Pumping Lemma exercises

The Pumping Lemma

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

How to apply PL to prove non-regularity of a language B

1. Assume that B is regular in order to obtain a contradiction
2. Use the PL to guarantee the existence of a pumping length p such that all strings of length p or greater in B can be pumped
3. Find a string s in B that has length p or greater but that cannot be pumped
4. Demonstrate that s cannot be pumped by considering all ways of dividing s into x, y, and z (taking Condition 3 of the PL into account when convenient), and for each such division, finding a value of i where $xy^iz \notin B$

Note: Step 4 often involves grouping the various ways of dividing s into several cases and analyzing them individually.

The existence of s contradicts the PL if B were regular. Hence B cannot be regular.

Pumping Lemma Exercises

Use the Pumping Lemma to show the following languages are nonregular

- $E = \{0^i 1^j \mid i > j\}$
- $F = \{0^n 1^m 0^n \mid m, n \geq 0\}$
- $G = \{0^m 1^n \mid m \neq n\}$

The Language E

- Let $E = \{0^i 1^j \mid i > j\}$

- Proof
  - Assume that E is regular
  - Let p be the pumping length for E
  - Let $s = 0^{p+1}1^p$
  - Then s can be split into xyz, satisfying the conditions of the PL
  - By condition 3, y consists of only 0s
  - Let us examine the string xyyz to see whether it can be in E.

The Language E

- Let $E = \{0^i 1^j \mid i > j\}$
- Proof
  - Let us examine the string $xyyz$ to see whether it can be in E.
  - Adding an extra copy of y increases the number of 0s
  - But E contains all strings in 0^*1^* that have more 0s than 1s, so increasing the number of 0s will still give a string in E. No contradiction occurs.
  - We need try something else – pumping down
  - Let us consider the string $xy^0z = xz$.
  - Removing string y decreases the number of 0s in s.
  - Recall that s has just one more 0 than 1.
  - Therefore, xz cannot have more 0s than 1s, so it cannot be a member of E.
  - Thus, we obtain a contradiction.

The Language F

Let $F = \{0^n 1^m 0^n \mid m, n \geq 0\}$

- Proof
  - Assume that F is regular
  - Let p be the pumping length for F
  - Let $s = 0^p 1 0^p$
  - Then s can be split into xyz, satisfying the conditions of the PL
  - We will split this as $x = 0^a$, $y = 0^b$, $z = 0^c 1 0^p$, where $a+b+c = p$ and $b \geq 1$.
  - Per Condition 1 of the pumping lemma, we may let $i = 0$ to yield a string in F and satisfying the other two conditions.
  - Observe that string $xy^0z = 0^{a+c} 1 0^p$ where $a+c < p$ since $b \geq 1$ and $a+b+c=0$.
  - Since the number of 0s before and after the 1 are non-equal, xy^0z is not in F.
  - Therefore F does not satisfy the conditions of the PL and is not regular.

The Language G

- Let $G = \{0^m 1^n \mid m \neq n\}$

- Proof I: without using the PL (easier)
  - Let G' denote the complement of G
  - Observe that $G' \cap 0^* 1^* = \{0^k 1^k \mid k \geq 0\}$
  - If G were regular, then G' would be regular and so would $G' \cap 0^* 1^*$
  - But we already know that $\{0^k 1^k \mid k \geq 0\}$ isn't regular, so G cannot be regular

The Language G

- Let $G = \{0^m 1^n \mid m \neq n\}$
- Proof II: using the PL (much trickier)
  - Assume that G is regular
  - Let p be the pumping length given by the PL
  - Observe that $p!$ is divisible by all integers from 1 to p
    - $p! = p(p-1)(p-2)\ldots..1$
  - The string $s = 0^p 1^{p+p!} \in G$, and $|s| \geq p$
  - Thus the PL implies that s can be divided as xyz with $x= 0^a$, $y = 0^b$ and $z = 0^c 1^{p+p!}$, where $b \geq 1$ and $a+b+c = p$.
  - Let s' be the string $xy^{i+1}z$, where $i = p!/b$.
  - Then $y^i = 0^{p!}$ so $y^{i+1} = 0^{b+p!}$, and so $s' = 0^{a+b+c+p!} 1^{p+p!}$.
  - That gives $s' = 0^{p+p!} 1^{p+p!} \notin G$, a contradiction.