# 6.2 Orthogonal Sets

A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ in $\mathbb{R}^n$ is said to be an <u>orthogonal set</u> if $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$.

<u>Ex.</u> Verify that $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set.

We have

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = (1)(2) + (-1)(1) + (1)(-1) = 0,$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (1)(0) + (-1)(1) + (1)(1) = 0,$$

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (2)(0) + (1)(1) + (-1)(1) = 0.$$

So we indeed have an orthogonal set.

<u>Theorem</u> If $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then $S$ is linearly independent and hence is a basis for $\mathrm{Span}(S)$.

An <u>orthogonal</u> <u>basis</u> for a subspace $H$ of $\mathbb{R}^n$ is a basis for $H$ that is also an orthogonal set.

<u>Theorem</u> Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace $H$ of $\mathbb{R}^n$. For each $\vec{y}$ in $H$, the weights in the linear combination

$$\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$$

are given by

$$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j} \qquad (j = 1, \ldots, p).$$

<u>Ex.</u> Let $\vec{y} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$. Find the weights of the linear combination $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$ where $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are the vectors from the previous example.

Here we have $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$c_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{(7)(1)+(-2)(-1)+(4)(1)}{(1)(1)+(-1)(-1)+(1)(1)} = \frac{13}{3}$$

$$c_2 = \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{(7)(2)+(-2)(1)+(4)(-1)}{(2)(2)+(1)(1)+(-1)(-1)} = \frac{8}{6} = \frac{4}{3}$$

$$c_3 = \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{(7)(0)+(-2)(1)+(4)(1)}{(0)(0)+(1)(1)+(1)(1)} = \frac{2}{2} = 1$$

$$\Rightarrow \quad \vec{y} = \frac{13}{3}\vec{u}_1 + \frac{4}{3}\vec{u}_2 + \vec{u}_3 .$$

## Orthogonal Projections

Let $\vec{u}$ be a nonzero vector in $\mathbb{R}^n$. Given another $\vec{y}$ in $\mathbb{R}^n$ we want to write

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} = \alpha\vec{u}$ for some scalar $\alpha$ and $\vec{u}\cdot\vec{z} = 0$.

Note that $\vec{y} - \hat{y} = \vec{z}$, so

$$\begin{aligned} 0 &= (\vec{y} - \hat{y})\cdot\vec{u} \\ &= (\vec{y} - \alpha\vec{u})\cdot\vec{u} \\ &= \vec{y}\cdot\vec{u} - \alpha(\vec{u}\cdot\vec{u}) \quad \Rightarrow \quad \alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} . \end{aligned}$$

$$\Rightarrow \quad \hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

We call $\hat{y}$ the orthogonal projection of $\vec{y}$ onto $L = \text{Span}\{\vec{u}\}$. We often denote it by $\text{proj}_L(\vec{y})$.

Ex. Let $\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find $\hat{y}$ and write $\vec{y}$ in the form $\hat{y} + \vec{z}$.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{(7)(4) + (6)(2)}{(4)(4) + (2)(2)} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \vec{y} = \underbrace{\begin{bmatrix} 8 \\ 4 \end{bmatrix}}_{\hat{y}} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\vec{z}}$$

## Orthonormal Sets

A set $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors. If $H$ is a subspace spanned by such a set, then $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an **orthonormal basis** for $H$.

Ex. The set $\mathcal{E} = \{\hat{e}_1, \dots, \hat{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

A matrix whose columns form an orthonormal set is called an <u>orthonormal</u> <u>matrix</u>.

<u>Theorem</u> An $m \times n$ matrix $U$ is an orthonormal matrix if and only if $U^T U = I$.

<u>Theorem</u> Let $U$ be an $m \times n$ orthonormal matrix, and let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Then

(a) $\| U\vec{x} \| = \| \vec{x} \|$

(b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

(c) $(U\vec{x}) \cdot (U\vec{y}) = 0$ if and only if $\vec{x} \cdot \vec{y} = 0$.

<u>Ex.</u> Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Notice that $U$ is an orthonormal matrix and

$$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also,

$$\| U\vec{x} \| = \left\| \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{11},$$

$$\| \vec{x} \| = \left\| \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} \right\| = \sqrt{11}.$$