PROJEKTIVNA GEOMETRIJA

- Relacija $\sim$ na $\mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\}$:

$$X = (x_1, \dots, x_n) \sim (y_1, \dots, y_n) = Y \iff (\exists \lambda \in \mathbb{R} \setminus \{0\}) \text{ tako da } (x_1, \dots, x_n) = \lambda (y_1, \dots, y_n)$$

$$X = \lambda Y$$

$\sim$ je relacija ekvivalencije

$(x_1 : x_2 : \dots : x_{n+1})$ ← klasa ekvivalencije (homogene koordinate)

Primer: $(2,2) \sim (5,5)$ jer $(2,2) = \frac{2}{5}(5,5)$

$(1:1)$ ← jedna tačka ekvivalencije ("=" jedna projektivna tačka) kojoj pripada npr. $(5,5)$

- Afine i homogene koordinate:

$$\underset{\text{afine}}{P(x,y)} \longleftrightarrow \underset{\text{homogene}}{P(x_1 : x_2 : x_3)}$$

, veza: $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$, $x_3 \neq 0$

(beskonačnim ne odgovara ni jedna konačna (x,y))

Primer: $A(2,3) \to A(2:3:1)$ ali i

$A(2,3) \to A(-4:-6:-2)$

$A(2,3) \to A(1 : \frac{3}{2} : \frac{1}{2})$

- $\mathbb{RP}^n := \mathbb{R}^{n+1} \setminus \{\overset{\text{bez koord. početka}}{\vec{0}}\} / \sim$

- projektivna prava $\mathbb{RP}^1$

$$\begin{aligned}
\mathbb{RP}^1 &= \{(x_1 : x_2) \mid x_1, x_2 \in \mathbb{R},\ x_1^2 + x_2^2 \neq 0\} \\
&= \{(x_1 : 1) \mid x_1 \in \mathbb{R}\} \cup \{(x_1 : 0) \mid x_1 \in \mathbb{R} \setminus \{0\}\} \\
&= \{(x_1 : 1) \mid x_1 \in \mathbb{R}\} \cup \{(1:0)\}
\end{aligned}$$

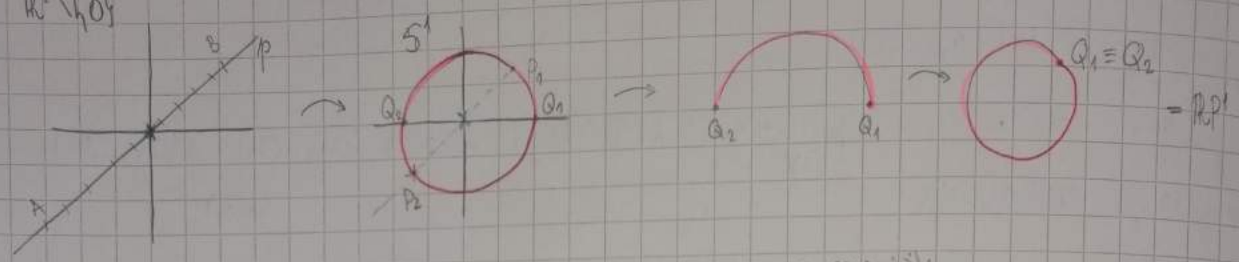
homogene koordinate tačaka koje "nastaju" od konačnih afinih

beskonačna tačka (njene homogene koordinate) (ne možemo "vratiti" u konačnu tačku: $\frac{1}{0}$!)

$\mathbb{RP}^1 := \mathbb{R}^2 \setminus \{(0,0)\} / \sim$

Skica kako dolazimo do $\mathbb{R}P^1$ – krećemo od $\mathbb{R}^2 \setminus \{0\}$:

$\mathbb{R}^2 \setminus \{0\}$



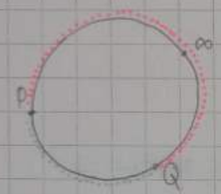
posmatramo prave kroz koordinatni početak, sve tačke sa iste prave imaju iste homogene koordinate jer: $\vec{A} = \lambda \vec{B}$

na ovoj kružnici imamo predstavnika svake prave - klase relacije $\sim$. Čak dva predstavnika

možemo se ograničiti samo na gornju polukružnicu. Treba imati u vidu da su krajnje tačke iz iste klase!

Dakle, $\mathbb{R}P^1$ možemo zamisliti kao kružnicu.

Primetimo da krećući se po projektivnoj pravoj od tačke P do tačke Q imamo dva različita puta!



$\mathbb{R}P^1 =$ afina prava $\cup \{\infty\} = {}_{\infty}\!\longleftrightarrow_{\infty} \cup \{\infty\} =$ ◯ ∞

- Projektivna ravan $\mathbb{R}P^2$

$$\mathbb{R}P^2 := \mathbb{R}^3 \setminus \{0\} / \sim$$

$$\mathbb{R}P^2 = \{ (x_1 : x_2 : x_3) \mid x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 \neq 0 \}$$

$$= \{ (x_1 : x_2 : 1) \mid x_1, x_2 \in \mathbb{R} \} \cup \{ (x_1 : x_2 : 0) \mid x_1, x_2 \in \mathbb{R},\ x_1^2 + x_2^2 \neq 0 \}$$

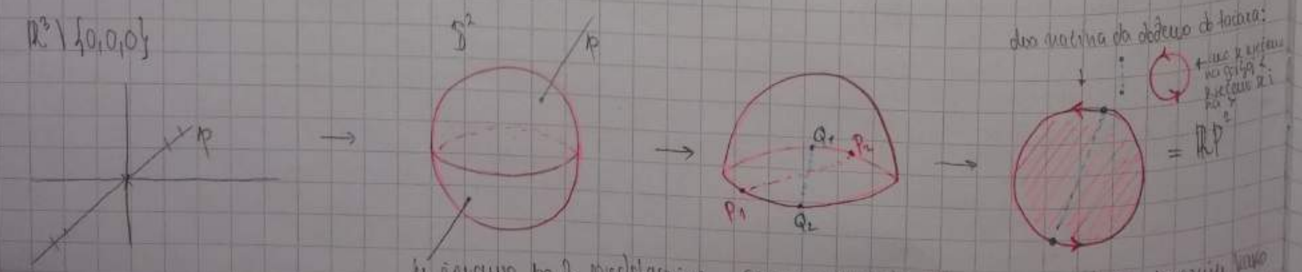
tačke koje "nastaju" od konačnih afinih

homogene koordinate tačaka sa beskonačne prave

Slično kao i u $\mathbb{R}P^1$, podela na ovakve skupove opravdava naziv dopunjena / proširena afina prava / ravan za $\mathbb{R}P^1$ / $\mathbb{R}P^2$.

Skica:

$\mathbb{R}^3 \setminus \{0,0,0\}$



tu imamo po 2 predstavnika svake klase (prave)

samo na "ekvatoru" imamo po dva predstavnika

strelice pokazuju kako "spajamo" iste tačke sa tog

② a) Odrediti u homogenim koordinatama, jednačinu prave $p = AB$, $A(1, \frac{5}{2})$ i $B(-3,0)$.

b) Pokazati da tačka $C(-1:5:3)$ pripada pravoj p.

c) Odrediti tačku D takvu da važi $\mathcal{H}(A,B;C,D)$.

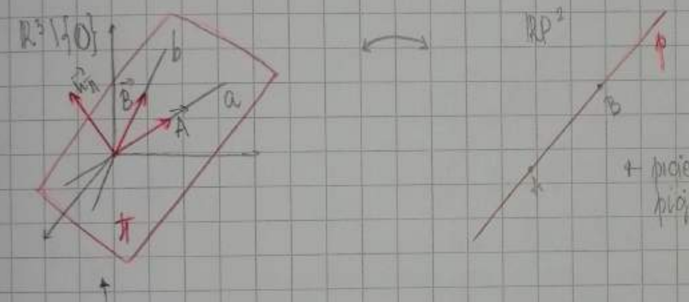
a) $A(1,\frac{5}{2}) \to A(1:\frac{5}{2}:1) \to A(2:5:2)$ ← jer $(1,\frac{5}{2},1) = \frac{1}{2}(2,5,2)$

$B(-3,0) \to B(-3:0:1)$

$$\vec{p} = \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 5 & 2 \\ -3 & 0 & 1 \end{vmatrix} = (5,-8,15) \to p[5:-8:15]$$

$p: 5x_1 - 8x_2 + 15x_3 = 0$

Zašto je opravdana f-la $\vec{p} = \vec{A} \times \vec{B}$?



+ projektivna prava koja sadrži projektivne tačke A i B

afina ravan $\pi = \pi(a,b)$ - ravan kroz koordinatni početak je jedinstveno određena vektorom normale $\vec{n}_\pi = \vec{p} = \vec{A} \times \vec{B}$

b) $C(-1:5:3) \in p$? , $p = AB$

I način: zamenom $5(-1) - 8 \cdot 5 + 3 \cdot 15 = 0 \Rightarrow C \in p$ Dokazati!

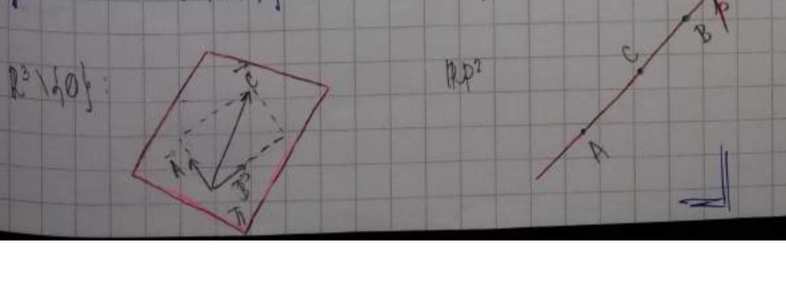
II način: $(\exists \alpha, \beta \in \mathbb{R}?)$ t.d. $\vec{C} = \alpha\vec{A} + \beta\vec{B}$

$(-1:5:3) = \alpha(2:5:2) + \beta(-3:0:1)$

$$\Longleftrightarrow \begin{cases} -1 = 2\alpha - 3\beta & \to -1 = 2 - 3 \ ✓ \\ 5 = 5\alpha & \to \alpha = 1 \\ 3 = 2\alpha + \beta & \to \beta = 1 \end{cases}$$

$\Rightarrow C \in p$

jer $\vec{C} \in \mathcal{L}(\vec{A},\vec{B})$, tj. omotaču A i B, tj. ravni vektora, $\vec{C} = \vec{A} \times \vec{B}$



A, B, C, D: $\vec{C} = \alpha\vec{A} + \beta\vec{B}$
$\vec{D} = \gamma\vec{A} + \delta\vec{B}$

- dvorazmera tačaka A, B, C, D je: $(A, B; C, D) = \dfrac{\beta}{\alpha} : \dfrac{\delta}{\gamma}$
- $\mathcal{H}(A, B; C, D) \Leftrightarrow (A, B; C, D) = -1$

+ A, B, C, D pripadaju istoj pravoj

- sva af. jesu projektivna, prj. nisu af.
- sva af. čuvaju razmeru, prj ne
- prj. čuvaju dvorazmeru – razmeru razmera
- dvorazmera je projektivna invarijanta

c) $D = ?$ za $\mathcal{H}(A, B; C, D)$

$\vec{D} = \gamma\vec{A} + \delta\vec{B}$

$-1 = (A, B; C, D) = \dfrac{\beta}{\alpha} : \dfrac{\delta}{\gamma} = \dfrac{1}{1} : \dfrac{\delta}{\gamma} \Rightarrow -1 = \dfrac{\delta}{\gamma} \Rightarrow \delta = -\gamma$

$\vec{D} = \gamma(\vec{A} - \vec{B}) = \gamma(5, 5, 1) \longrightarrow D(5:5:1)$

① Date su prave $a: 3x_1 - 2x_2 - x_3 = 0$, $b: x_1 - 2x_2 + x_3 = 0$, $c: 2x_1 - 3x_2 + x_3 = 0$, $d: x_1 - x_2 = 0$.

Izračunati dvorazmeru (a, b, c, d).

I način: kao za tačke

$a[3:-2:-1]$, $b[1:-2:1]$, $c[2:-3:1]$, $d[1:-1:0]$

- $\vec{c} = \alpha\vec{a} + \beta\vec{b}$

$$\Leftrightarrow \begin{cases} 2 = 3\alpha + \beta \\ -3 = -2\alpha - 2\beta \\ 1 = -\alpha + \beta \end{cases}$$

$2 = 3\alpha + \beta$
$-1 = -4\alpha \rightarrow \alpha = \frac{1}{4}$
$1 = -\alpha + \beta \rightarrow \beta = \frac{5}{4}$

- $\vec{d} = \gamma\vec{a} + \delta\vec{b}$

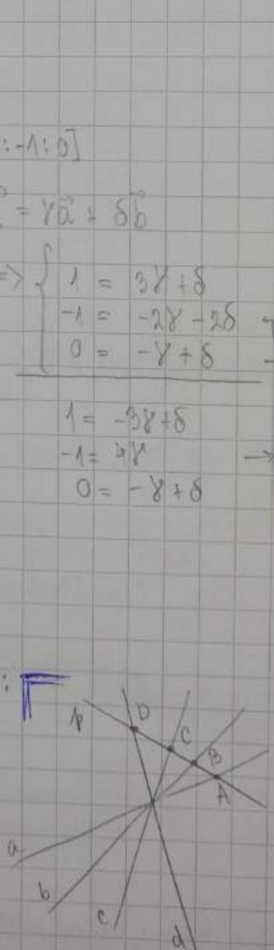
$$\Leftrightarrow \begin{cases} 1 = 3\gamma + \delta \\ -1 = -2\gamma - 2\delta \\ 0 = -\gamma + \delta \end{cases}$$

$1 = -3\gamma + \delta$
$-1 = 4\gamma \rightarrow \gamma = \frac{1}{4}$
$0 = -\gamma + \delta \longrightarrow \delta = \frac{1}{4}$

$(a, b; c, d) = \dfrac{\beta}{\alpha} : \dfrac{\delta}{\gamma} = \dfrac{\frac{5}{4}}{\frac{1}{4}} : \dfrac{\frac{1}{4}}{\frac{1}{4}} = 5$

II način: što je i definicija dvorazmere 4 prave:

$(a, b; c, d) := (A, B; C, D)$ Dokazati!



p – proizvoljna

prave a, b, c, d pripadaju istoj tački, tj. seku se u njoj, ako je to beskonačna tačka, onda su prave a, b, c, d paralelne

$(a, b; c, d) := (A, B; C, D)$

③ U proširenoj afinoj ravni date su prave $m: x_1 + 2x_2 - x_3 = 0$ i $n: 2x_1 - x_2 + 3x_3 = 0$.

a) Odrediti presek P pravih m i n

b) Odrediti pravu k koja sadrži tačke P i $C(-1:5:3)$ i pravu l koja sadrži tačku P i paralelna je pravoj $p: 5x - 8y + 15 = 0$.

c) Odrediti dvorazmeru pravih (m, n, k, l). Domaći!

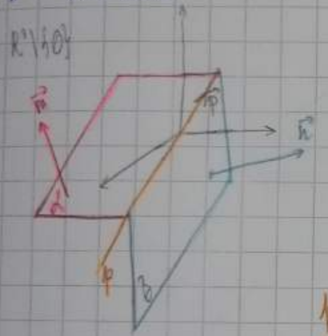
a) $P = m \times n$ ← presek dve prave

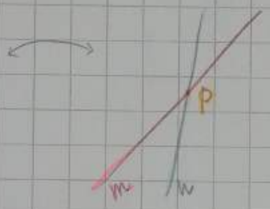
$\cdot$ ← unija dve prave

$$\vec{P} = \vec{m} \times \vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix} = (5, -5, 5) \longrightarrow P(1 : -1 : 1)$$

Ovo zašto:

$\mathbb{R}^3 \setminus \{0\}$





$p = \alpha \cap \beta$

$$\left.\begin{array}{l} \vec{p} \perp \vec{m} \\ \vec{p} \perp \vec{n} \end{array}\right\} \Rightarrow \vec{p} = \lambda \vec{m} \times \vec{n}$$

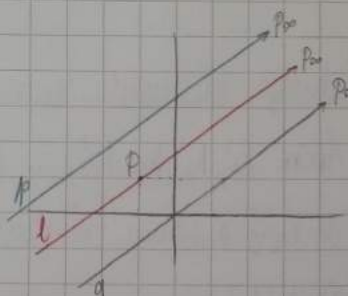
p presečna prava, njen vektor $\perp$ i na $\vec{m}$ i na $\vec{n}$

b) $C(-1:5:3)$, $k = ?$

$$\vec{k} = \vec{P} \times \vec{C} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ -1 & 5 & 3 \end{vmatrix} = (2, -2, 4) \longrightarrow k[1 : -1 : 2]$$

$l = ?$, $l \ni P$, $l \parallel p: 5x - 8y + 15 = 0$

$l \parallel p \Rightarrow l$ i p imaju istu zajedničku tačku u beskonačnost



$P_\infty = p \times u_\infty$, $u_\infty: x_3 = 0 \longrightarrow u_\infty[0:0:1]$ ← beskonačna prava

$$\vec{P_\infty} = \vec{p} \times \vec{u_\infty} = \begin{vmatrix} i & j & k \\ 5 & -8 & 15 \\ 0 & 0 & 1 \end{vmatrix} = (-8, -5, 0) \longrightarrow P_\infty(-8 : -5 : 0)$$

→ Skicirajmo P_∞, tj. pravu kroz koordinatni početak čiji je P_∞ koef. pravca:

$P_\infty \in q: ax_1 + bx_2 = 0$

$-8a - 5b = 0$

$b = -\frac{8}{5}a$

$q: ax_1 - \frac{8}{5}ax_2 = 0 \,/ .x_3 \,/ :a$

$x - \frac{8}{5}y = 0$

$\boxed{y = \frac{5}{8}x}$

$$\vec{l} = \vec{P_\infty} \times \vec{P} = \begin{vmatrix} i & j & k \\ -8 & -5 & 0 \\ 1 & -1 & -1 \end{vmatrix} = (5, 8, 13) \longrightarrow l[5 : -8 : 13]$$

c) $(m,n;k,l) = ?$

• $\vec{k} = \alpha\vec{m} + \beta\vec{n}$

$$\Leftrightarrow \begin{cases} 1 = \alpha + 2\beta \\ -1 = 2\alpha - \beta \\ 2 = -\alpha + 3\beta \end{cases}$$

$1 = \alpha + 2\beta$
$3 = 5\beta \longrightarrow \beta = \frac{3}{5}$
$2 = -\alpha + 3\beta \longrightarrow \alpha = -\frac{1}{5}$

• $\vec{l} = \gamma\vec{m} + \delta\vec{n}$

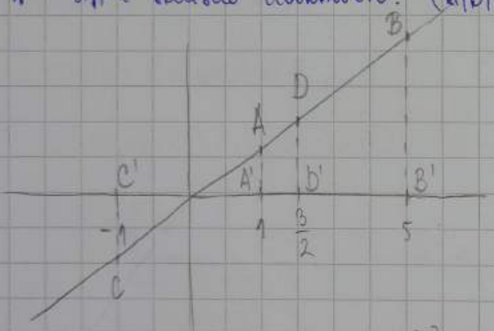
$$\Leftrightarrow \begin{cases} 5 = \gamma + 2\delta \\ -8 = 2\gamma - \delta \\ 13 = -\gamma + 3\delta \end{cases}$$

$5 = \gamma + 2\delta$
$18 = 5\delta \longrightarrow \delta = \frac{18}{5}$
$13 = -\gamma + 3\delta \longrightarrow \gamma = -\frac{11}{5}$

$$(m,n;k,l) = \frac{\beta}{\alpha} : \frac{\delta}{\gamma} = \frac{\frac{3}{5}}{-\frac{1}{5}} : \frac{\frac{18}{5}}{-\frac{11}{5}} = -3 : -\frac{18}{11} = \frac{11}{6}$$

4) Date su kolinearne tačke $A(1,1)$, $B(5,5)$, $C(-1,-1)$, $D(\frac{3}{2},\frac{3}{2})$. Koristeći afini smisao dvorazmere, izračunati $(A,B;C,D)$.

afini smisao dvorazmere: $(A,B;C,D) = \frac{\vec{AC}}{\vec{CB}} : \frac{\vec{AD}}{\vec{DB}}$

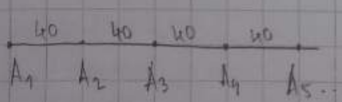


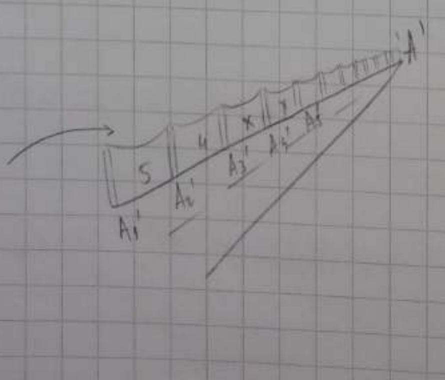
$$(A,B;C,D) = (A',B';C',D') = \frac{\vec{A'C'}}{\vec{C'B'}} : \frac{\vec{A'D'}}{\vec{D'B'}} = \frac{-2}{6} : \frac{\frac{1}{2}}{\frac{7}{2}} = -\frac{1}{3} \cdot 7 = -\frac{7}{3}$$

5) Podnožja telefonskih stubova $A_1, A_2, A_3, A_4, \ldots$ su u stvarnosti međusobno udaljena 40m i kolinearna. Tačke $A_1', A_2', A_3', A_4', \ldots$ su njihove slike na perspektivnom crtežu, a $A' = \lim_{n\to\infty} A_n'$ je tačka na crtežu u kojoj ti stubovi iščezavaju. Ako je $A_1'A_2' = 5cm$, $A_2'A_3' = 4cm$, odrediti:

a) $A_3'A_4'$

b) $A_4'A'$ domaći!





a) $(A_1, A_2; A_3, A_4) = (A_1', A_2'; A_3', A_4')$

$$(A_1, A_2; A_3, A_4) = \frac{\overrightarrow{A_1A_3}}{\overrightarrow{A_3A_2}} : \frac{\overrightarrow{A_1A_4}}{\overrightarrow{A_4A_2}} = \frac{80}{-40} : \frac{120}{-80} = \frac{4}{3}$$

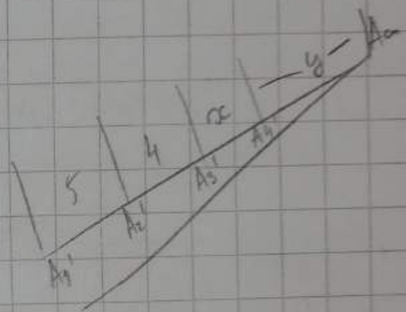
$$(A_1', A_2'; A_3', A_4') = \frac{\overrightarrow{A_1'A_3'}}{\overrightarrow{A_3'A_2'}} : \frac{\overrightarrow{A_1'A_4'}}{\overrightarrow{A_4'A_2'}} = \frac{9}{-4} : \frac{9+x}{-(4+x)} = \frac{9(4+x)}{4(9+x)}$$

$$\frac{4}{3} = \frac{9(4+x)}{4(9+x)}$$

$$27(4+x) = 16(9+x)$$

$$11x = 4 \cdot 9 \quad \Rightarrow \quad \boxed{x = \frac{36}{11}}$$

b) $A_3 = S(A_2, A_4) \Rightarrow (A_2, A_4, A_3, A_\infty) = -1$



$$(A_2', A_4', A_3', A_\infty') = \frac{\overrightarrow{A_2'A_3'}}{\overrightarrow{A_3'A_4'}} : \frac{\overrightarrow{A_2'A_\infty'}}{\overrightarrow{A_\infty'A_4}} = \frac{4}{\frac{36}{11}} : \frac{\frac{80}{11}+y}{-y}$$

$$-1 = \frac{-44y}{36\left(\frac{80}{11}+y\right)} \Rightarrow \frac{36 \cdot 80}{11} = 8y$$

$$\Rightarrow \boxed{y = \frac{360}{11}}$$

⑥ Dala je matrica $P=\begin{pmatrix}2&0&1\\1&2&2\\1&1&0\end{pmatrix}$ i tačke $A(1,0)$, $B(0,1)$. Odrediti i skicirati sliku $A'B'C'$ trougla ΔABC pri projektivnom preslikavanju f čija je matrica P ako:

$[f]=P$

a) $C(2,0)$

b) $C(0,0)$

c) $C(-1,0)$

$\lambda X' = PX \quad , \lambda \in \mathbb{R}\setminus\{0\}$

koordinate slike — koordinate polazne tačke

$A(1,0) \to A(1:0:1)$

$B(0,1) \to B(0:1:1)$

$f(\Delta ABC) = \Delta A'B'C'$?

a) $C(2,0) \to C(2:0:1)$

- $A' = f(A)$:

$$\lambda_A A' = PA = \begin{pmatrix}2&0&1\\1&2&2\\1&1&0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\3\\1\end{pmatrix}$$

$$\lambda_A = 1 \longrightarrow A'(3:3:1) \to A'(3,3)$$

- $B' = f(B)$

$$\lambda_B B' = PB = \begin{pmatrix}2&0&1\\1&2&2\\1&1&0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\4\\1\end{pmatrix}$$

$$\lambda_B = 1 \longrightarrow B'(1:4:1) \longrightarrow B'(1,4)$$

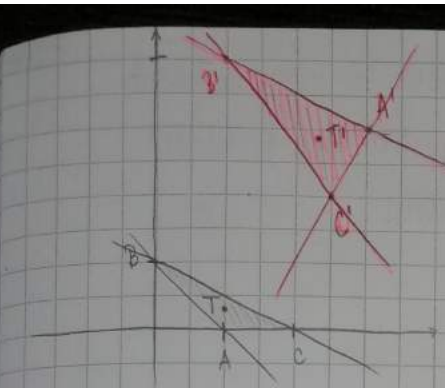
- $C' = f(C)$

$$\lambda_C C' = PC = \begin{pmatrix}2&0&1\\1&2&2\\1&1&0\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}5\\4\\2\end{pmatrix}$$

$$\lambda_C = 2 \longrightarrow C'\left(\tfrac{5}{2}:2:1\right) \longrightarrow C'\left(\tfrac{5}{2},2\right)$$

$$T = \tfrac{1}{3}(A+B+C) = \left(1,\tfrac{1}{3}\right) \longrightarrow T\left(1:\tfrac{1}{3}:1\right) = (3:1:3)$$

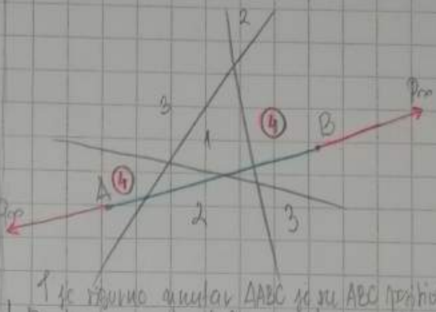
$$\lambda T' = PT = \begin{pmatrix}2&0&1\\1&2&2\\1&1&0\end{pmatrix}\begin{pmatrix}3\\1\\3\end{pmatrix} = \begin{pmatrix}9\\11\\4\end{pmatrix} \longrightarrow T'\left(\tfrac{9}{4},\tfrac{11}{4}\right)$$



← Znamo gde se slikaju temena, kao i prave određene temenima.

Ukoliko želimo da odredimo u šta se slika unutrašnjost trougla, trebaju nam još neka znanja, jer projektivni trougao na afinoj slici može izgledati neobično.

Trotemenik deli projektivnu ravan na 4 konveksne oblasti



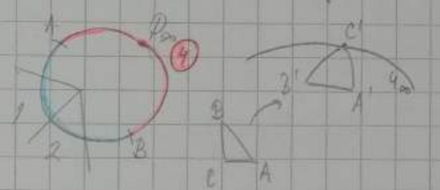
← afine oblasti označene istim brojevima su ista projektivna oblast

Ako krenemo od A do B preko konačne duži AB sečemo oblast 2 i 1, ukoliko od A do B idemo preko beskonačne tačke, ne izlazimo iz oblasti 4.

1 je ravno unutar ΔABC jer su ABC pozitivne!

Da bi smo odredili unutrašnjost ΔABC, preslikaćemo neku tačku, npr. T.

Ali sad moramo da ocenimo!



naš novi Δ dodiruje ∞ pravu u C' i onda B'C' ∥ A'C'

b) $C(0,0) \to C(0:0:1)$

$$\lambda C' = PC = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$\lambda = 1 \to C'(1:2:0)$ ← beskonačna tačka!

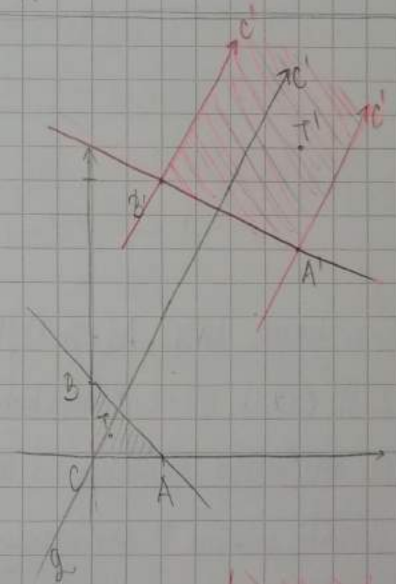
$C' \in u_\infty$, C' je kraj. pravca neke prave

$C' \in g: ax + by + c = 0$

$g: ax_1 + bx_2 = 0$

$a \cdot 1 + b \cdot 2 = 0$

$a = -2b \to g:[-2b:b:0] \to g:[2:-1:0]$



$g: 2x_1 - x_2 = 0 \quad /:x_3$

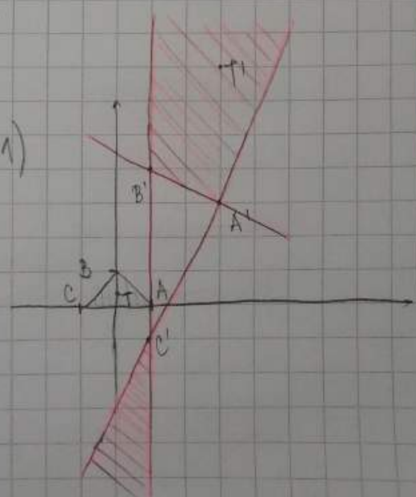
$2x - y = 0$

$T(\frac{1}{3}, \frac{1}{3}) \to T(1:1:3)$

$$\lambda T' = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} \longrightarrow T'(\tfrac{5}{2}, \tfrac{9}{2})$$

c) $C(-1,0) \to C(-1:0,1)$

$$\lambda C' = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \longrightarrow C'(1:-1:1) \to C'(1,-1)$$

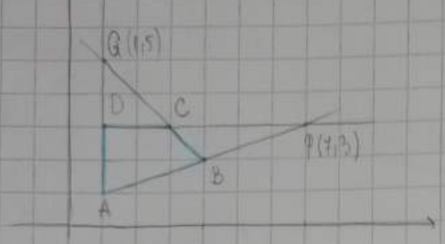


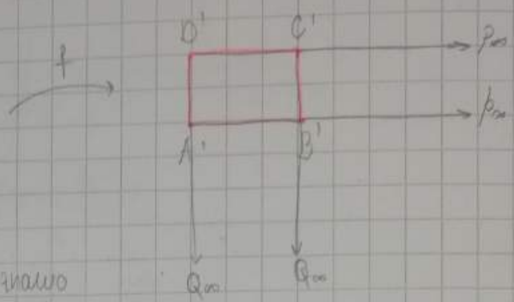
$T(0, \frac{1}{3}) \to T(0:\frac{1}{3}:1)$

$$\lambda T' = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} \longrightarrow T'(3,7)$$

7. U procesu rektifikacije tačke $A(1,1)$, $B(4,2)$, $C(3,3)$, $D(1,3)$ preslikane su redom u temena pravougaonika $ABCD$, preslikavanjem f. Bez određivanja projektivnog preslikavanja f odrediti jednačinu prave u koja se preslikala u beskonačno daleku pravu.

$ABCD \xrightarrow{f}$ pravougaonik $A'B'C'D'$





$Q = (A \times D) \times (B \times C)$ ← sa slike znamo

$P = (A \times B) \times (D \times C)$ ← P i Q znaće to treba izračunati

$A'B' \parallel C'D' \Rightarrow P_\infty = A'B' \times C'D' \in u_\infty$

$A'D' \parallel B'C' \Rightarrow Q_\infty = A'D' \times B'C' \in u_\infty$

$u? \xrightarrow{f} u_\infty = P_\infty \times Q_\infty$

$u = f^{-1}(P_\infty) \times f^{-1}(Q_\infty) = P \times Q$, gde $P = AB \times DC$, $Q = AD \times BC$

$$\vec{u} = \vec{P} \times \vec{Q} = \begin{vmatrix} i & j & k \\ 7 & 3 & 1 \\ 1 & 5 & 1 \end{vmatrix} = (-2, -6, 32) \longrightarrow u[-2:-6:32]$$

8. Dati su trapez $ABCD$, $A(-3,-1)$, $B(3,-1)$, $C(1,1)$, $D(-1,1)$ i pravougaonik $A'B'C'D'$ $A'(-2,-1)$, $B'(2,-1)$, $C'(2,1)$, $D'(-2,1)$. Projektivno preslikavanje f ravni $\mathbb{R}P^2$ slika trapez u pravougaonik.

a) Bez određivanja preslikavanja f odrediti koja se prava slika u beskonačno daleko.

b) Odrediti matricu P preslikavanja f.

a) $A'D' \times B'C' = Q_\infty$ $\quad$ $A'B' \times C'D' = P_\infty$

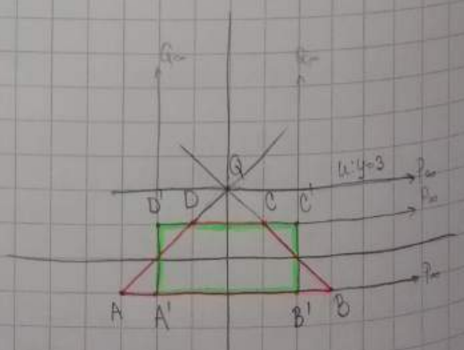
$AD \times BC = Q$ $\quad$ $AB \times CD = P_\infty$ $\Big\}$ + jer su to iste prave

$Q \longmapsto Q_\infty$ $\quad$ $P_\infty \longmapsto P_\infty$

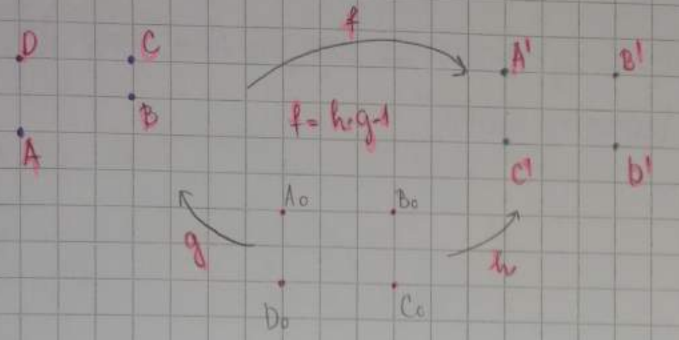
$QP_\infty \longmapsto Q_\infty P_\infty = u_\infty$

$u = QP_\infty$

$u: y = 3$



b)



$f: \lambda X' = [f] X$

$g: \lambda X' = [g] X$

$h: \lambda X' = [h] X$

$$[f] = [h \cdot g^{-1}] \overset{\text{homomorfizam}}{=} [h] \cdot [g]^{-1}$$

$$h: \begin{array}{llll} A_0 & (1:0:0) & \xmapsto{h} & A'(-2:-1:1) \\ B_0 & (0:1:0) & \xmapsto{h} & B'(2:-1:1) \\ C_0 & (0:0:1) & \xmapsto{h} & C'(2:1:1) \\ D_0 & (1:1:1) & \xmapsto{h} & D'(-2:1:1) \end{array}$$

$$\vec{D'} = \alpha \vec{A'} + \beta \vec{B'} + \gamma \vec{C'}$$

$$\Leftrightarrow \begin{cases} -2 = -2\alpha + 2\beta + 2\gamma & \mathrm{I} \\ 1 = -\alpha - \beta + \gamma & \mathrm{II} \\ 1 = \alpha + \beta + \gamma & \mathrm{III} \end{cases}$$

$\mathrm{II}+\mathrm{III}: \quad 2 = 2\gamma \longrightarrow \gamma = 1$

$\mathrm{II}: \quad \alpha = -\beta \qquad \longrightarrow \alpha = 1$

$\mathrm{I} \quad -4 = 4\beta \longrightarrow \beta = -1$

$$[h] = \begin{pmatrix} -2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$
$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \alpha\vec{A'} & \beta\vec{B'} & \gamma\vec{C'} \end{array}$$

$$g: \begin{array}{llll} A_0 & (1:0:0) & \xmapsto{g} & A(-3:-1:1) \\ B_0 & (0:1:0) & \xmapsto{g} & B(3:-1:1) \\ C_0 & (0:0:1) & \xmapsto{g} & C(1:1:1) \\ D_0 & (1:1:1) & \xmapsto{g} & D(-1:1:1) \end{array}$$

$$\vec{D} = \delta \vec{A} + \eta \vec{B} + \nu \vec{C}$$

$$\Leftrightarrow \begin{cases} -1 = -3\delta + 3\eta + \nu & \mathrm{I} \\ 1 = -\delta - \eta + \nu & \mathrm{II} \\ 1 = \delta + \eta + \nu & \mathrm{III} \end{cases}$$

$\mathrm{II}+\mathrm{III}: \quad 2 = 2\nu \longrightarrow \nu = 1$

$\mathrm{II}: \quad \delta = -\eta \qquad \longrightarrow \delta = \frac{1}{3}$

$\mathrm{I}: \quad -2 = 6\eta \longrightarrow \eta = -\frac{1}{3}$

$$[g] = \begin{pmatrix} -1 & -1 & 1 \\ -\frac{1}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -3 & -3 & 3 \\ -1 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix}$$
$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \frac{1}{3}\vec{A} & -\frac{1}{3}\vec{B} & \vec{C} \end{array}$$

$$[g]^{-1} = \frac{1}{\det[g]} \left(\operatorname{adj}[g]\right)^T = -\frac{3}{4} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{2}{3} \end{pmatrix}^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$[f] = [h] \circ [g]^{-1} = \begin{pmatrix} -2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \circ \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

⑨ Dokazati da je u matrici $P=(p_{ij})$, projektivnog preslikavanja dopunjene afine ravni $\overline{\mathbb{R}}^2$, element $p_{33}=0$ akko koordinatni početak slika beskonačno daleko.

$p_{33}=0 \iff f(O)=O' \in u_\infty$

$O(0,0)=(0:0:1)$

$\lambda X' = PX$

$$\lambda\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \lambda O' = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_{13} \\ p_{23} \\ p_{33} \end{pmatrix} \iff p_{33}=0$$

znamo da je slika beskonačno daleko

⑩ a) Ako je $\lambda X' = PX$ projektivna transformacija tačaka, dokazati da se prave slikaju po pravilu $\lambda u' = P^{-T}u$. $\quad (P^{-T} = (P^T)^{-1} = (P^{-1})^T)$

b) Koristeći dokazano pod a), preslikati pravu $u: x_1+2x_2+x_3=0$ preslikavanjem čije je P data u ⑥

c) Proveriti rezultat b) preslikavanjem dve tačke $A, B \in u$.

a) $\lambda X' = PX$ ← formula za tačke

$\lambda u' = P^{-T}u$ ← formula za prave

$u: ax_1 + bx_2 + cx_3 = 0$

$u: u^T X = 0$, gde $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $u=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$u^T X = 0$

$u^T \lambda P^{-1} X' = 0$ ← $\left\{ P^{-1} / \lambda X' = PX \atop \lambda P^{-1}X' = X \right.$

$\lambda u^T P^{-1} X' = 0$

$(\lambda P^{-T} u)^T X' = 0$ ← transponovanje menja redosled: $(AB)^T = B^T A^T$

$u' = \lambda P^{-T} u \ /:\lambda$

$\frac{1}{\lambda} u' = P^{-T} u$

$\mu u' = P^{-T} u$

b) $P = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

minor koji odgovara: $P\begin{pmatrix} 2 & 0 & 1 \\ 1 & \boxed{2} & \boxed{2} \\ 1 & \boxed{1} & \boxed{0} \end{pmatrix}$

zbir indeksa

$u: x_1 + 2x_2 - x_3 = 0 \qquad A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = -2,\ A_{12}, \dots, A_{33}$

$$P^{-1} = \frac{1}{\det P} (\operatorname{adj} P)^T = -\frac{1}{5} \begin{pmatrix} -2 & 2 & -1 \\ 1 & -1 & -2 \\ -2 & -3 & 4 \end{pmatrix}^{T} \Big/^{T}$$

$$P^{-T} = -\frac{1}{5} \begin{pmatrix} -2 & 2 & -1 \\ 1 & -1 & -2 \\ -2 & -3 & 4 \end{pmatrix}$$

$$\lambda u' = P^{-T} u$$

$$\lambda u' = -\frac{1}{5} \begin{pmatrix} -2 & 2 & -1 \\ 1 & -1 & -2 \\ -2 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix}$$

$$\lambda = -\frac{1}{5} \longrightarrow u' : [3 : 1 : -12]$$

c) $u: x_1 + 2x_2 - x_3 = 0$

$A(1:0:1) \in u \quad , \quad B(0:1:2) \in u$

$$\lambda A' = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \longrightarrow A'(3:3:1)$$

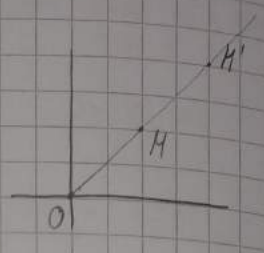
$u': 3x_1 + x_2 - 12x_3 = 0$

$A' \in u'?: \quad 3 \cdot 3 + 3 - 12 = 0 \quad ✓$

$$\lambda B' = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \longrightarrow B'(2:6:1)$$

$B' \in u'?: \quad 3 \cdot 2 + 6 - 12 = 0 \quad ✓$

• Homotetija u odnosu na koordinatni početak sa koeficijentom λ:



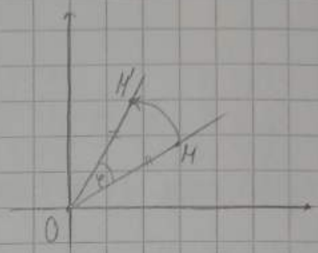
$$\mathcal{H}_{O,\lambda}: \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}$$

$$\lambda\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} = \begin{pmatrix}\lambda & 0 & 0\\0 & \lambda & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$ + zapis projektivnog preslikavanja

$\mathcal{H}_{O,\lambda}(M) = M' \Leftrightarrow \vec{OM'} = \lambda\vec{OM}$

matrica fiksira ∞ pravu pa je to afino preslikavanje

• Rotacija oko koordinatnog početka za ugao φ:

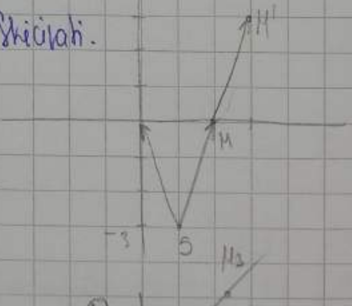


$$\mathcal{R}_{O,\varphi}: \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}$$

$$\lambda\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

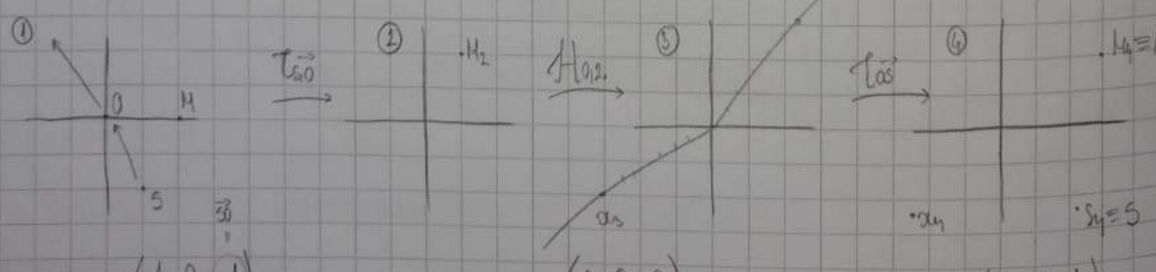
(11) a) Odrediti u afinim i homogenim koordinatama formule homotetije sa koeficijentom 2 u odnosu na tačku $S(1,-3)$. Odrediti sliku M' tačke $M(2,0)$. Skicirati.

b) Odrediti u afinim i homogenim koordinatama formule rotacije za $90°$, u pozitivnom smeru oko tačke $C(2,3)$. Odrediti sliku tačke $M(4,4)$. Skicirati.



a) $\mathcal{H}_{S,2}(M) = ?$, $M' = ?$, $S(1,-3)$

$\mathcal{H}_{S,2} = \tau_{\vec{OS}} \cdot \mathcal{H}_{O,2} \cdot \tau_{\vec{SO}}$

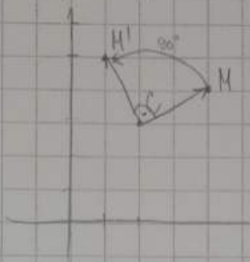


$$\tau_{\vec{SO}}: \lambda X_2 = \begin{array}{c} \begin{matrix} & & \vec{SO} \end{matrix} \\ \begin{pmatrix}1 & 0 & -1\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix} \end{array} X_1 \qquad \mathcal{H}_{O,2}: \lambda X_3 = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix} X_2 \qquad \tau_{\vec{OS}}: \lambda X_4 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & -3\\0 & 0 & 1\end{pmatrix} X_3$$

$$[\mathcal{H}_{S,2}] = [\tau_{\vec{OS}}][\mathcal{H}_{O,2}][\tau_{\vec{SO}}] = \begin{pmatrix}1 & 0 & 1\\0 & 1 & -3\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0 & 1\\0 & 2 & -3\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0 & -1\\0 & 2 & 3\\0 & 0 & 1\end{pmatrix}$$

$$\lambda M' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \longrightarrow M'(3:3:1) \longrightarrow M'(3,3)$$

b) $R_{C,\frac{\pi}{2}} = ?$, $M' ?$



$$R_{C,\frac{\pi}{2}} = t_{\vec{OC}} \cdot R_{O,\frac{\pi}{2}} \cdot t_{\vec{CO}}$$

$$t_{\vec{CO}}: \lambda X_2 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} X_1 \qquad R_{O,\frac{\pi}{2}}: \lambda X_3 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} X_2 \qquad t_{\vec{OC}}: \lambda X_4 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} X_3$$

$$R_{C,\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda M' = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \longrightarrow M'(1:5:1) \longrightarrow M'(1,5)$$

(12) Ako je O koordinatni početak, napisati kompoziciju skaliranja $S_{O,2,1}$ i rotacije $R_{O,45°}$ u oba redosleda i pokazati da te transformacije komutiraju. Odrediti sliku kvadrata sa temenima $(\pm 1, \pm 1)$ u oba slučaja i skicirati.

$$S_{O,2,1}: \lambda \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad R_{O,45°}: \lambda \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$S_{O,2,1} \cdot R_{O,45°} \neq R_{O,45°} \cdot S_{O,2,1}$$

$$[S_{O,2,1} \cdot R_{O,45°}] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{O,45°} \cdot S_{O,2,1}] = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ transformacije ne komutiraju

$A(-1,-1) \longrightarrow A(-1:-1:1)$

$B(1,-1) \longrightarrow B(1:-1:1)$

$C(1,1) \longrightarrow C(1:1:1)$

$D(-1,1) \longrightarrow D(-1:1:1)$

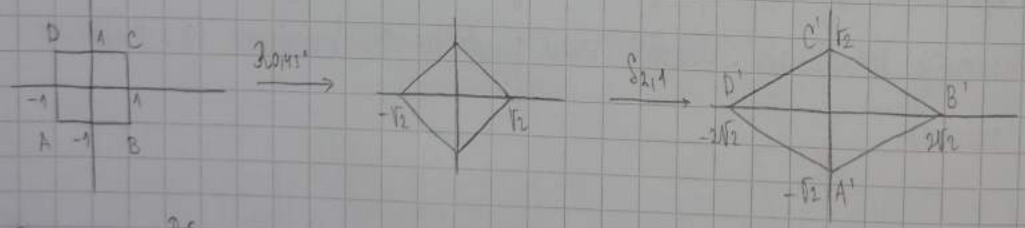
$S \cdot R$:

$$\lambda A' = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix} \longrightarrow A'(0:-\sqrt{2}:1) \longrightarrow A'(0,-\sqrt{2})$$

$$\lambda B' = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \\ 1 \end{pmatrix} \longrightarrow B'(2\sqrt{2},0)$$

$$\lambda C' = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 1 \end{pmatrix} \longrightarrow C'(0,\sqrt{2})$$

$$\lambda D' = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ 0 \\ 1 \end{pmatrix} \longrightarrow D'(-2\sqrt{2},0)$$



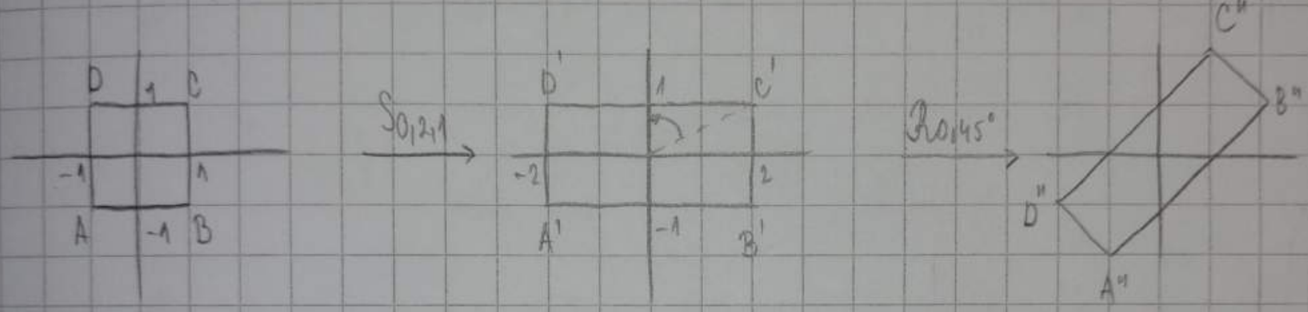
$R \cdot S$: $ABCD \xmapsto{R \cdot S} A''B''C''D''$

$$\lambda A'' = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{3\sqrt{2}}{2} \\ 1 \end{pmatrix} \longrightarrow A''\left(-\frac{\sqrt{2}}{2} : -\frac{3\sqrt{2}}{2} : 1\right) \longrightarrow A''\left(-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$$

$$\lambda B'' = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{pmatrix} \longrightarrow B''\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

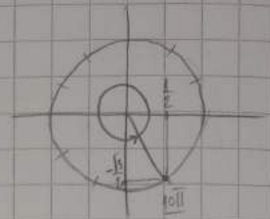
$$\lambda C'' = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{pmatrix} \longrightarrow C''\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\lambda D'' = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{pmatrix} \longrightarrow D''\left(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$



KRETANJA PROSTORA

① Napisati matricu rotacije za $\phi = \frac{10\pi}{6}$ oko: a) Ox ose
b) Oy ose
c) Oz ose



a) $[R_x(\frac{10\pi}{6})] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

b) $[R_y(\frac{10\pi}{6})] = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$

c) $[R_z(\frac{10\pi}{6})] = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

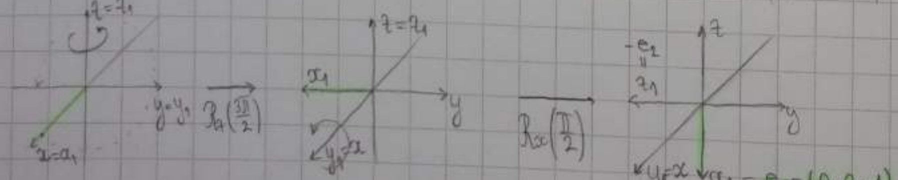
② i) Napisati kompoziciju, tim redom, rotacije oko Oz ose za $\psi = \frac{3\pi}{2}$ i rotacije oko Ox ose za $\phi = \frac{\pi}{2}$ kao: a) svetskih rotacija b) sopstvenih rotacija

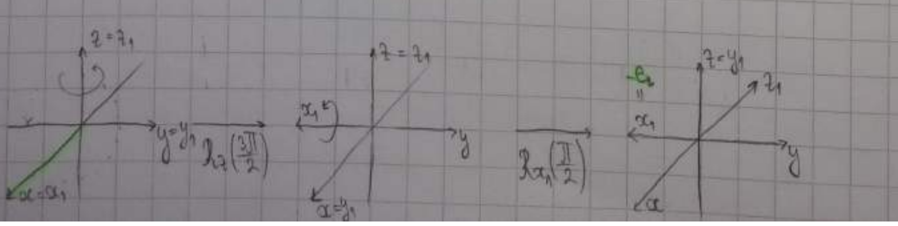
ii) Napisati kompoziciju, tim redom, rotacije oko Oy ose za $\frac{\pi}{2}$ i rotacije oko Ox ose za $\phi = \pi$ kao: a) svetskih rotacija b) sopstvenih rotacija

U svim slučajevima nacrtati početni reper i krajnji reper.

i) a) $[R_x(\frac{\pi}{2}) \cdot R_z(\frac{3\pi}{2})]_e = R_x(\frac{\pi}{2}) \cdot R_z(\frac{3\pi}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

b) $[R_{x_1}(\frac{\pi}{2}) \cdot R_z(\frac{3\pi}{2})]_e = R_z(\frac{3\pi}{2}) \cdot R_x(\frac{\pi}{2}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

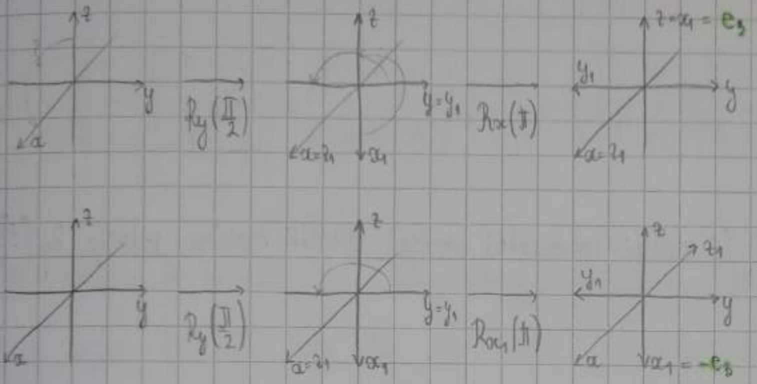




$$\boxed{[R_{x_1}(\phi)\cdot R_{y_1}(\theta)\cdot R_{z_1}(\psi)] = R_z(\psi)\cdot R_y(\theta)\cdot R_x(\phi)}$$ + u svetskom reperu se posmatraju sve matrice

ii) a) $[R_x(\pi)\cdot R_y(\frac{\pi}{2})]_e = R_x(\pi)\cdot R_y(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

b) $[R_{x_1}(\pi)\cdot R_y(\frac{\pi}{2})]_e = R_y(\frac{\pi}{2})\cdot R_{x_1}(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$



(3) Rodrigezovom formulom izvesti matrice rotacija oko koordinatnih osa.

$$R_p(\phi) = pp^T + \cos\phi\cdot(E - pp^T) + \sin\phi\cdot p_\times$$

$\|p\| = 1$, ako imamo p koja ne prolazi kroz (0,0) prvo je transliramo, pa onda rotacija, pa onda vratimo

$$p_\times = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}$$

$$p \times q = \begin{vmatrix} i & j & k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$ + krsivamo + uig se

a) rotacija za ϕ oko Ox ose

$p = (1,0,0)$, $p_\times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $pp^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1\ 0\ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$R_p(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos\phi\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) + \sin\phi\cdot\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

b) $\vec{p}=(0,1,0)$, $p_\times = \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}$, $pp^T = \begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0) = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$

$$R_p(\phi) = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \cos\phi\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right) + \sin\phi\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix} = \begin{pmatrix}\cos\phi&0&\sin\phi\\0&1&0\\-\sin\phi&0&\cos\phi\end{pmatrix}$$

c) $\vec{p}=(0,0,1)$, $p_\times = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$, $pp^T = \begin{pmatrix}0\\0\\1\end{pmatrix}(0\ 0\ 1) = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$

$$R_p(\phi) = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} + \cos\phi\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right) + \sin\phi\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}\cos\phi&-\sin\phi&0\\\sin\phi&\cos\phi&0\\0&0&1\end{pmatrix}$$

4) Prava p ima vektor pravca $\vec{p} = \frac{\sqrt{3}}{3}(1,1,1)$ (i sadrži koordinatni početak). Odrediti maticu rotacije $R_p\left(\frac{2\pi}{3}\right)$.

$|\vec{p}| = 1$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$pp^T = \left(\frac{\sqrt{3}}{3}\right)^2\begin{pmatrix}1\\1\\1\end{pmatrix}(1\ 1\ 1) = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$$\cos\varphi\,(E - pp^T) = -\frac{1}{2}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right) = -\frac{1}{2}\begin{pmatrix}\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}&\frac{1}{6}&\frac{1}{6}\\\frac{1}{6}&-\frac{1}{3}&\frac{1}{6}\\\frac{1}{6}&\frac{1}{6}&-\frac{1}{3}\end{pmatrix}$$

$$\sin\varphi\cdot p_\times = \frac{\sqrt{3}}{2}\begin{pmatrix}0&-\frac{\sqrt{3}}{3}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}&0&-\frac{\sqrt{3}}{3}\\-\frac{\sqrt{3}}{3}&\frac{\sqrt{3}}{3}&0\end{pmatrix}$$

$$R_p\left(\frac{2\pi}{3}\right) = \begin{pmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix} + \begin{pmatrix}-\frac{1}{3}&\frac{1}{6}&\frac{1}{6}\\\frac{1}{6}&-\frac{1}{3}&\frac{1}{6}\\\frac{1}{6}&\frac{1}{6}&-\frac{1}{3}\end{pmatrix} + \frac{\sqrt{3}}{2}\begin{pmatrix}0&-\frac{\sqrt{3}}{3}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}&0&-\frac{\sqrt{3}}{3}\\-\frac{\sqrt{3}}{3}&\frac{\sqrt{3}}{3}&0\end{pmatrix} = \begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix} + \begin{pmatrix}0&-\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&0\end{pmatrix} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

5) Rodrigezovom formulom odrediti maticu rotacije $R_p(\phi)$ ako je:

a) $p = \frac{1}{3}(1,2,2)$, $\phi = \frac{3\pi}{2}$

b) $p = \frac{\sqrt{2}}{2}(1,1,0)$, $\phi = \frac{\pi}{3}$

a) $\|\vec{p}\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1$ ✓ , $\cos\phi = 0$, $\sin\phi = -1$

$p_\times = \begin{pmatrix}0&-\frac{2}{3}&\frac{2}{3}\\\frac{2}{3}&0&-\frac{1}{3}\\-\frac{2}{3}&\frac{1}{3}&0\end{pmatrix}$, $pp^T = \frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}(1\ 2\ 2) = \frac{1}{9}\begin{pmatrix}1&2&2\\2&4&4\\2&4&4\end{pmatrix}$

$$R_p\left(\frac{3\pi}{2}\right) = \frac{1}{9}\begin{pmatrix}1&2&2\\2&4&4\\2&4&4\end{pmatrix} + 0 - \begin{pmatrix}0&-\frac{2}{3}&\frac{2}{3}\\ \frac{2}{3}&0&-\frac{1}{3}\\ -\frac{2}{3}&\frac{1}{3}&0\end{pmatrix} = \frac{1}{9}\begin{pmatrix}1&8&-4\\-4&4&7\\8&1&4\end{pmatrix}$$

b) $\|\vec p\| = \sqrt{\frac{2}{4}+\frac{2}{4}+0} = 1$ ✓ , $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$p_\times = \begin{pmatrix}0&0&\frac{\sqrt{2}}{2}\\0&0&-\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\end{pmatrix} \quad , \quad pp^T = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1&1&0\end{pmatrix} = \begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$$

$$R_p\left(\frac{\pi}{3}\right) = \frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix} + \frac{1}{2}\cdot\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}\right) + \frac{\sqrt{3}}{2}\begin{pmatrix}0&0&\frac{\sqrt{2}}{2}\\0&0&-\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3&1&\sqrt{6}\\1&3&-\sqrt{6}\\-\sqrt{6}&\sqrt{6}&2\end{pmatrix}$$

$\left(\frac{2\pi}{3}\right).$

⑥ Za matricu A proveriti da je ortogonalna, determinanta jednaka 1, a zatim odrediti jedinični vektor p i ugao $\phi \in [0,\pi]$ tako da $A = R_p(\phi)$. $\boxed{\text{A2AngleAxis}}$

a) $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ b) $\begin{pmatrix}1&8&-4\\-4&4&7\\8&1&4\end{pmatrix}$ c) $\frac{1}{4}\begin{pmatrix}3&1&\sqrt{6}\\1&3&-\sqrt{6}\\-\sqrt{6}&\sqrt{6}&2\end{pmatrix}$

a)
$$AA^T = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \checkmark$$

$\det A = 1$ ✓

$R_p\,\vec p = \lambda\vec p$

$R_p\,\vec p = \vec p \quad , \lambda = 1$

$(R_p - E)\cdot\vec p = 0 \quad , p = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\begin{pmatrix}-1&0&1\\1&-1&0\\0&1&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$-a + c = 0 \longrightarrow (-1,0,1)\cdot(a,b,c) = 0$
$a - b = 0 \longrightarrow (1,-1,0)\cdot(a,b,c) = 0$
$b - c = 0$

$$\vec p = (-1,0,1)\times(1,-1,0) = \begin{vmatrix}i&j&k\\-1&0&1\\1&-1&0\end{vmatrix} = (1,1,1)$$

$\|\vec p\| = \sqrt{1+1+1} = \sqrt{3}$

$$\boxed{\vec p = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}}$$

$\vec{u} \perp \vec{p}$, npr. $\vec{u} = (-1,0,1)$ ← u sistemu imamo dva takva

$\vec{u}' = R_p(\phi)\cdot u$

$u'^T = R_p(\phi)\cdot u^T = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\cos\phi = \dfrac{\vec{u}\circ\vec{u}'}{\|\vec{u}\|\|\vec{u}'\|} = \dfrac{-1+0+0}{2\cdot 2} = -\dfrac{1}{2} \Rightarrow \phi = \dfrac{2\pi}{3} \vee \phi = \dfrac{4\pi}{3}$

$[\vec{u},\vec{u}',\vec{p}] = \begin{vmatrix}-1&0&1\\1&-1&0\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{vmatrix} = \frac{1}{\sqrt3} > 0 \Rightarrow \boxed{\phi = \frac{2\pi}{3}}$

$\boxed{A = R_p\left(\frac{2\pi}{3}\right) = R_{-p}\left(\frac{4\pi}{3}\right)}$

b) $AA^T = E$ ✓ $\det A = 1$ ✓ $\Rightarrow$ matrica rotacije $\Rightarrow$ jedna sopstvena vrednost je 1

$\vec{p}$ ← sopstveni vektor koji odgovara sopstvenoj vrednosti $\lambda = 1$

$R_p\cdot\vec{p} = \lambda\vec{p}$

$R_p\cdot\vec{p} = \vec{p}$, $\lambda = 1$

$(R_p - E)\vec{p} = 0$, $\vec{p} = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$\frac{1}{9}\begin{pmatrix}-8&8&4\\-4&-5&7\\8&1&-5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

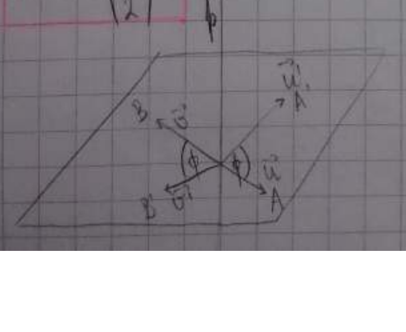
$\Leftrightarrow \begin{cases}-8a+8b+4c=0 & \longrightarrow (-8,8,4)\circ(a,b,c)=0\\ -4a-5b+7c=0 & \longrightarrow (-4,-5,7)\circ(a,b,c)=0\\ 8a+b-5c=0\end{cases}$ ← traži se vektor $\vec{p}$ ortogonalan na $(-8,8,4)$ i $(-4,-5,7)$

$p^T = (-8,8,4)\times(-4,-5,7) = \begin{vmatrix}i&j&k\\-8&8&4\\-4&-5&7\end{vmatrix} = (36,72,72)$

$p^T = (1,2,2)$

$\|\vec{p}\| = \sqrt{1+4+4} = \sqrt9 = 3$

$\boxed{\vec{p} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}}$



$\vec{u}'$ - proizvoljan vektor $\perp$ na $\vec{p}$

u sistemu imamo dva takva

jedan takav je $\vec{u}' = (-8,8,-4)$

tj. $\vec{u} = (-2,2,-1)$

$\vec{w} = R_p(\phi) u$

$$\vec{u_1} = R_p(\phi)\, \vec{u} = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ -4 & 4 & 7 \\ 8 & 1 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 18 \\ 9 \\ -18 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\cos\phi = \frac{\vec{u} \circ \vec{w}}{\|\vec{u}\| \cdot \|\vec{w}\|} = \frac{-4+2+2}{3 \cdot 3} = 0 \Rightarrow \phi = \frac{\pi}{2} \quad \vee \quad \phi = \frac{3\pi}{2}$$

$$[\vec{u}, \vec{w}, \vec{p}] = \begin{vmatrix} -2 & 2 & 1 \\ 2 & 1 & -2 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix} = -4 - 4 - 1 = -9 < 0 \Rightarrow \boxed{\phi = \frac{3\pi}{2}}$$

$$\boxed{A = R_p\left(\frac{3\pi}{2}\right) = R_{-p}\left(\frac{\pi}{2}\right)}$$

v) $AA^T = E$ ✓ $\det A = 1$ ✓

$$(R_p - E)\,\vec{p} = 0 \quad , \vec{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} -1 & 1 & \sqrt{6} \\ 1 & -1 & -\sqrt{6} \\ -\sqrt{6} & \sqrt{6} & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p^+ = (-1, 1, \sqrt{6}) \times (1, -1, -\sqrt{6}) = \begin{vmatrix} i & j & k \\ -1 & 1 & \sqrt{6} \\ -\sqrt{6} & \sqrt{6} & -2 \end{vmatrix} = (-8, -8, 0)$$

$p^+ = (1, 1, 0)$

$\|\vec{p}\| = \sqrt{1+1+0} = \sqrt{2}$

$$\boxed{\vec{p} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$$\vec{u_1} = R_p(\phi)\,\vec{u} = \frac{1}{4}\begin{pmatrix} 3 & 1 & \sqrt{6} \\ 1 & 3 & -\sqrt{6} \\ -\sqrt{6} & \sqrt{6} & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ \sqrt{6} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ -4 \\ 4\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \sqrt{6} \end{pmatrix}$$

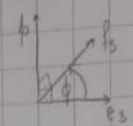
$$\cos\phi = \frac{\vec{u} \circ \vec{w}}{\|\vec{u}\| \cdot \|\vec{w}\|} = \frac{-1-1+6}{\sqrt{8} \cdot \sqrt{8}} = \frac{4}{8} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3} \quad \vee \quad \phi = \frac{5\pi}{3}$$

$$[\vec{u}\,\vec{w}, \vec{p}] = \begin{vmatrix} i & j & k \\ -1 & 1 & \sqrt{6} \\ 1 & -1 & \sqrt{6} \end{vmatrix} = 2\sqrt{6} + 2\sqrt{6} + 0 > 0 \Rightarrow \boxed{\phi = \frac{\pi}{3}}$$

$$\boxed{A = R_p\left(\frac{\pi}{3}\right) = R_{-p}\left(\frac{5\pi}{3}\right)}$$

7. a) Odrediti vektor i ugao (mere) rotacije oko koja rotira vektor $e_3=(0,0,1)$ u $f_3=\left(\frac{\sqrt6}{4},\frac{\sqrt6}{4},\frac12\right)$.

b) Odrediti matricu te rotacije.



e_3 i f_3 ← moraju da budu iste norme

a) $R_p(\phi)=?$, $p\perp e_3,f_3$

$$\vec p=\vec e_3\times\vec f_3=\begin{vmatrix} i & j & k\\ 0 & 0 & 1\\ \frac{\sqrt6}{4} & \frac{\sqrt6}{4} & \frac12\end{vmatrix}=\left(-\frac{\sqrt6}{4},\ \frac{\sqrt6}{4},\ 0\right)$$

$$\|\vec p\|=\sqrt{2\cdot\frac{6}{16}}=\frac{\sqrt3}{2}$$

$$\vec p=\frac{2}{\sqrt3}\left(-\frac{\sqrt6}{4},\frac{\sqrt6}{4},0\right)=\left(-\frac{\sqrt2}{2},\frac{\sqrt2}{2},0\right)\quad,\quad \boxed{\vec p=\left(-\frac{\sqrt2}{2},\frac{\sqrt2}{2},0\right)}$$

$$\cos\phi=\frac{\vec e_3\cdot\vec f_3}{\|\vec e_3\|\cdot\|\vec f_3\|}=\frac{0+0+\frac12}{1\cdot1}=\frac12\ \Rightarrow\ \phi=\frac{\pi}{3}\ \vee\ \phi=\frac{5\pi}{3}$$

$$[\vec e_3,\vec f_3,\vec p]=\begin{vmatrix}0 & 0 & 1\\ \frac{\sqrt6}{4} & \frac{\sqrt6}{4} & \frac12\\ -\frac{\sqrt2}{2} & \frac{\sqrt2}{2} & 0\end{vmatrix}=0-0+\frac{\sqrt3}{2}>0\ \Rightarrow\ \boxed{\phi=\frac{\pi}{3}}$$

$$\Rightarrow R_p\left(\frac{2\pi}{3}\right)=R_{-p}\left(\frac{4\pi}{3}\right)$$

b) $[R_p(\phi)]=?$

$\cos\phi=\frac12$, $\sin\phi=\frac{\sqrt3}{2}$

$$p_\times=\begin{pmatrix}0 & -p_3 & p_2\\ p_3 & 0 & -p_1\\ -p_2 & p_1 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & \frac{\sqrt2}{2}\\ 0 & 0 & \frac{\sqrt2}{2}\\ -\frac{\sqrt2}{2} & -\frac{\sqrt2}{2} & 0\end{pmatrix}$$

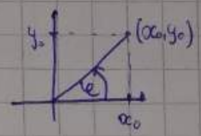
$$pp^T=\begin{pmatrix}-\frac{\sqrt2}{2}\\ \frac{\sqrt2}{2}\\ 0\end{pmatrix}\begin{pmatrix}-\frac{\sqrt2}{2} & \frac{\sqrt2}{2} & 0\end{pmatrix}=\frac12\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$R_p\left(\frac{\pi}{3}\right)=pp^T+\cos\phi\,(E-pp^T)+\sin\phi\cdot p_\times=\begin{pmatrix}\frac34 & -\frac14 & \frac{\sqrt6}{4}\\ -\frac14 & \frac34 & \frac{\sqrt6}{4}\\ -\frac{\sqrt6}{4} & -\frac{\sqrt6}{4} & \frac12\end{pmatrix}$$

↑ primetimo f_3 jer je to $f(e_3)$

A2 Euler

$$\operatorname{arctan2}(\overset{y}{y_0}, \overset{x}{x_0}) = \varphi$$



$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

želimo: $A = R_z(\psi) R_y(\theta) R_x(\phi)$

1° $a_{31} \neq \pm 1 \longrightarrow$
$$\theta = \arcsin(-a_{31})$$
$$\psi = \operatorname{arctan2}(a_{21}, a_{11})$$
$$\phi = \operatorname{arctan2}(a_{32}, a_{33})$$

2° $a_{31} = -1 \longrightarrow$
$$\theta = \arcsin(-a_{31}) = \frac{\pi}{2}$$
$$\phi = 0$$
$$\psi = \operatorname{arctan2}(-a_{12}, a_{22})$$
$$A = \begin{pmatrix} 0 & \sin(\phi-\psi) & \cos(\phi-\psi) \\ 0 & \cos(\phi-\psi) & -\sin(\phi-\psi) \\ -1 & 0 & 0 \end{pmatrix}$$

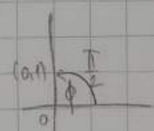
3° $a_{31} = 1 \longrightarrow$
$$\theta = \arcsin(-a_{31}) = -\frac{\pi}{2}$$
$$\phi = 0$$
$$\psi = \operatorname{arctan2}(-a_{12}, a_{22})$$
$$A = \begin{pmatrix} 0 & -\sin(\phi+\psi) & -\cos(\phi+\psi) \\ 0 & \cos(\phi+\psi) & -\sin(\phi+\psi) \\ 1 & 0 & 0 \end{pmatrix}$$

---

(8) Odrediti Eulerove kutove matrica iz (6) zadatka, tj. predstaviti te matrice u obliku $R_z(\psi) R_y(\theta) R_x(\phi)$. $\boxed{\text{A2Euler}}$

a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$a_{31} = 0 \neq \pm 1 \longrightarrow$
$$\theta = \arcsin 0 = 0$$
$$\phi = \operatorname{arctan2}(1, 0) = \frac{\pi}{2}$$
$$\psi = \operatorname{arctan2}(1, 0) = \frac{\pi}{2}$$



$$A = R_z\left(\frac{\pi}{2}\right) R_y(0) R_x\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

b) $A = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ -4 & 4 & 7 \\ 8 & 1 & 4 \end{pmatrix}$

jer $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$a_{31} = \frac{8}{9} \neq \pm 1 \longrightarrow \theta = \arcsin\left(-\frac{8}{9}\right) \longrightarrow \sin\theta = -\frac{8}{9}, \quad \cos\theta = \pm\sqrt{1 - \left(\frac{8}{9}\right)^2} = + \frac{\sqrt{17}}{9}$

$\psi = \operatorname{arctan2}\left(-\frac{4}{9}, \frac{1}{9}\right) \longrightarrow \cos\psi = \frac{\frac{1}{9}}{\sqrt{\left(\frac{1}{9}\right)^2 + \left(\frac{4}{9}\right)^2}} = \frac{\frac{1}{9}}{+\frac{\sqrt{17}}{9}} = \frac{1}{\sqrt{17}}$

$\sin\psi = \frac{-\frac{4}{9}}{\frac{\sqrt{17}}{9}} = -\frac{4}{\sqrt{17}}$

$\phi = \operatorname{arctan2}\left(\frac{1}{9}, \frac{4}{9}\right) \longrightarrow \cos\phi = \frac{\frac{4}{9}}{\frac{\sqrt{17}}{9}} = \frac{4}{\sqrt{17}}$

$\sin\phi = \frac{\frac{1}{9}}{\frac{\sqrt{17}}{9}} = \frac{1}{\sqrt{17}}$

$$A = R_z(\psi) R_y(\theta) R_x(\phi) = \begin{pmatrix} \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{17}}{9} & 0 & -\frac{8}{9} \\ 0 & 1 & 0 \\ \frac{8}{9} & 0 & \frac{\sqrt{17}}{9} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{\sqrt{17}} & -\frac{1}{\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \end{pmatrix}$$

c) $A = \frac{1}{4}\begin{pmatrix} 3 & 1 & \sqrt{6} \\ 1 & 3 & -\sqrt{6} \\ -\sqrt{6} & \sqrt{6} & 2 \end{pmatrix}$

$a_{31} = -\frac{\sqrt{6}}{4} \neq \pm 1 \longrightarrow \theta = \arcsin\left(\frac{\sqrt{6}}{4}\right) \longrightarrow \sin\theta = \frac{\sqrt{6}}{4}, \quad \cos\theta = \pm\sqrt{1 - \left(-\frac{\sqrt{6}}{4}\right)^2} = + \frac{\sqrt{10}}{4}$

$\psi = \operatorname{arctan2}\left(\frac{1}{4}, \frac{3}{4}\right) \longrightarrow \cos\psi = \frac{\frac{3}{4}}{\sqrt{\frac{3^2 + 1^2}{4^2}}} = \frac{3}{\sqrt{10}}$

$\sin\psi = \frac{1}{\sqrt{10}}$

$\phi = \operatorname{arctan2}\left(\frac{\sqrt{6}}{4}, \frac{2}{4}\right) \longrightarrow \cos\phi = \frac{2}{\sqrt{2^2 + \sqrt{6}^2}} = \frac{2}{\sqrt{10}}$

$\sin\phi = \frac{\sqrt{6}}{\sqrt{10}} = \frac{\sqrt{3}}{\sqrt{5}}$

$$A = R_z(\psi) R_y(\theta) R_x(\phi) = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{10}}{4} & 0 & \frac{\sqrt{6}}{4} \\ 0 & 1 & 0 \\ -\frac{\sqrt{6}}{4} & 0 & \frac{\sqrt{10}}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{10}} & -\frac{\sqrt{3}}{\sqrt{5}} \\ 0 & \frac{\sqrt{3}}{\sqrt{5}} & \frac{2}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{\sqrt{6}}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & \frac{1}{2} \end{pmatrix}$$

⑨ Odrediti bar dve trojke Ojlerovih uglova matrica (pošto je $a_{31} = \pm 1$ ti uglovi nisu jedinstveni)

a) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{pmatrix}$

a) $a_{31} = -1 \longrightarrow \theta = \arcsin(-(-1)) = \frac{\pi}{2}$

$$\phi_1 = 0$$

$$\psi_1 = \operatorname{arctan2}(0, -1) = \pi$$

$$A = \begin{pmatrix} 0 & \sin(\phi - \psi) & \cos(\phi - \psi) \\ 0 & \cos(\phi - \psi) & -\sin(\phi - \psi) \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} \sin(\phi - \psi) = 0 \\ \cos(\phi - \psi) = -1 \end{cases} \longrightarrow \phi - \psi = \pi + 2k\pi, k \in \mathbb{Z}$$

$$\phi_2 = \frac{\pi}{4}$$

$$\psi_2 = -\frac{3\pi}{4} \underset{+2k\pi}{=} \frac{5\pi}{4}$$

provera: $A = R_z(\psi_1) R_y(\theta) R_x(\phi_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

b) $a_{31} = 1 \longrightarrow \theta = \arcsin(-1) = -\frac{\pi}{2}$

$$\phi_1 = 0$$

$$\psi_1 = \operatorname{arctan2}(0, -1) = \pi$$

$$A = \begin{pmatrix} 0 & -\sin(\phi + \psi) & -\cos(\phi + \psi) \\ 0 & \cos(\phi + \psi) & -\sin(\phi + \psi) \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} \sin(\phi + \psi) = 0 \\ \cos(\phi + \psi) = -1 \end{cases} \longrightarrow \phi + \psi = \pi + 2k\pi, k \in \mathbb{Z}$$

$$\phi_2 = \frac{\pi}{2}$$

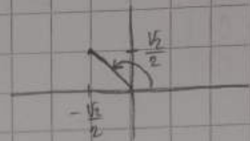
$$\psi_2 = \frac{\pi}{2}$$

provera: $A = R_z(\psi_1) R_y(\theta) R_x(\phi_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

c) $a_{31} = -1 \longrightarrow \theta = \arcsin(-(-1)) = \frac{\pi}{2}$

$\phi_1 = 0$

$\psi_1 = \operatorname{arctan2}\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$



$$\cos\psi_1 = \frac{-\frac{\sqrt{2}}{2}}{\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2}} = -\frac{\sqrt{2}}{2}$$

$$\sin\psi_1 = \frac{\frac{\sqrt{2}}{2}}{\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2}} = \frac{\sqrt{2}}{2}$$

$$A = \begin{pmatrix} 0 & \sin(\phi-\psi) & \cos(\phi-\psi) \\ 0 & \cos(\phi-\psi) & -\sin(\phi-\psi) \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{cases} \sin(\phi-\psi) = -\frac{\sqrt{2}}{2} \\ \cos(\phi-\psi) = -\frac{\sqrt{2}}{2} \end{cases} \Rightarrow \phi-\psi = \frac{5\pi}{4} + 2k\pi, \; k \in \mathbb{Z}$$

$\phi_2 = \frac{5\pi}{4}$

$\psi_2 = 0$

provera: $A = R_z(\psi_1) R_y(\theta) R_x(\phi_1) = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{pmatrix}$

$A = R_z(\psi_2) R_y(\theta) R_x(\phi_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{pmatrix}$

$\mathbb{H} = \{ q = ix + jy + kz + w \mid i^2 = j^2 = k^2 = -1,\ ij = k = -j\cdot i,\ x,y,z,w \in \mathbb{R} \}$

$\vec{v} = \mathrm{Im}\, q = ix + jy + kz \quad , \quad \mathrm{Re}\, q = w$

$\mathbb{H} \cong \mathbb{R}^4 \quad , \quad \mathrm{Im}\, \mathbb{H} \cong \mathbb{R}^3$

čisto imaginarni kvaternioni

Osobine: 1) $q + q_1$

2) $[v_1, 0]\cdot[v_2, 0] = [\overbrace{v_1 \times v_2}^{\text{vektorski}}, \overbrace{-\langle v_1, v_2 \rangle}^{\text{skalarni}}]$

3) $[v_1; w_1]\cdot[v_2, w_2] = [v_1 \times v_2 + w_2 v_1 + w_1 v_2, -\langle v_1, v_2\rangle + w_1 w_2]$

4) $\bar{q} = \overline{[v, w]} = [-v, w] = -ix - jy - kz + w$

5) $\|q\|^2 = \bar{q}q = q\bar{q}$

6) $q^{-1} = \dfrac{\bar{q}}{\|q\|^2}$

Konjugovanje: $C_q(p) = q\cdot p \cdot q^{-1}$

7) $(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$; $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$

8) $\|q_1 \cdot q_2\| = \|q_1\| \cdot \|q_2\|$

---

(10) Dati su kvaternioni $q = 2i + 2k + 1$, $q_1 = i - j$.

a) Izračunati $q + q_1$, qq_1, $q_1 q$, q^{-1}, q_1^{-1}

b) proveriti $qq_1 = [\vec{v}, w][\vec{v}_1, w_1] = [\vec{v} \times \vec{v}_1 + w\vec{v}_1 + w_1\vec{v}, ww_1 - \langle \vec{v}, \vec{v}_1 \rangle]$

c) proveriti $(qq_1)^{-1} = q_1^{-1} q^{-1}$

d) proveriti $\overline{qq_1} = \bar{q}_1 \bar{q}$

e) proveriti $|qq_1| = |q||q_1|$

a) $q + q_1 = 3i - j + 2k + 1$

$qq_1 = (2i + 2k + 1)(i - j) = -2 - 2k + 2j + 2i + i - j = 3i + j - 2k - 2$

$q_1 q = (i - j)(2i + 2k + 1) = -2 - 2j + i + 2k - 2i - j = -i - 3j + 2k - 2$

$q^{-1} = \dfrac{\bar{q}}{\|q\|^2} = \dfrac{-2i - 2k + 1}{\sqrt{2^2 + 2^2 + 1^2}^2} = \dfrac{1}{9}(-2i + 2k + 1)$

$q_1^{-1} = \dfrac{\bar{q}_1}{\|q_1\|^2} = \dfrac{-i + j}{\sqrt{1^2 + 1^2}^2} = \dfrac{1}{2}(-i + j)$

b) $\vec{v} \times \vec{v_1} = \begin{vmatrix} i & j & k \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 2i + 2j - 2k$

$\langle \vec{v}, \vec{v_1} \rangle = \langle (2,0,2), (1,-1,0) \rangle = 2$

$qq_1 = [v,w][v_1,w_1] = [(2,2,-2) + 0\cdot(2,0,2) + 1(1,-1,0),\ 1\cdot 0 - 2] = [(3,1,-2), -2] = 3i + j - 2k - 2$ ✓

c) $(qq_1)^{-1} = \dfrac{\overline{qq_1}}{\|qq_1\|^2} = \dfrac{-3i - j + 2k - 2}{18} = \dfrac{1}{18}(-3i - j + 2k - 2)$ ✓

$q_1^{-1} q^{-1} = \dfrac{-i+j}{2} \cdot \dfrac{-2i - 2k + 1}{9} = \dfrac{1}{18}(-2 - 2j - i + 2k - 2i + j) = \dfrac{1}{18}(-3i - j + 2k - 2)$

d) $\overline{qq_1} = -3i - j + 2k - 2$ ✓

$\overline{q_1}\,\overline{q} = (-i+j)(-2i - 2k + 1) = -3i - j + 2k - 2$

e) $\|q q_1\| = \sqrt{9+1+4+4} = \sqrt{18}$ ✓

$\|q\|\|q_1\| = \sqrt{4+4+1} \cdot \sqrt{1+1} = \sqrt{9 \cdot 2} = \sqrt{18}$

(11) Množeći kvaternione odrediti matricu konjugacije C_q jediničnim kvaternionom $q = \frac{1}{\sqrt{2}}k + \frac{1}{\sqrt{2}}$. Koju on rotaciju predstavlja?

$q^{-1} = \dfrac{\overline{q}}{\|q\|^2} = \dfrac{-\frac{1}{\sqrt{2}}k + \frac{1}{\sqrt{2}}}{1} = -\dfrac{1}{\sqrt{2}}k + \dfrac{1}{\sqrt{2}}$

$p = ix + jy + kz + w$

$C_q(p) = \left(\frac{1}{\sqrt{2}}\right)^2 (k+1)(ix + jy + kz + w)(-k+1) = \frac{1}{2}(jx - iy - z + kw + ix + jy + kz + w)(-k+1) =$

$= \frac{1}{2}(\cancel{ix} - jy + kz + w + jx - iy + \cancel{z} - \cancel{kw} + jx - iy - \cancel{z} + \cancel{kw} + ix + jy + kz + w) =$

$= \frac{1}{2}(-2iy + 2jx + 2kz + 2w) = -iy + jx + kz + w$

$C_q : (x,y,z,w) \longrightarrow (-y, x, z, w)$ ← linearno preslikavanje, tj. $C_q : (x,y,z) \to (-y, x, z)$

$$C_q : \begin{pmatrix} -y \\ x \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ C_q(i) & C_q(j) & C_q(k) & C_q(w) \end{array}$$

$$C_q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_z\left(\frac{\pi}{2}\right)$$

$$\boxed{[C_q] = R_z\left(\frac{\pi}{2}\right)}$$

---

q tako da $\|q\| = 1$

$q = \vec{v} + w$

Neka je $\vec{v'} = \lambda \vec{v}$, $\lambda > 0$ i $\|\vec{v}\| = 1$

$q = A \cdot \vec{v} + w = [\sin\alpha \cdot \vec{v'}, \cos\alpha]$ $\quad\Leftrightarrow\quad$ $q = [\sin\frac{\beta}{2} v', \cos\frac{\beta}{2}]$

$C_q = R_{v'}(2\alpha)$ $\qquad\qquad$ $C_q = R_{v'}(\beta)$

---

(12) Koju rotaciju predstavlja konjugacija C_q kvaternionom $q = i + 2k + 2$?

$\|q\| = 3 \longrightarrow q = \frac{1}{3}(i + 2k + 2) = \frac{1}{3}(\underbrace{i + 2k}_{v}) + \frac{2}{3}$

$\|v\| = \frac{\sqrt{5}}{3} \longrightarrow q = \frac{\sqrt{5}}{3} \cdot \frac{1}{\sqrt{5}}(i + 2k) + \frac{2}{3}$

$q = [\frac{\sqrt{5}}{3} p, \frac{2}{3}]$, $\boxed{p = \frac{1}{\sqrt{5}}(i + 2k)}$

$C_q = R_p(\beta)$

$\cos\frac{\beta}{2} = \frac{2}{3}$, $\sin\frac{\beta}{2} = \sqrt{1 - \cos^2\frac{\beta}{2}} = \frac{\sqrt{5}}{3}$

$\arccos\frac{2}{3} = \frac{\beta}{2}$!!! (pogledati znak za $\sin\frac{\beta}{2}$)

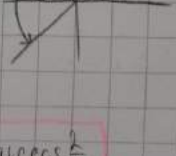
$\beta = 2\arccos\frac{2}{3}$

$C_q = R_p(2\arccos\frac{2}{3})$

$C_q = R_{-p}(2\pi - 2\arccos\frac{2}{3})$

$\arccos[-1,1] \to [0,\pi]$

da smo imali $-\frac{\sqrt{5}}{3}$, tj. $\sin\frac{\beta}{2} < 0$: $\frac{\beta}{2} = -\theta_1 = \underbrace{2\pi - \theta_1}_{\theta_2}$



imaju isti cos:
$\cos\theta_1 = \cos\theta_2$
$\sin\theta_1 > 0$
$\sin\theta_2 < 0$

predstavlja rotaciju $R_p(\phi)$, gde je $p = (\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}})$, a $\boxed{\phi = 2\arccos\frac{2}{3}}$

(13) Odrediti kvaternione koji odgovaraju rotacijama iz zadataka 3, 4, 5.

$\boxed{C_q = C_{-q}}$

a) $R_x(\phi) = R_p(\phi)$, $p = (1,0,0) = \vec{i}$

$\longrightarrow \quad q = [\sin\frac{\phi}{2}(1,0,0), \cos\frac{\phi}{2}] = \sin\frac{\phi}{2}\, i + \cos\frac{\phi}{2}$

b) $R_y(\phi) = R_p(\phi)$, $p = (0,1,0)$

$\longrightarrow \quad q = [\sin\frac{\phi}{2}(0,1,0), \cos\frac{\phi}{2}] = \sin\frac{\phi}{2}\cdot j + \cos\frac{\phi}{2}$

c) $R_z(\phi) = R_p(\phi)$, $p = (0,0,1)$

$\longrightarrow \quad q = [\sin\frac{\phi}{2}(0,0,1), \cos\frac{\phi}{2}] = \sin\frac{\phi}{2}\cdot k + \cos\frac{\phi}{2}$

d) $R_p(\phi)$, $p = \frac{\sqrt{3}}{3}(1,1,1)$, $\phi = \frac{2\pi}{3}$
← jeste jedinični

$\sin\frac{\phi}{2} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\phi}{2} = \cos\frac{\pi}{3} = \frac{1}{2}$

$\longrightarrow \quad q = [\frac{1}{2}(1,1,1), \frac{1}{2}] = \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k + \frac{1}{2}$ ← q jeste jedinični

e) $R_p(\phi)$, $p = \frac{1}{3}(1,2,2)$, $\phi = \frac{3\pi}{2}$
← jeste jedinični

$\sin\frac{\phi}{2} = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\phi}{2} = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$\longrightarrow \quad q = [\frac{\sqrt{2}}{2}\cdot\frac{1}{3}(1,2,2), -\frac{\sqrt{2}}{2}] = \frac{\sqrt{2}}{6}i + \frac{\sqrt{2}}{3}j + \frac{\sqrt{2}}{3}k - \frac{\sqrt{2}}{2}$ ← q jeste jedinični

f) $R_p(\phi)$, $p = \frac{\sqrt{2}}{2}(1,1,0)$, $\phi = \frac{\pi}{3}$

$\sin\frac{\phi}{2} = \sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\phi}{2} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\longrightarrow \quad q = [\frac{1}{2}\cdot\frac{\sqrt{2}}{2}(1,1,0), \frac{\sqrt{3}}{2}] = \frac{\sqrt{2}}{4}i + \frac{\sqrt{2}}{4}j + \frac{\sqrt{3}}{2}$

$\|q\| = \sqrt{\frac{2}{16} + \frac{2}{16} + \frac{3}{4}} = \sqrt{\frac{16}{16}} = 1$ ← jeste jedinični

(14) Kompozicije rotacija iz zadatka 2. predstaviti množenjem kvaterniona, a zatim ih predstaviti kao rotacije oko ose.

C_q - odgovara nekim svetskim rotacijama: $\boxed{C_{q_1} C_q = C_{q_1 q}}$

i) $R_x(\frac{\pi}{2}) \cdot R_z(\frac{3\pi}{2}) = ?$

$C_q = R_z(\frac{3\pi}{2})$: $p = (0,0,1)$, $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{3\pi}{4} = \frac{-\sqrt{2}}{2}$

$\longrightarrow q = [\frac{\sqrt{2}}{2}(0,0,1), -\frac{\sqrt{2}}{2}]$

$C_{q_1} = R_x(\frac{\pi}{2})$: $p = (1,0,0)$, $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\longrightarrow q_1 = [\frac{\sqrt{2}}{2}(1,0,0), \frac{\sqrt{2}}{2}]$

$R_x(\frac{\pi}{2}) \cdot R_z(\frac{3\pi}{2}) \longrightarrow C_{q_1} C_q = C_{q_1 q}$

$$q_1 q = \frac{\sqrt{2}}{2}(i+1)\frac{\sqrt{2}}{2}(k-1) = \frac{1}{2}(-j-i+k-1) = \frac{1}{2}(-i-j+k-1)$$

$$q_1 q = [\frac{1}{2}(-1,-1,1), -\frac{1}{2}] = [\frac{\sqrt{3}}{2} \cdot \underbrace{\frac{1}{\sqrt{3}}(-1,-1,1)}, -\frac{1}{2}]$$

$$\left.\begin{array}{l} \sin\frac{\phi}{2} = \frac{\sqrt{3}}{2} \\ \cos\frac{\phi}{2} = -\frac{1}{2} \end{array}\right\} \rightarrow \phi = 2\arccos(-\frac{1}{2})$$

$$\phi = 2 \cdot \frac{2\pi}{3} = \boxed{\frac{4\pi}{3}}$$

$\arccos[-1,1] \rightarrow [0,\pi]$

ako je $\left\{\begin{array}{l} \sin > 0 \text{, onda } \phi \in [0,\pi] \text{, zadržavamo } \arccos \\ \sin < 0 \text{, onda } 2\pi - \arccos \end{array}\right.$

$\boxed{R_x(\frac{\pi}{2}) \cdot R_z(\frac{3\pi}{2}) = R_p(\frac{4\pi}{3})}$, $\boxed{p = \frac{1}{\sqrt{3}}(-1,-1,1)}$

ii) $R_x(\pi) \cdot R_y(\frac{\pi}{2}) = ?$

$C_q = R_y(\frac{\pi}{2})$: $p = (0,1,0)$, $\sin\frac{\phi}{2} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\phi}{2} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\rightarrow q = [\frac{\sqrt{2}}{2}(0,1,0), \frac{\sqrt{2}}{2}] = \frac{\sqrt{2}}{2}j + \frac{\sqrt{2}}{2}$

$C_{q_1} = R_x(\pi)$: $p = (1,0,0)$, $\sin\frac{\phi}{2} = 1$, $\cos\frac{\phi}{2} = 0$

$\rightarrow q = [1(1,0,0), 0] = i$

$$q_1 q = i \cdot \frac{\sqrt{2}}{2}(j+1) = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k = [\underbrace{\frac{\sqrt{2}}{2}(1,0,1)}_{p}, 0]$$

$$\left.\begin{array}{l} \sin\phi = 1 \\ \cos\phi = 0 \end{array}\right\} \Rightarrow \boxed{\phi = \pi}$$

$\boxed{R_x(\pi) \cdot R_y(\frac{\pi}{2}) = R_p(\pi)}$, $\boxed{p = \frac{\sqrt{2}}{2}(1,0,1)}$

(15) a) Odrediti linearnu interpolaciju između tačaka $C_1(0,3,2)$ i $C_2(6,9,-1)$.

b) Odrediti k-ti frejm (4x4 matricu), ako animacija traje 3 sekunde, sa 60 frejmova u sekundi.

a) $c(0)=C_1 \xrightarrow{c(t)} C_2=c(1)$

$C(t) = C_1 + t\cdot \underbrace{\overrightarrow{C_1C_2}}_{C_2-C_1}$

$$\boxed{C(t) = (1-t)C_1 + tC_2 \quad , t\in[0,1]}$$

$C(0)=C_1, \; C(1)=C_2$

b) $3\times 60 = 180$

$C_k = \left(1-\frac{k}{180}\right)C_1 + \frac{k}{180}C_2 \quad , k=\overline{0,180}$ ← k-ti frejm

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} & A_{3\times3} & & c_x \\ & & & c_y \\ & & & c_z \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$\rightarrow C(t) = (x(t), y(t), z(t))$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & c_x(t) \\ 0 & 1 & 0 & c_y(t) \\ 0 & 0 & 1 & c_z(t) \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

↑ tačka koju želimo da preslikamo npr. C_1

$t = \frac{0}{180}, \frac{1}{180}, \dots, \frac{180}{180}$

* Pri kretanju od C_1 do C_2 gde se nalazi C u 60. frejmu?

$C_{60} = \left(1-\frac{60}{180}\right)C_1 + \frac{60}{180}C_2 = \frac{2}{3}C_1 + \frac{1}{3}C_2 = (2,5,1)$

$$C'_{1,60} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$ ← slika tačke C_1 u 60 frejmu

(16) Odrediti 4x4 matricu k-tog frejma interpolacije između položaja $(C_1(1,2,3), Id)$ i položaja $(C_2(3,3,5), R_y(-\frac{3\pi}{2}))$. Animacija treba da traje 5 sekundi sa 24 frejma u sekundi, a translacija i rotacija se dešavaju istovremeno. Interpolaciju "orijentacije" uraditi

a) primenom rotacije oko O_y ose, u ovom slučaju odrediti matricu 80-og frejma.

b) primenom funkcija $Slerp(q_1, q_2, t_m, t)$, $Q2AngleAxis(q)$ i $Rodrigez(p, \phi)$.

$(C_1(1,2,3), Id) \to (C_2(3,3,5), R_y(-\frac{3\pi}{2}))$

↑ ako je $> \pi$ uzimamo manji ugao

a) $R_y(-\frac{3\pi}{2}) = R_y(\frac{\pi}{2})$

$5s \times 24 f_s = 120 f$

$A_k = R_y(\frac{k}{120} \cdot \frac{\pi}{2})$, $k = \overline{0, 120}$

$C_k = (1 - \frac{k}{120}) C_1 + \frac{k}{120} C_2$

$$[f_k] = \begin{pmatrix} & A_k & & c_{kx} \\ & & & c_{ky} \\ & & & c_{kz} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$C_k = (c_{kx}, c_{ky}, c_{kz})$ $\quad C_{80} = (1 - \frac{80}{120}) C_1 + \frac{80}{120} C_2 = \frac{1}{3}(1,2,3) + \frac{2}{3}(3,3,5) = (\frac{7}{3}, \frac{8}{3}, \frac{11}{3})$

$k = 80$:

$$[f_{80}] = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & \frac{7}{3} \\ 0 & 1 & 0 & \frac{8}{3} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & \frac{11}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$A_{80} = R_y(\frac{80}{120} \cdot \frac{\pi}{2}) = R_y(\frac{\pi}{3})$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$

b) $q_1 = \pm 1 \xrightarrow{\text{npr.}} q_1 = +1$

$q_2 = [\sin\frac{\pi}{4}(0,1,0), \cos\frac{\pi}{4}] = \frac{\sqrt{2}}{2} j + \frac{\sqrt{2}}{2}$ ← kvaternion koji odg. rotaciji oko y za $\frac{\pi}{2}$, to znamo

Algoritam:

1° $q_k = slerp(q_1, q_2, 5, \frac{k}{24})$

2° $(p_k, \varphi_k) = Q2AngleAxis(q_k)$

3° $A_k = Rodrigez(p_k, \varphi_k)$

4° $[f_k] = (A_k |\ )$

(17) Odrediti 4x4 matricu k-tog frejma interpolacije između položaja $(C_1(0,0,5), R_z(\frac{5\pi}{6}))$ i položaja $(C_2(3,4,5), R_z(\frac{\pi}{2})R_x(\frac{\pi}{2}))$. Rotacija i translacija se dešavaju istovremeno i traju 10 sekundi sa 24 frejma u sekundi. Koristiti f-je $Slerp(q_1, q_2, t_m, t)$, $Q2AngleAxis(q)$ i $Rodrigez(p, \phi)$.

$$\left(C_1(0,0,5); R_z\left(\frac{5\pi}{6}\right)\right) \longrightarrow \left(C_2(3,4,5); R_z\left(\frac{\pi}{2}\right)\cdot R_x\left(\frac{\pi}{2}\right)\right)$$

$10 \times 24 = 240$

$$q_1 = \left[\sin\frac{5\pi}{12}(0,0,1), \cos\frac{5\pi}{12}\right] = \sin\frac{5\pi}{12}k + \cos\frac{5\pi}{12}$$

$$q_2 = \left[\sin\frac{\pi}{4}(0,0,1), \cos\frac{\pi}{4}\right]\cdot\left[\sin\frac{\pi}{4}(1,0,0), \cos\frac{\pi}{4}\right] = \left(\frac{\sqrt{2}}{2}k + \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}\right) = \frac{1}{2}(i+j+k+1)$$

$\longrightarrow q_k = slerp\left(q_1, q_2, 5, \frac{k}{24}\right)$

$\longrightarrow (p_k, \varphi_k) = Q2AngleAxis(q_k)$

$\longrightarrow A_k = Rodrigez(p_k, \varphi_k)$

$\longrightarrow [f_k] = \left(\begin{array}{c|c} A_k & \end{array}\right)$

(18) (GluLookAt funkcija)

a) Date su koordinate pozicije kamere $E(1,2,3)$, koordinate centra scene $C(3,4,-1)$ i koordinate UP vektora $f_2 = (2,-1,4)$ (on označava pravac "iznad kamere"). Odrediti 4x4 matricu koja zadaje položaj (poziciju i orijentaciju) kamere.

Napomena: koordinatni sistem $\vec{f_1}, \vec{f_2}, \vec{f_3}$ kamere je kao u OpenGL-u: pozitivne orijentacije i kamera "gleda" duž negativne z ose kamere.

b) Proveriti da je matrica "orijentacije" A ortogonalna i važi $\det A = 1$.

a) $\vec{f_3} = -\vec{EC} = (-2,-2,4)$

$\vec{up} \in \mathcal{L}(\vec{f_3}, \vec{f_2})$, $(\vec{up} \perp f_1)$

$\vec{up} = (2,-1,4)$ $(= \vec{f_2})$ $\leftarrow$ $\vec{up} = \vec{f_3} \times \vec{f_2}$, tj ne mora neki biti $= f_2$, tj nije nek $\vec{up} \perp f_3$

$\vec{f_1} \perp \vec{f_2}, \vec{f_3}, \vec{up}$

$[\vec{f_1}, \vec{f_2}, \vec{f_3}]$ - o.n. baza

jer $\vec{f_1} \perp \vec{f_3}$ i $\vec{up}$:

$$\vec{f_1} = \vec{up} \times \vec{f_3} = \vec{up} \times (-\vec{EC}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 4 \\ -2 & -2 & 4 \end{vmatrix} = (4,-16,-6) = 2(2,-8,-3)$$

$$\|\vec{f_1}\| = \sqrt{16+256+36} = 2\sqrt{77}$$

$\vec{f_1} = \frac{1}{\sqrt{77}}(2,-8,-3)$

$\|\vec{f_3}\| = \sqrt{2^2+2^2+4^2} = 2\sqrt{6}$

$\vec{f_3} = \frac{1}{\sqrt{6}}(-1,-1,2)$ + ako nemamo $E \neq C$, $\vec{f_3} = \vec{f_2} \times \vec{f_1}$

$\vec{f_2} = \vec{f_3} \times \vec{f_1} = \frac{1}{\sqrt{21}}(2,-1,4)$

bitan redosled, da bude poz. orijentisano

$$f = \begin{pmatrix} \frac{2}{\sqrt{77}} & \frac{2}{\sqrt{21}} & -\frac{1}{\sqrt{6}} & 1 \\ -\frac{8}{\sqrt{77}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{6}} & 2 \\ -\frac{3}{\sqrt{77}} & \frac{4}{\sqrt{21}} & \frac{2}{\sqrt{6}} & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ f_1 & f_2 & f_3 & E \end{matrix}$$

b) $AA^T = E$ - dovoljno da se pokaže da je $[\vec{f_1}, \vec{f_2}, \vec{f_3}]$ - on baza.

$\det A = 1$ - dovoljno da $\det A > 0$, izračunamo bez korena

(17) bez slerp-a: (preko Ojlerovih uglova) $10 \times 24 = 240$

$$C_k = \left(1 - \frac{k}{240}\right)C_1 + \frac{k}{240}C_2 = \frac{k}{240}(3,4,0) + (0,0,5) = \left(\frac{k}{80}, \frac{k}{60}, 5\right)$$

$\psi_1 = \frac{5\pi}{6}$ $\quad \psi_2 = \frac{\pi}{2}$

$\theta_1 = 0$ $\quad \theta_1 = 0$

$\phi_1 = 0$ $\quad \phi_2 = \frac{\pi}{2}$

$$\psi_k = \left(1 - \frac{k}{240}\right)\psi_1 + \frac{k}{240}\psi_2 = \frac{k}{240}\left(\frac{\pi}{2} - \frac{5\pi}{6}\right) + \frac{5\pi}{6} = \frac{5\pi}{6} - \frac{k}{240} \cdot \frac{\pi}{3}$$

$\theta_k = 0$

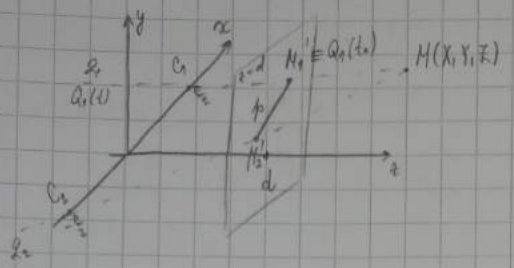
$$\phi_k = \left(1 - \frac{k}{240}\right)\phi_1 + \frac{k}{240}\phi_2 = \frac{k}{240} \cdot \frac{\pi}{2}$$

$$A_k = \underbrace{\begin{pmatrix} \cos\psi_k & -\sin\psi_k & 0 \\ \sin\psi_k & \cos\psi_k & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_z(\psi_k)} \underbrace{\begin{pmatrix} \cos\theta_k & 0 & \sin\theta_k \\ 0 & 1 & 0 \\ -\sin\theta_k & 0 & \cos\theta_k \end{pmatrix}}_{R_y(\theta_k)} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_k & -\sin\phi_k \\ 0 & \sin\phi_k & \cos\phi_k \end{pmatrix}}_{R_x(\phi_k)} = \begin{pmatrix} \cos\psi_k & -\sin\psi_k\cos\phi_k & \sin\psi_k\sin\phi_k \\ \sin\psi_k & \cos\psi_k\cos\phi_k & -\cos\psi_k\sin\phi_k \\ 0 & \sin\phi_k & \cos\phi_k \end{pmatrix}$$

$$[f_k] = \left(\begin{array}{c|c} A_k & \begin{matrix} \frac{k}{80} \\ \frac{k}{60} \\ 5 \end{matrix} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right)$$

① (a) Izvesti formule projektovanja iz dve tačke $C_{1,2} = (\pm\frac{e}{2}, 0, 0)$ na ravan $z=d$.

(b) Izvesti formule za paralaksu p.



a) $q_1 = q_1(C_1, M)$

$\vec{v}_{q_1} = \vec{C_1M} = (X - \frac{e}{2}, Y, Z)$

$$q_1 : \frac{x - \frac{e}{2}}{X - \frac{e}{2}} = \frac{y}{Y} = \frac{z}{Z} = t$$

+ C_1 tačka kroz koju prava prolazi
+ $\vec{C_1M}$

$$q_1 : \begin{cases} x = t(X - \frac{e}{2}) + \frac{e}{2} \\ y = tY \\ z = tZ \end{cases}$$

$q_1 \cap \{z = d\} \Rightarrow tZ = d$

$t_0 = \frac{d}{Z}$

$M_1'\left(\frac{d}{Z}(X - \frac{e}{2}) + \frac{e}{2}, \frac{d}{Z}Y, \frac{d}{Z} \cdot Z\right)$

$M_2'\left(\frac{d}{Z}(X + \frac{e}{2}) - \frac{e}{2}, \frac{d}{Z}Y, d\right)$

b) $p = M_1'M_2' = \frac{d}{Z}\frac{e}{2} - \frac{e}{2} - \left(-\frac{d}{Z}\frac{e}{2} + \frac{e}{2}\right) = e\left(\frac{d}{Z} - 1\right)$

② Odrediti udaljenost objekta Z.

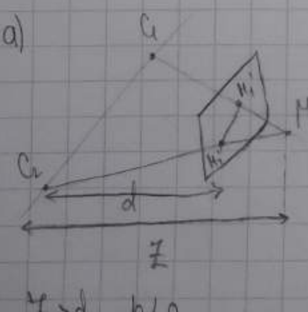
a) $d = 76cm = 760mm$
$e = 63mm$
$p = -60 mm$

b) $d = 76cm$
$e = 63mm$
$p = 100 mm$

paralelni pogled: $Z > d$, $p < 0$

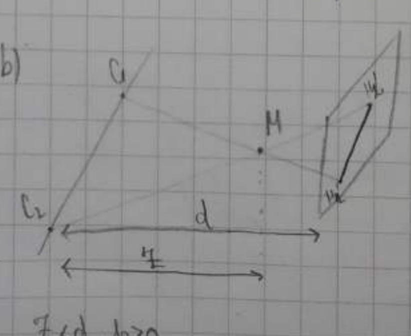
ukršteni pogled: $Z < d$, $p > 0$

a)



$Z > d$, $p < 0$

b)



$Z < d$, $p > 0$

a) $p = c\left(\frac{d}{z} - 1\right)$

b) $z = \frac{63\cdot 700}{100+63} = 294\text{ mm} = 29.4\text{ cm}$

$\frac{p}{c} + 1 = \frac{d}{z}$

$z = \frac{cd}{p+c} = \frac{63\cdot 760}{-60+63} = 21\cdot 760 = 15960\text{ mm} \approx 16\text{ m}$

③ Odrediti centar kamere C, $T = \begin{pmatrix} -2 & 3 & 0 & 7 \\ -3 & 0 & 3 & -6 \\ 1 & 0 & 0 & -2 \end{pmatrix}$.

$C(c_1 : c_2 : c_3 : c_4)$

$\vec{TC} = \vec{0}$

$$\begin{pmatrix} -2 & 3 & 0 & 7 \\ -3 & 0 & 3 & -6 \\ 1 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(c_1 : c_2 : c_3 : c_4) = \left(+ \begin{vmatrix} 3 & 0 & 7 \\ 0 & 3 & -6 \\ 0 & 0 & -2 \end{vmatrix} : - \begin{vmatrix} -2 & 0 & 7 \\ -3 & 3 & -6 \\ 1 & 0 & -2 \end{vmatrix} : + \begin{vmatrix} -2 & 3 & 7 \\ -3 & 0 & -6 \\ 1 & 0 & -2 \end{vmatrix} : - \begin{vmatrix} -2 & 3 & 0 \\ -3 & 0 & 3 \\ 1 & 0 & 0 \end{vmatrix} \right)$$

$$= \left(3(-6+0) + 7(0+0) : 2(-6) - 7(-3) : -3(6+6) : -3\cdot 3\right)$$

$$= (-18 : 9 : -36 : -9) = (-2 : 1 : -4 : -1)$$

$\boxed{C(2, -1, 4)}$

④ a) Odrediti dekompoziciju $T_0 = KA$ kamere iz prethodnog zadatka.

b) Kolika je žižna daljina d kamere?

c) Skicirati kameru i zastor u odnosu na svetski koordinatni sistem.

a) $\boxed{T_0^{-1} = QR}$

$R^{-1}Q^{-1} = T_0 = KA$

g.trougaona ort

$\boxed{A = Q^{-1} = Q^T}$

$T_0 = KA \,/\, A^T$

$T_0 A^T = K$

$\boxed{K = T_0 Q}$

$T_0^{-1} = \frac{1}{\det T_0} \cdot \operatorname{adj} T_0^T$

$$\det T_0 = \begin{vmatrix} -2 & 3 & 0 \\ -3 & 0 & 3 \\ 1 & 0 & 0 \end{vmatrix} = -3\begin{vmatrix} -3 & 3 \\ 1 & 0 \end{vmatrix} = -3(-3) = 9$$

$$T_0^T = \begin{pmatrix} -2 & -3 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$\operatorname{adj} T_0 = \begin{pmatrix} +\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} -3 & 1 \\ 3 & 0 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} -2 & -3 \\ 0 & 3 \end{vmatrix} \\ +\begin{vmatrix} -3 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} & +\begin{vmatrix} -2 & -3 \\ 3 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 9 \\ 3 & 0 & 6 \\ 0 & 3 & 9 \end{pmatrix}$$

$$T_0^{-1} = \frac{1}{9} \begin{array}{c} \begin{pmatrix} 0 & 0 & 9 \\ 3 & 0 & 6 \\ 0 & 3 & 9 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ t_1 & t_2 & t_3 \end{array} \end{array}$$

$t_1 = (0, \frac{1}{3}, 0)$

$t_2 = (0, 0, \frac{1}{3})$

$t_3 = (1, \frac{2}{3}, 1)$

Gram-Šmit: tražimo Q

$$q_1 = t_1 = (0, \tfrac{1}{3}, 0)$$

$$q_2 = t_2 - \frac{\langle t_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 = (0, 0, \tfrac{1}{3}) - 0 \cdot q_1 = (0, 0, \tfrac{1}{3})$$

$$q_3 = t_3 - \frac{\langle t_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2 - \frac{\langle t_3, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 = (1, \tfrac{2}{3}, 1) - \frac{\frac{1}{3}}{\frac{1}{9}} (0, 0, \tfrac{1}{3}) - \frac{\frac{2}{3}}{\frac{1}{9}} (0, \tfrac{1}{3}, 0) = (1, 0, 0)$$

$[q_1, q_2, q_3]$ je ortogonalna

$$q_1 = \frac{q_1}{\|q_1\|} = (0, 1, 0)$$

$$q_2 = \frac{q_2}{\|q_2\|} = (0, 0, 1)$$

$$q_3 = \frac{q_3}{\|q_3\|} = (1, 0, 0)$$

$$Q = \begin{array}{c} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ q_1 & q_2 & q_3 \end{array} \end{array}$$

$$\boxed{A = Q^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

$$K = T_0 Q = \begin{pmatrix} -2 & 3 & 0 \\ -3 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{K = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{pmatrix}}$$

b) $d = 3$

c) x-osa kamere je prva kolona A, tj. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

y-osa kamere je druga kolona A, tj. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

z-osa kamere je treća kolona A, tj. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Senzor je ravan $z=3$ u sistemu kamere (x, y, z)