Permutations of a set with repeated elements.

$\boxed{\text{Ex}}$ How many distinguishable orderings of letters in Mississippi are there?

Steps
1. Choose 4 positions for S's
2. Choose 4 positions for I's
3. Choose 2 positions for P's
4. Choose 1 position for M.

(The last step does not increase the count).

So, we have $\binom{11}{4}$ for step 1

$\binom{7}{4}$ for Step 2, $\binom{3}{2}$ for step 3.

total: $\binom{11}{4}\binom{7}{4}\binom{3}{2} = \frac{11!}{4!\,7!}\,\frac{7!}{4!\,3!}\,\frac{3!}{2!\,1!} = \frac{11!}{4!\,4!\,2!}$

Thm (no proof)

Suppose a collection ~~of~~ consists of n objects of which

n_1 are of type 1 and are indistinguishable;

n_2 are of type 2 and are indistinguishable;

...

n_k are of type k and are indistinguishable

Suppose that $n_1 + n_2 + \dots + n_k = n$

Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}=$$

$$=\frac{n!}{n_1!\,n_2!\,n_3!\,n_4!\cdots n_k!}$$

r- Combinations with repetition allowed.

Q: How many ways are there to choose r elements without ordering from a set of n elements if repetition is allowed?

Def An r-~~repetition~~ combination $\boxed{309}$ with repetition allowed (or multiset ~~of~~ size r), chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed.

Ex. 3-combinations ~~from~~ X with repetitions allowed. $X = \{1, 2, 3, 4\}$

A possible solution:

1) combinations with 1, 1
2) all additional combinations with 1, 2
3) additional combinations with 1, 3 or 1, 4.

4) Additional combinations with 2, 2

5) Additional combinations with 2, 3 or 2, 4

6) Additional combinations with 3, 3 or 3, 4

7) an additional combination with 4, 4.

The complete list

1) $\{1,1,1\}$, $\{1,1,2\}$, $\{1,1,3\}$, $\{1,1,4\}$

2) $\{1,2,2\}$, $\{1,2,3\}$, $\{1,2,4\}$

3) $\{1,3,3\}$, $\{1,3,4\}$, $\{1,4,4\}$

4) $\cancel{\{2,2,2\}}$ $\{2,2,2\}$, $\{2,2,3\}$, $\{2,2,4\}$

5) $\{2,3,3\}$, $\{2,3,4\}$, $\{2,4,4\}$

6) $\{3,3,3\}$, $\{3,3,4\}$, $\{3,4,4\}$

7) $\{4,4,4\}$

How do we organize such counts in general?

Think of elements as categories

So now we have four categories.

X - indicates an element chosen from a given category

| - separate categories.

A <u>selection</u> is a string such as

| x x | | x

this means:

none from 1

2 from 2

none from 3

1 from 4

Sample Other choices.

1	2	3	4	
x |	x |	x |		{1,2,3}
|	x |	x x |		{2,3,3}
x x |	x |	|		{1,1,2}
|	x |	|	x x	{2,4,4}

The count is the # of strings of 6 : 3 bars and 3 crosses.

6 slots to fill. Placing 3 crosses leaves no choices for bars.

$\binom{6}{3}$ is the final count.

Thm The number of r-combinations with repetition allowed that can be selected from a set of n elements is

$$\binom{r+n-1}{r}$$

Ex. Counting triples (i, j, k)

with $1 \le i \le j \le k \le n$ $(*)$

Given an integer $n \ge 0$, find the # of triples i, j, k, not necessarily distinct, and such that $(*)$ holds.

$n = 4$ $\quad 1 \mid 2 \mid 3 \mid 4$ $\quad$ 3 bars

Solution. A triple: $n-1$ vertical bars and 3 crosses.

Thus the count is $\binom{3+(n-1)}{3} =$

$$= \binom{n+2}{3} = \frac{(n+2)!}{3!\,(n-1)!} = \frac{n(n+1)(n+2)}{6}.$$

$$= \frac{n!\,(n+1)(n+2)}{3!\,(n-3)!\,(n-2)(n-1)} = \binom{n}{3}\frac{(n+1)(n+2)}{(n-2)(n-1)}$$

Ex. 1) How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

assuming $x_i \geq 0$?

$10 = 1+1+1+1+1+1+$
$+1+1+1+1.$

Solution. 4 categories. Place ten 1's into them.

3 bars separating the categories

10 crosses. Count $\binom{10+3}{10} = \frac{13!}{10!\,3!} =$

$$= \frac{11 \cdot 12 \cdot 13}{6} = 2 \cdot 11 \cdot 13 = 286.$$

2) Same as above, but assuming $x_i \geq 1$.

Solution. 1 cross goes into each category. Remaining 6 can be