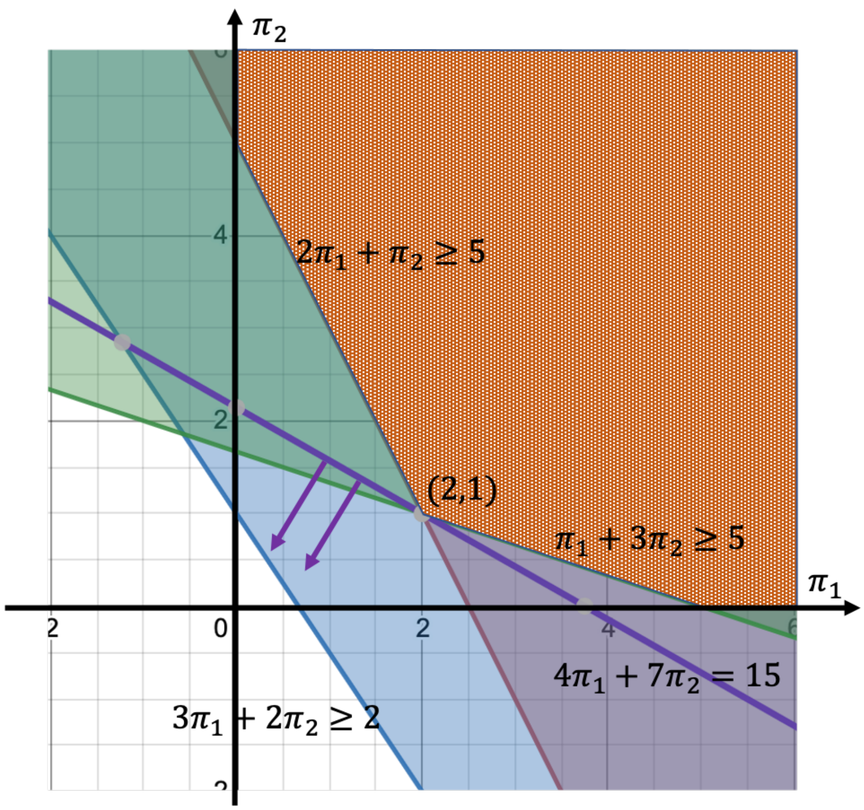
MAT3007
Assignment 4 Solution

Problem 1

1. The dual problem is:

$$\begin{aligned} \min \quad & 4\pi_1 + 7\pi_2 \\ \text{s.t.} \quad & 2\pi_1 + \pi_2 \geq 5 \\ & 3\pi_1 + 2\pi_2 \geq 2 \\ & \pi_1 + 3\pi_2 \geq 5 \\ & \pi_1, \pi_2 \geq 0. \end{aligned}$$

2. The dual problem can be illustrated with the following figure:



The shaded orange area represents the dual feasible region and the purple line is an iso-profit line (objective function). As the improving direction is pointed out by the arrows, the optimal dual solution is $\pi_1^*, \pi_2^*) = (2, 1)$. The optimal value of the dual problem is 15.

3. Both π_1^* and π_2^* are positive. The first and third constraints are tight and there is a slack in the second constraint in the dual problem. According to complementary slackness,

both of the primal constraints are tight at the optimal solution and x_2^* should be zero, i.e., we have

$$\begin{aligned} 2x_1 + x_3 &= 4 \\ x_1 + 3x_3 &= 7 \end{aligned}$$

By solving the set of equalities above, we obtain the optimal primal solution $(x_1^*, x_2^*, x_3^*) = (1, 0, 2)$.

Problem 2

1. Let x_{ij} denote the flow on edge (i, j). Let Δ denote the flow from imaginary node 0 to 1. The LP formulation of this problem is as follows:

$$\begin{array}{ll} \text{maximize} & \Delta \\ \text{s.t.} & x_{12} - x_{23} - x_{24} = 0 \\ & x_{13} + x_{23} - x_{34} = 0 \\ & \Delta - x_{12} - x_{13} = 0 \\ & x_{24} + x_{34} - \Delta = 0 \\ & 0 \leq x_{12} \leq 8 \\ & 0 \leq x_{13} \leq 7 \\ & 0 \leq x_{23} \leq 2 \\ & 0 \leq x_{24} \leq 4 \\ & 0 \leq x_{34} \leq 12 \end{array}$$

The optimal solutions are $x_{12} = 6, x_{13} = 7, x_{23} = 2, x_{24} = 4, x_{34} = 9$, and the objective value is 13 at optimal.

The MATLAB code is attached as follows:

```
cvx_begin quiet
variables d x12 x13 x23 x24 x34
maximize d
subject to
x12-x23-x24==0;
x13+x23-x34==0;
d-x12-x13==0;
x24+x34-d==0;
0<=x12<=8;
0<=x13<=7;
0<=x23<=2;
0<=x24<=4;
0<=x34<=12;
cvx_end
cvx_optval
```

The Python code is attached as follows:

```
import cvxpy as cp
from cvxopt import *

M = 100
# set of nodes
#    s   2  3  t
W = [[0, 8, 7, 0], # s
[0, 0, 2, 4],  # 2
[0, 0, 0, 12], # 3
[M, 0, 0, 0]]  # t

n = len(W[0])   # number of nodes

# model
e = cp.Variable((n, n))
objective = cp.Maximize(e[3,0])
constraints = [sum(e[:,i]) == sum(e[i,:]) for i in range(n)]
constraints += [e >= 0]
constraints += [e[i,j] <= W[i][j] for i in range(n) for j in range(n)]

prob = cp.Problem(objective, constraints)
result = prob.solve()
```

2. Dual problem:

$$\begin{array}{ll} \text{minimize} & 8z_{12} + 7z_{13} + 2z_{23} + 4z_{24} + 12z_{34} \\ \text{s.t.} & z_{12} \geq y_1 - y_2 \\ & z_{13} \geq y_1 - y_3 \\ & z_{23} \geq y_2 - y_3 \\ & z_{24} \geq y_2 - y_4 \\ & z_{34} \geq y_3 - y_4 \\ & y_1 - y_4 = 1 \\ & z_{ij} \geq 0 \end{array}$$

Solving the problem using MATLAB, we get $y = (1, 1, 0, 0)$, $z = (0, 1, 1, 1, 0)$ and the optimal value is 13.

The MATLAB code is attached as follows:

```
cvx_begin quiet
variables z12 z13 z23 z24 z34 y1 y2 y3 y4
minimize 8.*z12+7.*z13+2.*z23+4.*z24+12.*z34
subject to
```

```
y3-y4==1;
z12+y1-y3>=0;
z13+y2-y3>=0;
z23-y1+y2>=0;
z24-y1+y4>=0;
z34-y2+y4>=0;
z12>=0;
z13>=0;
z23>=0;
z24>=0;
z34>=0;
cvx_end
cvx_optval
```

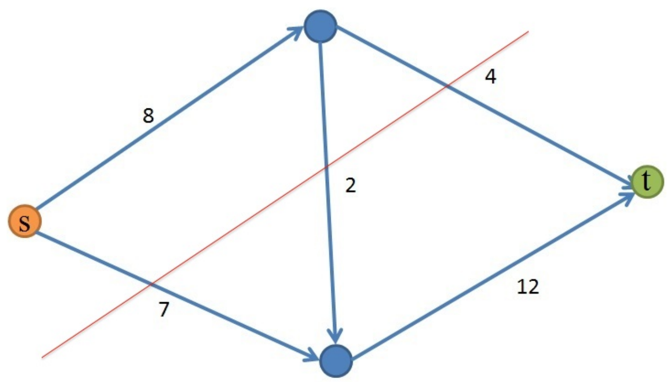
The Python code is attached as follows. Notice that the dual formulation in the code is derived differently, but it is still valid.

```
z = cp.Variable((n, n))
y = cp.Variable(n)
E = []          # set of edges
for i in range(n):
  for j in range(n):
    if (W[i][j] != 0)and(W[i][j] != M):
      E.append((i,j))

objective_d = cp.Minimize(sum(W[i][j]*z[i,j] for (i,j) in E))
constraints_d = [z[i,j] >= 0 for (i,j) in E]
constraints_d += [z[i,j] >= y[j] - y[i] for (i,j) in E]
constraints_d += [y[3] - y[0] == 1]

prob_d = cp.Problem(objective_d, constraints_d)
result_d = prob_d.solve()
```

The z_{ij} variables represent whether or not each edge is in the minimum cut. The cut is shown in the picture below.



Problem 3

Dual problem for model (1) is:

$$
\begin{aligned}
(D_1)\quad \min \quad & b^\top \pi \\
\text{s.t.} \quad & A^\top \pi \geq c \\
& \pi \geq 0
\end{aligned}
$$

Dual problem for model (2) is:

$$
\begin{aligned}
(D_2)\quad \min \quad & b'^\top \pi \\
\text{s.t.} \quad & A^\top \pi \geq c \\
& \pi \geq 0
\end{aligned}
$$

We can see that (D_1) and (D_2) have exactly the same feasible region because they have the same constraints.

We know model (1) is unbounded, according to the duality theory (D_1) should be feasible.

Because (D_1) and (D_2) have the same feasible region, (D_2) is also infeasible. If (D_2) is infeasible, and model (2) is feasible, according to the duality theory, model (2) cannot have a finite optimal solution. Model (2) can only be unbounded.

Problem 4

First, we show that the two systems can't both have solutions. If so, we have

$$0 = y^\top Ax \leq y^\top b < 0,$$

which is a contradiction.

Second, we show that if the second system is infeasible, then the first system must be feasible. We consider the following pair of linear optimization problems:

$$
\begin{aligned}
\min \quad & b^\top y \\
\text{s.t.} \quad & A^\top y = 0 \\
& y \geq 0.
\end{aligned}
$$

The dual of this problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

If the second system does not have a solution, then the primal problem can't attain negative objective value. In the meantime, $y = 0$ is always a feasible solution for the primal problem with objective value 0. Therefore, $y = 0$ must be an optimal solution to the primal problem. Then by the strong duality theorem, the dual problem must also be feasible. Thus, the result is proved.

Problem 5

Since π_1^* and π_3^* are strictly positive, according to the complementary slackness, we obtain that the first and third constraints of the primal problem hold as equalities.

In addition, the formulation of the dual problem is:

$$\begin{aligned} \min \quad & 9\pi_1 + 2\pi_2 + 4\pi_3 & \\ \text{s.t.} \quad & \pi_1 + \pi_2 - \pi_3 \geq -1 & : x_1 \geq 0 \\ & \pi_1 + \pi_2 + \pi_3 \geq -1 & : x_2 \geq 0 \\ & 2\pi_1 - \pi_2 + \pi_3 \geq 4 & : x_3 \geq 0 \\ & \pi_1, \pi_2, \pi_3 \geq 0. & \end{aligned}$$

Plugging in the dual optimal solution, we can find that the second constraint in the dual problem does not hold at equality. Therefore, $x_2^* = 0$, and we can rewrite the first and third constraints of the primal problem as:

$$\begin{aligned} x_1 + 2x_3 &= 9 \\ -x_1 + x_3 &= 4. \end{aligned}$$

Solving the set of linear equations above, we obtain the primal optimal solution as:

$$x_1^* = 1/3, x_2^* = 0, x_3^* = 13/3.$$