1.3 Vector Equations

A matrix with a single column is called a vector (or column vector).

Ex. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ← vectors with two entries

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ← vectors with three entries

Two vectors are equal if and only if their corresponding entries are equal.

We will denote a vector using an arrow or as a bolded letter, e.g. $\vec{u}$ or $\mathbf{u}$. We will most often resort to using the arrow.

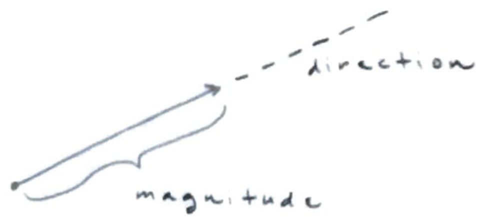
We add/subtract two vectors by adding/subtracting corresponding entries. We scale a vector by scaling all entries.

Ex. $\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+4 \\ -1+3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$2\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2(1) \\ 2(-5) \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$

Geometry of vectors

A vector is a quantity that has both a magnitude and a direction. We represent this geometrically as an arrow whose length corresponds to magnitude and direction is determined by where the arrow is pointing



Note: Vectors with 2 entries live in the xy-plane, vectors with 3 entries live in xyz-space, vectors with n entries live in n-dimensional space (denoted $\mathbb{R}^n$).

Algebraic Properties of $\mathbb{R}^n$

For all vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$ and all scalars c and d

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(iii) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

(iv) $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$

(v) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(vi) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

(vii) $c(d\vec{u}) = (cd)\vec{u}$

(viii) $1\vec{u} = \vec{u}$

Note: $\vec{0}$ denotes the vector of all zeroes.

Linear Combinations

Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p$$

is called a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ with weights $c_1, c_2, \ldots, c_p$.

Vector Equations

A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{b}$$

in the variables $x_1, x_2, \ldots, x_n$ has the same solution set as the linear system whose augmented matrix is

$$[\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n \quad \vec{b}].$$

One of the key questions in linear algebra is if a given vector is a linear combination of a fixed set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of vectors. The collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is denoted by $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$, i.e.,

is an element of

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \{c_1\vec{v}_1 + \ldots + c_p\vec{v}_p : c_i \in \mathbb{R}\}$$