Turing Machines, part II

Complexity Theory

Computability Theory

Automata Theory

In last lecture, we saw…

- Informal description of TM
- Formal definition of TM
- How TM computes
  - Changes in *configurations*
- *Turing recognizable and Turing decidable languages*

Formal definition of TM

$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where

1. Q is the set of states
2. Σ is the input alphabet not containing the *blank symbol* $\sqcup$
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$
4. $\delta: Q \times \Gamma \rightarrow Q \times T \times \{L,R\}$ is the transition function
5. $q_0 \in Q$ is the start state
6. $q_{accept} \in Q$ is the accept state
7. $q_{reject} \in Q$ is the reject state, where $q_{reject} \neq q_{accept}$

Formalization of how TM computes

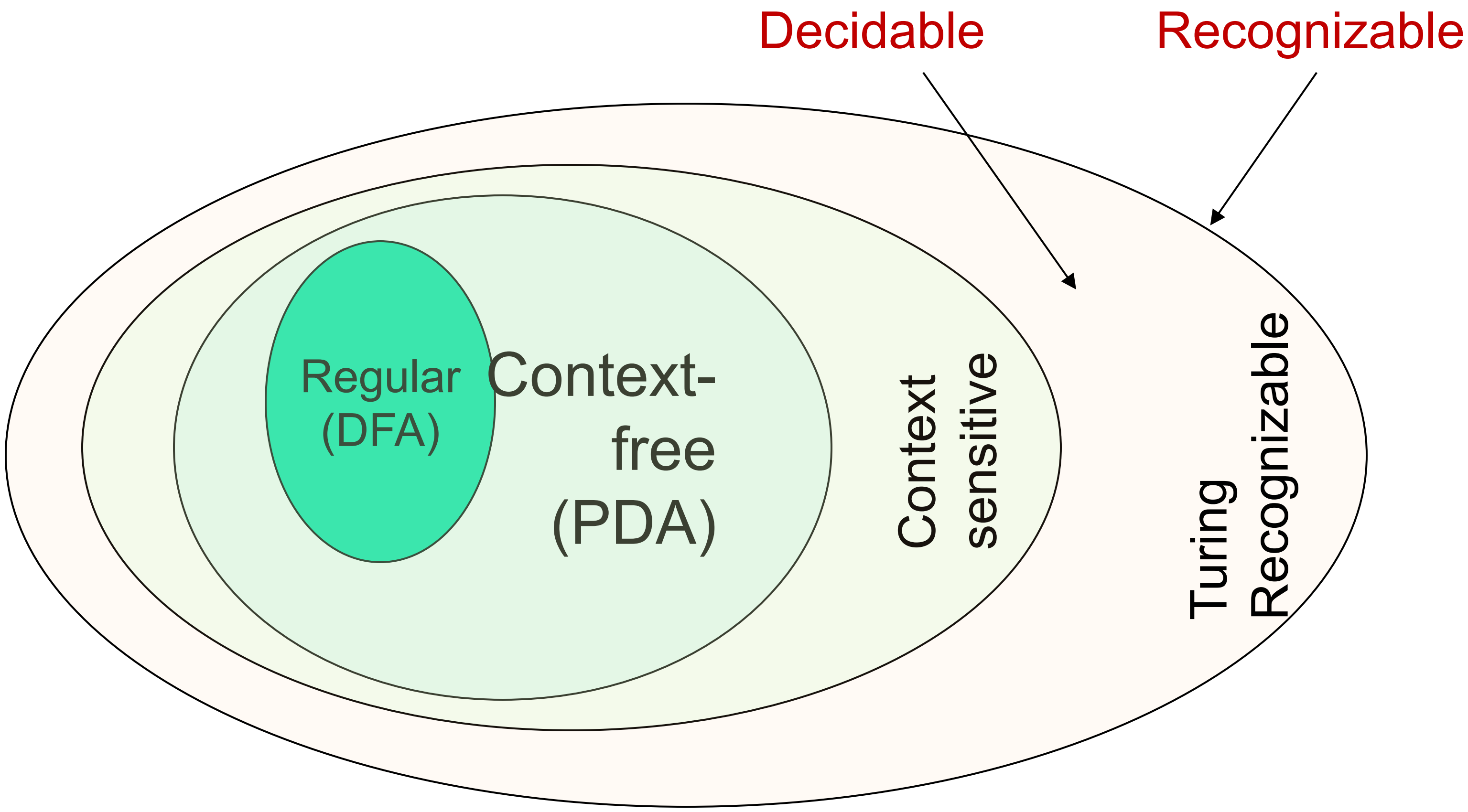
- The **start configuration** of M on input w is the configuration q_0w
- In an **accepting configuration**, the state of the configuration is q_{accept}
- In a **rejecting configuration**, the state of the configuration is q_{reject}
- Accepting and rejecting configurations are **halting configurations**
- A TM M **accepts** input w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists, where

1. C_1 is the start configuration of M on input w,
2. Each C_i yields C_{i+1}, and
3. C_k is an accepting configuration

Turing recognizable and Turing decidable languages

- The collection of strings that M accepts is the **language of M**, or the **language recognized by M**, denoted by L(M)
- A language is called **Turing-recognizable** if some Turing machine recognizes it
  - Aka Recursively enumerable language
- When we start a TM on an input, three outcomes are possible:
  - accept
  - reject
  - loop (does not halt)
- A TM M can fail to accept an input by entering the q_{reject} state and rejecting, or by looping.
- Sometimes distinguishing a machine that is looping from one that is merely taking a long time is difficult.
- For this reason, we may prefer TMs that halt on all inputs; such machines never loop. These machines are called **deciders**.
- A language is called **Turing-decidable** if some language decides it.
  - Aka recursive language

Language of Turing Machines



Today, we will look at

Examples of Turing Machines

Note:

We will mostly work with only higher-level descriptions, which are essentially a “shorthand” for formal (state diagram-based) descriptions.

Example 1:
(is the length a power of two?)

- Turing machine M_2 that decides

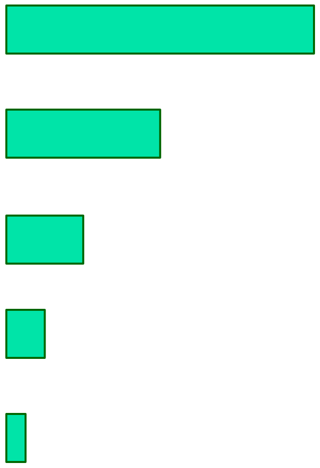
$$A = \{0^{2^n} \mid n \geq 0\},$$

the language consisting of all strings of 0s whose length is a power of 2.

First a high-level description of M_2

M_2 = *"On input string w:*

1. *Sweep left to right across the tape, crossing off every other 0.*
2. *If in stage 1 the tape contained a single 0, accept.*
3. *If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, reject.*
4. *Return the head to the left-hand end of the tape.*
5. *Go to stage 1."*

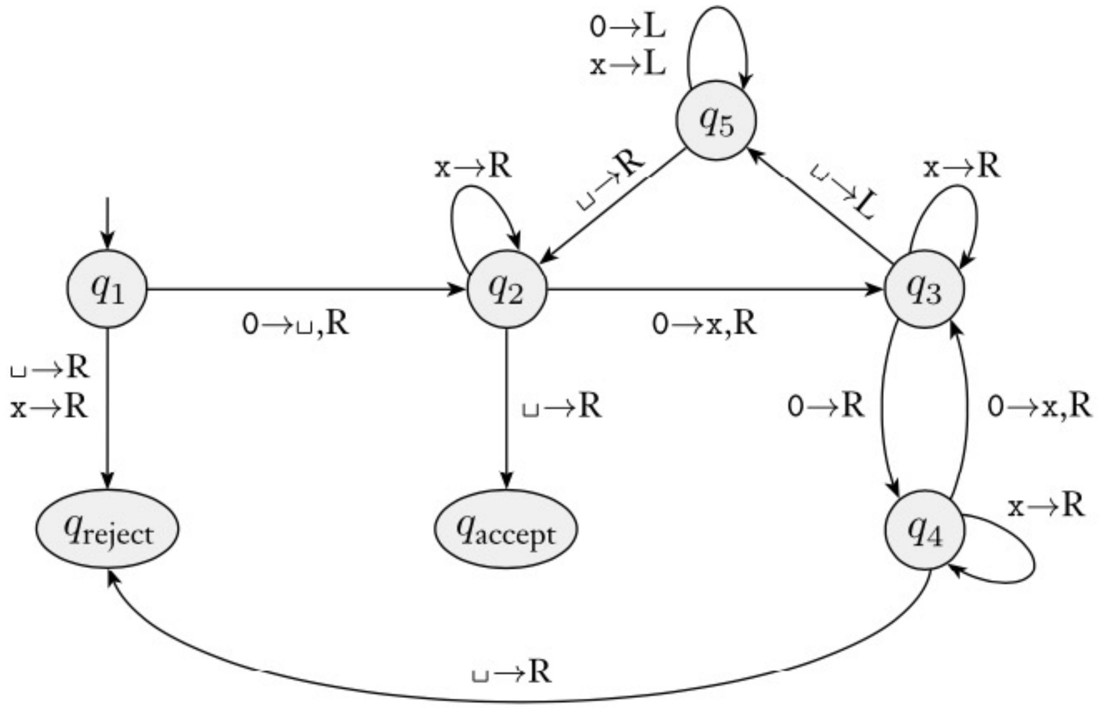


Each iteration of stage 1 cuts the number of 0s in half

Formal description of M_2

- $M_2 = (Q,\Sigma,\Gamma,\delta,q_1,q_{accept},q_{reject})$
  - $Q = \{q_1,\ldots,q_5, q_{accept},q_{reject}\}$
  - $\Sigma = \{0\}$
  - $\Gamma = \{0,x,␣\}$
  - δ (described with a state diagram in next slide)
  - The start, accept, and reject states are q_1, q_{accept}, and q_{reject}.

State diagram of M_2



This machine begins by writing a blank symbol over the leftmost 0 on the tape so that it can find the left-hand end of the tape in stage 4.

Sample run of M_2 on input 0000

1

$q_1 0000$

$\sqcup q_2 000$

$\sqcup \mathtt{x} q_3 00$

$\sqcup \mathtt{x} 0 q_4 0$

$\sqcup \mathtt{x} 0 \mathtt{x} q_3 \sqcup$

$\sqcup \mathtt{x} 0 q_5 \mathtt{x} \sqcup$

$\sqcup \mathtt{x} q_5 0 \mathtt{x} \sqcup$

$\sqcup q_5 \mathtt{x} 0 \mathtt{x} \sqcup$

$q_5 \sqcup \mathtt{x} 0 \mathtt{x} \sqcup$

$\sqcup q_2 \mathtt{x} 0 \mathtt{x} \sqcup$

$\sqcup \mathtt{x} q_2 0 \mathtt{x} \sqcup$

$\sqcup \mathtt{x} \mathtt{x} q_3 \mathtt{x} \sqcup$

$\sqcup \mathtt{x} \mathtt{x} \mathtt{x} q_3 \sqcup$

$\sqcup \mathtt{x} \mathtt{x} q_5 \mathtt{x} \sqcup$

$\sqcup \mathtt{x} q_5 \mathtt{x} \mathtt{x} \sqcup$

$\sqcup q_5 \mathtt{x} \mathtt{x} \mathtt{x} \sqcup$

$q_5 \sqcup \mathtt{x} \mathtt{x} \mathtt{x} \sqcup$

$\sqcup q_2 \mathtt{x} \mathtt{x} \mathtt{x} \sqcup$

$\sqcup \mathtt{x} q_2 \mathtt{x} \mathtt{x} \sqcup$

$\sqcup \mathtt{x} \mathtt{x} q_2 \mathtt{x} \sqcup$

$\sqcup \mathtt{x} \mathtt{x} \mathtt{x} q_2 \sqcup$

$\sqcup \mathtt{x} \mathtt{x} \mathtt{x} \sqcup q_{\text{accept}}$

2

Example 2:
(the example from last lecture: is the left the same as the right?)

- Turing Machine M_1 for testing membership in the language

$$B = \{w\#w \mid w \in \{0,1\}^*\}$$

Recall the high-level description of M_1

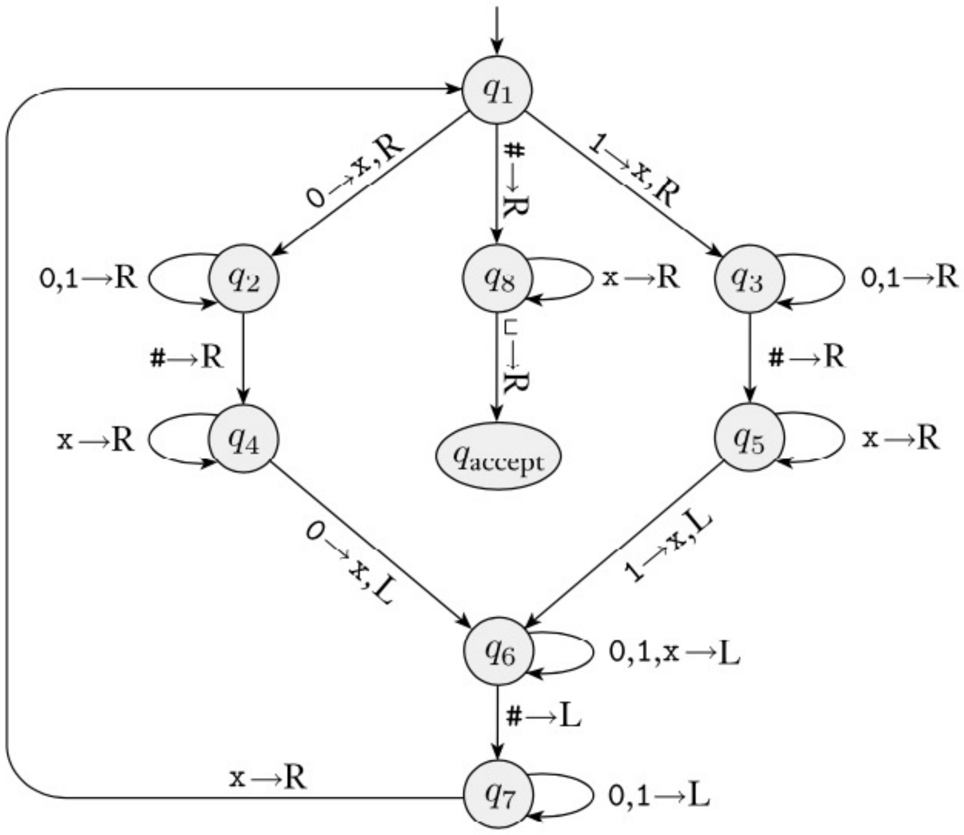
M_1 = "*on input string w:*

1. *Zig Zag across the tape to corresponding positions on either side of # to check whether the inner positions contain the same symbol. If they don't, or if no # is found, reject. Cross off symbols as they are checked to keep track of which symbols correspond.*
2. *When all symbols to the left of # have been crossed off, check for any remaining symbols on the right of #. If any symbols remain, reject; otherwise accept.*"

Formal description of M_1

- $M_1 = (Q, \Sigma, \Gamma, \delta, q_1, q_{accept}, q_{reject})$
  - $Q = \{q_1, \ldots q_8, q_{accept}, q_{reject}\}$
  - $\Sigma = \{0,1,\#\}$, and $T = \{0,1,\#,x,\sqcup\}$
  - δ (described with a state diagram in next slide)
  - The start, accept, and reject states are $q_{1,}$ q_{accept}, and q_{reject}.

State diagram of M_1



Example 3:
(let us do some arithmetic)

- Turing machine M_3 that decides the language

$$C = \{a^i\, b^j\, c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$$

High-level description of M_3

M_3 = *"On input string w:*

1. *Scan the input from left to right to determine whether it is a member of $a^+b^+c^+$ and reject if it isn't.*
2. *Return the head of the left-hand end of the tape.*
3. *Cross off an a and scan to the right until a b occurs. Shuttle between the b's and the c's, crossing off one of each until all b's are gone. If all c's have been crossed off and some b's remain, reject.*
4. *Restore the crossed off b's and repeat stage 3 if there is another a to cross off. If all a's have been crossed off, determine whether all c's have been crossed off. If yes, accept; otherwise reject."*

$C = \{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$

Some notes on M_3

- Stage 1
  - Operates much like a FA
  - No writing necessary as head moves from left to right
  - Keeps track by using its states to determine whether the input is in the proper form
- Stage 2
  - One subtle issue here is how to find the left-hand end of the input tape
  - One solution is to use a special symbol to mark (e.g. the blank symbol was used in M_2)
  - Another solution is to take advantage of the definition of TM (prevent left move when it is on the “cliff”)
- Stages 3 and 4
  - Have straightforward implementation and
  - use several states each

Example 4:
(let us solve the *element distinctness problem*)

- Given a list of strings over {0,1} separated by #s, design a Turning machine M_4 that would accept if all the strings are different. The language is

$$E = \{ \#x_1\#x_2\#\ldots\#x_l \mid \text{each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for each } i \neq j \}$$

- Machine M_4 works by comparing x_1 with x_2 through x_l, then by comparing x_2 with x_3 through x_l, and so on.

High-level description of M_4

M_4 = "On input w:

1. *Place a mark on top of the leftmost tape symbol. If that symbol was a blank, accept. If that symbol was a #, continue with the next stage. Otherwise, reject.*
2. *Scan right to the next # and place a second mark on top of it. If no # is encountered before a blank symbol, only x_1 was present, so accept.*
3. *By zig-zagging, compare the two strings to the right of the marked #s. If they are equal, reject.*
4. *Move the rightmost of the two marks to the next # symbol to the right. If no # symbol is encountered before a blank symbol, move the leftmost mark to the next # to its right and the rightmost mark to the # after that. This time, if no # is available for the rightmost mark, all the strings have been compared, so accept.*
5. *Go to stage 3."*

Notes on M_4

- M_4 illustrates the technique of marking tape symbols
  - In stage 2, the machine places a mark above the symbol #
  - In the actual implementation, the machine has two different symbols, # and #`, in its tape alphabet.
  - In general, we may want to place marks over various symbols on the tape. To do so, we merely include versions of all these tape symbols with dots in the tape alphabet.