2.3 Characterizations of Invertible Matrices

Theorem (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. Then the following are equivalent.

(a) A is an invertible matrix.

(b) A is row equivalent to I_n.

(c) A has n pivot positions

(d) The equation $A\vec{x} = \vec{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one

(g) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $T(\vec{x}) = A\vec{x}$ is onto.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

Note. Let $T(\vec{x}) = A\vec{x}$ be a linear transformation with A an $n \times n$ matrix. If A is invertible, then we say T is invertible. In this case, $S(\vec{x}) = A^{-1}\vec{x}$ is the inverse of T.

Ex. Is the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ invertible?

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \xrightarrow[R3+5R1 \to R3]{R2-3R1 \to R2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R3+R2 \to R3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \leftarrow 3 \text{ pivots!}$$

Since A has three pivot positions, we know that A is invertible.