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# MINIMAL TOPOLOGIES ON THE SEMIGROUPS OF MATRIX UNITS

**Markiian KHYLYNSKYI, Pavlo KHYLYNSKYI**

*Ivan Franko National University of Lviv,*
*Universitetska str., 1, 79000 Lviv, Ukraine*
*e-mail: khymarkiyan@gmail.com, khypavlo@gmail.com*

We describe minimal topologies in some class of semigroup topologies on the semigroups of matrix units.

*Key words:* topology, semigroup topology, minimal topology, semigroups of matrix units

## 1. Introduction, motivation and main definitions

In this paper all topological spaces are assumed to be Hausdorff.

A *topological semigroup* is a Hausdorff topological space together with a continuous semigroup operation . If $S$ is a semigroup and $\tau$ is a topology on $S$ such that $(S, \tau)$ is a topological semigroup, then we shall call $\tau$ *semigroup topology* on $S$. A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation. Topological semigroup $(S, \tau)$ is said to be *minimal* if no semigroup topology on $S$ is strictly contained in $\tau$. If $(S, \tau)$ is minimal topological semigroup, then $\tau$ is called *minimal semigroup topology.*

The concept of minimal topological groups was introduced independently in the early 1970's by Doitchinov [2] and Stephenson [6]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time. More than 20 years earlier L. Nachbin [5] had studied minimality in the context of division rings, and B. Banaschewski [1] investigated minimality in the more general setting of topological algebras.

Let $\lambda$ be a nonempty set. By $B_\lambda$ we denote the set $\lambda \times \lambda \cup \{0\}$ endowed with the following semigroup operation:

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & b = c; \\ 0, & b \neq c \end{cases}$$

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*2010 Mathematics Subject Classification*: 18A22 and $(a,b)\cdot 0 = 0\cdot(a,b) = 0\cdot 0 = 0$, for each $a,b,c,d\in\lambda$. The semigroup $B_\lambda$ is called the *semigroup of* $\lambda\times\lambda$*-matrix units*. The semitopological and topological semigroup of matrix units was investigated in [3].

A *directed graph* (or just *digraph*) $D$ consists of a nonempty set $V(D)$ of elements called *vertices* and a set $A(D)$ of ordered pairs of vertices called *arcs*. We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of $D$. The *order* (*size*) of $D$ is the cardinality of the vertex (arc) set of $D$. For an arc $(u,v)$ the first vertex $u$ is its *tail* and the second vertex $v$ is its *head*. If a tail and a head of arc coincide, then this arc is called a *loop*. The head and tail of an arc are its *end-vertices*. A vertex $v$ is a *source*(*source*) if $v$ is not a head(tail) for any arc. A digraph $H$ is a *subdigraph* of a digraph $D$ if $V(H)\subseteq V(D)$, $A(H)\subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that $H$ is induced by $X = V(H)$ and call $H$ an induced subdigraph of $D$.

A *walk* in $D$ is an alternating sequence $W = x_1a_1x_2a_2x_3\ldots x_{k-1}a_{k-1}x_k$ of vertices $x_i$ and arcs $a_j$ from $D$ such that the tail of $a_i$ is $x_i$ and the head of $a_i$ is $x_{i+1}$ for every $i = 1,2,\ldots,k-1$. The *length* of a walk is the number of its arcs. When the arcs of $W$ are defined from the context or simply unimportant, we will denote $W$ by $x_1x_2\ldots x_k$. If the vertices of $W$ are distinct, $W$ is a *path*. If the vertices $x_1,x_2,\ldots,x_{k-1}$ are distinct and $x_1 = x_k$, $W$ is a *cycle*. A walk (path, cycle) $W$ is a *Hamilton* (or *hamiltonian*) walk (path, cycle) if $W$ contains all vertices of $D$.

Let $\{D_i\}_{i\in I}$ be a family of digraphs. The digraph $(\bigsqcup\limits_{i\in I} V(D_i), \bigsqcup\limits_{i\in I} A(D_i))$ is called *disjoint union* of this family and denoted by $\bigoplus\limits_{i\in I} D_i$. If $D$ is a digraph and $\mathcal{R}$ is an equivalence relation on $V(D)$. Then the *quotient digraph* has vertex set $V/\mathcal{R}$ and arc set $\{([a]_\mathcal{R},[b]_\mathcal{R}) \mid (a,b)\in A(D)\}$.

## 2. Compositional families

If $(B_\lambda,\tau)$ is a semitopological semigroup, then any nonzero element of $B_\lambda$ is an isolated point of $(B_\lambda,\tau)$ [3, Lemma 2]. Therefore the following lemma is true.

**Lemma 1.** *Let $(B_\lambda,\tau)$ be a topological semigroup and $A$ be a closed subset $A$ of $(B_\lambda,\tau)$ which doesn't contain* $0$. *Then any subset of $A$ is closed.*

For $A\subseteq B_\lambda$ and $\alpha,\beta\in\lambda$ we denote

$$\begin{aligned}
{}_\alpha A_\beta &= \{(\beta,\gamma) \mid (\alpha,\gamma)\in A\};\\
{}^\alpha_\beta A &= \{(\gamma,\beta) \mid (\gamma,\alpha)\in A\};\\
\mathrm{pr}_1(A) &= \{\alpha \mid (\alpha,\beta)\in A\};\\
\mathrm{pr}_2(A) &= \{\beta \mid (\alpha,\beta)\in A\}.
\end{aligned}$$

**Lemma 2.** *Let $\tau$ be a topology on $B_\lambda$ and any nonzero element of $B_\lambda$ is isolated in $(B_\lambda,\tau)$. The semigroup operation is continues on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$ if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for every $\alpha,\beta\in\lambda$ and every closed subset $A$ of $(B_\lambda,\tau)$ which doesn't contain* $0$.

*Proof.* ($\Rightarrow$) Let $A$ be a closed subset $A$ of $(B_\lambda,\tau)$ which doesn't contain 0. Lemma 1 implies that the sets ${}_\alpha A_\alpha$ and ${}^\alpha_\alpha A$ are closed. By the continuity of operation, the maps $\lambda_{(\alpha,\beta)}: B_\lambda \to B_\lambda$ and $\rho_{(\beta,\alpha)}: B_\lambda \to B_\lambda$ defined by the formulas $\lambda_{(\alpha,\beta)}(x) = (\alpha,\beta)\cdot x$ and $\rho_{(\beta,\alpha)}(x) = x\cdot(\beta,\alpha)$ are continuous. Therefore the sets

$$ {}_\alpha A_\beta = (\lambda_{(\alpha,\beta)})^{-1}({}_\alpha A_\alpha) $$

and

$$ {}^\alpha_\beta A = (\rho_{(\beta,\alpha)})^{-1}({}^\alpha_\alpha A) $$

are closed in the topological space $(B_\lambda,\tau)$.

($\Leftarrow$) Since every nonzero point of $B_\lambda$ is isolated, we need check the continuity of operation only in the cases of $(\alpha,\beta)\cdot 0$ and $0\cdot(\alpha,\beta)$. Let $U$ be a neighborhood of 0. Denote by $A$ the closed set $B_\lambda \setminus U$. Then the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed. Denote by $V$ and $W$ the neighborhoods of zero $B_\lambda \setminus {}_\alpha A_\beta$ and $B_\lambda \setminus {}^\alpha_\beta A$, respectively. Therefore $\{(\alpha,\beta)\}\cdot V \subseteq U$ and $W\cdot\{(\alpha,\beta)\} \subseteq U$.

Lemma 2 implies the following corollary.

**Corollary 1.** *Let $\tau$ be a topology on $B_\lambda$. $(B_\lambda,\tau)$ is a semitopological semigroup if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for every $\alpha,\beta\in\lambda$ and every closed subset $A$ of $(B_\lambda,\tau)$ which doesn't contain* 0.

**Lemma 3.** *Let $A, B \subseteq B_\lambda$ and $(\alpha,\beta)\in B_\lambda$. The element $(\alpha,\beta)\notin A\cdot B$ if and only if $\mathrm{pr}_2({}_\alpha A_\alpha)\cap \mathrm{pr}_1({}^\beta_\beta B) = \varnothing$.*

*Proof.* ($\Rightarrow$) Suppose that there exists $\gamma \in \mathrm{pr}_2({}_\alpha A_\alpha)\cap \mathrm{pr}_1({}^\beta_\beta B)$ then $(\alpha,\gamma)\in A$ and $(\gamma,\beta)\in B$. Therefore $(\alpha,\gamma)\cdot(\gamma,\beta) = (\alpha,\beta)\in A\cdot B$, a contradiction.

($\Leftarrow$) Suppose that $\mathrm{pr}_2({}_\alpha A_\alpha)\cap \mathrm{pr}_1({}^\beta_\beta B) = \varnothing$ and $(\alpha,\beta)\in A\cdot B$. Then there exist $\gamma\in\lambda$ such that $(\alpha,\gamma)\in A$ and $(\gamma,\beta)\in B$. Hence $\gamma \in \mathrm{pr}_2({}_\alpha A_\alpha)\cap \mathrm{pr}_1({}^\beta_\beta B)$, a contradiction.

We will call elements of the set $\{A\times B \mid A, B\subseteq B_\lambda\}$ by rectangles.

**Definition 1.** *A nonempty family $\mathcal{F}$ of rectangles is called compositional if for $A\times B\in\mathcal{F}$ there exists $C\subseteq\lambda$ such that $A\times(\lambda\setminus C)\in\mathcal{F}$ and $C\times B\in\mathcal{F}$.*

**Lemma 4.** *Let $\tau$ be a semigroup topology on $B_\lambda$, then the family of all closed rectangles of $(B_\lambda,\tau)$ is compositional.*

*Proof.* Since every point of $(B_\lambda,\tau)$ is closed, the family of all closed rectangles is not empty. Let $A\times B$ be a closed subset of $(B_\lambda,\tau)$, then $B_\lambda\setminus(A\times B)$ is a neighborhood of 0. By the continuity of operation at the point $(0,0)$ there exist neighborhoods of zero $U$ and $V$ such that $U\cdot V\subseteq B_\lambda\setminus(A\times B)$. Lemma 3 implies that for each $\alpha\in A$ the neighborhood $V$ doesn't contain the set $\mathrm{pr}_2({}_\alpha V_\alpha)\times B$. Therefore, the neighborhood $V$ doesn't contain the set $(\bigcup\limits_{\alpha\in A}\mathrm{pr}_2({}_\alpha U_\alpha))\times B$. Denote this set by $C$.

Let $C\neq\lambda$. By the De Morgan's laws, $\lambda\setminus C = \bigcap\limits_{\alpha\in A}\lambda\setminus(\mathrm{pr}_2({}_\alpha U_\alpha))$. The neighborhood $U_1$ doesn't contain $A\times(\lambda\setminus C)$. Lemma 1 implies that the sets $A\times(\lambda\setminus C)\in S$ and $C\times B\in S$ are closed.

If $C=\lambda$, then, by Lemma 1, the subset $(\lambda\setminus\{\varphi\})\times B$ for some $\varphi\in\lambda$ is closed. Lemma 2 implies that subset $A\times\{\varphi\}$ is closed.

Hence the family of all closed rectangles of $(B_\lambda, \tau)$ is a compositional.

Let $\mathcal{F}$ be compositional family. Denote

$$C = \mathcal{F} \cup \{{}_\alpha A_\beta, {}^\alpha_\beta A \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$$

and

$$P_\mathcal{F} = \{B_\lambda \setminus B \mid B \in C\} \cup \{\{(\alpha, \beta)\} \mid (\alpha, \beta) \in B_\lambda\} \cup \{\varnothing\}.$$

**Proposition 1.** *For every compositional family $\mathcal{F}$ topology generated by the subbase $P_\mathcal{F}$ is the smallest semigroup topology such that elements of $\mathcal{F}$ are closed.*

*Proof.* First we shall show that topology generated by the subbase $P_\mathcal{F}$ is semigroup. Since

$$\begin{aligned}
{}_\alpha(\bigcup_{i\in I} A_i)_\beta &= \bigcup_{\iota\in I} {}_\alpha(A_i)_\beta, \\
{}^\alpha_\beta(\bigcup_{i\in I} A_i) &= \bigcup_{\iota\in I} {}^\alpha_\beta(A_i), \\
{}_\alpha(\bigcap_{i\in I} A_i)_\beta &= \bigcap_{\iota\in I} {}_\alpha(A_i)_\beta, \\
{}^\alpha_\beta(\bigcap_{i\in I} A_i) &= \bigcap_{\iota\in I} {}^\alpha_\beta(A_i),
\end{aligned}$$

by Lemma 2, the semigroup operation is continues on $(B_\lambda \times B_\lambda) \setminus \{(0,0)\}$.

The continuity of the operation in the point $(0,0)$ can be verify only for elements of the subbase. Let $U$ be a neighborhood of 0 such that $U \in P_\mathcal{F}$. Consider possible cases:

(1) $B_\lambda \setminus U = A \times B \in \mathcal{F}$, then there exists $C \subset \lambda$ such that $A \times (\lambda \setminus C)$ and $C \times B$ are closed subsets of $(B_\lambda, \tau)$. Thus $B_\lambda \setminus A \times (\lambda \setminus C)$ and $B_\lambda \setminus C \times B$ are neighborhoods of 0 and

$$(B_\lambda \setminus A \times (\lambda \setminus C)) \cdot (B_\lambda \setminus C \times B) \subseteq B_\lambda \setminus A \times B = U.$$

(2) $B_\lambda \setminus U = {}_\alpha(A \times B)_\beta = \{\beta\} \times B$ for some $\alpha \in \lambda, \beta \in A$ and $A \times B \in \mathcal{F}$. There exists $C \subseteq \lambda$ such that the sets $A \times (B_\lambda \setminus C)$ and $C \times B$ are closed. Since the set ${}_\alpha(A \times B)_\beta = \{\beta\} \times (B_\lambda \setminus C)$ is closed, the set $(A \cup \{\beta\}) \times (B_\lambda \setminus C)$ is closed. Then $B_\lambda \setminus (A \cup \{\beta\}) \times (B_\lambda \setminus C)$ and $B_\lambda \setminus C \times B$ are neighborhoods of 0 and

$$(B_\lambda \setminus (A \cup \{\beta\})) \cdot (B_\lambda \setminus C \times B) \subseteq B_\lambda \setminus (\{\beta\} \times B) = U.$$

(3) The case $B_\lambda \setminus U = {}^\alpha_\beta(A \times B) = A \times \{\beta\}$ is proved similarly.

Let $\tau$ be a topology on $B_\lambda$ such elements of $\mathcal{F}$ are closed in the topological space $(B_\lambda, \tau)$. Then, by Lemma 2, elements of $\{{}_\alpha(A \times B)_\beta, {}^\alpha_\beta(A \times B) \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$ are closed and, hence their complements are open. Since every nonzero point of $(B_\lambda, \tau)$ is isolated, elements of $P_\mathcal{F}$ are open in the topological space $(B_\lambda, \tau)$.

The topology generated by the subbase $P_\mathcal{F}$ will be called topology generated by the compositional family $\mathcal{F}$ and denoted by $\tau_\mathcal{F}$.

Proposition 1 and Lemma 4 imply the following corollary.

**Corollary 2.** *Every minimal semigroup topology on $B_\lambda$ is generated by some compositional family.*

Let $A$ and $B$ be sets. We will denote

$$A =^* B \text{ if a set } A \triangle B \text{ is finite ;}$$
$$A \subseteq^* B \text{ if a set } A \setminus B \text{ is finite .}$$

Note that there are semigroup topologies such that not generated by compositional families. For example, a topology generated by the base $\{\{(\alpha,\alpha) \mid \alpha \in A\} \cup \{0\} \mid A =^* \lambda\} \cup \{\{(\alpha,\beta)\} \mid \alpha,\beta \in \lambda\}$.

Observe that a semigroup topology can be generated by distinct compositional families. Let $\tau$ be a semigroup topology generated by some compositional family on $B_\lambda$. By $\mathrm{Com}(\tau)$ denote the set of all compositional families such that generate the topology $\tau$.

**Proposition 2.** *Let $\tau_1$ and $\tau_2$ be semigroup topologies on $B_\lambda$ generated by compositional families. A topology $\tau_1$ is weaker than a topology $\tau_2$ if and only if there exist compositional families $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F}_2 \in \mathrm{Com}(\tau_2)$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

*Proof.* ($\Rightarrow$) If $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F} \in \mathrm{Com}(\tau_2)$, then the family $\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}$ is compositional and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since the topology $\tau_1$ is weaker than the topology $\tau_2$, any element of $\mathcal{F}_1$ is a closed set in the topology $\tau_2$. Therefore, the family $\mathcal{F}_2$ generate the topology $\tau_2$.

($\Leftarrow$) Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, every closed set in the topology $\tau_1$ is closed in the topology $\tau_2$. Hence the topology $\tau_1$ is weaker than the topology $\tau_2$.

**Lemma 5.** *Let $\tau$ be semigroup topology on $B_\lambda$. If the set $A \times B$ is closed in the topological space $(B_\lambda, \tau)$ and $C =^* A$, $D =^* B$, then the set $C \times D$ is closed in the topological space $(B_\lambda, \tau)$.*

*Proof.* By Lemma 1, the set $(A \cap C) \times (B \cap D)$ is closed in the topological space $(B_\lambda, \tau)$. Then, by Lemma 2, the sets $(A \cap C) \times \{\alpha\}$ and $\{\beta\} \times (B \cap D)$ are closed for all $\alpha \in D \setminus B, \beta \in C \setminus A$. Since the sets $D \setminus B$ and $C \setminus A$ are finite, the sets $(A \cap C) \times (D \setminus B)$ and $(C \setminus A) \times (B \cap D)$ are closed. The set $(C \setminus A) \times (D \setminus B)$ is finite and therefore closed. Hence the set $C \times D$ is closed in the topological space $(B_\lambda, \tau)$.

## 3. Compositional digraphs

A compositional family $\mathcal{F}$ can be represented in the form of a digraph with loops $D(\mathcal{F})$. The vertexes of digraph is the set

$$V(D(\mathcal{F})) = \{A \mid A \times (\lambda \setminus B) \in \mathcal{F} \text{ or } B \times (\lambda \setminus A) \in \mathcal{F}\}$$

and $(A, B) \in A(D(\mathcal{F}))$ if $A \times (\lambda \setminus B) \in \mathcal{F}$.

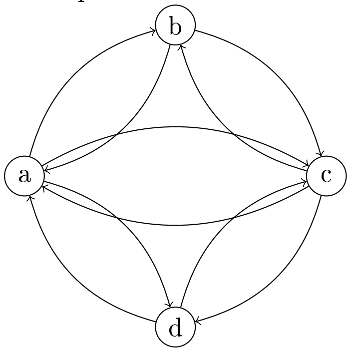
**Definition 2.** *A digraph with loops $D = (V, A)$ is called compositional if for all $(a, b) \in A$ there exists $c \in V$ such that $(a, c) \in A$ and $(c, b) \in A$.*

**Proposition 3.** *For any compositional family $\mathcal{F}$ the digraph $D(\mathcal{F})$ is compositional and any compositional digraph $D = (V, A)$ such that $V \subseteq \mathcal{P}(\lambda)$ determines some compositional family.*

**Example 1.** The digraph $D$ with $V(D) = \{a, b, c, d\}$ and

$$A(D) = \{(a, b), (b, a), (b, c), (c, b), (c, d), (d, c), (a, d), (d, a), (a, c), (c, a)\}$$

is compositional.

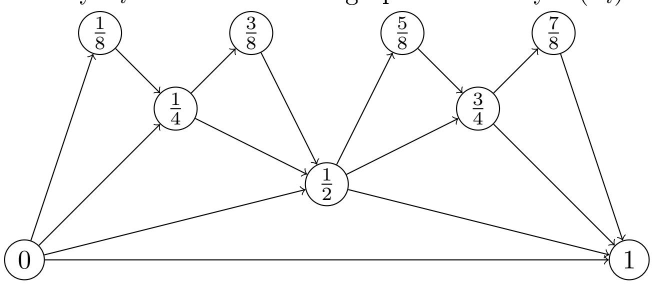



**Example 2.** Let $\mathbb{Z}[\frac{1}{2}]$ be the set of all dyadic rationals in $[0, 1]$. The digraph $\mathcal{U}$ with $V(\mathcal{U}) = \mathbb{Z}[\frac{1}{2}]$ and

$$A(\mathcal{U})\{(v, u) \mid v = \frac{k}{2^n} \text{ and } u = v + \frac{1}{m} \text{ for some } m \geqslant n\}$$

is compositional. Indeed, if $(u, v)$ is an arc of $\mathcal{U}$, then $(u, \frac{u+v}{2})$ and $(\frac{u+v}{2}, v)$ are arcs of $\mathcal{U}$.

By $\mathcal{U}_i$ denoted the subdigraph induced by $V(\mathcal{U}_i) = \{u \in \mathbb{Z}[\frac{1}{2}] \mid u = \frac{k}{2^n} \text{ for } n \leqslant i\}$.




**Proposition 4.** *There exists a hamiltonian path in $\mathcal{U}_i$ for each $i \in \mathbb{N}$.*

*Proof.* Consider the sequence of vertices $\frac{1}{2^i}, \frac{2}{2^i}, \frac{3}{2^i}, \dots, \frac{2^i-1}{2^i}, \frac{2^i}{2^i}$. Observe that $v_{i+1} - v_i = \frac{1}{2^i}$. Thus $(v_i, v_{i+1})$ is an arc of $\mathcal{U}_i$ and hence this sequence is a path. Since the path contains all vertices of $\mathcal{U}_i$, this is a hamiltonian path.

Proposition 4 implies the following corollary.

**Corollary 3.** *Arbitrary quotient digraph of $\mathcal{U}$ has a finite cycle.*

**Proposition 5.** *Let $\{D_i\}_{i \in I}$ be a collection of compositional digraphs and $D = (\bigsqcup\limits_{i \in I} V(D_i), \bigsqcup\limits_{i \in I} A(D_i))$. Arbitrary quotient digraph of $D$ is compositional.*

*Proof.* Let $\mathcal{R}$ be an equivalence relation on $\bigsqcup\limits_{i \in I} V_\alpha$ and $([a], [b]) \in V(D/\mathcal{R})$. Then $(a, b) \in \bigsqcup\limits_{i \in I} A_i$ and hence $(a, b) \in A_j$ for some $j \in I$. Therefore, there exist $(a, c), (c, b) \in V(D_j)$. Thus $([a], [c]), ([c], [b]) \in V(D/\mathcal{R})$.

**Lemma 6.** *Let $D$ be a digraph and $(a, b)$ be a arc of $D$. Then there exists an equivalence relation $\mathcal{R}$ on $V(\mathcal{U})$ such that $D/\mathcal{R}$ is isomorphic to a subdigraph of $D$ which contains $(a, b)$.*

*Proof.* Construct by induction equivalence classes of the relation $\mathcal{R}$. For each $v \in V(D)$ define $[v] = \varnothing$. Let $(u, w)$ is an arc of $\mathcal{U}_0$. Add vertex $u$ to $[a]$ and vertex $w$ to $[b]$. Suppose that all vertexes of $\mathcal{U}_i$ are added to equivalence classes. Take arbitrary vertex $v \in V(\mathcal{U}_{i+1}) \setminus V(\mathcal{U}_i)$. There exist two arcs $(u, v), (v, w) \in A(\mathcal{U}_{i+1}) \setminus A(\mathcal{U}_i)$ and $u \in [x]$, $w \in [y]$. Since the digraph $D$ is compositional, there exist arcs $(x, z), (z, y) \in A(D)$. Then add the vertex $v$ to $[z]$.

**Proposition 6.** *Any compositional digraph with size $\kappa$ is isomorphic to quotient digraph of $\bigoplus\limits_{k\in\kappa} \mathcal{U}$.*

*Proof.* Let $D$ be compositional digraph with size $\kappa$. Consider a collection $\{\mathcal{U}^{(a,b)}\}_{(a,b)\in A(\mathcal{U})}$ where $\mathcal{U}^{(a,b)}$ is isomorphic to $\mathcal{U}$ for each $(a, b) \in A(\mathcal{U})$. By lemma 6, for arbitrary $\mathcal{U}^{(a,b)}$ there exist an equivalence relation $\mathcal{R}_{(a,b)}$ and an isomorphism $f_{(a,b)}$ between $\mathcal{U}^{(a,b)}/\mathcal{R}_{(a,b)}$ and subdigraph of $D$. For $u \in V(\mathcal{U}^{(a,b)})$ and $v \in V(U^{(c,d)})$ define the equivalence relation $u\mathcal{R}v$ if $f_{(a,b)}([u]_{(a,b)}) = f_{(c,d)}([v]_{(c,d)})$. Hence $\bigoplus\limits_{k\in\kappa} \mathcal{U}/\mathcal{R}$ is isomorphic to $D$.

Corollary 3 and Proposition 6 imply the following corollaries.

**Corollary 4.** *Any compositional graph has a finite cycle or contains $\mathcal{U}$ as subgraph.*

**Corollary 5.** *Any finite compositional graph has a finite cycle.*

## 4. Main result

**Proposition 7.** *Let $A$ be subset of $\lambda$, then a topology $\tau$ generated by the compositional family $\{A \times (\lambda \setminus A)\}$ is minimal.*

*Proof.* Let $\tau_1$ be a weaker topology than the topology $\tau$ and $B \times (\lambda \setminus C)$ be a closed set in the topological space $(B_\lambda, \tau_1)$. By Lemma 4, there exists $D \subseteq \lambda$ such that the sets $B \times (\lambda \setminus D)$ and $D \times (\lambda \setminus C)$ are closed in the topological space $(B_\lambda, \tau_1)$. Since $\tau_1$ is weaker than $\tau$, $D \subseteq^* A$ and $(\lambda \setminus D) \subseteq^* (\lambda \setminus A)$. Therefore $D =^* A$. Again, by Lemma 4, there exists $F \subseteq \lambda$ such that the sets $D \times (\lambda \setminus F)$ and $F \times (\lambda \setminus C)$ are closed in the topological space $(B_\lambda, \tau_1)$. Hence $F \subseteq^* A$ and $(\lambda \setminus F) \subseteq^* (\lambda \setminus A)$ and then $F =^* A$. By Lemma 5, the set $(A \times (\lambda \setminus A)$ is closed in the topological space $(B_\lambda, \tau_1)$.

Proposition 7 generalizes [3, Theorem 5].

**Proposition 8.** *Let $\mathcal{F}$ be a compositional family. If there exists a finite subdigraph $H$ of $D(\mathcal{F})$ which doesn't contain sink or source, then there exists $A \subseteq \lambda$ such that the set $A \times (\lambda \setminus A)$ is closed in the topological space $(B_\lambda, \tau_\mathcal{F})$.*

*Proof.* Let $V(D) = \{A_1, \ldots, A_n\}$ and $H$ doesn't contain sink. For each $A_i \in V(H)$ there exits $A_j \in V(H)$ such that $A_i \times (\lambda \setminus A_j)$. Observe that

$$\lambda \setminus (A_1 \cup \ldots \cup A_n) = (\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n).$$

If $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) = \varnothing$, then $A_1 \cup \ldots \cup A_n = \lambda$ and hence, by Lemma 2, for each $\alpha \in \lambda$ the set $(\lambda \setminus \{\alpha\}) \times \{\alpha\}$ is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. Let $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) \neq \varnothing$. By Lemma 1, the set $A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$ is closed for any $A_i \in V(H)$. Hence

$$A_1 \cup \ldots \cup A_n \times \lambda \setminus (A_1 \cup \ldots \cup A_n) = \bigcup_{i=1}^{n} A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$$

is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. The case with source is proved similarly.

Propositions 7 and 8 imply the following corollary.

**Corollary 6.** *If $D(\mathcal{F})$ has a finite cycle, then $\mathcal{F}$ is a singleton or generates a nonminimal topology.*

Corollaries 3 and 6 imply the following theorem.

**Theorem 1.** *Let $\tau$ be a semigroup topology on $B_\lambda$ generated by compositional family $\mathcal{F}$ such that $D(\mathcal{F})$ doesn't contain subgraph isomorphic to $\mathcal{U}$. The topology $\tau$ is minimal if and only if $\tau$ is generated by singleton compositional family.*

**Problem 1.** *Is there a minimal semigroup topology on $B_\lambda$ generated by a composition family $\mathcal{F}$ such that $D(\mathcal{F})$ is isomorphic to $\mathcal{U}$?*

## References

1. B. Banaschewski, Minimal topological algebras, Math. Ann. 211 (1974), 107–114.
2. D. Doitchinov, Produits de groupes topologiques minimaux, Bull. Sci. Math. (2) 97 (1972), 59–64.
3. O. V. Gutik, and K. P. Pavlyk, On topological semigroups of matrix units, Semigroup Forum 71 (2005), no. 3, 389-400.
4. J. Bang-Jensen, G. Gutin, Digraphs - theory, algorithms and applications ?????????????.
5. L. Nachbin, On strictly minimal topological division rings, Bull. Amer. Math. Soc. 55 (1949), 1128–1136.
6. R. M. Stephenson, Jr., Minimal topological groups, Math. Ann. 192 (1971), 193–195.

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# МІНІМАЛЬНІ ТОПОЛОГІЇ НА НАПІВГРУПІ МАТРИЧНИХ ОДИНИЦЬ

**Маркіян ХИЛИНСЬКИЙ, Павло ХИЛИНСЬКИЙ**

*Львівський національний університет ім. І. Франка,
м. Львів, 79000, вул. Університетська, 1
e-mail: khymarkiyan@gmail.com, khypavlo@gmail.com*

*Ключові слова:* напівградка, топологія, напівгрупа матричних одиниць