MAT3007
Assignment 9 Solution

Problem 1

1. The formulation is presented as follows:

<u>Indices & Sets</u>
$i \in I$ types of gasoline
$j \in J$ compartment in the truck

<u>Data</u>
K_j capacity of compartment j [gallons]
D_i demand for gasoline of type i [gallons]
u_i upper bound on shortage of gasoline of type i [gallons]
c_i shortage cost for gasoline of type i [\$/gallon]

<u>Decision Variables</u>
x_{ij} amount of gasoline of type i put in compartment j [gallons]
y_{ij} takes value 1 if gasoline of type i is put in compartment j and takes value 0 otherwise
z_i shortage of gasoline of type i [gallons]

<u>Formulation</u>

$$\begin{aligned}
\min_{x,y,z} \quad & \sum_{i \in I} c_i z_i \\
\text{s.t.} \quad & \sum_{i \in I} y_{ij} = 1 \qquad j \in J & (1)\\
& x_{ij} \le K_j y_{ij} \qquad i \in I, j \in J & (2)\\
& \sum_{j \in J} x_{ij} + z_i \ge D_i \qquad i \in I & (3)\\
& 0 \le z_i \le u_i \qquad i \in I & (4)\\
& y_{ij} \in \{0,1\} \qquad i \in I, j \in J & (5)\\
& x_{ij} \ge 0 \qquad i \in I, j \in J. & (6)
\end{aligned}$$

The MATLAB codes are attached as follows:

```
I = 3
J = 5
K = [2500,3000,1400,1600,3200]
D = [2800,4200,5000]
U = [500,400,300]
```

```
C = [10,6,8]
K_mat = diag(repmat(K,1,3))
A1_mat = [1,1,1]
A2_mat = [1;1;1;1;1]

cvx_solver gurobi
cvx_begin quiet
variable x(I,J)
variable y(I,J) binary
variable z(I)

minimize C*z
subject to
x >= 0;
z <= U';
z >= 0;
A1_mat*y == ones(1,5);
vec(x') <= K_mat * vec(y');
x*A2_mat + z >= D';
cvx_end
cvx_optval
```

The Python codes are attached as follows:

```
import cvxpy as cp
from cvxopt import *

I = 3
J = 5
K = [2500,3000,1400,1600,3200]
D = [2800,4200,5000]
U = [500,400,300]
C = [10,6,8]

x = cp.Variable((I,J))
y = cp.Variable((I,J),boolean=True)
z = cp.Variable(I)

constraints = [sum(y[i,j] for i in range(I)) == 1 for j in range(J)]
constraints += [x[i,j] <= K[j]*y[i,j] for j in range(J) for i in range(I)]
constraints += [sum(x[i,j] for j in range(J)) + z[i] == D[i] for i in range(I)]
constraints += [x >= 0]
constraints += [z >= 0]
```

```
constraints += [z[i] <= U[i] for i in range(I)]

objective = cp.Minimize(sum(C[i]*z[i] for i in range(I)))

prob = cp.Problem(objective, constraints)
result = prob.solve()
```

2. The solutions are:

Compartment	Gasoline	Capacity
1	2500 gallons of premium	2500
2	1800 gallons of super	3000
3	1400 gallons of premium	1400
4	1600 gallons of regular	1600
5	3200 gallons of regular	3200

Gasoline fulfillment:

	Demand	Delivered	Shortage
regular	5000	4800	200
premium	4200	3900	300
super	2800	2800	0

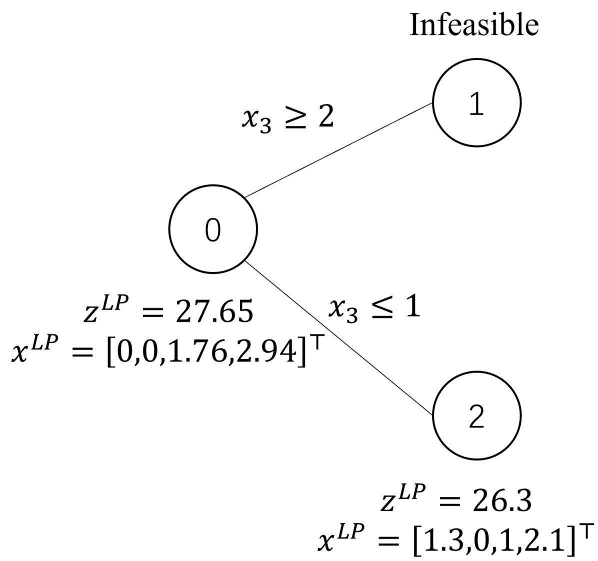
The shortage cost is \$3400.

Problem 2
First we solve

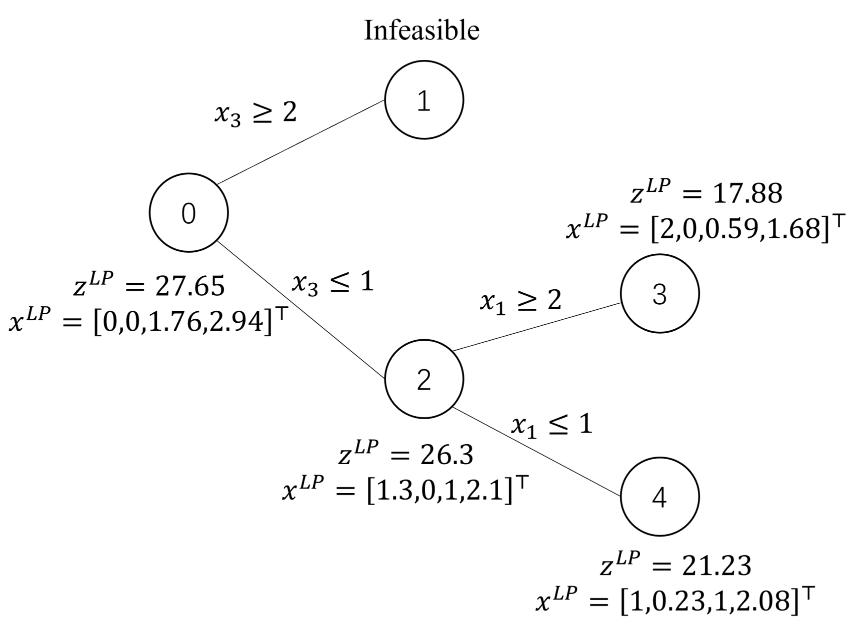
$$\begin{aligned}\max_{x} \quad & 2x_1 + 3x_2 + 4x_3 + 7x_4 \\ \text{s.t.} \quad & 4x_1 + 6x_2 - 2x_3 + 8x_4 = 20 \\ & x_1 + 2x_2 - 6x_3 + 7x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geq 0.\end{aligned}$$

We obtain, when solving this LP relaxation, $z^{LP} = 27.65$ and $x_0^{LP} = [0, 0, 1.76, 2.94]^\top$.

We form two subproblems via $x_3 \leq 1$ and $x_3 \geq 2$. Solving them yields the following B&B tree:

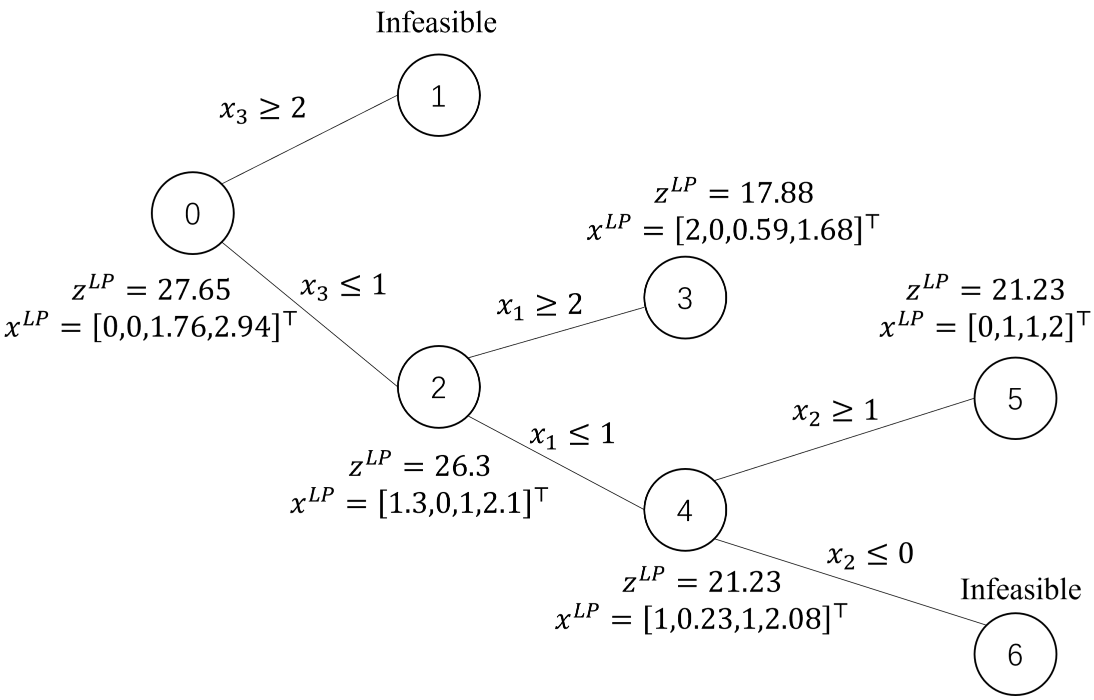


From node #2 we can form subproblems via $x_1 \leq 1$ and $x_1 \geq 2$ (we could have also branched on x_4: $x_4 \leq 2$ and $x_4 \geq 3$.) Solving these two subproblems yields the updated tree displayed below:



Using a best-first search rule, we select node #4 because $z_4^{LP} = 21.23 > 17.88 = z_3^{LP}$ and we are maximizing.

We form two new subproblems via $x_2 \leq 0$ ($x_2 = 0$) and $x_2 \geq 1$. Solving them yields the following continuation of the tree.



We finally have an incumbent solution in node #5; i.e., $z^* = 21$ and $x^* = [0, 1, 1, 2]^\top$.

We now examine unfathomed nodes. There is only one such node: #3. Its $z_3^{LP} = 17.88 < z^* = 21$. Thus, we can fathom node #3, and the algorithm is complete. $x^* = [0, 1, 1, 2]^\top$ and $z^* = 21$ is an optimal solution.

Problem 3

1. The linear programming formulation is shown as follows:

Indices & Sets

$t \in \mathcal{T}$	planning horizon, $\mathcal{T} = \{1, 2, \ldots, T\}$

Data

e_t	existing capacity in year t [MW]
d_t	demand for the required capacity in year t [MW]
c_t^g	natural gas power plants unit capacity cost in year t [1/MW]
c_t^n	nuclear power plants unit capacity cost in year t [1/MW]
Δg_t	upper bound for natural gas power plants capacity change in year t [1/MW]
Δn_t	upper bound for nuclear power plants capacity change in year t [1/MW]

Decision Variables

x_t	natural gas power plants capacity in year t [MW]
y_t	nuclear power plants capacity in year t [MW]

z takes value 1 if nuclear power plant capacity is incorporated and takes value 0 otherwise

<u>Formulation</u>

$$
\begin{aligned}
\min_{x,y} \quad & \sum_{t\in\mathcal{T}} (c_t^g x_t + c_t^n y_t) \\
\text{s.t.} \quad & e_t + x_t + y_t \geq d_t \qquad t \in \mathcal{T} && (7)\\
& y_t \geq 0.2(e_t + x_t + y_t) \qquad t \in \mathcal{T} && (8)\\
& x_{t+1} - x_t \leq \Delta g_t \qquad t = 1, \ldots, T-1 && (9)\\
& x_{t+1} - x_t \geq -\Delta g_t \qquad t = 1, \ldots, T-1 && (10)\\
& y_{t+1} - y_t \leq \Delta n_t \qquad t = 1, \ldots, T-1 && (11)\\
& y_{t+1} - y_t \geq -\Delta n_t \qquad t = 1, \ldots, T-1 && (12)\\
& x, y \geq 0. && (13)
\end{aligned}
$$

2. We need to replace constraint (8) by the following constraints involving indicator z:

$$
\begin{aligned}
& y_t \leq M_1 z \qquad t \in \mathcal{T} && (14)\\
& y_t \geq 0.2(e_t + x_t + y_t) - M_2(1-z) \qquad t \in \mathcal{T}. && (15)
\end{aligned}
$$

We set up two big M parameters, M_1 and M_2. Reasonably small values for those big M parameters are:

$$M_1 = \max_{t\in\mathcal{T}}\{d_t - e_t\} \qquad\qquad M_2 = 0.2\max_{t\in\mathcal{T}}\{d_t\}. \tag{16}$$

3. $z_a^* \geq z_b^*$, as the feasible solutions for the formulation in part 1 are all feasible for the formulation in part 2. However, the reverse is not true. Therefore, we can consider the formulation in part 2 a relaxation of the formulation in part 1.

Problem 4

1. From branching we know that every child node is going to yield a smaller LP relaxation objective value. Therefore, we have

$$
\begin{aligned}
& z_0^{LP} \geq z_1^{LP} && z_0^{LP} \geq z_2^{LP}\\
& z_1^{LP} \geq z_5^{LP} && z_1^{LP} \geq z_6^{LP}\\
& z_2^{LP} \geq z_3^{LP} && z_2^{LP} \geq z_4^{LP}\\
& z_3^{LP} \geq z_7^{LP} && z_3^{LP} \geq z_8^{LP}
\end{aligned}
$$

From the best-search rule, the earlier explored node should have a larger z^{LP}; therefore, we can obtain the following inequalities:

$$
\begin{aligned}
& z_2^{LP} \geq z_1^{LP} && z_1^{LP} \geq z_3^{LP}\\
& z_1^{LP} \geq z_4^{LP} && z_3^{LP} \geq z_4^{LP}\\
& z_3^{LP} \geq z_5^{LP} && z_3^{LP} \geq z_6^{LP}.
\end{aligned}
$$

If we summarize the inequalities above, we can obtain:

$$z_0^{LP} \geq z_2^{LP} \geq z_1^{LP} \geq z_3^{LP} \geq z_4^{LP}$$

$$z_3^{LP} \geq z_5^{LP} \qquad z_3^{LP} \geq z_6^{LP} \qquad z_3^{LP} \geq z_7^{LP} \qquad z_3^{LP} \geq z_8^{LP}$$

2. The numerical value of the x_2^{LP} in the LP relaxation solution should be within the range of $5 < x_2^{LP} < 6$