#48 $(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$

$$\prod_{i=1}^{4}(1-t^i)$$

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$$\prod_{i=n}^{2n} \frac{n-i+1}{n+i}$$

$$= \prod_{j=n-1}^{2n-1} \frac{n-j}{n+j+1}$$

#3

a. $$\frac{1(1+1)((2\times 1)+1)}{6} = \frac{2\times 3}{6} = 1$$

b. $$\sum_{i=1}^{k} i^2 = \frac{k(k+1)((2\times k)+1}{6}$$

c. $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)((2\times(k+1))+1)}{6}$$

d. $$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$\therefore \sum_{i=1}^{k+1} i^2 = k+1\left(\frac{k(2k+1)}{6} + (k+1)\right)$$

$$\therefore \sum_{i=1}^{k+1} i^2 = k+1\left(\frac{k(2k+1)+6k+6}{6}\right) = (k+1)\left(\frac{2k^2+7k+6}{6}\right)$$

$$\therefore (k+1)\left(\frac{2(k+1)+1)(k+2)}{6}\right) = \sum_{i=1}^{k+1} i^2$$

$\therefore$ $p(k+1)$ is true.

9. For every integer $n \geq 3$

$$4^3 + 4^4 + 4^5 + \dots + 4^n = \frac{4(4^n - 16)}{3}$$

base case:

$$4^3 = \frac{4(4^3 - 16)}{3} = \frac{4(48)}{3}$$

$$4^3 = \frac{4(4^3 - 4^2)}{3} = \frac{4(\cancel{3} \times 16)}{\cancel{3}}$$

$$= 4 \times 6 = 64$$

2)

$$4^3 + (4)^4 + 4^5 + \dots + 4^n + 4^{n+1} = \frac{4(4^{n+1} - 16)}{3}$$

$$\rightarrow 4^3 + 4^4 + 4^5 + \dots + 4^n + 4^{n+1}$$

$$\rightarrow \frac{4(4^n - 16)}{3} + 4^{n+1}$$

$$\rightarrow \frac{4(4^n - 16) + 3(4^{n+1})}{3}$$

$$\rightarrow \frac{4^2 \cdot 4^n - 64}{3} = \frac{4^{n+2} - 64}{3}$$

$$\rightarrow \frac{4(4^{n+1} - 16)}{3}$$

$$4^3 + 4^4 + 4^5 + \dots = \frac{4(4^n - 16)}{3}$$

10 prove each statement in 8-23 by Mathematical induction

$n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

$F(n) = n^3 - 7n + 3$

$\therefore -3$ is divisible by 3

$F(1) = 1 - 7(1) + 3 = -3$

Hence, $f(1)$ is true.

$F(k) = k^3 - 7k + 3$ is divisble by 3 or

$k^3 - 7k + 3 = 3m$, $m \in N$

$$F(k+1) = (k+1)^3 - 7(k+1) + 3$$
$$= k^3 + 1 + 3k(k+1) - 7k - 7 + 3$$
$$= k^3 - 7k + 3 + 3[k(k-1) - 2]$$
$$= 3m + 3(k(k+1) - 2)$$

which is divible by 3

Thus $F(k+1)$ is true whenever $F(k)$ is True.

$\therefore$ by principon of mathematical induction $F(n)$ is true for all natural numbers n

$\rightarrow n^3 - 7n + 3$ is divisible by 3 for all $n \geq 0$

#18 prove each statement in 8–23 by mathematical induction.

$5^n + 9 < 6^n$, for each integer $n \geq 2$

$P(n)$: $5^n + 9 < 6^n \quad \forall\, n \geq 2$

$P(2)$: $5^2 + 9 = 34 < 3r = 6^2$. So, $p(2)$ is True

Suppose $p(k)$ is True is $5^k + 9 < 6^k$

$P(k+1)$: $5^{k+1} + 9 = 5^k \cdot 5 + 9 = 5^k \cdot 5 + 9 \cdot 5 - 9 \cdot 5 + 9$

$= 5(5^k + 9) + 9(1-5)$

$< 5 \cdot 6^k - 4 \cdot 9 < 6 \cdot 6^k = 6^{k+1}$

Thus, $p(k+1)$ is true

Hence the induction $p(n)$ is True

#6 suppose that $f_0, f_1, f_2, \ldots$ is a sequence defined as follows:

$$f_0 = 5, f_1 = 16,$$

$$f_k = 7f_{k-1} - 10f_{k-2}$$ for every integer $k \geq 2$

$n = 0$ in $p(n)$

$$f_0 = 3 \cdot 2^0 + 2 \cdot 5^0$$
$$= 3 \cdot 1 + 2 \cdot 1$$
$$= 3 + 2 = 5$$

$\rightarrow$ $p(n)$ is true for $n = 0$

Let $n = 1$

$$f_1 = 3 \cdot 2^1 + 2 \cdot 5^1$$
$$= 3 \cdot 2 + 2 \cdot 5$$
$$= 6 + 10 = 16$$

$\rightarrow$ $p(1)$ is true $n = 1$

$$f_{k+1} = 7f_k - 10f_{k-1} = 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{k-1} + 2 \cdot 5^{k-1})$$
$$= 21 \cdot 2^k + 14 \cdot 5^k - 30 \cdot 2^{k-1} - 20 \cdot 5^{k-1}$$
$$21 \cdot 2^k + 14 \cdot 5^k - 15 \cdot 2^k - 4 \cdot 5^k$$
$$3 \cdot 2 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$$

$p(n)$ is true for $n = k+1$

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$$n = p_1^{e1} p_2^{e2} p_3^{e3} \cdots p_k^{ek}$$

$k+1$ can be written as,

$$k+1 = a.b = (p_1, p_2, p_3 \cdots p_n)$$
$$(q_1, q_2, q_3 \cdots q_r)$$

so the integer $k+1$ can be expressed as product of prime numbers even if it is not a prime number

Hence, $P(k+1)$ is also true

We can conclude that the statement $P(n)$ is true for all n.

#14 Step 1

$a_1 = 1$ & $a_2 = 3$ are odd

product of a_1 & $a_2 = a_1 \cdot a_2$

$(1)(3) = 3$

product of two odd integers is odd.

Result is true for $a_1 = 1$ & $a_2 = 3$

Step 2

$a_1 = 1$, $a_2 = 3$ & $a_3 = 5$ are odd

∴ product of $(a_1 ; a_2) \cdot a_3 = (a_1 \cdot a_2) \cdot a_3$

$= 1 \cdot 3 \cdot 5$

→ product of odd integers is odd $= 15 =$ odd

→ Result is true for $a_1 = 1$, $a_2 = 3$, $a_3 = 5$

Step 3

product of $a_1, a_2, \cdots a_n = (a_1, a_2 \ldots a_{k-1})$

$= (1 \cdot 3 \cdot 5 \cdots k-2)$

$=$ odd

Step 4 $= (a_1 \cdots a_2 \cdots a_3 \cdots a_n \cdot a_{n+1}$

$= (1 \cdot 2 \cdot 3 \cdots k) \cdot (k+2)$

$=$ odd $\cdot$ odd $=$ odd

→ Any product of two or more odd integers is odd

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a. case I ($r \le 0$) For this case consider $n = 1$, then $n > r$

case II: ($r > 0$) Let p, q be two positive integers such that $r = \frac{p}{q}$, by definition of rational

Let $n = 2p$. multiply both sides of the inequality $1 < 2$ by p. we will get $p < 2p$ and multiply both sides of the inequality $1 < q$ by $2p$ to obtain $2p < 2pq = nq$ and hence $p < nq$, and so, by transitivity of order $p < nq$. So dividing both sides by q gives that $\frac{p}{q} < n$ or $r < n$.

Hence $r < n$

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b. Let p, q be two integers such that $q \neq 0$ and $r = \frac{p}{q}$, by definition of rational Now $-r = -\frac{p}{q} = \frac{-p}{q}$, then $-p$ and q are two integers such that $q \neq 0$ and $-r$ is rational.

$\longrightarrow$

C. r is a rational number, then $-r$ is also rational. Again for each rational $-r$, there is an integer n such that $n > -r$

This implies $-n < r$. Since $m = -n$ is also an integer and hence $m < r$