$A = (a_{ij})_{n\times n}$ $x = (x_1, x_2, \ldots, x_n)^T$ $b = (b_1, b_2, \ldots, b_n)^T$

Gauss

$A^{(1)}x = b^{(1)}$ 2 n x_1 $A^{(2)}x = b^{(2)}$

$a^{(1)}_{11}x_1 + a^{(1)}_{12}x_2 + \ldots + a^{(1)}_{1n} = b^{(1)}_1$

$a^{(2)}_{22}x_2 + \ldots + a^{(2)}_{2n}x_n = b^{(2)}_2$

$\ldots\ldots\ldots\ldots$

$a^{(2)}_{n2}x_2 + \ldots + a^{(2)}_{nn}x_n = b^{(2)}_n$

3 n x_2 $A^{(3)}x = b^{(3)}$ $n-1$ $A^{(n)}x = b^{(n)}$

$a^{(1)}_{11}x_1 + a^{(1)}_{12}x_2 + a^{(1)}_{13}x_3 + \ldots + a^{(1)}_{1n} = b^{(1)}_1$

$a^{(2)}_{22}x_2 + a^{(2)}_{23}x_3 + \ldots + a^{(2)}_{2n}x_n = b^{(2)}_2$

$a^{(3)}_{33}x_3 + \ldots + a^{(3)}_{3n}x_n = b^{(3)}_2$

$\ldots\ldots\ldots\ldots$

$a^{(2)}_{nn}x_n = b^{(n)}_n$

$$\{\ l_{ik} = a^{(k)}_{ik}/a^{(k)}_{kk}(i = k+1, k+2, \ldots, n)a^{(k+1)}_{ij} = a^{(k)}_{ij} - l_{ik}a^{(k)}_{kj}(i, j = k+1, k+2, \ldots, n)b^{(k+1)}_i = b^{(k)}_i - l_{ik}b^{(k)}_k(i = k+1, k+2, \ldots$$

(1)

$a^{(n)}_{nn} \neq 0$

$$\{\ x_n = b^{(n)}_n/a^{(n)}_{nn}x_k = [b^{(k)}_k - \sum_{j=k+1}^{n} a^{(k)}_{kj}x_j]/a^{(k)}_{kk}(k = n-1, n-2, \ldots, 1)$$

(2)

$a^{(i)}_{ii} \neq 0 (i = 1, 2, \ldots, n-1)$ A

Theorem .1 $a^{(i)}_{ii} \neq 0\ (i = 1, 2, \ldots, n-1)$ A $D_i \neq 0\ (i = 1, 2, \ldots, k;\ k0)$. D_k

$$D_k = a^{(1)}_{11} \cdots a^{(1)}_{1k} \quad a^{(k)}_{kk} = a^{(1)}_{11}a^{(2)}_{22} \cdots a^{(k)}_{kk}$$

(3)

$AD_i \neq$

$a_{ii}^{(i)} \neq$
$0(i =$
$1, 2, \ldots, n)$
A
$|a_{i_1,j_1}| =$
$\max_{1i,jn} |a_{ij}| \neq$
0
$a_{i_1,j_1} n-$
1
$_{11} a_{12} \cdots a_{1n}$
$a_{21} \cdots a_{2n}$

a_{nn}
1
y_2

y_n
1
b_2

b_n
$y_1, y_2, \ldots, y_n x_1, x_2, \ldots, x_n$
$x_1, x_2, \ldots, x_n$
$_{11} a_{12} \cdots a_{1n}$
$a_{21} \cdots a_{2n}$

a_{nn}
1
x_2

x_n
1
b_2

b_n

A

$A = LU$

(4)

$LU n A D_i \neq 0 (i = 1, 2, \ldots, n-1) A U$

$A A^{(1)} b b^{(1)}$

$A^{(2)} = L_1^{-1} A^{(1)}, b^{(2)} = L_1^{-1} b^{(1)}, L_1^{-1} =$

1

$-l_{21} 1$

$-l_{31} 0 1$

$-l_{n1} 0 0 \cdots 1$

$, l_{i1} = \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} (i = 2, 3, \ldots, n)$

$n-1$

$\{ L_{n-1}^{-1} L_{n-2}^{-1} \cdots L_2^{-1} L_1^{-1} A^{(1)} = A^{(n)} = U$

$L_{n-1}^{-1} L_{n-2}^{-1} \cdots L_2^{-1} L_1^{-1} b^{(1)} = b^{(n)}$

$L = L_1 L_2 \ldots L_n LU = A^{(1)} = A$

$l_{21} 1$

$l_{31} l_{32} 1$

$l_{n1} l_{n2} \cdots l_{n,n-1} 1$

$u_{11} u_{12} u_{13} \cdots u_{1n}$

$u_{22} u_{23} \cdots u_{2n}$

$u_{33} \cdots u_{3n}$

u_{nn}

$Ax = LUx = b$

$$\{ \; Ly = b Ux = y, \{ \; y_1 = b_1 y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j, \{ \; x_n = y_n / u_{nn} x_i = (y_i - \sum_{j=i+1}^{n} u_{ij} x_j)$$

(5)

LU $\Leftrightarrow$
$D_i \neq 0 (i = 1, 2, \ldots, n-1)$

Eg.1

$\Rightarrow D_1 = 1, D_2 = 0$

$A^{(2)} =$
123
00−2
0−1−3

LU

Eg.2

$\Rightarrow D_1 = 1, D_2 = 0$

$A^{(2)} =$
123
00−2
00−3
$= U$

LU

Eg.3

$\Rightarrow D_1 = 1, D_2 = -1$

$A \longrightarrow$
123
0−1−2
0−2−3
$2 \longrightarrow$ 123
0−1−2
001
$= U$

LU

$u_{ii} u_{ii}$

Theorem .2

A P 1 L U

$PA = LU$

Theorem .3

n A

$$A = LDL^T \tag{6}$$

L D 0.

AD_iU

$_{21} 1$

$l_{n1} l_{n2} \cdots 1$

$_{11}$

u_{22}

u_{nn}

$_{12}/u_{11} \cdots u_{1n}/u_{11}$

$1 \cdots u_{2n}/u_{22}$

1

$= LDU_0$

$A = A^T_0 = U_0^T DL^T$

LU

$^T_0 = L, L^T = U_0$

$AA = LDL^T$

$\bar{D} = \mathrm{diag}(\sqrt{u_{11}}, \sqrt{u_{22}}, \ldots, \sqrt{u_{nn}})$

$\bar{D}^2 L^T = (L\bar{D})(L\bar{D})^T \equiv GG^T$

G **Cholesky**

Theorem .4

n A

$$A = GG^T \tag{7}$$

G G

$G = (g_{ij})_{n\times n}$

$k = 1, 2, \ldots, n$

$$\{\ a_{kk} \leftarrow g_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} g_{kj}^2} a_{kj} \leftarrow g_{kj} = \frac{a_{kj} - \sum_{i=1}^{j-1} g_{ki} g_{ji}}{g_{jj}} (k = j+1, j+2, \ldots, n) \tag{8}$$

Theorem .5

$Ax = b$ A

$Ax = g$

$$b_1 c_1 a_2 b_2 c_2 \quad a_{n-1} b_{n-1} c_{n-1} a_n b_n x_1 x_2 x_{n-1} x_n = g_1 g_2 g_{n-1} g_n \tag{9}$$

A

$$\{\ |b_1| > |c_1|\ |b_i| > |a_i| + |c_i|\ |b_n| > |a_n|\ (i = 1, 2, \ldots, n-1) \tag{10}$$

$A Ax = g$

LU

$_1 c_1$

$a_2 b_2 c_2$

$a_{n-1} b_{n-1} c_{n-1}$

$a_n b_n$

$_2 1$

$|x-x_0| x \to x_0$ $Ax = b$ Ab

Def $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}$ $N(x) = \|x\|$

$\|x\| 0, \|x\| = 0 \Leftrightarrow x = 0 \in \mathbf{R}^n$

$\forall \alpha \in \mathbf{R}, \|\alpha x\| = |\alpha| \|x\|$

$\forall y = (y_1, y_2, \ldots, y_n)^T \in \mathbf{R}^n, \|x + y\| \|x\| + \|y\|$

$N(x) = \|x\|$ x $\mathbf{R}^n$

$\|x\|_1 = \sum_{i=1}^{n} |x_i| (1-)$

$\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (2-)$

$\|x\|_\infty = \max_{1in} \{|x_i|\} (\infty$

$\|\cdot\|_s, \|\cdot\|_t$ $\mathbf{R}^n$ x x $m, M > 0$

$$m\|x\|_s \|x\|_t M \|x\|_s, \forall x \in \mathbf{R}^n. \quad (12)$$

def $\{x^{(k)}\}$ $\mathbf{R}^n$ $x^* \in \mathbf{R}^n$ $\lim_{k\to\infty} x_j^{(k)} = x_j^* (j = 1, 2, \ldots, n)$ $\{x^{(k)}\}$ x^* **th** $\lim_{k\to\infty} x^{(k)} = x^* \Leftrightarrow \lim_{k\to\infty} \|x^{(k)} - x^*\| = 0$

Def $A \in$
$\mathbf{R}^{n\times n} N(A) =$
$\|A\|$
$\|A\| 0, \|A\| =$
$0 \Leftrightarrow$
$A =$
$0 \in$
$\mathbf{R}^{n\times n}$
$\forall \alpha \in$
$\mathbf{R}, \|\alpha A\| =$
$|\alpha| \|A\|$
$\forall A, B \in$
$\mathbf{R}^{n\times n}, \|A+$
$B\| \|A\|+$
$\|B\|$
$\forall A, B \in$
$\mathbf{R}^{n\times n}, \|AB\| \|A\| \|B\|$
$N(A) =$
$\|A\| \quad A$
$\mathbf{R}^{n\times n}$
$\|A\|_1 =$
$\max_{1 j n} \sum_{i=1}^{n} |a_{ij}| ()$

$\|A\|_\infty =$
$\max_{1 i n} \sum_{j=1}^{n} |a_{ij}| ()$

$\|A\|_2 =$
$\sqrt{\lambda_{\max}(A^T A)} ()$

$\|A\|_{\mathrm{F}} =$
$(\sum_{i,j=1}^{n} a_{ij}^2)^{\frac{1}{2}} (F)$
$A =$
$(a_{ij})_{n\times n}, \lambda_{\max}(A^T A) A^T A$
$\|Ax\|_s \|A\| \|x\|_s \forall A \in$
$\mathbf{R}^{n\times n}, x \in$
$\mathbf{R}^n$
def $\{A^{(k)}\} \mathbf{R}^{n\times n} A \in$
$\mathbf{R}^{n\times n}$
$\lim_{k\to\infty} a_{ij}^{(k)} =$
$a_{ij} (i, j =$
$1, 2, \ldots, n)$
$\{A^{(k)}\} A$ **th**
$\lim_{k\to\infty} A^{(k)} =$
$A \Leftrightarrow$
$\lim_{k\to\infty} \|A^{(k)} -$
$A\| =$
0