The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

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MAT 3007 – Optimization

Exercise Sheet 6

Problem 1 (An Unconstrained Optimization Problem): (approx. *25* pts)
Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^2} \; f(x) := x_1^4 + \frac{2}{3}x_1^3 + \frac{1}{2}x_1^2 - 2x_1^2 x_2 + \frac{4}{3}x_2^2.$$

a) Calculate all stationary points of the mapping f and investigate whether the stationary points are local maximizer, local minimizer, or saddle points.

b) Create a 3D or contour plot of the function using `MATLAB` or `Python` and decide whether the problem possesses a global solution or not.

Problem 2 (Circle Fitting): (approx. *25* pts)
Suppose that the m points $a_1, a_2, \dots, a_m \in \mathbb{R}^n$ are given. In this exercise, we want to find a circle with center $x \in \mathbb{R}^n$ and radius r that best fits the m points, i.e., we want to determine x and r such that

$$\|x - a_i\| \approx r \quad \forall \; i = 1, \dots, m.$$

Since these approximate equations can be inconsistent, x and r are recovered as global solutions of the following nonlinear least-squares problem:

$$\min_{x,r} \; f(x,r) := \sum_{i=1}^m (\|x - a_i\|^2 - r^2)^2. \tag{1}$$

a) Consider the related optimization problem

$$\min_{y \in \mathbb{R}^{n+1}} \; g(y) := \sum_{i=1}^m (\|a_i\|^2 - b_i^\top y)^2, \quad b_j := \begin{pmatrix} 2a_j \\ -1 \end{pmatrix}, \quad j = 1, \dots, m \tag{2}$$

and show/verify the following statements:

– For all $(x, r) \in \mathbb{R}^n \times \mathbb{R}$ it holds that $g((x^\top, \|x\|^2 - r^2)^\top) = f(x, r)$.

– Let $y^* \in \mathbb{R}^{n+1}$ be a global solution of (2) and set $\bar{y} = (y_1^*, \dots, y_n^*)^\top$. Show that we have $y_{n+1}^* \le \|\bar{y}\|^2$.

  Hint: Assume that the result is wrong, i.e., we have $y_{n+1}^* > \|\bar{y}\|^2$. Can you then find a point $z \in \mathbb{R}^{n+1}$ with $g(z) < g(y^*)$?

– Given the global minimizer y^* of (2), can you construct a global solution (x^*, r^*) of the initial problem (1)?

b) Assume that the matrix $B^\top = (b_1, b_2, \dots, b_m) \in \mathbb{R}^{n+1 \times m}$ has full row rank. Show that problem (2) has a unique strict local minimizer and compute it.

c) Write a `MATLAB` or `Python` code to calculate a solution (x^*, r^*) of problem (1) for a given set of points $A = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^{n \times m}$. Test your code on the following dataset:

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad a_6 = \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}.$$

Visualize your solution and the points a_1–a_6 using an appropriate plot.

Problem 3 (KKT Conditions – I): (approx. *10* pts)
Let us consider the optimization problem

$$\begin{array}{ll} \min_{x \in \mathbb{R}^3} & f(x) := 2x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 - 6x_1 - 7x_2 - 8x_3 - 9 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 1, \quad x_1 - x_2^2 = 0. \end{array}$$

Write down the KKT conditions for this problem.

Problem 4 (KKT Conditions – II): (approx. *20* pts)
Consider the problem

$$\min_{x \in \mathbb{R}^3} \ 2x_1x_2 + \frac{1}{2}x_3^2 \quad \text{subject to} \quad 2x_1x_3 + \frac{1}{2}x_2^2 \leq 0, \quad 2x_2x_3 + \frac{1}{2}x_1^2 \leq 0.$$

a) Write down the KKT conditions for this problem.

b) Investigate whether the point $x^* = (0,0,0)^\top$ is a KKT point satisfying the KKT conditions.

Problem 5 (Projection Onto a Ball): (approx. *20* pts)
Let $m \in \mathbb{R}^n$ and $r > 0$ be given and define the ball $C := \{x \in \mathbb{R}^n : \|x - m\| \leq r\}$. In this exercise, we want to compute the projection $\mathcal{P}_C(x)$ for $x \in \mathbb{R}^n$, i.e., we want solve the optimization problem

$$\min_{y \in \mathbb{R}^n} \ \frac{1}{2}\|y - x\|^2 \quad \text{subject to} \quad \|y - m\|^2 \leq r^2. \tag{3}$$

a) Write down the KKT conditions for problem (3).

b) Show that the KKT conditions have a unique solution and calculate the corresponding KKT pair explicitly.

Sheet 6 is due on **Apr, 13th**. Submit your solutions before **Apr, 13th**, **12:00 pm (noon)**.