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Questions

Mark Philip Roach - Thesis Defence

Committee Members:

Mark Iwen^{1*} Longxiu Huang¹ Ekaterina Rapinchuk¹ Guowei Wei²

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* Dissertation/Thesis Director

¹Department of Mathematics, CMSE, MSU

²Department of Mathematics, Biochemistry & Molecular Biology, Electrical and
Computer Engineering, MSU

Code/slides available at <https://github.com/MarkPhilipRoach>



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Near-field Ptychography

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- Ptychography is a particular form of masked phase retrieval in which the collections of masks are generated by taking one physical mask and shifting it in space
- Near-field Ptychography is when the distance between the lens, object, and detector are small (microscopic imaging).

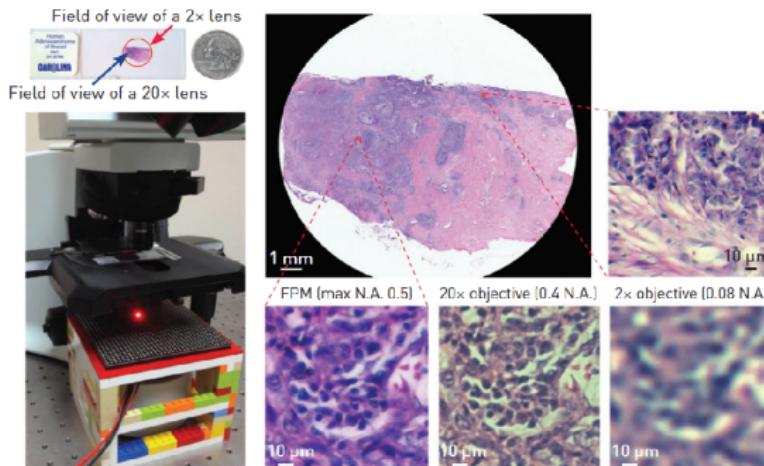


Figure: "Wide-field, high-resolution Fourier ptychographic microscopy" - G. Zheng et al., 2013



Near-field Ptychographic Measurements

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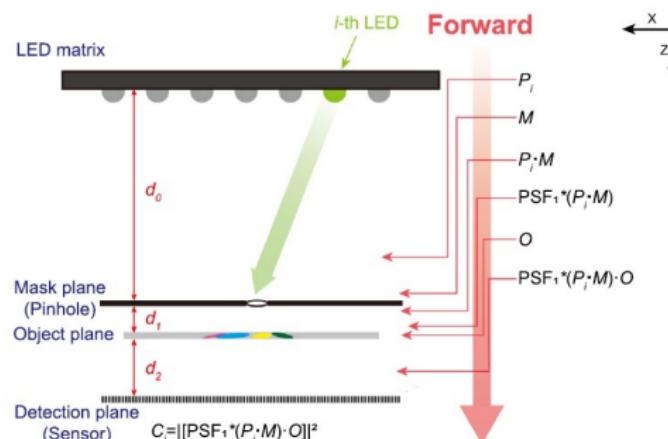
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- Let $\mathbf{x} \in \mathbb{C}^d$ denote the unknown object
- Let \mathbf{m} denote the known mask, \mathbf{p} the known point spread function (PSF).
- The noisy near-field ptychographic measurements will be of the form

$$Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2 + N_{k,\ell}, \quad (k, \ell) \in \mathcal{K} \times \mathcal{L} \subseteq [d] \times [d].$$

- \mathcal{K} : set of shifts, \mathcal{L} : set of frequencies
- Circular shift: $(S_k \mathbf{m})_n = m_{n+k \bmod d}$
- Discrete convolution: $(\mathbf{u} * \mathbf{v})_n = \sum_{k=0}^{d-1} u_k v_{n-k \bmod d}$
- Pointwise (Hadamard) product: $(\mathbf{u} \circ \mathbf{v})_n = u_n v_n$





Reducing To Inner Product Form

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- Consider noiseless measurements $Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2, (k, \ell) \in [d] \times [2\delta - 1]$
- By letting $Y_{k,\ell} = Y_{-k,\ell+k}, \check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}, (\tilde{\mathbf{p}})_n = p_{-n}$, one can show that

$$Y_{k,\ell} = |\langle \check{\mathbf{m}}_\ell^{(p,m)}, S_k \mathbf{x} \rangle|^2, (k, \ell) \in [d] \times [2\delta - 1].$$

- Fast Phase Retrieval from Local Correlation Measurements - Iwen, M., Viswanathan, A., Wang, Y.* analyzes measurements of this form, by using a lifted linear system involving a block circulant matrix $\tilde{\mathbf{M}}$ and $\delta \ll d$ supported masks.
- Let $D = d(2\delta - 1)$. We then define the block circulant matrix $\tilde{\mathbf{M}} \in \mathbb{C}^{D \times D}$ via

$$\tilde{\mathbf{M}} := \begin{pmatrix} \tilde{\mathbf{M}}_0 & \tilde{\mathbf{M}}_1 & \dots & \tilde{\mathbf{M}}_{\delta-1} & 0 & 0 & \dots & 0 \\ 0 & \tilde{\mathbf{M}}_0 & \tilde{\mathbf{M}}_1 & \dots & \tilde{\mathbf{M}}_{\delta-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \tilde{\mathbf{M}}_1 & \dots & \tilde{\mathbf{M}}_{\delta-1} & 0 & 0 & 0 & \dots & \tilde{\mathbf{M}}_0 \end{pmatrix}$$

where the matrices $\tilde{\mathbf{M}}_k \in \mathbb{C}^{(2\delta-1) \times (2\delta-1)}$ are defined entry-wise by

$$(\tilde{\mathbf{M}}_k)_{ij} := \begin{cases} (\tilde{\mathbf{m}}_i)_k \overline{(\tilde{\mathbf{m}}_i)_{j+k}}, & 0 \leq j \leq \delta - k \\ (\tilde{\mathbf{m}}_i)_k (\tilde{\mathbf{m}}_i)_{j+k-2\delta}, & 2\delta - 1 + k \leq j \leq 2\delta - 2 \wedge k < \delta \\ 0, & \text{otherwise} \end{cases}$$



Condition Number Bound

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- Then it is shown that $\text{vec}(\mathbf{Y}) = \check{\mathbf{M}}\mathbf{z}$ for $\mathbf{z} \in \mathbb{C}^d$ being a portion of $\text{vec}(\mathbf{x}\mathbf{x}^*)$
- Then $\mathbf{z} = \check{\mathbf{M}}^{-1}\text{vec}(\mathbf{Y})$ and we reshape \mathbf{z} to recover $\widehat{\mathbf{X}}$ whose non-zero entries are estimates of the $\mathbf{x}\mathbf{x}^*$
- Angular synchronization is performed on $\widehat{\mathbf{X}}$ to recover \mathbf{x}_{est} .
- In the paper, it is shown that exponential masks $\check{\mathbf{m}}_{\ell}^{(fpr)}$ defined by

$$(\check{\mathbf{m}}_{\ell}^{(fpr)})_n = \begin{cases} \frac{e^{-(n+1)/a}}{\sqrt[4]{2\delta-1}} \cdot e^{\frac{2\pi i n \ell}{2\delta-1}}, & n \in [\delta] \\ 0, & \text{otherwise} \end{cases}, \quad a := \max \left\{ 4, \frac{\delta-1}{2} \right\},$$

lead to lifted linear systems that are well-conditioned and allow for recovery guarantees.

Theorem (Theorem 4 - M. Iwen., et. al.)

For collection of masks $\check{\mathbf{m}}_{\ell}^{(fpr)}$, the condition number has the bound

$$\kappa := \kappa(\check{\mathbf{M}}) < \max \left\{ 144e^2, \frac{9e^2(\delta-1)^2}{4} \right\} \leq C\delta^2, \quad C \in \mathbb{R}^+.$$

Furthermore, $\check{\mathbf{M}}$ can be inverted in $O(\delta \cdot d \log d)$ -time.



Admissible Selection of PSF and Mask

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- We would like to show there are realistic \mathbf{p} and \mathbf{m} with $\check{\mathbf{m}}_\ell^{(fpr)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}$.
- One can show that it is impossible to choose a \mathbf{p} and \mathbf{m} that are independent of ℓ that accomplishes this, however we can approximate it up to a phase factor.

Lemma (Admissible Selection of PSF and Mask)

Let $\mathbf{p}, \mathbf{m} \in \mathbb{C}^d$ have entries given by

$$p_n := e^{-\frac{2\pi i n^2}{2\delta-1}}, \quad \text{and} \quad m_n := \begin{cases} \frac{e^{-n+1}/a}{\sqrt[4]{2\delta-1}} \cdot e^{\frac{2\pi i n^2}{2\delta-1}}, & n \in [\delta]_0 \\ 0, & \text{otherwise} \end{cases},$$

where $a := \max \left\{ 4, \frac{\delta-1}{2} \right\}$. Then for all $\ell \in [2\delta-1]_0$, $\check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}$ satisfies

$$\check{\mathbf{m}}_\ell^{(p,m)} = e^{\frac{2\pi i \ell^2}{2\delta-1}} \cdot \check{\mathbf{m}}_{2\ell \bmod 2\delta-1}^{(fpr)},$$

- Let $\tilde{\mathbf{M}}^{(fpr)}$ and $\tilde{\mathbf{M}}^{(p,m)}$ be our lifted linear measurement matrices obtained by $\check{\mathbf{m}}_\ell^{(fpr)}$ and $\check{\mathbf{m}}_\ell^{(p,m)}$, respectively.
- Then $\tilde{\mathbf{M}}^{(p,m)} = \tilde{\mathbf{P}} \tilde{\mathbf{M}}^{(fpr)}$, where \mathbf{P} is a $D \times D$ block diagonal permutation matrix.
- Thus $\tilde{\mathbf{M}}^{(p,m)}$ and $\tilde{\mathbf{M}}^{(fpr)}$ have the same singular values and

$$\kappa\left(\tilde{\mathbf{M}}^{(p,m)}\right) = \kappa\left(\tilde{\mathbf{M}}^{(fpr)}\right) \leq C\delta^2,$$



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Magnitude Error Bound

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- For $\mathbf{x} \in \mathbb{C}^d$, we write its n^{th} entry as $x_n =: |x_n|e^{i\theta_n}$
- Let $\mathbf{x}^{(\text{mag})} := (|x_0|, \dots, |x_{d-1}|)^T, \mathbf{x}^{(\theta)} := (e^{i\theta_0}, e^{i\theta_1}, \dots, e^{i\theta_{d-1}})^T$
- We may then decompose \mathbf{x} as $\mathbf{x} = \mathbf{x}^{(\text{mag})} \circ \mathbf{x}^{(\theta)}$

Lemma (Lemma 3 - M. Roach, et al.)

Let \mathbf{x}_{est} be decomposed $\mathbf{x}_{\text{est}} = \mathbf{x}_{\text{est}}^{(\text{mag})} \circ \mathbf{x}_{\text{est}}^{(\theta)}$. Then, we have that

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x} - e^{i\phi} \mathbf{x}_{\text{est}}\|_2 \leq \|\mathbf{x}\|_\infty \min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)}\|_2 + \|\mathbf{x}^{(\text{mag})} - \mathbf{x}_{\text{est}}^{(\text{mag})}\|_2.$$

- We now utilize a restated result from *Phase Retrieval from Local Measurements: Improved Robustness via Eigenvector Based Angular Synchronization* - Iwen, M., Preskitt, B., Saab, R., Viswanathan, A.

Lemma (Lemma 3 - M. Iwen, et al.)

Let $\sigma_{\min}(\widetilde{\mathbf{M}}^{(p,m)})$ denote the smallest singular value of the lifted measurement matrix $\widetilde{\mathbf{M}}^{(p,m)}$. Then $\|\mathbf{x}^{(\text{mag})} - \mathbf{x}_{\text{est}}^{(\text{mag})}\|_2 \leq C \sqrt{\frac{\|\mathbf{N}\|_F}{\sigma_{\min}(\widetilde{\mathbf{M}}^{(p,m)})}}.$



Angular Error Bound

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- Apply lemma to our error bound and utilize the conditioning of \mathbf{p}, \mathbf{m} .

Theorem (Theorem 3 - M. Roach, et al.)

Let \mathbf{p} and \mathbf{m} be our admissible PSF, mask pair. Then, we have that

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x} - e^{i\phi} \mathbf{x}_{\text{est}}\|_2 \leq \|\mathbf{x}\|_\infty \min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)}\|_2 + C \sqrt{d\delta \|\mathbf{N}\|_F}.$$

- In *On Recovery Guarantees for Angular Synchronization* - Filbir, Krahmer, Melnyk, authors used weighted graph approach to prove bound on $\|\mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)}\|_2$.
- Let $\mathbf{X}, \widehat{\mathbf{X}}$ denote partial autocorrelation matrix corresponding to $\mathbf{x}, \mathbf{x}_{\text{est}}$
- Let $G = (V, E, \mathbf{W})$ be weighted graph, $V = [d]_0$, $E = \{(i, j) \mid i \neq j, |i - j| \bmod d < \delta\}$, and whose weight matrix \mathbf{W} is defined entrywise by

$$W_{i,j} = \begin{cases} |\widehat{X}_{i,j}|^2, & 0 < |i - j| \bmod d < \delta \\ 0, & \text{otherwise} \end{cases}.$$



Graph Theoretic Approach

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- Let \mathbf{D} denote the weighted degree matrix, $\mathbf{L}_G := \mathbf{D} - \mathbf{W}$.
- Let τ_G denote the spectral gap (second smallest eigenvalue) of \mathbf{L}_G .
- Since G is connected, τ_G is strictly positive.

Theorem (Corollary 3 - F. Filbir, F. Krahmer, O. Melnyk)

Let τ_G denote the spectral gap of the associated unnormalized Laplacian \mathbf{L}_G . Then we have that

$$\min_{\phi \in [0, 2\pi)} \left\| \mathbf{x}_{\text{est}}^{(\theta)} - e^{i\phi} \mathbf{x}^{(\theta)} \right\|_2 \leq C \sqrt{1 + \|\mathbf{x}_{\text{est}}\|_\infty} \cdot \frac{\|\mathbf{X} - \widehat{\mathbf{X}}\|_F}{\sqrt{\tau_G}}, \quad C \in \mathbb{R}^+.$$

- We have a bound for the Frobenius norm

Lemma (Lemma 6 - M. Roach, et al.)

Let \mathbf{p} and \mathbf{m} be our choice of PSF and mask. Then, $\|\mathbf{X} - \widehat{\mathbf{X}}\|_F \leq C\delta\|\mathbf{N}\|_F$.



Spectral Bound of Weighted Graph

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- We now seek a lower bound for our weighted spectral gap.

Theorem (General Weighted Spectral Gap Bound)

Let $G = (V, E, \mathbf{W})$ be a weighted graph and let W_{\min} and W_{\max} be the minimum and maximum value of any its (nonzero) weights. Then

$$\tau_G \geq \frac{2 \cdot (W_{\min})^2}{W_{\max}(n - 1) \cdot \text{diam}(G_{\text{unw}})},$$

where $G_{\text{unw}} = (V, E)$ is the unweighted counterpart of G , and $\text{diam}(G_{\text{unw}})$ is the corresponding diameter.

Lemma (NFP Weighted Spectral Gap Bound)

Let $|\mathbf{x}_{\text{est}}|_{\min}$ denote the smallest magnitude of any entry in \mathbf{x}_{est} . For our graph G , we have that

$$\tau_G \geq \frac{|\mathbf{x}_{\text{est}}|_{\min}^4}{\|\mathbf{x}_{\text{est}}\|_{\infty}^2} \frac{4(\delta - 1)}{d^2}.$$



Recovery Guarantee Theorem

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- We now apply the Frobenius bound and spectral gap bound to our existing error bound to obtain our main result.

Theorem (Theorem 1 - M. Roach, et al.)

Choose $\delta \in [d]_0$ such that $2\delta - 1$ divides d . One can construct a PSF $\mathbf{p} \in \mathbb{C}^d$ and a mask $\mathbf{m} \in \mathbb{C}^d$ with $\text{supp}(\mathbf{m}) \subseteq [\delta]_0$ such that we can return an estimate $\mathbf{x}_{\text{est}} \in \mathbb{C}^d$ of \mathbf{x} satisfying

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{\text{est}} - e^{i\phi} \mathbf{x}\|_2 \leq C \left(\|\mathbf{x}\|_\infty \frac{d \sqrt{\delta} \sqrt{\|\mathbf{x}_{\text{est}}\|_\infty^2 + \|\mathbf{x}_{\text{est}}\|_\infty^3}}{|\mathbf{x}_{\text{est}}|_{\min}^2} \cdot \|\mathbf{N}\|_{\text{F}} + \sqrt{d\delta \|\mathbf{N}\|_{\text{F}}} \right).$$

Here $C \in \mathbb{R}^+$ is an absolute constant.



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SNR vs. Reconstruction Error

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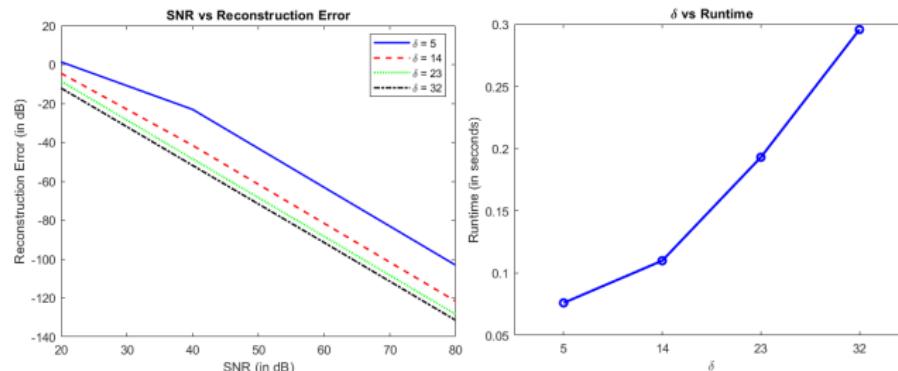
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- Evaluation of algorithm with Gaussian \mathbf{x} for proposed $\mathbf{p}, \mathbf{m}, d = 945$.
- Left: Reconstruction error vs SNR for various $\delta = |\text{supp}(\mathbf{m})|$.
- Right: Runtime as a function of δ .
- Then the signal-to-noise ratio is given by $\text{SNR} = 10 \log_{10} \left(\frac{\|\mathbf{Y} - \mathbf{N}\|_F}{\|\mathbf{N}\|_F} \right)$.
- We measure the reconstruction error by $10 \log_{10} \left(\frac{\min_{\phi} \|\mathbf{x} - e^{i\phi} \mathbf{x}_{\text{est}}\|_2^2}{\|\mathbf{x}\|_2^2} \right)$ and we plot based on varying levels of signal-to-noise ratio.



Real World Application

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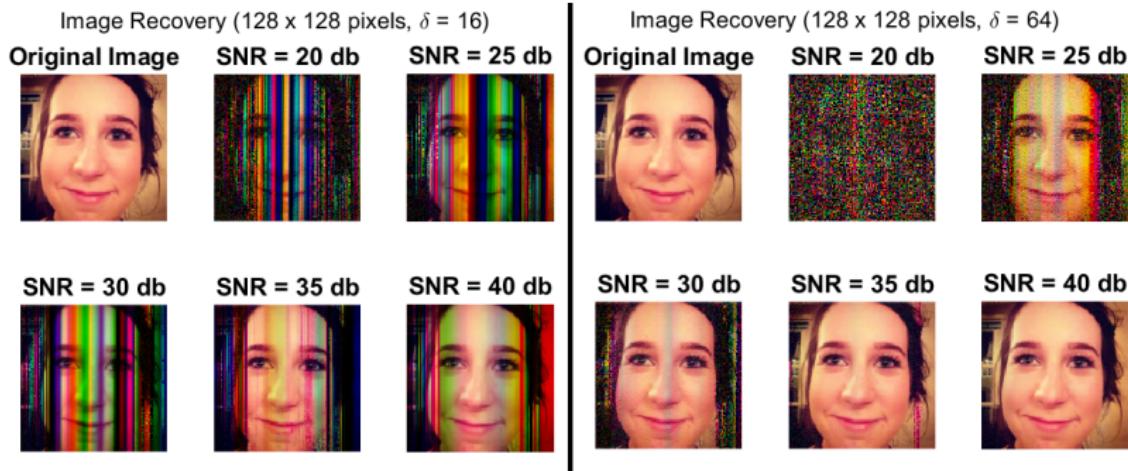
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- NFP BlockPR algorithm applied to 128×128 pixel color image
- Each color channel applied separately and then combined to form final image
- $d = 128^2 = 16384$ with two delta levels applied and varying noise



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Far-field Ptychography

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- Far-field Ptychography is when there is a large enough distance between the lens, object, and detector to obtain magnitude-square Fourier transform measurements

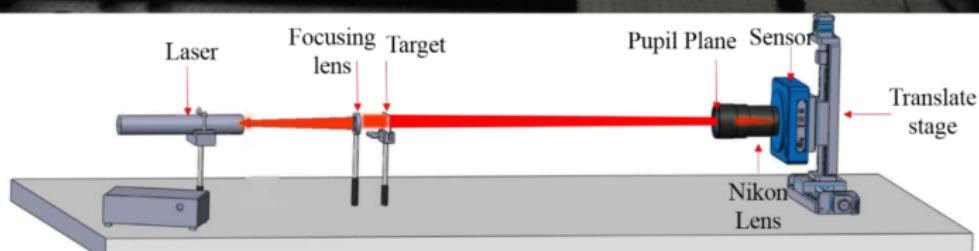
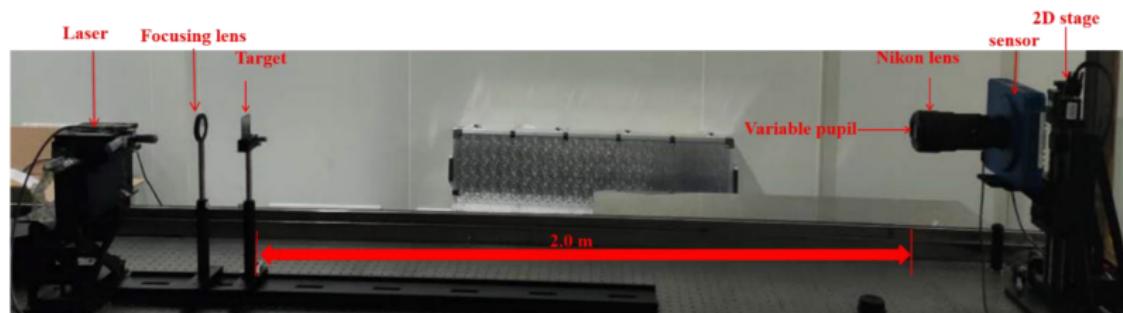


Figure: "Analysis, Simulations, and Experiments for Far-Field Fourier Ptychography Imaging Using Active Coherent Synthetic-Aperture" - Yang et al., 2022



Blind Ptychography

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- Let $\mathbf{x}, \mathbf{m} \in \mathbb{C}^d$ denote the unknown object and mask, respectively.
- Suppose that we have d^2 noisy ptychographic measurements of the form

$$(\mathbf{Y})_{\ell,k} = |(\mathbf{F}(\mathbf{x} \circ S_k \mathbf{m}))_\ell|^2 + (\mathbf{N})_{\ell,k}, \quad (\ell, k) \in [d]_0 \times [d]_0.$$

- Can rewrite the measurements as

$$\left(\mathbf{Y}^T \mathbf{F}^T \right)_k = d \cdot (\mathbf{x} \circ S_k \bar{\mathbf{x}}) * (\tilde{\mathbf{m}} \circ S_{-k} \tilde{\mathbf{m}}) + \left(\mathbf{N}^T \mathbf{F}^T \right)_k$$

- Fix k . Let $\mathbf{y}' = \left(\mathbf{Y}^T \mathbf{F}^T \right)_k$, $\mathbf{f} = \sqrt{d}(\tilde{\mathbf{m}} \circ S_{-k} \tilde{\mathbf{m}})$, $\mathbf{g} = \sqrt{d}(\mathbf{x} \circ S_k \bar{\mathbf{x}})$, $\mathbf{n} = \left(\mathbf{N}^T \mathbf{F}^T \right)_k$
- This is now a noisy blind deconvolution problem i.e.

$$\mathbf{y}' = \mathbf{f} * \mathbf{g} + \mathbf{n}.$$



Blind Deconvolution - Part I

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- Strohmer et.al., demonstrated successful recovery by utilizing reasonable assumptions
- Let $\mathbf{y} = \mathbf{f} * \mathbf{g} + \mathbf{n}$, with \mathbf{f}, \mathbf{g} unknown, \mathbf{n} noise
- Assume $\mathbf{f} = \begin{bmatrix} \mathbf{h} \\ \mathbf{0}_{d-K} \end{bmatrix}$, $\mathbf{g} = \mathbf{C}\bar{\mathbf{x}}$ for some matrix $\mathbf{C} \in \mathbb{C}^{d \times N}$, $N \ll d$.
- Let $\mathbf{F}_d \in \mathbb{C}^{d \times d}$ be DFT matrix, $\mathbf{B} \in \mathbb{C}^{d \times K}$ denote the first K columns of \mathbf{F}_d .
- Then we have that

$$\mathbf{y} = \mathbf{B}\mathbf{h} \circ \overline{\mathbf{A}\mathbf{x}} + \mathbf{e},$$

where $\mathbf{y} = \frac{1}{\sqrt{d}}\widehat{\mathbf{y}'}$, $\bar{\mathbf{A}} = \mathbf{F}_d \mathbf{C} \in \mathbb{C}^{d \times N}$, and $\mathbf{e} = \frac{1}{\sqrt{d}}\mathbf{F}_d \mathbf{n}$ represents noise.

- If $(\mathbf{h}_0, \mathbf{x}_0)$ is a solution, then so is $(\alpha\mathbf{h}_0, \alpha^{-1}\mathbf{x}_0)$ for any non-zero α . Thus we assume $\|\mathbf{f}\|_2$ known.



Blind Deconvolution - Part II

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- We define the matrix-valued linear operator $\mathcal{A} : \mathbb{C}^{K \times N} \longrightarrow \mathbb{C}^d$ by

$$\mathcal{A}(\mathbf{Z}) := \{\mathbf{b}_\ell^* \mathbf{Z} \mathbf{a}_\ell\}_{\ell=1}^d,$$

where \mathbf{b}_k denotes the k -th column of \mathbf{B}^* , and \mathbf{a}_k is the k -th column of \mathbf{A}^* .

- Then $\mathbf{y} = \mathcal{A}(\mathbf{h}\mathbf{x}^*) + \mathbf{e}$.
- Define corresponding adjoint operator $\mathcal{A}^* : \mathbb{C}^d \longrightarrow \mathbb{C}^{K \times N}$, given by

$$\mathcal{A}^*(\mathbf{z}) := \sum_{k=1}^d \mathbf{z}_k \mathbf{b}_k \mathbf{a}_k^*.$$

- For any given $\mathbf{Z} \in \mathbb{C}^{K \times N}$, $\mathbb{E}(\mathcal{A}^*(\mathcal{A}(\mathbf{Z}))) = \mathbf{Z}$.
- Leading singular vectors of $\mathcal{A}^*(\mathbf{y}) \approx (\mathbf{h}, \mathbf{x})$.
- Apply gradient descent using gradients of $F(\mathbf{h}, \mathbf{x}) := \|\mathcal{A}(\mathbf{h}\mathbf{x}^*) - \mathbf{y}\|^2$



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Blind Ptychography Algorithm

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- Let $\mathbf{y}_{(k)} = \frac{1}{\sqrt{d}} \cdot (\mathbf{F}(\widetilde{\mathbf{FY}}^T))_k, \mathbf{f}_{(k)} = \tilde{\mathbf{m}} \circ S_{-k} \bar{\mathbf{m}}$ ($\|\mathbf{m}\|_2$ known)
- Let $\mathbf{x} = \mathbf{Cx}'$ for some known matrix $\mathbf{C} \in \mathbb{C}^{d \times N}, \mathbf{x}' \in \mathbb{C}^N$.

Lemma

Let $\mathbf{A} \in \mathbb{C}^{m \times n}, \mathbf{B} \in \mathbb{C}^{n \times p}, \mathbf{C} \in \mathbb{C}^{m \times q}, \mathbf{D} \in \mathbb{C}^{q \times p}$. Then we have that

$$(\mathbf{AB}) \circ (\mathbf{CD}) = (\mathbf{A} \bullet \mathbf{C})(\mathbf{B} \odot \mathbf{D}),$$

where \circ is the Hadamard product, \odot is the standard Khatri-Rao product, \bullet is the transposed Khatri-Rao product.

- Let $\mathbf{g}_{(k)} = \mathbf{x} \circ S_k \bar{\mathbf{x}} = \mathbf{Cx}' \circ S_k \bar{\mathbf{C}} \bar{\mathbf{x}}'$.
- Then $\mathbf{g}_{(k)} = \mathbf{C}'_{(k)} \mathbf{x}''$ where $\mathbf{C}' \in \mathbb{C}^{d \times N^2}, \mathbf{x}'' \in \mathbb{C}^{N^2}$ are given by

$$\mathbf{C}'_{(k)} = \mathbf{C} \bullet S_k \bar{\mathbf{C}}, \mathbf{x}'' = \mathbf{x}' \odot \bar{\mathbf{x}}', \quad 0 \leq k \leq K, d - K + 1 \leq k \leq d.$$



Performing Series of Blind Deconvolutions

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- Perform $2\delta - 1$ blind deconvolutions, with $\mathbf{y}_{(k)}, \mathbf{f}_{(k)}, \mathbf{g}_{(k)}, \mathbf{C}$, as above
- We obtain $2\delta - 1$ estimates for

$$\mathbf{x}' \odot \bar{\mathbf{x}}' = \text{vec}(\mathbf{x}'(\mathbf{x}')^*).$$

- Use angular synchronization to solve for $2\delta - 1$ estimates \mathbf{x}'_{est}
- Thus solve for $2\delta - 1$ estimates

$$\mathbf{x}_{\text{est}}^i = \mathbf{C}\mathbf{x}'_{\text{est}}, \quad i \in [2\delta - 1].$$

- Use these estimates $\mathbf{x}_{\text{est}}^i$ to compute $2\delta - 1$ estimates $\mathbf{m}_{\text{est}}^j, j \in [2\delta - 1]$.
- Let $\alpha^i = \frac{\|\mathbf{m}_{\text{est}}^i\|_2}{\|\mathbf{m}\|_2}$. Then $\mathbf{x}_{\text{est}}^i = \alpha^i \mathbf{x}_{\text{est}}^i, \mathbf{m}_{\text{est}}^i = \frac{1}{\alpha^i} \mathbf{m}_{\text{est}}^i, \quad \forall i \in [2\delta - 1]$.



Computing the Mask

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- From our original measurements, it can be further shown that

$$\mathbf{F}_d \left(\mathbf{Y}^T \mathbf{F}_d^T \right)_k = d \cdot \mathbf{F}_d(\mathbf{x} \circ S_k \bar{\mathbf{x}}) \circ \mathbf{F}_d(\tilde{\mathbf{m}} \circ S_{-k} \tilde{\bar{\mathbf{m}}}) + \mathbf{F}_d \left(\mathbf{N}^T \mathbf{F}_d^T \right)_k$$

- Assuming noiseless scenario, then we have that

$$\mathbf{F}_d(\tilde{\mathbf{m}} \circ S_{-k} \tilde{\bar{\mathbf{m}}}) = \frac{1}{d} \frac{\mathbf{F}_d \left(\mathbf{Y}^T \mathbf{F}_d^T \right)_k}{\mathbf{F}_d(\mathbf{x} \circ S_k \bar{\mathbf{x}})}$$

- So once we have an estimate for \mathbf{x} , we compute the $2\delta - 1$ pointwise divisions, then $2\delta - 1$ inverse Fourier transforms.
- These become the diagonals of a matrix which will estimate $\tilde{\mathbf{m}}\tilde{\mathbf{m}}^*$. Thus compute angular synchronization and finally a reversal.



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- Ideally, we would want to select the estimates which generate the minimum error for each \mathbf{x} and \mathbf{m} i.e. find

$$\text{Min Shift}^{(x)} = \underset{\mathbf{x}_{\text{est}}^i}{\operatorname{argmin}} \|\mathbf{x} - \mathbf{x}_{\text{est}}^i\|_2^2, \quad \text{Min Shift}^{(m)} = \underset{\mathbf{m}_{\text{est}}^j}{\operatorname{argmin}} \|\mathbf{m} - \mathbf{m}_{\text{est}}^j\|_2^2$$

for $i, j \in [2\delta - 1]$.

- However that implies prior knowledge of \mathbf{x} and \mathbf{m} .
- Instead, compute $(2\delta - 1)^2$ estimates of the Fourier measurements by

$$(\mathbf{Y}_{\text{est}}^{i,j})_{\ell,k} = |(\mathbf{F}(\mathbf{x}_{\text{est}}^i \circ S_k \mathbf{m}_{\text{est}}^j))_\ell|^2, \quad i, j \in [2\delta - 1].$$

- We then compute the associated error

$$(i', j') = \underset{(i,j)}{\operatorname{argmin}} \frac{\|\mathbf{Y}_{\text{est}}^{i,j} - \mathbf{Y}\|_F^2}{\|\mathbf{Y}\|_F^2}, \quad i, j \in [2\delta - 1]$$

- Let $\mathbf{x}_{\text{est}} = \mathbf{x}_{\text{est}}^{i'}, \mathbf{m}_{\text{est}} = \mathbf{m}_{\text{est}}^{j'}$.



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SNR vs. Reconstruction Error

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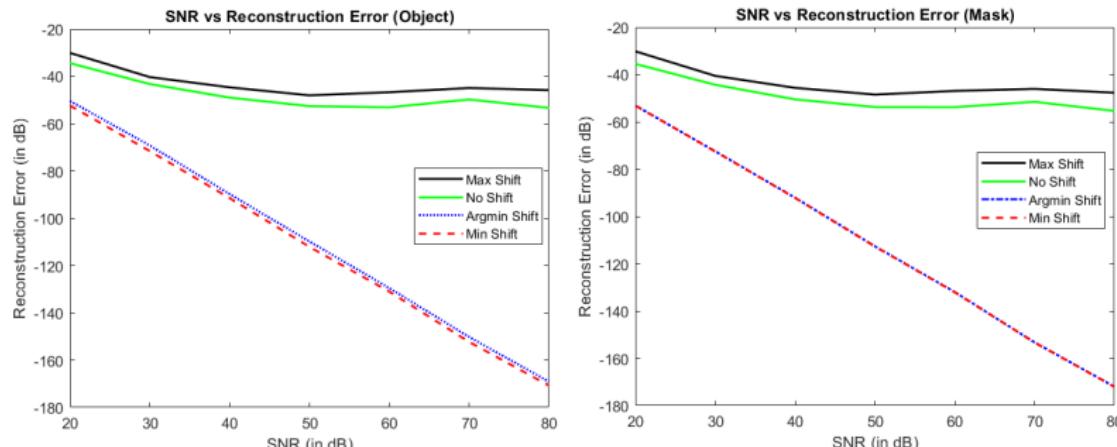
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- $d = 2^6, K = \delta = \log_2 d, N = 6, \mathbf{C}$ complex Gaussian, 100 simulations
- Max shift refers to the maximum possible error.
- Min shift refers to the minimum possible error.
- Argmin Shift refers to the choice of object and mask chosen by our algorithm.



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Johnson-Lindenstrauss Lemma

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Questions

- We consider real-world problems with geometric input data X and want to consider metric embeddings $X \rightarrow Y$ where $\dim(Y) \ll \dim(X)$
- Working with lower-dimensional embedded data typically results in efficiency gains, in terms of memory, running time, and/or other resources but reduction in accuracy

Lemma (Johnson-Lindenstrauss Lemma)

Let $\epsilon \in (0, 1)$ and $X \subset \mathbb{R}^d$ be arbitrary with $|X| = n > 1$. There exists $f : X \rightarrow \mathbb{R}^m$ with $m = O(\epsilon^{-2} \log n)$ such that $\forall \mathbf{x} \in X, \forall \mathbf{y} \in X$

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2$$

- $1 - \epsilon, 1 + \epsilon$, are referred to as the **distortion**, $Dist(f)$, of the embedding f
- Let X denote training data. Lemma states distance between training data preserved up to small distortion.
- Want distance between training data and all \mathbb{R}^d (testing data) to be preserved up to small distortion



Terminal Embedding Theorem

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- Work by Elkin et.al showed that y can be taken as an arbitrary point in \mathbb{R}^d with $m = O(\log n)$ and distortion $\approx \sqrt{10}$.

Theorem (Theorem 1.1 - Narayanan, Nelson)

Let $\epsilon \in (0, 1)$ and $X \subset \mathbb{R}^d$ be arbitrary with $|X| = n > 1$. There exists $f : X \longrightarrow \mathbb{R}^m$ with $m = O(\epsilon^{-2} \log n)$ such that $\forall x \in X, \forall y \in \mathbb{R}^d$

$$(1 - \epsilon) \|x - y\|_2 \leq \|f(x) - f(y)\|_2 \leq (1 + \epsilon) \|x - y\|_2$$

- This embedding is called a **terminal embedding** with multiplicative factor on the right hand side referred to as the **terminal distortion**.
- If the points in X are mapped to \mathbb{R}^m well, which occurs with high probability, then our final terminal embedding is guaranteed to have low terminal distortion as a map from all of \mathbb{R}^d to \mathbb{R}^m .



Outer Extension, ϵ -Convex Hull Distortion

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Definition (Outer extension)

For $f : X \rightarrow \mathbb{R}^m$ and $X \subset Z$, we say $g : Z \rightarrow \mathbb{R}^{m'}$ is an **outer extension** of f if $m' \geq m$, and $g(\mathbf{x})$ being a zero-padding of $f(\mathbf{x})$, for $\mathbf{x} \in X$.

- We define our terminal embedding by $f_{\text{Ext}}(\mathbf{u}) = \begin{cases} (f(\mathbf{u}), 0), & \mathbf{u} \in X \\ f^{(\mathbf{u})}(\mathbf{u}), & \mathbf{u} \in \mathbb{R}^d \setminus X \end{cases}$ which is an outer extension on X , and $f^{(\mathbf{u})}$ function depending on \mathbf{u}
- The goal is to specify how to embed points $\mathbf{x} \in X$ and then how to embed points $\mathbf{u} \notin X$ such that we obtain low distortion.

Definition (ϵ -convex hull distortion)

For $T \subset \mathcal{S}^{d-1}$ and $\epsilon \in (0, 1)$, we say that $\Pi \in \mathbb{R}^{m \times d}$ provides **ϵ -convex hull distortion** for T if

$$|\|\Pi \mathbf{x}\|_2 - \|\mathbf{x}\|_2| < \epsilon, \quad \forall \mathbf{x} \in \text{conv}(T),$$

where $\mathcal{S}^{d-1} \subset \mathbb{R}^d$ unit sphere, $\text{conv}(T)$ convex hull of T



Construction of Terminal Embedding

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Lemma (Lemma 3.2 - Narayanan, Nelson)

Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$ be distinct. Let $Y = \{\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|_2} : i \neq j\}$. Suppose Π provides an ϵ -convex hull distortion for Y . Then $\forall \mathbf{u} \in \mathbb{R}^d$, \exists outer extension f_{Ext} with $f(\mathbf{x}_i) = \Pi \mathbf{x}_i$, $f^{(\mathbf{u})}$ defined below, which provides $(1 + \epsilon)$ -distortion

- By choosing $\Pi \in \mathbb{R}^{m \times d}$ to have i.i.d sub-Gaussian entries, we get with high probability, a matrix providing ϵ -convex hull distortion for Y .

Lemma

Let Π be an ϵ -convex hull distortion for Y . Then \mathbf{u}' exists for all \mathbf{u} , and \mathbf{u}' can be found with semi-definite programming in polynomial time.



Terminal Embedding of a Finite Set

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Algorithm Terminal Embedding of a Finite Set

Input: $\epsilon \in (0, 1)$, $X \subset \mathbb{R}^N$, $|X| =: n$, $S \subset \mathbb{R}^N$, $|S| =: n'$, $S \cap X = \emptyset$, $m \in \mathbb{N}$ with $m < N$, a random matrix Gaussian entries, $\Phi \in \mathbb{R}^{m \times N}$, $\Pi := \frac{1}{\sqrt{m}}\Phi$

Output: A terminal embedding of X , $f \in \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$, evaluated on S

for $\mathbf{u} \in S$ do

1) Compute $\mathbf{x}_{NN} := \operatorname{argmin}_{\mathbf{x} \in X} \|\mathbf{u} - \mathbf{x}\|_2$

2) Solve constrained minimization problem to compute $\mathbf{u}' \in \mathbb{R}^m$

$$\text{Minimize } h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \|\mathbf{z}\|_2^2 + 2\langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$$

subject to $\|\mathbf{z}\|_2 \leq \|\mathbf{u} - \mathbf{x}_{NN}\|_2, \quad \forall \mathbf{x} \in X$

$$|\langle \mathbf{z}, \Pi(\mathbf{x} - \mathbf{x}_{NN}) \rangle - \langle \mathbf{u} - \mathbf{x}_{NN}, \mathbf{x} - \mathbf{x}_{NN} \rangle| \leq \epsilon \|\mathbf{u} - \mathbf{x}_{NN}\|_2 \|\mathbf{x} - \mathbf{x}_{NN}\|_2$$

3) Compute $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$ at \mathbf{u} via

$$f(\mathbf{u}) := \begin{cases} (\Pi \mathbf{u}, 0), & \mathbf{u} \in X \\ (\Pi \mathbf{x}_{NN} + \mathbf{u}', \sqrt{\|\mathbf{u} - \mathbf{x}_{NN}\|_2^2 - \|\mathbf{u}'\|_2^2}), & \mathbf{u} \in S \end{cases}$$

end for



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Measuring Compressive Nearest Neighbor Classification Accuracy

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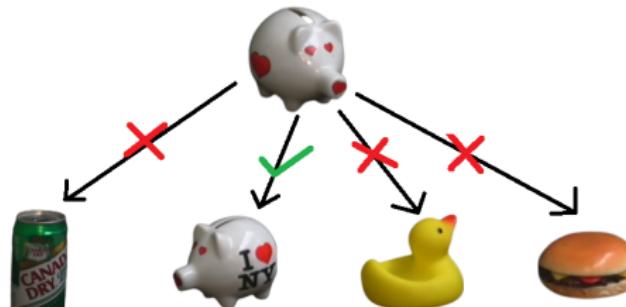
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- Let $\epsilon \in (0, 1)$, labeled data set $\mathcal{D} \subset \mathbb{R}^N$: Training set $X \subset \mathcal{D}$ with $|X| =: n$, Test set $S \subset \mathcal{D}$ with $|S| =: n'$, such that $S \cap X = \emptyset$.
- Let $m \ll N$ and fix $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$ such that

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2, \quad \mathbf{x}, \mathbf{y} \in X$$

- For $\mathbf{x} \in X, \mathbf{u} \in S$, compute $f(\mathbf{x}), f(\mathbf{u})$ using previous Algorithm
- For each $\mathbf{u} \in S$, compute $\mathbf{z} = \operatorname{argmin}_{\mathbf{y} \in X} \|f(\mathbf{u}) - f(\mathbf{y})\|_2$
- If $\text{Label}(\mathbf{u}) = \text{Label}(\mathbf{z})$, this is deemed a successful classification
- Track success rate as percentage





COIL-100 Dataset

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- COIL-100 dataset: 100 objects, 128×128 -pixel color images of 100 objects, 72 evenly spaced rotations
- Grayscale, vectorized, 7,200 total datapoints in \mathbb{R}^N with $N = 128^2 = 16,384$
- Training data: evenly distributed, evenly rotated collection of all 100 objects



COIL-100 Classification Simulations

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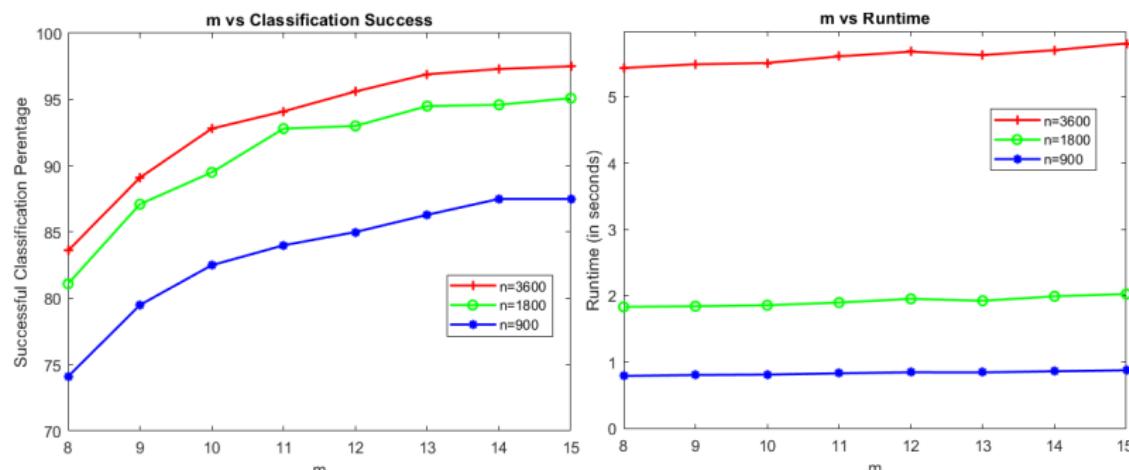
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- This figure compares compressive NN classification accuracies, and the associated classification run times averaged over all $\mathbf{u} \in S$.
- Three different training data set sizes $n = |X| \in \{900, 1800, 3600\}$ were fixed as the embedding dimension $m + 1$ varied for each of the first two subfigures.



Comparison Figure

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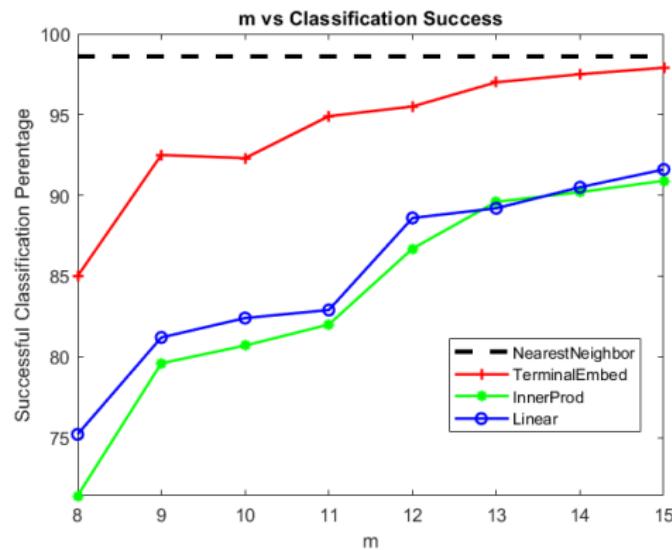
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- **Nearest Neighbor:** Find the nearest neighbor in the original space
- **TerminalEmbed:** $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \|\mathbf{z}\|_2^2 + 2\langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$
- **InnerProd:** $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$
- **Linear:** Embed into the space linearly i.e. $f(\mathbf{u}) = (\Pi \mathbf{u}, 0)$



Compressed Classification from Phaseless Measurements

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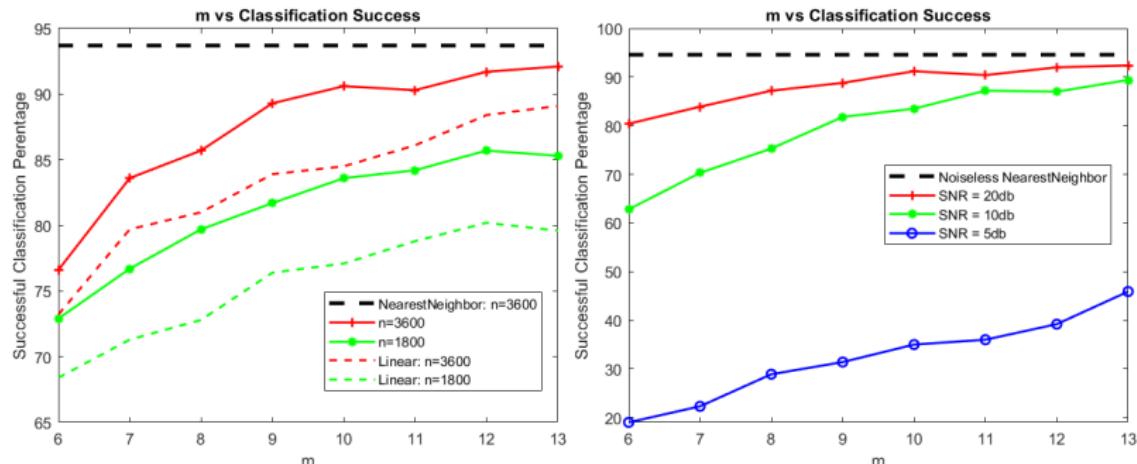
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- Sub-sampled NFP measurements $|(\mathbf{p} * (\mathbf{S}_k \mathbf{m} \circ \mathbf{x}))_\ell|^2$, $(k, \ell) \in [d] \times [1]$.
- \mathbf{x} = vectorized, grayscaled, COIL-100 images.
- Left: Comparing NearestNeighbor, TerminalEmbed, & Linear.
- Right: Additive noise applied to testing data i.e. $\mathbf{u} \rightarrow \mathbf{u} + \mathbf{n}$.



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Reach

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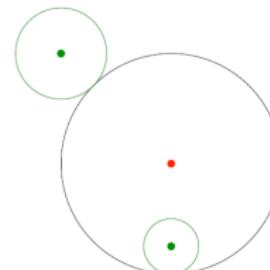
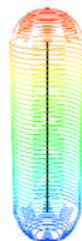
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Definition

For a subset $S \subset \mathbb{R}^N$ of Euclidean space, the **reach** τ_S is

$$\tau_S := \sup \left\{ t \geq 0 \mid \forall \mathbf{x} \in \mathbb{R}^n \text{ s.t. } d(\mathbf{x}, S) < t, \mathbf{x} \text{ has a unique closest point in } S \right\}.$$

- All points within distance τ_S of S have unique nearest Euclidean neighbors on S
- $\tau_S = \infty \Leftrightarrow S \subseteq \mathbb{R}^N$ is closed and convex.
- Larger $\tau_S \Rightarrow S$ simple (flatter, non-self-intersecting, etc.)



$\text{reach}(\text{line segment}) = \infty$

$\text{reach}(\text{sphere}) = \text{radius of sphere}$



Definition

The **Gaussian width** of a set $T \subset \mathbb{R}^N$ is $w(T) := \mathbb{E} \sup_{\mathbf{x} \in T} \langle \mathbf{g}, \mathbf{x} \rangle$, where \mathbf{g} is a random vector with N i.i.d. mean 0 and variance 1 Gaussian entries.

- Gaussian width can replace cardinality as measure of set complexity.
- If T is unit ℓ_2 ball in a N -dimensional subspace, then $w(T) \approx \sqrt{N}$

Definition

We define the **unit secants of $T \subset \mathbb{R}^N$** to be

$$S_T := \overline{U((T - T) \setminus \{\mathbf{0}\})} = \left\{ \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|_2} \mid \mathbf{x}, \mathbf{y} \in T, \mathbf{x} \neq \mathbf{y} \right\}.$$

- Let $\mathcal{M} \hookrightarrow \mathbb{R}^N$ be a compact d -dimensional submanifold of \mathbb{R}^N with boundary $\partial\mathcal{M}$, finite reach $\tau_{\mathcal{M}}$, and volume $V_{\mathcal{M}}$.
- τ_i = reach of i^{th} connected component of $\partial\mathcal{M}$ as submanifold of \mathbb{R}^N .
- Set $\tau := \min_i \{\tau_{\mathcal{M}}, \tau_i\}$, let $V_{\partial\mathcal{M}} = \text{Vol}(\partial\mathcal{M})$, and $\omega_d = \text{Vol}(B_d(1))$.



Main Result

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Theorem (The Main Result)

Let $d \geq 2$. Define $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$. Fix $\epsilon \in (0, 1)$ and define $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$. Then, there exists a map $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c(\ln(\beta_M) + 4d)/\epsilon^2$ that satisfies

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2,$$

for all $\mathbf{x} \in M$ and $\mathbf{y} \in \mathbb{R}^N$. Here $c \in \mathbb{R}^+$ is an absolute constant.

The following theorem bounds the Gaussian width of a smooth submanifold of \mathbb{R}^N in terms of its dimension, reach, and volume.

Theorem (Theorem 2.1 - Iwen, Roach)

Let $d \geq 2$. Define $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$, $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$. Then we have that

$$w(S_M) \leq 8\sqrt{2} \sqrt{\ln(\beta_M) + 4d}.$$



Proof of Main Result

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- $\Phi \in \mathbb{C}^{m \times N}$ with independent, isotropic, sub-Gaussian rows provides ϵ -convex hull distortion for S_M .
- This forms the basis for our outer bi-Lipschitz extension f

Theorem (Theorem 3.2 - Iwen, Roach)

Let $M \subset \mathbb{R}^N$ and $\epsilon \in (0, 1)$. There exists a map $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c \left(\frac{w(S_M)}{\epsilon} \right)^2$ that satisfies

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2^2,$$

for all $\mathbf{x} \in M$ and $\mathbf{y} \in \mathbb{R}^N$, where $c \in \mathbb{R}^+$ is an absolute constant.

Proof of Main Result.

Theorem 3.2 proves that there exists a terminal embedding $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c \left(\frac{w(S_M)}{\epsilon} \right)^2$. Theorem 2.1 bounds $w(S_M)$. Apply both theorems to complete proof. □



References - Near-field Ptychography

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References - Blind Ptychography via Blind Deconvolution

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References - Terminal Manifold Embeddings

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Questions

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Questions

- Code/slides available at <https://github.com/MarkPhilipRoach>



Any questions?



Angular Synchronization

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Example (Angular Synchronization)

Let $d = 4, \delta = 2$. Then $\widehat{\mathbf{X}}$ is given by

$$\widehat{\mathbf{X}} = \begin{pmatrix} |(\mathbf{x}_{\text{est}})_0|^2 & (\mathbf{x}_{\text{est}})_0 \overline{(\mathbf{x}_{\text{est}})_1} & 0 & (\mathbf{x}_{\text{est}})_0 \overline{(\mathbf{x}_{\text{est}})_3} \\ (\mathbf{x}_{\text{est}})_1 \overline{(\mathbf{x}_{\text{est}})_0} & |(\mathbf{x}_{\text{est}})_1|^2 & (\mathbf{x}_{\text{est}})_1 \overline{(\mathbf{x}_{\text{est}})_2} & 0 \\ 0 & (\mathbf{x}_{\text{est}})_2 \overline{(\mathbf{x}_{\text{est}})_1} & |(\mathbf{x}_{\text{est}})_2|^2 & (\mathbf{x}_{\text{est}})_2 \overline{(\mathbf{x}_{\text{est}})_3} \\ (\mathbf{x}_{\text{est}})_3 \overline{(\mathbf{x}_{\text{est}})_0} & 0 & (\mathbf{x}_{\text{est}})_3 \overline{(\mathbf{x}_{\text{est}})_2} & |(\mathbf{x}_{\text{est}})_3|^2 \end{pmatrix}.$$

One may verify that the lead eigenvector is $\mathbf{u} = (e^{i\theta_0} \ e^{i\theta_1} \ e^{i\theta_2} \ e^{i\theta_3})^T$ and therefore

$$\mathbf{x}_{\text{est}} = \sqrt{\text{diag}(\widehat{\mathbf{X}})} \circ \mathbf{u} = (|(\mathbf{x}_{\text{est}})_0|e^{i\theta_0} \ |(\mathbf{x}_{\text{est}})_1|e^{i\theta_1} \ |(\mathbf{x}_{\text{est}})_2|e^{i\theta_2} \ |(\mathbf{x}_{\text{est}})_3|e^{i\theta_3})^T.$$

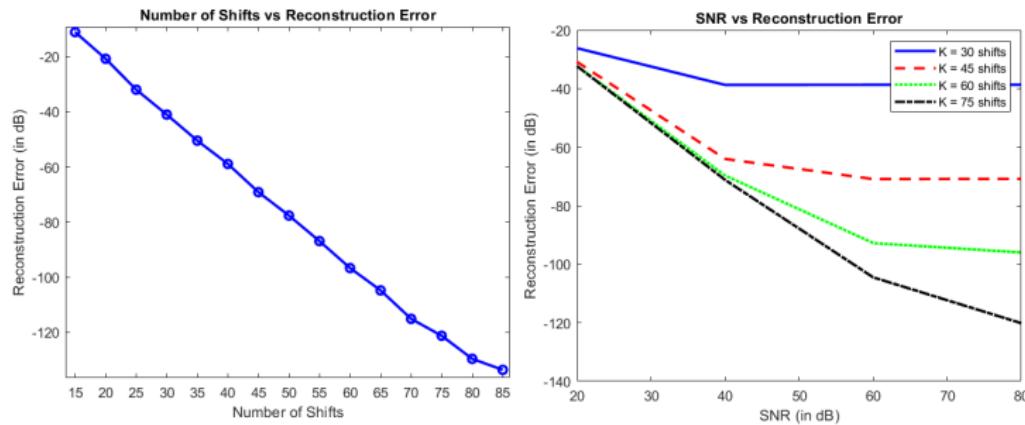


Near Field Ptychography via Wirtinger Flow

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- Want to be able to assume \mathbf{p} is a low pass filter and be able to take fewer shifts.
- In "Phase Retrieval via Wirtinger Flow: Theory and Algorithms - Candes, Li, Soltanolkotabi", quadratic equations of the form $y_n = |\langle \mathbf{m}_\ell, \mathbf{x} \rangle|^2$, $n = 1, 2, \dots, N$ are solved by applying a gradient descent like algorithm



Plotting reconstruction error with $d = 102$, $\mathcal{L} = [d]_0$, and number of iterations $T = 2000$.

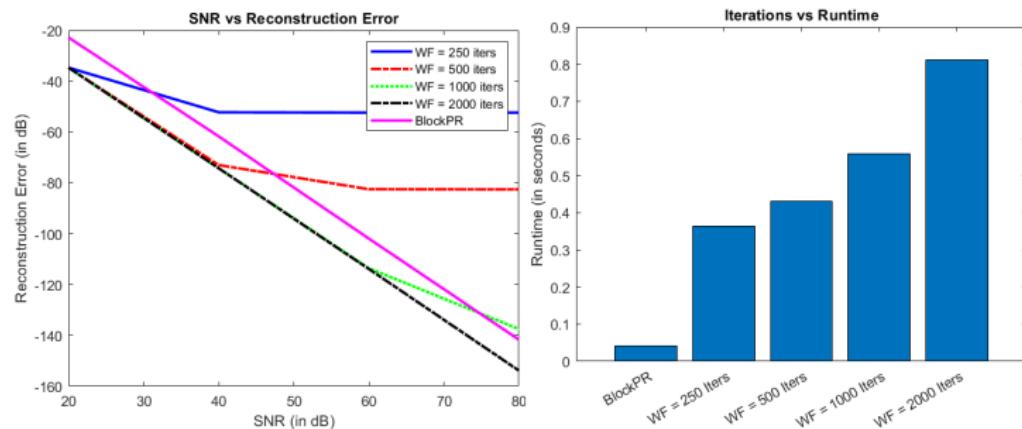
Left: Reconstruction error vs the number of total shifts K for fixed SNR = 80.
Right: Reconstruction error vs SNR for various numbers of shifts K .



Comparison Between Algorithms

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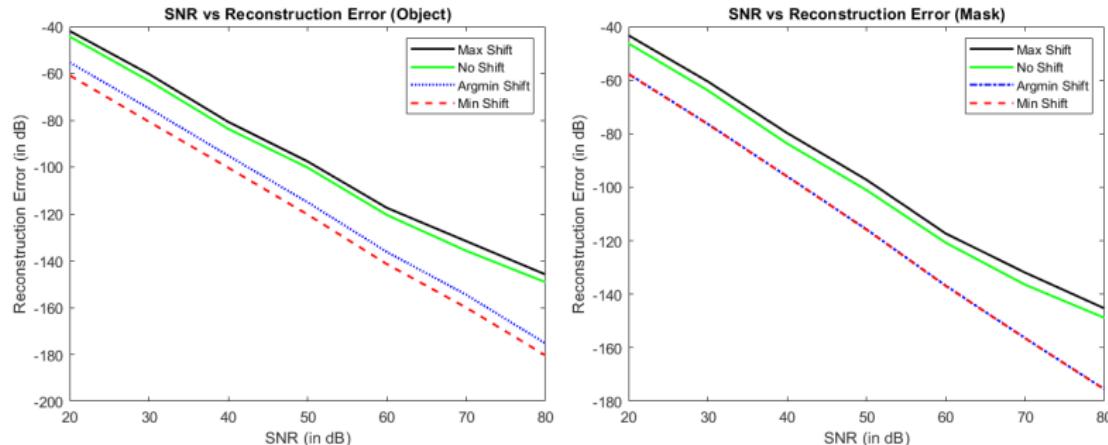
- A comparison of BlockPR and Wirtinger Flow algorithms for the proposed PSF and mask with $\delta = 26$ and $d = 102$.
- Left: Reconstruction error vs SNR for various numbers of Wirtinger Flow iterations.
- Right: The corresponding average runtimes.



SNR vs. Reconstruction Error

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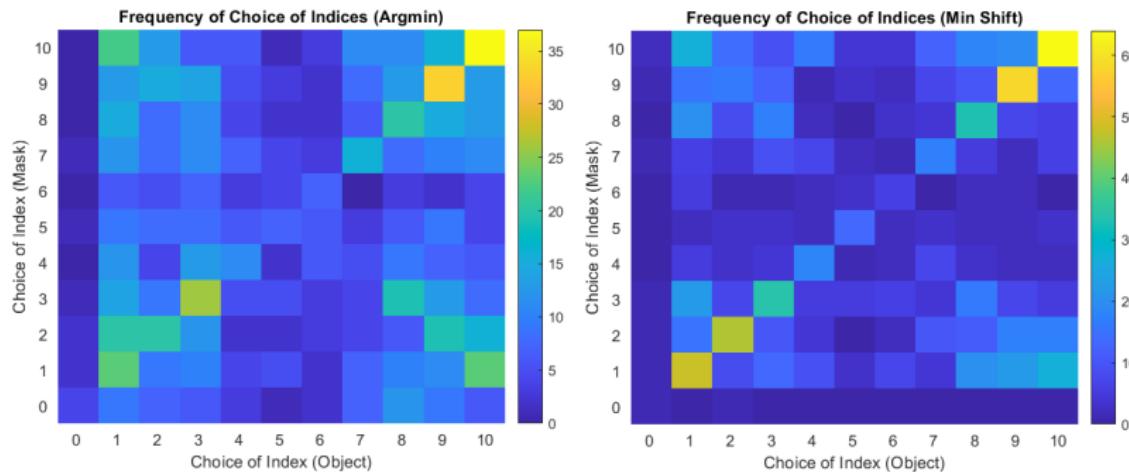
- $d = 2^6, K = \delta = \log_2 d, N = 4, \mathbf{C}$ complex Gaussian, 100 simulations
- Max shift refers to the maximum possible error.
- Min shift refers to the minimum possible error.
- Argmin Shift refers to the choice of object and mask chosen by our algorithm.



Frequency of Indices

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- $d = 2^6, \delta = 6, N = 4, \mathbf{C}$ complex Gaussian, 1000 simulations
- Left: Frequency of indices being chosen to compute ($\text{Argmin Shift}^{(x)}, \text{Argmin Shift}^{(m)}$)
- Right: Frequency of indices being chosen to compute ($\text{Min Shift}^{(x)}, \text{Min Shift}^{(m)}$).



MNIST

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A 4x10 grid of handwritten digits from the MNIST dataset. The digits are arranged in four rows and ten columns. The digits are drawn in black ink on a white background. They represent the digits 0 through 9.

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9

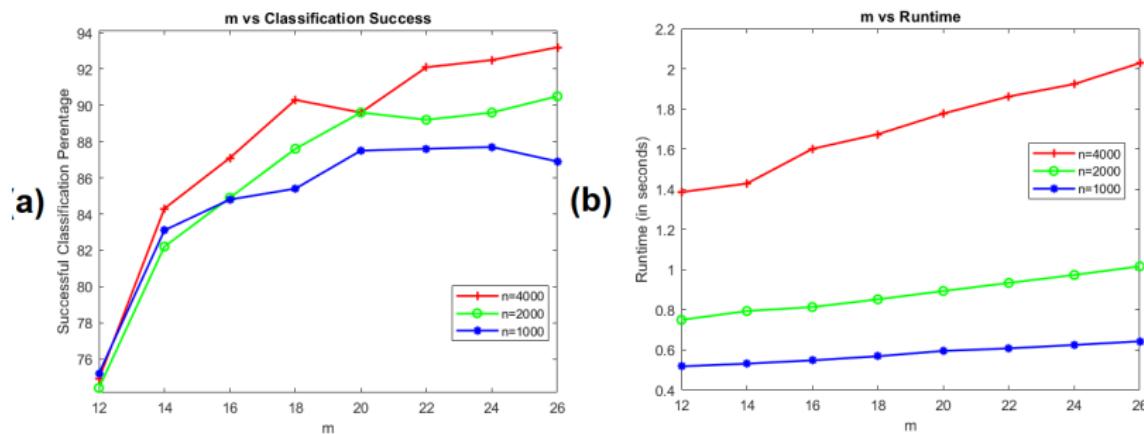
- The MNIST data set consists of 60,000 training images of 28×28 -pixel grayscale hand-written images of the digits 0 through 9.
- Thus, MNIST has 10 labels to correctly classify between, and $N = 28^2 = 784$.
- For all experiments involving the MNIST dataset, $n/10$ digits of each type are selected uniformly at random to form the training set X , for a total of n vectorized training images in \mathbb{R}^{784} .
- Then, 100 digits of each type are randomly selected from those not used for training in order to form the test set S , leading to a total of $n' = 1000$ vectorized test images in \mathbb{R}^{784} .



MNIST Classification Simulations

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- This figure compares compressive NN classification accuracies, and the associated classification run times averaged over all $\mathbf{u} \in S$.
- Three different training data set sizes $n = |X| \in \{1000, 2000, 4000\}$ were fixed as the embedding dimension $m + 1$ varied for each of the first two subfigures.
- Recall that the test set size is always fixed to $n' = 1000$.



Lemma

The terminal embedding is non-linear

Proof.

- Let $X \subset \mathbb{R}^d$ be arbitrary
- Suppose for contradiction that $f : X \rightarrow \mathbb{R}^m, d > m$ is a linear embedding with constant terminal distortion.
- By the Rank-Nullity theorem, $\dim(\ker(f)) \geq d - m \geq 1$.
- This means $\exists y \in \ker(f) \setminus \{0\}$
- Let $x \in X$ be arbitrary. Since f is a linear embedding and $x - y \in \mathbb{R}^d$

$$\begin{aligned}\|x - (x - y)\|_2 &\leq \|f(x) - f(x - y)\|_2 \\ \Rightarrow 0 < \|y\|_2 &\leq \|f(x) - f(x) + f(y)\|_2 = \|f(y)\|_2 = 0\end{aligned}$$

- Thus we have arrived at a contradiction





Distortion Simulations

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Theorem (Theorem 1.1 - Narayanan, Nelson)

Let $\epsilon \in (0, 1)$ and $X \subset \mathbb{R}^d$ be arbitrary with $|X| = n > 1$. There exists $f : X \rightarrow \mathbb{R}^m$ with $m = O(\epsilon^{-2} \log n)$ such that $\forall x \in X, \forall y \in \mathbb{R}^d$

$$(1 - \epsilon) \|x - y\|_2 \leq \|f(x) - f(y)\|_2 \leq (1 + \epsilon) \|x - y\|_2$$

Definition

To compute the effective distortions of a given (terminal) embedding of training data X , $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$, over all available test and train data $X \cup S$ we use

$$\text{MaxDist}_f = \max_{\mathbf{x} \in X} \max_{\mathbf{u} \in S \cup X \setminus \{\mathbf{x}\}} \frac{\|f(\mathbf{u}) - f(\mathbf{x})\|_2}{\|\mathbf{u} - \mathbf{x}\|_2},$$

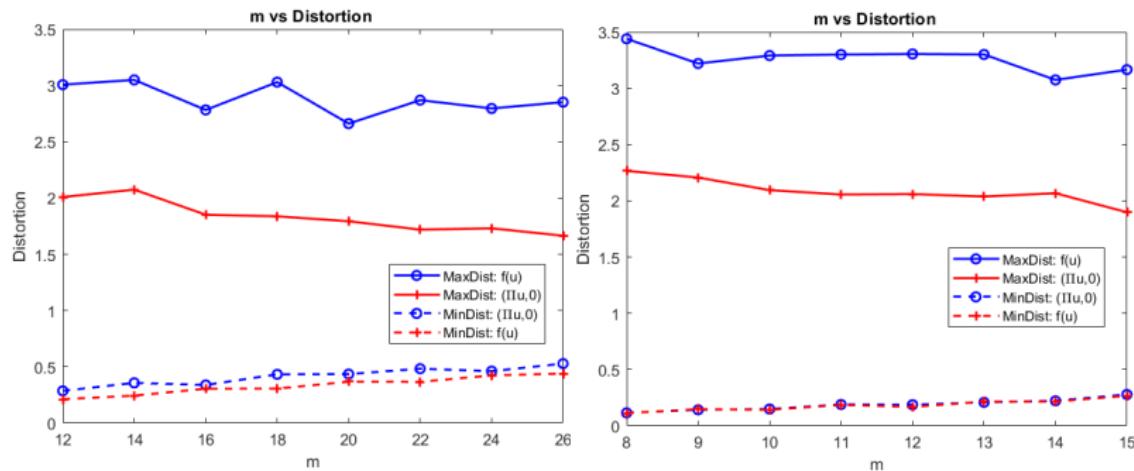
$$\text{MinDist}_f = \min_{\mathbf{x} \in X} \min_{\mathbf{u} \in S \cup X \setminus \{\mathbf{x}\}} \frac{\|f(\mathbf{u}) - f(\mathbf{x})\|_2}{\|\mathbf{u} - \mathbf{x}\|_2}.$$



Distortion Simulations

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- Left figure compares MaxDist_f and MinDist_f for the nonlinear f versus its component linear embedding $\mathbf{u} \mapsto (\Pi \mathbf{u}, 0)$ as m varies for a fixed embedded training set size of $n = 4000$.
- Right figure compares MaxDist_f and MinDist_f for COIL-100 dataset, for the nonlinear f versus its component linear embedding $\mathbf{u} \mapsto (\Pi \mathbf{u}, 0)$ as m varies for a fixed embedded training set size of $n = 3600$.



Measuring Compressive Nearest Neighbor Classification Accuracy

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Algorithm Measuring Compressive Nearest Neighbor Classification Accuracy

Input: $\epsilon \in (0, 1)$, A labeled data set $\mathcal{D} \subset \mathbb{R}^N$ split into two disjoint subsets: A training set $X \subset \mathcal{D}$ with $|X| =: n$, and a test set $S \subset \mathcal{D}$ with $|S| =: n'$, such that $S \cap X = \emptyset$. A compressive dimension $m < N$.

Output: Successful Nearest Neighbor Classification Percentage for Data Embedded in \mathbb{R}^{m+1}

Fix $f : \mathbb{R}^N \rightarrow \mathbb{R}^{m+1}$, an embedding of the training data $X \subset \mathbb{R}^N$ into \mathbb{R}^{m+1} satisfying

$$(1 - \epsilon) \|\mathbf{x} - \mathbf{y}\|_2 \leq \|f(\mathbf{x}) - f(\mathbf{y})\|_2 \leq (1 + \epsilon) \|\mathbf{x} - \mathbf{y}\|_2$$

for all $\mathbf{x}, \mathbf{y} \in X$. [Note: this can either be a JL-embedding of X , or a stronger terminal embedding of X .]

% Embed the training data into \mathbb{R}^{m+1} .

for $\mathbf{x} \in X$ **do**

 Compute $f(\mathbf{x})$ using previous Algorithm

end for



Measuring Compressive Nearest Neighbor Classification Accuracy

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Algorithm Measuring Compressive Nearest Neighbor Classification Accuracy

% Classify the test data using its embedded distance in \mathbb{R}^{m+1} .

p = 0

for $\mathbf{u} \in S$ **do**

 Compute $f(\mathbf{u})$ using, e.g., Algorithm 1

 Compute $\mathbf{x} = \underset{\mathbf{y} \in X}{\operatorname{argmin}} \|f(\mathbf{u}) - f(\mathbf{y})\|_2$

if $\text{Label}(\mathbf{u}) = \text{Label}(\mathbf{x})$ **then**

$p = p + 1$

end if

end for

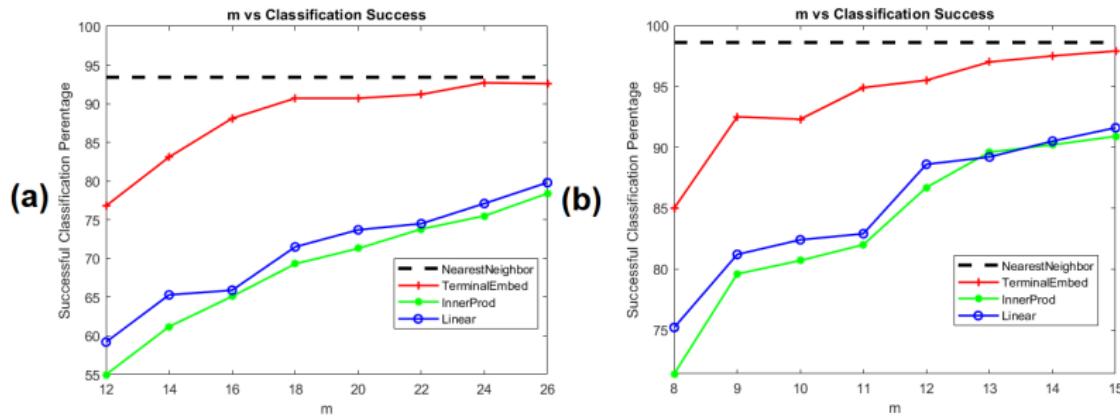
Output the Successful Classification Percentage = $\frac{p}{n'} \times 100\%$



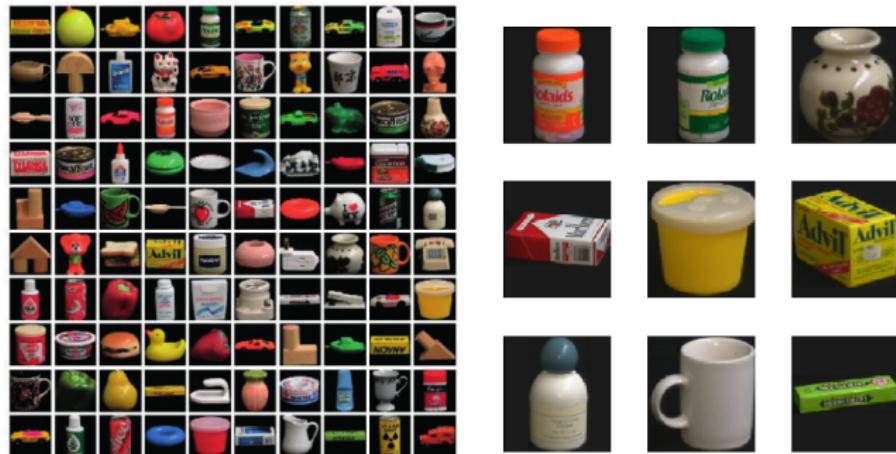
Comparison Figures

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- Left: MNIST, Right: COIL-100
- Nearest Neighbor:** Find the nearest neighbor in the original space
- TerminalEmbed:** $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \|\mathbf{z}\|_2^2 + 2\langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$
- InnerProd:** $h_{\mathbf{u}, \mathbf{x}_{NN}}(\mathbf{z}) := \langle \Pi(\mathbf{u} - \mathbf{x}_{NN}), \mathbf{z} \rangle$
- Linear:** Embed into the space linearly i.e. $f(\mathbf{u}) = (\Pi \mathbf{u}, 0)$



- The COIL-100 data set is a collection of 128×128 -pixel color images of 100 objects, each photographed 72 times where the object has been rotated by 5 degrees each time to get a complete rotation.
- However, only the green color channel of each image is used herein for simplicity. Thus, herein COIL-100 consists of 7,200 total vectorized images in \mathbb{R}^N with $N = 128^2 = 16,384$, where each image has one of 100 different labels (72 images per label).



- For all experiments involving this COIL-100 data set, $n/100$ training images are down sampled from each of the 100 objects' rotational image sequences.
- Thus, the training sets each contain $n/100$ vectorized images of each object, each photographed at rotations of $\approx 36000/n$ degrees (rounded to multiples of 5). The resulting training data sets therefore all consist of n vectorized images in $\mathbb{R}^{16,384}$.
- After forming each training set, 10 images of each type are then randomly selected from those not used for training in order to form the test set S , leading to a total of $n' = 1000$ vectorized test images in $\mathbb{R}^{16,384}$ per experiment.



Gaussian Width of the Unit Secants

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The following theorem bounds the Gaussian width of a smooth submanifold of \mathbb{R}^N in terms of its dimension, reach, and volume.

Theorem (Theorem 2.1 - Iwen, Roach)

Let $M \hookrightarrow \mathbb{R}^N$ be a compact d -dimensional submanifold of \mathbb{R}^N with boundary ∂M , finite reach τ_M , and volume V_M . Enumerate the connected components of ∂M and let τ_i be the reach of the i^{th} connected component of ∂M as a submanifold of \mathbb{R}^N . Set $\tau := \min_i\{\tau_M, \tau_i\}$, let $V_{\partial M}$ be the volume of ∂M , and denote the volume of the d -dimensional Euclidean ball of radius 1 by ω_d . Next,

- ① if $d = 1$, define $\alpha_M := \frac{20V_M}{\tau} + V_{\partial M}$, else
- ② if $d \geq 2$, define $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$.

Finally, define $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$. Then, the Gaussian width of $U((M - M) \setminus \{\mathbf{0}\})$ satisfies

$$w(S_M) = w\left(\overline{U((M - M) \setminus \{\mathbf{0}\})}\right) \leq 8\sqrt{2} \sqrt{\ln(\beta_M) + 4d}.$$



Main Result

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Theorem (The Main Result)

Let $\mathcal{M} \hookrightarrow \mathbb{R}^N$ be a compact d -dimensional submanifold of \mathbb{R}^N with boundary $\partial\mathcal{M}$, finite reach $\tau_{\mathcal{M}}$, and volume $V_{\mathcal{M}}$. Enumerate the connected components of $\partial\mathcal{M}$ and let τ_i be the reach of the i^{th} connected component of $\partial\mathcal{M}$ as a submanifold of \mathbb{R}^N . Set $\tau := \min_i \{\tau_{\mathcal{M}}, \tau_i\}$, let $V_{\partial\mathcal{M}}$ be the volume of $\partial\mathcal{M}$, and denote the volume of the d -dimensional Euclidean ball of radius 1 by ω_d . Next,

- ① if $d = 1$, define $\alpha_{\mathcal{M}} := \frac{20V_{\mathcal{M}}}{\tau} + V_{\partial\mathcal{M}}$, else
- ② if $d \geq 2$, define $\alpha_{\mathcal{M}} := \frac{V_{\mathcal{M}}}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial\mathcal{M}}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$.

Finally, fix $\epsilon \in (0, 1)$ and define $\beta_{\mathcal{M}} := (\alpha_{\mathcal{M}}^2 + 3^d \alpha_{\mathcal{M}})$. Then, there exists a map $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c(\ln(\beta_{\mathcal{M}}) + 4d)/\epsilon^2$ that satisfies

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all $\mathbf{x} \in \mathcal{M}$ and $\mathbf{y} \in \mathbb{R}^N$. Here $c \in \mathbb{R}^+$ is an absolute constant.



Theorem 3.1, 3.2

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Theorem (Theorem 3.1 - Iwen, Roach)

Let $\mathcal{M} \subset \mathbb{R}^N$, $\epsilon \in (0, 1)$, and suppose that $\Phi \in \mathbb{C}^{m \times N}$ is an $\left(\frac{\epsilon^2}{2304}\right)$ -JL map of $S_{\mathcal{M}} + S_{\mathcal{M}}$ into \mathbb{C}^m . Then, there exists an outer bi-Lipschitz extension of $\Phi : \mathcal{M} \rightarrow \mathbb{C}^m$, $f : \mathbb{R}^N \rightarrow \mathbb{C}^{m+1}$, with the property that

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

holds for all $\mathbf{x} \in \mathcal{M}$ and $\mathbf{y} \in \mathbb{R}^N$.

Theorem (Theorem 3.2 - Iwen, Roach)

Let $\mathcal{M} \subset \mathbb{R}^N$ and $\epsilon \in (0, 1)$. There exists a map $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c \left(\frac{w(S_{\mathcal{M}})}{\epsilon} \right)^2$ that satisfies

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all $\mathbf{x} \in \mathcal{M}$ and $\mathbf{y} \in \mathbb{R}^N$, where $c \in \mathbb{R}^+$ is an absolute constant.



Proof of Theorem 3.1

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Theorem (Theorem 3.4 - Iwen, Roach)

Let $\mathcal{M} \subset \mathbb{R}^N$, $\epsilon \in (0, 1)$, and suppose that $\Phi \in \mathbb{C}^{m \times N}$ is an $(\frac{\epsilon^2}{4})$ -JL map of $S_{\mathcal{M}} + S_{\mathcal{M}}$ into \mathbb{C}^m . Then, Φ will also provide ϵ -convex hull distortion for $S_{\mathcal{M}}$.

Lemma (Lemma 3.4 - Iwen, Roach)

Let $\mathcal{M} \subset \mathbb{R}^N$ be non-empty, $\epsilon \in (0, 1)$, and suppose that $\Phi \in \mathbb{C}^{m \times N}$ provides ϵ -convex hull distortion for $S_{\mathcal{M}}$. Then, there exists an outer bi-Lipschitz extension of Φ , $f : \mathbb{R}^N \rightarrow \mathbb{C}^{m+1}$, with the property that

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq 24\epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

holds for all $\mathbf{x} \in \mathcal{M}$ and $\mathbf{y} \in \mathbb{R}^N$.

Proof of Theorem 3.1.

Apply Theorem 3.4 with $\epsilon \leftarrow \epsilon/24$ in order obtain $\epsilon/24$ -convex hull distortion for $S_{\mathcal{M}}$ via Φ . Then, apply Lemma 3.4. □



Proof of Theorem 3.2

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Corollary (Corollary 3.1 - Iwen, Roach)

Let $\mathcal{M} \subset \mathbb{R}^N$, $\epsilon, p \in (0, 1)$, and $\Phi \in \mathbb{R}^{m \times N}$ be an $m \times N$ matrix whose rows are independent, isotropic, and sub-Gaussian random vectors in \mathbb{R}^N . Furthermore, suppose that

$$m \geq \frac{c'}{\epsilon^2} \left(w(S_{\mathcal{M}}) + \sqrt{\ln(2/p)} \right)^2,$$

where c' is a constant depending only on the distribution of the rows of Φ . Then, with probability at least $1 - p$, $\frac{1}{\sqrt{m}}\Phi$ will be an ϵ -JL embedding of \mathcal{M} into \mathbb{R}^m and provide ϵ -convex hull distortion for $S_{\mathcal{M}}$.

Proof of Theorem 3.2.

- Apply Corollary 3.1 with $p = 1/2$ to demonstrate that a $\left[\frac{c''}{\epsilon^2} \left(w(S_{\mathcal{M}}) + \sqrt{\ln(4)} \right)^2 \right] \times N$ matrix with i.i.d. standard normal random entries can provide $(\epsilon/24)$ -convex hull distortion for $S_{\mathcal{M}}$, where c'' is an absolute constant.
- Apply Lemma 3.4 to finish the proof.





Main Result

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Theorem (The Main Result)

Let $d \geq 2$. Define $\alpha_M := \frac{V_M}{\omega_d} \left(\frac{41}{\tau}\right)^d + \frac{V_{\partial M}}{\omega_{d-1}} \left(\frac{81}{\tau}\right)^{d-1}$. Fix $\epsilon \in (0, 1)$ and define $\beta_M := (\alpha_M^2 + 3^d \alpha_M)$. Then, there exists a map $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c(\ln(\beta_M) + 4d)/\epsilon^2$ that satisfies

$$\left| \|f(\mathbf{x}) - f(\mathbf{y})\|_2^2 - \|\mathbf{x} - \mathbf{y}\|_2^2 \right| \leq \epsilon \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all $\mathbf{x} \in M$ and $\mathbf{y} \in \mathbb{R}^N$. Here $c \in \mathbb{R}^+$ is an absolute constant.

Proof of Main Result.

- Theorem 3.2 proves that there exists a map $f : \mathbb{R}^N \rightarrow \mathbb{C}^m$ with $m \leq c \left(\frac{w(S_M)}{\epsilon} \right)^2$.
- Theorem 2.1 bounds S_M .
- Apply both theorems to complete proof.





Embedding Circle in Two Dimensions

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Manifold in Original Space Manifold in Original Space Manifold in Original Space

