2.1 Matrix Operations

Recall that we often write a matrix in terms of its columns

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n].$$

Other times we may write a generalized matrix in terms of its entries

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Note: we often use i for row and j for column.

Sums and Scalar Multiples

This functions very similar to sums and scalar multiples of vectors. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be same size matrices. Then

$$A + B = [a_{ij} + b_{ij}]$$

$$rA = [ra_{ij}] \quad (r \text{ is a real number})$$

Theorem Let A, B, and C be matrices of the same size, and let r and s be scalars.

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $A + 0 = A$

zero matrix

(d) $r(A + B) = rA + rB$

(e) $(r + s)A = rA + sA$

(f) $r(sA) = (rs)A$

Matrix Multiplication

If A is $m \times n$ and B is $n \times p$ then

$$AB = A[\vec{b}_1 \; \vec{b}_2 \; \dots \; \vec{b}_p]$$

$$= [A\vec{b}_1 \; A\vec{b}_2 \; \dots \; A\vec{b}_p]$$ ← i.e., columns of AB are the matrix-vector products $A\vec{b}_i$.

It is really important to note here that the number of columns in A matches the number of rows in B.

Ex. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Then

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array}$$

$$AB = \left[A\vec{b}_1 \quad A\vec{b}_2 \quad A\vec{b}_3 \right]$$

$$A\vec{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

$$\Rightarrow \quad AB = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Note: If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Theorem Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products make sense.

(a) $A(BC) = (AB)C$

(b) $A(B+C) = AB + AC$

(c) $(B+C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$ for any scalar r

(e) $I_m A = A I_n = A$ where $I_k = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ is the $k \times k$ identity matrix.

In general, $AB \neq BA$.

Powers of a Matrix

If A is an $n \times n$ matrix and k is a positive integer, then

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_{k \text{ times}}.$$

We take $A^0 = I_n$ by definition.

Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are the corresponding rows of A.

Ex. If $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 7 \end{bmatrix}$, then

$$A^T = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 6 & 7 \end{bmatrix}.$$

Theorem Let A and B denote matrices whose sizes are appropriate for the following to make sense.

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$ for any scalar r

(d) $(AB)^T = B^T A^T$.