$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\substack{n\to\infty \\ \max\Delta x_k \to 0}} \sum_{k=1}^{n} \Delta x_k f(x_k)$$

(1)

$$I = \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

(2)

$\int_a^b f(x)\,\mathrm{d}x =$
$\lim_{\substack{n\to\infty \\ \max\Delta x_k \to 0}} \sum_{k=1}^{n} \Delta x_k f(x_k) \approx$
$\sum_{k=1}^{n} \Delta x_k f(x_k)$
$\approx \sum_{k=0}^{n} A_k f(x_k)()$

$mm+$

$1m$

$$\{\ f(x) = x^{\mu}, \sum_{k=0}^{n} A_k x_k^{\mu} = \frac{1}{\mu+1}(b^{\mu+1} - a^{\mu+1})(\mu m) f(x) = x^{m+1}, \sum_{k=0}^{n} A_k x_k^{m+1} \neq \frac{1}{m+2}(b^{m+2} - a^{m+2})$$

(3)

$$\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b \sum_{k=0}^{n} l_k(x) f(x_k)\,\mathrm{d}x = \sum_{k=0}^{n} [\int_a^b l_k(x)\,\mathrm{d}x] f(x_k) = \sum_{k=0}^{n} A_k f(x_k)$$
(4)

$$A_k = \int_a^b l_k(x)\,\mathrm{d}x$$

$$R(f) = \int_a^b [f(x) - L_n(x)]\,\mathrm{d}x = \int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_{n+1}(x)\,\mathrm{d}x \xi = \xi(x) \in [a,b]$$
(5)

Theorem .1
??

n
Newton-Cotes
$[a,b]$ n $h = \frac{b-a}{n}$ $x_k = a + kh (k = 0, 1, \ldots, n)$
$n =$
$\sum_{k=0}^{n} [\int_a^b l_k(x)\,\mathrm{d}x] f(x_k) = \sum_{k=0}^{n} [\int_a^b \prod_{j=0 j\neq k}^{n} \frac{x - x_j}{x_k - x_j}\,\mathrm{d}x] f(x_k)$
$= \sum_{k=0}^{n} [h \int_0^n \prod_{j=0 j\neq k}^{n} \frac{t-j}{k-j}\,\mathrm{d}t] f(x_k) = \sum_{k=0}^{n} [\frac{h}{\prod_{j=0 j\neq k}^{n} (k-j)} \int_0^n \prod_{j=0 j\neq k}^{n} (t-j)\,\mathrm{d}t] f(x_k)$
$= \sum_{k=0}^{n} [\frac{h(-1)^{n-k}}{k!(n-k)!} \int_0^n \prod_{j=0 j\neq k}^{n} (t-j)\,\mathrm{d}t] f(x_k)$
$= (b-a) \sum_{k=0}^{n} c_k^{(n)} f(x_k)$
Newton-Cotes $c_k^{(n)}$

$$c_k^{(n)} = \frac{h}{b-a} \int_0^n \prod_{j=0 j\neq k}^{n} \frac{t-j}{k-j}\,\mathrm{d}t = \frac{(-1)^{n-k}}{nk!(n-k)!} \int_0^n \prod_{j=0 j\neq k}^{n} (t-j)\,\mathrm{d}t$$
(6)

$$A_k = (b-a) c_k^{(n)}$$
(7)

$\mu = 0$ $\sum_{k=0}^{n} A_k = b - a$ $A_k = (b-a) c_k^{(n)}$

$$\sum_{k=0}^{n} c_k^{(n)} = 1$$
(8)

$c_k^{(n)}$ $u = n - t$

$$c_k^{(n)} = c_{n-k}^{(n)}$$
(9)

$x = a + th$

$[x_k, x_{k+1}] x_{k+\frac{1}{4}}, x_{k+\frac{1}{2}}, x_{k+\frac{3}{4}}$

$_n =$

$\frac{h}{90}[7f(a)+$

$32\sum_{k=0}^{n-1} f(x_{k+\frac{1}{4}})+$

$12\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}})$

$+$

$32\sum_{k=0}^{n-1} f(x_{k+\frac{3}{4}})+$

$14\sum_{k=1}^{n-1} f(x_k)+$

$7f(b)]$

$R_{C_n} =$

$I-$

$C_n =$

$-\frac{2(b-a)}{945}(\frac{h}{4})^6 f^{(6)}(\eta) \eta \in$

$[a, b]$

$I_n h \to$

0

$$\frac{I - I_n}{h^p} \to C(C)$$

$I_n p$

n

$h^2 = -\frac{1}{12}\sum_{k=0}^{n-1} h f''(\eta_k) h \to 0 \longrightarrow -\frac{1}{12}\int_a^b f''(x)\,\mathrm{d}x \Rightarrow \lim_{h\to 0}\frac{I-T_n}{h^2} = -\frac{1}{12}[f'(b)-f'(a)] \Rightarrow I-T_n \approx -\frac{h^2}{12}[f'(b)-f'(a)] I-S_n \approx -\frac{1}{180}(\frac{h}{2})^4[f'''(b)-f'''(a)] I-C_n \approx$

$\frac{1}{4}\frac{1}{16}\frac{1}{64}$

$[a,b] n T_n n+1 n [x_k, x_{k+1}] x_{k+\frac{1}{2}} = \frac{1}{2}(x_k + x_{k+1}) (k = 0, 1, \ldots, n-1)$

$k = \frac{h}{2}[f(x_k) + f(x_{k+1})]$

$\Rightarrow$

$T'_k = \frac{h}{4}[f(x_k) + 2f(x_{k+\frac{1}{2}}) + f(x_{k+1})]$

$2n = \frac{h}{4}\sum_{k=0}^{n-1}[f(x_k) + f(x_{k+1})] + \frac{h}{2}\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}})$

$=$

$\frac{1}{2}T_n + \frac{h}{2}\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}})$

$f''(x)[a,b](\eta_n \approx \eta_{2n})$

T_n

$$R_{T_{2n}} = \frac{I-T_n}{I-T_{2n}} = \frac{-\dfrac{b-a}{12}h^2 f''(\eta_n)}{-\dfrac{b-a}{12}(\dfrac{h}{2})^2 f''(\eta_{2n})} \approx 4 \Rightarrow I - T_{2n} \approx \frac{1}{3}(T_{2n} - T_n)() \Rightarrow I \approx T_{2n} + \frac{1}{3}(T_{2n} - T_n)() I \approx S_{2n} + \frac{1}{15}(S_{2n} - S_n) I \approx C_{2n} + \frac{1}{63}(C_{2n} - C_n)$$

$T_{2n} I \dfrac{1}{3}(T_{2n} - T_n)$

$|T_{2n} - T_n| < \varepsilon$

Romberg

$\dfrac{1}{3}(T_{2n}-T_n)T_{2n}T_{2n}\tilde{T}$

$$\tilde{T}=\frac{4}{3}T_{2n}-\frac{1}{3}T_n$$

$\eta =$
$\frac{4}{3}T_{2n}-$
$\frac{1}{3}T_n$

$\overrightarrow{\overrightarrow{C}}_n =$
$\frac{16}{15}S_{2n}-$
$\frac{1}{15}S_n$

$\overrightarrow{\overrightarrow{R}}_n =$
$\frac{64}{63}C_{2n}-$
$\frac{1}{64}C_n$
??
Romberg

$$I=\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x\varepsilon=10^{-7}$$

$f(0) =$
$1f(1) =$
0.8414710

$$T_2^0=\frac{1}{2}[f(0)+f(1)]=0.9207355$$

$\left|T_2^{10}-T_2^{9}\right|\varepsilon T_2^{10}=$
$0.94608312^{10}+$
12^3+
1
$kT_2^kS_2^{k-1}C_2^{k-2}R_2^{k-3}$

Richardson

Theorem .5

$f(x) \in C^\infty[a,b]$

$$T(h) = I + \alpha_1 h^2 + \alpha_2 h^4 + \ldots + \alpha_k h^{2k} + \ldots \tag{16}$$

α_k $(k = 1, 2, \ldots)$ h

??

$$T(\frac{h}{2}) = I + \frac{\alpha_1}{4} h^2 + \frac{\alpha_2}{16} h^4 + \frac{\alpha_3}{64} h^6 + \ldots \tag{17}$$

??

??

$T_1(h)$

$_1(h) = \frac{4}{3}T(\frac{h}{2}) - \frac{1}{3}T(h) = S_n(Simpson)$

$T_1(h) = I + \beta_1 h^4 + \beta_2 h^6 + \beta_3 h^8 + \ldots$

$T_2(h)$

$_2(h) = \frac{16}{15}T_1(\frac{h}{2}) - \frac{1}{15}T_1(h) = C_n(Cotes)$

$T_2(h) = I + \gamma_1 h^6 + \gamma_2 h^8 + \gamma_3 h^{10} + \ldots$

$T_3(h)$

$_3(h) = \frac{64}{63}T_2(\frac{h}{2}) - \frac{1}{63}T_2(h) = R_n(Romberg)$

$T_3(h) = I + \eta_1 h^8 + \eta_2 h^{10} + \eta_3 h^{12} \ldots$

$T_0(h) = T(h) m (m = 1, 2, \ldots)$

$$T_m(h) = \frac{4^m}{4^m - 1} T_{m-1}(\frac{h}{2}) - \frac{1}{4^m - 1} T_{m-1}(h) \tag{18}$$

$$T_m(h) = I + \delta_1 h^{2(m+1)} + \delta_2 h^{2(m+2)} + \ldots \tag{19}$$

Richardson $m = 3m4$

??

$f(x)[x_k, x_{k+1}]x_{k+\frac{1}{2}}$

$$f(x) = f_{x+\frac{1}{2}} + (x - x_{k+\frac{1}{2}})f'_{k+\frac{1}{2}} + \frac{(x - x_{k+\frac{1}{2}})^2}{2!}f''_{k+\frac{1}{2}} + \frac{(x - x_{k+\frac{1}{2}})^3}{3!}f^{(3)}_{k+\frac{1}{2}} + \cdots$$

$f(x_k)f(x_{k+1})[x_k, x_{k+1}]$

$$\frac{h}{2}[f(x_k) + f(x_{k+1})] = hf_{k+\frac{1}{2}} + \frac{h}{2!}(\frac{h}{2})^2 f''_{k+\frac{1}{2}} + \frac{h}{4!}(\frac{h}{2})^4 f^{(4)}_{k+\frac{1}{2}} + \cdots$$

$$T(h) = \frac{h}{2}\sum_{k=0}^{n-1}[f(x_k) + f(x_{k+1})] = h\sum_{k=0}^{n-1} f_{k+\frac{1}{2}} + \frac{h^3}{2! \times 2^2}\sum_{k=0}^{n-1} f''_{k+\frac{1}{2}} + \cdots$$

$[x_k, x_{k+1}]$

$$I = \int_a^b f(x)\,\mathrm{d}x = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(x)\,\mathrm{d}x = h\sum_{k=0}^{n-1} f_{k+\frac{1}{2}} + \frac{h^3}{3! \times 2^2}\sum_{k=0}^{n-1} f''_{k+\frac{1}{2}} + \cdots$$

$T(h)Ih\sum_{k=0}^{n-1} f_{k+\frac{1}{2}}$

$$T(h) = I + \frac{h^3}{2! \times 6}\sum_{k=0}^{n-1} f''_{k+\frac{1}{2}} + \frac{h^5}{4! \times 20}\sum_{k=0}^{n-1} f^{(4)}_{k+\frac{1}{2}} + \cdots$$

$f''(x)f(x)[a, b]$

$$f'(b) - f'(a) = h\sum_{k=0}^{n-1} f''_{k+\frac{1}{2}} + \frac{h^3}{3! \times 2^2}\sum_{k=0}^{n-1} f^{(4)}_{k+\frac{1}{2}} + \frac{h^5}{5! \times 2^4}\sum_{k=0}^{n-1} f^{(6)}_{k+\frac{1}{2}} + \cdots$$

$T(h)Ih\sum_{k=0}^{n-1} f''_{k+\frac{1}{2}}$

$$T(h) = I + \frac{h^2}{2! \times 6}[f'(b) - f'(a)] - \frac{h^5}{4! \times 30}\sum_{k=0}^{n-1} f^{(4)}_{k+\frac{1}{2}} + \cdots$$

$h\sum_{k=0}^{n-1} f^{(4)}_{k+\frac{1}{2}} h\sum_{k=0}^{n-1} f^{(6)}_{k+\frac{1}{2}}$

??

$$I_L = (b-a)f(a)$$

$$I_M = (b-a)f(\frac{a+b}{2})$$

$$I_R = (b-a)f(b)$$

$f(x) x =$
$a x =$
$b x = \frac{a+b}{2}$
$_0) +$
$f'(x_0)(x-$
$x_0) +$
$\frac{f''}{2}(x_0)(x-$
$x_0)^2 +$
$\dots$

$\int_a^b f(x)\,\mathrm{d}x \approx$
$(b-$
$a)f(a) +$
$f'(a)\int_a^b (x-$
$a)\,\mathrm{d}x =$
$(b-$
$a)f(a) +$
$\frac{f'(\xi)}{2}(b-$
$a)^2 \xi \in$
(a,b)

$\Rrightarrow$
$R_L =$
$\frac{f'(\xi)}{2}(b-$
$a)^2 \xi \in$
(a,b)

$\int_a^b f(x)\,\mathrm{d}x \approx$
$(b-$
$a)f(b) +$
$f'(b)\int_a^b (x-$
$b)\,\mathrm{d}x =$
$(b-$
$a)f(a) -$
$\frac{f'(\xi)}{2}(b-$
$a)^2 \xi \in$
(a,b)

$\Rrightarrow$
$R_R =$
$-\frac{f'(\xi)}{2}(b-$
$a)^2 \xi \in$
(a,b)

$\int_a^b f(x)\,\mathrm{d}x \approx$
$(b-$
$a)f(\frac{a+b}{2}) +$
$f'(\frac{a+b}{2})\int_a^b (x-$
$\frac{a+b}{2})\,\mathrm{d}x +$
$\frac{f''(\frac{a+b}{2})}{2}\int_a^b (x-$
$\frac{a+b}{2})^2\,\mathrm{d}x$

$=$
$(b-$
$a)f(\frac{a+b}{2}) +$
$\frac{f''(\xi)}{24}(b-$
$a)^3 \xi \in$
(a,b)

$\Rrightarrow$
$R_M =$

$\int_a^b \omega(x)f(x)\,\mathrm{d}x \approx$
$\sum\limits_{k=0}^{n} A_k f(x_k) =$
$\sum\limits_{k=0}^{n} [\int_a^b \omega(x) l_k(x)\,\mathrm{d}x] f(x_k)$

$R =$
$\int_a^b \omega(x)f(x)\,\mathrm{d}x -$
$\sum\limits_{k=0}^{n} A_k f(x_k) =$
$\int_a^b \omega(x)\frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)\,\mathrm{d}x$
$n+$
$1 n 2n+$
2**??**
$f(x) =$
$1, x, x^2, \ldots, x^{2n+1} 2n+$
1
??
$2n+$
1**??**
Gauss $x_k(k =$
$0, 1, \ldots, n) A_k$
Eg.
$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx$
$A_0 f(x_0)+$
$A_1 f(x_1) A_0, A_1 x_0, x_1$
$_0+$
A_1
$f(x) =$
$x, 0 =$
$A_0 x_0 +$
$A_1 x_1$
$f(x) =$
$x^2, \frac{2}{3} =$
$A_0 x_0^2 +$
$A_1 x_1^2$
$f(x) =$
$x^3, 0 =$
$A_0 x_0^3 +$
$A_1 x_1^3$
$f(x) =$
x^4
Gauss $f(x) \in$
$C[a,b]$ $f(x) \in$
$C^{2n+2}[a,b]$ **Gauss**

$$R = \int_a^b \omega(x)f(x)\,\mathrm{d}x - \sum_{k=0}^{n} A_k f(x_k) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\int_a^b \omega(x)\omega_{n+1}^2(x)\,\mathrm{d}x \xi \in (a,b)$$

(20)

$2n+$
$1 H_{2n+1}(x)$

$$R = \int_a^b \omega(x)[f(x) - H_{2n+1}(x)]\,\mathrm{d}x = \int_a^b \omega(x)\frac{f^{(2n+2)}(\eta)}{(2n+2)!}\omega_{n+1}^2(x)\,\mathrm{d}x$$

Theorem .6

x_k $(k = 0, 1, \dots, n)$ *Gauss* $[a, b]$ $n+1$ $\omega_{n+1}(x)$ n $p(x)$ $\omega(x)$

$$\int_a^b \omega(x)\omega_{n+1}(x)p(x)\,\mathrm{d}x = 0 \tag{21}$$

$f(x) = \omega_{n+1}(x)p(x)$ $2n+1$ $2n+1$

$$\int_a^b \omega(x)\omega_{n+1}(x)p(x)\,\mathrm{d}x = \sum_{k=0}^{n} A_k\omega_{n+1}(x_k)p(x_k) = 0$$

$2n+1$ $f(x)$ n $p(x), q(x)$

$$f(x) = p(x)\omega_{n+1}(x) + q(x)$$

$\int_a^b \omega(x)f(x)\,\mathrm{d}x = \int_a^b \omega(x)p(x)\omega_{n+1}(x)\,\mathrm{d}x + \int_a^b \omega(x)q(x)\,\mathrm{d}x$ $=$ $\int_a^b \omega(x)q(x)\,\mathrm{d}x = \sum_{k=0}^{n} A_k q(x_k)$ (n) $=$ $\sum_{k=0}^{n} A_k f(x_k)$

$2n+1$

$2n+1$ $f(x) = l_k^2(x)$ $\int_a^b \omega(x)l_k^2(x)\,\mathrm{d}x = \sum_{k=0}^{n} A_i l_k^2(x_i) = A_k > 0$

Gauss-Legendre
$[a, b] = [-1, 1], \omega(x) = 1 x_k n+1 p_{n+1}(x)$
$_{n+1}(x) = \frac{1}{(n+1)!2^{n+1}} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} (x^2 - 1)^{n+1}$

$A_k = \int_{-1}^{1} l_k(x) \, \mathrm{d}x = \frac{2}{(1-x_k^2)[p'_{n+1}(x_k)]^2} (k = 0, 1, \ldots, n)$

$R = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3} f^{(2n+2)}(\xi) \xi \in (-1, 1)$

Gauss-
$[a, b] = [-1, 1], \omega(x) = (1 - x^2)^{-\frac{1}{2}} x_k n+1$
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) \, \mathrm{d}x \approx \sum_{k=0}^{n} A_k f(x_k)$

$A_k = \frac{\pi}{n+1}, x_k = \cos(\frac{2k+1}{2n+2}\pi) (k = 0, 1, \ldots, n)$

$R = \frac{\pi}{2^{2n+1}(2n+2)!} f^{(2n+2)}(\xi) \xi \in (-1, 1)$

Gauss-Laguerre
$[a, b] = [0, +\infty), \omega(x) = \mathrm{e}^{-x} x_k n+1 L_{n+1}(x)$
$\int_{0}^{+\infty} \mathrm{e}^{-x} f(x) \, \mathrm{d}x \approx \sum_{k=0}^{n} A_k f(x_k)$

$L_{n+1}(x) = \mathrm{e}^{x} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} (x^{n+1} \mathrm{e}^{-x})$

$A_k = \frac{[(n+1)!]^2}{x_k [L'_{n+1}(x_k)]^2} (k = 0, 1, \ldots, n)$

$R = \frac{[(n+1)!]^2}{(2n+2)!} f^{(2n+2)}(\xi) \xi \in [0, +\infty)$

Gauss-Hermite
$(-\infty, +\infty), \omega(x) = \mathrm{e}^{-x^2} x_k n+1 H_{n+1}(x)$
$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} f(x) \, \mathrm{d}x \approx \sum_{k=0}^{n} A_k f(x_k)$

$H_{n+1}(x) = (-1)^{n+1} \mathrm{e}^{x^2} = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} (e^{-x^2})$

$A_k = \frac{2^{n+2}(n+1)!}{[H'_{n+1}(x_k)]^2} \sqrt{\pi} (k = 0, 1, \ldots, n)$

$f'(x_0)$
$_0+$
$h) =$
$f(x_0)+$
$hf'(x_0)+$
$\frac{h^2}{2!}f''(\xi)$

$\Rightarrow$
$f'(x_0) =$
$\frac{f(x_0+h)-f(x_0)}{h}+$
$O(h)$

$\Rightarrow$
$f'(x_0) \approx$
$\frac{f(x_0+h)-f(x_0)}{h}$

$f'(x_0) \approx$
$\frac{f(x_0)-f(x_0-h)}{h}$

$f'(x_0) \approx$
$\frac{f(x_0+h)-f(x_0-h)}{2h}$
hh
$_0\pm$
$h) =$
$f(x_0)\pm$
$hf'(x_0)+$
$\frac{h^2}{2!}f''(x_0)\pm$
$\frac{h^3}{3!}f'''(x_0)+$
$\frac{h^4}{4!}f^{(4)}(x_0)\pm$
$\cdots$

$\Rightarrow$
$G(h) =$
$\frac{f(x_0+h)-f(x_0-h)}{2h} =$
$f'(x_0)+$
$[\frac{h^2}{3!}f^{(3)}(x_0)+$
$\frac{h^4}{5!}f^{(5)}(x_0)+$
$\frac{h^6}{7!}f^{(7)}(x_0)+$
$\cdots]$

$$\{ G_1(h) = G(h) G_{m+1}(h) = \frac{4^m G_m(\frac{h}{2}) - G_m(h)}{4^m - 1} m = 1, 2, \ldots \tag{22}$$

$mf'(x)$
$'(x_0)-$
$G_{m+1}(h) =$
$O(h^{2(m+1)})$

$[a,b] n x_k = a + kh (k = 1, 2, \ldots, n), h = \dfrac{b-a}{n}$

$_{k+1})- f(x_{k-1}) = \int_{x_{k-1}}^{x_{k+1}} f'(x)\,\mathrm{d}x (k = 1, 2, \ldots, n-1)$

$_{k+1})- f(x_{k-1}) \approx \frac{h}{3}[f'(x_{k-1}) + 4f'(x_k) + f'(x_{k+1})] (k = 1, 2, \ldots, n-1)$

$m_k \approx f'(x_k) m_0 = f'(x_0), m_n = f'(x_n) n-1 n-1$

$$\{\ m_{k-1} + 4m_k + m_{k+1} = \frac{3}{h}[f(x_{k+1}) - f(x_{k-1})] (k = 1, 2, \ldots, n-1) m_0 = f'(x_0) m_n = f'(x_n) \tag{23}$$

Lagrange

$_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x) \xi \in (x_0, x_n)$

$\Rightarrow f'(x) = L_n'(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)' + \frac{\omega_{n+1}(x)}{(n+1)!}\frac{\mathrm{d}}{\mathrm{d}x} f^{(n+1)}(\xi)$

$f^{(n+1)(\xi)} \omega_{n+1}(x_k) = 0 x_k$

$$f'(x_k) - L_n'(x_k) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x_k)' \tag{24}$$

$|f'(x)|\, M, x \in (a, b)$

$|f'(x_k) - L_n'(x_k)|\, M \frac{(b-a)^n}{(n+1)!} n \to \infty \longrightarrow 0 \Rightarrow f'(x_k) \approx L_n'(x_k)$