Proof of EFT Optimality (by Contradiction)

Assume the input intervals (tasks) are sorted by finish time:

$$f_1 \le f_2 \le \cdots \le f_n$$

Let $G = (g_1, g_2, \ldots, g_k)$ be the set of intervals selected by the Greedy-EFT algorithm. Each g_i is chosen to finish as early as possible without overlapping with the previously selected intervals. Hence, G is sorted by finish time:

$$f_{g_1} \le f_{g_2} \le \cdots \le f_{g_k}$$

Goal: Prove that G is optimal (i.e., of maximum possible size).

Proof (by contradiction): Suppose G is not optimal. Let $\mathcal{O} = (o_1, o_2, \ldots, o_m)$ be an optimal solution with $m > k$, that agrees with G for as many intervals as possible (i.e., maximal overlap from the beginning of both solutions). Assume both G and $\mathcal{O}$ are sorted by finish time.

Let r be the first index where $g_r \neq o_r$ (i.e., $g_1 = o_1, \ldots, g_{r-1} = o_{r-1}$, but $g_r \neq o_r$).

We now construct a new solution $\mathcal{O}'$ defined as:

$$\mathcal{O}' = (o_1, \ldots, o_{r-1}, g_r, o_{r+1}, \ldots, o_m)$$

Since Greedy-EFT chose g_r over o_r, it must have an earlier finish time:

$$f_{g_r} \le f_{o_r}$$

Moreover, since g_r was selected to be compatible with g_{r-1} (which equals o_{r-1}), and $o_{r+1}, \ldots, o_m$ were compatible with o_r, then replacing o_r with g_r maintains feasibility: - g_r does not overlap with o_{r-1} (same as g_{r-1}) - $f_{g_r} \le f_{o_r}$ the rest of $\mathcal{O}$ remains non-overlapping

Thus, $\mathcal{O}'$ is also a feasible schedule with the same size m but agrees with G for r steps instead of $r - 1$ — a contradiction to the maximality of our original overlap.

Conclusion: Our assumption was false. Hence, the greedy algorithm produces an optimal solution.