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MINIMAL TOPOLOGIES ON THE SEMIGROUPS OF MATRIX UNITS

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We describe minimal topologies in some class of semigroup topologies on the semigroups of matrix units.

Key words: topology, semigroup topology, minimal topology, semigroups of matrix units

1. Introduction, motivation and main definitions

In this paper all topological spaces are assumed to be Hausdorff.

A *topological semigroup* is a Hausdorff topological space together with a continuous semigroup operation . If S is a semigroup and τ is a topology on S such that (S,τ) is a topological semigroup, then we shall call τ *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation. Topological semigroup (S,τ) is said to be *minimal* if no semigroup topology on S is strictly contained in τ. If (S,τ) is minimal topological semigroup, then τ is called *minimal semigroup topology.*

The concept of minimal topological groups was introduced independently in the early 1970's by Doitchinov [2] and Stephenson [6]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time. More than 20 years earlier L. Nachbin [5] had studied minimality in the context of division rings, and B. Banaschewski [1] investigated minimality in the more general setting of topological algebras.

Let λ be a nonempty set. By B_λ we denote the set $\lambda\times\lambda\cup\{0\}$ endowed with the following semigroup operation:

$$(\alpha,\beta)\cdot(\gamma,\delta)=\begin{cases}(\alpha,\delta), & \beta=\gamma;\\ 0, & \beta\neq\gamma\end{cases}$$

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and $(\alpha,\beta)\cdot 0 = 0\cdot(\alpha,\beta) = 0\cdot 0 = 0$, for each $\alpha,\beta,\gamma,\delta \in \lambda$. The semigroup B_λ is called the *semigroup of* $\lambda\times\lambda$*-matrix units*. The semitopological and topological semigroup of matrix units was investigated in [3].

A *directed graph* (or just *digraph*) D consists of a nonempty set $V(D)$ of elements called *vertices* and a set $A(D)$ of ordered pairs of vertices called *arcs*. We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D. The *order* (*size*) of D is the cardinality of the vertex (arc) set of D. For an arc (u,v) the first vertex u is its *tail* and the second vertex v is its *head*. If a tail and a head of arc coincide, then this arc is called a *loop*. The head and tail of an arc are its *end-vertices*. A vertex v is a *source*(*source*) if v is not a head(tail) for any arc. A digraph H is a *subdigraph* of a digraph D if $V(H)\subseteq V(D)$, $A(H)\subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $X = V(H)$ and call H an induced subdigraph of D.

A *walk* in D is an alternating sequence $W = x_1a_1x_2a_2x_3\dots x_{k-1}a_{k-1}x_k$ of vertices x_i and arcs a_j from D such that the tail of a_i is x_i and the head of a_i is x_{i+1} for every $i = 1,2,\dots,k-1$. The *length* of a walk is the number of its arcs. When the arcs of W are defined from the context or simply unimportant, we will denote W by $x_1x_2\dots x_k$. If the vertices of W are distinct, W is a *path*. If the vertices $x_1,x_2,\dots,x_{k-1}$ are distinct and $x_1 = x_k$, W is a *cycle*. A walk (path, cycle) W is a *Hamilton* (or *hamiltonian*) walk (path, cycle) if W contains all vertices of D.

Let $\{D_i\}_{i\in I}$ be a family of digraphs. The digraph $(\bigsqcup_{i\in I} V(D_i), \bigsqcup_{i\in I} A(D_i))$ is called *disjoint union* of this family and denoted by $\bigoplus_{i\in I} D_i$. If D is a digraph and $\mathcal{R}$ is an equivalence relation on $V(D)$. Then the *quotient digraph* $D/\mathcal{R}$ has vertex set $V/\mathcal{R}$ and arc set $\{([a]_\mathcal{R},[b]_\mathcal{R}) \mid (a,b)\in A(D)\}$.

2. Compositional families

If (B_λ,τ) is a semitopological semigroup, then any nonzero element of B_λ is an isolated point of (B_λ,τ) [3, Lemma 2]. Therefore the following lemma is true.

Lemma 1. *Let (B_λ,τ) be a topological semigroup and A be a closed subset A of (B_λ,τ) which doesn't contain the zero 0. Then any subset of A is closed.*

For $A\subseteq B_\lambda\setminus\{0\}$ and $\alpha,\beta\in\lambda$ we denote

$$\begin{aligned} {}_\alpha A_\beta &= \{(\beta,\gamma) \mid (\alpha,\gamma)\in A\};\\ {}^\alpha_\beta A &= \{(\gamma,\beta) \mid (\gamma,\alpha)\in A\};\\ \mathrm{proj}_1(A) &= \{\alpha \mid (\alpha,\beta)\in A\};\\ \mathrm{proj}_2(A) &= \{\beta \mid (\alpha,\beta)\in A\}. \end{aligned}$$

Lemma 2. *Let τ be a topology on B_λ and any nonzero element of B_λ is isolated in (B_λ,τ). The semigroup operation is continuous on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$ if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for all $\alpha,\beta\in\lambda$ and any closed subset A of (B_λ,τ) which doesn't contain the zero 0.*

Proof. ($\Rightarrow$) Let A be a closed subset of (B_λ, τ) which doesn't contain the zero 0. By the continuity of operation, the maps $\lambda_{(\alpha,\beta)} : B_\lambda \to B_\lambda$ and $\rho_{(\beta,\alpha)} : B_\lambda \to B_\lambda$ defined by the formulas $\lambda_{(\alpha,\beta)}(x) = (\alpha, \beta) \cdot x$ and $\rho_{(\beta,\alpha)}(x) = x \cdot (\beta, \alpha)$ are continuous. Therefore the sets

$$_\alpha A_\beta = (\lambda_{(\alpha,\beta)})^{-1}(A)$$

and

$$^\alpha_\beta A = (\rho_{(\beta,\alpha)})^{-1}(A)$$

are closed in the topological space (B_λ, τ).

($\Leftarrow$) Since every nonzero point of B_λ is isolated, we check the continuity of semigroup operation only in the cases of $(\alpha, \beta) \cdot 0$ and $0 \cdot (\alpha, \beta)$. Let U be a neighborhood of the zero 0 and $A = B_\lambda \setminus U$. Then the sets $_\alpha A_\beta$ and $^\alpha_\beta A$ are closed. Denote by V and W the neighborhoods of zero $B_\lambda \setminus {_\alpha A_\beta}$ and $B_\lambda \setminus {^\alpha_\beta A}$, respectively. Simple calculations imply that $\{(\alpha, \beta)\} \cdot V \subseteq U$ and $W \cdot \{(\alpha, \beta)\} \subseteq U$.

Lemma 2 implies the following corollary.

Corollary 1. *Let τ be a topology on B_λ. Then (B_λ, τ) is a semitopological semigroup if and only if the sets $_\alpha A_\beta$ and $^\alpha_\beta A$ are closed for all $\alpha, \beta \in \lambda$ and any closed subset A of (B_λ, τ) such that $0 \notin A$.*

Lemma 3. *Let $A, B \subseteq B_\lambda$ and $(\alpha, \beta) \in B_\lambda$. The element $(\alpha, \beta) \notin A \cdot B$ if and only if* $\mathrm{proj}_2({_\alpha A_\alpha}) \cap \mathrm{proj}_1({^\beta_\beta B}) = \varnothing$.

Proof. ($\Rightarrow$) Suppose that $(\alpha, \beta) \notin A \cdot B$ and there exists $\gamma \in \mathrm{proj}_2({_\alpha A_\alpha}) \cap \mathrm{proj}_1({^\beta_\beta B})$. Then $(\alpha, \gamma) \in A$ and $(\gamma, \beta) \in B$. Therefore $(\alpha, \gamma) \cdot (\gamma, \beta) = (\alpha, \beta) \in A \cdot B$, a contradiction.

($\Leftarrow$) Suppose that $\mathrm{proj}_2({_\alpha A_\alpha}) \cap \mathrm{proj}_1({^\beta_\beta B}) = \varnothing$ and $(\alpha, \beta) \in A \cdot B$. Then there exist $\gamma \in \lambda$ such that $(\alpha, \gamma) \in A$ and $(\gamma, \beta) \in B$. Hence $\gamma \in \mathrm{proj}_2({_\alpha A_\alpha}) \cap \mathrm{proj}_1({^\beta_\beta B})$, a contradiction.

For any nonempty subsets A and B of λ we shall call $A \times B$ by a *rectangle* of B_λ.

Definition 1. *A nonempty family $\mathcal{F}$ of rectangles of B_λ is called compositional if for $A \times B \in \mathcal{F}$ there exists $C \subset \lambda$ such that $A \times (\lambda \setminus C) \in \mathcal{F}$ and $C \times B \in \mathcal{F}$.*

Lemma 4. *If τ is a semigroup topology on B_λ, then the family of all closed rectangles of (B_λ, τ) is compositional.*

Proof. Since every point of (B_λ, τ) is a closed subset, the family of all closed rectangles of B_λ is not empty. Let $A \times B$ be a closed subset of (B_λ, τ). Then $B_\lambda \setminus (A \times B)$ is a neighborhood of the zero 0. Since the semigroup operation is continuous at $(0, 0)$ there exist neighborhoods of zero U and V such that $U \cdot V \subseteq B_\lambda \setminus (A \times B)$. Lemma 3 implies that for each $\alpha \in A$ the neighborhood V doesn't contain the set $\mathrm{proj}_2({_\alpha U_\alpha}) \times B$. Therefore, the neighborhood V doesn't contain the set $\left(\bigcup\limits_{\alpha \in A} \mathrm{proj}_2({_\alpha U_\alpha})\right) \times B$. Denote the set $\bigcup\limits_{\alpha \in A} \mathrm{proj}_2({_\alpha U_\alpha})$ by C.

Suppose that $C \neq \lambda$. By De Morgan's laws, $\lambda \setminus C = \bigcap\limits_{\alpha \in A} \lambda \setminus \mathrm{proj}_2({_\alpha U_\alpha})$. The neighborhood U doesn't contain the set $A \times (\lambda \setminus C)$. Lemma 1 implies that the sets $A \times (\lambda \setminus C)$ and $C \times B$ are closed.

If $C=\lambda$, then by Lemma 1 the set $(\lambda\setminus\{\varphi\})\times B$ for some $\varphi\in\lambda$ is closed. Lemma 2 implies that set $A\times\{\varphi\}$ is closed.

Hence the family of all closed rectangles of (B_λ,τ) is compositional.

Let $\mathcal{F}$ be compositional family. Denote

$$\mathcal{C}=\mathcal{F}\cup\{{}_\alpha A_\beta, {}^\alpha_\beta A \mid A\in\mathcal{F} \text{ and } \alpha,\beta\in\lambda\}\cup\{\{(\alpha,\beta)\}\mid(\alpha,\beta)\in B_\lambda\setminus\{0\}\}$$

and

$$P_{\mathcal{F}}=\{B_\lambda\setminus B\mid B\in\mathcal{C}\}\cup\{\{(\alpha,\beta)\}\mid(\alpha,\beta)\in B_\lambda\setminus\{0\}\}\cup\{\varnothing\}.$$

Proposition 1. *For every compositional family $\mathcal{F}$ the topology generated by the subbase $P_{\mathcal{F}}$ is the smallest semigroup topology on B_λ such that elements of $\mathcal{F}$ are closed.*

Proof. Let τ be the topology generated by $P_{\mathcal{F}}$. Since each nonzero point in (B_λ,τ) is clopen subset, the topological space (B_λ,τ) is Hausdorff.

First we shall show that topology generated by the subbase $P_{\mathcal{F}}$ is semigroup. For any $A\in\mathcal{C}$ the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are elements of $\mathcal{C}$. Since

$$\begin{aligned}
{}_\alpha(\bigcup_{i\in I}A_i)_\beta &= \bigcup_{\iota\in I}{}_\alpha(A_i)_\beta,\\
{}^\alpha_\beta(\bigcup_{i\in I}A_i) &= \bigcup_{\iota\in I}{}^\alpha_\beta(A_i),\\
{}_\alpha(\bigcap_{i\in I}A_i)_\beta &= \bigcap_{\iota\in I}{}_\alpha(A_i)_\beta,\\
{}^\alpha_\beta(\bigcap_{i\in I}A_i) &= \bigcap_{\iota\in I}{}^\alpha_\beta(A_i),
\end{aligned}$$

for arbitrary family $\{A_i\}_{i\in I}$ subsets of $B_\lambda\setminus\{0\}$, by Lemma 2, the semigroup operation is continuous on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$.

The continuity of the operation in the point $(0,0)$ can be verify only for elements of the subbase. Let U be a neighborhood of 0 such that $U\in P_{\mathcal{F}}$. Consider possible cases.

(1) If $B_\lambda\setminus U=A\times B\in\mathcal{F}$, then there exists $C\subset\lambda$ such that $A\times(\lambda\setminus C)$ and $C\times B$ are closed subsets of (B_λ,τ). Thus $B_\lambda\setminus(A\times(\lambda\setminus C))$ and $B_\lambda\setminus(C\times B)$ are neighborhoods of the zero 0 and
$$(B_\lambda\setminus(A\times(\lambda\setminus C)))\cdot(B_\lambda\setminus(C\times B))\subseteq B_\lambda\setminus(A\times B)=U.$$

(2) If $B_\lambda\setminus U={}_\alpha(A\times B)_\beta=\{\beta\}\times B$ for some $\alpha\in A,\beta\in\lambda$ and $A\times B\in\mathcal{F}$, then there exists $C\subset\lambda$ such that the sets $A\times(\lambda\setminus C)$ and $C\times B$ are closed in (B_λ,τ). The set ${}_\alpha(A\times(\lambda\setminus C))_\beta=\{\beta\}\times(\lambda\setminus C)$ is closed in (B_λ,τ). Hence $B_\lambda\setminus(\{\beta\}\times(\lambda\setminus C))$ and $B_\lambda\setminus(C\times B)$ are neighborhoods of the zero 0 and
$$(B_\lambda\setminus(\{\beta\}\times(\lambda\setminus C)))\cdot(B_\lambda\setminus(C\times B))\subseteq B_\lambda\setminus(\{\beta\}\times B)=U.$$

(3) In the case $B_\lambda\setminus U={}^\alpha_\beta(A\times B)=A\times\{\beta\}$ the proof of the continuity of the semigroup operation is similar.

(4) Suppose that $B_\lambda \setminus U = \{(\alpha,\beta)\}$ for some $(\alpha,\beta) \in B_\lambda \setminus \{0\}$. Since family $\mathcal{C}$ is not empty, there exists $A \times B \in \mathcal{C}$. Then there exists $C \subset \lambda$ such that the sets $A \times (\lambda \setminus C)$ and $C \times B$ are closed in (B_λ, τ). Fix any $\varphi \in A$ and $\psi \in B$. Therefore the sets ${}_\varphi(A \times (\lambda \setminus C))_\alpha$ and ${}^\psi_\beta(C \times B)$ are closed in (B_λ, τ). Thus $B_\lambda \setminus ({}_\varphi(A \times (\lambda \setminus C))_\alpha)$ and $B_\lambda \setminus ({}^\psi_\beta(C \times B))$ are neighborhoods of the zero 0 and

$$(B_\lambda \setminus ({}_\varphi(A \times (\lambda \setminus C))_\alpha)) \cdot (B_\lambda \setminus ({}^\psi_\beta(C \times B))) \subseteq B_\lambda \setminus \{(\alpha,\beta)\} = U.$$

Let τ be a semigroup topology on B_λ such that elements of $\mathcal{F}$ are closed in the topological space (B_λ, τ). Then all nonzero points are closed. By Lemma 2, elements of $\{{}_\alpha(A \times B)_\beta, {}^\alpha_\beta(A \times B) \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$ are closed and, hence their complements are open. Since every nonzero point of (B_λ, τ) is isolated, elements of $P_\mathcal{F}$ are open in the topological space (B_λ, τ).

The topology generated by the subbase $P_\mathcal{F}$ is called *topology generated by the compositional family* $\mathcal{F}$ and denoted by $\tau_\mathcal{F}$.

Proposition 1 and Lemma 4 imply the following corollary.

Corollary 2. *Every minimal semigroup topology on B_λ is generated by some compositional family.*

For arbitrary sets A and B we denote

$$A \subseteq^* B \text{ if a set } A \setminus B \text{ is finite;}$$
$$A =^* B \text{ if a set } A \triangle B \text{ is finite.}$$

Remark 1. Note that there are semigroup topologies on B_λ such that not generated by compositional families. In particular, a topology generated by the base

$$\mathcal{B} = \{\{(\alpha,\alpha) \mid \alpha \in A\} \cup \{0\} \mid A =^* \lambda\} \cup \{\{(\alpha,\beta)\} \mid \alpha,\beta \in \lambda\}.$$

Remark 2. Observe that a semigroup topology on B_λ can be generated by distinct compositional families. For example, for arbitrary sets $A \subset \lambda$ and $B \subset A$ the following compositional families $\mathcal{F}_1 = \{A \times (\lambda \setminus A)\}$ and $\mathcal{F}_2 = \{A \times (\lambda \setminus A), B \times (\lambda \setminus A)\}$ generate the same semigroup topology on B_λ.

Let τ be a semigroup topology on B_λ generated by some compositional family. By $\mathrm{Com}(\tau)$ denote the set of all compositional families such that generate the topology τ.

Proposition 2. *Let τ_1 and τ_2 be semigroup topologies on B_λ generated by compositional families. A topology τ_1 is weaker than a topology τ_2 if and only if there exist compositional families $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F}_2 \in \mathrm{Com}(\tau_2)$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

Proof. ($\Rightarrow$) If $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F} \in \mathrm{Com}(\tau_2)$, then the family $\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}$ is compositional and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since the topology τ_1 is weaker than the topology τ_2, any element of $\mathcal{F}_1$ is a closed set in (B_λ, τ_2). Therefore, the family $\mathcal{F}_2$ generate the topology τ_2.

($\Leftarrow$) Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, every closed set in (B_λ, τ_1) is closed in (B_λ, τ_2). Then the topology τ_1 is weaker than the topology τ_2.

Lemma 5. *Let τ be a semigroup topology on B_λ. If the set $A \times B$ is closed in (B_λ, τ) and $C =^* A$, $D =^* B$, then the set $C \times D$ is closed in (B_λ, τ).*

Proof. By Lemma 1, the set $(A \cap C) \times (B \cap D)$ is closed in (B_λ, τ). Lemma 2 implies that the sets $(A \cap C) \times \{\alpha\}$ and $\{\beta\} \times (B \cap D)$ are closed for all $\alpha \in D \setminus B, \beta \in C \setminus A$. Since the sets $D \setminus B$ and $C \setminus A$ are finite, the sets $(A \cap C) \times (D \setminus B)$ and $(C \setminus A) \times (B \cap D)$ are closed. The set $(C \setminus A) \times (D \setminus B)$ is finite and therefore closed. Hence the set $C \times D$ is closed in (B_λ, τ).

3. Compositional digraphs

A compositional family $\mathcal{F}$ can be represented in the form of a digraph with loops $D(\mathcal{F})$. The vertices of the digraph is the set

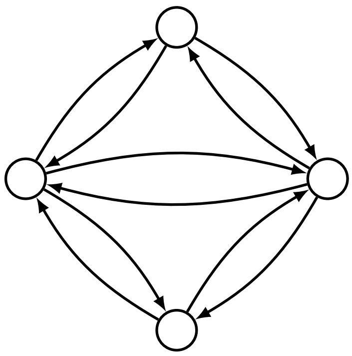
$$V(D(\mathcal{F})) = \{A \mid A \times (\lambda \setminus B) \in \mathcal{F} \text{ or } B \times (\lambda \setminus A) \in \mathcal{F}\}$$

and $(A, B) \in A(D(\mathcal{F}))$ if $A \times (\lambda \setminus B) \in \mathcal{F}$.

Definition 2. *A digraph with loops $D = (V, A)$ is called compositional if for all $(u, v) \in A$ there exists $w \in V$ such that $(u, w) \in A$ and $(w, v) \in A$.*

Proposition 3. *For any compositional family $\mathcal{F}$ the digraph $D(\mathcal{F})$ is compositional and any compositional digraph D such that vertices of D are subsets of λ determines some compositional family.*

Example 1. The following digraphs are compositional.



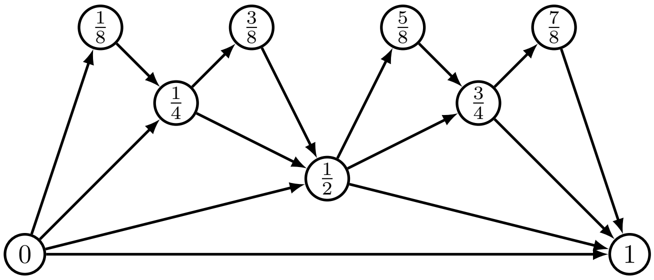
Example 2. Let $\mathbb{Z}[\frac{1}{2}]$ be the set of all dyadic rationals in $[0, 1]$. The digraph $\mathcal{U}$ with $V(\mathcal{U}) = \mathbb{Z}[\frac{1}{2}]$ and

$$A(\mathcal{U}) = \left\{(v, u) \mid v = \frac{k}{2^n} \text{ and } u = v + \frac{1}{2^m} \text{ for some } m \geqslant n\right\}$$

is compositional. Indeed, if (u, v) is an arc of $\mathcal{U}$, then $(u, \frac{u+v}{2})$ and $(\frac{u+v}{2}, v)$ are arcs of $\mathcal{U}$.

For each $i \in \mathbb{N}$ by $\mathcal{U}_i$ denoted the subdigraph of $\mathcal{U}$ induced by $V(\mathcal{U}_i) = \{u \in \mathbb{Z}[\frac{1}{2}] \mid u = \frac{k}{2^n} \text{ for } n \leqslant i\}$.

Proposition 4. *There exists a hamiltonian path in $\mathcal{U}_i$ for each $i \in \mathbb{N}$.*



Proof. Consider the sequence of vertices $W = 0, \frac{1}{2^i}, \frac{2}{2^i}, \frac{3}{2^i}, \ldots, \frac{2^i-1}{2^i}, 1$. Observe that $v_{j+1} - v_j = \frac{1}{2^i}$ for arbitrary $v_j, v_{j+1} \in W$. Thus (v_j, v_{j+1}) is an arc of $\mathcal{U}_i$ and hence this sequence is a path. Since the path contains all vertices of $\mathcal{U}_i$, W is a hamiltonian path.

Proposition 4 implies the following corollary.

Corollary 3. *Arbitrary quotient digraph of $\mathcal{U}$ has a finite cycle.*

Proposition 5. *Let $\{D_i\}_{i\in I}$ be a collection of compositional digraphs. Arbitrary quotient digraph of $D = \bigoplus_{i\in I} D_i$ is compositional.*

Proof. Let $\mathcal{R}$ be an equivalence relation on $V(D)$ and $([u],[v]) \in A(D/\mathcal{R})$. Then there exist $u', v' \in V(D)$ such that $(v', u') \in A(V)$. Thus (v', u') is an arc of V_j for some $j \in I$. Since digraph V_j is compositional, there exist arcs (u', w) and (w, v') of V_j. Hence $([u'],[w]), ([w],[v']) \in A(D/\mathcal{R})$.

Lemma 6. *Let D be a compositional digraph and (u, v) be an arc of D. Then there exists an equivalence relation $\mathcal{R}$ on $V(\mathcal{U})$ such that $\mathcal{U}/\mathcal{R}$ is isomorphic to a subdigraph of D which contains (u, v).*

Proof. For each $w \in V(D)$ define a set C_w of vertices of $\mathcal{U}$ by induction. For $i = 0$ add vertex 0 to C_u and vertex 1 to C_v.

Suppose that all vertexes of $\mathcal{U}_n$ are added to sets $\{C_w\}_{w\in V(D)}$. Take arbitrary vertex $y \in V(\mathcal{U}_{n+1}) \setminus V(\mathcal{U}_n)$. There exist two arcs $(x, y), (y, z) \in A(\mathcal{U}_{n+1}) \setminus A(\mathcal{U}_n)$ and $x \in C_r$, $z \in C_t$ for some $r, t \in V(D)$. Since the digraph D is compositional, there exist arcs $(r, s), (s, t) \in A(D)$. Then add the vertex y to C_s.

Nonempty elements of $\{C_w\}_{w\in V(D)}$ provide a partition of $V(\mathcal{U})$ which determines an equivalence relation $\mathcal{R}$. Let H be an subdigraph of D induced by $\{w \in V(D) \mid C_w \neq \varnothing\}$. Then the map $f : H \to \mathcal{U}/\mathcal{R}$ defined by the formula $f(w) = C_w$ is an isomorphism.

Proposition 6. *Any compositional digraph with size κ is isomorphic to quotient digraph of $\bigoplus_{k\in\kappa} \mathcal{U}$.*

Proof. Let D be compositional digraph with size κ. Consider a collection $\{\mathcal{U}^{(u,v)}\}_{(u,v)\in A(D)}$ where $\mathcal{U}^{(u,v)}$ is a digraph isomorphic to $\mathcal{U}$ for each $(u, v) \in A(D)$. By lemma 6, for arbitrary $\mathcal{U}^{(u,v)}$ there exist an equivalence relation $\mathcal{R}_{(u,v)}$ and an isomorphism $f_{(u,v)}$ between $\mathcal{U}^{(u,v)}/\mathcal{R}_{(u,v)}$ and subdigraph H of D which contains (u, v).

Define the equivalence relation $\mathcal{R}$ on $\bigoplus_{k\in\kappa}\mathcal{U}$ in the next way $u\mathcal{R}v$ if $f_{(s,t)}([u]_{\mathcal{R}_{(s,t)}}) = f_{(x,y)}([v]_{\mathcal{R}_{(x,y)}})$ for $u \in V(\mathcal{U}^{(s,t)})$ and $v \in V(\mathcal{U}^{(x,y)})$. Now define the map $f : \left(\bigoplus_{k\in\kappa}\mathcal{U}\right)/\mathcal{R} \to D$ defined by the formula $f([w]) = f_{(s,t)}([w]_{\mathcal{R}_{(s,t)}})$ where $w \in \mathcal{U}^{(s,t)}$. Then f is an isomorphism.

Corollary 3 and Proposition 6 imply the following corollaries.

Corollary 4. *Any compositional digraph has a finite cycle or contains $\mathcal{U}$ as subdigraph.*

Corollary 5. *Any finite compositional digraph has a finite cycle.*

4. Main result

Proposition 7. *If $A \subset \lambda$, then a topology τ generated by the compositional family $\{A \times (\lambda \setminus A)\}$ is minimal.*

Proof. Let τ_1 be a weaker topology than the topology τ and $B\times(\lambda\setminus C)$ be a closed set in (B_λ, τ_1). By Lemma 4, there exists $D \subseteq \lambda$ such that the sets $B \times (\lambda \setminus D)$ and $D \times (\lambda \setminus C)$ are closed in (B_λ, τ_1). Since τ_1 is weaker than τ, $D \subseteq^* A$ and $(\lambda\setminus D) \subseteq^* (\lambda\setminus A)$. Therefore $D =^* A$. Again, by Lemma 4, there exists $F \subseteq \lambda$ such that the sets $D \times (\lambda \setminus F)$ and $F\times(\lambda\setminus C)$ are closed in (B_λ, τ_1). Hence $F \subseteq^* A$ and $(\lambda\setminus F) \subseteq^* (\lambda\setminus A)$ and then $F =^* A$. By Lemma 5, the set $(A \times (\lambda \setminus A)$ is closed in the topological space (B_λ, τ_1).

Proposition 7 generalizes [3, Theorem 5].

Proposition 8. *Let $\mathcal{F}$ be a compositional family. If there exists a finite subdigraph H of $D(\mathcal{F})$ which doesn't contain sink or source, then there exists $A \subseteq \lambda$ such that the set $A \times (\lambda \setminus A)$ is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$.*

Proof. Let $V(D) = \{A_1, \ldots, A_n\}$ and H doesn't contain sink. For each $A_i \in V(H)$ there exits $A_j \in V(H)$ such that $A_i \times (\lambda \setminus A_j)$. Observe that

$$\lambda \setminus (A_1 \cup \ldots \cup A_n) = (\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n).$$

If $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) = \varnothing$, then $A_1 \cup \ldots \cup A_n = \lambda$ and hence, by Lemma 2, for each $\alpha \in \lambda$ the set $(\lambda \setminus \{\alpha\}) \times \{\alpha\}$ is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. Let $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) \neq \varnothing$. By Lemma 1, the set $A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$ is closed for any $A_i \in V(H)$. Hence

$$A_1 \cup \ldots \cup A_n \times \lambda \setminus (A_1 \cup \ldots \cup A_n) = \bigcup_{i=1}^{n} A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$$

is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. The case with source is proved similarly.

Propositions 7 and 8 imply the following corollary.

Corollary 6. *If $D(\mathcal{F})$ has a finite cycle, then $\mathcal{F}$ is a singleton or generates a nonminimal topology.*

Corollaries 3 and 6 imply the following theorem.

Theorem 1. *Let τ be a semigroup topology on B_λ generated by compositional family $\mathcal{F}$ such that $D(\mathcal{F})$ doesn't contain subgraph isomorphic to $\mathcal{U}$. The topology τ is minimal if and only if τ is generated by singleton compositional family.*

Problem 1. *Is there a minimal semigroup topology on B_λ generated by a composition family $\mathcal{F}$ such that $D(\mathcal{F})$ is isomorphic to $\mathcal{U}$?*

References

1. B. Banaschewski, Minimal topological algebras, Math. Ann. 211 (1974), 107–114.
2. D. Doitchinov, Produits de groupes topologiques minimaux, Bull. Sci. Math. (2) 97 (1972), 59–64.
3. O. V. Gutik, and K. P. Pavlyk, On topological semigroups of matrix units, Semigroup Forum 71 (2005), no. 3, 389-400.
4. J. Bang-Jensen and G. Gutin. Digraphs: Theory, Algorithms and Applications. Springer-Verlag, London, 2nd edition, 2009.
5. L. Nachbin, On strictly minimal topological division rings, Bull. Amer. Math. Soc. 55 (1949), 1128–1136.
6. R. M. Stephenson, Jr., Minimal topological groups, Math. Ann. 192 (1971), 193–195.

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МІНІМАЛЬНІ ТОПОЛОГІЇ НА НАПІВГРУПІ МАТРИЧНИХ ОДИНИЦЬ

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Ключові слова: напівградка, топологія, напівгрупа матричних одиниць