Vrije Universiteit Amsterdam

E_EOR3_FTR

Financial Engineering

Assignment part 2

Authors:
Daniël van Hanswijk (2726843)
Mark de Kwaasteniet (2649271)
Martijn van Welsenes (2713315)
Sem Wierdsma (2670919)

Tutor:
Paolo Gorgi

May 21, 2021

Contents

1 volatility modeling with SV and indirect inference **1**

1.1 Estimate by indirect inference an SV model for the log-returns of the S&P500 index . . . 1

1.2 For the model estimated in Question 1 Obtain and plot the filtered volatility and compare with the GARCH(1,1) model. . . . 2

1.3 Estimate the SV model by indirect inference using the sample mean and autocovariance function as set of auxiliary statistics. . . . 4

1.4 Filtered Volatility with Different Auxiliary Statistics . . . 4

1.5 Difference in Auxiliary Statistics . . . 6

1.6 Estimation of an SV-AR(2) Model . . . 7

2 Dynamic regression and CAPM **8**

2.1 Estimate the betas of the CAPM model for each of the three assets using OLS. 8

2.2 Comment on the statement of a colleague . . . 9

2.3 Estimate by ML a CAPM model with an observation-driven dynamic coefficient β_t for each of the assets. . . . 10

2.4 Exposition obtained from CAPM model . . . 10

2.5 Time-varying β_t from CCC model . . . 11

2.6 Parameter-driven dynamic regression . . . 12

1 volatility modeling with SV and indirect inference

We are interested in estimating the parameters of a Stochastic Volatility model as an alternative modeling approach to GARCH models. Consider the S&P500 index contained in the file S&P500.txt to answer the following questions.

1.1 Estimate by indirect inference an SV model for the log-returns of the S&P500 index

We are going to estimate by indirect inference an SV model for the log-returns of the S&P 500 index. By making use of the sample variance of y_t, sample kurtosis of y_t and autocorrelation of absolute log returns $|y_t|$ as auxiliary statistics and considering $H = 30T$. To do this we are first going to start with the observation (1) and transition equations (2) of the parameter-driven Stochastic Volatility model.

$$y_t = \sigma_t \epsilon_t, \tag{1}$$

where $(\epsilon_t)_{t\in\mathbb{Z}}$ is an $NID(0, 1)$ random sequence. The peculiarity of the SV model comes from the specification of the transition equation.

$$\sigma_t^2 = \exp(f_t), \quad f_t = \omega + \beta f_{t-1} + \eta_t \tag{2}$$

where $(\eta_t)_{t\in\mathbb{Z}}$ is an $NID(0, \sigma^2)$ and independent of $(\epsilon_t)_{t\in\mathbb{Z}}$. The ω, β and σ_η^2 determine certain moments of y_t. Namely the unconditional variance of y_t, unconditional kurtosis and the autocovariance in squared log-returns $|y_t|$. The variance (3), kurtosis (4) and autocorrelation of absolute log-returns (5) are specified by the following equations.

$$\sigma_f^2 = \mathbb{V}ar(f_t) \tag{3}$$

$$k_u = \frac{\mathbb{E}\left(y_t^4\right)}{\mathbb{E}\left(y_t^2\right)^2} \tag{4}$$

$$\rho_f(l) = \mathbb{C}ov(f_t, f_{t-l})/\mathbb{V}ar(f_t) \tag{5}$$

The sample variance, kurtosis and autocorrelation give the following values $(5.474, 10.605, 0.305)$. The results show that the unconditional distribution of y_t generated by the SV model is not normal distributed because the kurtosis of y_t is bigger than 3.

Next we choose the length of the H value that will be used to obtain the simulated moments. In this case is the H value: $H = 30T$. Where T is the sample size. We also generate the errors matrix e that contains $H \times 2$ normal random draws with $N(0, 1)$

At the third step we are going to choose the initial parameter values for the numerical optimization which are $(\omega_0, \beta_0, \sigma_{0\eta}^2) = (\omega_0,\ 0.9,\ 0.1)$. To obtain the initial value of ω_0 we can use the following equitation.

$$\omega_0 = \log(\mathbb{V}ar(f_t)) * (1 - \beta) \tag{6}$$

Furthermore, we need to minimize the criterion function by making use of the maximum likelihood function. In order to obtain the true parameter vector θ_0, we need to find the value of θ that makes the vector of auxiliary statistics of the observed data ($\hat{B_T}$) as close as possible to the vector of the auxiliary statistics of the simulated data ($\tilde{B}_H(\theta)$). This will result in the following function

$$\hat{\theta}_{TH} = \arg\min d\big(\hat{B_T}, \tilde{B}_H(\theta)\big) \tag{7}$$

where $d\big(\hat{B_T}, \tilde{B}_H(\theta)\big)$ denotes the quadratic distance between the two vectors of auxiliary statistics.

The last step is estimating the parameters. This give the following estimates:

$$(\hat{\omega}, \hat{\beta}, \hat{\sigma_\eta^2}) = (0.0604, 0.9365, 0.1857) \tag{8}$$

1.2 For the model estimated in Question 1 Obtain and plot the filtered volatility and compare with the GARCH(1,1) model.

In this section we obtain and plot the filtered volatility. Also we compare the filtered volatility obtained from the SV model with the conditional variance σ_t^2 estimated from a GARCH(1,1) model.

To obtain the filtered estimate of f_t we use a for loop minimizing the filter function at each time period. We do this by making use of the following equation.

$$\hat{f}_t = \arg\min \left\{ y_t^2 \exp(-f_t) + 3f_t + \frac{(f_t - \omega - \beta f_{t-1})^2}{\sigma_\eta^2} \right\} \tag{9}$$

With $t = 1, ..., T$ and with a given initial value of f_1. The conditional variance from a GARCH(1,1) model is estimated by making use of the GARCH equations,

$$y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad \forall t \in \mathbb{Z}, \tag{10}$$

Where $\omega > 0, \alpha_1 \geq 0, \beta_1 \geq 0$ are the parameters to be estimated. The estimates are found by optimizing the likelihood function. For the GARCH(1,1) the simplified log-likelihood is.

$$L(y_1, ..., y_T, \theta) = \sum_{t=2}^{T} -\frac{1}{2} \left(\log \sigma_t^2 + \frac{y_t^2}{\sigma_t^2} \right), \tag{11}$$

Where the summations starts from $t = 2$ because the updating equation for σ_t^2 goes back one time step.

To assure that the parameters remain positive during the optimization we use the logarithm link function to give the optimization function the logarithm of the parameters and let the function calculate the exponent again. This way the optimizer can give any value to the function and the final results simply need to be exponentiated (with the link function) again to obtain the proper estimates.

For the S&P500 data the initial values $\theta_0 = (\omega_0, \alpha, \beta) = (\omega_0, 0.01, 0.9)$ are used. To obtain the initial value of ω we can use the unconditional variance, because the unconditional variance of a GARCH(1,1) model is given by

$$\mathbb{V}\text{ar}\,(y_t) = \omega / \,(1 - \alpha - \beta)\,. \tag{12}$$

Inverting this equation gives an initial value of $\omega_0 = \mathbb{V}\text{ar}(y_t)(1 - \alpha - \beta)$. Furthermore, we initialize the updating equation with values for σ_1. these variances can be set to the unconditional variance of the log-returns, $\mathbb{V}\text{ar}(y_t)$.

Doing the optimization gives the following parameter estimates

$$\left(\hat{\omega}_1, \hat{\alpha}_1, \hat{\beta}_1\right) \approx (0.2523, 0.2054, 0.7595)\,. \tag{13}$$

The estimated conditional variance can now be calculated by using the found parameter estimates and plugging them into the updating equation (10). The results are shown in Figure (1). As shown in the figures we can clearly see clusters of volatility in the simulated time series. Whenever the log-returns fluctuate, the filtered stochastic volatility has a peak. The GARCH models volatility shows significantly higher peaks compared to the volatility generated by the indirect inference SV model.

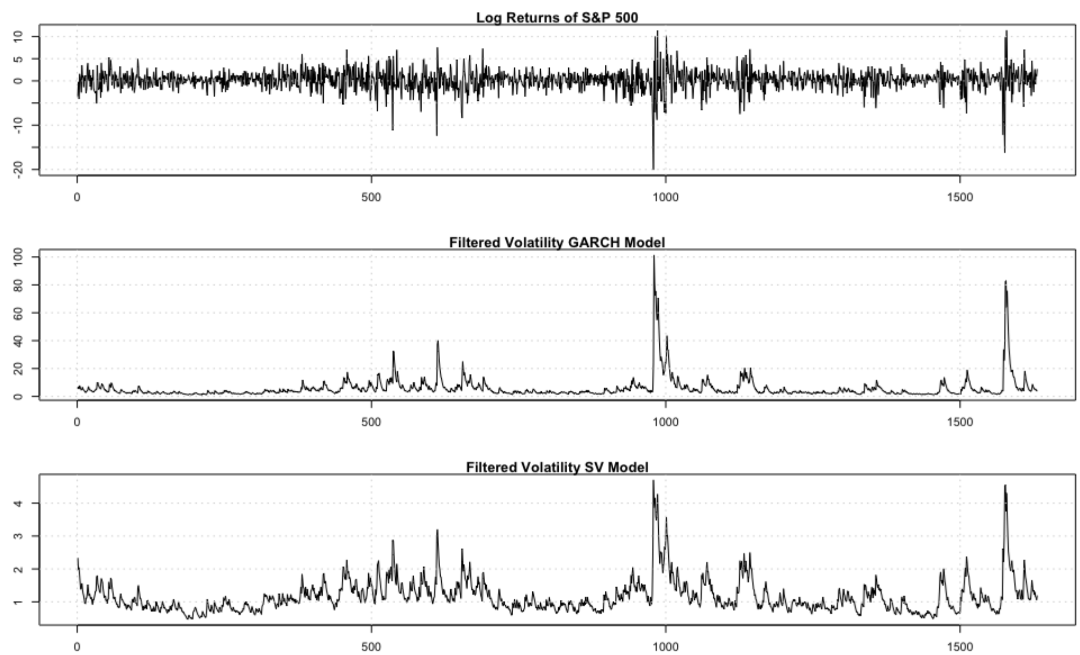

Figure 1: The filtered volatility of both the SV and the GARCH model.

1.3 Estimate the SV model by indirect inference using the sample mean and autocovariance function as set of auxiliary statistics.

In this section we will estimate the SV model by indirect inference using the sample mean and autocovariance function as auxiliary statistics. To estimate the SV model by indirect inference we first obtain the sample mean of $|y_t|$.

$$\hat{s} = T^{-1} \sum_{t=1}^{T} |y_t| \tag{14}$$

The next step is to obtain the autocovariance function. This is done by the following equation with a lag of 15 of the autocovariance function.

$$\gamma_{y_t} = T^{-1} \sum_{t=1}^{T} (|y_t| - \hat{s})(|y_{t-l}| - \hat{s}), \quad l = 0, 1, ..., 15. \tag{15}$$

Next we choose the length of the H value that will be used to obtain the simulated moments. In this case is the H value: $H = 30T$. Where T is the sample size. We also generate the errors matrix e that contains $H \times 2$ normal random draws with $N(0, 1)$

At the fourth step we are choosing the initial parameter values for the numerical optimization which are $(\omega_0, \beta_0, \sigma^2_{0\eta}) = (\omega_0,\ 0.9,\ 0.1)$. To obtain the initial value of ω_0 we can use the following equitation.

$$\omega_0 = \log(\mathbb{V}ar(f_t)) * (1 - \beta) \tag{16}$$

Furthermore, we to minimize the criterion function by making use of the maximum likelihood function. In order to obtain the true parameter vector θ_0, we need to find the value of θ that makes the vector of auxiliary statistics of the observed data $(\hat{B_T})$ as close as possible to the vector of the auxiliary statistics of the simulated data $(\tilde{B}_H(\theta))$. This will result in the following function

$$\hat{\theta}_{TH} = \arg\min d\big(\hat{B_T}, \tilde{B}_H(\theta)\big) \tag{17}$$

Next we need to obtain the parameters estimates using the link function. The last step is estimating the parameters. This give the following estimates:

$$(\hat{\omega}, \hat{\beta}, \hat{\sigma^2_\eta}) = (0.0774, 0.9301, 0.1568) \tag{18}$$

These values are very similar to the model estimated with the aforementioned auxiliary statistics. We see that the β has decreased slightly compared with the values before.

1.4 Filtered Volatility with Different Auxiliary Statistics

We would like to estimate the filtered path of the time-varying volatility, σ_t^2. In order to obtain the time-varying volatility of the models, we will be using the approximate Maximum

Likelihood filter. Using the same sequential maximization procedure from Section 1.2. In this procedure we want to find the values for σ_t^2 that maximizes the following probability:

$$p(y_t, \sigma_t^2 | \sigma_{t-1}^2) \tag{19}$$

where y_t denotes the log-returns of the S&P500. Which will result in the following log-likelihood function:

$$\log p(y_t, \sigma_t^2 | \sigma_{t-1}^2) = \log p(y_t | \sigma_{t-1}^2) + \log p(\sigma_t^2 | \sigma_{t-1}^2) \tag{20}$$

where the conditional density $p(y_t|\sigma_t^2)$ is determined by the observation equation (1) which is normally distributed with $N(0, \sigma_t^2)$ and the density function $p(\sigma_t^2|\sigma_{t-1}^2)$is determined by transition equation (2) which is log-normally distributed with $\log -N(\omega + \beta \log \sigma_{t-1}^2, \sigma_\eta^2)$.

Since we can both maximize with respect to σ_t and f_t, because $f_t = \log(\sigma_t^2)$, we can use the following equation to estimate the values of f_t that minimize the following function.

$$\hat{f}_t = \arg\min \left\{ y_t^2 \exp(-f_t) + 3f_t + \frac{(f_t - \omega - \beta f_{t-1})^2}{\sigma_\eta^2} \right\} \tag{21}$$

Figure 2 shows the filtered volatility's of both the model. Model 1 is the model that is estimated using the sample variance of y_t, the sample kurtosis of y_t and first-order autocorrelation of absolute log-returns $|y_t|$ as auxiliary statistics. Model 2 is the model that is estimated using the sample mean of $|y_t|$ and 15 lags of the autocovariance function of $|y_t|$ as auxiliary statistics. The difference between the filtered volatility's of the two models is barely noticeable using eyeball econometrics. This is not surprising, since the $\hat{\theta}$ of both models were very similar. When we look at the parameter estimates, we see that the β of model 2 is slightly smaller than the β of model 1, resulting in a lower f_t and thus indicating that it is likely that the volatility of model 2 is slightly smaller compared to model 1.

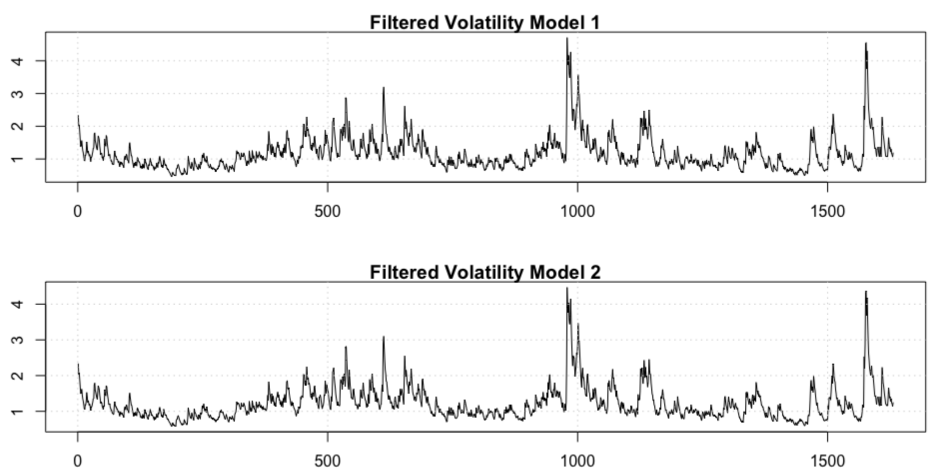

Figure 2: This figure show the filtered volatility of model 1 and model 2 using indirect inference to estimate the parameters.

1.5 Difference in Auxiliary Statistics

"Would you consider the auxiliary statistics proposed in Question 1 or the auxiliary statistics proposed in Question 2?"

We would consider the auxiliary statistics proposed in Question 1.3. The auxiliary statistics from Question 1.3, describe the autocovariance structure of the absolute log returns better than the auxiliary statistics of Question 1. If we look at the ACF and PACF in figure 3 of the squared log-returns of the S&P500 we see that there is autocorrelation present, therefore using the parameters of a AR(15) model provides us with a natural alternative compared to the raw moments of data, i.e. the sample variance, sample kurtosis and the first-order autocorrelation. A feature of good auxiliary statistics is that they have to be rich in the sense that they need to characterize the different values of the parameters. Thus, including 17 statistics in model 2, instead of 3 statistics in model 1, will probably result in a more accurate process since there are more quadratic distances between the auxiliary moments that need to minimized. Therefore, we think the auxiliary statistics from question 1.3 will describe the moments of y_t more precisely compared to the auxiliary statistics of question 1.

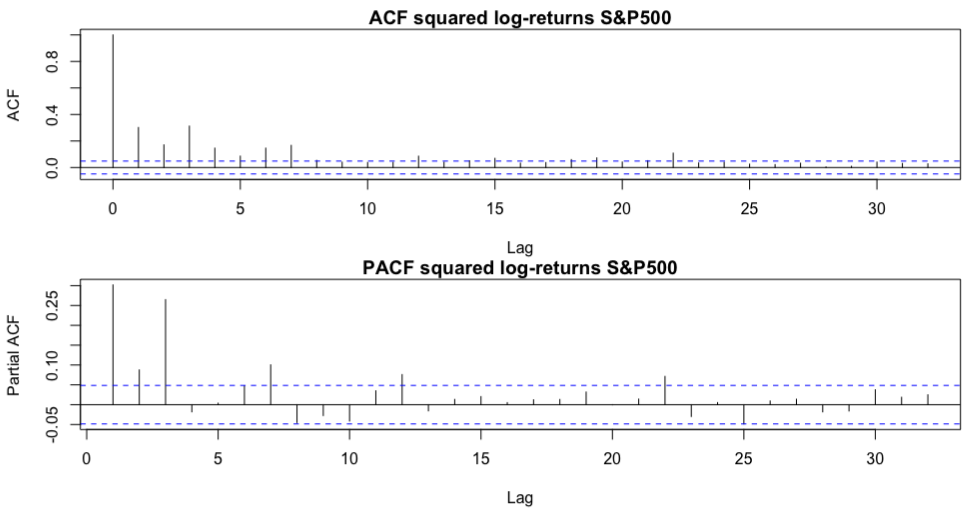

Figure 3: ACF and PACF of the squared log returns of the S&P500

1.6 Estimation of an SV-AR(2) Model

To estimate the SV-AR(2) model, we use the auxiliary statistics proposed in Question 1.3. We still consider $H = 30T$. In order to estimate the SV-AR(2) model by indirect inference, we need to specify the observation and transition equation. The observation equation will remain the same, i.e. observation equation 1 in section 1.1. The difference in estimating the SV-AR(2) model lies in the transition equation. Transition equation 2 will transform into a second order autoregressive equation, shown in equation 22.

$$f_t = \omega + \beta_0 f_{t-1} + \beta_1 f_{t-2} + \eta_t \tag{22}$$

In order to obtain the above mentioned parameters, we are going to compare two time series based on auxiliary statistics. The first set of data will contain the properties of the observed data, this will be compared with the second set of simulated data that is based on the very same auxiliary statistics. In order to obtain the true parameter vector θ_0, we need to find the value of θ that makes the vector of auxiliary statistics of the observed data $(\hat{B_T})$ as close as possible to the vector of the auxiliary statistics of the simulated data $(\tilde{B}_H(\theta))$. This will result in the following function

$$\hat{\theta}_{TH} = \arg\min d\big(\hat{B_T}, \tilde{B}_H(\theta)\big) \tag{23}$$

where $d\big(\hat{B_T}, \tilde{B}_H(\theta)\big)$ denotes the quadratic distance between the two vectors of auxiliary statistics.

By estimating the SV-AR(2) Model, we find the following parameters.

$$\hat{\theta}_{TH} = (\omega, \beta_0, \beta_1, \sigma^2_\eta) = (0.0874, 0.7580, 0.7596, 0.2150) \tag{24}$$

2 Dynamic regression and CAPM

Three risky assets have been identified as potential investment: Microsoft (MSFT), Bank of America (BAC), and Exxon Mobil (XOM).

You are expected to study the exposition of these three assets to the market risk. The weekly prices of the financial assets are in the files MSFT.txt, BAC.txt and XOM.txt. You can use the S&P500 as market proxy. The weekly S&P500 index is in the dataset market.txt. Furthermore, you can assume that the risk free rate is equal to zero $r^f = 0$. *Use these datasets to answer the following questions.*

2.1 Estimate the betas of the CAPM model for each of the three assets using OLS.

We want to estimate the betas of the Capital Asset Pricing Model (CAPM) for the assets: Microsoft (MSFT), Bank of America (BAC), and Exxon Mobil (XOM). The CAPM model describes how financial markets price assets, described by the following equation:

$$R_i = r^f + \beta_i(R^m - r^f) \tag{25}$$

Here R_i is the expected return of stock i, r^f is the risk-free rate, R^m is the expected market return, and β_i is the beta of stock i. Figure 4 shows the log-returns of the potential investment assets.

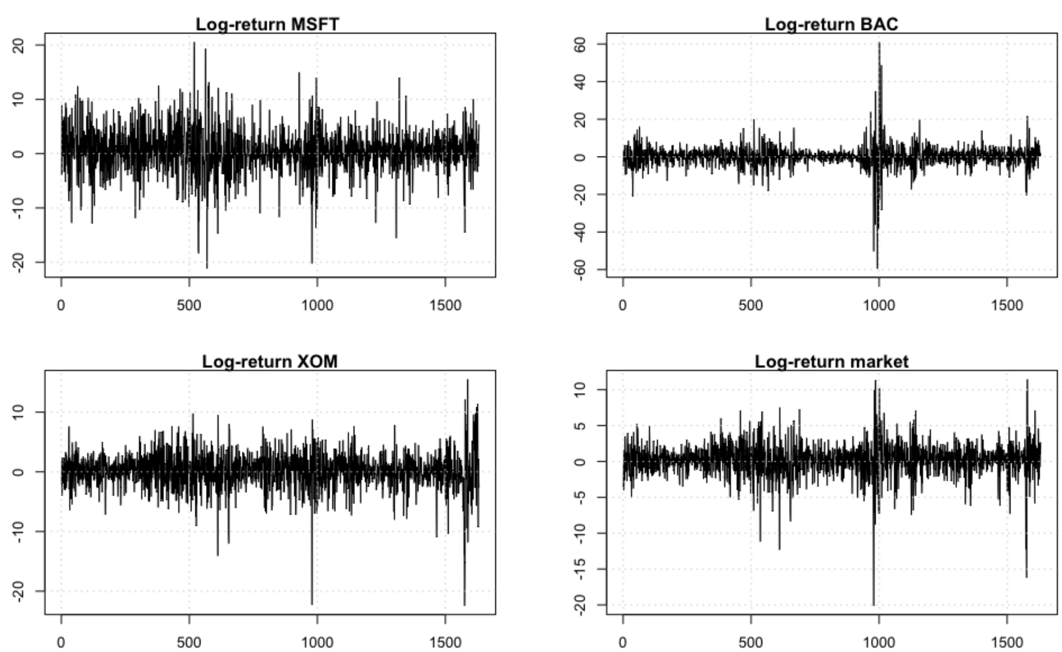

Figure 4: Log-returns of each asset and the market

With the assumption of $r^f = 0$, Equation (25) simplifies to:

$$R_i = \beta_i R^m \tag{26}$$

The beta value provides a measure of exposition to the systematic risk of a stock. If $\beta = 1$, then the stock is as risky as the market and therefore it has the same expected return; If $\beta < 1$, then the stock has less systematic risk than the market and therefore it has a lower expected return; And finally, if $\beta > 1$, then the stock has more systematic risk than the market and therefore it has a higher expected return.

We can estimate the value of β using Ordinary Least Squares (OLS). The estimation of β is then obtained by:

$$\hat{\beta}_{OLS} = \left(X^T X\right)^{-1} X^T y \tag{27}$$

Where X is the expected market return and y the expected return of an asset. Which in this case can be rewritten as:

$$\hat{\beta}_{OLS} = \frac{\mathbb{C}ov(r_t^m, r_{i,t})}{\mathbb{V}ar(r_t^m)} = \frac{\sigma_{m,i}}{\sigma_m^2} \tag{28}$$

Table 1 shows the estimations of β. From this table we see that the Bank of America has the largest measure of exposition to the systematic risk, thus being the most risky of the three. Exxon Mobil has the lowest value of β, even lower than 1, therefore being even less risky than the market. Microsoft has a beta value of approximately 1, thus being approximately as risky as the market.

	$\hat{\beta}_{OLS}$
MSFT	1.0067
BAC	1.5562
XOM	0.7932

Table 1: Estimations of β of the CAPM models for each asset using OLS

2.2 Comment on the statement of a colleague

A colleague made the following statement:

"The current portfolio of the bank has an high exposition to the market risk. Therefore we should be careful and invest on the asset with lowest exposition to the market risk".

The colleague suggests to invest in Exxon Mobile. From Section 2.1 we have seen that XOM indeed has the lowest exposition to the market risk of the three potential investments. Therefore, if the goal is to lower the overall exposition to the market risk of the portfolio, then investing in XOM is a good suggestion. This will, however, result in a lower expected return of the portfolio.

2.3 Estimate by ML a CAPM model with an observation-driven dynamic coefficient β_t for each of the assets.

We can also find an observation-driven dynamic coefficient β_t, which is time varying. A method is to estimate such a CAPM model using Maximum Likelihood (ML).

For the observation-driven regression we use the following updating equation of β_t:

$$\beta_t = \omega + \phi\beta_{t-1} + \alpha(y_{t-1} - \beta_{t-1}x_{t-1})x_{t-1} \tag{29}$$

where ω, ϕ and α, determine the dynamic properties of β_t. Using ML we estimate these properties as well as the conditional variance, σ^2. The ML estimator is then given by:

$$\hat{\theta}_T = \arg\max_{\theta\in\Phi} L_T(\theta) \tag{30}$$

Where $\theta = (\omega, \phi, \alpha, \sigma^2)^T$ and $L_T(\theta)$ is the log-likelihood function. The estimates of the the model parameters are shown in Table 2.

	$\hat{\omega}$	$\hat{\phi}$	$\hat{\alpha}$	$\hat{\sigma}^2$
MSFT	0.0073	0.9932	0.0012	11.1253
BAC	0.0244	0.9818	0.0027	17.2288
XOM	0.0157	0.9811	0.0020	5.8565

Table 2: Estimations CAPM model parameters by ML

Now having obtained the model parameter estimates, we can use Equation (29) to obtain the filtered coefficient β_t. The estimated dynamic coefficient β_t for all three assets is shown in Figure 5. Where the initialization of β_1 is done by:

$$\beta_1 = \frac{\hat{\omega}}{1 - \hat{\phi}} \tag{31}$$

2.4 Exposition obtained from CAPM model

The exposition of the market at the next time step is obtained by using the updating equation (29) one step further and plugging in x_T and y_T. The following table shows the exposition for each assest.

Asset	β_{T+1}
MSFT	0.9624361
BAC	1.293192
XOM	0.9203983

Table 3: Prediction of the observation-driven β at the next time step

We see that the β_{T+1} for XOM is still lower than 1. So, investing in XOM will remain to decrease exposure to systematic risk, but the reduction is smaller compared to exposition calculated from simply using the OLS estimate.

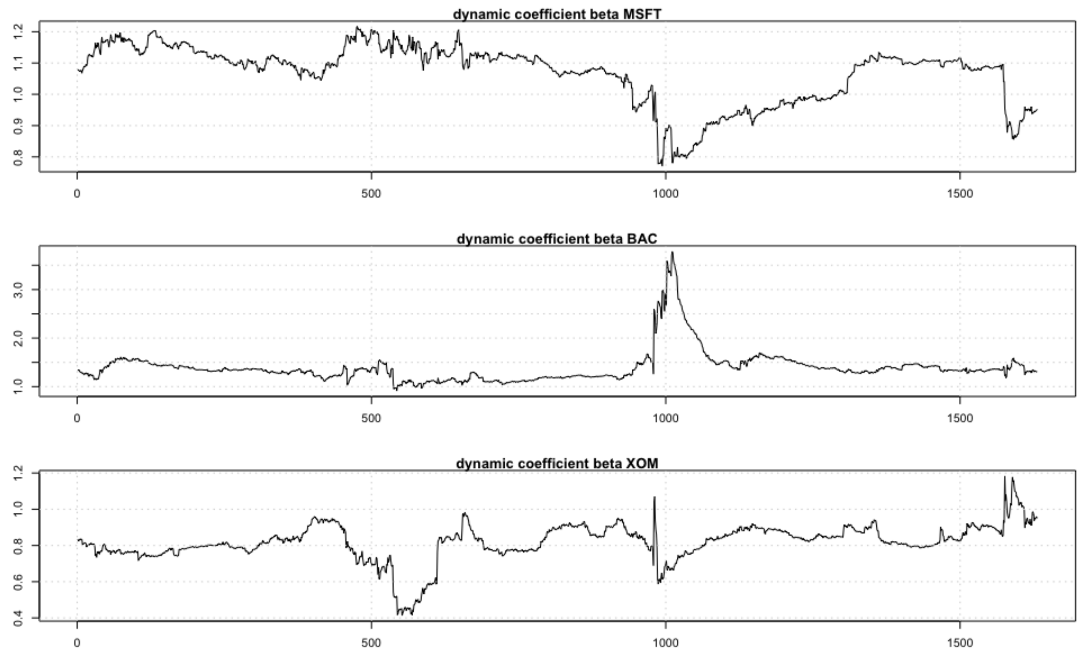

Figure 5: Time-varying β_t using on the CAPM model

2.5 Time-varying β_t from CCC model

In this section we estimate the time-varying β_t by using a bivarite GARCH model for the variance. The model used is a bivariate CCC model, with a equation by equation approach. A GARCH(1,1) model for each assest (including the market) is calculated to obtain the conditional volatility. The standardized residuals

$$\hat{\epsilon}_{it} = \frac{y_{it}}{\hat{\sigma}_{it}}, \tag{32}$$

are then used to calculate the covariance matrix. Using the conditional covariance $\sigma_{m,i,t}$ of the market with asset i and the conditional variance of the market $\sigma_{m,t}^2$, the time-varying β_t is obtained by adjusting the CAPM equation (28) to depend on the time as well:

$$\beta_{i,t} = \frac{\sigma_{m,i,t}}{\sigma_{m,t}^2}. \tag{33}$$

The results for the different assets are shown in Figure 6. Comparing these time-varying parameter-driven β_t with observation-driven β_t obtain from the maximum likelihood estimation (figure 5) we clearly see some differences. Firstly, the observation driven β_t is much smoother and has less variation over time. Furthermore, some of the global features do not overlap. For example the dip in observation driven β_t of XOM around index 600 is not present in the β_t from the CCC model. While the peak around 1000 of the BAC is clearly visible in both estimations.

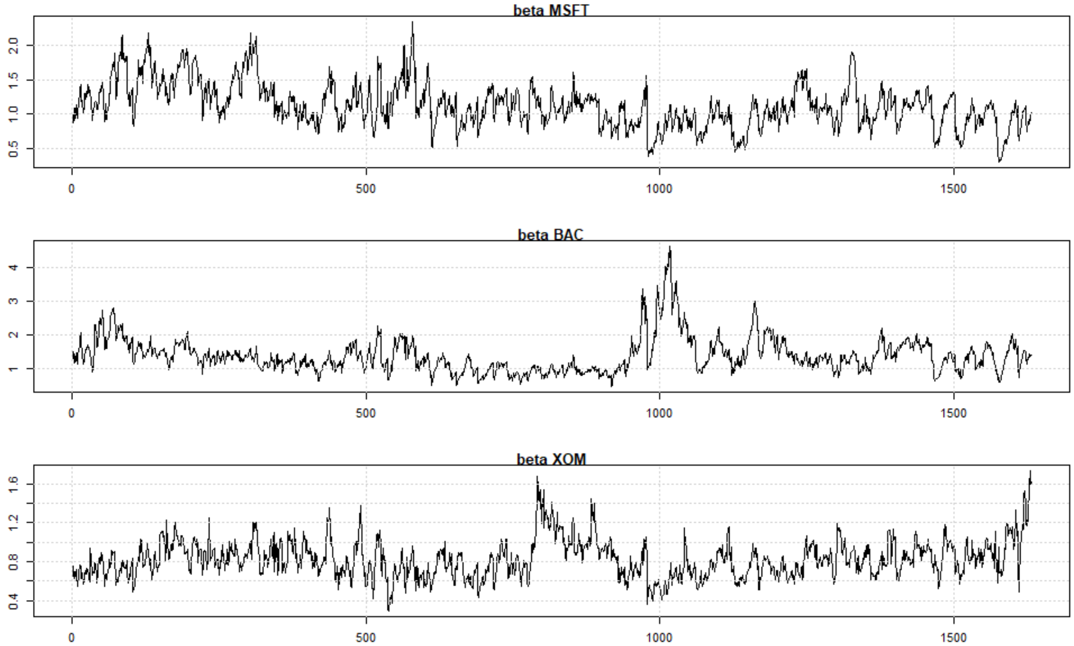

Figure 6: Time-varying β_t based on the bivariate CCC model

2.6 Parameter-driven dynamic regression

Another approach to estimate β_t is a dynamic regression by indirect inference. The model in particular is

$$\beta_t = \exp(f_t), \quad f_t = \alpha_0 + \alpha_1 f_{t-1} + \eta_t, \quad \{\eta_t\} \sim NID\left(0, \sigma_\eta^2\right), \tag{34}$$

where we have used an exponential link function for β_t to ensure it is always positive. The parameters, for each asset i, to be estimated are $\theta_i = (\alpha_{i,0}, \alpha_{i,1}, \sigma_{i,\eta}, \sigma_{i,\epsilon})$, with $\sigma_{i,\epsilon}$ the standard deviation of the errors of the linear regression model $r_{i,t} = \beta_{i,t} r_t^m + \epsilon_{i,t}$. The model does not specify a equation for r_t^m so it cannot be simulated and we restrict the size of the simulated paths to be T and use the observed r_t^m for each repetition. The auxiliary statistic that we used are defined as

$$\hat{B}_T = \begin{bmatrix} \hat{\beta} \\ \hat{s}^2 \\ \hat{\gamma}_0 \\ \vdots \\ \hat{\gamma}_p \end{bmatrix}, \quad \tilde{B}_H(\theta) = \frac{1}{M} \begin{bmatrix} \sum_{i=1}^M \tilde{\beta}_i(\theta) \\ \sum_{i=1}^M \tilde{s}_{y,i}^2(\theta) \\ \sum_{i=1}^M \tilde{\gamma}_{0,i}(\theta) \\ \vdots \\ \sum_{i=1}^M \tilde{\gamma}_{p,i}(\theta) \end{bmatrix}. \tag{35}$$

Here $\hat{s}^2 = (1/T)\sum_{t=1}^T (r_{i,t} - \hat{\beta} r_t^m)^2$ and for the maximum lag in the autocovariance we used $p = 15$. The estimated parameters are then given by the indirect inference estimate of θ:

$$\hat{\theta}_{TH} = \arg\min_{\theta \in \Theta} d\left(\hat{B}_T, \tilde{B}_H(\theta)\right). \tag{36}$$

The parameter estimates we find for each asset are shown in table 4. For the estimate of σ_ϵ^2 of BAC we find an extremely low value. This is not what we expect and would mean that there is no error in the BAC and the market. Especially in contrast to the other assests where the deviations are large.

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\sigma}_\eta^2$	$\hat{\sigma}_\epsilon^2$
MSFT	-73.2348	0.0026	9e-4	10.2713
BAC	-26.2151	0.2823	0.0288	1.39e-10
XOM	-31.109	0.3725	0.0201	5.7938

Table 4: Estimations of the dynamic CAPM model parameters by indirect inference