MAT3007
Assignment 5 Solution

Problem 1
1. The primal linear program formulation:

$$\begin{aligned} \max \quad & 200x_1 + 300x_2 + 100x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + 6x_3 \leq 100 \\ & 2x_1 + 4x_2 + 8x_3 \leq 120 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The decision variables x_1, x_2, x_3 represent the acres of land planting wheat, corn, and alfalfa, respectively.

The MATLAB code is attached as follows:

```
cvx_begin quiet
variables x1 x2 x3
maximize 200*x1+300*x2+100*x3
subject to
3*x1 + 2*x2 + 6*x3 <= 100;
2*x1 + 4*x2 + 8*x3 <= 120;
x1 >= 0;
x2 >= 0;
x3 >= 0;
cvx_end
cvx_optval
```

The Python code is attached as follows:

```
x = cp.Variable(3)
objective = cp.Maximize(200*x[0] + 300*x[1] + 100*x[2])
constraints = [3*x[0] + 2*x[1] + 6*x[2] <= 100]
constraints += [2*x[0] + 4*x[1]+ 8*x[2] <= 120]
constraints += [x >= 0]

prob = cp.Problem(objective, constraints)
result = prob.solve()
```

Solving the primal linear program, we obtain an optimal solution $x_1^* = 20, x_2^* = 20, x_3^* = 0$, which means to maximize the profit, Farmer Clink should plant 20 acres of wheat and 20 acres of corn. The optimal profit Farmer Clink can obtain is \$10,000.

2. The dual linear program formulation:

$$\begin{array}{rl} \min & 100\pi_1 + 120\pi_2 \\ \text{s.t.} & 3\pi_1 + 2\pi_2 \geq 200 \\ & 2\pi_1 + 4\pi_2 \geq 300 \\ & 6\pi_1 + 8\pi_2 \geq 100 \\ & \pi_1, \pi_2 \geq 0. \end{array}$$

The decision variables π_1, π_2 represent the shadow price of the worker and fertilizer constraints, respectively.

The MATLAB code is attached as follows:

```
cvx_begin quiet
variables p1 p2
maximize 100*p1+120*p2
subject to
3*p1 + 2*p2 >= 200;
2*p1 + 4*p2 >= 300;
6*p1 + 8*p2 >= 100;
p1 >= 0;
p2 >= 0;
cvx_end
cvx_optval
```

The Python code is attached as follows:

```
p = cp.Variable(2)

objective_d = cp.Minimize(100*p[0] + 120*p[1])
constraints_d = [3*p[0] + 2*p[1] >= 200]
constraints_d += [2*p[0] + 4*p[1] >= 300]
constraints_d += [6*p[0] + 8*p[1] >= 100]
constraints += [p >= 0]

prob_d = cp.Problem(objective_d, constraints_d)
result_d = prob_d.solve()
```

Solving the dual linear program, we obtain an optimal solution $\pi_1^* = 25, \pi_2^* = 62.5$. The optimal value matches that of the primal problem, $10,000.

3. If one more worker is added, the optimal solution is still feasible and the optimal basis remains the same. Since $\pi_1^* = 25$, it means that if Farmer Clink get 1 more worker, he

can get \$25 more profit.

4. If one more ton of fertilizer is added, the optimal solution is still feasible and the optimal basis remains the same. Since $\pi_2^* = 62.5$, it means that if Farmer Clink get 1 more ton of fertilizer, he can get \$62.5 more profit.

5. If c_3 becomes \$600, the current reduced cost of column 3 is

$$\bar{c}_3 = c_3 - c_B A_B^{-1} A_{\cdot 3} = -600 - [-200 \quad -300] \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = 50 > 0,$$

which means it is still not beneficial to enter x_3 into the basis, i.e., it is not profitable to plant alfalfa, even with a \$500 subsidy.

6. If we consider soybeans, it is equivalent to introduce a variable, x_4, and we can rewrite the primal linear program as:

$$\begin{aligned} \max \quad & 200x_1 + 300x_2 + 100x_3 + 250x_4 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + 6x_3 + 3x_4 \leq 100 \\ & 2x_1 + 4x_2 + 8x_3 + 5x_4 \leq 120 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

We can compute the reduced cost for soybeans:

$$\bar{c}_4 = c_4 - c_B A_B^{-1} A_{\cdot 4} = -250 - [-200 \quad -300] \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 137.5 > 0,$$

which means it is not beneficial to enter x_4 into the basis, i.e., Farmer Clink should not consider planting soybeans.

Problem 2

1. Transforming the LP to the standard for:

$$\begin{aligned} \min \quad & -4x_1 - x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 = 6 \\ & 5x_1 + 3x_2 + x_4 = 15 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Step 1. The initial simplex tableau is:

-4	-1	0	0	0
3	1	1	0	6
5	3	0	1	15

Step 2. Since $\bar{c}_1 < 0$ we select x_1 to enter the basis. According to MRT, x_3 exits the basis. Update the tableau, we obtain:

0	1/3	4/3	0	8
1	1/3	1/3	0	2
0	4/3	-5/3	1	5

Step 3. All reduced costs are greater than 0, so we reach the optimality, where the optimal solution is $x_1^* = 2, x_2^* = 0$ and the optimal value is 8.

2. The dual problem formulation is:

$$\begin{aligned} \min \quad & 6\pi_1 + 15\pi_2 \\ \text{s.t.} \quad & 3\pi_1 + 5\pi_2 \geq 4 \\ & \pi_1 + 3\pi_2 \geq 1 \\ & \pi_1, \pi_2 \geq 0. \end{aligned}$$

Since the primal optimal solution is $x_1^* = 2, x_2^* = 0$ and there is a positive slack for the second constraint, according to the complementarity condition, we obtain:

$$\begin{aligned} & \pi_2^* = 0 \\ & 3\pi_1^* + 5\pi_2^* = 4. \end{aligned}$$

Therefore, the optimal dual solution is $\pi_1^* = 4/3, \pi_2^* = 0$.

3. Suppose the change of c_2 is denoted by λ. For the current basis to be optimal, we need to have the reduced costs for the non-basic variable x_2 to remain nonnegative:

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B A_B^{-1} A_{\cdot 2} \\ &= -1 + \lambda - [-4 \ \ 0] \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \lambda + 1/3. \end{aligned}$$

Therefore, to keep $\bar{c}_2 \geq 0$, we need to have $\lambda \geq -1/3$.

4. Suppose the change of b_2 is denoted by λ. For the current basis to be optimal, we need to have the new basic solution feasible:

$$\begin{aligned} x_B^* + \lambda A_B^{-1} e_2 &= \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 5 + \lambda \end{bmatrix}. \end{aligned}$$

Therefore, for the new basic solution to be feasible, $5 + \lambda \geq 0$, i.e., $\lambda \geq -5$.