G00

Group ID Number

The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Midterm Exam – Solutions

MAT 3007 – Optimization

Spring Semester 2022

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March 26th, 2022

Points	
E01	23
E02	17
E03	18
E04	12
E05	16
E06	14
Tot.	100

Please read the following instructions carefully:

– You have **90 minutes** to complete the exam.
  Examination Time: **10:00 to 11:30 am**. (No early leave is allowed).
  Solutions need to be submitted before **11:45 am** on **Blackboard**.
– This is an **open book** and **open note exam**. You can check and use any form of hard-copy materials during the exam.
  You are not allowed to use electronic devices other than accessing the exam sheets and zoom. In particular, it is not allowed to check materials on electronic devices. In addition, you cannot search anything online, use your keyboard, or have any form of communication with other students and parties during the examination period.
– Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results. Write down all necessary steps when answering the questions.
– Any violation of the exam policies will be considered as cheating and reported. Consequences of such a violation include zero points for the midterm exam and corresponding disciplinary actions.

Zoom rules and regulations:

– You can only use two devices during the exam: your computer for accessing exam sheets and connecting to zoom and another device (phone or iPad/tablet) for an extra zoom connection. Other devices should be placed in an inaccessible area and turned off. Your camera must be positioned in a way so that the proctors can see you, your desk/table, and your computer screen.
– All instant messaging apps (WeChat, WhatsApp, etc.) **must be closed**. This will be checked during the exam.
– During the exam, you must keep your camera on and mute yourself at all time. Please ensure that you have good/stable internet connection.
– You **cannot leave your seat** at any time during the exam. No one should enter the room where you are seated.
– The exam starts after the designated starting time (10:00 am on March 26th). At the end of the exam, the proctor will notify you to stop writing. Please scan/take pictures of your exam answers and submit the files via Email. (You are allowed to use your phone during this period of the exam to scan and submit your solutions).
– The deadline of receiving your submission is 11:45 am on March 26th, 2022. Late submissions will not be accepted. The submission is single-attempt. Please make sure to submit the right files.
– You are not allowed to leave the zoom meeting before your submission has been received and verified. The proctor will announce your name when you can leave the meeting.
– In case there are emergencies/problems, first use the “raise hand” function in zoom. Wait for an approval from the proctor and closely follow his/her instructions.

Exercise 1 (Simplex Method): ($\mathit{23}$ points)

We consider the linear problem:

$$
\begin{array}{lccccccccl}
\text{minimize} & & & x_2 & - & \frac{1}{2}x_3 & + & x_4 & & \\
\text{subject to} & x_1 & + & x_2 & + & \beta x_3 & & & = & 1 \\
 & & & & & x_3 & - & x_4 & = & -\gamma \\
 & x_1 & & & + & 2x_3 & - & x_4 & \leq & 0 \\
 & & & & & & & & & \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}
\tag{1}
$$

Use the two-phase simplex method to solve this linear program. State the found optimal solution and optimal objective function value of problem (1).

Solution: After transforming (1) into standard form, the associated auxiliary problem is given by:

$$
\begin{array}{lccccccccccccccl}
\text{minimize} & y_1 & + & y_2 & + & y_3 & & & & & & & & \\
\text{subject to} & x_1 & + & x_2 & + & \beta x_3 & + & y_1 & & & & & = & 1 \\
 & & & & - & x_3 & + & x_4 & + & y_2 & & & = & \gamma \\
 & x_1 & + & 2x_3 & - & x_4 & + & s_1 & + & y_3 & & & = & 0 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & s_1, & & y_1, \; y_2, \; y_3 & \geq & 0.
\end{array}
$$

4pts: 2pts for correct variables (1pt for slack, 1pt for auxiliary) + 1pt for objective coefficients + 1pt for adjusted constraints.

Hence, we can choose $(0,0,0,0,0,1,\gamma,0)^\top$ as initial BFS; the reduced costs are given by $(-2,-1,-1-\beta,0,-1)$. This yields the following initial simplex tableau:

B	-2	-1	$-1-\beta$	0	-1	0	0	0	$-1-\gamma$
6	1	1	β	0	0	1	0	0	1
7	0	0	-1	1	0	0	1	0	γ
8	1	0	2	-1	1	0	0	1	0

5pts: 1pt for BFS + 2pts for reduced costs + 2pts for remaining correct initial tableau.

The pivot column is $\{1\}$; the pivot row is $\{8\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	0	-1	$3-\beta$	-2	1	0	0	2	$-1-\gamma$
6	0	1	$\beta-2$	1	-1	1	0	-1	1
7	0	0	-1	1	0	0	1	0	γ
1	1	0	2	-1	1	0	0	1	0

2pts: for correct tableau update.

The pivot column is $\{2\}$; the pivot row is $\{6\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	0	0	1	-1	0	1	0	1	$-\gamma$
2	0	1	$\beta-2$	1	-1	1	0	-1	1
7	0	0	-1	1	0	0	1	0	γ
1	1	0	2	-1	1	0	0	1	0

2pts: for correct tableau update.

Under the assumption $\gamma > 1$, the pivot column is $\{4\}$; the pivot row is $\{2\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B									
	0	1	$\beta-1$	0	-1	2	0	0	$1-\gamma$
4	0	1	$\beta-2$	1	-1	1	0	-1	1
7	0	-1	$1-\beta$	0	1	-1	1	1	$\gamma-1$
1	1	1	β	0	0	1	0	0	1

2pts: for correct tableau update.

Under the assumption $\beta \geq 1$, the pivot column is $\{5\}$; the pivot row is $\{7\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B									
	0	0	0	0	0	1	1	1	0
4	0	0	-1	1	0	0	1	0	γ
5	0	-1	$1-\beta$	0	1	-1	1	1	$\gamma-1$
1	1	1	β	0	0	1	0	0	1

2pts: for correct tableau update.

Hence, $(1, 0, 0, \gamma, \gamma-1)^\top$ is an initial BFS for the original problem (in standard form) with basis $B = \{4, 5, 1\}$ (1pt). We now calculate the reduced costs:

$$\bar{c}^\top = c^\top - c_B^\top A_B^{-1} A = c^\top - \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1-\beta & 0 & 1 \\ 1 & 1 & \beta & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -\frac{1}{2} & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1-\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

3pts: for correct reduces costs.

The reduced costs are nonnegative and we can stop with the solution $x^* = (1, 0, 0, \gamma, \gamma-1)^\top$ with optimal objective value γ. (2=1+1pts)

Correct transformations and start of phase I: 9pts (total); Correct simplex procedure to find initial BFS: 9pts (total); Check optimality and solution: 5pts (total);

Exercise 2 (Duality Theory): (*9+4+4=17* points)

a) Consider the following linear program:

$$\begin{array}{llllllllll} \text{minimize} & 5x_1 & + & x_2 & - & 4x_3 & & & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \geq & \delta \\ & & & 4x_2 & & & + & 8x_4 & \leq & \varepsilon \\ & x_1 & + & 6x_2 & - & x_3 & & & = & \varphi \\ & & & & & & & & & \\ & x_1 & \text{free,} & x_2 & \geq 0, & x_3 & \geq 0, & x_4 & \leq 0. & \end{array}$$

Write down the dual problem and the complementarity conditions of the linear program above.

b) Consider the following two linear programs:

$$z_1^* = \begin{array}{ll} \text{maximize}_x & c^\top x \\ \text{subject to} & Ax \le b \end{array} \tag{2}$$

and

$$z_2^* = \begin{array}{ll} \text{minimize}_x & c^\top x \\ \text{subject to} & Ax \ge b. \end{array} \tag{3}$$

Assume that both model (2) and (3) are feasible. Assume further that model (2) has a finite optimal solution. Show that this implies that model (3) has a finite optimal solution.

c) Under the assumptions in part b), can you relate z_1^* and z_2^*? That is, can you show $z_1^* \le z_2^*$ or $z_1^* \ge z_2^*$? Or, can the result go "either way" depending on the specific values in the matrix A, the objective function vector c, and the right-hand side value b?

Solution:

a) The dual program is given by:

$$\begin{array}{lccccccc} \text{maximize} & \delta y_1 & + & \varepsilon y_2 & + & \varphi y_3 & & \\ \text{subject to} & y_1 & & & + & y_3 & = & 5 \\ & y_1 & + & 4y_2 & + & 6y_3 & \le & 1 \\ & y_1 & & & - & y_3 & \le & -4 \\ & y_1 & + & 8y_2 & & & \ge & 0 \\ & y_1 & \ge 0, & y_2 & \le 0, & y_3 & \text{free.} & \end{array}$$

The dual formulation is worth 4 pts. Correct objective coefficient 1 pt, correct signs for constraints 1 pt, correct signs of variables 1 pt, correct rest of coefficients 1 pt. If there is one error in any element of those four items, regardless how many elements there are in the item, the 1 pt for that item is taken off.

The optimality conditions are given by: x is feasible for the primal problem, y is feasible for the dual problem, and we have:

$$\begin{array}{lll} y_1 \cdot (x_1 + x_2 + x_3 + x_4 - \delta) = 0, & & x_1 \cdot (y_1 + y_3 - 5) = 0, \\ y_2 \cdot (4x_2 + 8x_4 - \varepsilon) = 0, & & x_2 \cdot (y_1 + 4y_2 + 6y_3 - 1) = 0, \\ y_3 \cdot (x_1 + 6x_2 - x_3 - \varphi) = 0, & & x_3 \cdot (y_1 - y_3 + 4) = 0, \\ & & x_4 \cdot (y_1 + 8y_2) = 0. \end{array}$$

The complementarity condition is worth 5 pts. Dual var/primal constraints 2 pt (the first two conditions each worth 1 pt), primal var/dual constraints 3 pts (the last three conditions each worth 1 pt). If they miss $x_1(y_1 + y_3 - 5) = 0$ or $y_3(x_1 + 6x_2 - x_3 - \phi) = 0$, it is still correct and no points are taken off.

b) We consider the duals of problem (2) and (3):

$$z_1^* = \begin{array}{ll} \text{maximize}_x & c^\top x \\ \text{subject to} & Ax \le b \end{array} \quad \Bigg| \quad \textbf{Dual:} \begin{array}{ll} \text{minimize} & b^\top y \\ \text{subject to} & A^\top y = c, \quad y \ge 0, \end{array} \tag{4}$$

and

$$z_2^* = \begin{array}{ll} \text{minimize}_x & c^\top x \\ \text{subject to} & Ax \ge b \end{array} \quad \Bigg| \quad \textbf{Dual:} \begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & A^\top y = c, \quad y \ge 0. \end{array} \tag{5}$$

Assume both primal models (2) and (3) are feasible, and suppose that (2) has a finite optimal solution. The latter implies that the dual model (4) has a finite optimal solution. This means model (5) is feasible because (4) and (5) have the same feasible region. Thus, both (3) and (5) are feasible. The only situation where the primal-dual pair can both have feasible solutions is that both of them have finite optimal solutions. Therefore, model (3) and (5) both have finite optimal solutions.

4 points in total for b). The correct derivation of the dual models is worth 1 pt. Pointing out that two dual models have the same feasible region is worth 1 pt. Stating the potential outcomes of primal-dual pair is worth 1 pt. Correct conclusion is worth 1 pt.

c) The feasible regions and the objective function expressions are identical for model (4) and (5), except that in (4) we minimize and in (5) we maximize. Let y_1^* and y_2^* be dual solutions of the problems (4) and (5), respectively. Then, by strong duality, we have

$$z_1^* = b^\top y_1^* \leq b^\top y_2^* = z_2^*.$$

Hence, it always holds that $z_1^* \leq z_2^*$.

We have $z_1^* \geq z_2^*$ if and only if $z_1^* = z_2^*$. This holds for example if $c = 0$ or if $b = 0$ or if $A \in \mathbb{R}^{m \times n}$ has full row rank. Indeed, every solution of the linear system $A^\top y = c$ needs to satisfy $AA^\top y = Ac$. However, since $AA^\top$ is invertible, this uniquely characterizes y via $y = (AA^\top)^{-1} Ac$ in such case. The feasible set in (4) and (5) then reduces to a singleton.

4 points in total for c). Pointing out the same feasible region for two dual models is worth 2 pts. Pointing out $z_1^* \leq z_2^*$ because one is maximizing and the other is minimizing is worth 2 pts. Whether there is a special case discussion does not count towards points.

Exercise 3 (Sensitivity Analysis): (*3+4+4+4+3=18* points)

Consider the following linear program, where we try to maximize our profit of production with raw material constraints, and let $x_B = (x_1, x_2, x_3)$:

$$\begin{array}{llllllllll}
\text{maximize} & 2x_1 & + & 4x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & x_1 & + & 3x_2 & & & + & x_4 & \leq & 8 & \text{(raw material 1)} \\
 & 2x_1 & + & x_2 & & & & & \leq & 6 & \text{(raw material 2)} \\
 & & & x_2 & + & 4x_3 & + & x_4 & \leq & 6 & \text{(raw material 3)} \\
 & & & & & & & & & \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. &
\end{array}$$

We can compute

$$A_B^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{10} & \frac{1}{20} & \frac{1}{4} \end{bmatrix}$$

a) Is the current basis optimal? Explain why.

b) If the availability of only one of the raw materials can be marginally increased, which one should you choose to increase to be the most beneficial? Explain.

c) For what range of values of b_1 (the amount of raw material 1 available) does the basis x_B remain optimal?

d) What is an optimal solution to the problem if $b_1 = 18$? Is this solution degenerate? Explain why.

e) If seven more units of raw material 1 can be made available (in addition to the current amount of eight), what is the maximum amount you should pay for those seven units? Explain.

Solution:

a) We check the reduced costs. When transferring the problem into standard form, the objective function coefficients are adjusted to $c^\top = (-2, -4, -1, -1, 0, 0, 0)$ and we have:

$$\begin{aligned}
\bar{c}_N^\top = c_N^\top - c_B^\top A_B^{-1} A_N &= \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{10} & \frac{1}{20} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{11}{10} & -\frac{9}{20} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{27}{20} & -\frac{11}{5} & -\frac{9}{20} & -\frac{1}{4} \end{pmatrix}.
\end{aligned}$$

We see that the reduced costs are nonnegative and hence, the basis $B = \{1, 2, 3\}$ is optimal.

3 pts in total. Correct conclusion (it is optimal) is worth 1 pt. Correct reduced cost is worth 1 pt. Pointing out the link between all nonnegative reduced costs and optimality is worth 1 pt.

b) The dual solution is given by $(y^*)^\top = c_B^T A_B^{-1} = (-\frac{11}{10}, -\frac{9}{20}, -\frac{1}{4})^\top$. Local sensitivity suggests that the change of the objective function will be given by $y_i^* \Delta b_i$, $i = 1, \ldots, 3$ (under the assumption that y^* is unique). Hence, adjusting $b_1 = 8$ will have the most effect.

4 pts in total. Correct conclusion (it should be raw material 1) is worth 1 pt. Correct dual value calculation is worth 1 pt. Pointing out the economic meaning (shadow price) of the dual value is worth 2 pts.

c) We change b to $b + \lambda e_1$. Then, the basis B remains optimal if $\tilde{x}_B = A_B^{-1}(b + \lambda e_1) \geq 0$. We have:

$$A_B^{-1}(b + \lambda e_1) = \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{10} & \frac{1}{20} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 8 + \lambda \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{5} + 2 \\ \frac{2\lambda}{5} + 2 \\ -\frac{\lambda}{10} + 1 \end{pmatrix}.$$

This gives the range $\lambda \in [-5, 10]$.

4 pts in total. Writing down the $\tilde{x}_B = A_B^{-1}(b + \lambda e_1) \geq 0$ condition is worth 2 pts. Getting -5 and 10 correct is worth 1 pt each.

d) $b_1 = 18$ requires to set $\lambda = 10$. Based on part c), we obtain the optimal solution $z^* = (x^*, s^*) = (0, 6, 0, 0, 0, 0, 0)$. Since two components of z_B^* are zero, this is a degenerate solution.

4 pts in total. Correct optimal solution is worth 2 points. Pointing out the definition of degeneracy (there are basic variables equal to 0) is worth 1 pt. Correct conclusion is worth 1 pt.

e) Since $7 < 10$, according to the conclusion in c), the current basis remains optimal and we can use the local sensitivity analysis here. The optimal dual value equals to the change of objective value given a unit change of the right-hand side. The objective function increases by $-7y_1^* = \frac{77}{10} = -c_B^\top(-\frac{7}{5}, \frac{14}{5}, -\frac{7}{10})^\top$. Hence, as long as those 7 additional units are bought for less than 7.7 (currency units), we can make a profit.

3 pts in total. Pointing out 7 is within the range where local sensitivity analysis can be applied is worth 1 pt. Calculating the correct number 7.7 is worth one point. Having the right conclusion (maximum amount to pay) is worth 1 pt.

Exercise 4 (Cruise Allocation Problem): (*12* points)

Consider a cruise line company that runs a certain cruise route. There are $N+1$ stops at the route. We use stop 0 to denote the starting point and stop N to denote the final stop. We call the jth leg in the route to be the leg between stop $j-1$ and stop j. There is a demand d_{ij} of passengers who want to embark during leg i to leg j (e.g., d_{34} represents the amount of passengers who want to travel on leg 3 and leg 4, or in other words, from stop 2 to stop 4). We denote such demands by $i \to j$. The unit ticket price for demand $i \to j$ is p_{ij}. Moreover, the capacity of the cruise is C, which means that at any moment (on any leg) there cannot be more than C passengers on the cruise.

Write a linear optimization formulation to decide how many passengers for each type of demand to accept, in order to maximize the total revenue of the firm. (Note that the total demand may exceed the total capacity, i.e., if we accept all demands, then there could be more than C passengers on board, which is not feasible. Therefore we may be only able to accept a portion of the demand.)

Solution: Let x_{ij} denote the number of demand $i \to j$ we accept (2pts). Then the optimization problem can be formulated as

$$\begin{array}{lllr} \text{maximize} & \sum_{1\le i\le j\le N} p_{ij}x_{ij} & & \text{(2pts)} \\ \text{subject to} & \sum_{(i,j):i\le \ell\le j} x_{ij} \le C & \forall\ 1\le \ell\le N & \text{(6pts)} \\ & x_{ij}\le d_{ij} & \forall\ 1\le i\le j\le N & \text{(1pts)} \\ & x_{ij}\ge 0 & \forall\ 1\le i\le j\le N & \text{(1pts)}. \end{array}$$

Note: 1) It is fine to ignore the integral constraint in the answer (it is fine to include it too). 2) Only writing down the last two constraints will not receive any points. 3) Any other valid formulation is also fine.

Exercise 5 (Properties of Optimization Problems): (*4+4+4+4=16* points)

For each of the following cases, find an example of an optimization problem satisfying the corresponding description or explain why such an example can not exist.

a) A continuous optimization problem with exactly two optimal solutions. (4pts)

b) A linear optimization problem in standard form with 3 variables and 2 constraints, in which there is a unique optimal solution such that all 3 variables are nonzero. (4pts)

c) A primal-dual pair of linear optimization problems in which the primal has a unique optimal solution while the dual problem has an infinite number of optimal solutions. (4pts)

d) A linear optimization problem that has unique optimal solution but the solution is degenerate. (4pts)

Solution:

a) An example can be:

$$\text{maximize } x^2 \quad \text{subject to} \quad x\in[-1,1].$$

Note: Any valid example will get 4 points. Otherwise gets 0 point.

b) This is not possible (1pt). By the LP fundamental theorem, an LP with two constraints must have an optimal solution with at most two non-zero entries. (3pts)

c) An example can be:

$$\textbf{Primal:} \quad \begin{array}{ll} \text{minimize} & x \\ \text{subject to} & x = 0 \\ & x \geq 0 \end{array} \qquad \textbf{Dual:} \quad \begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & y \leq 1 \end{array}$$

Note: Any valid example will get 4 points. Otherwise gets 0 point.

d) An example can be:

$$\text{minimize } x \quad \text{subject to} \quad x = 0, \quad x \geq 0.$$

Note: Any valid example will get 4 points. Otherwise gets 0 point.

Exercise 6 (Duality and Feasibility): (*5+9=14* points)

Consider the following linear optimization problem:

$$\begin{array}{lllll} \text{maximize} & b^\top z & & & \\ \text{subject to} & A^\top z \leq 0, & & z \leq \mathbb{1}, & \end{array} \tag{6}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ are given and $\mathbb{1} \in \mathbb{R}^m$ is the vector of all ones, i.e., we have $\mathbb{1}_i = 1$ for all i.

a) Derive the dual of problem (6). (5pts)

b) Prove that the following two statements are equivalent: (9pts)

- There exists a point $x \in \mathbb{R}^n$ satisfying $Ax = b$ and $x \geq 0$.
- The point $z = 0$ is an optimal solution of problem (6).

Solution: *Part a).* The dual of problem (6) is given by

$$\begin{array}{llll} \text{minimize} & \mathbb{1}^\top y & & \\ \text{subject to} & Ax + y & = & b \\ & x, y & \geq & 0. \end{array} \tag{7}$$

5pts in total. 2pts for the correct dimension and form of the dual variable $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. 1pt for objective function coefficient. 1pt for the equality constraint. 1pt for nonnegativity constraints. If the format of the dual is wrong: minus 2pts – rest depends on whether the logic is correct. (Constraints and coefficients don't need to be simplified).

Part b). Suppose that there exists x with $Ax = b$ and $x \geq 0$. Then, the point $(x^*, y^*) := (x, 0)$ is feasible for problem (7) with $\mathbb{1}^\top y^* = 0$ (1pt). Due to $\mathbb{1}^\top y \geq 0$ for every feasible y, this implies that (x^*, y^*) is an optimal solution of problem (7) (1pt). By strong duality, the primal problem (6) then needs to possess an optimal solution z^* as well and we have $b^\top z^* = \mathbb{1}^\top y^* = 0$ (2pts). Hence, the point $z = 0$ is an(other) optimal solution of (6) in this case (1pt).

Note: the feasibility result for the simplex auxiliary problem shown in the lectures can be cited to simplify things. Core step: application of the strong duality theorem to ensure that 0 is the optimal function value of the primal problem.

Conversely, let $z = 0$ be a solution of (6). By strong duality, we can again infer that there exists a solution (x^*, y^*) of the dual problem (7) with $\mathbb{1}^\top y^* = 0$ (2pts). As before, this can only occur if $y^* = 0$ which requires x^* to satisfy $Ax^* = b$ and $x^* \geq 0$ (2pts).

Core step: $\mathbb{1}^\top y^* = 0$ ensures $y^* = 0$ and implies feasibility.

Note: If the derived dual in a) is wrong and massively simplifies the proof (e.g., if the objective is just "0"), then the maximum points achievable in part b) is limited to 5pts!