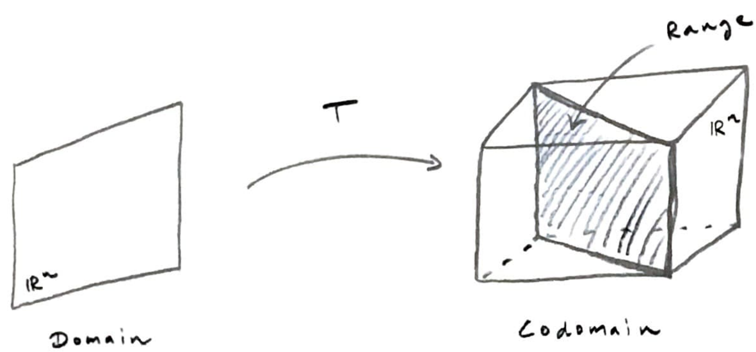
1.8 Intro to Linear Transformations

A <u>transformation</u> T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector $T(\vec{x})$ in $\mathbb{R}^m$. We call $\mathbb{R}^n$ the domain of T, and $\mathbb{R}^m$ the codomain of T. In this case we write $T : \mathbb{R}^n \to \mathbb{R}^m$.

The collection of all $\vec{y}$ in $\mathbb{R}^m$ which are the image of an $\vec{x}$ in $\mathbb{R}^n$ (i.e., $T(\vec{x}) = \vec{y}$) is called the range of T.



Matrix Transformations

Let A be an $m \times n$ matrix and consider the transformation $T(\vec{x}) = A\vec{x}$. Then T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

<u>Ex</u> Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

Define $T(\vec{x}) = A\vec{x}$.

(a) Find $T(\vec{u})$

$$T(\vec{u}) = A\vec{u}$$

$$= \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1) + (-1)(-3) \\ 2(3) + (-1)(5) \\ 2(-1) + (-1)(7) \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

(b) Find an $\vec{x}$ in $\mathbb{R}^2$ whose image under T is $\vec{b}$.

Solve $T(\vec{x}) = \vec{b}$, i.e., solve $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$ ← Note that this solution is unique.

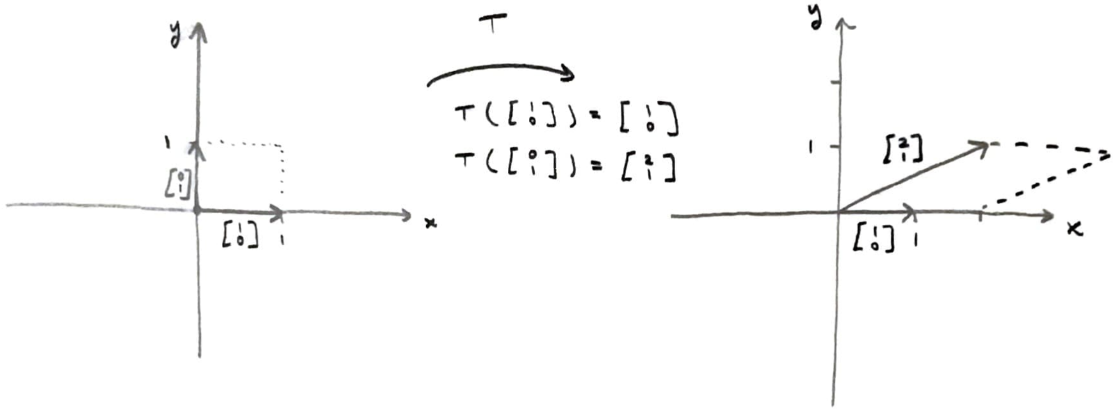
(c) Is $\vec{c}$ in the range of T?

we are asking if $A\vec{x} = \vec{c}$ has a solution.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

we have a pivot in the augmented column so $\vec{c}$ is not in the range of T.

<u>Ex.</u> Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$.



Here T is called a shear transformation.

<u>Linear Transformations</u>

A transformation T is <u>linear</u> if

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T,

(ii) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in the domain of T.

Note that $T(\vec{0}) = T(\vec{0} + \vec{0})$ and, if T is linear,

$$T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0}).$$

It follows that $T(\vec{0}) = \vec{0}$ for a linear transformation. Moreover, for a linear transformation

$$T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$$

and, in general,

$$T\left(\sum_{i=1}^{p} c_i \vec{v}_i \right) = \sum_{i=1}^{p} c_i T(\vec{v}_i).$$

<u>Ex.</u> All matrix transformations are linear transformations!