# 6.4 The Gram - Schmidt Process

The **Gram - Schmidt process** is an algorithm for producing an orthogonal basis for any nonzero subspace of $\mathbb{R}^n$.

**Theorem** Given a basis $\{\vec{x}_1, \dots, \vec{x}_p\}$ for a nonzero subspace $H$ of $\mathbb{R}^n$, define

$$\vec{v}_1 = \vec{x}_1,$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}$$

Then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal basis for $H$. In addition,

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\} \text{ for } 1 \le k \le p.$$

One should notice that we are utilizing iterative projections to build an orthogonal basis.

Ex. Let $H = \left\{ \begin{bmatrix} 2x - y + z \\ y + 3z \\ x - 2y \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$. Find an orthogonal basis for H.

Notice that $H = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

We know use the Gram-Schmidt process

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \frac{(-1)(2) + (1)(0) + (-2)(1)}{(2)^2 + (0)^2 + (1)^2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \frac{-4}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 1 \\ -6/5 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - \frac{(1)(2) + (3)(0) + (0)(1)}{(2)^2 + (0)^2 + (1)^2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$- \frac{(1)(3/5) + (3)(1) + (0)(-6/5)}{(3/5)^2 + (1)^2 + (-6/5)^2} \begin{bmatrix} 3/5 \\ 1 \\ -6/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{9}{7} \begin{bmatrix} 3/5 \\ 1 \\ -6/5 \end{bmatrix} = \begin{bmatrix} -4/7 \\ 12/7 \\ 8/7 \end{bmatrix}$$

So $\quad H = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 1 \\ -6/5 \end{bmatrix}, \begin{bmatrix} -4/7 \\ 12/7 \\ 8/7 \end{bmatrix} \right\}$.

<u>Note:</u> It is evident that in the previous example, $H = \mathbb{R}^3$. So $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is also an orthogonal basis for $H$. We didn't point this out in the problem as we wanted to apply the Gram-Schmidt process.