香港中文大學（深圳）
The Chinese University of Hong Kong

CSC3100 Data Structures
Lecture 20: Minimum spanning tree

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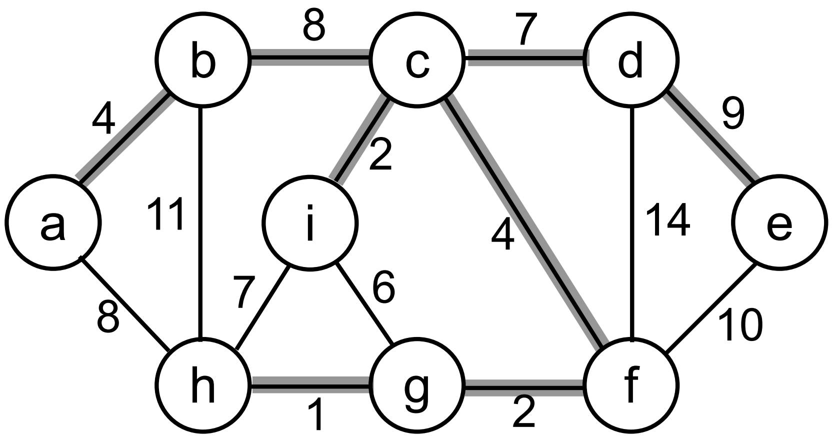
Outline

- What is minimum spanning tree (MST)?
  - Definition and applications
- A generic approach
  - Theoretic proof
  - Prim's algorithm
  - Kruskal's algorithm

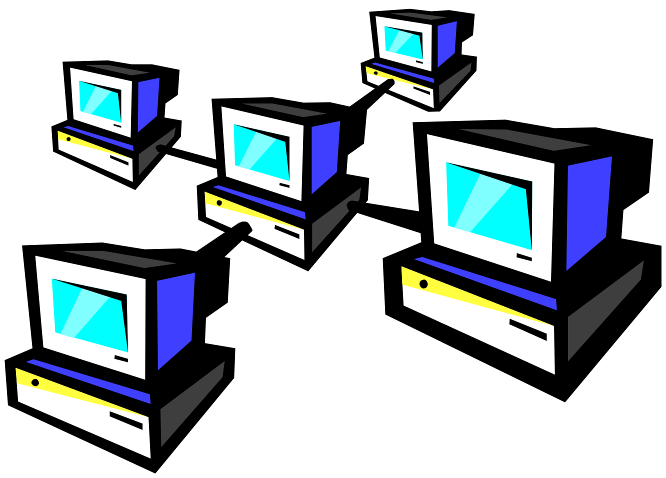
Minimum spanning trees

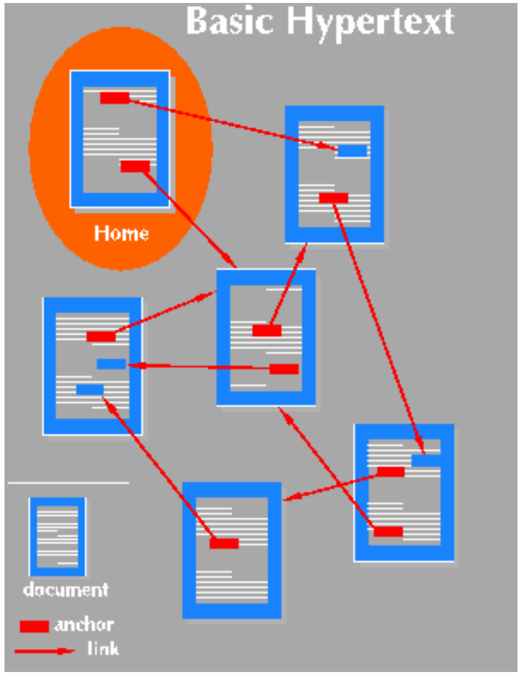
- Spanning tree
  - A tree (i.e., connected, acyclic graph) which contains all the vertices of the graph
- Minimum spanning tree (MST)
  - Spanning tree with the minimum sum of weights
  - If a graph is not connected, then there is an MST for each connected component of the graph



Applications of MST

- Find the least expensive way to connect a set of houses, cities, terminals, computers, etc.

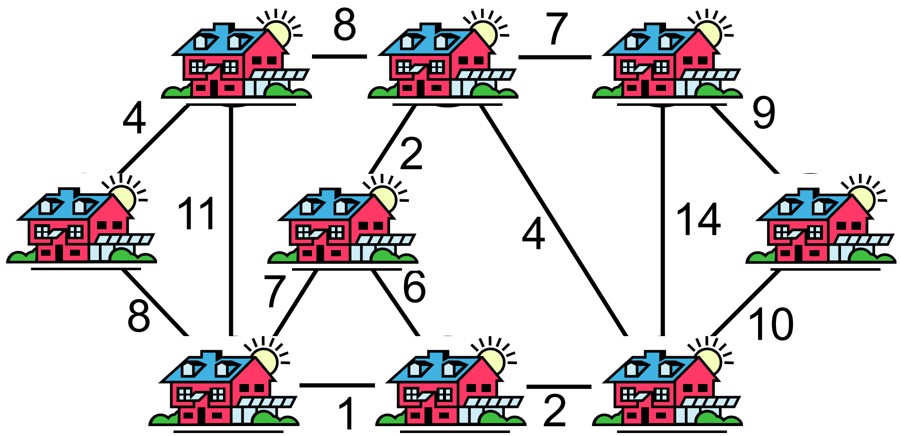




Example

Problem

- A town has a set of houses and a set of roads
- A road connects 2 and only 2 houses
- A road connecting houses u and v has a cost w(u, v)



Goal: Build enough (and no more) roads such that:

1. Everyone stays connected

   i.e., can reach every house from all other houses

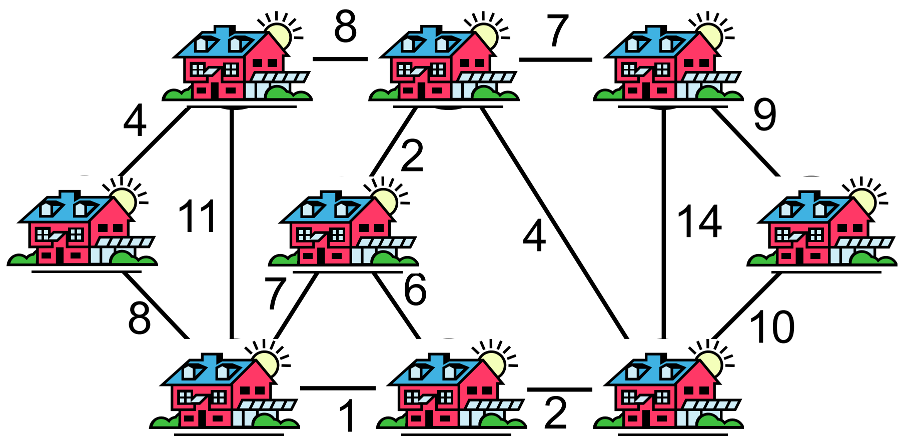
2. Total cost is minimum

Minimum spanning trees (MSTs)

- A connected, undirected graph
  - Vertices = houses
  - Edges = roads
- A weighted w(u, v) on each edge $(u, v) \in E$

Find $T \subseteq E$ such that:

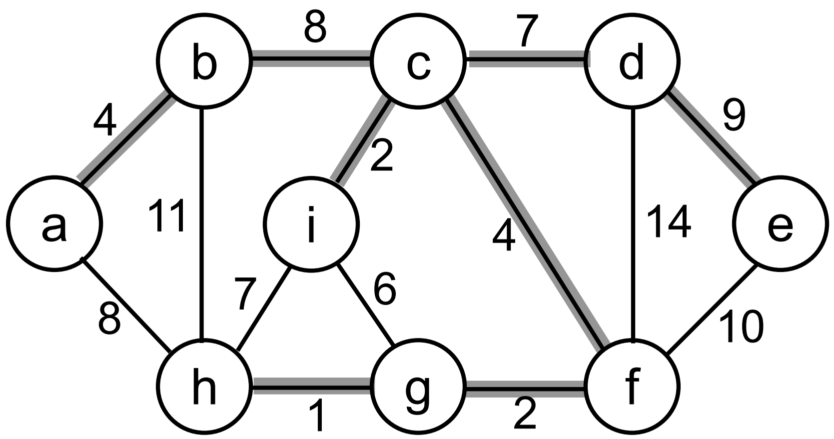
1. T connects all vertices
2. $w(T) = \Sigma_{(u,v)\in T}\, w(u, v)$ is minimized



Properties of MST:
(1) MST is **not** unique; (2) MST has no cycles;

Growing an MST: generic approach

- Grow a set A of edges (initially empty)
- Incrementally add edges to A such that they would belong $\boxed{\text{to an MST}}$



Idea: **add only "safe" edges**

- An edge (u, v) is **safe** for A, if and only if $A \cup \{(u, v)\}$ is also a subset of **some** MST

Generic MST algorithm

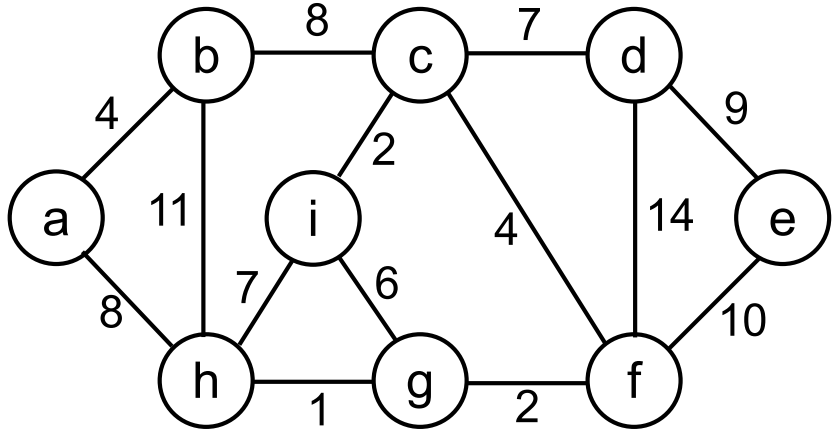
$A \leftarrow \varnothing$

while A is not a spanning tree

  do find an edge (u, v) that is safe for A

    $A \leftarrow A \cup \{(u, v)\}$

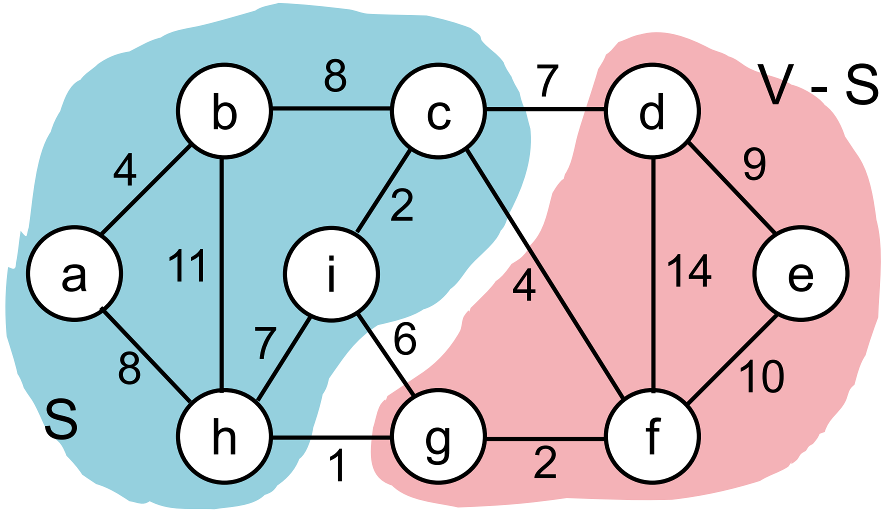
return A



- How do we find safe edges?

Finding safe edges

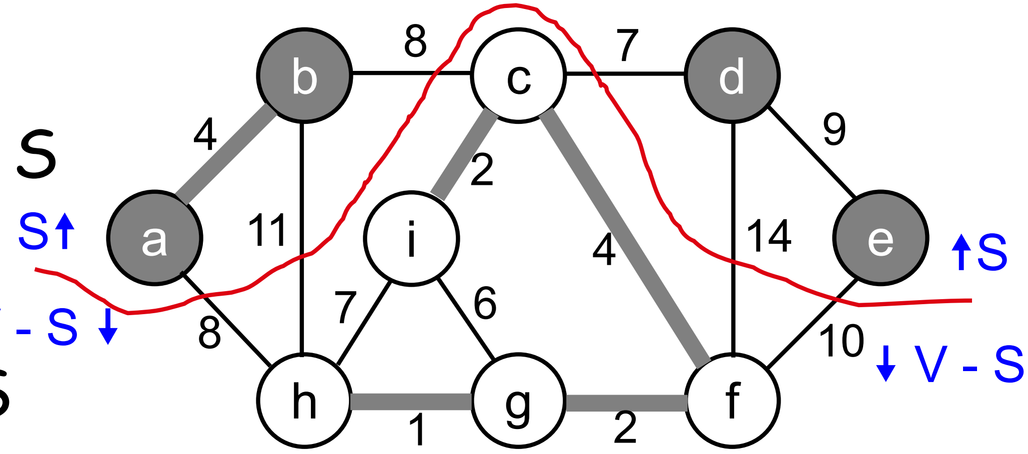
- Let's look at edge (h, g)
  - Is it safe for A initially?



- Yes. Why?
  - Let $S \subset V$ be any set of vertices that includes h but not g (so that g is in V - S)
  - In any MST, there has to be one edge (at least) that connects S with V - S
  - Why not choose the edge with minimum weight (h, g)?

Definitions

- A **cut** (S, V - S) is a partition of vertices into two disjoint sets S and V - S



- An edge **crosses** the cut (S, V - S) if one endpoint is in S and the other in V – S

- A cut **respects** a set A of edges ⇔ no edge in A crosses the cut

- An edge is a **light edge** crossing a cut ⇔ its weight is minimum over all edges crossing the cut
  - Note that for a given cut, there can be > 1 light edges crossing it

Theorem

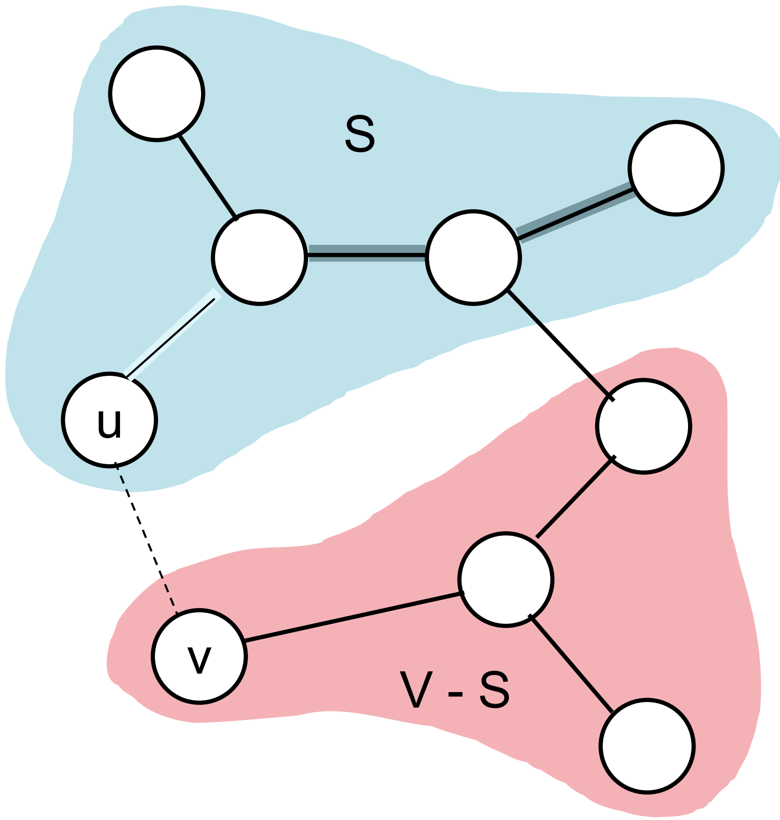
- Let A be a subset of some MST, $(S, V - S)$ be a **cut** that respects A, and (u, v) be a **light edge** crossing $(S, V-S)$. Then (u, v) is safe for A.

Proof:

- Let T be an MST that includes A
  - edges in A are shaded

- Case1: If T includes (u,v), then it would be safe for A

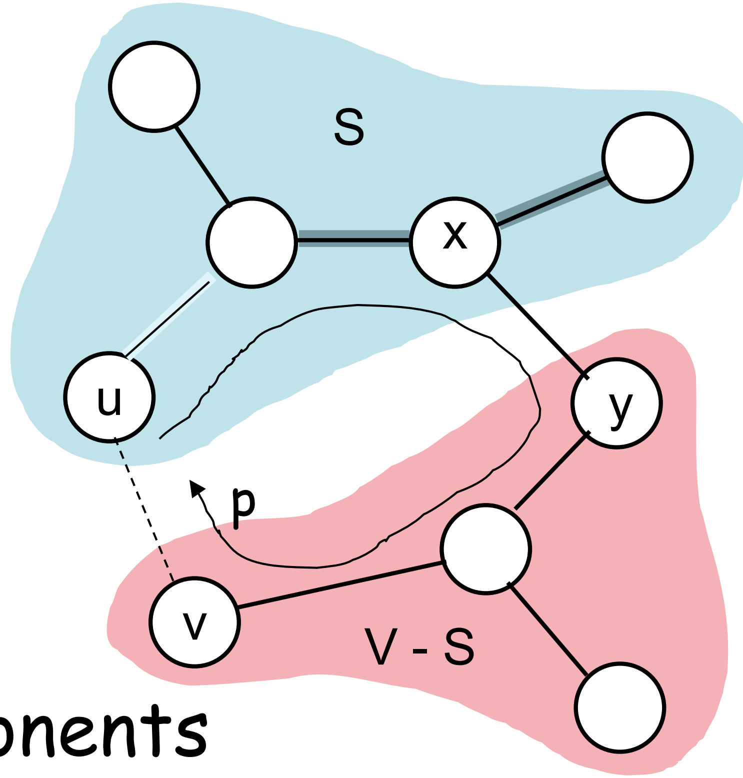
- Case2: Suppose T does not include the edge (u, v)
  - **Idea**: construct another MST T' that includes $A + \{(u, v)\}$



Theorem: proof

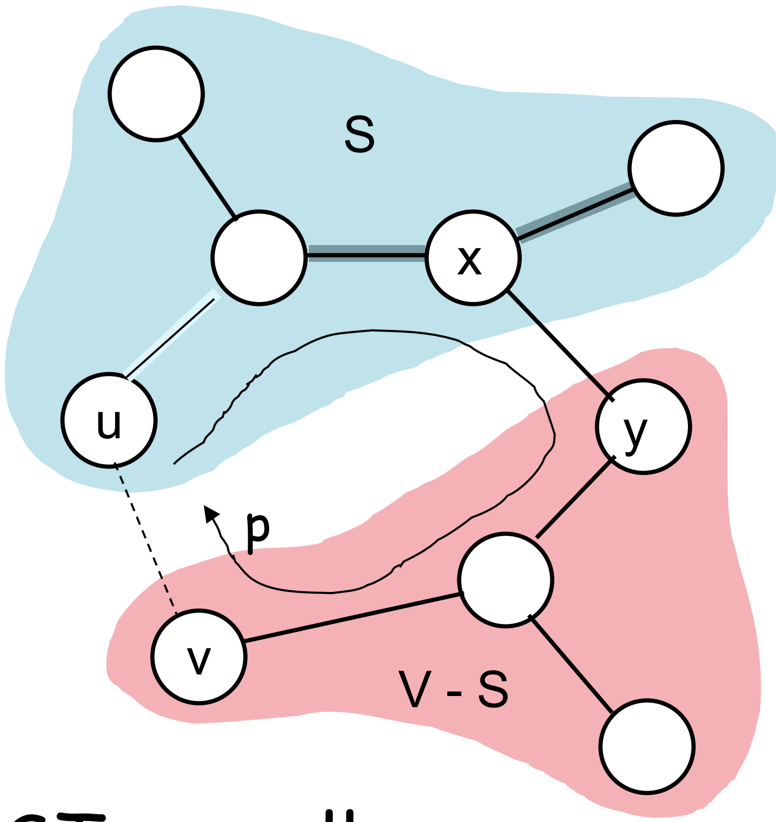
- T contains a unique path p between u and v
- Path p must cross the cut (S, V - S) at least once: let (x, y) be that edge
- Let's remove (x, y) $\Rightarrow$ breaks T into two components
- Adding (u, v) reconnects the components

  $T' = T - \{(x, y)\} + \{(u, v)\}$



Theorem: proof

$T' = T - \{(x, y)\} + \{(u, v)\}$



Have to show that T' is an MST:

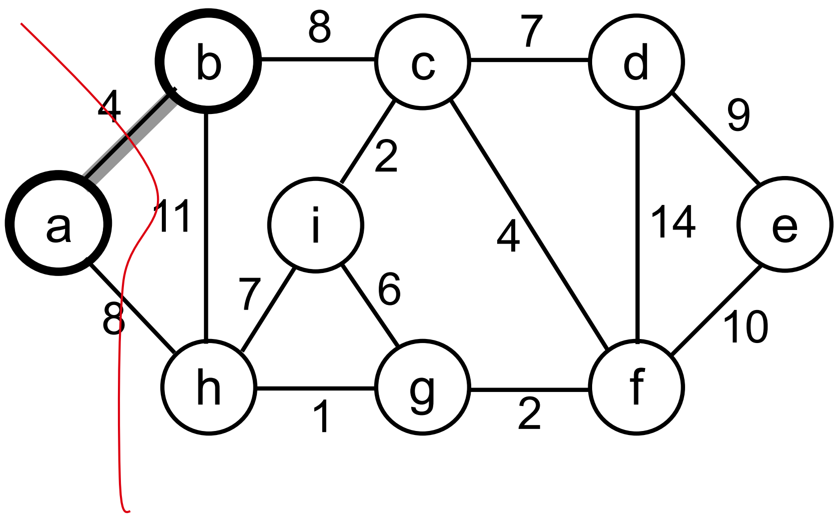
- (u, v) is a light edge
  $\Rightarrow w(u, v) \le w(x, y)$

- $w(T') = w(T) - w(x, y) + w(u, v)$
  $\le w(T)$

- Since T is an MST,
  $w(T) \le w(T') \Rightarrow T'$ must be an MST as well

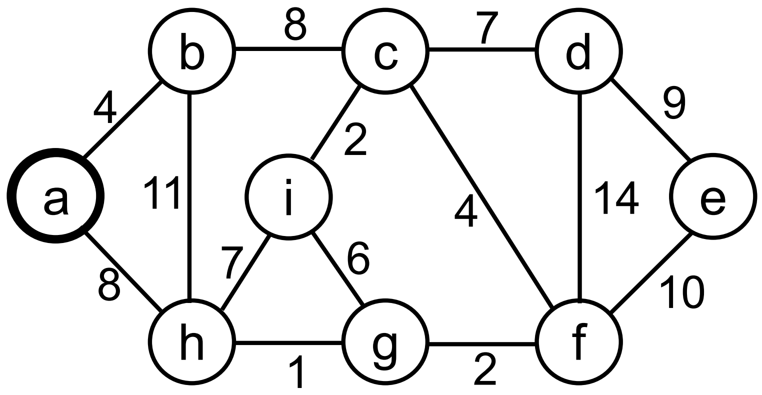
Prim's algorithm

- The edges in set A always <u>form a single tree</u>
- Starts from an arbitrary "root": $V_A = \{a\}$
- At each step:
  - Find a light edge crossing $(V_A, V - V_A)$
  - Add this edge to A
  - Repeat until the tree spans all vertices



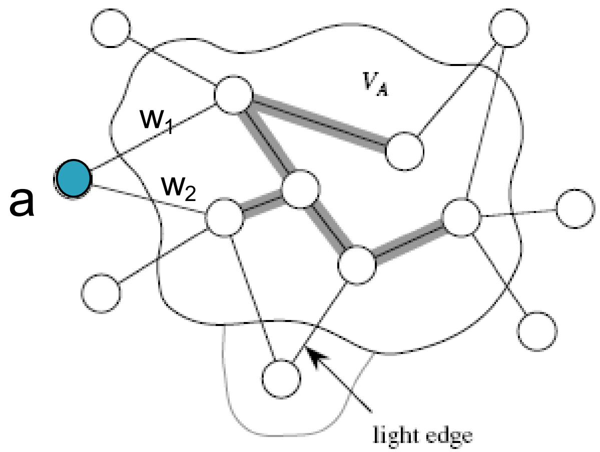
How to find light edges quickly?

- Use a priority queue Q to include vertices not in the tree, i.e., ($V - V_A$)
  - V_A = {a}, Q = {b, c, d, e, f, g, h, i}



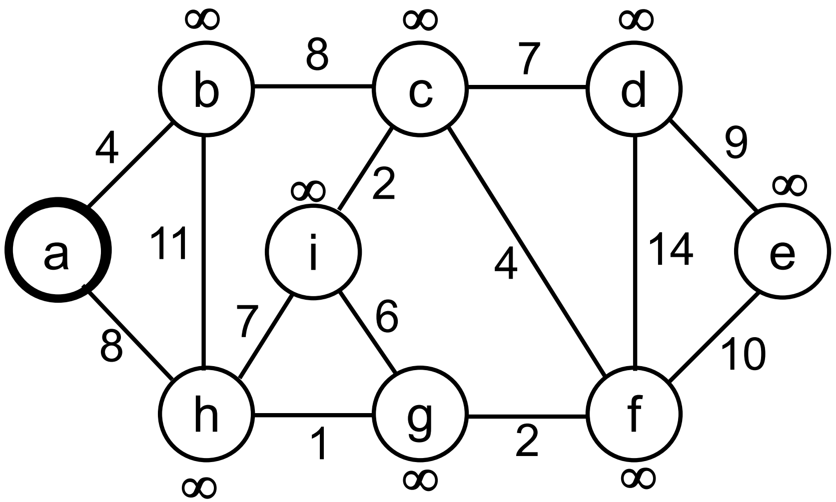
- Associate a key to each vertex v in Q:
  - key[v] = minimum weight of any edge (u, v) connecting v to V_A
  - key[v] = ∞, if v is not adjacent to any vertices in V_A

key[a] = min(w_1, w_2)



- After adding a new vertex to V_A, update the weights of all vertices adjacent to it
  - E.g., after adding a, k[b]=4 and k[h]=8

Example

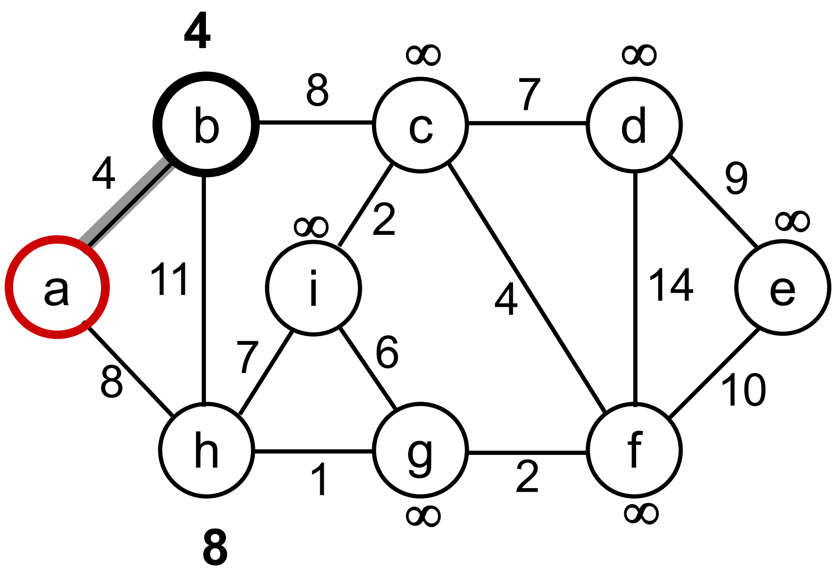


0 ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Q = {a, b, c, d, e, f, g, h, i}

$V_A = \varnothing$

Extract-MIN(Q) ⇒ a



key [b] = 4 π [b] = a

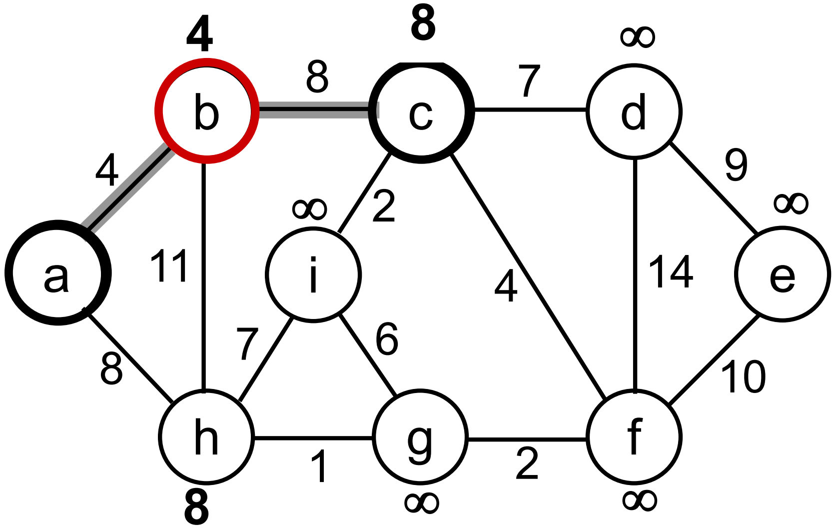
key [h] = 8 π [h] = a

4 ∞ ∞ ∞ ∞ ∞ **8** ∞

Q = {b, c, d, e, f, g, h, i} V_A = {a}

Extract-MIN(Q) ⇒ b

Example



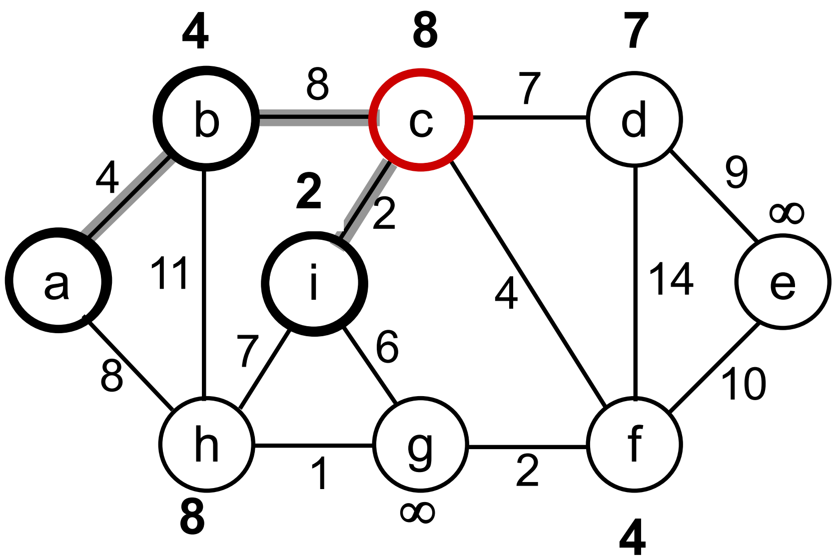
key [c] = 8 π [c] = b

key [h] = 8 π [h] = a unchanged

8 ∞ ∞ ∞ ∞ 8 ∞

Q = {c, d, e, f, g, h, i} V_A = {a, b}

Extract-MIN(Q) ⇒ c



key [d] = 7 π [d] = c

key [f] = 4 π [f] = c

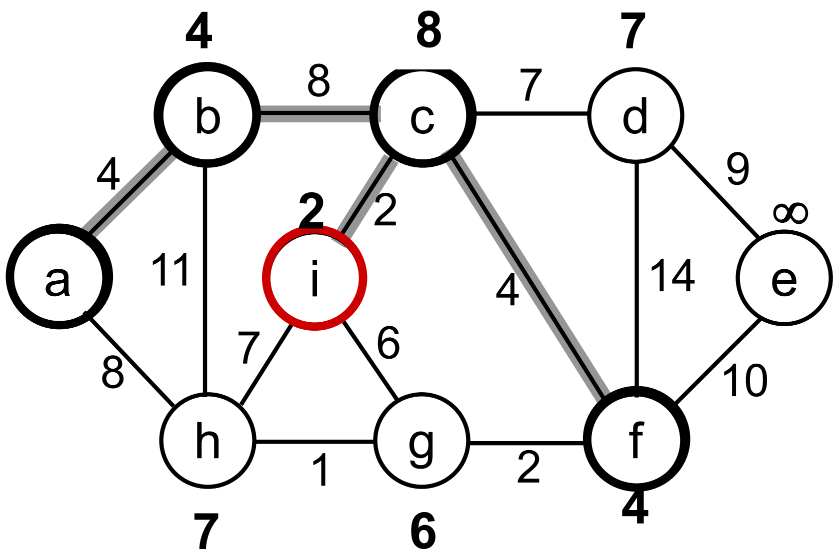
key [i] = 2 π [i] = c

7 ∞ 4 ∞ 8 2

Q = {d, e, f, g, h, i} V_A = {a, b, c}

Extract-MIN(Q) ⇒ i

Example



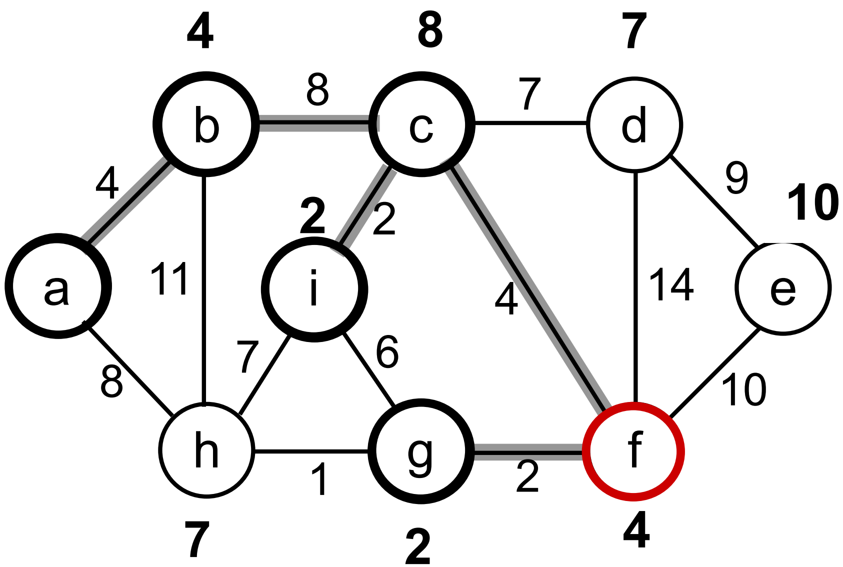
key [h] = 7 π [h] = i

key [g] = 6 π [g] = i

7 ∞ 4 6 7

Q = {d, e, f, g, h} V_A = {a, b, c, i}

Extract-MIN(Q) ⇒ f



key [g] = 2 π [g] = f

key [d] = 7 π [d] = c unchanged

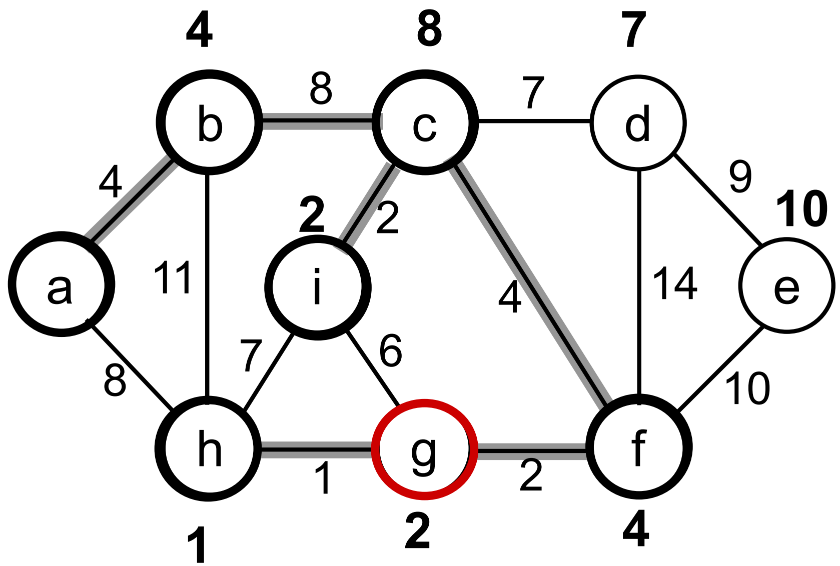
key [e] = 10 π [e] = f

7 10 2 7

Q = {d, e, g, h} V_A = {a, b, c, i, f}

Extract-MIN(Q) ⇒ g

Example

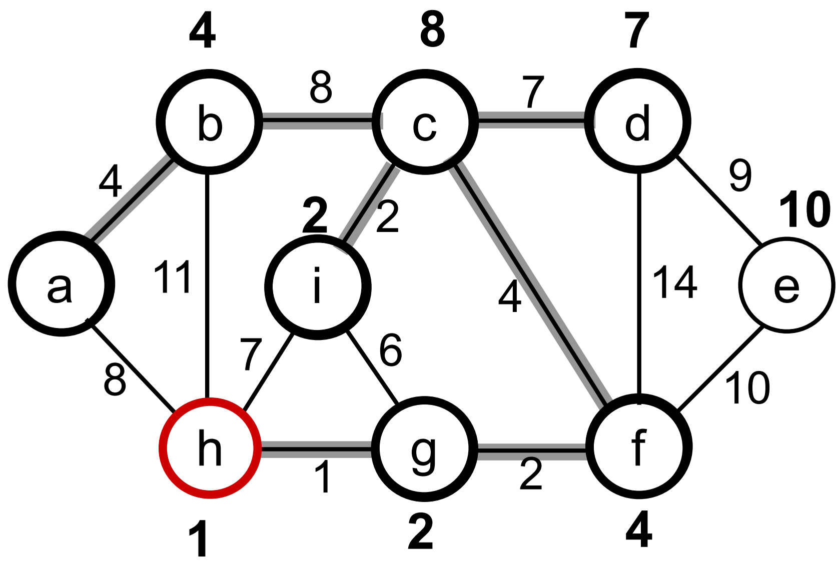


key [h] = 1 π [h] = g

7 10 1

Q = {d, e, h} V_A = {a, b, c, i, f, g}

Extract-MIN(Q) $\Rightarrow$ h

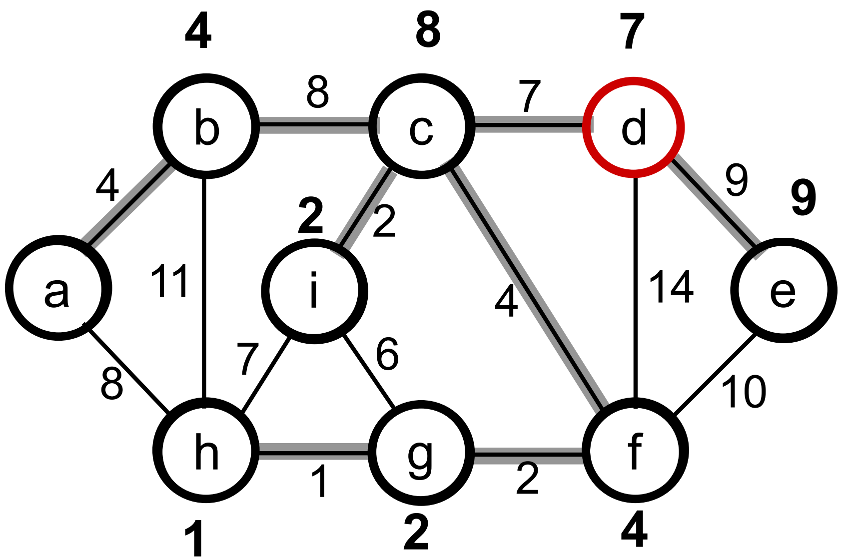


7 10

Q = {d, e} V_A = {a, b, c, i, f, g, h}

Extract-MIN(Q) $\Rightarrow$ d

Example



key [e] = 9 π [e] = d

9

Q = {e} V_A = {a, b, c, i, f, g, h, d}

Extract-MIN(Q) $\Rightarrow$ e

Q = $\varnothing$ V_A = {a, b, c, i, f, g, h, d, e}

Prim(V, E, w, r)

Total time: O(VlgV + ElgV) = O(ElgV)

```
Q ← ∅
for each u ∈ V
    do key[u] ← ∞
        π[u] ← NIL
        INSERT(Q, u)
DECREASE-KEY(Q, r, 0)        ▶ key[r] ← 0
while Q ≠ ∅
        do u ← EXTRACT-MIN(Q)
            for each v ∈ Adj[u]
                do if v ∈ Q and w(u, v) < key[v]
                    then π[v] ← u
                        DECREASE-KEY(Q, v, w(u, v))
```

Lines 1–5: O(V) if Q is implemented as a min-heap

Line 6 (key[r] ← 0): O(lgV)

Line 7: Executed |V| times

Line 8: Takes O(lgV)

Lines 7–8: Min-heap operations: O(VlgV)

Line 9: Executed O(E) times

Line 10: Constant

Line 12: Takes O(lgV)

Lines 9–12: O(ElgV)

Prim(V, E, w, r)

Total time: $O(V\lg V + E\lg V + V^2) = O(E\lg V + V^2)$

```
Q ← ∅
for each u ∈ V
    do key[u] ← ∞
        π[u] ← NIL
        INSERT(Q, u)
DECREASE-KEY(Q, r, 0)        ▶ key[r] ← 0
while Q ≠ ∅
        do u ← EXTRACT-MIN(Q)
            for (j=0; j<|V|; j++)
                if (A[u][j]=1)
                    if v ∈ Q and w(u, v) < key[v]
                        then π[v] ← u
                            DECREASE-KEY(Q, v, w(u, v))
```

Lines 1–5: O(V) if Q is implemented as a min-heap

key[r] ← 0: O(lgV)

while Q ≠ ∅: Executed |V| times

EXTRACT-MIN(Q): Takes O(lgV)

Min-heap operations: O(VlgV)

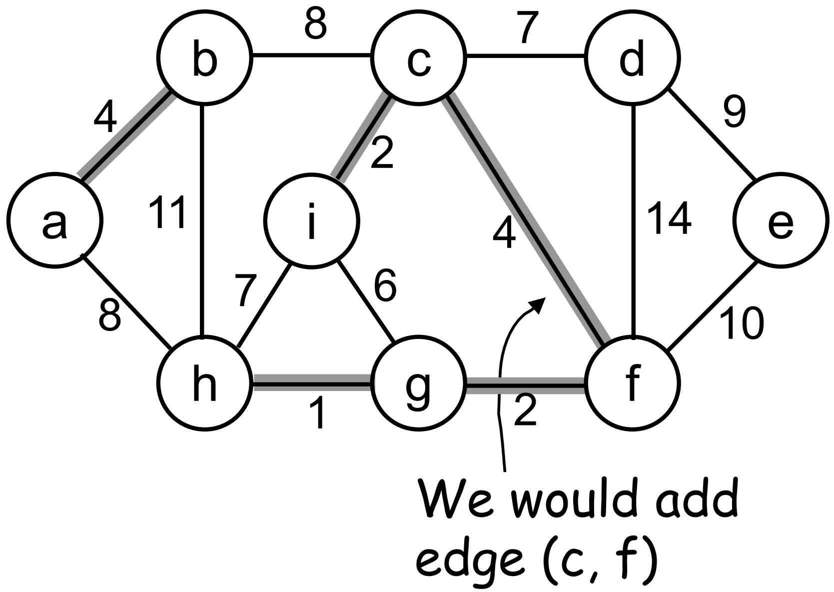
for (j=0; j<|V|; j++): Executed $O(V^2)$ times total

if (A[u][j]=1): Constant

DECREASE-KEY(Q, v, w(u, v)): Takes O(lgV) — O(ElgV)

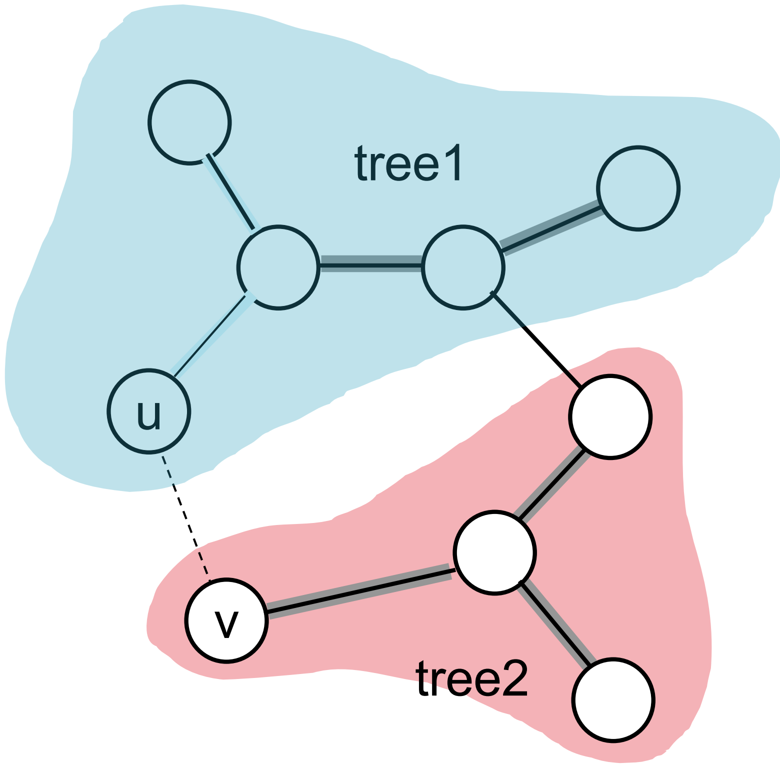
Kruskal's algorithm

- Start with each vertex being its own component
- Repeatedly merge two components into one by choosing the **light** edge that connects them
- Which components to consider at each iteration?
  - Scan the set of edges in monotonically increasing order by weight

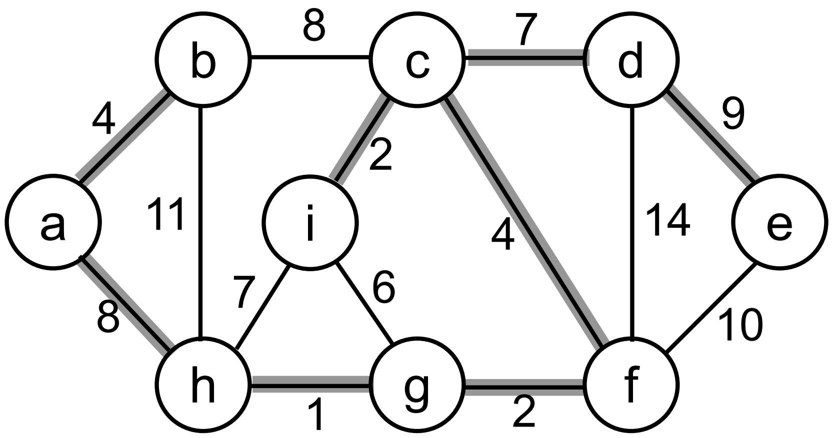


Kruskal's algorithm

- How is it different from Prim's algorithm?
  - Prim's algorithm grows one tree all the time
  - Kruskal's algorithm grows multiple trees (i.e., a forest) at the same time
  - Trees are merged together using **safe** edges



Example



1: (h, g)
2: (c, i), (g, f)
4: (a, b), (c, f)
6: (i, g)
7: (c, d), (i, h)
8: (a, h), (b, c)
9: (d, e)
10: (e, f)
11: (b, h)
14: (d, f)

{a}, {b}, {c}, {d}, {e}, {f}, {g}, {h}, {i}

1. Add (h, g) {g, h}, {a}, {b}, {c}, {d}, {e}, {f}, {i}
2. Add (c, i) {g, h}, {c, i}, {a}, {b}, {d}, {e}, {f}
3. Add (g, f) {g, h, f}, {c, i}, {a}, {b}, {d}, {e}
4. Add (a, b) {g, h, f}, {c, i}, {a, b}, {d}, {e}
5. Add (c, f) {g, h, f, c, i}, {a, b}, {d}, {e}
6. Ignore (i, g) {g, h, f, c, i}, {a, b}, {d}, {e}
7. Add (c, d) {g, h, f, c, i, d}, {a, b}, {e}
8. Ignore (i, h) {g, h, f, c, i, d}, {a, b}, {e}
9. Add (a, h) {g, h, f, c, i, d, a, b}, {e}
10. Ignore (b, c) {g, h, f, c, i, d, a, b}, {e}
11. Add (d, e) {g, h, f, c, i, d, a, b, e}
12. Ignore (e, f) {g, h, f, c, i, d, a, b, e}
13. Ignore (b, h) {g, h, f, c, i, d, a, b, e}
14. Ignore (d, f) {g, h, f, c, i, d, a, b, e}

Algorithm 1: a naive method

Assume vertices are 1, 2, …, n, and E >= V

```
Sort all the edges          ← O(ElogE)
for each v ∈ V
    map.add(v, {v})         } O(V)
for each edge (u, v) ∈ E
    setU = map.get(u), setV = map.get(v)
    isConnected = false                      } O(1)
    for each vertex w ∈ setU
        if w == v
            isConnected = true               } O(setU.size)
            break
    if isConnected == false
        setU = setU ∪ setV
        map.add(u, setU), map.add(v, setU)   } O(setV.size)
        R = R ∪ {(u, v)}
Output R
                                             } O(VE)
```

Can we do better?

Algorithm 2: using labels

- Using labels
  - A label means a connected component
  - Assign a unique label to each vertex initially
  - When merging two connected components, we always change the labels of vertices in the small component to the label of the large component
    - The cost of changing labels is smaller

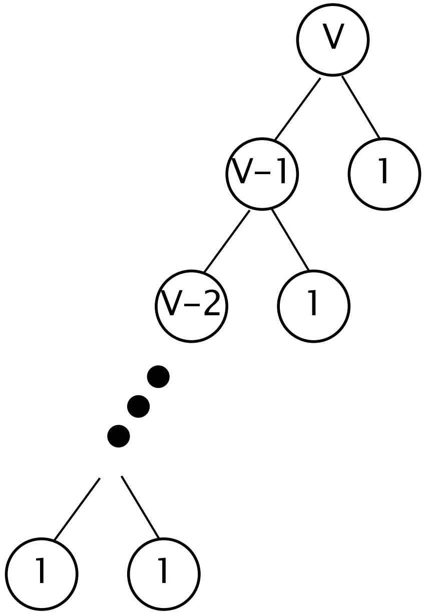
Algorithm 2: using labels

Assume vertices are 1, 2, …, n, and E >= V

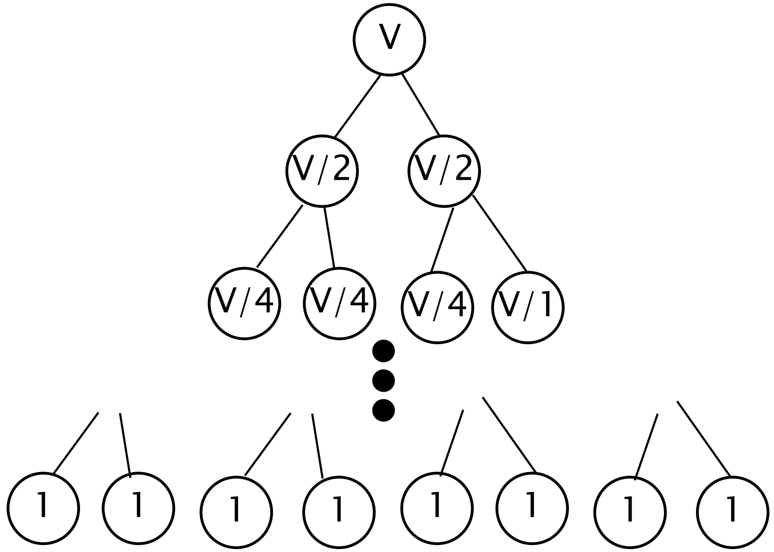
```
Sort all the edges  ←  O(ElogE)
for each v ∈ V
    label[v] = v                                  O(V)
    setArray[v] = {v}
for each edge (u, v) ∈ E
    uL = label[u], vL = label[v]
    if uL == vL  continue                         O(E)
    R.add((u, v))
   if setArray[uL].size >= setArray[vL].size                           ???
      for each vertex w ∈ setArray[vL]
            label[w] = uL                         O(setArray[vL].size)
            setArray[uL].add(w)
    else
        for each vertex w ∈ setArray[uL]
            label[w] = vL                         O(setArray[uL].size)
            setArray[vL].add(w)
Output R
```

What's the total times of changing the labels for all the vertices?

- Extreme case 1
  - Height: V-1
  - Change O(V) times



- Extreme case 2
  - Height: lgV
  - Change O(VlgV) times



Consider a specific vertex v: if v's label is changed, then the updated set setArray[vL] will be at least twice larger than the original set setArray[vL]. Hence, the number of times for changing labels for v is at most O(lgV).

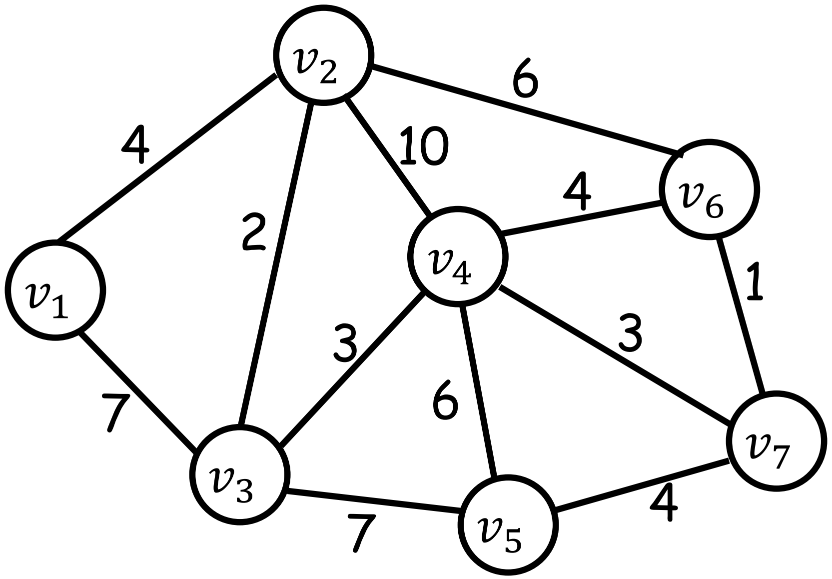
Algorithm 2: using labels

Assume vertices are 1, 2, …, n, and E >= V

```
Sort all the edges   ← O(ElogE)
for each v ∈ V
    label[v] = v                          O(V)
    setArray[v] = {v}
for each edge (u, v) ∈ E
    uL = label[u], vL = label[v]
    if uL == vL  continue                 O(E)
    R.add((u, v))
      if setArray[uL].size >= setArray[vL].size
        for each vertex w ∈ setArray[vL]
            label[w] = uL
            setArray[uL].add(w)
      else                                          O(VlogV)      O(ElgE)
        for each vertex w ∈ setArray[uL]
            label[w] = vL
            setArray[vL].add(w)
Output R
```

Exercises

- Given the following graph, find its MST using Prim's algorithm and Kruskal's algorithm respectively



Recommended reading

- Reading materials
  - Textbook Chapter 23
- Next lecture
  - Shortest paths, Chapters 24&25