5.3 Diagonalization

Consider a diagonal matrix

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}.$$

It is not hard to see that

$$D^k = \begin{bmatrix} d_1^k & & & 0 \\ & d_2^k & & \\ & & \ddots & \\ 0 & & & d_n^k \end{bmatrix}$$

for any $k \geq 1$. Suppose A is an $n \times n$ matrix and let P be an invertible matrix such that $A = PDP^{-1}$. Then, for $k \geq 1$,

$$\begin{aligned} A^k &= (PDP^{-1})^k \\ &= \underbrace{(PD\overbrace{P^{-1})(P}^{I}DP^{-1}) \cdots (PDP^{-1})}_{k \text{ times}} \\ &= PD^kP^{-1}. \end{aligned}$$

This makes calculating powers of A much less cumbersome. But when is A diagonalizable?

<u>Theorem</u> An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact,

$$A = PDP^{-1},$$

with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case,

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues.

<u>Diagonalization</u>

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

We will attempt to diagonalize A. First we find the eigenvalues of A.

$$\det\left(\begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)((-5-\lambda)(1-\lambda)+9)$$

$$-(3)((-3)(1-\lambda)+9)$$

$$+(3)(-9-(3)(-5-\lambda))$$

$$= -\lambda^3 - 3\lambda^2 + 4$$

$$= -(\lambda-1)(\lambda+2)^2 \overset{set}{=} 0$$

$$\Rightarrow \lambda = 1, -2$$

Next we find the eigenvectors.

$\underline{\lambda = 1}$: Solve $(A - I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Take $x_3 = 1$... Hence $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector.

$\underline{\lambda = -2}$: Solve $(A + 2I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Take $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$.. we then get two eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ which are linearly independent.

$$\Rightarrow P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\underline{Ex}$. Consider $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. Here we can verify the eigenvalues $\lambda = 1, -2$ with eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ respectively. Since we only have two eigenvectors the matrix

A is not diagonalizable.

<u>Theorem</u> An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.