3.2 Properties of Determinants

Theorem Let A be a square matrix.

(a) If a multiple of one row of A is added to another row to produce a matrix B, then

$$\det(B) = \det(A).$$

(b) If two rows of A are interchanged to produce a matrix B, then

$$\det(B) = -\det(A).$$

(c) If one row of A is multiplied by k to produce a matrix B, then

$$\det(B) = k\det(A).$$

Ex. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find $\det(B)$ if $\det(A) = 9$

and $B = \begin{bmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ 2g & 2h & 2i \end{bmatrix}$.

We have one row replacement and one

row scaling. Thus

$$\det(B) = 2\det(A)$$
$$= 2(9)$$
$$= 18.$$

<u>Theorem</u> Suppose a square matrix has been reduced to an echelon form U by using row replacements and row swaps. Then

$$\det(A) = (-1)^r \det(U)$$

where r is the total number of row swaps.

<u>Theorem</u> A square matrix A is invertible if and only if $\det(A) \neq 0$.

<u>Ex.</u> Let $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & 2 \\ 2 & -1 & 2 \end{bmatrix}$. Is A invertible?

We calculate the determinant of A.

$$\det(A) = (1)\det\left(\begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}\right) - (3)\det\left(\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}\right)$$

$$= (1)(10) - (3)(-6)$$

$$= 10 + 18$$

$$= 28 \neq 0$$

Thus A is invertible.

<u>Note</u>: The last theorem allows us to add

(r) $\det(A) \neq 0$

to the Invertible Matrix Theorem.

<u>Theorem</u> If A is an $n \times n$ matrix, then

$$\det(A^T) = \det(A).$$

<u>Theorem</u> If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

<u>Note</u>: In general, $\det(A+B) \neq \det(A) + \det(B)$

Ex Let $A = \begin{bmatrix} -3 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 7 \\ -6 & -8 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} -3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -6 & -8 \end{bmatrix} = \begin{bmatrix} -27 & -37 \\ -19 & -25 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \det(AB) &= (-27)(-25) - (-19)(-37) \\ &= 675 - 703 \\ &= -28 \end{aligned}$$

Notice that $\det(A) = -14$ and $\det(B) = 2$

So

$$\begin{aligned} \det(A)\det(B) &= (-14)(2) \\ &= -28 \\ &= \det(AB). \end{aligned}$$