3.1 Introduction to Determinants

Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have

$$\det(A) = ad - bc.$$

We will use this to extend the determinant to larger matrices.

For $n \geq 2$, the <u>determinant</u> of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij} \det A_{ij}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A, and A_{ij} is the submatrix obtained by deleting row i and column j. In symbols,

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n})$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

Ex Compute the determinant of $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

we have $A_{11} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, and $A_{13} = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$.

Then

$$\det(A) = (1)\det\left(\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\right) - (3)\det\left(\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$+ (4)\det\left(\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= (1)\left((-1)(1) - (1)(2)\right) - (3)\left((-2)(1) - (1)(1)\right)$$

$$+4\left((-2)(2) - (-1)(1)\right)$$

$$= (1)(-3) - (3)(-3) + 4(-3)$$

$$= -3 + 9 - 12$$

$$= -6$$

Let $C_{ij} = (-1)^{i+j}\det(A_{ij})$. This is called the <u>$(i,j)$-cofactor</u> of A. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

<u>Theorem</u> The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and the expansion down the jth column

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

Ex. Compute the determinant of $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ using a cofactor expansion down the second column.

$$\det(A) = -(3)\det\left(\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\right) + (-1)\det\left(\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\right)$$

$$- (2)\det\left(\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\right)$$

$$= -(3)\left((-2)(1)-(1)(1)\right) + (-1)\left((1)(1)-(4)(1)\right)$$

$$-(2)\left((1)(1)-(4)(-2)\right)$$

$$= -(3)(-3) + (-1)(-3) - (2)(9)$$

$$= 9 + 3 - 18$$

$$= -6.$$

Theorem If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.