1.1 Systems of Linear Equations

How would you approach solving the following system?

$$\begin{cases} 2x + y = 3, \\ x - 3y = 5. \end{cases}$$

One strategy would be to use substitution.

$$2x + y = 3 \implies y = 3 - 2x$$
$$\implies x - 3(3 - 2x) = 5$$
$$\implies 7x - 9 = 5$$
$$\implies x = 2$$
$$\implies y = -1$$

Another strategy would be to use elimination.

$$\begin{aligned} 2x + y &= 3 \\ x - 3y &= 5 \end{aligned} \implies \begin{aligned} 6x + 3y &= 9 \\ x - 3y &= 5 \end{aligned}$$

$$\underset{\text{add equations}}{\implies} 7x = 14$$

$$\implies x = 2$$

$$\implies y = -1$$

Consider now the system

$$\begin{cases} x + 3y - 2z = -9, \\ -2x - y + z = 2, \\ 3x + 2y - 3z = -7. \end{cases}$$

The strategies used before now become quite tedious.

Linear algebra is the branch of mathematics concerning linear equations and their various representations. This subject is central to almost all areas of mathematics.

Systems of Linear Equations

A <u>linear equation</u> in the variables $x_1, \ldots, x_n$ has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where $a_1, \ldots, a_n, b$ are real or complex numbers. A <u>system of linear equations</u> is a collection of one or more linear equations involving the same variables. The examples we saw above are examples of systems.

A <u>solution</u> of the system is a list $(s_1, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$ respectively. The set of all possible solutions is called the <u>solution set</u>. Two linear systems are called <u>equivalent</u> if they have the same solution set.

Note: A system of linear equations has

→ No solution, or

→ Exactly one solution, or

→ Infinitely many solutions.

<u>Matrix Notation</u>

One useful representation of a linear system is as a matrix. For example, the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

has matrix representation:

$$\underbrace{\begin{bmatrix} \overbrace{\begin{matrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{matrix}}^{\text{coefficient matrix}} & \begin{matrix} 0 \\ 8 \\ 10 \end{matrix} \end{bmatrix}}_{\text{augmented matrix}}$$

The <u>size</u> of a matrix tells us how many rows and columns it has. An $m \times n$ matrix (read m by n) has m rows and n columns.

<u>Solving Linear Systems</u>

We will be using matrix representations as an organizational tool to help us solve systems of linear equations. The method we will be using to actually solve these systems is elimination.

<u>Example</u>: Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

We will proceed using elimination.

$$\begin{aligned} x_1 - 2x_1 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \quad \underset{\text{Add eq. 3 to -5 times eq. 2}}{\Longrightarrow} \quad \begin{aligned} x_1 - 2x_1 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \leftarrow \text{New eq. 3} \end{aligned}$$

$$\underset{\text{Multiply eq. 2 by } 1/2}{\Longrightarrow} \quad \begin{aligned} x_1 - 2x_1 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \leftarrow \text{New eq. 2} \\ 10x_2 - 10x_3 &= 10 \end{aligned}$$

$$\underset{\text{Add eq 3 to -10 times eq. 2}}{\Longrightarrow} \quad \begin{aligned} x_1 - 2x_1 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \leftarrow \text{New eq. 3} \end{aligned}$$

$\Longrightarrow$ Multiply eq. 3 by 1/30

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \quad \leftarrow \text{New eq. 3} \end{aligned}$$

$\Longrightarrow$ Add eq. 2 to 4 times eq. 3

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 &= 0 \quad \leftarrow \text{New eq. 2} \\ x_3 &= -1 \end{aligned}$$

$\Longrightarrow$ Add eq. 1 to -1 times eq. 3

$$\begin{aligned} x_1 - 2x_2 &= 1 \quad \leftarrow \text{New eq. 1} \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

$\Longrightarrow$ Add eq. 1 to 2 times eq. 2

$$\begin{aligned} x_1 &= 1 \quad \leftarrow \text{New eq. 1} \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

Notice that at each stage we can see the manipulations encoded in the matrices

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

At each stage we are performing what are known as <u>elementary row operations</u>.

<u>Elementary Row Operations</u>

The following operations can be performed on the rows of an augmented matrix:

1. (Row replacement) Replace one row by the sum of itself and a multiple of another row.

2. (Row swap) Swap two rows

3. (Row scaling) Multiply all entries in a row by a nonzero constant.

In the next section we will talk more about these row operations.

Existence and Uniqueness of Solutions

Given a system of linear equations we want to ask the following questions:

1. Does the system have a solution?
   (that is, does a solution exist?)

2. If a solution exists, is it the only one?
   (that is, is the solution unique?)

A system with at least one solution is said to be <u>consistent</u>. Otherwise the system is said to be <u>inconsistent</u>.