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MINIMAL TOPOLOGIES ON THE SEMIGROUPS OF MATRIX UNITS

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We describe minimal topologies in some class of semigroup topologies on the semigroups of matrix units.

Key words: topology, semigroup topology, minimal topology, semigroups of matrix units

1. Introduction, motivation and main definitions

In this paper all topological spaces are assumed to be Hausdorff.

A *topological semigroup* is a Hausdorff topological space together with a continuous semigroup operation . If S is a semigroup and τ is a topology on S such that (S,τ) is a topological semigroup, then we shall call τ *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation. Topological semigroup (S,τ) is said to be *minimal* if no semigroup topology on S is strictly contained in τ. If (S,τ) is minimal topological semigroup, then τ is called *minimal semigroup topology.*

The concept of minimal topological groups was introduced independently in the early 1970's by Doitchinov [2] and Stephenson [5]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time. More than 20 years earlier L. Nachbin [4] had studied minimality in the context of division rings, and B. Banaschewski [1] investigated minimality in the more general setting of topological algebras.

Let λ be a nonempty set. By B_λ we denote the set $\lambda\times\lambda\cup\{0\}$ endowed with the following semigroup operation:

$$(a,b)\cdot(c,d)=\begin{cases}(a,d), & b=c;\\ 0, & b\neq c\end{cases}$$

2010 Mathematics Subject Classification: 18A22 and $(a,b)\cdot 0 = 0\cdot(a,b) = 0\cdot 0 = 0$, for each $a,b,c,d\in\lambda$. The semigroup B_λ is called the *semigroup of* $\lambda\times\lambda$*-matrix units.* The semitopological and topological semigroup of matrix units was investigated in [3].

A *directed graph* (or just *digraph*) D consists of a nonempty set $V(D)$ of elements called *vertices* and a set $A(D)$ of ordered pairs of vertices called *arcs.* We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D. The *order* (*size*) of D is the cardinality of the vertex (arc) set of D. For an arc (u,v) the first vertex u is its *tail* and the second vertex v is its *head.* If a tail and a head of arc coincide, then this arc is called a *loop.* The head and tail of an arc are its *end-vertices.* A vertex v is a *source*(*source*) if v is not a head(tail) for any arc. A digraph H is a *subdigraph* of a digraph D if $V(H)\subseteq V(D)$, $A(H)\subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. A *walk* in D is an alternating sequence $W = x_1a_1x_2a_2x_3\dots x_{k-1}a_{k-1}x_k$ of vertices x_i and arcs a_j from D such that the tail of a_i is x_i and the head of a_i is x_{i+1} for every $i=1,2,\dots,k-1$. The *length* of a walk is the number of its arcs. When the arcs of W are defined from the context or simply unimportant, we will denote W by $x_1x_2\dots x_k$. If the vertices of W are distinct, W is a *path.* If the vertices $x_1,x_2,\dots,x_{k-1}$ are distinct and $x_1=x_k$, W is a *cycle.*

Let $\{D_i\}_{i\in I}$ be a family of digraphs. The digraph $(\bigsqcup\limits_{i\in I} V(D_i), \bigsqcup\limits_{i\in I} A(D_i))$ is called *disjoint union* of this family and denoted by $\bigoplus\limits_{i\in I} D_i$. If D is a digraph and $\mathcal{R}$ is an equivalence relation on $V(D)$. Then the *quotient digraph* has vertex set $V/\mathcal{R}$ and arc set $\{([a]_\mathcal{R},[b]_\mathcal{R}) \mid (a,b)\in A(D)\}$.

2. Compositional families

If (B_λ,τ) is a semitopological semigroup, then any nonzero element of B_λ is an isolated point of (B_λ,τ) [3, Lemma 2]. Therefore the following lemma is true.

Lemma 1. *Let (B_λ,τ) be a topological semigroup and A be a closed subset A of (B_λ,τ) which doesn't contain 0. Then any subset of A is closed.*

Thus a semigroup topology on B_λ is determined by closed sets that don't contain 0.

For $A\subseteq B_\lambda$ and $\alpha,\beta\in\lambda$ we denote

$$\begin{aligned}
{}_\alpha A_\beta &= \{(\beta,\gamma) \mid (\alpha,\gamma)\in A\};\\
{}^\alpha_\beta A &= \{(\gamma,\beta) \mid (\gamma,\alpha)\in A\};\\
\mathrm{pr}_1(A) &= \{\alpha \mid (\alpha,\beta)\in A\};\\
\mathrm{pr}_2(A) &= \{\beta \mid (\alpha,\beta)\in A\}.
\end{aligned}$$

Lemma 2. *Let τ be a topology on B_λ and any nonzero element of B_λ is isolated in (B_λ,τ). The semigroup operation is continues on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$ if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for every $\alpha,\beta\in\lambda$ and every closed subset A of (B_λ,τ) which doesn't contain 0.*

Proof. ($\Rightarrow$) Let A be a closed subset A of (B_λ,τ) which doesn't contain 0. Lemma 1 implies that the sets ${}_\alpha A_\alpha$ and ${}^\alpha_\alpha A$ are closed. By the continuity of operation, the maps

$\lambda_{(\alpha,\beta)}: B_\lambda \to B_\lambda$ and $\rho_{(\beta,\alpha)}: B_\lambda \to B_\lambda$ defined by the formulas $\lambda_{(\alpha,\beta)}(x) = (\alpha,\beta)\cdot x$ and $\rho_{(\beta,\alpha)}(x) = x\cdot(\beta,\alpha)$ are continuous. Therefore the sets

$$_\alpha A_\beta = (\lambda_{(\alpha,\beta)})^{-1}(_\alpha A_\alpha)$$

and

$$^\alpha_\beta A = (\rho_{(\beta,\alpha)})^{-1}(^\alpha_\alpha A)$$

are closed in the topological space (B_λ, τ).

($\Leftarrow$) Since every nonzero point of B_λ is isolated, we need check the continuity of operation only in the cases of $(\alpha,\beta)\cdot 0$ and $0\cdot(\alpha,\beta)$. Let $U(0)$ be a neighborhood of 0. Denote by A the closed set $B_\lambda \setminus U(0)$. Then the sets $_\alpha A_\beta$ and $^\alpha_\beta A$ are closed. Denote by $U_1(0)$ and $U_2(0)$ the sets $B_\lambda \setminus {_\alpha A_\beta}$ and $B_\lambda \setminus {^\alpha_\beta A}$, respectively. Therefore $\{(\alpha,\beta)\}\cdot U_1(0) \subseteq U(0)$ and $U_2(0)\cdot\{(\alpha,\beta)\} \subseteq U(0)$.

Lemma 2 implies the following corollary.

Corollary 1. *Let τ be a topology on B_λ. (B_λ,τ) is a semitopological semigroup if and only if the sets $_\alpha A_\beta$ and $^\alpha_\beta A$ are closed for every $\alpha,\beta \in \lambda$ and every closed subset A of (B_λ,τ) which doesn't contain 0.*

Lemma 3. *Let $A, B \subseteq B_\lambda$ and $(\alpha,\beta) \in B_\lambda$. The element $(\alpha,\beta) \notin A\cdot B$ if and only if $\mathrm{pr}_2(_\alpha A_\alpha) \cap \mathrm{pr}_1(^\beta_\beta B) = \varnothing$.*

Proof. ($\Rightarrow$) Suppose that there exists $\gamma \in \mathrm{pr}_2(_\alpha A_\alpha) \cap \mathrm{pr}_1(^\beta_\beta B)$ then $(\alpha,\gamma) \in A$ and $(\gamma,\beta) \in B$. Therefore $(\alpha,\gamma)\cdot(\gamma,\beta) = (\alpha,\beta) \in A\cdot B$, a contradiction.

($\Leftarrow$) Suppose that $(\alpha,\beta) \in A\cdot B$. Then there exist $\gamma \in \lambda$ such that $(\alpha,\gamma) \in A$ and $(\gamma,\beta) \in B$. Hence $\gamma \in \mathrm{pr}_2(_\alpha A_\alpha) \cap \mathrm{pr}_1(^\beta_\beta B)$, a contradiction.

We will call elements of the set $\{A\times B \mid A, B \subseteq B_\lambda\}$ by rectangles.

Definition 1. *A nonempty family $\mathcal{F}$ of rectangles is called compositional if for $A\times B \in \mathcal{F}$ there exists $C \subseteq \lambda$ such that $A\times(\lambda\setminus C) \in \mathcal{F}$ and $C\times B \in \mathcal{F}$.*

Lemma 4. *Let τ be a semigroup topology on B_λ, then the family of all closed rectangles of (B_λ,τ) is compositional.*

Proof. Since every point of (B_λ,τ) is closed, the family of all closed rectangles is not empty. Let $A\times B$ be a closed subset of (B_λ,τ), then $B_\lambda \setminus (A\times B)$ is a neighborhood of 0. By the continuity of operation at the point $(0,0)$ there exist neighborhoods of zero U and V such that $U\cdot V \subseteq B_\lambda \setminus (A\times B)$. Lemma 3 implies that for each $\alpha \in A$ the neighborhood V doesn't contain the set $\mathrm{pr}_2(_\alpha V_\alpha)\times B$. Therefore, the neighborhood V doesn't contain the set $(\bigcup\limits_{\alpha\in A} \mathrm{pr}_2(_\alpha U_\alpha))\times B$. Denote this set by C.

Let $C \neq \lambda$. By the De Morgan's laws, $\lambda\setminus C = \bigcap\limits_{\alpha\in A} \lambda\setminus(\mathrm{pr}_2(_\alpha U_\alpha))$. The neighborhood U_1 doesn't contain $A\times(\lambda\setminus C)$. Lemma 1 implies that the sets $A\times(\lambda\setminus C) \in S$ and $C\times B \in S$ are closed.

If $C = \lambda$, then, by Lemma 1, the subset $(\lambda\setminus\{\varphi\})\times B$ for some $\varphi \in \lambda$ is closed. Lemma 2 implies that subset $A\times\{\varphi\}$ is closed.

Hence the family of all closed rectangles of (B_λ,τ) is a compositional.

Let $\mathcal{F}$ be compositional family. Denote

$$C = \mathcal{F} \cup \{{}_{\alpha}A_{\beta}, {}^{\alpha}_{\beta}A \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$$

and

$$P_{\mathcal{F}} = \{B_{\lambda} \setminus B \mid B \in C\} \cup \{\{(\alpha, \beta)\} \mid (\alpha, \beta) \in B_{\lambda}\} \cup \{\varnothing\}.$$

Proposition 1. *For every compositional family $\mathcal{F}$ topology generated by the subbase $P_{\mathcal{F}}$ is the smallest semigroup topology such that elements of $\mathcal{F}$ are closed.*

Proof. First we shall show that topology generated by the subbase $P_{\mathcal{F}}$ is semigroup. Since

$$\begin{aligned} {}_{\alpha}(\bigcup_{i\in I} A_i)_{\beta} &= \bigcup_{\iota\in I} {}_{\alpha}(A_i)_{\beta}, \\ {}^{\alpha}_{\beta}(\bigcup_{i\in I} A_i) &= \bigcup_{\iota\in I} {}^{\alpha}_{\beta}(A_i), \\ {}_{\alpha}(\bigcap_{i\in I} A_i)_{\beta} &= \bigcap_{\iota\in I} {}_{\alpha}(A_i)_{\beta}, \\ {}^{\alpha}_{\beta}(\bigcap_{i\in I} A_i) &= \bigcap_{\iota\in I} {}^{\alpha}_{\beta}(A_i), \end{aligned}$$

by Lemma 2, the semigroup operation is continues on $(B_{\lambda} \times B_{\lambda}) \setminus \{(0, 0)\}$.

The continuity of the operation in the point $(0, 0)$ can be verify only for elements of the subbase. Let U be a neighborhood of 0 such that $U \in P_{\mathcal{F}}$. Consider possible cases:

(1) $B_{\lambda} \setminus U = A \times B \in \mathcal{F}$, then there exists $C \subset \lambda$ such that $A \times (\lambda \setminus C)$ and $C \times B$ are closed subsets of (B_{λ}, τ). Thus $B_{\lambda} \setminus A \times (\lambda \setminus C)$ and $B_{\lambda} \setminus C \times B$ are neighborhoods of 0 and

$$(B_{\lambda} \setminus A \times (\lambda \setminus C)) \cdot (B_{\lambda} \setminus C \times B) \subseteq B_{\lambda} \setminus A \times B = U.$$

(2) $B_{\lambda} \setminus U = {}_{\alpha}(A \times B)_{\beta} = \{\beta\} \times B$ for some $\alpha \in \lambda, \beta \in A$ and $A \times B \in \mathcal{F}$. There exists $C \subseteq \lambda$ such that the sets $A \times (B_{\lambda} \setminus C)$ and $C \times B$ are closed. Since the set ${}_{\alpha}(A \times B)_{\beta} = \{\beta\} \times (B_{\lambda} \setminus C)$ is closed, the set $(A \cup \{\beta\}) \times (B_{\lambda} \setminus C)$ is closed. Then $B_{\lambda} \setminus (A \cup \{\beta\}) \times (B_{\lambda} \setminus C)$ and $B_{\lambda} \setminus C \times B$ are neighborhoods of 0 and

$$(B_{\lambda} \setminus (A \cup \{\beta\})) \cdot (B_{\lambda} \setminus C \times B) \subseteq B_{\lambda} \setminus (\{\beta\} \times B) = U.$$

(3) The case $B_{\lambda} \setminus U = {}^{\alpha}_{\beta}(A \times B) = A \times \{\beta\}$ is proved similarly.

Let τ be a topology on B_{λ} such elements of $\mathcal{F}$ are closed in the topological space (B_{λ}, τ). Then, by Lemma 2, elements of $\{{}_{\alpha}(A \times B)_{\beta}, {}^{\alpha}_{\beta}(A \times B) \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$ are closed and, hence their complements are open. Since every nonzero point of (B_{λ}, τ) is isolated, elements of $P_{\mathcal{F}}$ are open in the topological space (B_{λ}, τ).

The topology generated by the subbase $P_{\mathcal{F}}$ will be called topology generated by the compositional family $\mathcal{F}$ and denoted by $\tau_{\mathcal{F}}$.

Proposition 1 and Lemma 4 imply the following corollary.

Corollary 2. *Every minimal semigroup topology on B_{λ} is generated by some compositional family.*

Let A and B be sets. We will denote

$$A =^* B \text{ if a set } A \triangle B \text{ is finite ;}$$
$$A \subseteq^* B \text{ if a set } A \setminus B \text{ is finite .}$$

Note that there are semigroup topologies such that not generated by compositional families, for example a topology generated by the base $\{\{(\alpha,\alpha) \mid \alpha \in A\} \cup \{0\} \mid A =^* \lambda\} \cup \{\{(\alpha,\beta)\} \mid \alpha,\beta \in \lambda\}$.

Observe that a semigroup topology can be generated by distinct compositional families. Let τ be a semigroup topology generated by some compositional family on B_λ. By $\mathrm{Com}(\tau)$ denote the set of all compositional families such that generate the topology τ.

Proposition 2. *Let τ_1 and τ_2 be semigroup topologies on B_λ generated by compositional families. A topology τ_1 is weaker than a topology τ_2 if and only if there exist compositional families $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F}_2 \in \mathrm{Com}(\tau_2)$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

Proof. ($\Rightarrow$) If $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F} \in \mathrm{Com}(\tau_2)$, then the family $\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}$ is compositional and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since the topology τ_1 is weaker than the topology τ_2, any element of $\mathcal{F}_1$ is a closed set in the topology τ_2. Therefore, the family $\mathcal{F}_2$ generate the topology τ_2.

($\Leftarrow$) Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, every closed set in the topology τ_1 is closed in the topology τ_2. Hence the topology τ_1 is weaker than the topology τ_2.

Lemma 5. *Let τ be semigroup topology on B_λ. If the set $A \times B$ is closed in the topological space (B_λ, τ) and $C =^* A$, $D =^* B$, then the set $C \times D$ is closed in the topological space (B_λ, τ).*

Proof. By Lemma 1, the set $(A \cap C) \times (B \cap D)$ is closed in the topological space (B_λ, τ). Then, by Lemma 2, the sets $(A \cap C) \times \{\alpha\}$ and $\{\beta\} \times (B \cap D)$ are closed for all $\alpha \in D \setminus B, \beta \in C \setminus A$. Since the sets $D \setminus B$ and $C \setminus A$ are finite, the sets $(A \cap C) \times (D \setminus B)$ and $(C \setminus A) \times (B \cap D)$ are closed. The set $(C \setminus A) \times (D \setminus B)$ is finite and therefore closed. Hence the set $C \times D$ is closed in the topological space (B_λ, τ).

3. Compositional digraphs

A compositional family $\mathcal{F}$ can be represented in the form of a digraph with loops $D(\mathcal{F})$. The vertexes of digraph is the set

$$V(D(\mathcal{F})) = \{A \mid A \times (\lambda \setminus B) \in \mathcal{F} \text{ or } B \times (\lambda \setminus A) \in \mathcal{F}\}$$

and $(A, B) \in A(D(\mathcal{F}))$ if $A \times (\lambda \setminus B) \in \mathcal{F}$.

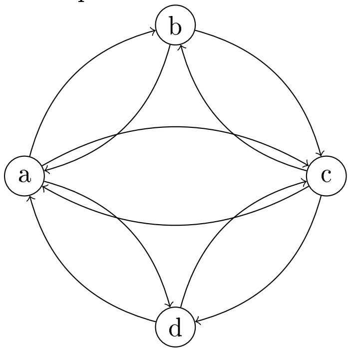
Definition 2. *A digraph with loops $D = (V, A)$ is called compositional if for all $(a, b) \in A$ there exists $c \in V$ such that $(a, c) \in A$ and $(c, b) \in A$.*

Proposition 3. *For any compositional family $\mathcal{F}$ the digraph $D(\mathcal{F})$ is compositional and any compositional digraph $D = (V, A)$ such that $V \subseteq \mathcal{P}(\lambda)$ determines some compositional family.*

Example 1. The digraph D with $V(D) = \{a, b, c, d\}$ and

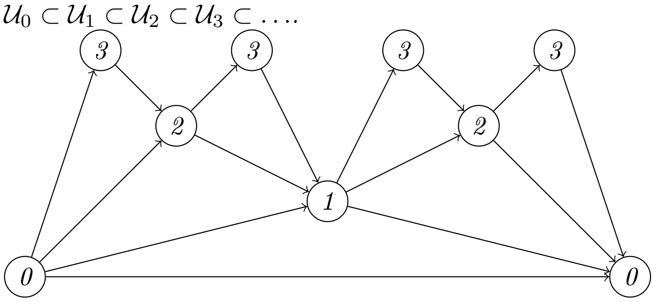
$$A(D) = \{(a, b), (b, a), (b, c), (c, b), (c, d), (d, c), (a, d), (d, a), (a, c), (c, a)\}$$

is compositional.



Definition 3. *The basis step is a graph $\mathcal{U}_0$ with two vertices and one arc. For the inductive step, assuming $\mathcal{U}_i$ is defined. Construct a graph v_{i+1} as follows: for any arc (a, b) of $\mathcal{U}_i$ that is not an arc of $\mathcal{U}_{i-1}$, attach a new vertex $v_{a,b}$ and arcs $(a, v_{a,b})$, $(v_{a,b}, b)$.*

Now define the universal graph $\mathcal{U}$ to be the union of the nested sequence of graphs $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset \ldots$.



Proposition 4. *There exists a hamiltonian path in any $\mathcal{U}_i$.*

Proof. We will prove by induction that, elements of $A(\mathcal{U}_i) \setminus A(\mathcal{U}_{i-1})$ form such path. For $\mathcal{U}_1$ statement is true. Suppose that statement is true for $\mathcal{U}_i$. Then elements of $A(\mathcal{U}_i) \setminus A(\mathcal{U}_{i-1})$ form a path $a_1 a_2 \ldots a_n$ in $\mathcal{U}_i$. For each a_i and a_{i+1} there exist arcs (a_i, b_i) and (b_i, a_{i+1}) in the $A(\mathcal{U}_i) \setminus A(\mathcal{U}_{i-1})$. Therefore, elements of $A(\mathcal{U}_{i+1}) \setminus A(\mathcal{U}_i)$ form a path $a_1 b_1 a_2 \ldots a_{n-1} b_{n-1} a_n$ where $b_j \in V(\mathcal{U}_{i+1}) \setminus V(\mathcal{U}_i)$.

Proposition 4 implies the following corollary.

Corollary 3. *Arbitrary quotient digraph of $\mathcal{U}$ has a finite cycle.*

Proposition 5. *Let $\{D_i\}_{i \in I}$ be a collection of compositional digraphs and $D = (\bigsqcup_{i \in I} V(D_i), \bigsqcup_{i \in I} A(D_i))$. Arbitrary quotient digraph of D is compositional.*

Proof. Let $\mathcal{R}$ be an equivalence relation on $\bigsqcup_{i \in I} V_\alpha$ and $([a], [b]) \in V(D/\mathcal{R})$. Then $(a, b) \in \bigsqcup_{i \in I} A_i$ and hence $(a, b) \in A_j$ for some $j \in I$. Therefore, there exist $(a, c), (c, b) \in V(D_j)$. Thus $([a], [c]), ([c], [b]) \in V(D/\mathcal{R})$.

Lemma 6. *Let D be a digraph and (a, b) be some arc of D. Then there exists an equivalence relation R on $V(\mathcal{U})$ such that $D/\mathcal{R}$ is isomorphic to a subdigraph of D which contains (a, b).*

Proof. Construct by induction equivalence classes of the relation $\mathcal{R}$. For each $v \in V(D)$ define $[v] = \varnothing$. Let (u, w) is an arc of $\mathcal{U}_0$. Add vertex u to $[a]$ and vertex w to $[b]$. Suppose that all vertexes of $\mathcal{U}_i$ are added to equivalence classes. Take arbitrary vertex $v \in V(\mathcal{U}_{i+1}) \setminus V(\mathcal{U}_i)$. There exist two arcs $(u, v), (v, w) \in A(\mathcal{U}_{i+1}) \setminus A(\mathcal{U}_i)$ and $u \in [m]$, $w \in [n]$. Since the digraph D is compositional, there exist arcs $(m, k), (k, n) \in A(D)$. Then add the vertex v to $[k]$.

Proposition 6. *Any compositional digraph with size κ is isomorphic to quotient digraph of $\bigoplus\limits_{k\in\kappa} \mathcal{U}$.*

Proof. Let D be compositional digraph with size κ. Consider a collection $\{\mathcal{U}^{(a,b)}\}_{(a,b)\in A(\mathcal{U})}$ where $\mathcal{U}^{(a,b)}$ is isomorphic to $\mathcal{U}$ for each $(a, b) \in A(\mathcal{U})$. By lemma 6, for arbitrary $\mathcal{U}^{(a,b)}$ there exist an equivalence relation $\mathcal{R}_{(a,b)}$ and an isomorphism $f_{(a,b)}$ between $\mathcal{U}^{(a,b)} / \mathcal{R}_{(a,b)}$ and subdigraph of D. Let $u \in V(\mathcal{U}^{(a,b)})$ and $v \in V(U^{(c,d)})$ then define $u\mathcal{R}v$ if $f_{(a,b)}([u]) = f_{(c,d)}([v])$. Hence $\bigoplus\limits_{k\in\kappa} \mathcal{U}/\mathcal{R}$ is isomorphic to D.

Corollary 3 and Proposition 6 imply the following corollaries.

Corollary 4. *Any compositional graph has a finite cycle or contains the universal graph as subgraph.*

Corollary 5. *Any finite compositional graph has a finite cycle.*

4. Main result

Proposition 7. *Let A be subset of λ, then a topology τ generated by the compositional family $\{A \times (\lambda \setminus A)\}$ is minimal.*

Proof. Let τ_1 be a weaker topology than the topology τ and $B \times (\lambda \setminus C)$ be a closed set in the topological space (B_λ, τ_1). By Lemma 4, there exists $D \subseteq \lambda$ such that the sets $B \times (\lambda \setminus D)$ and $D \times (\lambda \setminus C)$ are closed in the topological space (B_λ, τ_1). Since τ_1 is weaker than τ, $D \subseteq^* A$ and $(\lambda \setminus D) \subseteq^* (\lambda \setminus A)$. Therefore $D =^* A$. Again, by Lemma 4, there exists $F \subseteq \lambda$ such that the sets $D \times (\lambda \setminus F)$ and $F \times (\lambda \setminus C)$ are closed in the topological space (B_λ, τ_1). Hence $F \subseteq^* A$ and $(\lambda \setminus F) \subseteq^* (\lambda \setminus A)$ and then $F =^* A$. By Lemma 5, the set $(A \times (\lambda \setminus A)$ is closed in the topological space (B_λ, τ_1).

Proposition 7 generalizes [3, Theorem 5].

Proposition 8. *Let $\mathcal{F}$ be a compositional family. If there exists a finite subdigraph H of $D(\mathcal{F})$ which doesn't contain sink or source, then there exists $A \subseteq \lambda$ such that the set $A \times (\lambda \setminus A)$ is closed in the topological space $(B_\lambda, \tau_\mathcal{F})$.*

Proof. Let $V(D) = \{A_1, \ldots, A_n\}$ and H doesn't contain sink. For each $A_i \in V(H)$ there exits $A_j \in V(H)$ such that $A_i \times (\lambda \setminus A_j)$. Observe that

$$\lambda \setminus (A_1 \cup \ldots \cup A_n) = (\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n).$$

If $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) = \varnothing$, then $A_1 \cup \ldots \cup A_n = \lambda$ and hence, by Lemma 2, for each $\alpha \in \lambda$ the set $(\lambda \setminus \{\alpha\}) \times \{\alpha\}$ is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. Let $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) \neq \varnothing$. By Lemma 1, the set $A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$ is closed for any $A_i \in V(H)$. Hence

$$A_1 \cup \ldots \cup A_n \times \lambda \setminus (A_1 \cup \ldots \cup A_n) = \bigcup_{i=1}^{n} A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$$

is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. The case with source is proved similarly.

Propositions 7 and 8 imply the following corollary.

Corollary 6. *If $D(\mathcal{F})$ has a finite cycle, then $\mathcal{F}$ is a singleton or generates a nonminimal topology.*

Corollaries 3 and 6 imply the following theorem.

Theorem 1. *Let τ be a semigroup topology on B_λ generated by compositional family $\mathcal{F}$ such that $D(\mathcal{F})$ doesn't contain subgraph isomorphic to $\mathcal{U}$. The topology τ is minimal if and only if τ is generated by singleton compositional family.*

Problem 1. *Is there a minimal semigroup topology on B_λ generated by a composition family $\mathcal{F}$ such that $D(\mathcal{F})$ is isomorphic to $\mathcal{U}$?*

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МІНІМАЛЬНІ ТОПОЛОГІЇ НА НАПІВГРУПІ МАТРИЧНИХ ОДИНИЦЬ

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Ключові слова: напівградка, топологія, напівгрупа матричних одиниць