# Turing Machines, part III

Variants of Turing Machines

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## Variants of Turing Machines

Warm-up question of the day (4/4/22):

*What comes first to your mind when you hear the word **robust**?*

Send me your answer via Canvas email.

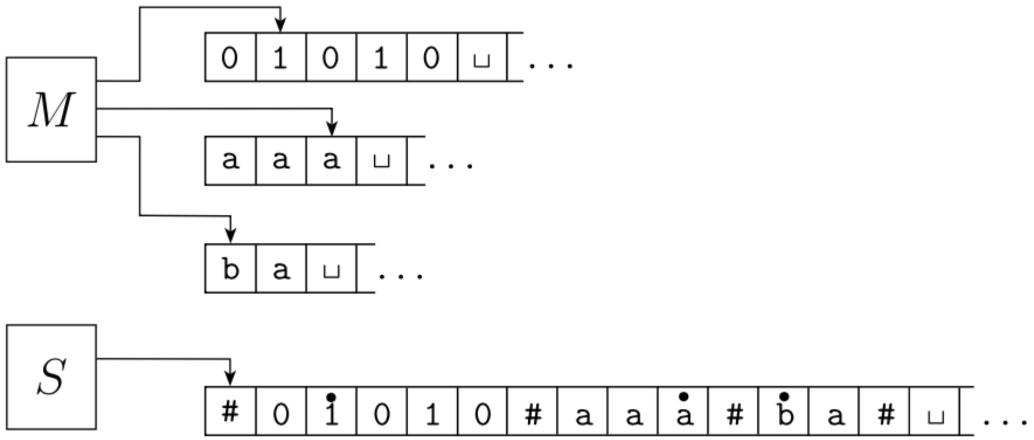
# Variants of Turing Machine

- The Turing Machine model the way we defined it thus far is robust
  - Finite Automata and PDA are (to a degree) robust too, but TMs are supremely so
- Quick example of TM robustness
  - Variation 0: tape head allowed to "stay", not just move Left/Right
  - Result: TM with only {L,R} can easily simulate {L,R,S} by doing L+R for an S
- We will see that several other more involved *variants* can also be nicely simulated

## Variation I: Multitape TM

- Theorem I: *Every multi-tape TM has an equivalent single-tape TM*
- Proof:
  - Show how to convert a multi-tape TM M to an equivalent single-tape TM S

# Variation I: Multitape TM



**Simulation Idea:**

- Say M has k tapes
- S simulates the effect of k tapes by storing their information on its single tape
- Delineate tapes: S uses the new symbol # as a delimiter to separate contents of the different tapes
- Track tape heads: S writes a dot above a tape symbol to mark the place where the head on that tape would be

## Variation I: Multitape TM Complete/formal simulation

$S =$ "On input $w = w_1 \cdots w_n$:

1. First $S$ puts its tape into the format that represents all $k$ tapes of $M$. The formatted tape contains

$$\#\overset{\bullet}{w}_1 w_2 \cdots w_n \#\overset{\bullet}{\sqcup}\#\overset{\bullet}{\sqcup}\# \cdots \#.$$

2. To simulate a single move, $S$ scans its tape from the first #, which marks the left-hand end, to the $(k+1)$st #, which marks the right-hand end, in order to determine the symbols under the virtual heads. Then $S$ makes a second pass to update the tapes according to the way that $M$'s transition function dictates.
3. If at any point $S$ moves one of the virtual heads to the right onto a #, this action signifies that $M$ has moved the corresponding head onto the previously unread blank portion of that tape. So $S$ writes a blank symbol on this tape cell and shifts the tape contents, from this cell until the rightmost #, one unit to the right. Then it continues the simulation as before."

## Variation II: Nondeterministic TMs

- Nondeterminism: at any point in a computation, the machine may proceed according to several possibilities.
- IOW, the transition function of a nondeterministic TM has the form

$$\delta: Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}).$$

- The computation of a nondeterministic TM can be viewed as a tree whose branches correspond to different possibilities of the machine.
- If some branch leads to the accept state, the machine accepts the input.

## Variation II: Nondeterministic TMs

- Theorem II: *Every nondeterministic TM has an equivalent deterministic TM*

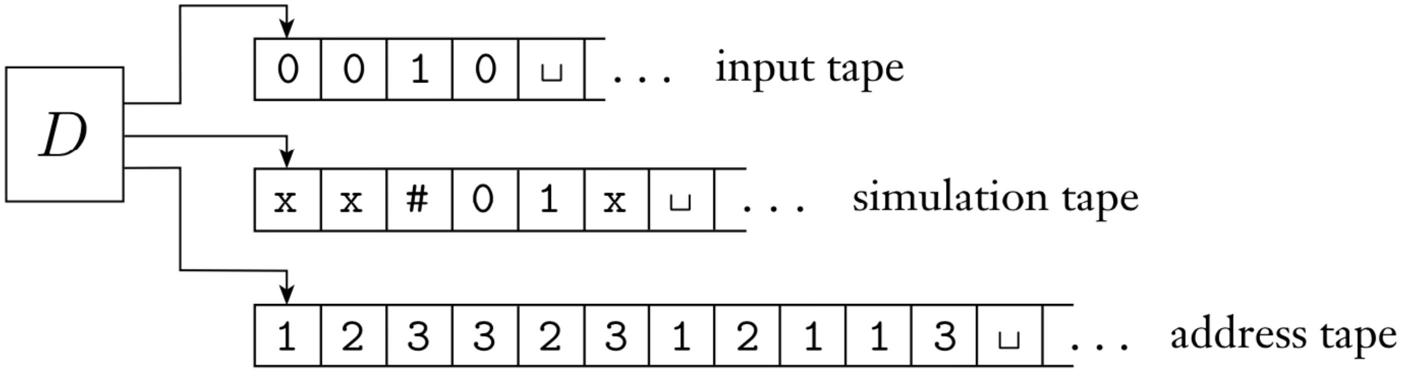
- Proof Idea
  - Simulate a NDTM N with a DTM D
  - The idea behind the simulation is to have D try all possible branches of N's nondeterministic computation.
  - If D ever finds the accept state on one of the branches, D accepts. Otherwise D's simulation will not terminate.
  - Use Breadth First Search to explore the tree.

# Variation II: Nondeterministic TMs: Details of the proof

- The simulating DTM D has three tapes

(this is equivalent to having a single tape, by Theorem I)

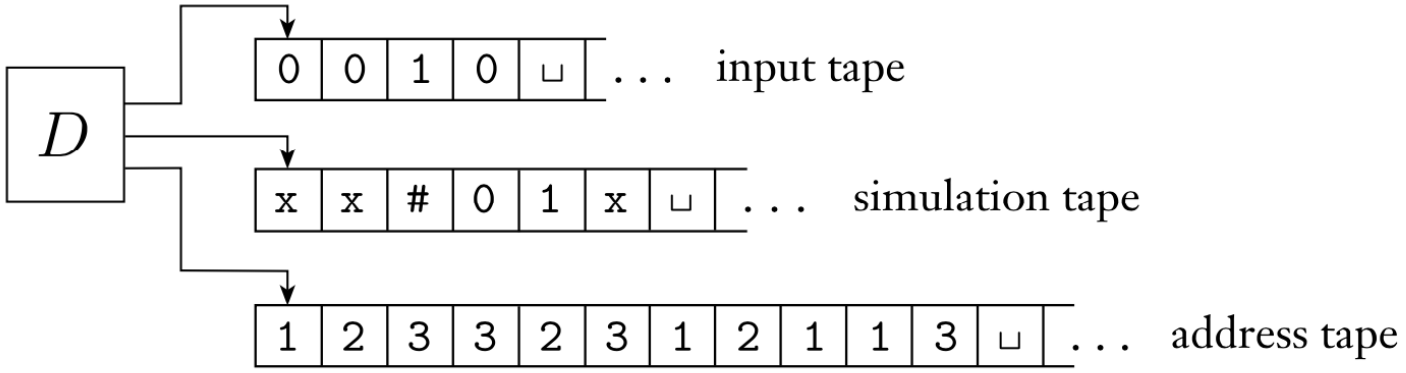
- Tape 1: contains the input string, never altered
- Tape 2: maintains a copy of N's tape on some branch of its ND computation
- Tape 3: keeps track of D's location in N's ND computation tree



D
0 0 1 0 ␣ . . . input tape
x x # 0 1 x ␣ . . . simulation tape
1 2 3 3 2 3 1 2 1 1 3 ␣ . . . address tape

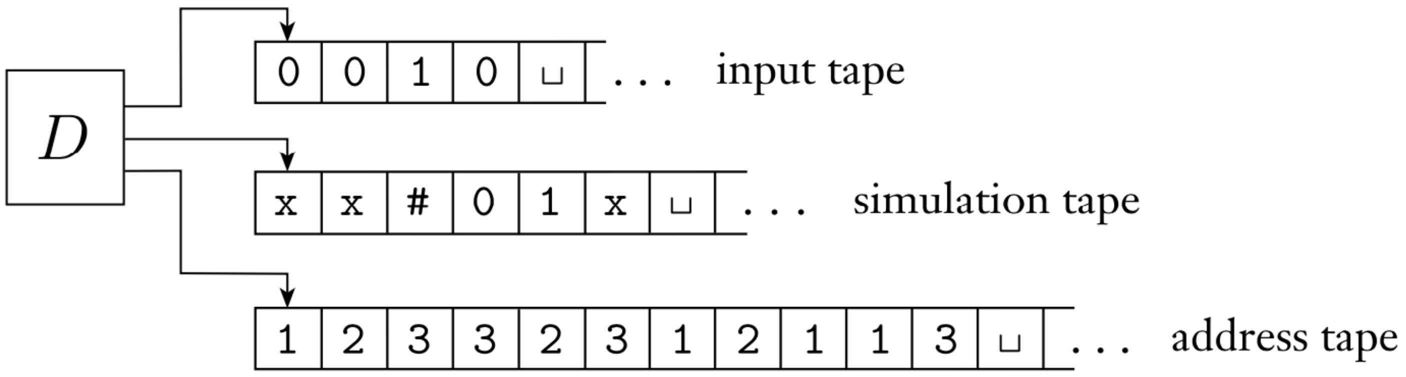
## Variation II: Nondeterministic TMs: Details of the proof

- Let us first consider the data representation on tape 3.
- Every node in the tree can have at most b children, where b is the size of the largest set of possible choices by N's transition function.
- To every node in the tree we assign an address that is a string over the alphabet $T_b = \{1,2,\ldots,b\}$
- E.g. we assign the address 231 to the node we arrive at by starting at the root, going to its 2nd child, going to that nodes 3rd child, and finally going to that node's 1st child



# Variation II: Nondeterministic TMs: Details of the proof

- Each symbol in the string tells us which choice to make next when simulating a step in one branch in N's computation
- Sometimes a symbol may not correspond to any choice if too few choices are available for a configuration. In that case, the address is invalid and doesn't correspond to any node.
- Tape 3 contains a string over $T_b$. It represents the branch of N's computation from the root to the node addressed by that string unless the address is invalid.
- The empty string is the address of the root of the tree.

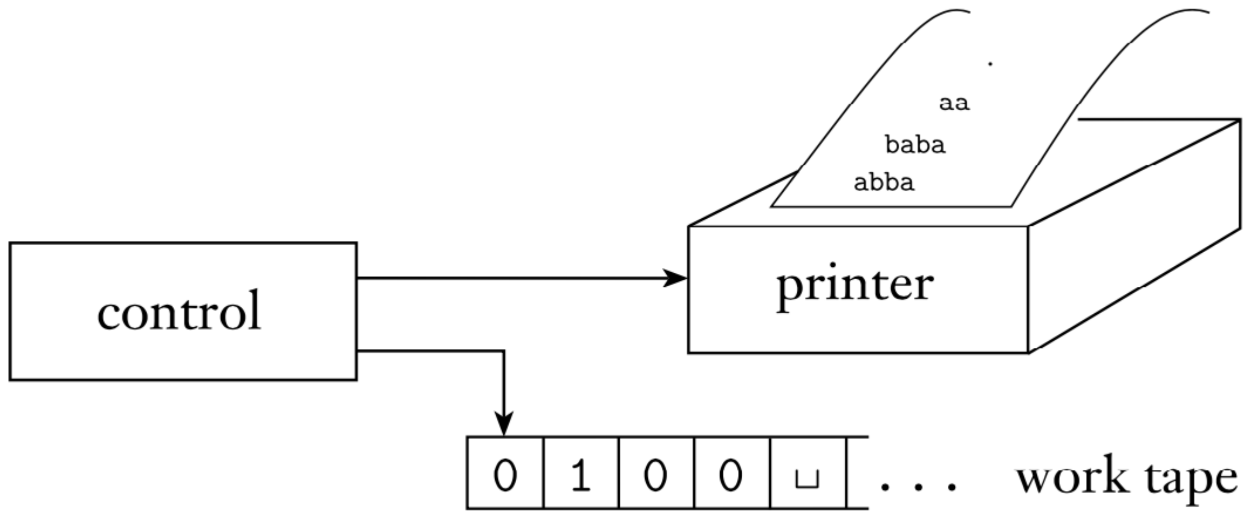


## Variation II: Nondeterministic TMs: Full description of D

1. Initially, tape 1 contains the input $w$, and tapes 2 and 3 are empty.
2. Copy tape 1 to tape 2 and initialize the string on tape 3 to be $\varepsilon$.
3. Use tape 2 to simulate $N$ with input $w$ on one branch of its nondeterministic computation. Before each step of $N$, consult the next symbol on tape 3 to determine which choice to make among those allowed by $N$'s transition function. If no more symbols remain on tape 3 or if this nondeterministic choice is invalid, abort this branch by going to stage 4. Also go to stage 4 if a rejecting configuration is encountered. If an accepting configuration is encountered, *accept* the input.
4. Replace the string on tape 3 with the next string in the string ordering. Simulate the next branch of $N$'s computation by going to stage 2.

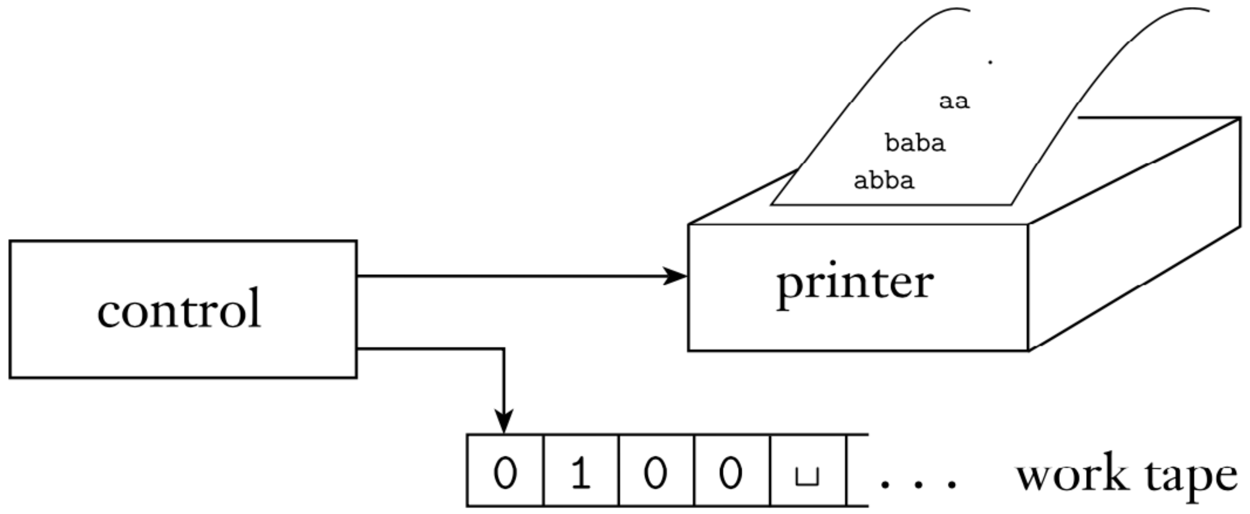
# Variation III: Enumerators

- Loosely defined, an enumerator is a TM with an attached printer
- The TM can use that printer as an output device to print strings
- Every time the TM wants to add a string to the list, it sends the string to the printer



# Variation III: Enumerators

- An enumerator E starts with a blank input on its work tape.
- If the enumerator doesn't halt, it may print an infinite list of strings.
- The language enumerated by E is the collection of all the strings that it eventually prints out.
- Moreover, E may generate the strings of the language in any order, possibly with repetitions.



# Variation III: Enumerators

- Theorem III: *A language is Turing-recognizable if and only if some enumerator enumerates it.*

- Proof:
  - First direction: we show that if we have an enumerator E that enumerates a language A, a TM recognizes A.

  The TM works in the following way.

  M = “On input w:

  1. Run E. Every time that E outputs a string, compare it with w.
  2. If w ever appears in the output of E, accept.”

  Clearly, M accepts those strings that appear on E’s list.

# Variation III: Enumerators

- Theorem III: *A language is Turing-recognizable if and only if some enumerator enumerates it.*

- Proof
  - The other direction: If TM $M$ recognizes language $A$, we can construct the following enumerator for $A$. Say that $s_1, s_2, s_3, \ldots$ is a list of all possible strings in $\Sigma^*$.
  - $E$ = "Ignore the input,

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1. Repeat the following for i = 1,2,3,…
    2. Run M for i steps on each input, $s_1, s_2, \ldots, s_i$
    3. If any computations accept, print out the corresponding $s_j$."
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If $M$ accepts a particular string $s$, eventually it will appear on the list generated by $E$. In fact, it will appear on the list infinitely many times because $M$ runs from the beginning on each string for each repetition of step 1. This procedure gives the effect of running $M$ in parallel on all possible input strings.

## Equivalence with other models

- We saw several variants of the TM model and their equivalence
- Many other models of general purpose computation have been proposed
- Some are very much like TMs, but others are quite different
- All share the essential feature of TMs – namely, unrestricted access to unlimited memory – distinguishing them from weaker models such as FA
- Remarkably, all models with that feature turn out to be equivalent in power, so long as they satisfy reasonable requirements (e.g. the ability to perform only a finite amount of work in a single step)
- This phenomenon is analogous to "equivalence of programming languages"
- This analogy has profound implication – definition of algorithm -- the subject of our next lecture!