5.1 Eigenvectors and Eigenvalues

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$.

Ex. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\vec{u},$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\vec{v}.$$

$\Longrightarrow$ $\vec{u}$ is an eigenvector (with $\lambda = -4$) and $\vec{v}$ is **not** an eigenvector.

Note: If $A\vec{x} = \lambda\vec{x}$, then notice

$$\begin{aligned} & A\vec{x} = \lambda\vec{x} \\ \Longleftrightarrow\ & A\vec{x} - \lambda\vec{x} = \vec{0} \\ \Longleftrightarrow\ & A\vec{x} - \lambda I\vec{x} = \vec{0} \\ \Longleftrightarrow\ & (A - \lambda I)\vec{x} = \vec{0} \end{aligned}$$

$\sim$ Homogeneous!

Ex. Let $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix}$. One eigenvalue of A

is $\lambda = -4$. Find all corresponding eigenvectors.

For the note, we find nonzero solutions

to $(A - (-4)I)\vec{x} = \vec{0}$

$$A - (-4)I = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -1 & 3 \\ -1 & 7 & 3 \\ 6 & 6 & 6 \end{bmatrix}$$

Now solve...

$$\begin{bmatrix} 7 & -1 & 3 & 0 \\ -1 & 7 & 3 & 0 \\ 6 & 6 & 6 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{array}{l} x_1 = -\frac{1}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 \text{ is free} \end{array}$$

$$\Longrightarrow \vec{x} = x_3 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Take $x_3 = 2$... Then $\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector

of A (including all other multiples).

we call the solution set of $(A - \lambda I)\vec{x} = \vec{0}$ the <u>eigenspace of A corresponding to λ</u>.

<u>Theorem</u> The eigenvalues of a triangular matrix are the entries on its main diagonal.

<u>Theorem</u> If $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent.

<u>Proof</u> Suppose that $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly dependent. Since $\vec{v}_1 \neq \vec{0}$, we know that one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index such that $\vec{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors. Then for some scalars $c_1, \ldots, c_p$ we have

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1}.$$

$$\Longrightarrow \quad c_1 A\vec{v}_1 + \cdots + c_p A\vec{v}_p = A\vec{v}_{p+1}$$

$$\Longrightarrow \quad c_1 \lambda_1 \vec{v}_1 + \cdots + c_p \lambda_p \vec{v}_p = \lambda_{p+1} \vec{v}_{p+1} \qquad (\star)$$

We also have $c_1 \lambda_{p+1} \vec{v}_1 + \cdots + c_p \lambda_{p+1} \vec{v}_p = \lambda_{p+1} \vec{v}_{p+1}$ $(\star\star)$.

Now compute $(\star) - (\star\star)$:

$$c_1 (\lambda_1 - \lambda_{p+1}) \vec{v}_1 + \cdots + c_p (\lambda_p - \lambda_{p+1}) \vec{v}_p = \vec{0}$$

Since $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are linearly independent we have $c_1(\lambda_1 - \lambda_{p+1}) = \cdots = c_p(\lambda_p - \lambda_{p+1}) = 0$. But $\lambda_i - \lambda_{p+1}$ is nonzero since the λ's are distinct. Thus $c_1 = \cdots = c_p = 0$. But this implies that $\vec{v}_{p+1} = \vec{0}$ which is impossible. Hence $\{\vec{v}_1, \ldots, \vec{v}_r\}$ must be linearly independent.

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