1.2 Row Reduction and Echelon Forms

This section focuses on matrices. In the stuff to follow, a nonzero row or column in a matrix means a row or column with at least one nonzero entry; a leading entry of a row refers to the leftmost nonzero entry in a nonzero row.

Echelon Forms

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties

1) All nonzero rows are above any rows of all zeroes.

2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3) All entries in a column below a leading entry are zeroes

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form)

4) The leading entry in each nonzero row is 1.

5) Each leading 1 is the only nonzero entry in its column.

Example The following matrix is in echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\blacksquare$ - leading entry

$*$ - nonzero entry

The following matrix is in reduced echelon form:

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$*$ - nonzero entry

In the last section we discussed elementary row operations. If two matrices can be manipulated into each other using elementary row operations then we say that the matrices are <u>row equivalent</u>

Any matrix may be row reduced (that is, transformed by elementary row operations) into a matrix that is in echelon form. A given matrix is row equivalent to infinitely many matrices in echelon form. However ...

Theorem A matrix is row equivalent to a unique matrix in reduced echelon form.

Pivot Positions and Row Reduction

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

The Row Reduction Algorithm: The row reduction algorithm consists of four steps. It produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form.

Step 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Step 2: Select a nonzero entry in the pivot column as a pivot. If necessary, swap rows to move this entry into the pivot position.

Step 3: Use row replacement operations to create zeroes in all positions below the pivot.

Step 4: Ignore the row containg the pivot position and ignore all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Step 5: Beginning with the rightmost pivot and working upward and to the left, create zeroes above each pivot. If a pivot is not 1, make it 1 using an appropriate row scaling.

Example Use row operations to reduce the following matrix to reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Note: whenever you row reduce a matrix you should always indicate the operations you use.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow[R1 \leftrightarrow R3]{\text{Swap rows 1 and 3}} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

Pivot column

$$\xrightarrow[R2 + (-1)R1 \rightarrow R2]{\text{Add row 2 to } -1 \text{ times row 1}} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

New pivot column, pivot in second position

$$\xrightarrow[R3 + (-3/2)R2 \rightarrow R3]{\text{Add row 3 to } -3/2 \text{ times row 2}} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

New pivot column

$$\xrightarrow[R1 + (-6)R3 \rightarrow R1]{\text{Add row 1 to } -6 \text{ times row 3}} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[R2 + (-2)R3 \rightarrow R2]{\text{Add row 2 to } -2 \text{ times row 3}} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R2 \rightarrow R2]{\text{Scale row 2 by } 1/2} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[\text{}]{\substack{\text{Add row 1 to} \\ \text{9 times row 2}}} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{3}R1 \to R1]{\substack{\text{Scale row 1} \\ \text{by } 1/3}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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Solutions of Linear Systems

The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the corresponding augmented matrix.

Example The augmented matrix

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in reduced echelon form and corresponds to the linear system

$$\begin{aligned} x_1 \quad - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

The variables x_1 and x_2 corresponding to the pivot columns in the matrix are called <u>basic variables</u>. The variable x_3 is called a <u>free variable</u>. The name free variable comes from the fact that x_3 can be selected as any value and this completely determines the values of the other variables.

In this case, we would write the solution as

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

The form that this solution is presented is called <u>parametric form</u>.

Existence and Uniqueness Questions

The reduced echelon form of a matrix is our key to determining the solution set of a linear system. However, a nonreduced echelon form can still answer the questions of existence and uniqueness of solutions.

Theorem (Existence and Uniqueness)

A linear system is consistent if and only if the rightmost column of the augmented matrix is <u>not</u> a pivot column - that is, if and only if an echelon form of the augmented matrix has <u>no</u> row of the form

$$[0 \quad 0 \quad \cdots \quad 0 \quad b] \qquad (b \neq 0)$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions when there is at least one free variable.