1.4 The Matrix Equation $A\vec{x} = \vec{b}$

If A is an $m \times n$ matrix, with columns $\vec{a}_1, \dots, \vec{a}_n$, and if $\vec{x}$ is in $\mathbb{R}^n$, then the product of A and $\vec{x}$, denoted by $A\vec{x}$, is the linear combination of the columns of A using the corresponding entries in $\vec{x}$ as weights.

$$A\vec{x} = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n .$$

<u>Note</u>: $A\vec{x}$ is only defined when

# of columns of A. = # of entries in $\vec{x}$.

<u>Ex.</u> (a)

$$\begin{bmatrix} 2 & 1 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (3)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (2)\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 6 \\ -4 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = (-2)\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} + (3)\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 8 \\ -2 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 13 \end{bmatrix}$$

The equation $A\vec{x} = \vec{b}$ is called a <u>matrix equation</u>.

<u>Theorem</u> If A is an $m \times n$ matrix, with columns $\vec{a}_1, \dots, \vec{a}_n$ and if $\vec{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n \ \ \vec{b}].$$

We also get the following fact:

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

This fact means that the questions "Is $\vec{b}$ in Span$\{\vec{a}_1, \dots, \vec{a}_n\}$?" is the same as "Is $A\vec{x} = \vec{b}$ consistent?"

A harder question is is whether the equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.

Ex. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\vec{x} = \vec{b}$ consistent for all possible b_1, b_2, b_3?

We row reduce the matrix:

$$\begin{bmatrix} 1 & 2 & 4 & b_1 \\ 0 & 1 & 5 & b_2 \\ -2 & -4 & -3 & b_3 \end{bmatrix} \xrightarrow{R3+2R1 \to R3} \begin{bmatrix} 1 & 2 & 4 & b_1 \\ 0 & 1 & 5 & b_2 \\ 0 & 0 & 5 & 2b_1 + b_3 \end{bmatrix}$$

Notice that we have a pivot in each column and row of the coefficient matrix. Thus there is a unique solution for all b_1, b_2, b_3.

Theorem Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

(a) For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

(b) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row

Computing $A\vec{x}$

To compute $A\vec{x}$ in general we do the following:

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Theorem If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

(a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

(b) $A(c\vec{u}) = c(A\vec{u})$