}, \dots, e^{i\theta_{d-1}})^T$ .
- We may then decompose  $\mathbf{x}$  as  $\mathbf{x} = \mathbf{x}^{(\text{mag})} \circ \mathbf{x}^{(\theta)}$ .

## Lemma (Lemma 3 - M. Roach, et al.)

Let  $\mathbf{x}_{\text{est}}$  be decomposed  $\mathbf{x}_{\text{est}} = \mathbf{x}_{\text{est}}^{(\text{mag})} \circ \mathbf{x}_{\text{est}}^{(\theta)}$ . Then, we have that

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x} - e^{i\phi} \mathbf{x}_{\text{est}}\|_2 \leq \|\mathbf{x}\|_\infty \min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)}\|_2 + \|\mathbf{x}^{(\text{mag})} - \mathbf{x}_{\text{est}}^{(\text{mag})}\|_2.$$

- We now utilize a restated result from *Phase Retrieval from Local Measurements: Improved Robustness via Eigenvector Based Angular Synchronization* - Iwen, M., Preskitt, B., Saab, R., Viswanathan, A.

## Lemma (Lemma 3 - M. Iwen, et al.)

Let  $\sigma_{\min}(\widetilde{\mathbf{M}}^{(p,m)})$  denote the smallest singular value of the lifted measurement matrix  $\widetilde{\mathbf{M}}^{(p,m)}$ . Then  $\|\mathbf{x}^{(\text{mag})} - \mathbf{x}_{\text{est}}^{(\text{mag})}\|_2 \leq C \sqrt{\frac{\|\mathbf{N}\|_F}{\sigma_{\min}(\widetilde{\mathbf{M}}^{(p,m)})}}$ .



# Angular Error Bound

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- Apply lemma to our error bound and utilize the conditioning of  $\mathbf{p}, \mathbf{m}$ .

## Theorem (Theorem 3 - M. Roach, et al.)

Let  $\mathbf{p}$  and  $\mathbf{m}$  be our admissible PSF, mask pair. Then, we have that

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x} - e^{i\phi} \mathbf{x}_{\text{est}}\|_2 \leq \|\mathbf{x}\|_\infty \min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)}\|_2 + C \sqrt{d\delta \|\mathbf{N}\|_F}.$$

- In *On Recovery Guarantees for Angular Synchronization* - Filbir, Krahmer, Melnyk, authors used weighted graph approach to prove bound on  $\|\mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)}\|_2$ .
- Let  $\mathbf{X}, \widehat{\mathbf{X}}$  denote partial autocorrelation matrix corresponding to  $\mathbf{x}, \mathbf{x}_{\text{est}}$ .
- Let  $G = (V, E, \mathbf{W})$  be weighted graph,  $V = [d]_0$ ,  $E = \{(i, j) \mid i \neq j, |i - j| \bmod d < \delta\}$ , and whose weight matrix  $\mathbf{W}$  is defined entrywise by

$$W_{i,j} = \begin{cases} |\widehat{X}_{i,j}|^2, & 0 < |i - j| \bmod d < \delta \\ 0, & \text{otherwise} \end{cases}.$$



# Graph Theoretic Approach

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- Let  $\mathbf{D}$  denote the weighted degree matrix,  $\mathbf{L}_G := \mathbf{D} - \mathbf{W}$ .
- Let  $\tau_G$  denote the spectral gap (second smallest eigenvalue) of  $\mathbf{L}_G$ .
- Since  $G$  is connected,  $\tau_G$  is strictly positive.

## Theorem (Corollary 3 - F. Filbir, F. Krahmer, O. Melnyk)

Let  $\tau_G$  denote the spectral gap of the associated unnormalized Laplacian  $\mathbf{L}_G$ . Then we have that

$$\min_{\phi \in [0, 2\pi)} \left\| \mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)} \right\|_2 \leq C \sqrt{1 + \|\mathbf{x}_{\text{est}}\|_\infty} \cdot \frac{\|\mathbf{X} - \widehat{\mathbf{X}}\|_F}{\sqrt{\tau_G}}, \quad C \in \mathbb{R}^+.$$

- We have a bound for the Frobenius norm

## Lemma (Lemma 6 - M. Roach, et al.)

Let  $\mathbf{p}$  and  $\mathbf{m}$  be our choice of PSF and mask. Then,  $\|\mathbf{X} - \widehat{\mathbf{X}}\|_F \leq C\delta\|\mathbf{N}\|_F$ .



# Spectral Bound of Weighted Graph

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- We now seek a lower bound for our weighted spectral gap.

## Theorem (General Weighted Spectral Gap Bound)

Let  $G = (V, E, \mathbf{W})$  be a weighted graph and let  $W_{\min}$  and  $W_{\max}$  be the minimum and maximum value of any its (nonzero) weights. Then

$$\tau_G \geq \frac{2 \cdot (W_{\min})^2}{W_{\max}(n - 1) \cdot \text{diam}(G_{unw})},$$

where  $G_{unw} = (V, E)$  is the unweighted counterpart of  $G$ , and  $\text{diam}(G_{unw})$  is the corresponding diameter.

## Lemma (NFP Weighted Spectral Gap Bound)

Let  $|\mathbf{x}_{\text{est}}|_{\min}$  denote the smallest magnitude of any entry in  $\mathbf{x}_{\text{est}}$ . For our graph  $G$ , we have that

$$\tau_G \geq \frac{|\mathbf{x}_{\text{est}}|_{\min}^4}{\|\mathbf{x}_{\text{est}}\|_{\infty}^2} \frac{4(\delta - 1)}{d^2}.$$



# Recovery Guarantee Theorem

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- We now apply the Frobenius bound and spectral gap bound to our existing error bound to obtain our main result.

## Theorem (Theorem 1 - M. Roach, et al.)

Choose  $\delta \in [d]_0$  such that  $2\delta - 1$  divides  $d$ . One can construct a PSF  $\mathbf{p} \in \mathbb{C}^d$  and a mask  $\mathbf{m} \in \mathbb{C}^d$  with  $\text{supp}(\mathbf{m}) \subseteq [\delta]_0$  such that we can return an estimate  $\mathbf{x}_{\text{est}} \in \mathbb{C}^d$  of  $\mathbf{x}$  satisfying

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{\text{est}} - e^{i\phi} \mathbf{x}\|_2 \leq C \left( \|\mathbf{x}\|_\infty \frac{d \sqrt{\delta} \sqrt{\|\mathbf{x}_{\text{est}}\|_\infty^2 + \|\mathbf{x}_{\text{est}}\|_\infty^3}}{|\mathbf{x}_{\text{est}}|_{\min}^2} \cdot \|\mathbf{N}\|_{\text{F}} + \sqrt{d\delta \|\mathbf{N}\|_{\text{F}}} \right).$$

Here  $C \in \mathbb{R}^+$  is an absolute constant.



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# SNR vs. Reconstruction Error

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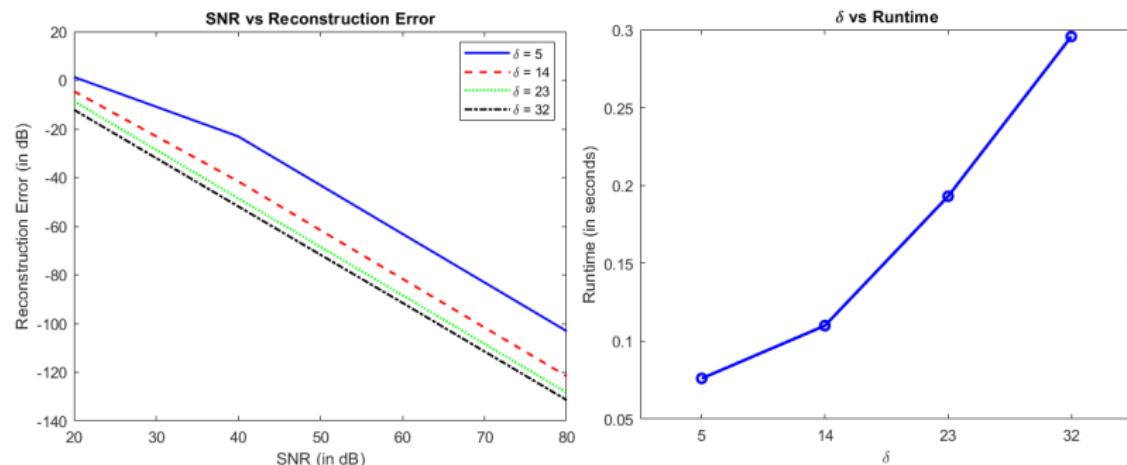
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- Evaluation of algorithm with Gaussian  $\mathbf{x}$  for proposed  $\mathbf{p}, \mathbf{m}, d = 945$ .
- Signal-to-noise ratio is given by  $SNR = 10 \log_{10} \left( \frac{\|\mathbf{Y} - \mathbf{N}\|_F}{\|\mathbf{N}\|_F} \right)$ .
- Reconstruction error given by  $10 \log_{10} \left( \frac{\min_{\phi} \|\mathbf{x} - e^{i\phi} \mathbf{x}_{est}\|_2^2}{\|\mathbf{x}\|_2^2} \right)$ .
- Left: Reconstruction error vs SNR for various  $\delta = |\text{supp}(\mathbf{m})|$ .
- Right: Runtime as a function of  $\delta$ .



# Real World Application

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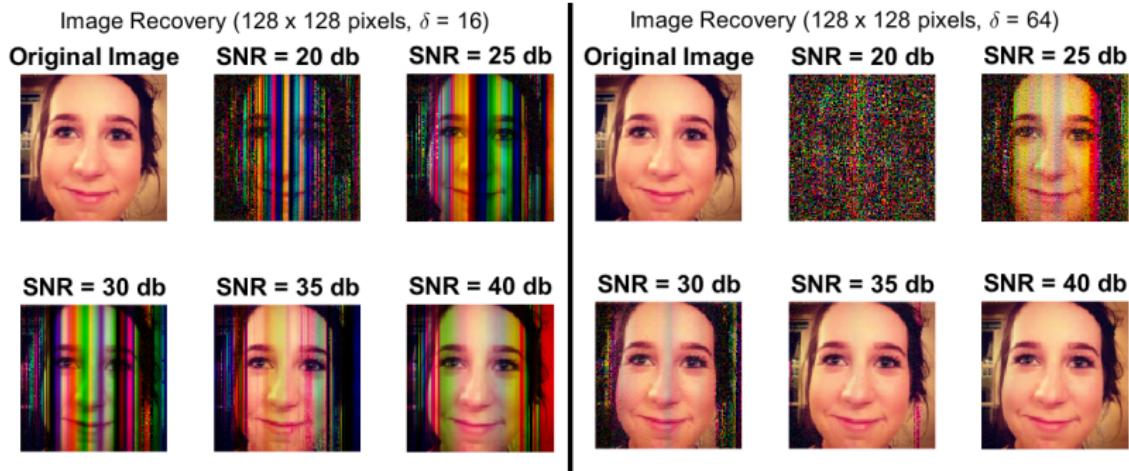
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- NFP BlockPR algorithm applied to  $128 \times 128$  pixel color image.
- Each color channel applied separately and then combined to form final image.
- $d = 128^2 = 16,384$  with two delta levels applied and varying noise.



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# Far-field Ptychography

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- Far-field Ptychography is when there is a large enough distance between the lens, object, and detector to obtain magnitude-square Fourier transform measurements.

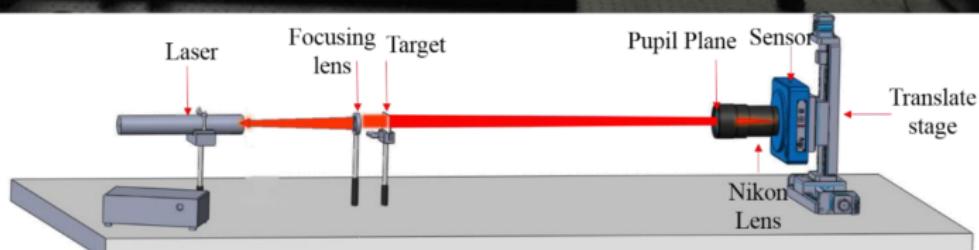
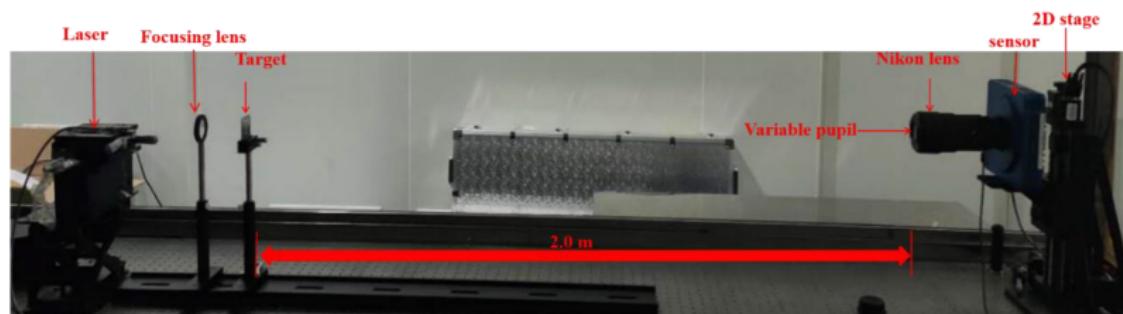


Figure: "Analysis, Simulations, and Experiments for Far-Field Fourier Ptychography Imaging Using Active Coherent Synthetic-Aperture" - Yang et al., 2022



# Blind Ptychography

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- Let  $\mathbf{x}, \mathbf{m} \in \mathbb{C}^d$  denote the unknown object and mask, respectively.
- Suppose that we have  $d^2$  noisy ptychographic measurements of the form

$$(\mathbf{Y})_{\ell,k} = |(\mathbf{F}(\mathbf{x} \circ S_k \mathbf{m}))_\ell|^2 + (\mathbf{N})_{\ell,k}, \quad (\ell, k) \in [d] \times [d].$$

- Can rewrite the measurements as

$$\left( \mathbf{Y}^T \mathbf{F}^T \right)_k = d \cdot (\mathbf{x} \circ S_k \bar{\mathbf{x}}) * (\tilde{\mathbf{m}} \circ S_{-k} \tilde{\mathbf{m}}) + \left( \mathbf{N}^T \mathbf{F}^T \right)_k.$$

- Fix  $k$ . Let  $\mathbf{y}' = \left( \mathbf{Y}^T \mathbf{F}^T \right)_k$ ,  $\mathbf{f} = \sqrt{d}(\tilde{\mathbf{m}} \circ S_{-k} \tilde{\mathbf{m}})$ ,  $\mathbf{g} = \sqrt{d}(\mathbf{x} \circ S_k \bar{\mathbf{x}})$ ,  $\mathbf{n} = \left( \mathbf{N}^T \mathbf{F}^T \right)_k$
- This is now a noisy blind deconvolution problem i.e.

$$\mathbf{y}' = \mathbf{f} * \mathbf{g} + \mathbf{n}.$$



# Blind Deconvolution - Part I

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- Strohmer et.al., demonstrated successful recovery by utilizing reasonable assumptions.
- Let  $\mathbf{y}' = \mathbf{f} * \mathbf{g} + \mathbf{n}$ , with  $\mathbf{f}, \mathbf{g}$  unknown,  $\mathbf{n}$  noise.
- Assume  $\mathbf{f} = \begin{bmatrix} \mathbf{h} \\ \mathbf{0}_{d-K} \end{bmatrix}$ ,  $\mathbf{g} = \mathbf{C}\bar{\mathbf{x}}$  for some matrix  $\mathbf{C} \in \mathbb{C}^{d \times N}$ ,  $N \ll d$ .
- Let  $\mathbf{F} \in \mathbb{C}^{d \times d}$  be DFT matrix,  $\mathbf{B} \in \mathbb{C}^{d \times K}$  denote the first  $K$  columns of  $\mathbf{F}_d$ .
- Then we have that

$$\mathbf{y} = \mathbf{B}\mathbf{h} \circ \overline{\mathbf{A}\mathbf{x}} + \mathbf{e},$$

where  $\mathbf{y} = \frac{1}{\sqrt{d}}\widehat{\mathbf{y}'}$ ,  $\bar{\mathbf{A}} = \mathbf{F}\mathbf{C} \in \mathbb{C}^{d \times N}$ , and  $\mathbf{e} = \frac{1}{\sqrt{d}}\mathbf{F}\mathbf{n}$  represents noise.

- If  $(\mathbf{h}_0, \mathbf{x}_0)$  is a solution, then so is  $(\alpha\mathbf{h}_0, \alpha^{-1}\mathbf{x}_0)$  for any non-zero  $\alpha$ . Thus we assume  $\|\mathbf{f}\|_2$  known.
- Measurements:

$$(\mathbf{Y})_{\ell,k} = |(\mathbf{F}(\mathbf{x} \circ S_k \mathbf{m}))_\ell|^2 + (\mathbf{N})_{\ell,k}.$$

$(\mathbf{x}, \mathbf{m})$  solution  $\rightarrow (\alpha\mathbf{x}, \alpha^{-1}\mathbf{m})$  solution,  $\forall \alpha \in \mathbb{R}$



# Blind Deconvolution - Part II

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- We define the matrix-valued linear operator  $\mathcal{A} : \mathbb{C}^{K \times N} \longrightarrow \mathbb{C}^d$  by

$$\mathcal{A}(\mathbf{Z}) := \{\mathbf{b}_\ell^* \mathbf{Z} \mathbf{a}_\ell\}_{\ell=1}^d,$$

where  $\mathbf{b}_k$  denotes the  $k$ -th column of  $\mathbf{B}^*$ , and  $\mathbf{a}_k$  is the  $k$ -th column of  $\mathbf{A}^*$ .

- Then  $\mathbf{y} = \mathcal{A}(\mathbf{h}\mathbf{x}^*) + \mathbf{e}$ .
- Define corresponding adjoint operator  $\mathcal{A}^* : \mathbb{C}^d \longrightarrow \mathbb{C}^{K \times N}$ , given by

$$\mathcal{A}^*(\mathbf{z}) := \sum_{k=1}^d \mathbf{z}_k \mathbf{b}_k \mathbf{a}_k^*.$$

- For any given  $\mathbf{Z} \in \mathbb{C}^{K \times N}$ ,  $\mathbb{E}(\mathcal{A}^*(\mathcal{A}(\mathbf{Z}))) = \mathbf{Z}$ .
- Leading singular vectors of  $\mathcal{A}^*(\mathbf{y}) \approx (\mathbf{h}, \mathbf{x})$ .
- Apply gradient descent using gradients of  $F(\mathbf{h}, \mathbf{x}) := \|\mathcal{A}(\mathbf{h}\mathbf{x}^*) - \mathbf{y}\|^2$ .



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# Blind Ptychography Algorithm

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- Let  $\mathbf{y}_{(k)} = \frac{1}{\sqrt{d}} \cdot (\mathbf{F}(\widetilde{(\mathbf{F}\mathbf{Y})^T}))_k$ ,  $\mathbf{f}_{(k)} = \tilde{\mathbf{m}} \circ S_{-k} \bar{\mathbf{m}}$ . ( $\|\mathbf{m}\|_2$  known)
- Let  $\mathbf{x} = \mathbf{Cx}'$  for some known matrix  $\mathbf{C} \in \mathbb{C}^{d \times N}$ ,  $\mathbf{x}' \in \mathbb{C}^N$ .

## Lemma

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{C}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{C}^{m \times q}$ ,  $\mathbf{D} \in \mathbb{C}^{q \times p}$ . Then we have that

$$(\mathbf{AB}) \circ (\mathbf{CD}) = (\mathbf{A} \bullet \mathbf{C})(\mathbf{B} \odot \mathbf{D}),$$

where  $\circ$  is the Hadamard product,  $\bullet$  is the standard Khatri-Rao product,  $\odot$  is the transposed Khatri-Rao product.

- Let  $\mathbf{g}_{(k)} = \mathbf{x} \circ S_k \bar{\mathbf{x}} = \mathbf{Cx}' \circ S_k \bar{\mathbf{C}} \bar{\mathbf{x}'}$ .
- Then  $\mathbf{g}_{(k)} = \mathbf{C}'_{(k)} \mathbf{x}''$  where  $\mathbf{C}' \in \mathbb{C}^{d \times N^2}$ ,  $\mathbf{x}'' \in \mathbb{C}^{N^2}$  are given by

$$\mathbf{C}'_{(k)} = \mathbf{C} \bullet S_k \bar{\mathbf{C}}, \quad \mathbf{x}'' = \mathbf{x}' \odot \bar{\mathbf{x}'}, \quad 0 \leq k \leq K, d - K + 1 \leq k \leq d.$$



## Definition

Let  $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{m \times n}$  and  $\mathbf{B} = (B_{i,j}) \in \mathbb{C}^{p \times q}$ . Then the **Kronecker product**  $\mathbf{A} \otimes \mathbf{B} \in \mathbb{C}^{mp \times nq}$  is defined by

$$(\mathbf{A} \otimes \mathbf{B})_{n(i-1)+k, q(j-1)+\ell} = A_{i,j} B_{k,\ell}.$$

## Definition

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  and  $\mathbf{B} \in \mathbb{C}^{p \times n}$  with columns  $\mathbf{a}_i, \mathbf{b}_i$  for  $i \in [n]$ . Then the **Khatri-Rao product**  $\mathbf{A} \bullet \mathbf{B} \in \mathbb{C}^{mp \times n}$  is defined by

$$\mathbf{A} \bullet \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1 \ \mathbf{a}_2 \otimes \mathbf{b}_2 \dots \mathbf{a}_n \otimes \mathbf{b}_n].$$

## Definition

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  and  $\mathbf{B} \in \mathbb{C}^{m \times p}$  be matrices with rows  $\mathbf{a}_i, \mathbf{b}_i$  for  $i \in [m]$ . Then the **transposed Khatri-Rao product** (or **face-splitting product**), denoted  $\odot$ , is the matrix whose rows are Kronecker products of the columns of  $\mathbf{A}$  and  $\mathbf{B}$  i.e. the rows of  $\mathbf{A} \odot \mathbf{B} \in \mathbb{C}^{m \times np}$  are given by

$$(\mathbf{A} \odot \mathbf{B})_i = \mathbf{a}_i \otimes \mathbf{b}_i, \quad i \in [m].$$



# Performing Series of Blind Deconvolutions

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- Perform  $2\delta - 1$  blind deconvolutions, with  $\mathbf{y}_{(k)}, \mathbf{f}_{(k)}, \mathbf{g}_{(k)}, \mathbf{C}$ , as stated.
- We obtain  $2\delta - 1$  estimates for

$$\mathbf{x}' \odot \bar{\mathbf{x}}' = \text{vec}(\mathbf{x}'(\mathbf{x}')^*)^T.$$

- Reshape, use angular synchronization to solve for  $2\delta - 1$  estimates  $\mathbf{x}'_{\text{est}}$ .
- Thus solve for  $2\delta - 1$  estimates

$$\mathbf{x}_{\text{est}}^i = \mathbf{C}\mathbf{x}'_{\text{est}}, \quad i \in [2\delta - 1].$$

- Use these estimates  $\mathbf{x}_{\text{est}}^i$  to compute  $2\delta - 1$  estimates  $\mathbf{m}_{\text{est}}^j, j \in [2\delta - 1]$ .
- Let  $\alpha^i = \frac{\|\mathbf{m}_{\text{est}}^i\|_2}{\|\mathbf{m}\|_2}$ . Then  $\mathbf{x}_{\text{est}}^i = \alpha^i \mathbf{x}_{\text{est}}^i, \mathbf{m}_{\text{est}}^i = \frac{1}{\alpha^i} \mathbf{m}_{\text{est}}^i, \quad \forall i \in [2\delta - 1]$ .



# Computing the Mask

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- From our original measurements, it can be further shown that

$$\mathbf{F}\left(\mathbf{Y}^T \mathbf{F}^T\right)_k = d \cdot \mathbf{F}(\mathbf{x} \circ S_k \bar{\mathbf{x}}) \circ \mathbf{F}(\tilde{\mathbf{m}} \circ S_{-k} \tilde{\mathbf{m}}) + \mathbf{F}\left(\mathbf{N}^T \mathbf{F}^T\right)_k.$$

- Assuming noiseless scenario, then we have that

$$\mathbf{F}(\tilde{\mathbf{m}} \circ S_{-k} \tilde{\mathbf{m}}) = \frac{1}{d} \frac{\mathbf{F}\left(\mathbf{Y}^T \mathbf{F}^T\right)_k}{\mathbf{F}(\mathbf{x} \circ S_k \bar{\mathbf{x}})}.$$

- So once we have an estimate for  $\mathbf{x}$ , we compute the  $2\delta - 1$  pointwise divisions, then  $2\delta - 1$  inverse Fourier transforms.
- These become the diagonals of a matrix which will estimate  $\tilde{\mathbf{m}}\tilde{\mathbf{m}}^*$ . Thus compute angular synchronization and finally a reversal.



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- Ideally, we would want to select the estimates which generate the minimum error for each  $\mathbf{x}$  and  $\mathbf{m}$  i.e. find

$$\text{Min Shift}^{(x)} = \underset{\mathbf{x}_{\text{est}}^i}{\operatorname{argmin}} \|\mathbf{x} - \mathbf{x}_{\text{est}}^i\|_2^2, \quad \text{Min Shift}^{(m)} = \underset{\mathbf{m}_{\text{est}}^j}{\operatorname{argmin}} \|\mathbf{m} - \mathbf{m}_{\text{est}}^j\|_2^2,$$

for  $i, j \in [2\delta - 1]$ .

- However that implies prior knowledge of  $\mathbf{x}$  and  $\mathbf{m}$ .
- Instead, compute  $(2\delta - 1)^2$  estimates of the Fourier measurements by

$$(\mathbf{Y}_{\text{est}}^{i,j})_{\ell,k} = |(\mathbf{F}(\mathbf{x}_{\text{est}}^i \circ S_k \mathbf{m}_{\text{est}}^j))_\ell|^2, \quad i, j \in [2\delta - 1].$$

- We then compute the associated error

$$(i', j') = \underset{(i,j)}{\operatorname{argmin}} \frac{\|\mathbf{Y}_{\text{est}}^{i,j} - \mathbf{Y}\|_F^2}{\|\mathbf{Y}\|_F^2}, \quad i, j \in [2\delta - 1].$$

- Let  $\mathbf{x}_{\text{est}} = \mathbf{x}_{\text{est}}^{i'}, \mathbf{m}_{\text{est}} = \mathbf{m}_{\text{est}}^{j'}$ .



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# SNR vs. Reconstruction Error

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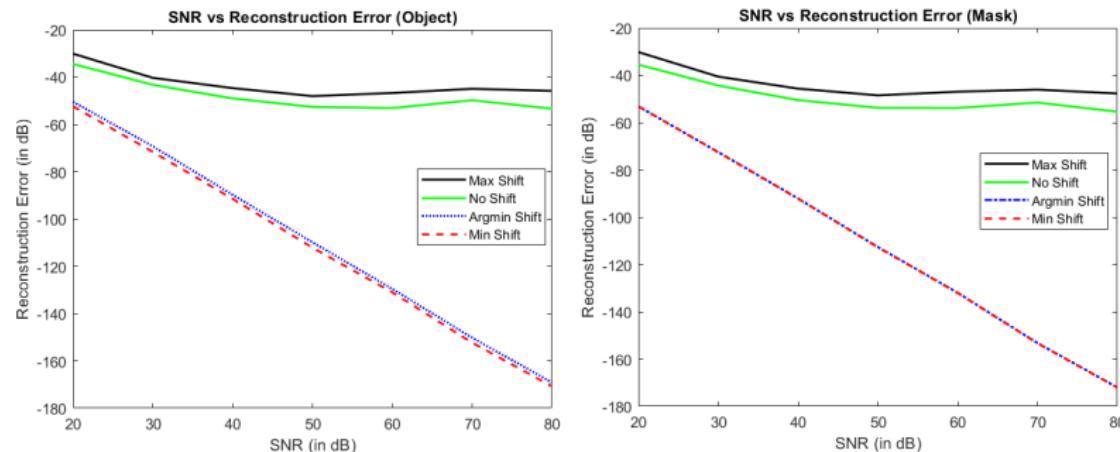
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- $d = 2^6, K = \delta = \log_2 d, N = 6, \mathbf{C}$  complex Gaussian, 100 simulations.
- Max shift := maximal possible error.
- Min shift := minimal possible error.
- No Shift := singular shift  $k = 0$ .
- Argmin Shift := choice of object and mask chosen by algorithm.



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# Johnson-Lindenstrauss Lemma

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- We consider real-world problems with geometric input data  $X$  and want to consider metric embeddings  $X \rightarrow Y$  where  $\dim(Y) \ll \dim(X)$ .
- Working with lower-dimensional embedded data typically results in efficiency gains, in terms of memory, running time, and/or other resources but reduction in accuracy.

## Lemma (Johnson-Lindenstrauss Lemma)

Let  $\epsilon \in (0, 1)$  and  $X \subset \mathbb{R}^d$  be arbitrary with  $|X| = n > 1$ . There exists  $f : X \rightarrow \mathbb{R}^m$  with  $m = O(\epsilon^{-2} \log n)$  such that  $\forall \mathbf{x} \in X, \forall \mathbf{y} \in X$ ,

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2.$$

- $1 - \epsilon, 1 + \epsilon$ , are referred to as the **distortion**,  $Dist(f)$ , of the embedding  $f$ .
- Let  $X$  denote training data. Lemma states distance between training data preserved up to small distortion.
- Want distance between training data and all  $\mathbb{R}^d$  (testing data) to be preserved up to small distortion.



# Terminal Embedding Theorem

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- Work by Elkin et.al showed that  $y$  can be taken as an arbitrary point in  $\mathbb{R}^d$  with  $m = O(\log n)$  and distortion  $\approx \sqrt{10}$ .

## Theorem (Theorem 1.1 - Narayanan, Nelson)

Let  $\epsilon \in (0, 1)$  and  $X \subset \mathbb{R}^d$  be arbitrary with  $|X| = n > 1$ . There exists  $f : X \longrightarrow \mathbb{R}^m$  with  $m = O(\epsilon^{-2} \log n)$  such that  $\forall \mathbf{x} \in X, \forall \mathbf{y} \in \mathbb{R}^d$ ,

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2.$$

- This embedding is called a **terminal embedding** with multiplicative factor on the right hand side referred to as the **terminal distortion**.
- If the points in  $X$  are mapped to  $\mathbb{R}^m$  well, which occurs with high probability, then our final terminal embedding is guaranteed to have low terminal distortion as a map from all of  $\mathbb{R}^d$  to  $\mathbb{R}^m$ .



# Outer Extension, $\epsilon$ -Convex Hull Distortion

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## Definition (Outer extension)

For  $f : X \rightarrow \mathbb{R}^m$  and  $X \subset Z$ , we say  $g : Z \rightarrow \mathbb{R}^{m'}$  is an **outer extension** of  $f$  if  $m' \geq m$ , and  $g(\mathbf{x})$  being a zero-padding of  $f(\mathbf{x})$ , for  $\mathbf{x} \in X$ .

- We define our terminal embedding by  $f_{\text{Ext}}(\mathbf{u}) = \begin{cases} (f(\mathbf{u}), 0), & \mathbf{u} \in X \\ f^{(\mathbf{u})}(\mathbf{u}), & \mathbf{u} \in \mathbb{R}^d \setminus X \end{cases}$  which is an outer extension on  $X$ , and  $f^{(\mathbf{u})}$  function depending on  $\mathbf{u}$
- The goal is to specify how to embed points  $\mathbf{x} \in X$  and then how to embed points  $\mathbf{u} \notin X$  such that we obtain low distortion.

## Definition ( $\epsilon$ -convex hull distortion)

For  $T \subset \mathcal{S}^{d-1}$  and  $\epsilon \in (0, 1)$ , we say that  $\Pi \in \mathbb{R}^{m \times d}$  provides  **$\epsilon$ -convex hull distortion** for  $T$  if

$$|\|\Pi \mathbf{x}\|_2 - \|\mathbf{x}\|_2| < \epsilon, \quad \forall \mathbf{x} \in \text{conv}(T),$$

where  $\mathcal{S}^{d-1} \subset \mathbb{R}^d$  unit sphere,  $\text{conv}(T)$  convex hull of  $T$



# Construction of Terminal Embedding

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## Lemma (Lemma 3.2 - Narayanan, Nelson)

Let  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$  be distinct. Let  $Y = \{\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|_2} : i \neq j\}$ . Suppose  $\Pi$  provides an  $\epsilon$ -convex hull distortion for  $Y$ . Then  $\forall \mathbf{u} \in \mathbb{R}^d$ ,  $\exists$  outer extension  $f_{\text{Ext}}$  with  $f(\mathbf{x}_i) = \Pi \mathbf{x}_i$ ,  $f^{(\mathbf{u})}$  defined below, which provides  $(1 + \epsilon)$ -distortion.

- By choosing  $\Pi \in \mathbb{R}^{m \times d}$  to have i.i.d sub-Gaussian ( $\mathbb{P}(|X| < t) \leq 2e^{-t^2/C^2}$ ) entries, we get with high probability, a matrix providing  $\epsilon$ -convex hull distortion for  $Y$ .
- Let  $\mathbf{u}'$  be the output of constrained minimization problem such that  $f^{(\mathbf{u})}$  is a function of  $\mathbf{u}'$

## Lemma

Let  $\Pi$  be an  $\epsilon$ -convex hull distortion for  $Y$ . Then  $\mathbf{u}'$  exists for all  $\mathbf{u}$ , and  $\mathbf{u}'$  can be found with semi-definite programming in polynomial time.



# Terminal Embedding of a Finite Set

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## Algorithm Terminal Embedding of a Finite Set

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**Input:**  $\epsilon \in (0, 1)$ ,  $X \subset \mathbb{R}^N$ ,  $|X| =: n$ ,  $S \subset \mathbb{R}^N$ ,  $|S| =: n'$ ,  $S \cap X = \emptyset$ ,  $m \in \mathbb{N}$  with  $m < N$ , a random matrix Gaussian entries,  $\Phi \in \mathbb{R}^{m \times N}$ ,  $\Pi := \frac{1}{\sqrt{m}}\Phi$

**Output:** A terminal embedding of  $X$ ,  $f \in \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$ , evaluated on  $S$

for  $\mathbf{u} \in S$  do

1) Compute  $\mathbf{x}_{NN} := \operatorname{argmin}_{\mathbf{x} \in X} \|\mathbf{u} - \mathbf{x}\|_2$

2) Solve constrained minimization problem to compute  $\mathbf{u}' \in \mathbb{R}^m$

$$\text{Minimize } h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \|\mathbf{z}\|_2^2 + 2\langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$$

subject to  $\|\mathbf{z}\|_2 \leq \|\mathbf{u} - \mathbf{x}_{NN}\|_2, \quad \forall \mathbf{x} \in X$

$$|\langle \mathbf{z}, \Pi(\mathbf{x} - \mathbf{x}_{NN}) \rangle - \langle \mathbf{u} - \mathbf{x}_{NN}, \mathbf{x} - \mathbf{x}_{NN} \rangle| \leq \epsilon \|\mathbf{u} - \mathbf{x}_{NN}\|_2 \|\mathbf{x} - \mathbf{x}_{NN}\|_2$$

3) Compute  $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$  at  $\mathbf{u}$  via

$$f(\mathbf{u}) := \begin{cases} (\Pi \mathbf{u}, 0), & \mathbf{u} \in X \\ (\Pi \mathbf{x}_{NN} + \mathbf{u}', \sqrt{\|\mathbf{u} - \mathbf{x}_{NN}\|_2^2 - \|\mathbf{u}'\|_2^2}), & \mathbf{u} \in S \end{cases}$$

end for

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# Measuring Compressive Nearest Neighbor Classification Accuracy

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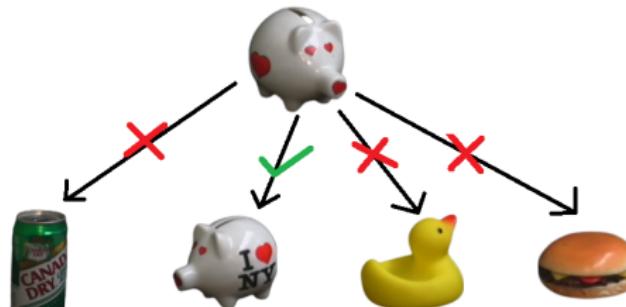
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- Let  $\epsilon \in (0, 1)$ , labeled data set  $\mathcal{D} \subset \mathbb{R}^N$ : Training set  $X \subset \mathcal{D}$  with  $|X| =: n$ , Test set  $S \subset \mathcal{D}$  with  $|S| =: n'$ , such that  $S \cap X = \emptyset$ .
- Let  $m \ll N$  and fix  $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$  such that

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2, \quad \mathbf{x}, \mathbf{y} \in X$$

- For  $\mathbf{x} \in X, \mathbf{u} \in S$ , compute  $f(\mathbf{x}), f(\mathbf{u})$  using previous Algorithm.
- For each  $\mathbf{u} \in S$ , compute  $\mathbf{z} = \operatorname{argmin}_{\mathbf{y} \in X} \|f(\mathbf{u}) - f(\mathbf{y})\|_2$ .
- If  $\text{Label}(\mathbf{u}) = \text{Label}(\mathbf{z})$ , this is deemed a successful classification.
- Track success rate as percentage.





# COIL-100 Dataset

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- COIL-100 dataset: 100 objects,  $128 \times 128$ -pixel color images of 100 objects, 72 evenly spaced rotations.
- Grayscale, vectorized, 7,200 total datapoints in  $\mathbb{R}^N$  with  $N = 128^2 = 16,384$ .
- Training data: evenly distributed, evenly rotated collection of all 100 objects.



# COIL-100 Classification Simulations

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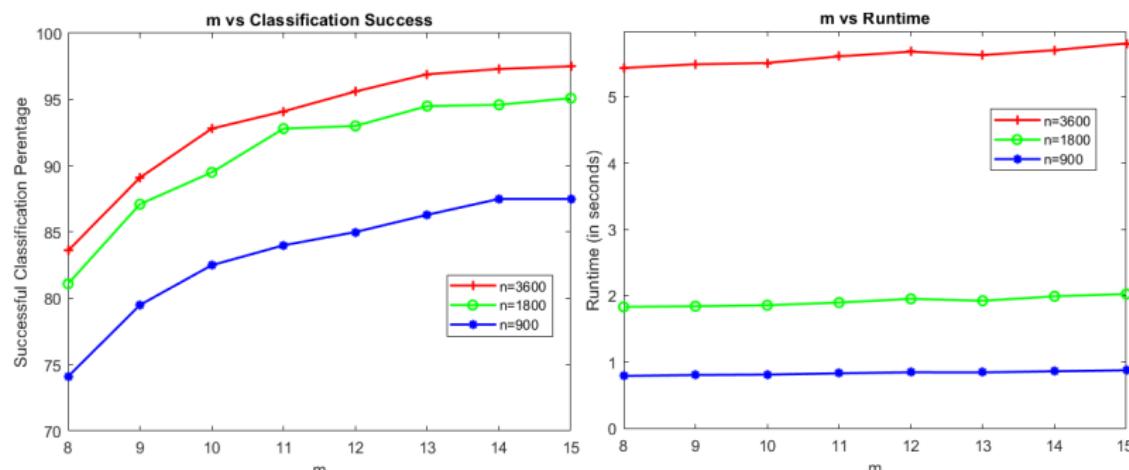
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- This figure compares compressive NN classification accuracies, and the associated classification run times averaged over all  $\mathbf{u} \in S$ .
- Three different training data set sizes  $n = |X| \in \{900, 1800, 3600\}$  were fixed as the embedding dimension  $m + 1$  varied for each of the first two subfigures.



# Comparison Figure

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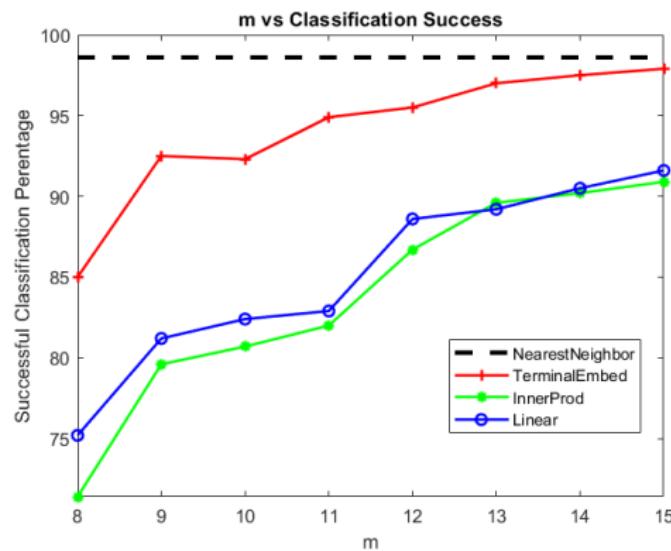
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- **Nearest Neighbor:** Find the nearest neighbor in the original space.
- **TerminalEmbed:**  $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \|\mathbf{z}\|_2^2 + 2\langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$ .
- **InnerProd:**  $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$ .
- **Linear:** Embed into the space linearly i.e.  $f(\mathbf{u}) = (\Pi \mathbf{u}, 0)$ .



# Compressed Classification from Phaseless Measurements (without reconstruction)

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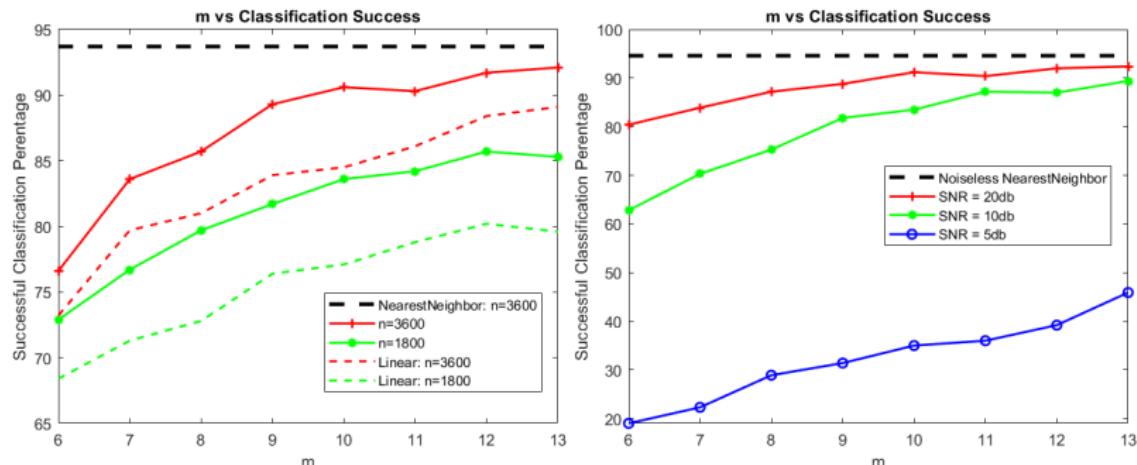
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- Sub-sampled NFP measurements  $|(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2$ ,  $(k, \ell) \in [d] \times [1]$ .
- $\mathbf{x}$  = vectorized, grayscaled, COIL-100 images.
- Left: Comparing NearestNeighbor, TerminalEmbed, & Linear.
- Right: Additive noise applied to testing data i.e.  $\mathbf{u} \rightarrow \mathbf{u} + \mathbf{n}$ .



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## Definition

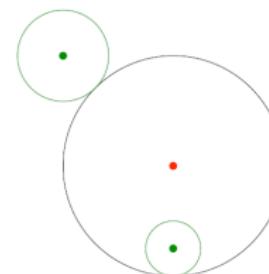
For a subset  $S \subset \mathbb{R}^N$  of Euclidean space, the **reach**  $\tau_S$  is

$$\tau_S := \sup \left\{ t \geq 0 \mid \forall \mathbf{x} \in \mathbb{R}^n \text{ s.t. } d(\mathbf{x}, S) < t, \mathbf{x} \text{ has a unique closest point in } S \right\}.$$

- Larger  $\tau_S \Rightarrow S$  simple (flatter, non-self-intersecting, etc.)
- $\tau_S = \infty \Leftrightarrow S \subseteq \mathbb{R}^N$  is closed and convex.



$\text{reach}(\text{line segment}) = \infty$



$\text{reach}(\text{sphere}) = \text{radius of sphere}$



## Definition

The **Gaussian width** of a set  $T \subset \mathbb{R}^N$  is  $w(T) := \mathbb{E} \sup_{\mathbf{x} \in T} \langle \mathbf{g}, \mathbf{x} \rangle$ , where  $\mathbf{g}$  is a random vector with  $N$  i.i.d. mean 0 and variance 1 Gaussian entries.

- Gaussian width can replace cardinality as measure of set complexity.

## Definition

We define the **unit secants of  $T \subset \mathbb{R}^N$**  to be

$$S_T := \overline{U((T - T) \setminus \{\mathbf{0}\})} = \left\{ \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|_2} \mid \mathbf{x}, \mathbf{y} \in T, \mathbf{x} \neq \mathbf{y} \right\}.$$

- Let  $\mathcal{M} \hookrightarrow \mathbb{R}^N$  be a compact  $d$ -dimensional submanifold of  $\mathbb{R}^N$  with boundary  $\partial \mathcal{M}$ , finite reach  $\tau_{\mathcal{M}}$ , and volume  $V_{\mathcal{M}}$ .
- $\tau_i$  = reach of  $i^{\text{th}}$  connected component of  $\partial \mathcal{M}$  as submanifold of  $\mathbb{R}^N$ .
- Set  $\tau := \min_i \{\tau_{\mathcal{M}}, \tau_i\}$ , let  $V_{\partial \mathcal{M}} = \text{Vol}(\partial \mathcal{M})$ , and  $\omega_d = \text{Vol}(B_d(1))$ .



# Main Result

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## Theorem (The Main Result)

Let  $d \geq 2$ . Define  $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$ . Fix  $\epsilon \in (0, 1)$  and define  $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$ . Then, there exists a map  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c(\ln(\beta_M) + 4d)/\epsilon^2$  that satisfies

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2,$$

for all  $\mathbf{x} \in M$  and  $\mathbf{y} \in \mathbb{R}^N$ . Here  $c \in \mathbb{R}^+$  is an absolute constant.

The following theorem bounds the Gaussian width of a smooth submanifold of  $\mathbb{R}^N$  in terms of its dimension, reach, and volume.

## Theorem (Theorem 20 - Iwen, Schmidt, Tavakoli)

Let  $d \geq 2$ . Define  $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$ ,  $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$ . Then we have that

$$w(S_M) \leq 8\sqrt{2} \sqrt{\ln(\beta_M) + 4d}.$$



# Proof of Main Result

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- $\Phi \in \mathbb{C}^{m \times N}$  with independent, isotropic ( $\mathbb{E}[\mathbf{u}\mathbf{u}^T] = \mathbf{I}$ ), sub-Gaussian rows provides  $\epsilon$ -convex hull distortion for  $S_{\mathcal{M}}$ .
- This forms the basis for our outer extension  $f$ .

## Theorem (Theorem 3.2 - Iwen, Roach)

Let  $\mathcal{M} \subset \mathbb{R}^N$  and  $\epsilon \in (0, 1)$ . There exists a map  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c \left( \frac{w(S_{\mathcal{M}})}{\epsilon} \right)^2$  that satisfies

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2,$$

for all  $\mathbf{x} \in \mathcal{M}$  and  $\mathbf{y} \in \mathbb{R}^N$ , where  $c \in \mathbb{R}^+$  is an absolute constant.

## Proof of Main Result.

Theorem 3.2 proves that there exists a terminal embedding  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c \left( \frac{w(S_{\mathcal{M}})}{\epsilon} \right)^2$ . Theorem 2.1 bounds  $w(S_{\mathcal{M}})$ . Apply both theorems to complete proof. □



# References - Near-field Ptychography

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- Iwen, M., Perlmutter, M., Roach, M.P., **Toward Fast and Provably Accurate Near-field Ptychographic Phase Retrieval**, *Sampling Theory, Signal Processing, and Data Analysis*, Vol 21, 2022
- Filbir, F., Krahmer, F., Melnyk, O.  
**On Recovery Guarantees for Angular Synchronization**  
*Fourier Anal Appl* 27, 31, 2021
- Iwen, M., Preskitt, B., Saab, R., Viswanathan, A.  
**Phase Retrieval from Local Measurements: Improved Robustness via Eigenvector Based Angular Synchronization**  
*Applied and Computational Harmonic Analysis*, Vol. 48, Issue 1, pp. 415-444, 2020
- Iwen, M., Viswanathan, A., Wang, Y.  
**Fast Phase Retrieval from Local Correlation Measurements**  
*SIAM J. Imaging Sciences*, Vol. 9, No. 4, pp. 1655 - 1688, 2016
- Zhang, H., Jiang, S., Liao, J., Deng, J., Liu, J., Zhang, Y., Zheng, G.  
**Near-field Fourier Ptychography: Super-Resolution Phase Retrieval via Speckle Illumination**  
*Opt. Express* 277498-512, 2019



# References - Blind Ptychography via Blind Deconvolution

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Questions

- Perlmutter, M., Merhi, S., Viswanathan, A., Iwen, M.,  
**Inverting Spectrogram Measurements via Aliased Wigner Distribution Deconvolution and Angular Synchronization**  
<https://arxiv.org/abs/1907.10773>
- Iwen, M., Merhi, S., Perlmutter, M.,  
**Lower Lipschitz Bounds for Phase Retrieval from Locally Supported Measurements**  
<https://arxiv.org/abs/1806.08262>
- Merhi, S.,  
**PhD Thesis**  
<https://d.lib.msu.edu/etd/47915/datasream/OBJ/View/>
- Li, X., Ling, S., Strohmer, T., Wei, K.,  
**Rapid, Robust, and Reliable Blind Deconvolution via Nonconvex Optimization**  
<https://arxiv.org/abs/1606.04933>
- Ahmed, A., Recht, B., Romberg, J.,  
**Blind Deconvolution using Convex Programming**  
<https://arxiv.org/abs/1211.5608>



# References - Terminal Manifold Embeddings

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- Narayanan, S., Nelson, J.,  
**Optimal terminal dimensionality reduction in Euclidean space**  
<https://arxiv.org/abs/1810.09250>
- Iwen, M., Tavakoli, A., Schmidt, B.,  
**Lower Bounds on the Low-Distortion Embedding Dimension of Submanifolds of  $\mathbb{R}^n$**   
<https://arxiv.org/abs/2105.13512>
- Iwen, M., Tavakoli, A., Schmidt, B.,  
**On Fast Johnson-Lindenstrauss Embeddings of Compact Submanifolds of  $\mathbb{R}^N$  with Boundary**  
<https://arxiv.org/abs/2110.04193>
- Iwen, M., Roach, M.P.,  
**On Outer Bi-Lipschitz Extensions of Linear Johnson-Lindenstrauss Embeddings of Low-Dimensional Submanifolds of  $\mathbb{R}^N$**   
<https://arxiv.org/abs/2206.03376>



# Questions

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- Code/slides available at <https://github.com/MarkPhilipRoach>



Any questions?

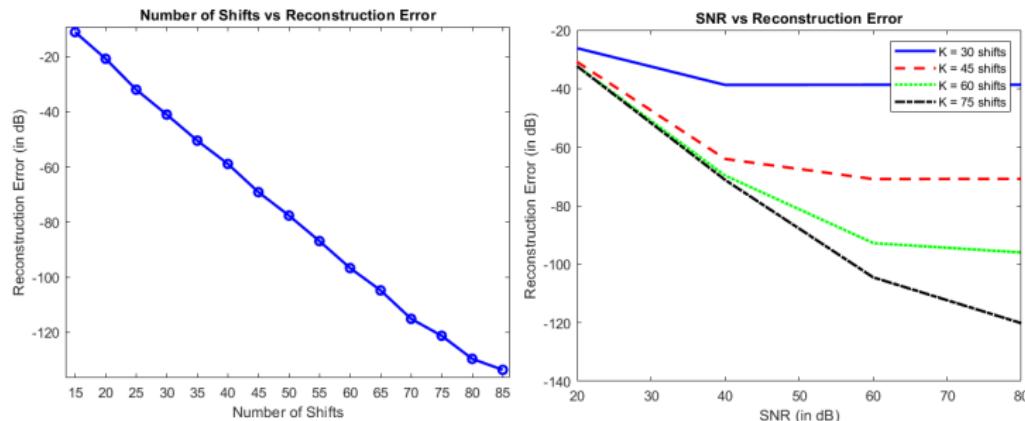


# Near Field Ptychography via Wirtinger Flow

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- Want to be able to assume  $\mathbf{p}$  is a low pass filter and be able to take fewer shifts.
- In "Phase Retrieval via Wirtinger Flow: Theory and Algorithms - Candes, Li, Soltanolkotabi", quadratic equations of the form  $y_n = |\langle \mathbf{m}_\ell, \mathbf{x} \rangle|^2$ ,  $n = 1, 2, \dots, N$  are solved by applying a gradient descent like algorithm



Plotting reconstruction error with  $d = 102$ ,  $\mathcal{L} = [d]_0$ , and number of iterations  $T = 2000$ .

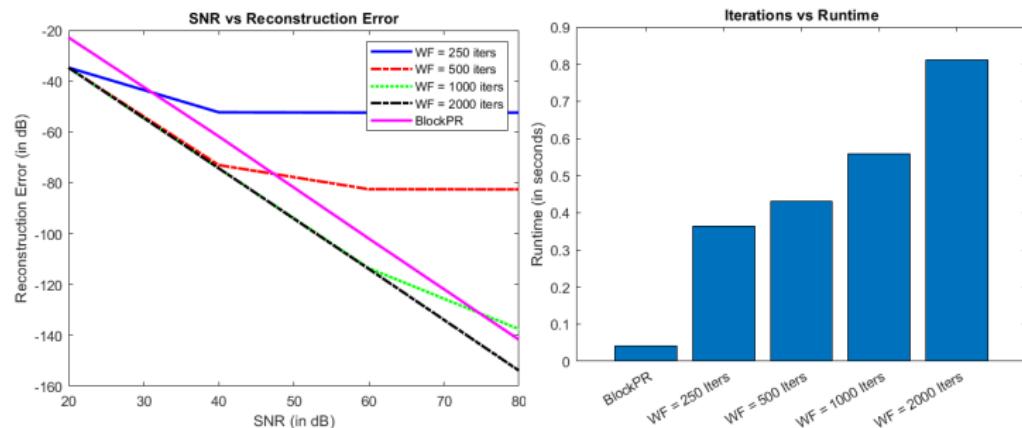
Left: Reconstruction error vs the number of total shifts  $K$  for fixed SNR = 80.  
Right: Reconstruction error vs SNR for various numbers of shifts  $K$ .



# Comparison Between Algorithms

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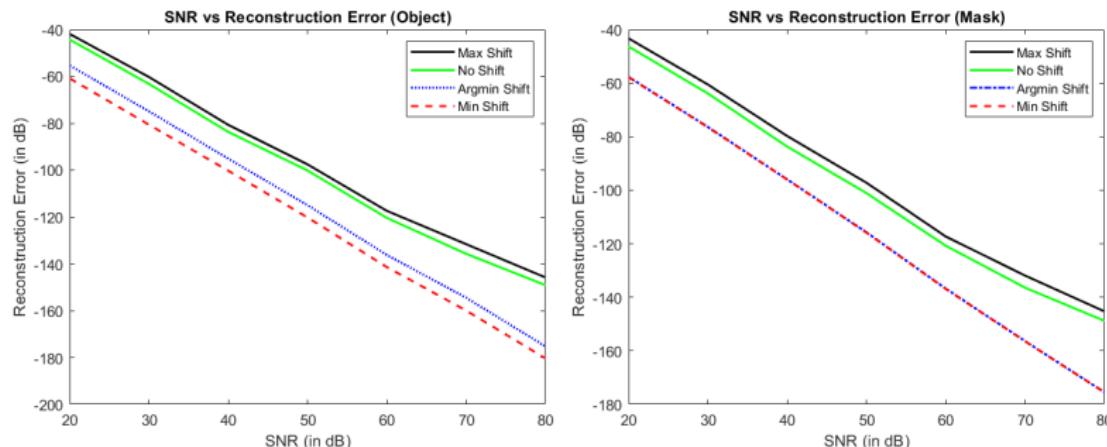
- A comparison of BlockPR and Wirtinger Flow algorithms for the proposed PSF and mask with  $\delta = 26$  and  $d = 102$ .
- Left: Reconstruction error vs SNR for various numbers of Wirtinger Flow iterations.
- Right: The corresponding average runtimes.



# SNR vs. Reconstruction Error

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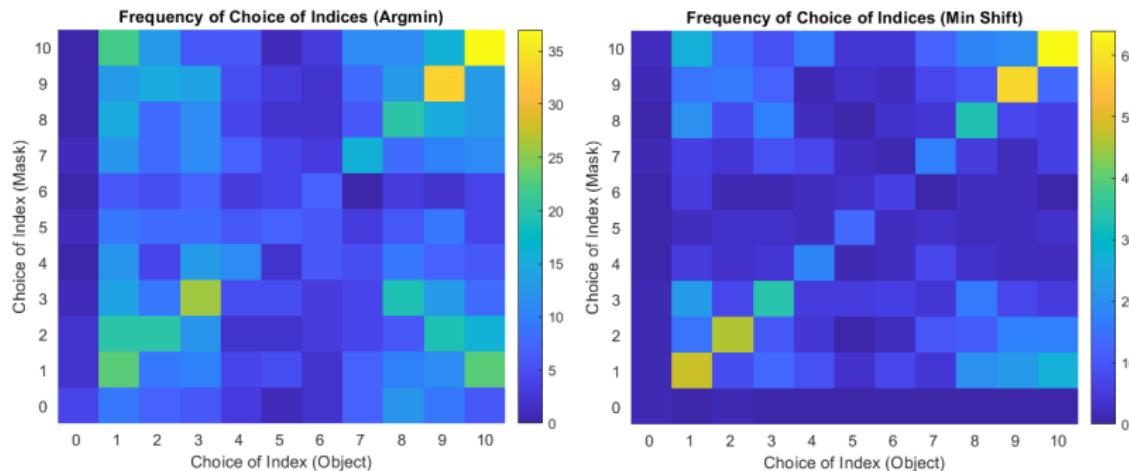
- $d = 2^6, K = \delta = \log_2 d, N = 4, \mathbf{C}$  complex Gaussian, 100 simulations
- Max shift refers to the maximum possible error.
- Min shift refers to the minimum possible error.
- Argmin Shift refers to the choice of object and mask chosen by our algorithm.



# Frequency of Indices

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- $d = 2^6, \delta = 6, N = 4, \mathbf{C}$  complex Gaussian, 1000 simulations
- Left: Frequency of indices being chosen to compute ( $\text{Argmin Shift}^{(x)}, \text{Argmin Shift}^{(m)}$ )
- Right: Frequency of indices being chosen to compute ( $\text{Min Shift}^{(x)}, \text{Min Shift}^{(m)}$ ).



# MNIST

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A 4x10 grid of handwritten digits from the MNIST dataset. The digits are arranged in four rows and ten columns. The digits are drawn in black ink on a white background. They represent the digits 0 through 9.

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9

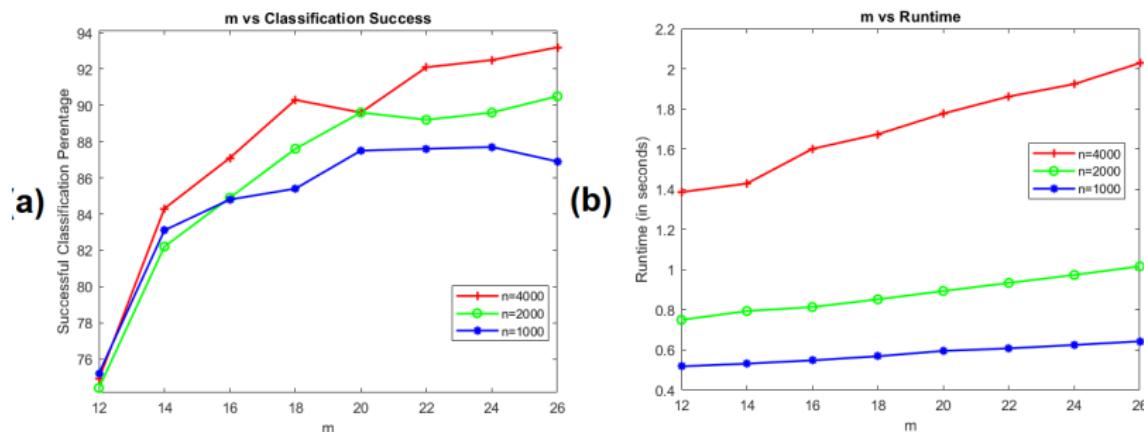
- The MNIST data set consists of 60,000 training images of  $28 \times 28$ -pixel grayscale hand-written images of the digits 0 through 9.
- Thus, MNIST has 10 labels to correctly classify between, and  $N = 28^2 = 784$ .
- For all experiments involving the MNIST dataset,  $n/10$  digits of each type are selected uniformly at random to form the training set  $X$ , for a total of  $n$  vectorized training images in  $\mathbb{R}^{784}$ .
- Then, 100 digits of each type are randomly selected from those not used for training in order to form the test set  $S$ , leading to a total of  $n' = 1000$  vectorized test images in  $\mathbb{R}^{784}$ .



# MNIST Classification Simulations

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- This figure compares compressive NN classification accuracies, and the associated classification run times averaged over all  $\mathbf{u} \in S$ .
- Three different training data set sizes  $n = |X| \in \{1000, 2000, 4000\}$  were fixed as the embedding dimension  $m + 1$  varied for each of the first two subfigures.
- Recall that the test set size is always fixed to  $n' = 1000$ .



## Lemma

*The terminal embedding is non-linear*

## Proof.

- Let  $X \subset \mathbb{R}^d$  be arbitrary
- Suppose for contradiction that  $f : X \rightarrow \mathbb{R}^m, d > m$  is a linear embedding with constant terminal distortion.
- By the Rank-Nullity theorem,  $\dim(\ker(f)) \geq d - m \geq 1$ .
- This means  $\exists y \in \ker(f) \setminus \{0\}$
- Let  $x \in X$  be arbitrary. Since  $f$  is a linear embedding and  $x - y \in \mathbb{R}^d$

$$\begin{aligned}\|x - (x - y)\|_2 &\leq \|f(x) - f(x - y)\|_2 \\ \Rightarrow 0 < \|y\|_2 &\leq \|f(x) - f(x) + f(y)\|_2 = \|f(y)\|_2 = 0\end{aligned}$$

- Thus we have arrived at a contradiction





# Distortion Simulations

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## Theorem (Theorem 1.1 - Narayanan, Nelson)

Let  $\epsilon \in (0, 1)$  and  $X \subset \mathbb{R}^d$  be arbitrary with  $|X| = n > 1$ . There exists  $f : X \rightarrow \mathbb{R}^m$  with  $m = O(\epsilon^{-2} \log n)$  such that  $\forall x \in X, \forall y \in \mathbb{R}^d$

$$(1 - \epsilon) \|x - y\|_2 \leq \|f(x) - f(y)\|_2 \leq (1 + \epsilon) \|x - y\|_2$$

## Definition

To compute the effective distortions of a given (terminal) embedding of training data  $X$ ,  $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$ , over all available test and train data  $X \cup S$  we use

$$\text{MaxDist}_f = \max_{\mathbf{x} \in X} \max_{\mathbf{u} \in S \cup X \setminus \{\mathbf{x}\}} \frac{\|f(\mathbf{u}) - f(\mathbf{x})\|_2}{\|\mathbf{u} - \mathbf{x}\|_2},$$

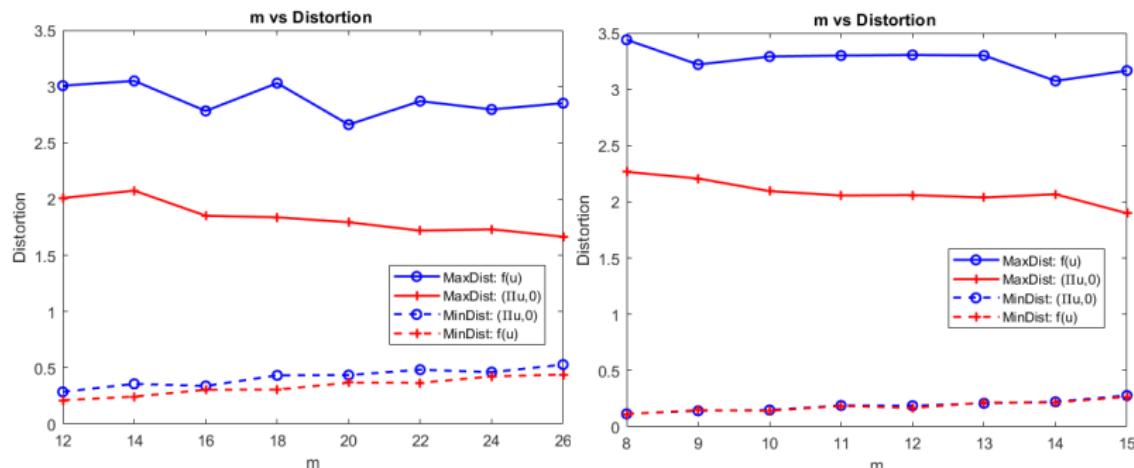
$$\text{MinDist}_f = \min_{\mathbf{x} \in X} \min_{\mathbf{u} \in S \cup X \setminus \{\mathbf{x}\}} \frac{\|f(\mathbf{u}) - f(\mathbf{x})\|_2}{\|\mathbf{u} - \mathbf{x}\|_2}.$$



# Distortion Simulations

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- Left figure compares  $\text{MaxDist}_f$  and  $\text{MinDist}_f$  for the nonlinear  $f$  versus its component linear embedding  $\mathbf{u} \mapsto (\Pi \mathbf{u}, 0)$  as  $m$  varies for a fixed embedded training set size of  $n = 4000$ .
- Right figure compares  $\text{MaxDist}_f$  and  $\text{MinDist}_f$  for COIL-100 dataset, for the nonlinear  $f$  versus its component linear embedding  $\mathbf{u} \mapsto (\Pi \mathbf{u}, 0)$  as  $m$  varies for a fixed embedded training set size of  $n = 3600$ .



# Measuring Compressive Nearest Neighbor Classification Accuracy

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## Algorithm Measuring Compressive Nearest Neighbor Classification Accuracy

---

**Input:**  $\epsilon \in (0, 1)$ , A labeled data set  $\mathcal{D} \subset \mathbb{R}^N$  split into two disjoint subsets: A training set  $X \subset \mathcal{D}$  with  $|X| =: n$ , and a test set  $S \subset \mathcal{D}$  with  $|S| =: n'$ , such that  $S \cap X = \emptyset$ . A compressive dimension  $m < N$ .

**Output:** Successful Nearest Neighbor Classification Percentage for Data Embedded in  $\mathbb{R}^{m+1}$

Fix  $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$ , an embedding of the training data  $X \subset \mathbb{R}^N$  into  $\mathbb{R}^{m+1}$  satisfying

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2$$

for all  $\mathbf{x}, \mathbf{y} \in X$ . [Note: this can either be a JL-embedding of  $X$ , or a stronger terminal embedding of  $X$ .]

% Embed the training data into  $\mathbb{R}^{m+1}$ .

**for**  $\mathbf{x} \in X$  **do**

    Compute  $f(\mathbf{x})$  using previous Algorithm

**end for**



# Measuring Compressive Nearest Neighbor Classification Accuracy

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## Algorithm Measuring Compressive Nearest Neighbor Classification Accuracy

---

% Classify the test data using its embedded distance in  $\mathbb{R}^{m+1}$ .

**p** = 0

**for**  $\mathbf{u} \in S$  **do**

    Compute  $f(\mathbf{u})$  using, e.g., Algorithm 1

    Compute  $\mathbf{x} = \underset{\mathbf{y} \in X}{\operatorname{argmin}} \|f(\mathbf{u}) - f(\mathbf{y})\|_2$

**if**  $\text{Label}(\mathbf{u}) = \text{Label}(\mathbf{x})$  **then**

$p = p + 1$

**end if**

**end for**

---

Output the Successful Classification Percentage =  $\frac{p}{n'} \times 100\%$

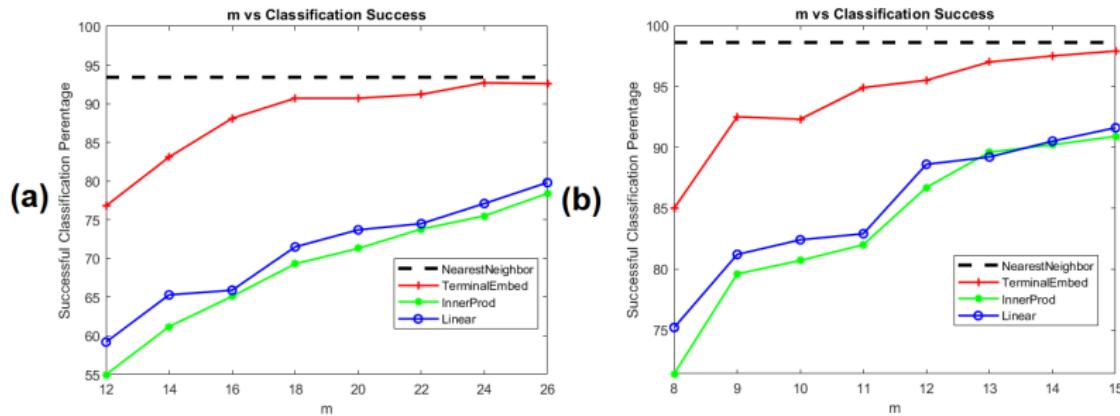
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# Comparison Figures

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- Left: MNIST, Right: COIL-100
- Nearest Neighbor:** Find the nearest neighbor in the original space
- TerminalEmbed:**  $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \|\mathbf{z}\|_2^2 + 2\langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$
- InnerProd:**  $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$
- Linear:** Embed into the space linearly i.e.  $f(\mathbf{u}) = (\Pi \mathbf{u}, 0)$



- The COIL-100 data set is a collection of  $128 \times 128$ -pixel color images of 100 objects, each photographed 72 times where the object has been rotated by 5 degrees each time to get a complete rotation.
- However, only the green color channel of each image is used herein for simplicity. Thus, herein COIL-100 consists of 7,200 total vectorized images in  $\mathbb{R}^N$  with  $N = 128^2 = 16,384$ , where each image has one of 100 different labels (72 images per label).



- For all experiments involving this COIL-100 data set,  $n/100$  training images are down sampled from each of the 100 objects' rotational image sequences.
- Thus, the training sets each contain  $n/100$  vectorized images of each object, each photographed at rotations of  $\approx 36000/n$  degrees (rounded to multiples of 5). The resulting training data sets therefore all consist of  $n$  vectorized images in  $\mathbb{R}^{16,384}$ .
- After forming each training set, 10 images of each type are then randomly selected from those not used for training in order to form the test set  $S$ , leading to a total of  $n' = 1000$  vectorized test images in  $\mathbb{R}^{16,384}$  per experiment.



# Gaussian Width of the Unit Secants

Thesis  
Defense

Mark Philip  
Roach

The following theorem bounds the Gaussian width of a smooth submanifold of  $\mathbb{R}^N$  in terms of its dimension, reach, and volume.

## Theorem (Theorem 2.1 - Iwen, Roach)

Let  $M \hookrightarrow \mathbb{R}^N$  be a compact  $d$ -dimensional submanifold of  $\mathbb{R}^N$  with boundary  $\partial M$ , finite reach  $\tau_M$ , and volume  $V_M$ . Enumerate the connected components of  $\partial M$  and let  $\tau_i$  be the reach of the  $i^{\text{th}}$  connected component of  $\partial M$  as a submanifold of  $\mathbb{R}^N$ . Set  $\tau := \min_i\{\tau_M, \tau_i\}$ , let  $V_{\partial M}$  be the volume of  $\partial M$ , and denote the volume of the  $d$ -dimensional Euclidean ball of radius 1 by  $\omega_d$ . Next,

- ① if  $d = 1$ , define  $\alpha_M := \frac{20V_M}{\tau} + V_{\partial M}$ , else
- ② if  $d \geq 2$ , define  $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$ .

Finally, define  $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$ . Then, the Gaussian width of  $U((M - M) \setminus \{\mathbf{0}\})$  satisfies

$$w(S_M) = w\left(\overline{U((M - M) \setminus \{\mathbf{0}\})}\right) \leq 8\sqrt{2} \sqrt{\ln(\beta_M) + 4d}.$$



# Main Result

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## Theorem (The Main Result)

Let  $\mathcal{M} \hookrightarrow \mathbb{R}^N$  be a compact  $d$ -dimensional submanifold of  $\mathbb{R}^N$  with boundary  $\partial\mathcal{M}$ , finite reach  $\tau_{\mathcal{M}}$ , and volume  $V_{\mathcal{M}}$ . Enumerate the connected components of  $\partial\mathcal{M}$  and let  $\tau_i$  be the reach of the  $i^{\text{th}}$  connected component of  $\partial\mathcal{M}$  as a submanifold of  $\mathbb{R}^N$ . Set  $\tau := \min_i \{\tau_{\mathcal{M}}, \tau_i\}$ , let  $V_{\partial\mathcal{M}}$  be the volume of  $\partial\mathcal{M}$ , and denote the volume of the  $d$ -dimensional Euclidean ball of radius 1 by  $\omega_d$ . Next,

- ① if  $d = 1$ , define  $\alpha_{\mathcal{M}} := \frac{20V_{\mathcal{M}}}{\tau} + V_{\partial\mathcal{M}}$ , else
- ② if  $d \geq 2$ , define  $\alpha_{\mathcal{M}} := \frac{V_{\mathcal{M}}}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial\mathcal{M}}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$ .

Finally, fix  $\epsilon \in (0, 1)$  and define  $\beta_{\mathcal{M}} := (\alpha_{\mathcal{M}}^2 + 3^d \alpha_{\mathcal{M}})$ . Then, there exists a map  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c(\ln(\beta_{\mathcal{M}}) + 4d)/\epsilon^2$  that satisfies

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all  $\mathbf{x} \in \mathcal{M}$  and  $\mathbf{y} \in \mathbb{R}^N$ . Here  $c \in \mathbb{R}^+$  is an absolute constant.



## Theorem 3.1, 3.2

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### Theorem (Theorem 3.1 - Iwen, Roach)

Let  $\mathcal{M} \subset \mathbb{R}^N$ ,  $\epsilon \in (0, 1)$ , and suppose that  $\Phi \in \mathbb{C}^{m \times N}$  is an  $\left(\frac{\epsilon^2}{2304}\right)$ -JL map of  $S_{\mathcal{M}} + S_{\mathcal{M}}$  into  $\mathbb{C}^m$ . Then, there exists an outer bi-Lipschitz extension of  $\Phi : \mathcal{M} \rightarrow \mathbb{C}^m$ ,  $f : \mathbb{R}^N \rightarrow \mathbb{C}^{m+1}$ , with the property that

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

holds for all  $\mathbf{x} \in \mathcal{M}$  and  $\mathbf{y} \in \mathbb{R}^N$ .

### Theorem (Theorem 3.2 - Iwen, Roach)

Let  $\mathcal{M} \subset \mathbb{R}^N$  and  $\epsilon \in (0, 1)$ . There exists a map  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c \left( \frac{w(S_{\mathcal{M}})}{\epsilon} \right)^2$  that satisfies

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all  $\mathbf{x} \in \mathcal{M}$  and  $\mathbf{y} \in \mathbb{R}^N$ , where  $c \in \mathbb{R}^+$  is an absolute constant.



## Proof of Theorem 3.1

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### Theorem (Theorem 3.4 - Iwen, Roach)

Let  $\mathcal{M} \subset \mathbb{R}^N$ ,  $\epsilon \in (0, 1)$ , and suppose that  $\Phi \in \mathbb{C}^{m \times N}$  is an  $(\frac{\epsilon^2}{4})$ -JL map of  $S_{\mathcal{M}} + S_{\mathcal{M}}$  into  $\mathbb{C}^m$ . Then,  $\Phi$  will also provide  $\epsilon$ -convex hull distortion for  $S_{\mathcal{M}}$ .

### Lemma (Lemma 3.4 - Iwen, Roach)

Let  $\mathcal{M} \subset \mathbb{R}^N$  be non-empty,  $\epsilon \in (0, 1)$ , and suppose that  $\Phi \in \mathbb{C}^{m \times N}$  provides  $\epsilon$ -convex hull distortion for  $S_{\mathcal{M}}$ . Then, there exists an outer bi-Lipschitz extension of  $\Phi$ ,  $f : \mathbb{R}^N \rightarrow \mathbb{C}^{m+1}$ , with the property that

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq 24\epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

holds for all  $\mathbf{x} \in \mathcal{M}$  and  $\mathbf{y} \in \mathbb{R}^N$ .

### Proof of Theorem 3.1.

Apply Theorem 3.4 with  $\epsilon \leftarrow \epsilon/24$  in order obtain  $\epsilon/24$ -convex hull distortion for  $S_{\mathcal{M}}$  via  $\Phi$ . Then, apply Lemma 3.4. □



## Proof of Theorem 3.2

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### Corollary (Corollary 3.1 - Iwen, Roach)

Let  $\mathcal{M} \subset \mathbb{R}^N$ ,  $\epsilon, p \in (0, 1)$ , and  $\Phi \in \mathbb{R}^{m \times N}$  be an  $m \times N$  matrix whose rows are independent, isotropic, and sub-Gaussian random vectors in  $\mathbb{R}^N$ . Furthermore, suppose that

$$m \geq \frac{c'}{\epsilon^2} \left( w(S_{\mathcal{M}}) + \sqrt{\ln(2/p)} \right)^2,$$

where  $c'$  is a constant depending only on the distribution of the rows of  $\Phi$ . Then, with probability at least  $1 - p$ ,  $\frac{1}{\sqrt{m}}\Phi$  will be an  $\epsilon$ -JL embedding of  $\mathcal{M}$  into  $\mathbb{R}^m$  and provide  $\epsilon$ -convex hull distortion for  $S_{\mathcal{M}}$ .

### Proof of Theorem 3.2.

- Apply Corollary 3.1 with  $p = 1/2$  to demonstrate that a  $\left[ \frac{c''}{\epsilon^2} \left( w(S_{\mathcal{M}}) + \sqrt{\ln(4)} \right)^2 \right] \times N$  matrix with i.i.d. standard normal random entries can provide  $(\epsilon/24)$ -convex hull distortion for  $S_{\mathcal{M}}$ , where  $c''$  is an absolute constant.
- Apply Lemma 3.4 to finish the proof.





# Main Result

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## Theorem (The Main Result)

Let  $d \geq 2$ . Define  $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$ . Fix  $\epsilon \in (0, 1)$  and define  $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$ . Then, there exists a map  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c(\ln(\beta_M) + 4d)/\epsilon^2$  that satisfies

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all  $\mathbf{x} \in M$  and  $\mathbf{y} \in \mathbb{R}^N$ . Here  $c \in \mathbb{R}^+$  is an absolute constant.

## Proof of Main Result.

- Theorem 3.2 proves that there exists a map  $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$  with  $m \leq c \left( \frac{w(S_M)}{\epsilon} \right)^2$ .
- Theorem 2.1 bounds  $S_M$ .
- Apply both theorems to complete proof.





# Embedding Circle in Two Dimensions

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Manifold in Original Space Manifold in Original Space Manifold in Original Space

