2.9 Dimension and Rank

Let H be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for H. Remember that the vectors in $\mathcal{B}$ are linearly independent. Thus, if $\vec{x}$ is in H and we have two representations

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$$

and

$$\vec{x} = d_1\vec{b}_1 + \cdots + d_p\vec{b}_p,$$

then

$$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \cdots + (c_p - d_p)\vec{b}_p.$$

We must have $c_i - d_i = 0$ for all i and hence $c_i = d_i$ for each i. This shows that all $\vec{x}$ in H are represented in a unique way in terms of the elements in the basis.

For each $\vec{x}$ in H, the <u>coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$</u> are the weights $c_1, \ldots, c_p$ such that $\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$ and the vector in $\mathbb{R}^p$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the <u>coordinate vector of $\vec{x}$ relative to B.</u>

<u>Ex.</u> Let $B = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \end{bmatrix} \right\}$.

(a) Suppose $[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Calculate $\vec{x}$.

(b) Let $\vec{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$. Calculate $[\vec{x}]_B$.

If $[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then

$$\begin{aligned} \vec{x} &= (1)\vec{b}_1 + (-1)\vec{b}_2 \\ &= \begin{bmatrix} 1 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -11 \end{bmatrix} \end{aligned}$$

Let $\vec{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$. To find $[\vec{x}]_B$ we solve

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2$$

$$\begin{bmatrix} 1 & -2 & -3 \\ -4 & 7 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\Longrightarrow [\vec{x}]_B = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

Dimension of a Subspace

The **dimension** of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{0\}$ is defined to be 0.

Note: The **rank** of a matrix A, denoted by $\operatorname{rank}(A)$, is the dimension of the column space of A.

$$\rightsquigarrow \operatorname{rank}(A) = \dim \operatorname{Col}(A)$$

Theorem If a matrix A has n columns, then

$$\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = n.$$

Theorem Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Theorem (Invertible Matrix Theorem cont.)

Let A be an $n \times n$ matrix. Then the following are all equivalent to

(a) A is an invertible matrix.

(m) The columns of A form a basis of $\mathbb{R}^n$.

(n) $\mathrm{Col}(A) = \mathbb{R}^n$.

(o) $\mathrm{rank}(A) = n$.

(p) $\dim \mathrm{Nul}(A) = 0$.

(q) $\mathrm{Nul}(A) = \{0\}$.