Section 15.2 – Limits and Continuity

THEOREM 12.2 **Limit Laws for Functions of Two Variables**

Let L and M be real numbers and suppose that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$.

Assume c is a constant, and m and n are integers.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} c\, f(x, y) = c\, L$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x, y)\, g(x, y) = L\, M$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$
7. **Fractional power** If m and n have no common factors and $n \neq 0$, then $\lim_{(x,y)\to(a,b)} (f(x, y))^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

- Evaluate the following limits:
    - $\lim_{(x,y)\to(2,-1)} (xy^8 - 3x^2y^3)$

- $\lim_{(x,y)\to(2,0)} \left(\frac{x^2-3xy^2}{x+y}\right)$

DEFINITION **Interior and Boundary Points**

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R (Figure 12.40).

A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

DEFINITION **Open and Closed Sets**

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

- Evaluate the following limits at boundary points.
  - $\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x}$

- If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.
- Use two paths to prove that the following limits do not exist.
  - $\lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^3+y^2}$

- $\lim\limits_{(x,y)\to(0,0)} \frac{y^3+x^3}{xy^2}$

- We say f is continuous at the point (a, b) if:
  - f is defined at (a, b).
  - $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists
  - $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

- At what points of $\mathbb{R}^2$ are the following functions continuous?
  - $f(x, y) = x^2 + 2xy - y^3$

- $H(x,y) = \frac{e^x+e^y}{e^{xy}-1}$

- At what points of $\mathbb{R}^2$ are the following composite functions continuous?
    - $g(x,y) = \ln(x-y)$
    - $f(x,y) = \ln(x^2+y^2)$

- Find the limit, if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2 + x^2y}{x^2 + y^2}$$