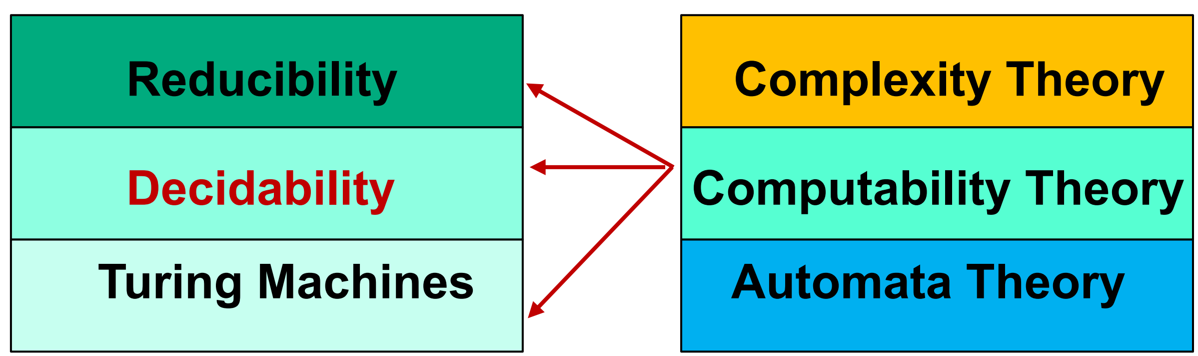
Decidability



In the last lecture…

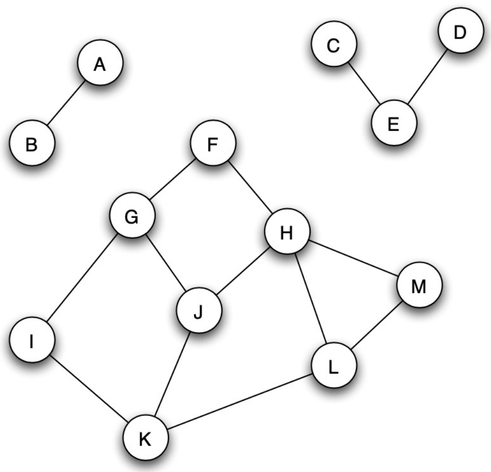
- Discussed the definition of algorithm
  - Church-Turing thesis
- Established terminology for describing TMs
  - Format and notation:
    - Encoding in terms of strings
- Looked at an example

Example (from last lecture)

- Let A be the language consisting of all strings representing undirected graphs that are connected. That is,

  A = {<G> | G is a connected undirected graph}

- The following (next slide) is a high-level description of a TM M that decides A



TM that decides A

$M =$ "On input $\langle G \rangle$, the encoding of a graph G:

1. Select the first node of G and mark it.

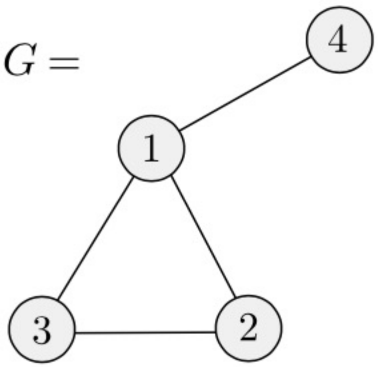
2. Repeat the following stage until no new nodes are marked:

3. For each node in G, mark it if it is attached by an edge to a node that is already marked.

4. Scan all the nodes of G to determine whether they all are marked. If they are, *accept*; otherwise, *reject*."

Just a bit of implementation detail on M…

$G =$



$\langle G \rangle =$

`(1,2,3,4)((1,2),(2,3),(3,1),(1,4))`

Encoding

Some details of M…

- Input check:
  - node list (distinct elements)
  - edge list (pairs drawn from node list)
- Stags 1 -- 3:
  - Markings
- Stage 4:
  - Scanning

Today's lecture: Decidability

- Our objective is to explore the limits of algorithmic solvability
  - Certain problems can be solved algorithmically, and others cannot
- Why bother study unsolvability?
  1. Practice

     ➔ (Re)formulation of problem
  2. Perspective

     ➔ A glimpse of the unsolvable may stimulate imagination

Decidable languages

- Will look at decidable problems concerning
  - Finite automata
    - Acceptance
    - Emptiness
    - Equivalence
  - Context-free grammars
    - Generation
    - Emptiness

- Will cover results on FA today, and those on CFG next lecture

1) Finite Automata: Acceptance Problem (DFA)

Let:

A_{DFA}= {<B,w> | B is a DFA that accepts input string w}

(**Note**: we choose to represent computation problems by languages. In the case above, the problem of testing whether a DFA B accepts an input w is the same as the problems of testing whether <B,w> is a member of the language A_{DFA})

Theorem: A_{DFA} is a decidable language

Proof

We simply need to present a TM M that decides A_{DFA}

M = "On input $\langle B, w \rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

A few implementation details…

- <B,w>
  - A reasonable representation of B may be its 5 components (Q, Σ, δ, q_0 and F)
- Simulation
  - M may do this directly

2) Finite Automata: Acceptance Problem (NFA)

We can prove a similar theorem for NFA

Let:

A_{NFA} = {<B,w> | B is an NFA that represents input string w}

Theorem:

A_{NFA} is a decidable language

Proof

- We present a TM N that decides A_{NFA}.
- Instead of making N simulate an NFA, we will make it use M (the DFA) as a subroutine.

$N =$ "On input $\langle B, w \rangle$, where B is an NFA and w is a string:
1. Convert NFA B to an equivalent DFA C, using the procedure for this conversion given in Theorem 1.39.
2. Run TM M from Theorem 4.1 on input $\langle C, w \rangle$.
3. If M accepts, *accept*; otherwise, *reject*."

Thm 1.39: *every NFA has an equivalent DFA*

Thm 4.1: *A_{DFA} is decidable*

3) Regular expression: Generation

We can prove similar result for determining whether a regular expression generates a given string.

Let:

A_{REX} = {<R,w> | R is a regular expression that generates string w}

Theorem:

A_{REX} is a decidable language

Proof

The following TM P decides A_{REX}

$P =$ "On input $\langle R, w\rangle$, where R is a regular expression and w is a string:

1. Convert regular expression R to an equivalent NFA A by using the procedure for this conversion given in Theorem 1.54.
2. Run TM N on input $\langle A, w\rangle$.
3. If N accepts, *accept*; if N rejects, *reject*."

Thm 1.54: *a language is regular iff some regular expression describes it*

What did we observe so far?

- The previous three results illustrate that, for decidability purposes, it is equivalent to present the TM with a DFA, an NFA or a regular expression because the machine can convert one form of encoding to another.

- Next we see two different kinds of problems concerning FA:
  - Emptiness testing
  - Equivalence of two DFAs

4) Finite Automata: Emptiness

Let:

E_{DFA} = {<A> | A is a DFA and L(A) = ∅}

Theorem:

E_{DFA} is a decidable language

Proof

- A DFA accepts some string iff reaching an accept state from the start state by traveling along the arrows of the DFA is possible.
- To test this condition, we can design a TM T that uses a marking algorithm similar to the example on *connected graphs* we saw at the beginning of this lecture.

$T =$ "On input $\langle A \rangle$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise, *reject*."

5) Finite Automata: Equivalence

Let:

$EQ_{DFA} = \{\langle A,B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

Theorem:

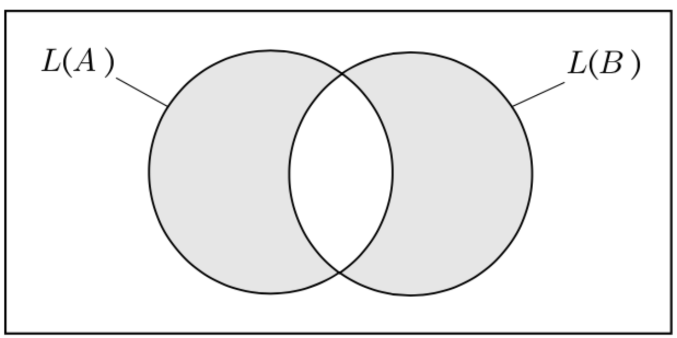
EQ_{DFA} is a decidable language

Proof

- To prove this theorem, we use the previous theorem (emptiness)
- We construct a new DFA C from A and B, where C accepts only those strings that are accepted by either A or B but not by both.
- Thus if A and B recognize the same language, C will accept nothing.
- The language L(C) is the symmetric difference between L(A) and L(B)

$$L(C) = (L(A) \cap L(B)^{--}) \cup (L(A)^{--} \cap L(B))$$

Notation:
X^{--} denotes complement of X



F = "On input $\langle A, B\rangle$, where A and B are DFAs:

1. Construct DFA C as described.
2. Run TM T from Theorem 4.4 on input $\langle C\rangle$.
3. If T accepts, *accept*. If T rejects, *reject*."

Thm 4.4: *E_{DFA} is a decidable language*