2.8 Subspaces of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

(i) The zero vector is in H.

(ii) For each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u}+\vec{v}$ is in H.

(iii) For each $\vec{u}$ in H and each scalar c, the vector $c\vec{u}$ is in H.

Ex. If $\vec{v}_1, \vec{v}_2$ are in $\mathbb{R}^n$ and $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, then H is a subspace of $\mathbb{R}^n$. Why?

(i) $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$ is in H.

(ii) Let $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2$ and $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2$.

Then

$$\vec{u}+\vec{v} = (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2$$

which is in H.

(iii) Let $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2$ and c a scalar.

Then

$$c\vec{u} = ca_1\vec{v}_1 + ca_2\vec{v}_2$$

which is in H.

In general, if $\vec{v}_1, \ldots, \vec{v}_p$ are vectors in $\mathbb{R}^n$

then $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of $\mathbb{R}^n$.

Column Space and Null Space

The <u>column space</u> of a matrix A is the set $\text{Col}(A)$ of all linear combinations of the columns of A. That is, if

$$A = [\vec{a}_1 \;\cdots\; \vec{a}_n]$$

then $\text{Col}(A) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$. This is a subspace of $\mathbb{R}^m$ if A is $m \times n$.

<u>Fact</u> $\text{Col}(A) = \mathbb{R}^m$ if the columns of A span $\mathbb{R}^m$.

<u>Ex.</u> Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ Is $\vec{b}$ in $\text{Col}(A)$?

This is the same question as "Is $\vec{b}$ in the span of the columns of A?"

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\vec{b}$ is in Col(A).

The <u>null space</u> of a matrix A is the set Nul(A) of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$.

<u>Theorem</u> The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

<u>Basis for a Subspace</u>

A <u>basis</u> for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

<u>Ex.</u> The vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ form a basis for $\mathbb{R}^n$. We call it the <u>standard</u> <u>basis</u> <u>for</u> <u>$\mathbb{R}^n$</u>.

<u>Ex.</u> Find a basis for the null space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

We solve $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}}$$

$$= x_2 \vec{u} + x_3 \vec{v} + x_5 \vec{w}$$

$$= \text{Span}\{\vec{u}, \vec{v}, \vec{w}\}$$

So $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $Nul(A)$.

Theorem The pivot columns of a matrix A form a basis for the column space of A.

Ex. Let $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$. Find a basis for $Col(A)$.

$$\begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \right\}$$ is a basis for $Col(A)$.