1.5 Solution Sets of Linear Systems

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix and $\vec{0}$ is the zero vector in $\mathbb{R}^m$.

Q: Does the matrix equation $A\vec{x} = \vec{0}$ have a solution?

A: Yes! Take $\vec{x} = \vec{0}$ (in $\mathbb{R}^n$).

The solution $\vec{x} = \vec{0}$ is called the **trivial solution**.

A solution $\vec{x} \neq \vec{0}$ is called a **nontrivial solution**.

Fact: The homogeneous matrix equation $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Ex. Let $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$. Does the equation $A\vec{x} = \vec{0}$ have a nontrivial solution?

We row reduce...

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow[R3 - 2R1 \to R3]{R2 + R1 \to R2} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R3+3R2 \to R3} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R2 \to R2} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1-5R2 \to R1} \begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R1 \to R1} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{aligned} x_1 \quad -\tfrac{4}{3}x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 0 \end{aligned}$$

$$\Longrightarrow \begin{aligned} & x_1 = \tfrac{4}{3}x_3 \\ & x_2 = 0 \\ & x_3 \text{ is free} \end{aligned}$$

$$\Longrightarrow \vec{x} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

So $A\vec{x} = \vec{0}$ has a nontrivial solution (take $x_3 = 1$ for example).

The presentation of $\vec{x}$ in the previous example is called <u>parametric vector form</u>

<u>Nonhomogeneous Linear Systems</u>

<u>Ex.</u> Describe all solutions of $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \quad &= 2 \\ 0 &= 0 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= -1 + \tfrac{4}{3}x_3 \\ x_2 &= 2 \\ x_3 &\text{ is free} \end{aligned}$$

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{A particular} \\ \text{solution to} \\ A\vec{x} = \vec{b}}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \underbrace{x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{\substack{\uparrow \\ \text{Solution to} \\ A\vec{x} = \vec{0}}}$$

Theorem Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + \vec{v}_h$, where $\vec{v}_h$ is any solution of the homogeneous equation $A\vec{x} = \vec{0}$.