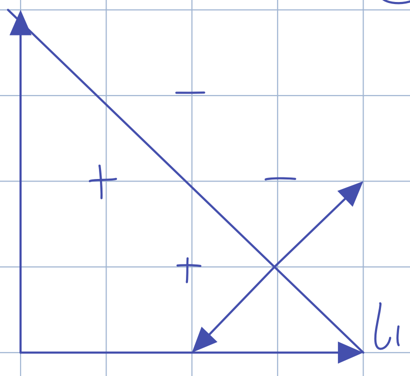
Support Vector Machine (SVM)

Binary Classification

$d=2$



SVM (Find Linear Classifier With Largest Margin)

Dataset

$D = \underline{\{(x_1, y_1), (x_2, y_n), \ldots, (x_n, y_n)\}}$

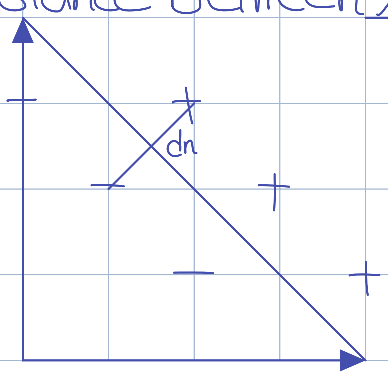
$\underline{x_i} \in \mathbb{R}^d \qquad y \in \{-1, +1\}$

Decision Rule

$\hat{y} = \text{sign}(\underline{w^T x} + b)$

$\underline{w} \in \mathbb{R}^d \qquad b \in \mathbb{R}$

Distance between $\underline{x_n}$ & l



$d_n = \text{distance}(\underline{x_n}, l)$

$= \dfrac{\boxed{w^T x_n + b}}{\|\boxed{w}\|}$

$w^T x + b = 0$

$d = 2$

$\underline{w} = (w_1, w_2)$

$\underline{x_n} = (x_{1n}, x_{2n})$

$$\text{dist}(\underline{x_n}, l) = \frac{|w_1 x_{1n} + w_2 x_{2n} + b|}{(w_1^2 + w_2^2)^{\frac{1}{2}}}$$

Margin of l wrt D

$$\rho(l) = \min_{1 \leq n \leq N} \text{dist}(\underline{x_n}, l)$$

SVM: Find Classifier l with largest margin

Suppose line l : $\underline{w^T x} + b = 0$

Perfectly classifies D

$\hat{y}_n = \text{sign}(\underline{w^T x_n} + b)$

$\hat{y}_n = y_n \quad n = 1, 2, \ldots, N$

Claim: $y_n(\underline{w^T x_n} + b) > 0$

if $y_n = +1$, $\hat{y}_n = +1 \longrightarrow \underline{w^T x_n} + b > 0$

$y_n = -1$, $\hat{y}_n = -1 \longrightarrow \underline{w^T x_n} + b < 0$

$y_n(\underline{w}^T \underline{x}_n + b) = |\underline{w}^T \underline{x}_n + b|$

$\text{dist}(\underline{x}_n, L) = \dfrac{y_n(\underline{w}^T \underline{x}_n + b)}{\|\underline{w}\|}$

Find $\underline{w}$ & b s.t.

$$\max \left\{ \min_{1 \le n \le N} \text{dist}(\underline{x}_n, L) \right\}$$

$$\left.\begin{aligned} x_1 + x_2 - 1 &= 0 \\ 5x_1 + 5x_2 - 5 &= 0 \end{aligned}\right\} \text{Same Line}$$

Need to normalize

~~Idea 1,~~ Difficult To Optimize

$\cancel{\|\underline{w}\| = 1}$

Idea 2,

$$\delta = \min_{1 \le n \le N} y_n(\underline{w}^T \underline{x}_n + b)$$

Always possitive

$\delta > 0$

Without Loss Of Generality

Assume $\delta = 1$

Claim for any L that perfectly classifies D, we can set $\delta = 1$ without loss of generality

Proof, given $(\underline{w}, b)$, suppose $\delta \neq 1$

$$\delta = \min_{1 \le n \le N} y_n(\underline{w}^T x_n + b)$$

Define

$$\underline{\tilde{w}} = \frac{w}{\delta}, \quad \tilde{b} = \frac{b}{\delta}$$

That represent the same line

$L' : \underline{\tilde{w}^T x} + \tilde{b} = 0$ Is the same line as L

$$\tilde{\delta} = \min_{1 \le n \le N} y_n(\underline{\tilde{w}^T x_n} + \tilde{b}) = \frac{\delta}{\delta} = 1$$

$$= \min_{1 \le n \le N} y_n\left(\frac{w^T}{\delta}\underline{x_n} + \frac{b}{\delta}\right) = \frac{1}{\delta} \min_{1 \le n \le N} y_n(\underline{w}^T x_n + b)$$

Margin

$$\rho(L) = \min_{1 \le n \le N} \frac{|w_1 x_{1n} + w_2 x_{2n} + b|}{(w_1^2 + w_2^2)^{\frac{1}{2}}}$$

$$= \frac{\delta}{\|w\|} = \frac{1}{\|w\|}$$

Constrant

$$\min_{1 \le n \le N} y_n(\underline{w^T x_n} + b) = 1$$

Maximize

$$\max \frac{1}{\|w\|} \longrightarrow \text{mimize } \underline{w^T w}$$

$$\min_{1 \le n \le N} y_n(\underline{w^T x_n} + b) = 1$$

$$y_n(\underline{w^T x_n} + b) \geq 1 \qquad n = 1, 2, \ldots, N$$

With equality for at least one value of n

mimize $\underline{w^T w}$

$$y_n(\underline{w^T x_n} + b) \geq 1 \qquad n = 1, 2, \ldots, N$$

~~With equality for at least one value of n~~

Problem

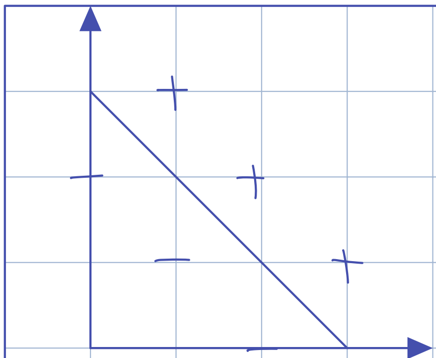
Not Standard

Relaxed
Problem

$$y_n(\underline{w^T x_n} + b) \geq 2 \qquad n = 1, 2, \ldots, n$$

Exact Problem

$$\underline{\tilde{w}} = \frac{w}{2} \qquad \tilde{b} = \frac{b}{2}$$



$w = (5, 5) \quad b = -5$

$y_n(w^T x_n + b) > 0 \quad n = 1, 2, \ldots, 6$

$\min_{1 \le n \le 6} y_n(w^T x_n + b) = 5$

$y_n(w^T x_n + b) \geq 1$

$\tilde{w} = \frac{w}{5} = (1, 1) \quad \tilde{b} = \frac{b}{5} = 1$

Equality For At Least One

SVM

$D = \{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$

Find (w, b)

Minimize $w^T w$

Subject to $y_n(w^T x_n + 1) \geq 1$

$n = 1, 2, 3, \ldots, N$

Quadratic Programming

Dual Formulation

Identity Support Vector

Non Linear Transforms

Quadratic Programming

Variables: $u \in \mathbb{R}^L$

Objective Minimize

$$\boxed{\underbrace{\frac{1}{2} u^T Q u}_{\text{Quadratic Term}} + \underbrace{p^T u}_{\text{Linear Term}}}$$

$Q \in \mathbb{R}^{L \times L}$

$p \in \mathbb{R}^{L \times 1}$

Linear Inequality Constraints

$a_m^T u \geq c_m \qquad m = 1, 2, \ldots, M$

$a_m \in \mathbb{R}^L, \quad c_m \in \mathbb{R}$

Equality Constraints

$b_m^T u = d_m \qquad m = 1, 2, \ldots, N$

$l_i \leq u_i \leq h_i$

$i = 1, 2, \ldots, L$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_L \end{bmatrix}$$

SVM

$$\underset{w,\,b}{\text{minimize}} \ \frac{1}{2} w^T w$$

Subject to $y_n(w^T x_n + 1) \geq 1$

$n = 1, 2, 3, \ldots, N$

Specify u, Q, p, $\{a_m, c_m\}_{m=1}^{M}$, $\{b_m, d_m\}_{m=1}^{N}$, $\{l_i, u_i\}_{i=1}^{L}$

$$u = \begin{bmatrix} w \\ b \end{bmatrix} \in \mathbb{R}^{d+1}$$

Q, p : $\frac{1}{2} w^T w = \frac{1}{2} u^T Q u + \overset{0}{p^T} u$

$$\frac{1}{2} w^T w = \frac{1}{2} \begin{bmatrix} w^T & b \end{bmatrix} Q \begin{bmatrix} w \\ b \end{bmatrix}$$

$$Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Inequality Constraints

$\{a_m, c_m\}_{m=1}^{M}$

$a_m^T u \geq c_m$

SVM $y_n \underline{w^T x_n} + y_n b \geq 1 \qquad n = 1, 2, \dots, N$

$M = N$

$$y_n \underline{x_n^T w} + y_n b \geq 1$$

$$\begin{bmatrix} y_n \underline{x_n^T} & y_n \end{bmatrix} \begin{bmatrix} w \\ b \end{bmatrix} \geq 1$$

$a_n^T = \begin{bmatrix} y_n x_n^T & y_n \end{bmatrix}$

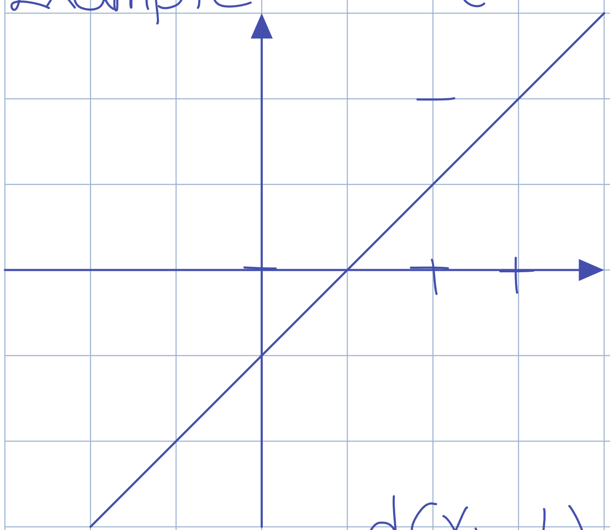
$c_m = 1$

No Equality Constraint

$l_i = -\infty \qquad h_i = \infty$

$i = 1, 2, \dots, L$

Example $\qquad L: x_1 - x_2 - 1 = 0$



$\underline{x_1} = (0, 0) \quad y_1 = -1$

$\underline{x_2} = (2, 2) \quad y_2 = -1$

$\underline{x_3} = (2, 0) \quad y_3 = +1$

$\underline{x_4} = (3, 0) \quad y_4 = +1$

$$d(x_1, L) = \frac{1}{\sqrt{2}}$$

$$d(x_2, L) = d(x_3, L) = 2^{-\frac{1}{2}}$$

SVM

$$\frac{1}{2}(w_1^2 + w_2^2) \longleftarrow \text{minimize}$$

s.t. $y_n(w^T x_n + b) \geq 1$

$n=1, x_1 = (0, 0)$

$-b \geq 1$

$n=2, x_2 = (2, 2)$

$(-1)(2w_1 + 2w_2 + b) \geq 1$

$n=3, x_3 = (2, 0)$

$(2w_1 + b) \geq 1$

$n=4, x_4 = (3, 0)$

$(3w_1 + b) \geq 1$

1.	$-b \geq 1$	$b \leq -1$	$b=-1$
2.	$-(2w_1 + 2w_2 + b) \geq 1$	$b \leq -1$	
3.	$2w_1 + b \geq 1$	$b \geq -1$	
4.	$3w_1 + b \geq 1$	$b \geq -2$	

$1 + 3 \longrightarrow 2w_1 \geq 2, \; w_1 \geq 1$

$2 + 3 \longrightarrow -2w_2 \geq 2, \; w_2 \leq -1$

minimize

$$\frac{1}{2}(w_1^2 + w_2^2)$$

$w_1^* = 1$

$w_2^* = -1$

$$\underline{w}^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$b = -1$

$C: x_1 - x_2 - 1 = 0$

$\rho(C) = 2^{-\frac{1}{2}}$

x_1, x_2, x_3 are at minimum distance from C } Support Vector

Dual Formulation

Automatically Identify Support Vector