The Pumping Lemma for Regular Languages

Revised schedule ahead

- Wed Feb 9: nonregular languages, part I
- Fri Feb 11 (today):
  - Pumping Lemma
  - HW3 in, HW4 out
- Mon Feb 14: In-class exercises on PL + Hints on HW4
- Wed Feb 16: Examples, solutions for HW1--HW3
- Fri Feb 18: Review for Mid-term 1
  - HW4 in
- Mon Feb 21: HOLIDAY (no class)
- Wed Feb 23: Mid-term 1 (in-class)

The Pumping Lemma

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Note

- When s is divided into xyz, either x or z maybe ε, but Condition 2 says that $y \neq \varepsilon$
- Condition 3 states that the pieces x and y together have length at most p

- Today we will see how to apply the PL to prove that a given language is not regular.

How to apply PL to prove non-regularity of a language B

1. Assume that B is regular in order to obtain a contradiction
2. Use the PL to guarantee the existence of a pumping length p such that all strings of length p or greater in B can be pumped
3. Find a string s in B that has length p or greater but that cannot be pumped
4. Demonstrate that s cannot be pumped by considering all ways of dividing s into x, y, and z (taking Condition 3 of the PL into account when convenient), and for each such division, finding a value of i where $xy^iz \notin B$

Note: Step 4 often involves grouping the various ways of dividing s into several cases and analyzing them individually.

The existence of s contradicts the PL if B were regular. Hence B cannot be regular.

Example 1

- Let $B = \{0^n1^n \mid n \geq 0\}$

- Proof
  - Assume that B is regular
  - Let p be the pumping length
  - Choose s to be 0^p1^p
  - Because s is a member of B and s has length more than p, the PL guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$, the string xy^iz is in B

  - We consider three cases to show that this result is impossible

Example 1

1. The string y consists of only 0s.
   - In this case, the string $xyyz$ has more 0s than 1s, and so is not a member of B, violating Condition 1 of the PL. This is a contradiction.
2. The string y consists of only 1s.
   - This also gives a contradiction
3. The string y consists of both 0s and 1s.
   - In this case, the string $xyyz$ may have the same number of 0s and 1s, but they will be out of order with some 1s before 0s. Hence it is not a member of B, leading to a contradiction

Example 2

- Let $C = \{w \mid w$ has equal number of 0s and 1s$\}$

- Proof
  - Assume C is regular
  - Let p be the pumping length
  - Let s be $0^p 1^p$
  - Consider Condition 3 of the PL.
    - It states that $|xy| \leq p$
    - That restriction suggests that the string $s = 0^p 1^p$ can not be pumped.
    - If $|xy| \leq p$, then y must consist only of 0s, so $xyyz \notin C$.
    - Therefore, s can not be pumped.

Alternative proof for Example 2

- We can rely on our knowledge that $B = \{0^n 1^n \mid n>=0\}$ is nonregular.

- Consider $D = \{0^* 1^*\}$
- If C were regular, then $C \cap D$ would also be regular, since D is regular and regular languages are closed under intersection
- But $C \cap D = B$, and we know B is non-regular
- Therefore, C must be non-regular

Example 3

- Let $F = \{ww \mid w \in \{0, 1\}^*\}$

- Proof
  - Assume F is regular
  - Let p be the pumping length
  - Let s be the string 0^p10^p1
  - Because s is a member of F and s has length more than p, the PL guarantees that s can be divided into the three pieces $s = xyz$ satisfying the three conditions of the Lemma.
  - We show this is impossible.

Example 3

- Condition 3 (i.e. $|xy| \leq p$) is crucial here because without it we could pump s if we let x and z be the empty string ε
- With Condition 3 the proof follows because y must consist of only 0s, so

$xyyz \notin F$.

- Note that we choose $s = 0^p10^p1$ to be a string that exhibits the "essence" of the non-regularity of F.
- For example, 0^p1^p is a member of F, but it fails to demonstrate a contradiction because it can be pumped.

Example 4

- Let $D = \{1^{n^2} \mid n >= 0\}$
- In words: D contains all strings of 1s whose length is a perfect square

- Proof
  - Assume D is regular
  - Let p be the pumping length
  - Let s be the string 1^{p^2}
  - Because s is a member of D and s has length at least p, the PL guarantees that s can be divided into $s = xyz$, where for any $i \geq 0$, the string xy^iz is in D
  - We show that this outcome is impossible

Example 4

- A little thought about the sequence of perfect squares is helpful in this proof:
  - Consider 0, 1, 4, 9, 16, 25, 36, 49, ….
  - Note that the gap between successive members of this sequence grows as we go further in the sequence
  - That is, large members of this sequence cannot be “near” each other
- Now consider the two strings xyz and xy^2z
- These strings differ from each other by a single repetition of y, and consequently their lengths differ by the length of y

Example 4

- By Condition 3 of the PL,

  $|xy| <= p$

  and thus $|y| <= p$
- We have $|xyz| = p^2$ and so $|xy^2z| <= p^2 + p$
- But $p^2 + p < p^2 + 2p + 1 = (p+1)^2$
- Moreover, Condition 2 of the PL implies that y is not the empty string and so $|xy^2z| > p^2$
- Therefore, the length of xy^2z lies strictly between consecutive perfect squares p^2 and $(p+1)^2$
- Hence this length can not be a perfect square itself
- So we arrive at the contradiction $xy^2z \notin D$ and conclude that D is not regular