CSC3100 Data Structures
Lecture 22: Graph shortest path

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Outline

- We focus on weighted graphs
- Graphs with non-negative weights
  - Single-Source Shortest Path: Dijkstra's algorithm
- All-Pair Shortest Path: Floyd's algorithm
- Graphs with negative weights
  - Bellman-Ford algorithm

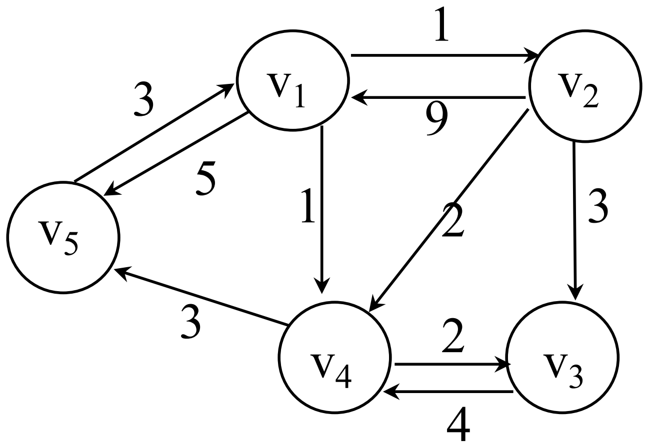
All pairs shortest path

- A representation: a weight matrix where

  W(i, j) = 0 if i = j

  W(i, j) = ∞ if there is no edge between i and j

  W(i, j) = "weight of edge"



	1	2	3	4	5
1	0	1	∞	1	5
2	9	0	3	2	∞
3	∞	∞	0	4	∞
4	∞	∞	2	0	3
5	3	∞	∞	∞	0

- Problem: find the shortest distance/path between every pair of vertices of a graph

A straightforward method

- A naïve method is to run a single-source shortest path algorithm for each vertex
  - Run Dijkstra's algorithm $|V|$ times
  - Dijkstra's algorithm's time complexity: $O(|E| \times \lg|V|)$
  - Total time cost: $O(|V| \times |E| \times \lg|V|)$

- Floyd's algorithm
  - Total time cost: $O(|V|^3)$
  - For dense graphs, Floyd's algorithm is faster
  - It is easier to implement

Floyd's algorithm

- We have shown principle of optimality applies to shortest path problems
- How can we define the shortest distance $d_{i,j}$ in terms of "smaller" problems?
- Main idea of Floyd's algorithm
  - One way is to restrict the paths to only include vertices from a restricted subset
  - Initially, the subset is empty
  - Then, it is incrementally increased until it includes all the vertices

The subproblems

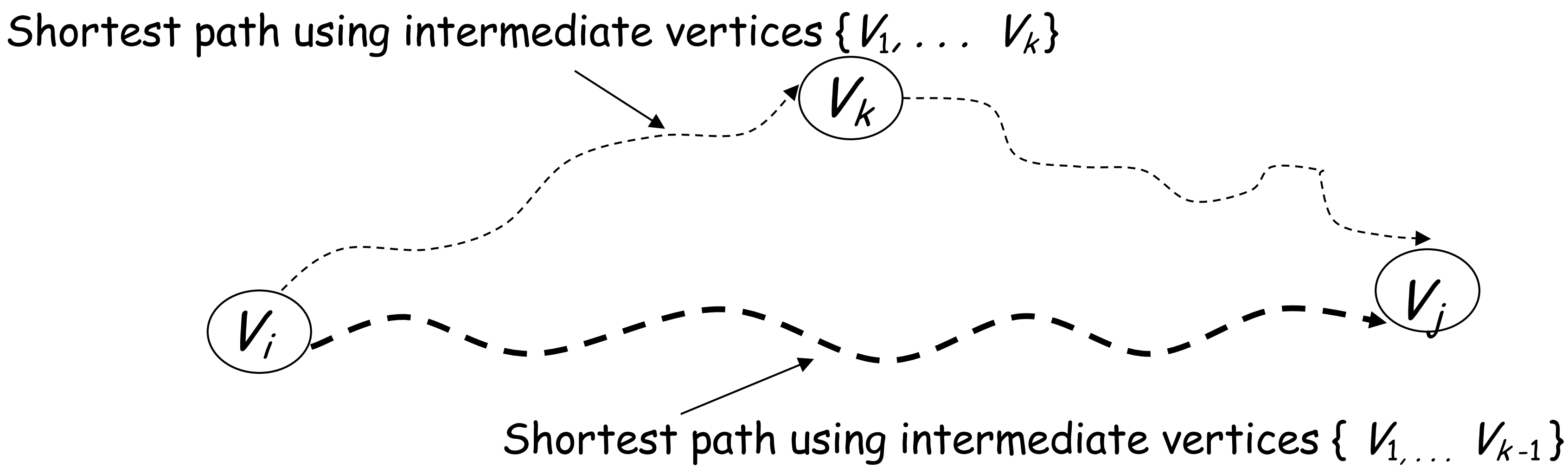
- Let $D^{(k)}[i, j]$ = weight of a shortest path from v_i to v_j using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices in the path
  - $D^{(0)} = W$
  - $D^{(n)} = D$ which is the goal matrix

- How do we compute $D^{(k)}$ from $D^{(k-1)}$?

The recursive definition:

Case 1: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices does not use v_k Then $D^{(k)}[i,j] = D^{(k-1)}[i,j]$

Case 2: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices does use v_k Then $D^{(k)}[i,j] = D^{(k-1)}[i,k] + D^{(k-1)}[k,j]$



The recursive definition

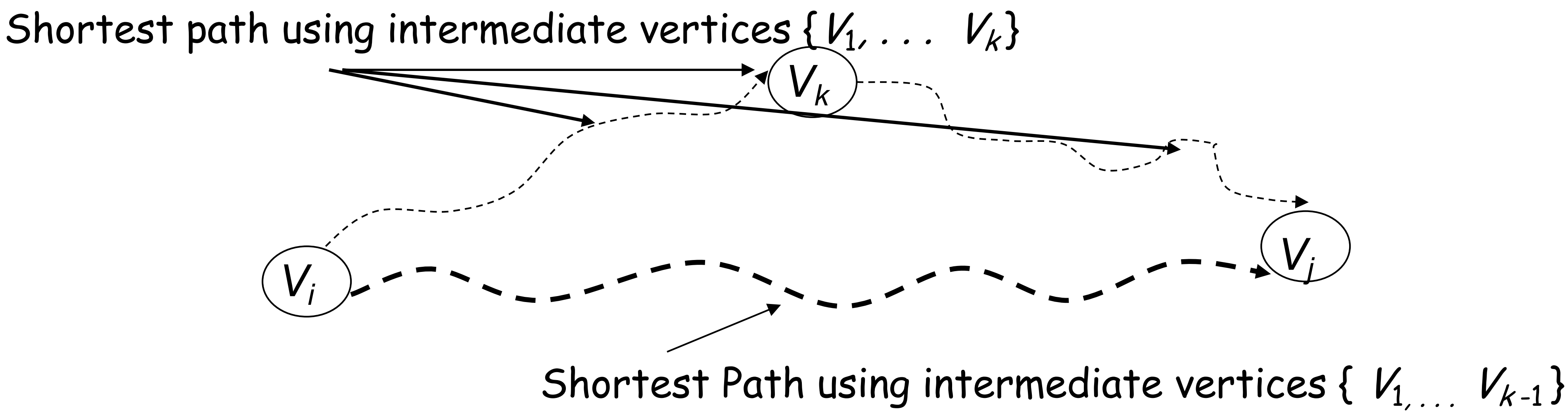
- Since

  $D^{(k)}[i, j] = D^{(k-1)}[i, j]$ or

  $D^{(k)}[i, j] = D^{(k-1)}[i, k] + D^{(k-1)}[k, j]$

  We conclude:

  $D^{(k)}[i, j] = \min\{ D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j] \}$



The pointer array P

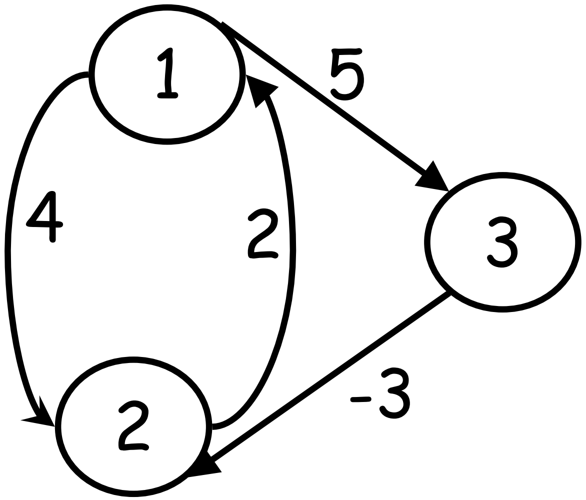
- How to recover the shortest paths?
  - We can use a pointer array P, which initially contains 0
  - Each time that a shorter path from *i* to *j* is found, the *k* that provided the minimum distance is saved
  - To print the intermediate nodes on the shortest path by a recursive procedure, which print the shortest paths from *i* and *k*, and from *k* to *j*

Floyd's algorithm using n+1 D matrices

Floyd//Computes shortest distance between all pairs of
//nodes, and saves P to enable finding shortest paths

1. $D^0 \leftarrow W$ // initialize D array to W[ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. for $k \leftarrow 1$ to n
4. do for $i \leftarrow 1$ to n
5. do for $j \leftarrow 1$ to n
6. if $(D^{k-1}[i, j] > D^{k-1}[i, k] + D^{k-1}[k, j])$
7. then $D^k[i, j] \leftarrow D^{k-1}[i, k] + D^{k-1}[k, j]$
8. $P[i, j] \leftarrow k$;
9. else $D^k[i, j] \leftarrow D^{k-1}[i, j]$

Example



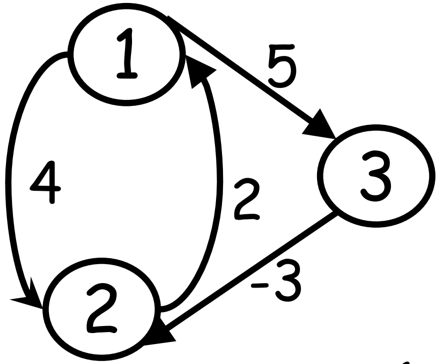
$W = D^0 =$

	1	2	3
1	0	4	5
2	2	0	∞
3	∞	-3	0

$P =$

	1	2	3
1	0	0	0
2	0	0	0
3	0	0	0

Example



D^0 =

	1	2	3
1	0	4	5
2	2	0	∞
3	∞	-3	0

k = 1
Vertex 1 can be intermediate node

D^1 =

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

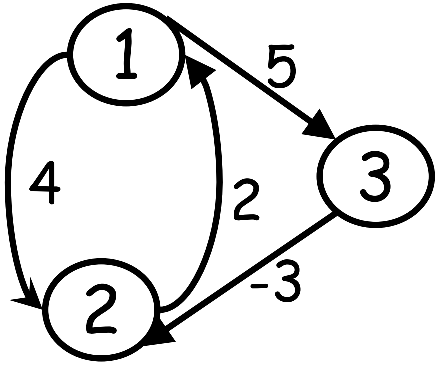
$D^1[2,3] = \min(D^0[2,3], D^0[2,1]+D^0[1,3])$
$= \min (\propto, 7)$
$= 7$

P =

	1	2	3
1	0	0	0
2	0	0	1
3	0	0	0

$D^1[3,2] = \min(D^0[3,2], D^0[3,1]+D^0[1,2])$
$= \min (-3,\propto)$
$= -3$

Example



$D^1 =$

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

k = 2
Vertices 1, 2 can be intermediate

$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

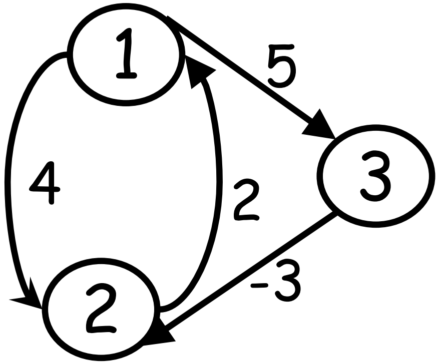
$D^2[1,3] = \min(D^1[1,3], D^1[1,2]+D^1[2,3])$
$= \min (5, 4+7)$
$= 5$

P =

	1	2	3
1	0	0	0
2	0	0	1
3	2	0	0

$D^2[3,1] = \min(D^1[3,1], D^1[3,2]+D^1[2,1])$
$= \min (\infty, -3+2)$
$= -1$

Example



$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

k = 3
Vertices 1, 2, 3 can be intermediate

$D^3 =$

	1	2	3
1	0	2	5
2	2	0	7
3	-1	-3	0

$D^3[1,2] = \min(D^2[1,2], D^2[1,3]+D^2[3,2])$
$= \min(4, 5+(-3))$
$= 2$

$P =$

	1	2	3
1	0	3	0
2	0	0	1
3	2	0	0

$D^3[2,1] = \min(D^2[2,1], D^2[2,3]+D^2[3,1])$
$= \min(2, 7+(-1))$
$= 2$

Floyd's algorithm using 2 *D* matrices

Floyd's algorithm

```
1. D ← W    // initialize D array to W[ ]
2. P ← 0    // initialize P array to [0]
3. for k ← 1 to n
       // Computing D′ from D
4.       do for i ← 1 to n
5.           do for j ← 1 to n
6.               if (D[ i, j ] > D[ i, k ] + D[ k, j ] )
7.                   then  D′[ i, j ] ← D[ i, k ] + D[ k, j ]
8.                         P[ i, j ] ← k;
9.                   else  D′[ i, j ] ← D[ i, j ]
10. Move D′ to D
```

Can we use only one D matrix?

- $D[i, j]$ depends only on elements in the k-th column and row of the distance matrix

- We will show that the k-th row and the k-th column of the distance matrix are unchanged when D^k is computed

- This means D can be calculated *in-place*

The main diagonal values

- Before we show that k-th row and column of D remain unchanged we show that the main diagonal remains 0

- $$\begin{aligned} D^{(k)}[j, j] &= \min\{ D^{(k-1)}[j, j], D^{(k-1)}[j, k] + D^{(k-1)}[k, j] \} \\ &= \min\{ 0, \ D^{(k-1)}[j, k] + D^{(k-1)}[k, j] \} \\ &= 0 \end{aligned}$$

The *k*th column

- k-th column of D^k is equal to the k-th column of D^{k-1}
- Intuitively true - a path from i to k will not become shorter by adding k to the allowed subset of intermediate vertices
- For all i, $D^{(k)}[i,k] =$

$$= \min\{ D^{(k-1)}[i,k],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,k] \}$$
$$= \min\{ D^{(k-1)}[i,k], D^{(k-1)}[i,k] + 0 \}$$
$$= D^{(k-1)}[i,k]$$

The *k*th row

- k-th row of D^k is equal to the kth row of D^{k-1}

For all j, $D^{(k)}[k,j] =$

$$= \min\{ D^{(k-1)}[k, j], D^{(k-1)}[k, k] + D^{(k-1)}[k, j] \}$$
$$= \min\{ D^{(k-1)}[k, j], 0 + D^{(k-1)}[k, j] \}$$
$$= D^{(k-1)}[k, j]$$

Question

- Can we claim that D^k equals to D^{k-1}, D^{k-2} ?
  - No, we can only claim
    - The 1-st row and 1-st column of D^1 equal to the 1-st row and 1-st column of D^0, respectively
    - The 2-nd row and 2-nd column of D^2 equal to the 2-nd row and 2-nd column of D^1, respectively
    - ……

Floyd's algorithm using a single D

Floyd

1. $D \leftarrow W$ // initialize D array to W[ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. for $k \leftarrow 1$ to n
4. do for $i \leftarrow 1$ to n
5. do for $j \leftarrow 1$ to n
6. if ($D[i, j] > D[i, k] + D[k, j]$)
7. then $D[i, j] \leftarrow D[i, k] + D[k, j]$
8. $P[i, j] \leftarrow k$;

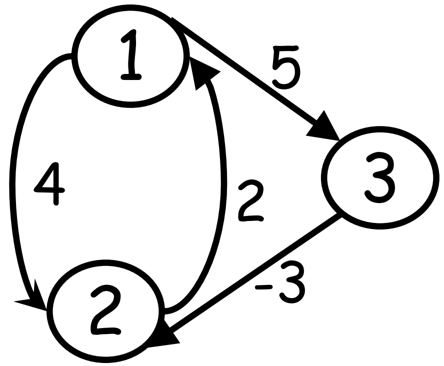
$O(|V|^3)$

Total time cost: $O(|V|^3)$

Printing intermediate nodes on shortest path from q to r

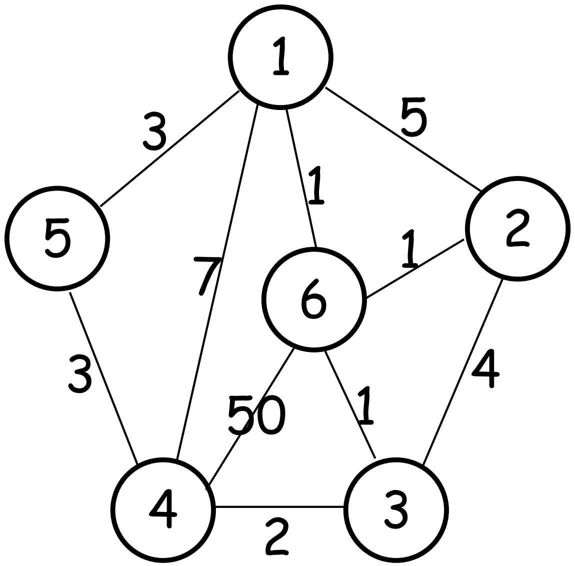
```
path(index q, r)
  if (P[ q, r ] != 0)
        path(q, P[q, r])
        println( "v"+ P[q, r])
        path(P[q, r], r)
         return;
  //no intermediate nodes
  else return
```

$$P = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 0 \\ \hline 2 & 0 & 0 & 1 \\ \hline 3 & 2 & 0 & 0 \\ \hline \end{array}$$



Before calling path check $D[q, r] < \infty$, and print node q, after the call to path print node r

Example

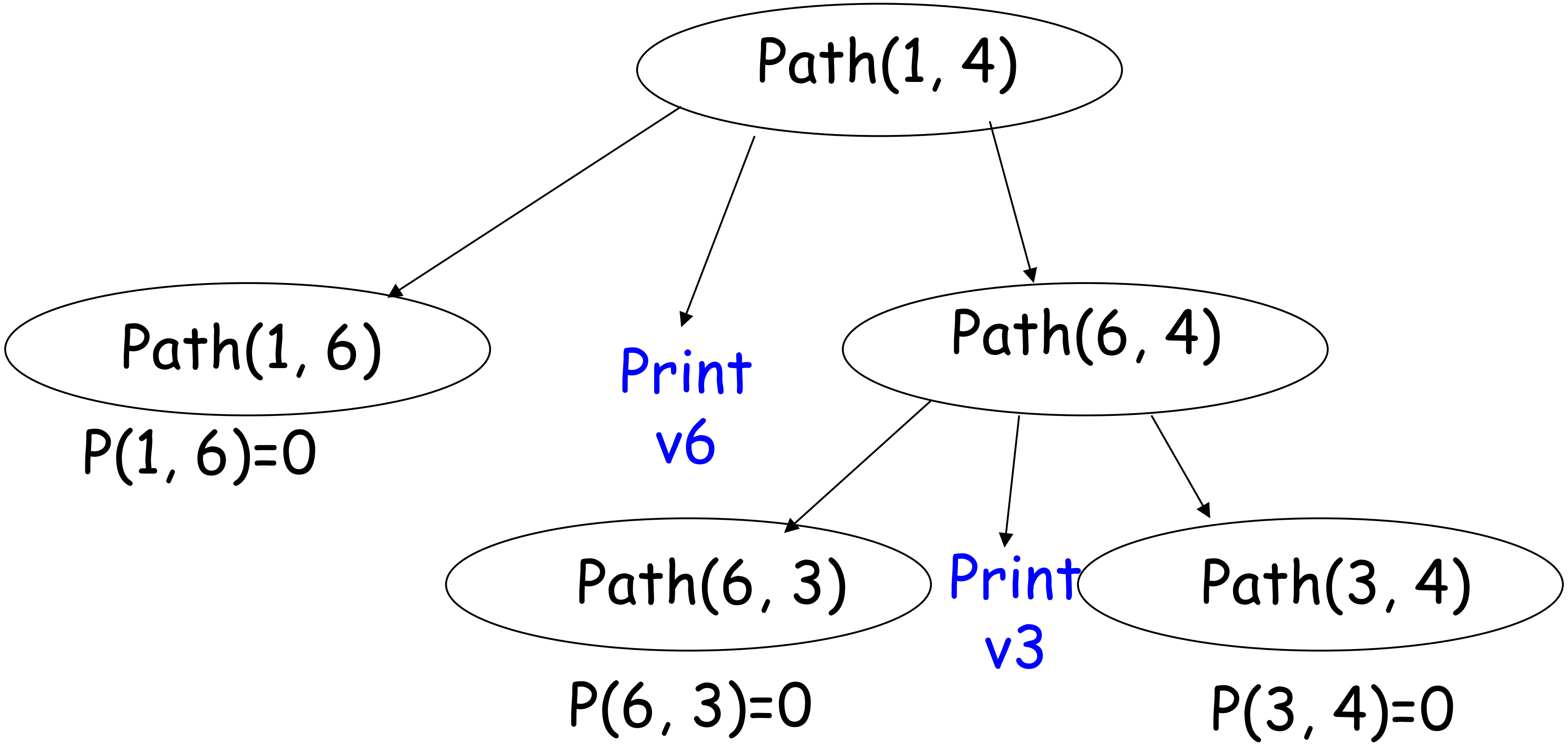


$D^6 =$

	1	2	3	4	5	6
1	0	2(6)	2(6)	4(6)	3	1
2	2(6)	0	2(6)	4(6)	5(6)	1
3	2(6)	2(6)	0	2	5(4)	1
4	4(6)	4(6)	2	0	3	3(3)
5	3	5(6)	5(4)	3	0	4(1)
6	1	1	1	3(3)	4(1)	0

The values in parenthesis are the non-zero P values

The call tree for Path(1, 4)



The intermediate nodes on the shortest path from v1 to v4 are v6, v3, so the shortest path is v1, v6, v3, v4.

Recommended reading

- Reading this week
  - Textbook Chapters 24-25
- Next lecture
  - Some java data structures