6.1 Inner Product, Length, and Orthogonality

Inner Product

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $\vec{u}^T\vec{v}$ is a 1×1 matrix which we write as a single (real) number without brackets. We call the number $\vec{u}^T\vec{v}$ the <u>inner</u> <u>product</u> of $\vec{u}$ and $\vec{v}$. We will often write $\vec{u}^T\vec{v}$ as $\vec{u}\cdot\vec{v}$. In this case, we refer to the inner product of $\vec{u}$ and $\vec{v}$ as the <u>dot</u> <u>product</u> of $\vec{u}$ and $\vec{v}$. If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

<u>Ex.</u> Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. Then

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = (1)(3) + (-1)(2) + (2)(-1)$$

$$= -1.$$

Theorem Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,

(b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$,

(c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$,

(d) $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$.

Length of a Vector

If $\vec{u}$ is a vector in $\mathbb{R}^n$ with

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

then the <u>length</u> <u>of</u> <u>$\vec{u}$</u> is

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

For any scalar c,

$$\|c\vec{u}\| = |c| \, \|\vec{u}\|.$$

A vector whose length is 1 is called a <u>unit</u> <u>vector</u>.

It is clear that for any nonzero vector $\vec{u}$ in $\mathbb{R}^n$, the vector $\frac{\vec{u}}{\|\vec{u}\|}$ is a unit vector. The process

$$\vec{u} \longmapsto \frac{\vec{u}}{\|\vec{u}\|}$$

is called <u>normalizing</u> $\vec{u}$.

<u>Ex.</u> Let $\vec{u} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$. Find a unit vector in the same direction as $\vec{u}$.

we have $\|\vec{u}\| = \sqrt{(2)^2 + (4)^2 + (-1)^2} = \sqrt{21}$

$$\Rightarrow \frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{21} \\ 4/\sqrt{21} \\ -1/\sqrt{21} \end{bmatrix}.$$

<u>Distance in $\mathbb{R}^n$</u>

For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the <u>distance</u> <u>between</u> <u>$\vec{u}$ and $\vec{v}$</u>, written as $\text{dist}(\vec{u}, \vec{v})$, is the length of $\vec{u} - \vec{v}$. That is,

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

<u>Ex.</u> Let $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Then

$$\text{dist}(\vec{u}, \vec{v}) = \left\| \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\| = \sqrt{17}.$$

Orthogonal Vectors

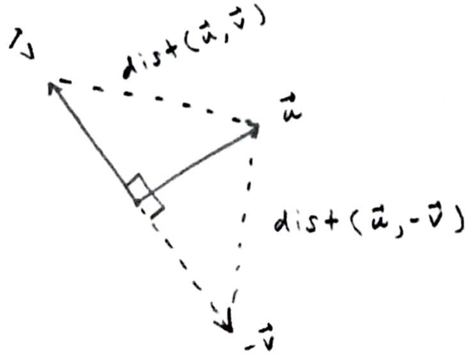
Notice that for $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$
\begin{aligned}
(\operatorname{dist}(\vec{u},-\vec{v}))^2 &= \|\vec{u}+\vec{v}\|^2 \\
&= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot(\vec{u}+\vec{v}) + \vec{v}\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u}\cdot\vec{v})
\end{aligned}
$$

and

$$(\operatorname{dist}(\vec{u},\vec{v}))^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v}).$$

we see that $\operatorname{dist}(\vec{u},-\vec{v}) = \operatorname{dist}(\vec{u},\vec{v})$ if and only if $\vec{u}\cdot\vec{v} = 0$.



Two vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> if $\vec{u}\cdot\vec{v} = 0$.

Note: In $\mathbb{R}^n$, orthogonal is the same as perpendicular.

Theorem (Pythagorean Theorem) Two vectors are orthogonal if and only if

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Orthogonal Complements

Let H be a subspace of $\mathbb{R}^n$. The orthogonal complement of H, denoted by $H^\perp$, is defined as

$$H^\perp = \{\vec{z} : \vec{x} \cdot \vec{z} = 0 \text{ for all } \vec{x} \in H\}.$$

Fact: (i) A vector $\vec{z}$ is in $H^\perp$ if and only if $\vec{z}$ is orthogonal to every vector in a set that spans H.

(ii) $H^\perp$ is a subspace of $\mathbb{R}^n$.

Theorem Let A be an $m \times n$ matrix Then

$$(\mathrm{Col}(A))^\perp = \mathrm{Nul}(A^T).$$