2.2 The Inverse of a Matrix

An $n \times n$ matrix A is <u>invertible</u> if there is an $n \times n$ matrix C such that

$$AC = CA = I_n \quad \leftarrow \text{$n \times n$ identity matrix}$$

We call C the <u>inverse of A</u> and write $C = A^{-1}$.

<u>Fact</u>. The inverse of a matrix is unique!

$$\underset{\substack{\uparrow \\ \text{another inverse}}}{B} = BI_n = B(AC) = (BA)C = I_nC = \underset{\substack{\uparrow \\ \text{original inverse}}}{C}$$

Not all $n \times n$ matrices are invertible (these are called singular matrices).

<u>Ex</u> Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Then for $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (-7)(2)+(3)(5) & (-5)(2)+(2)(5) \\ (-7)(-3)+(3)(-7) & (-5)(-3)+(2)(-7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} (2)(-7)+(-3)(-5) & (5)(-7)+(-7)(-5) \\ (2)(3)+(-3)(2) & (5)(3)+(-7)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

We call $ad-bc$ the **determinant** of A and write

$$\det(A) = ad - bc.$$

Ex. Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Since $\det(A) = (3)(6) - (4)(5) = -2 \neq 0$ we know that A is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Theorem If A is an invertible $n \times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Theorem (a) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

(b) If A and B are $n \times n$ invertible matrices, then so is AB and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(c) If A is an invertible matrix, then so is A^T and

$$(A^T)^{-1} = (A^{-1})^T.$$

Calculating A^{-1}

We will be using row reduction to calculate inverses.

Theorem An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of row operations that reduces A to I_n also transforms I_n to A^{-1},

$$[A \mid I_n] \xrightarrow{RREF} [I_n \mid A^{-1}]$$

Ex. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$