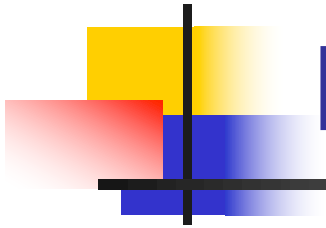
Pushdown Automata



Pushdown Automata (PDA)

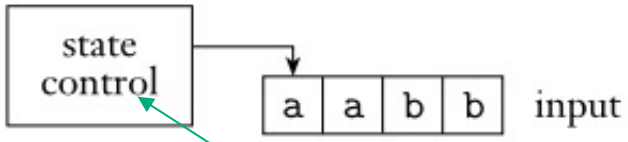
- PDA generalize nondeterministic finite automata (NFA) but have an extra component called a stack
- The stack provides additional memory beyond that available in NFA
- The stack allows PDA to recognize some nonregular languages

- PDA are equivalent in power to context-free grammars
  - This gives us two options for proving that a language L is context free:
  1) Give a CFG generating L
  2) Give a PDA recognizing L

NFA vs PDA schematic

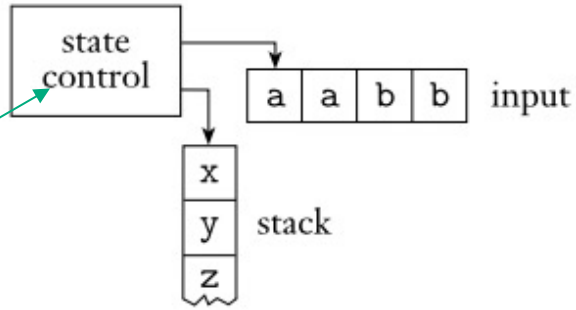
Regular languages

- NFA

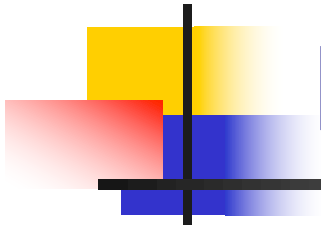


Context-free languages

- PDA

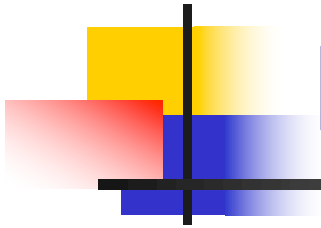


state control:
states and transition function



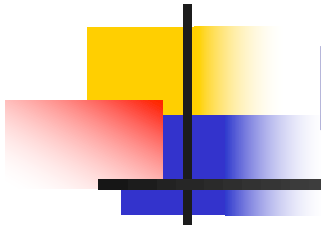
PDA

- A PDA can write symbols on the stack and read them back later
- Writing a symbol “pushes down” all the other symbols on the stack
- At any time, the symbol on the top of the stack can be read and removed.
  - The remaining symbols then move back up
- Writing a symbol on the stack is referred to as ***pushing*** the symbol, and removing a symbol is referred to as ***popping*** it.
- A stack is a “Last In First Out” (LIFO) storage device



Revisiting the language B = $\{0^n1^n \mid n \geq 0\}$

- Recall that a finite automaton is unable to recognize B because it cannot store very large numbers (unbounded information) in its finite memory
- A PDA is able to recognize this language because it can use its stack to store the number of 0s it has seen
- Thus, the unlimited nature of a stack allows the PDA to store numbers of unbounded size

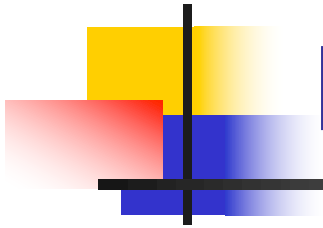


Revisiting the language B = $\{0^n1^n \mid n \geq 0\}$

Informal description of how PDA for the language B works

- Read symbols from the input
- As each 0 is read, push it onto the stack
- As soon as 1s are seen, pop a 0 off the stack for each 1 read
- If reading the input is finished exactly when the stack becomes empty of 0s, **accept**
- If
  - stack becomes empty while 1s remain OR
  - the 1s are finished while the stack still contains 0s OR
  - any 0s appear in the input following 1s

  then **reject** the input



Formal definition of PDA: the ingredients

- Similar to NFA, except for the stack
- The machine may use different alphabets for its input and stack, so we now have an input alphabet Σ and a stack alphabet Γ
- Recall that $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ and $\Gamma_\varepsilon = \Gamma \cup \{\varepsilon\}$
- Domain of the **transition function**: $Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon$
- Range of the **transition function**: $Q \times \Gamma_\varepsilon$
- **Nondeterminism** allowed: transition function δ returns a member of $P(Q \times \Gamma_\varepsilon)$

Formal definition of PDA: put together

DEFINITION 2.13

A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

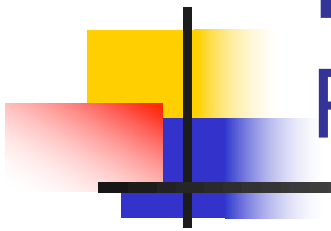
How a PDA computes

- A PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ computes as follows.
- It accepts input w if w can be written as $w = w_1 w_2 \ldots w_m$, where each $w_i \in \Sigma_\varepsilon$ and sequence of states $r_0, r_1, \ldots r_m \in Q$ and strings $s_0, s_1, \ldots, s_m \in \Gamma^*$ exist that satisfy the following three conditions. The strings s_i represent the sequence of stack contents that M has on the *accepting branch* of the computation.
  1. $r_0 = q_0$ and $s_0 = \varepsilon$
  2. For $i = 0, \ldots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$
  3. $r_m \in F$

Example 1:
Design PDA for the language $B = \{0^n1^n \mid n \geq 0\}$

A couple of notations before presenting the PDA

1) **Testing for empty stack**
   - The formal definition of a PDA contains no explicit mechanism to allow the PDA to test for an empty stack
   - The same effect can be attained by initially putting a special symbol (e.g., $) on the stack
2) **Short-hand notation**
   - for (state-input-stack) transitions we write a, b ⟶ c to signify that when the machine is reading an a from the input, it replaces the symbol b on the top of the stack with a c
   - any of a, b, and c may be ε
     - a = ε means machine may make this transition without reading any symbol from the input
     - b = ε means machine may make this transition without reading & popping any symbol from the stack
     - c = ε means machine does not write any symbol on the stack when going along this transition

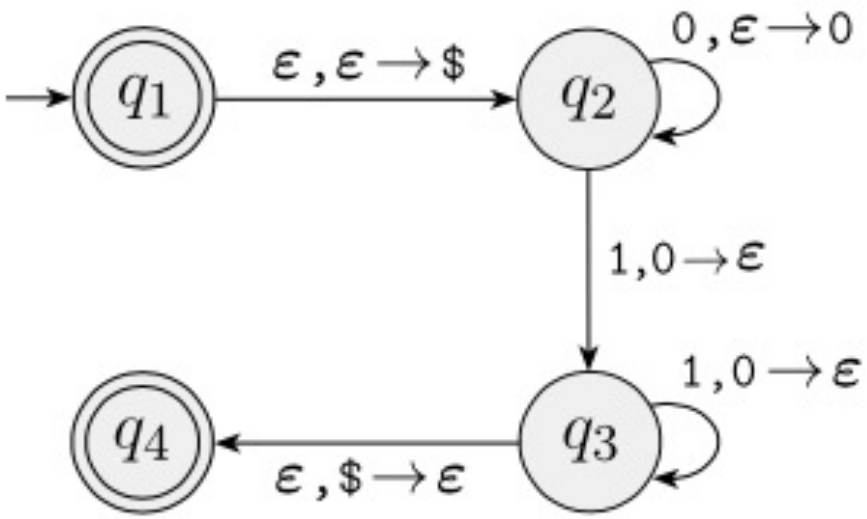


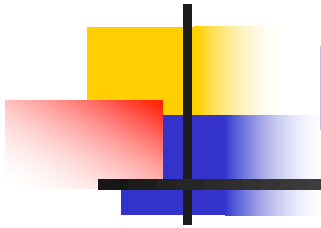
Example 1:
PDA for the language $B = \{0^n1^n \mid n \geq 0\}$

How would you go about designing a PDA for B?

How many states would you need?

Example 1:
PDA for the language $B = \{0^n1^n \mid n \geq 0\}$





Example 2

Design a PDA that recognizes the language

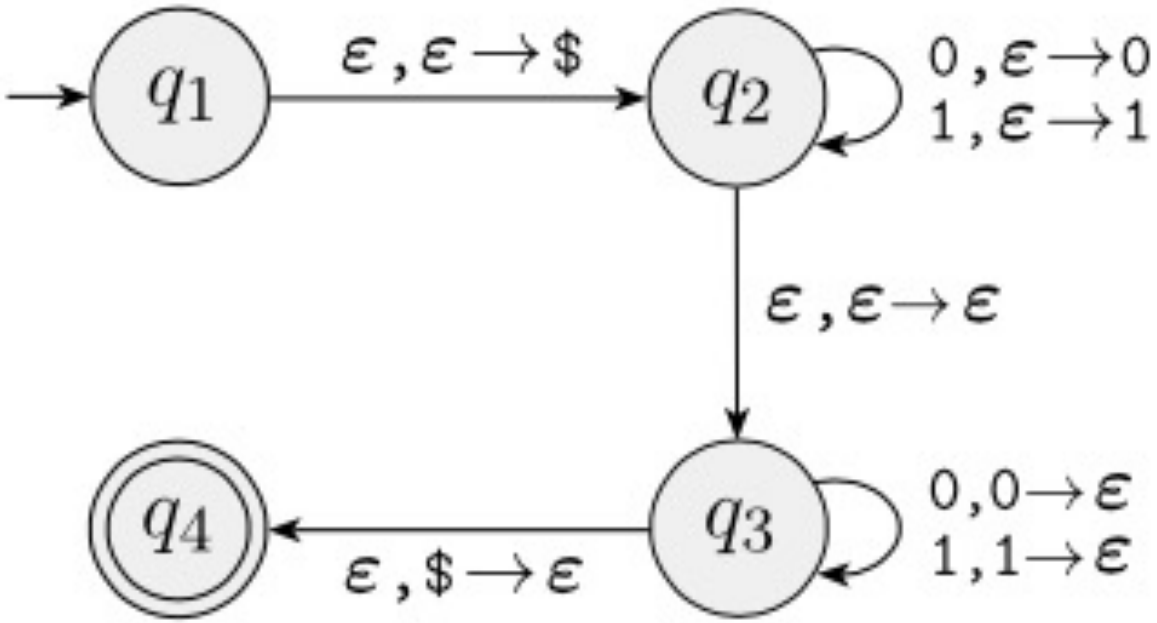
$$\{ww^R \mid w \{0, 1\}^*\}$$

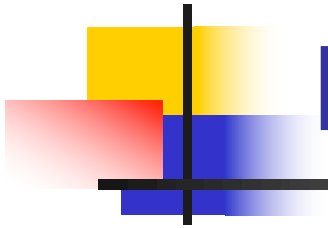
Recall that w^R means w written backwards

Note that this language is a palindrome

How would you go about designing the PDA?

Example 2 (palindrome)





Example 3

- Design a PDA that recognizes the language

$$D = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$

- How would you go about designing the PDA?

Example 3 (PDA for language D)