The Characteristic Equation

To compute eigenvalues of an $n \times n$ matrix A we exploit the invertible matrix theorem.

Note that

$$(A - \lambda I)\vec{x} = \vec{0} \text{ has a nontrivial solution}$$

$$\Updownarrow$$

$$A - \lambda I \text{ is not invertible}$$

$$\Updownarrow$$

$$\det(A - \lambda I) = 0$$

Thus to compute eigenvalues of A, we need to solve the equation $\det(A - \lambda I) = 0$. We call this equation the <u>characteristic equation</u>.

<u>Ex.</u> Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 9$$

$$= \lambda^2 + 4\lambda - 21$$

$$= (\lambda + 7)(\lambda - 3) \overset{\text{set}}{=} 0$$

$$\Rightarrow \lambda = -7 \text{ or } \lambda = 3.$$

Theorem Let A be an $n \times n$ matrix. Then A is invertible if and only if 0 is not an eigenvalue of A.

Note: The previous theorem allows us to add

(s) 0 is not an eigenvalue of A

to the invertible matrix theorem.

Similarity

Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible matrix P such that

$$P^{-1}AP = B \quad \text{or} \quad B = PAP^{-1}.$$

Theorem If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.