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MINIMAL TOPOLOGIES ON THE SEMIGROUPS OF MATRIX UNITS

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We describe minimal topologies in some class of semigroup topologies on the semigroups of matrix units.

Key words: topology, semigroup topology, minimal topology, semigroups of matrix units

1. Introduction, motivation and main definitions

In this paper all topological spaces are assumed to be Hausdorff.

A *topological semigroup* is a Hausdorff topological space together with a continuous semigroup operation . If S is a semigroup and τ is a topology on S such that (S, τ) is a topological semigroup, then we shall call τ *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation. Topological semigroup (S, τ) is said to be *minimal* if no semigroup topology on S is strictly contained in τ. If (S, τ) is minimal topological semigroup, then τ is called *minimal semigroup topology.*

The concept of minimal topological groups was introduced independently in the early 1970's by Doitchinov [2] and Stephenson [6]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time. More than 20 years earlier L. Nachbin [5] had studied minimality in the context of division rings, and B. Banaschewski [1] investigated minimality in the more general setting of topological algebras.

Let λ be a nonempty set. By B_λ we denote the set $\lambda \times \lambda \cup \{0\}$ endowed with the following semigroup operation:

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & b = c; \\ 0, & b \neq c \end{cases}$$

2010 Mathematics Subject Classification: 18A22 and $(a,b)\cdot 0 = 0\cdot(a,b) = 0\cdot 0 = 0$, for each $a,b,c,d\in\lambda$. The semigroup B_λ is called the *semigroup of* $\lambda\times\lambda$*-matrix units*. The semitopological and topological semigroup of matrix units was investigated in [3].

A *directed graph* (or just *digraph*) D consists of a nonempty set $V(D)$ of elements called *vertices* and a set $A(D)$ of ordered pairs of vertices called *arcs*. We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D. The *order* (*size*) of D is the cardinality of the vertex (arc) set of D. For an arc (u,v) the first vertex u is its *tail* and the second vertex v is its *head*. If a tail and a head of arc coincide, then this arc is called a *loop*. The head and tail of an arc are its *end-vertices*. A vertex v is a *source*(*source*) if v is not a head(tail) for any arc. A digraph H is a *subdigraph* of a digraph D if $V(H)\subseteq V(D)$, $A(H)\subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $X = V(H)$ and call H an induced subdigraph of D.

A *walk* in D is an alternating sequence $W = x_1a_1x_2a_2x_3\ldots x_{k-1}a_{k-1}x_k$ of vertices x_i and arcs a_j from D such that the tail of a_i is x_i and the head of a_i is x_{i+1} for every $i = 1,2,\ldots,k-1$. The *length* of a walk is the number of its arcs. When the arcs of W are defined from the context or simply unimportant, we will denote W by $x_1x_2\ldots x_k$. If the vertices of W are distinct, W is a *path*. If the vertices $x_1,x_2,\ldots,x_{k-1}$ are distinct and $x_1 = x_k$, W is a *cycle*. A walk (path, cycle) W is a *Hamilton* (or *hamiltonian*) walk (path, cycle) if W contains all vertices of D.

Let $\{D_i\}_{i\in I}$ be a family of digraphs. The digraph $(\bigsqcup_{i\in I} V(D_i), \bigsqcup_{i\in I} A(D_i))$ is called *disjoint union* of this family and denoted by $\bigoplus_{i\in I} D_i$. If D is a digraph and $\mathcal{R}$ is an equivalence relation on $V(D)$. Then the *quotient digraph* $D/\mathcal{R}$ has vertex set $V/\mathcal{R}$ and arc set $\{([a]_\mathcal{R},[b]_\mathcal{R}) \mid (a,b)\in A(D)\}$.

2. Compositional families

If (B_λ,τ) is a semitopological semigroup, then any nonzero element of B_λ is an isolated point of (B_λ,τ) [3, Lemma 2]. Therefore the following lemma is true.

Lemma 1. *Let (B_λ,τ) be a topological semigroup and A be a closed subset A of (B_λ,τ) which doesn't contain 0. Then any subset of A is closed.*

For $A\subseteq B_\lambda$ and $\alpha,\beta\in\lambda$ we denote

$$
\begin{aligned}
{}_\alpha A_\beta &= \{(\beta,\gamma) \mid (\alpha,\gamma)\in A\};\\
{}^\alpha_\beta A &= \{(\gamma,\beta) \mid (\gamma,\alpha)\in A\};\\
\mathrm{pr}_1(A) &= \{\alpha \mid (\alpha,\beta)\in A\};\\
\mathrm{pr}_2(A) &= \{\beta \mid (\alpha,\beta)\in A\}.
\end{aligned}
$$

Lemma 2. *Let τ be a topology on B_λ and any nonzero element of B_λ is isolated in (B_λ,τ). The semigroup operation is continues on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$ if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for every $\alpha,\beta\in\lambda$ and every closed subset A of (B_λ,τ) which doesn't contain 0.*

Proof. ($\Rightarrow$) Let A be a closed subset A of (B_λ, τ) which doesn't contain 0. Lemma 1 implies that the sets ${}_\alpha A_\alpha$ and ${}^\alpha_\alpha A$ are closed. By the continuity of operation, the maps $\lambda_{(\alpha,\beta)} : B_\lambda \to B_\lambda$ and $\rho_{(\beta,\alpha)} : B_\lambda \to B_\lambda$ defined by the formulas $\lambda_{(\alpha,\beta)}(x) = (\alpha, \beta) \cdot x$ and $\rho_{(\beta,\alpha)}(x) = x \cdot (\beta, \alpha)$ are continuous. Therefore the sets

$$ {}_\alpha A_\beta = (\lambda_{(\alpha,\beta)})^{-1}({}_\alpha A_\alpha) $$

and

$$ {}^\alpha_\beta A = (\rho_{(\beta,\alpha)})^{-1}({}^\alpha_\alpha A) $$

are closed in the topological space (B_λ, τ).

($\Leftarrow$) Since every nonzero point of B_λ is isolated, we need check the continuity of operation only in the cases of $(\alpha, \beta) \cdot 0$ and $0 \cdot (\alpha, \beta)$. Let U be a neighborhood of 0. Denote by A the closed set $B_\lambda \setminus U$. Then the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed. Denote by V and W the neighborhoods of zero $B_\lambda \setminus {}_\alpha A_\beta$ and $B_\lambda \setminus {}^\alpha_\beta A$, respectively. Therefore $\{(\alpha, \beta)\} \cdot V \subseteq U$ and $W \cdot \{(\alpha, \beta)\} \subseteq U$.

Lemma 2 implies the following corollary.

Corollary 1. *Let τ be a topology on B_λ. (B_λ, τ) is a semitopological semigroup if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for every $\alpha, \beta \in \lambda$ and every closed subset A of (B_λ, τ) which doesn't contain 0.*

Lemma 3. *Let $A, B \subseteq B_\lambda$ and $(\alpha, \beta) \in B_\lambda$. The element $(\alpha, \beta) \notin A \cdot B$ if and only if $\mathrm{pr}_2({}_\alpha A_\alpha) \cap \mathrm{pr}_1({}^\beta_\beta B) = \varnothing$.*

Proof. ($\Rightarrow$) Suppose that there exists $\gamma \in \mathrm{pr}_2({}_\alpha A_\alpha) \cap \mathrm{pr}_1({}^\beta_\beta B)$ then $(\alpha, \gamma) \in A$ and $(\gamma, \beta) \in B$. Therefore $(\alpha, \gamma) \cdot (\gamma, \beta) = (\alpha, \beta) \in A \cdot B$, a contradiction.

($\Leftarrow$) Suppose that $\mathrm{pr}_2({}_\alpha A_\alpha) \cap \mathrm{pr}_1({}^\beta_\beta B) = \varnothing$ and $(\alpha, \beta) \in A \cdot B$. Then there exist $\gamma \in \lambda$ such that $(\alpha, \gamma) \in A$ and $(\gamma, \beta) \in B$. Hence $\gamma \in \mathrm{pr}_2({}_\alpha A_\alpha) \cap \mathrm{pr}_1({}^\beta_\beta B)$, a contradiction.

We will call elements of the set $\{A \times B \mid A, B \subseteq B_\lambda\}$ by rectangles.

Definition 1. *A nonempty family $\mathcal{F}$ of rectangles is called compositional if for $A \times B \in \mathcal{F}$ there exists $C \subseteq \lambda$ such that $A \times (\lambda \setminus C) \in \mathcal{F}$ and $C \times B \in \mathcal{F}$.*

Lemma 4. *Let τ be a semigroup topology on B_λ, then the family of all closed rectangles of (B_λ, τ) is compositional.*

Proof. Since every point of (B_λ, τ) is closed, the family of all closed rectangles is not empty. Let $A \times B$ be a closed subset of (B_λ, τ), then $B_\lambda \setminus (A \times B)$ is a neighborhood of 0. By the continuity of operation at the point $(0, 0)$ there exist neighborhoods of zero U and V such that $U \cdot V \subseteq B_\lambda \setminus (A \times B)$. Lemma 3 implies that for each $\alpha \in A$ the neighborhood V doesn't contain the set $\mathrm{pr}_2({}_\alpha V_\alpha) \times B$. Therefore, the neighborhood V doesn't contain the set $\left(\bigcup\limits_{\alpha \in A} \mathrm{pr}_2({}_\alpha U_\alpha)\right) \times B$. Denote this set by C.

Let $C \neq \lambda$. By the De Morgan's laws, $\lambda \setminus C = \bigcap\limits_{\alpha \in A} \lambda \setminus (\mathrm{pr}_2({}_\alpha U_\alpha))$. The neighborhood U_1 doesn't contain $A \times (\lambda \setminus C)$. Lemma 1 implies that the sets $A \times (\lambda \setminus C) \in S$ and $C \times B \in S$ are closed.

If $C = \lambda$, then, by Lemma 1, the subset $(\lambda \setminus \{\varphi\}) \times B$ for some $\varphi \in \lambda$ is closed. Lemma 2 implies that subset $A \times \{\varphi\}$ is closed.

Hence the family of all closed rectangles of (B_λ, τ) is compositional.

Let $\mathcal{F}$ be compositional family. Denote

$$C = \mathcal{F} \cup \{_\alpha A_\beta, {}^\alpha_\beta A \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$$

and

$$P_\mathcal{F} = \{B_\lambda \setminus B \mid B \in C\} \cup \{\{(\alpha, \beta)\} \mid (\alpha, \beta) \in B_\lambda\} \cup \{\varnothing\}.$$

Proposition 1. *For every compositional family $\mathcal{F}$ topology generated by the subbase $P_\mathcal{F}$ is the smallest semigroup topology such that elements of $\mathcal{F}$ are closed.*

Proof. First we shall show that topology generated by the subbase $P_\mathcal{F}$ is semigroup. Since

$$\begin{aligned}
{}_\alpha(\bigcup_{i\in I} A_i)_\beta &= \bigcup_{\iota\in I} {}_\alpha(A_i)_\beta,\\
{}^\alpha_\beta(\bigcup_{i\in I} A_i) &= \bigcup_{\iota\in I} {}^\alpha_\beta(A_i),\\
{}_\alpha(\bigcap_{i\in I} A_i)_\beta &= \bigcap_{\iota\in I} {}_\alpha(A_i)_\beta,\\
{}^\alpha_\beta(\bigcap_{i\in I} A_i) &= \bigcap_{\iota\in I} {}^\alpha_\beta(A_i),
\end{aligned}$$

by Lemma 2, the semigroup operation is continues on $(B_\lambda \times B_\lambda) \setminus \{(0,0)\}$.

The continuity of the operation in the point $(0,0)$ can be verify only for elements of the subbase. Let U be a neighborhood of 0 such that $U \in P_\mathcal{F}$. Consider possible cases:

(1) $B_\lambda \setminus U = A \times B \in \mathcal{F}$, then there exists $C \subset \lambda$ such that $A \times (\lambda \setminus C)$ and $C \times B$ are closed subsets of (B_λ, τ). Thus $B_\lambda \setminus A \times (\lambda \setminus C)$ and $B_\lambda \setminus C \times B$ are neighborhoods of 0 and

$$(B_\lambda \setminus A \times (\lambda \setminus C)) \cdot (B_\lambda \setminus C \times B) \subseteq B_\lambda \setminus A \times B = U.$$

(2) $B_\lambda \setminus U = {}_\alpha(A \times B)_\beta = \{\beta\} \times B$ for some $\alpha \in \lambda, \beta \in A$ and $A \times B \in \mathcal{F}$. There exists $C \subseteq \lambda$ such that the sets $A \times (B_\lambda \setminus C)$ and $C \times B$ are closed. Since the set ${}_\alpha(A \times B)_\beta = \{\beta\} \times (B_\lambda \setminus C)$ is closed, the set $(A \cup \{\beta\}) \times (B_\lambda \setminus C)$ is closed. Then $B_\lambda \setminus (A \cup \{\beta\}) \times (B_\lambda \setminus C)$ and $B_\lambda \setminus C \times B$ are neighborhoods of 0 and

$$(B_\lambda \setminus (A \cup \{\beta\})) \cdot (B_\lambda \setminus C \times B) \subseteq B_\lambda \setminus (\{\beta\} \times B) = U.$$

(3) The case $B_\lambda \setminus U = {}^\alpha_\beta(A \times B) = A \times \{\beta\}$ is proved similarly.

Let τ be a topology on B_λ such elements of $\mathcal{F}$ are closed in the topological space (B_λ, τ). Then, by Lemma 2, elements of $\{_\alpha(A \times B)_\beta, {}^\alpha_\beta(A \times B) \mid A \in \mathcal{F} \text{ and } \alpha, \beta \in \lambda\}$ are closed and, hence their complements are open. Since every nonzero point of (B_λ, τ) is isolated, elements of $P_\mathcal{F}$ are open in the topological space (B_λ, τ).

The topology generated by the subbase $P_\mathcal{F}$ will be called topology generated by the compositional family $\mathcal{F}$ and denoted by $\tau_\mathcal{F}$.

Proposition 1 and Lemma 4 imply the following corollary.

Corollary 2. *Every minimal semigroup topology on B_λ is generated by some compositional family.*

Let A and B be sets. We will denote

$$A =^* B \text{ if a set } A \triangle B \text{ is finite };$$
$$A \subseteq^* B \text{ if a set } A \setminus B \text{ is finite }.$$

Note that there are semigroup topologies such that not generated by compositional families. For example, a topology generated by the base $\{\{(\alpha,\alpha) \mid \alpha \in A\} \cup \{0\} \mid A =^* \lambda\} \cup \{\{(\alpha,\beta)\} \mid \alpha,\beta \in \lambda\}$.

Observe that a semigroup topology can be generated by distinct compositional families. Let τ be a semigroup topology generated by some compositional family on B_λ. By $\mathrm{Com}(\tau)$ denote the set of all compositional families such that generate the topology τ.

Proposition 2. *Let τ_1 and τ_2 be semigroup topologies on B_λ generated by compositional families. A topology τ_1 is weaker than a topology τ_2 if and only if there exist compositional families $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F}_2 \in \mathrm{Com}(\tau_2)$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

Proof. ($\Rightarrow$) If $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F} \in \mathrm{Com}(\tau_2)$, then the family $\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}$ is compositional and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since the topology τ_1 is weaker than the topology τ_2, any element of $\mathcal{F}_1$ is a closed set in the topology τ_2. Therefore, the family $\mathcal{F}_2$ generate the topology τ_2.

($\Leftarrow$) Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, every closed set in the topology τ_1 is closed in the topology τ_2. Hence the topology τ_1 is weaker than the topology τ_2.

Lemma 5. *Let τ be semigroup topology on B_λ. If the set $A \times B$ is closed in the topological space (B_λ, τ) and $C =^* A$, $D =^* B$, then the set $C \times D$ is closed in the topological space (B_λ, τ).*

Proof. By Lemma 1, the set $(A \cap C) \times (B \cap D)$ is closed in the topological space (B_λ, τ). Then, by Lemma 2, the sets $(A \cap C) \times \{\alpha\}$ and $\{\beta\} \times (B \cap D)$ are closed for all $\alpha \in D \setminus B, \beta \in C \setminus A$. Since the sets $D \setminus B$ and $C \setminus A$ are finite, the sets $(A \cap C) \times (D \setminus B)$ and $(C \setminus A) \times (B \cap D)$ are closed. The set $(C \setminus A) \times (D \setminus B)$ is finite and therefore closed. Hence the set $C \times D$ is closed in the topological space (B_λ, τ).

3. Compositional digraphs

A compositional family $\mathcal{F}$ can be represented in the form of a digraph with loops $D(\mathcal{F})$. The vertices of digraph is the set

$$V(D(\mathcal{F})) = \{A \mid A \times (\lambda \setminus B) \in \mathcal{F} \text{ or } B \times (\lambda \setminus A) \in \mathcal{F}\}$$

and $(A, B) \in A(D(\mathcal{F}))$ if $A \times (\lambda \setminus B) \in \mathcal{F}$.

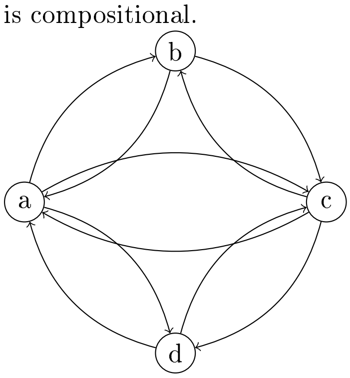
Definition 2. *A digraph with loops $D = (V, A)$ is called compositional if for all $(u, v) \in A$ there exists $w \in V$ such that $(u, w) \in A$ and $(w, v) \in A$.*

Proposition 3. *For any compositional family $\mathcal{F}$ the digraph $D(\mathcal{F})$ is compositional and any compositional digraph $D = (V, A)$ such that $V \subseteq \mathcal{P}(\lambda)$ determines some compositional family.*

Example 1. The digraph D with $V(D) = \{a, b, c, d\}$ and

$$A(D) = \{(a, b), (b, a), (b, c), (c, b), (c, d), (d, c), (a, d), (d, a), (a, c), (c, a)\}$$

is compositional.

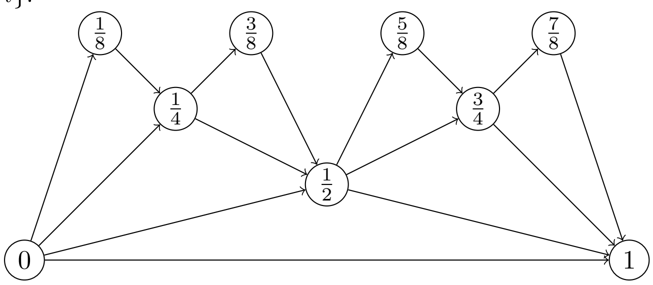


Example 2. Let $\mathbb{Z}[\frac{1}{2}]$ be the set of all dyadic rationals in $[0, 1]$. The digraph $\mathcal{U}$ with $V(\mathcal{U}) = \mathbb{Z}[\frac{1}{2}]$ and

$$A(\mathcal{U})\{(v, u) \mid v = \frac{k}{2^n} \text{ and } u = v + \frac{1}{2^m} \text{ for some } m \geqslant n\}$$

is compositional. Indeed, if (u, v) is an arc of $\mathcal{U}$, then $(u, \frac{u+v}{2})$ and $(\frac{u+v}{2}, v)$ are arcs of $\mathcal{U}$.

By $\mathcal{U}_i$ denoted the subdigraph of $\mathcal{U}$ induced by $V(\mathcal{U}_i) = \{u \in \mathbb{Z}[\frac{1}{2}] \mid u = \frac{k}{2^n} \text{ for } n \leqslant i\}$.



Proposition 4. *There exists a hamiltonian path in $\mathcal{U}_i$ for each $i \in \mathbb{N}$.*

Proof. Consider the sequence of vertices $W = 0, \frac{1}{2^i}, \frac{2}{2^i}, \frac{3}{2^i}, \ldots, \frac{2^i-1}{2^i}, 1$. Observe that $v_{j+1} - v_j = \frac{1}{2^i}$ for arbitrary $v_j, v_{j+1} \in W$. Thus (v_j, v_{j+1}) is an arc of $\mathcal{U}_i$ and hence this sequence is a path. Since the path contains all vertices of $\mathcal{U}_i$, W is a hamiltonian path.

Proposition 4 implies the following corollary.

Corollary 3. *Arbitrary quotient digraph of $\mathcal{U}$ has a finite cycle.*

Proposition 5. *Let $\{D_i\}_{i \in I}$ be a collection of compositional digraphs. Arbitrary quotient digraph of $D = \bigoplus\limits_{i \in I} D_i$ is compositional.*

Proof. Let $\mathcal{R}$ be an equivalence relation on $V(D)$ and $([u], [v]) \in A(D/\mathcal{R})$. Then there exist $u', v' \in V(D)$ such that $(v', u') \in A(V)$. Thus (v', u') is an arc of V_j for some

$j \in I$. Since digraph V_j is compositional, there exist arcs (u', w) and (w, v') of V_j. Hence $([u'],[w]),([w],[v']) \in A(D/\mathcal{R})$.

Lemma 6. *Let D be a digraph and (a,b) be a arc of D. Then there exists an equivalence relation $\mathcal{R}$ on $V(\mathcal{U})$ such that $\mathcal{U}/\mathcal{R}$ is isomorphic to a subdigraph of D which contains (a,b).*

Proof.

For each $v \in V(D)$ define a set C_v by induction.

Construct by induction equivalence classes of the relation $\mathcal{R}$. For each $v \in V(D)$ define $[v] = \varnothing$. Let (u,w) is an arc of $\mathcal{U}_0$. Add vertex u to $[a]$ and vertex w to $[b]$. Suppose that all vertexes of $\mathcal{U}_i$ are added to equivalence classes. Take arbitrary vertex $v \in V(\mathcal{U}_{i+1}) \setminus V(\mathcal{U}_i)$. There exist two arcs $(u,v),(v,w) \in A(\mathcal{U}_{i+1}) \setminus A(\mathcal{U}_i)$ and $u \in [x]$, $w \in [y]$. Since the digraph D is compositional, there exist arcs $(x,z),(z,y) \in A(D)$. Then add the vertex v to $[z]$.

Proposition 6. *Any compositional digraph with size κ is isomorphic to quotient digraph of $\bigoplus\limits_{k\in\kappa} \mathcal{U}$.*

Proof. Let D be compositional digraph with size κ. Consider a collection $\{\mathcal{U}^{(a,b)}\}_{(a,b)\in A(D)}$ where $\mathcal{U}^{(a,b)}$ is isomorphic to $\mathcal{U}$ for each $(a,b) \in A(D)$. By lemma 6, for arbitrary $\mathcal{U}^{(a,b)}$ there exist an equivalence relation $\mathcal{R}_{(a,b)}$ and an isomorphism $f_{(a,b)}$ between $\mathcal{U}^{(a,b)}/\mathcal{R}_{(a,b)}$ and subdigraph H of D which contains (a,b). Define the map

For $u \in V(\mathcal{U}^{(a,b)})$ and $v \in V(U^{(c,d)})$ define the equivalence relation $u\mathcal{R}v$ if $f_{(a,b)}([u]_{(a,b)}) = f_{(c,d)}([v]_{(c,d)})$. Hence $\bigoplus\limits_{k\in\kappa} \mathcal{U}/\mathcal{R}$ is isomorphic to D.

Corollary 3 and Proposition 6 imply the following corollaries.

Corollary 4. *Any compositional graph has a finite cycle or contains $\mathcal{U}$ as subgraph.*

Corollary 5. *Any finite compositional graph has a finite cycle.*

4. Main result

Proposition 7. *Let A be subset of λ, then a topology τ generated by the compositional family $\{A \times (\lambda \setminus A)\}$ is minimal.*

Proof. Let τ_1 be a weaker topology than the topology τ and $B \times (\lambda \setminus C)$ be a closed set in the topological space (B_λ, τ_1). By Lemma 4, there exists $D \subseteq \lambda$ such that the sets $B \times (\lambda \setminus D)$ and $D \times (\lambda \setminus C)$ are closed in the topological space (B_λ, τ_1). Since τ_1 is weaker than τ, $D \subseteq^* A$ and $(\lambda \setminus D) \subseteq^* (\lambda \setminus A)$. Therefore $D =^* A$. Again, by Lemma 4, there exists $F \subseteq \lambda$ such that the sets $D \times (\lambda \setminus F)$ and $F \times (\lambda \setminus C)$ are closed in the topological space (B_λ, τ_1). Hence $F \subseteq^* A$ and $(\lambda \setminus F) \subseteq^* (\lambda \setminus A)$ and then $F =^* A$. By Lemma 5, the set $(A \times (\lambda \setminus A)$ is closed in the topological space (B_λ, τ_1).

Proposition 7 generalizes [3, Theorem 5].

Proposition 8. *Let $\mathcal{F}$ be a compositional family. If there exists a finite subdigraph H of $D(\mathcal{F})$ which doesn't contain sink or source, then there exists $A \subseteq \lambda$ such that the set $A \times (\lambda \setminus A)$ is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$.*

Proof. Let $V(D) = \{A_1, \ldots, A_n\}$ and H doesn't contain sink. For each $A_i \in V(H)$ there exits $A_j \in V(H)$ such that $A_i \times (\lambda \setminus A_j)$. Observe that

$$\lambda \setminus (A_1 \cup \ldots \cup A_n) = (\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n).$$

If $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) = \varnothing$, then $A_1 \cup \ldots \cup A_n = \lambda$ and hence, by Lemma 2, for each $\alpha \in \lambda$ the set $(\lambda \setminus \{\alpha\}) \times \{\alpha\}$ is closed in the topological space $(B_\lambda, \tau_\mathcal{F})$. Let $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) \neq \varnothing$. By Lemma 1, the set $A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$ is closed for any $A_i \in V(H)$. Hence

$$A_1 \cup \ldots \cup A_n \times \lambda \setminus (A_1 \cup \ldots \cup A_n) = \bigcup_{i=1}^{n} A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$$

is closed in the topological space $(B_\lambda, \tau_\mathcal{F})$. The case with source is proved similarly.

Propositions 7 and 8 imply the following corollary.

Corollary 6. *If $D(\mathcal{F})$ has a finite cycle, then $\mathcal{F}$ is a singleton or generates a nonminimal topology.*

Corollaries 3 and 6 imply the following theorem.

Theorem 1. *Let τ be a semigroup topology on B_λ generated by compositional family $\mathcal{F}$ such that $D(\mathcal{F})$ doesn't contain subgraph isomorphic to $\mathcal{U}$. The topology τ is minimal if and only if τ is generated by singleton compositional family.*

Problem 1. *Is there a minimal semigroup topology on B_λ generated by a composition family $\mathcal{F}$ such that $D(\mathcal{F})$ is isomorphic to $\mathcal{U}$?*

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МІНІМАЛЬНІ ТОПОЛОГІЇ НА НАПІВГРУПІ МАТРИЧНИХ ОДИНИЦЬ

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Ключові слова: напівградка, топологія, напівгрупа матричних одиниць