1.9 The Matrix of a Linear Transformation

When we are given a description of a linear transformation T, then we often want to find a formula for $T(\vec{x})$. Consider the vector in $\mathbb{R}^n$:

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith position.}$$

Any vector $\vec{x}$ in $\mathbb{R}^n$ can be decomposed as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n.$$

If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots + x_n T(\vec{e}_n) \\ &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\vec{x}. \end{aligned}$$

This shows that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

We call the matrix

$$A = \left[T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_n) \right]$$

the <u>standard matrix for T.</u>

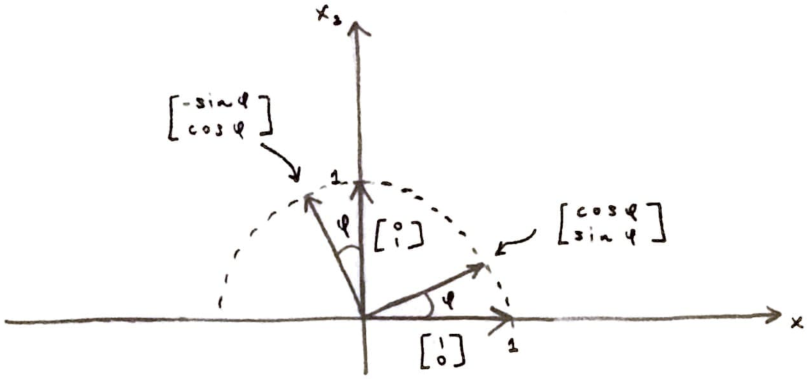
<u>Ex.</u> Find the standard matrix for the transformation $T(\vec{x}) = r\vec{x}$ with r a real number and for $\vec{x}$ in $\mathbb{R}^2$.

$$T(\vec{e}_1) = r\vec{e}_1 = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = r\vec{e}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$\Rightarrow \boxed{A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}$$

<u>Ex.</u> Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each vector in $\mathbb{R}^2$ about the origin through an angle φ, with counter-clockwise rotation for a positive angle. Find the standard matrix for this transformation.



$$\Longrightarrow \boxed{A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}$$

Existence and Uniqueness Questions

Linear transformations provide a new way to think about existence and uniqueness questions.

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be <u>onto</u> $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$. Equivalently, T is onto $\mathbb{R}^m$ if the range of T is all of the codomain. So the question "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question.

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be <u>one-to-one</u> if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$. Equivalently, T is one-to-one if, for each $\vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has either a unique solution or no solution. So the question

"Is T one-to-one?" is a uniqueness question.

Ex. Let T be the transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Is T onto? Is T one-to-one?

Notice that A is in echelon form. Since A has a pivot in every row we know that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^3$. Thus T maps $\mathbb{R}^5$ onto $\mathbb{R}^3$. However, since there are two pivotless columns, the equation $A\vec{x} = \vec{b}$ has two free variables and thus T is not one-to-one.

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

(b) T is one-to-one if and only if the columns of A are linearly independent.