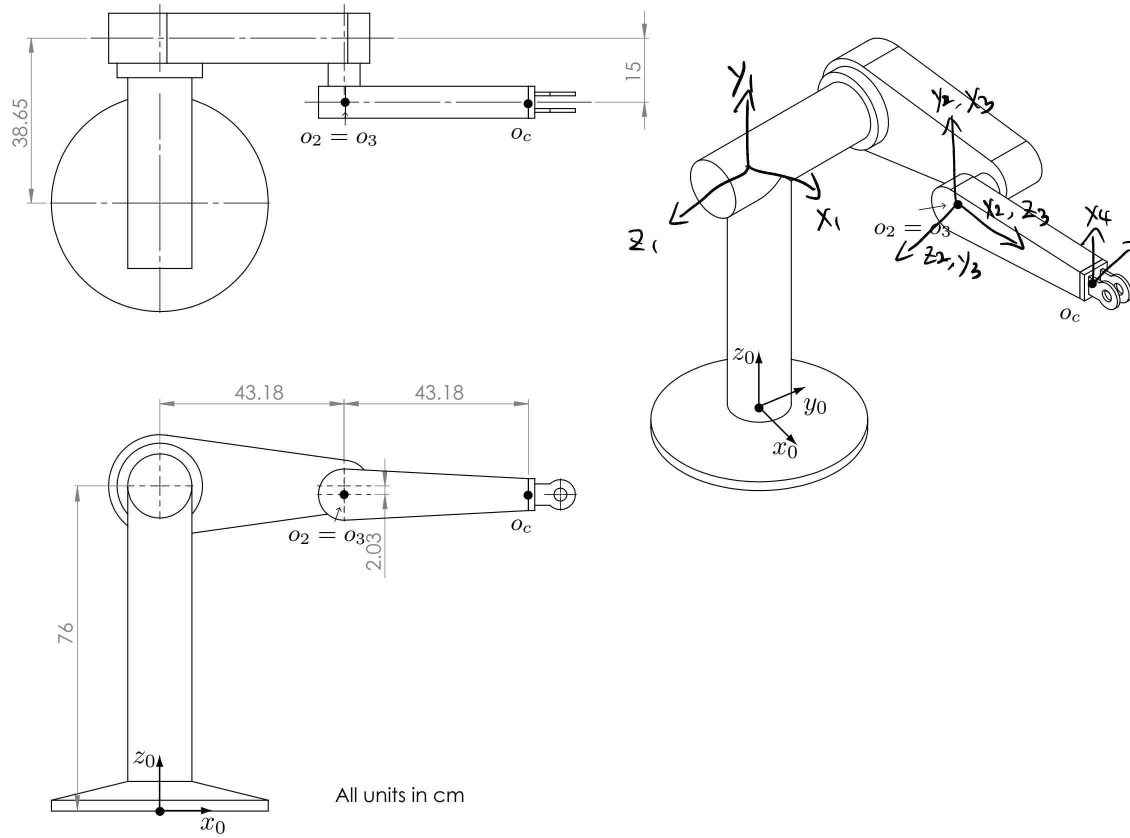


Figure 1: Schematics of the PUMA560 robot

By extracting the 3×3 matrix R_6^0 and the 3×1 vector o_6^0 from H_6^0, the forward kinematics computation is complete. Recall the geometric meaning of R_6^0 and o_6^0: the columns of R_6^0 are the unit axes of the end effector frame 6 represented in the coordinates of frame 0: $R_6^0 = [x_6^0 \mid y_6^0 \mid z_6^0]$; the vector o_n^0 is the origin of frame 6 expressed in the coordinates of frame 0.

2.2 Inverse Kinematics

Given a desired position $o_d^0 \in \mathbb{R}^3$ and desired orientation $R_d \in \mathrm{SO}(3)$ of the end effector, the inverse kinematics problem is to find $(\theta_1, \dots, \theta_6)$ such that $R_6^0(\theta_1, \dots, \theta_6) = R_d$ and $o_6^0(\theta_1, \dots, \theta_6) = o_d^0$.

Since the PUMA 560 has six links and a spherical wrist, one can solve the inverse kinematics problem by the technique of kinematic decoupling. In kinematic decoupling, the problem is divided in two parts: inverse position and inverse orientation.

Inverse position. The position of the wrist centre o_c only depends on the angles of the first three joints, $(\theta_1, \theta_2, \theta_3)$. The idea is to compute the desired location of the wrist centre and then find $(\theta_1, \theta_2, \theta_3)$ accordingly.

- Compute the vector

$$o_d^0 - R_d \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix}.$$

2.

Link	a_i	α_i	d_i	θ_i
1	0	$\frac{\pi}{2}$	d_1	θ_1^*
2	a_2	0	d_2	θ_2^*
3	0	$\frac{\pi}{2}$	0	θ_3^*
4	0	$-\frac{\pi}{2}$	d_4	θ_4^*
5	0	$\frac{\pi}{2}$	0	θ_5^*
6	0	0	d_6	θ_6^*

Link 1: $d_1 = 76$

Link 2: $a_2 = \sqrt{43.18^2 + 2.03^2} = 43.23$

$d_2 = -(38.65 - 15) = -23.65$

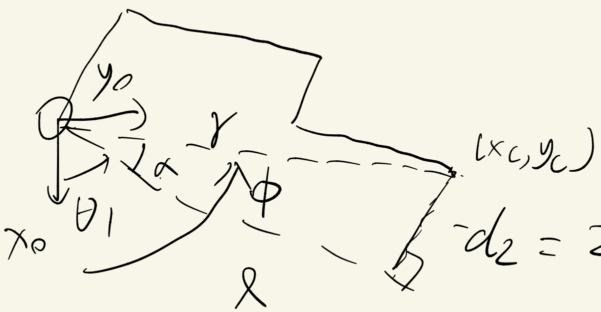
Link 4: $d_4 = 43.18$

Link 6: $d_6 = 20$

3.

$\theta_4 = \theta, \theta_5 = \phi, \theta_6 = \psi$

Find θ_1, Top view



$-d_2 = 23.65$

$r = \sqrt{x_c^2 + y_c^2}$

$\theta_1 = \phi - \alpha$

$\phi = \text{atan2}(y_c, x_c)$

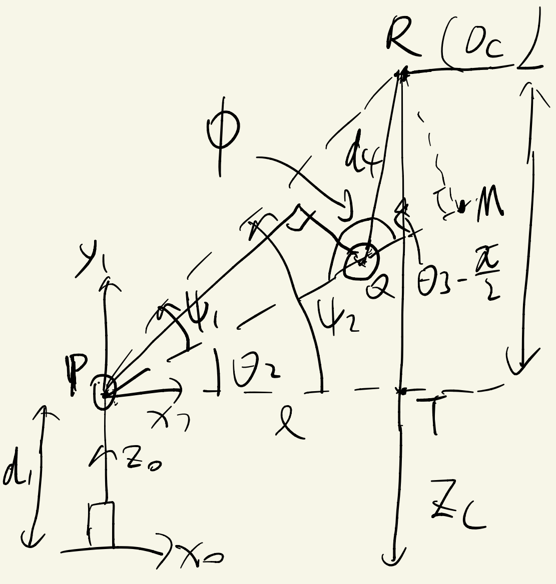
$\alpha = \text{atan2}(-d_2, l)$

$l^2 = r^2 - d_2^2$

$l = \sqrt{x_c^2 + y_c^2 - d_2^2}$

$\therefore \theta_1 = \text{atan2}(y_c, x_c) - \text{atan2}(-d_2, \sqrt{x_c^2 + y_c^2 - d_2^2})$

For θ_2, θ_3, Side view



$\theta_2 = \psi_2 - \psi_1$

$\phi + \theta_3 - \frac{\pi}{2} = \pi$

$\phi = \frac{3}{2}\pi - \theta_3$

$z_c - d_1 = s$ $\quad PQ = a_2$ from DH table

$RT = z_c - d_1 = s$

$PT = l$

$PT = \sqrt{x_c^2 + y_c^2 - d_2^2}$

$PR = \sqrt{(z_c - d_1)^2 + l^2}$

$$PR = \sqrt{(z_c - d_1)^2 + x_c^2 + y_c^2 - d_2^2}$$

$$PR = \sqrt{s^2 + \ell^2} = \sqrt{s^2 + (PT)^2}$$

$$s = (z_c - d_1), \quad \ell = PT = \sqrt{x_c^2 + y_c^2 - d_2^2}$$

Find θ_3 first:

$$PR^2 = d_4^2 + a_2^2 - 2 a_2 d_4 \cos(\phi)$$

$$s^2 + \ell^2 = d_4^2 + a_2^2 - 2 a_2 d_4 \cos\left(\frac{3}{2}\pi - \theta_3\right)$$

$$= d_4^2 + a_2^2 + 2 a_2 d_4 \sin(\theta_3)$$

$$\therefore \sin(\theta_3) = \frac{s^2 + \ell^2 - d_4^2 - a_2^2}{2 a_2 d_4}$$

$$= \frac{(z_c - d_1)^2 + x_c^2 + y_c^2 - d_2^2 - d_4^2 - a_2^2}{2 a_2 d_4}$$

$$\therefore \sin(\theta_3) = D$$

$$\therefore \theta_3 = \operatorname{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

Find $\theta_2 = \psi_2 - \psi_1$

$$\sin\left(\theta_3 - \frac{\pi}{2}\right) = \frac{RM}{d_4}$$

$$\therefore RM = d_4 \sin\left(\theta_3 - \frac{\pi}{2}\right)$$

$$\cos\left(\theta_3 - \frac{\pi}{2}\right) = \frac{QM}{d_4}$$

$$QM = d_4 \cos\left(\theta_3 - \frac{\pi}{2}\right)$$

$$\therefore \psi_1 = \operatorname{atan2}\left(d_4 \sin\left(\theta_3 - \frac{\pi}{2}\right), a_2 + d_4 \cos\left(\theta_3 - \frac{\pi}{2}\right)\right)$$

$$\psi_2 = \operatorname{atan2}(z_c - d_1, l)$$

$$= \operatorname{atan2}\left(z_c - d_1, \sqrt{x_c^2 + y_c^2 - d_2^2}\right)$$

$$\therefore \theta_2 = \psi_2 - \psi_1$$

$$= \operatorname{atan2}\left(z_c - d_1, \sqrt{x_c^2 + y_c^2 - d_2^2}\right) -$$

$$\operatorname{atan2}\left(d_4 \sin\left(\theta_3 - \frac{\pi}{2}\right), a_2 + d_4 \cos\left(\theta_3 - \frac{\pi}{2}\right)\right)$$

$$R_d = R_6^0 = R_3^0 R_6^3 = R_3^0(\theta_1, \theta_2, \theta_3)\, R_6^3(\theta_4, \theta_5, \theta_6)$$

$$R_6^3(\theta_4, \theta_5, \theta_6) = (R_3^0(\theta_1, \theta_2, \theta_3))^T R_d$$

$$\theta_1 = \operatorname{atan2}(y_c, x_c) - \operatorname{atan2}\left(-d_2, \sqrt{x_c^2 + y_c^2 - d_2^2}\right)$$

$$\theta_2 = \operatorname{atan2}\left(z_c - d_1, \sqrt{x_c^2 + y_c^2 - d_2^2}\right) - \operatorname{atan2}\left(d_4 \sin(\theta_3 - \tfrac{\pi}{2}), a_2 + d_4 \cos(\theta_3 - \tfrac{\pi}{2})\right)$$

$$\theta_3 = \operatorname{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

$$D = \frac{(z_c - d_1)^2 + x_c^2 + y_c^2 - d_2^2 - d_4^2 - a_2^2}{2 a_2 d_4}$$

$$H_i^{i-1} = \begin{bmatrix} C_{\theta i} & -S_{\theta i} C_{\alpha i} & S_{\theta i} S_{\alpha i} & a_i S_{\theta i} \\ S_{\theta i} & C_{\theta i} C_{\alpha i} & -C_{\theta i} S_{\alpha i} & a_i S_{\theta i} \\ 0 & S_{\alpha i} & C_{\alpha i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

if $S_\theta > 0$

$$\begin{cases} \theta_4 = \operatorname{atan2}\left(\sqrt{1 - a_{33}^2}, a_{33}\right) \\ \theta_5 = \operatorname{atan2}(a_{23}, a_{13}) \\ \theta_6 = \operatorname{atan2}(a_{32}, -a_{31}) \end{cases}$$

if $S_\theta < 0$

$$\begin{cases} \theta_4 = \operatorname{atan2}\left(-\sqrt{1 - a_{33}^2}, a_{33}\right) \\ \theta_5 = \operatorname{atan2}(-a_{23}, -a_{13}) \\ \theta_6 = \operatorname{atan2}(-a_{32}, a_{31}) \end{cases}$$