The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

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MAT 3007 – Optimization

Midterm Exam – Sample

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

- *The exam time is 90 minutes.*
- *There are six exercises on three sheets (including this sheet).*
- *The total number of achievable points is* 100 *points.*
- *Please abide by the honor codes of CUHK-SZ.*

Good Luck!

Exercise 1 (Simplex Method): (*20* points)
Use the two-phase simplex method to solve the following linear program:

$$\begin{array}{lrcrcrcl}
\text{minimize} & 4x_1 & + & x_2 & + & x_3 & & \\
\text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & = & 4 \\
 & 3x_1 & + & x_2 & + & x_3 & = & 3 \\
 & x_1, & & x_2, & & x_3 & \geq & 0.
\end{array}$$

Exercise 2 (Duality and Complementarity Conditions): (*15* points)
Continue to consider the linear program in the previous question:

$$\begin{array}{lrcrcrcl}
\text{minimize} & 4x_1 & + & x_2 & + & x_3 & & \\
\text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & = & 4 \\
 & 3x_1 & + & x_2 & + & x_3 & = & 3 \\
 & x_1, & & x_2, & & x_3 & \geq & 0.
\end{array}$$

a) Write down its dual problem.

b) Write down the complementarity conditions.

c) Use the complementarity conditions to compute the dual optimal solution.

Exercise 3 (Sensitivity Analysis): (*20* points)
Consider the following linear program:

$$\begin{array}{lrcrcrcrcl}
\text{maximize} & 2x_1 & + & 4x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & x_1 & + & 3x_2 & & & + & x_4 & \leq & 4 \\
 & 2x_1 & + & x_2 & & & & & \leq & 3 \\
 & & & x_2 & + & 4x_3 & + & x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B	0	0	0	$\frac{7}{20}$	$\frac{11}{10}$	$\frac{9}{20}$	$\frac{1}{4}$	$\frac{13}{2}$
2	0	1	0	$\frac{2}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	1
1	1	0	0	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	0	1
3	0	0	1	$\frac{3}{20}$	$-\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{2}$

a) What is the optimal solution and the optimal value of the problem?

b) What is the optimal solution to the dual problem?

c) In what range can we change the first right hand side number $b_1 = 4$ (the one appearing in the constraint $x_1 + 3x_2 + x_4 \leq 4$) so that the current optimal basis is still the optimal basis?

d) In what range can we change the fourth objective coefficient $c_4 = 1$ so that the current optimal solution is still remains optimal?

Exercise 4 (LP Formulation): (*15* points)
Consider two points in the plane $a = (x_1, y_1)^\top$ and $b = (x_2, y_2)^\top$. We define their ℓ_1-distance as

$$\|a - b\|_1 = |x_1 - x_2| + |y_1 - y_2|$$

(one can view this as the distance of two points if one can only go horizontally or vertically). Now assume that there are three towns located at $(0, 0)$, $(0, 5)$, and $(2, 2)$ and we want to build a post office (it can be built anywhere on the plane).

Formulate a linear programming problem to find the optimal location of the post office such that the maximum ℓ_1-distance between the three towns and the post office is minimized (only the formulation is required, you do not need to solve the problem.

Exercise 5 (Miscellenous): (*15* points)
State whether each of the following statements is *True* or *False*. For each true statement provide a short explanation or proof. For each false statement provide an appropriate counterexample. Only answers with full explanations will be graded. (Short answers of the form "true" or "false" will not be accepted).

a) For a linear optimization problem any optimal solution must be a basic feasible solution.

b) In one iteration of the simplex tableau, suppose there is a column with a negative reduced cost and all the elements in that column are non-positive. Then the LP must be unbounded.

c) Increasing the right hand side value (the b vector) of a standard LP will always increase the optimal value of the LP (suppose both problems are feasible and bounded).

d) If a linear program is unbounded, then the problem will still be unbounded if we add a constraint.

e) Consider a standard LP and its dual. If the dual has a feasible point with objective value 1, then any primal feasible point must have an objective value greater than or equal to 1.

Exercise 6 (Properties of Linear Programs): (*15* points)
Consider the following two linear optimization problems:

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & \tilde{c}^\top x \\ \text{subject to} & Ax = b\ . \\ & x \geq 0 \end{array}$$

Let $x^* = (x_1^*, ..., x_n^*)^\top$ be an optimal solution to the first LP with optimal value V and let $\tilde{x}^* = (\tilde{x}_1^*, ..., \tilde{x}_n^*)^\top$ be an optimal solution to the second LP with optimal value $\tilde{V}$ (thus we have assumed that both problems are feasible and have finite optimal solutions). Suppose c and $\tilde{c}$ only differ in their first component and we have $c_1 > \tilde{c}_1$ and $c_i = \tilde{c}_i$ for all $i = 2, ..., n$.

Prove $x_1^* \geq \tilde{x}_1^*$ and $V \geq \tilde{V}$.