香港中文大學(深圳)
The Chinese University of Hong Kong

(Materials of this lecture are NOT included in the midterm and final exams)

CSC3100 Data Structures
Lecture 16: Red-black tree

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Outline

- Definitions and examples
- Properties
- Operations
  - Insertion algorithm with three cases
  - Deletion algorithm (homework)

Red-black tree

- A “balanced” binary search tree
  - It guarantees an O(logn) running time for many operations, such as search, insertion, and deletion
- Overview
  - A binary search tree has an additional attribute for its nodes: color which can be either **red** or **black**
  - It restricts the way that nodes can be colored on any path from the root to a leaf
  - It ensures that no path is more than twice as long as any other path

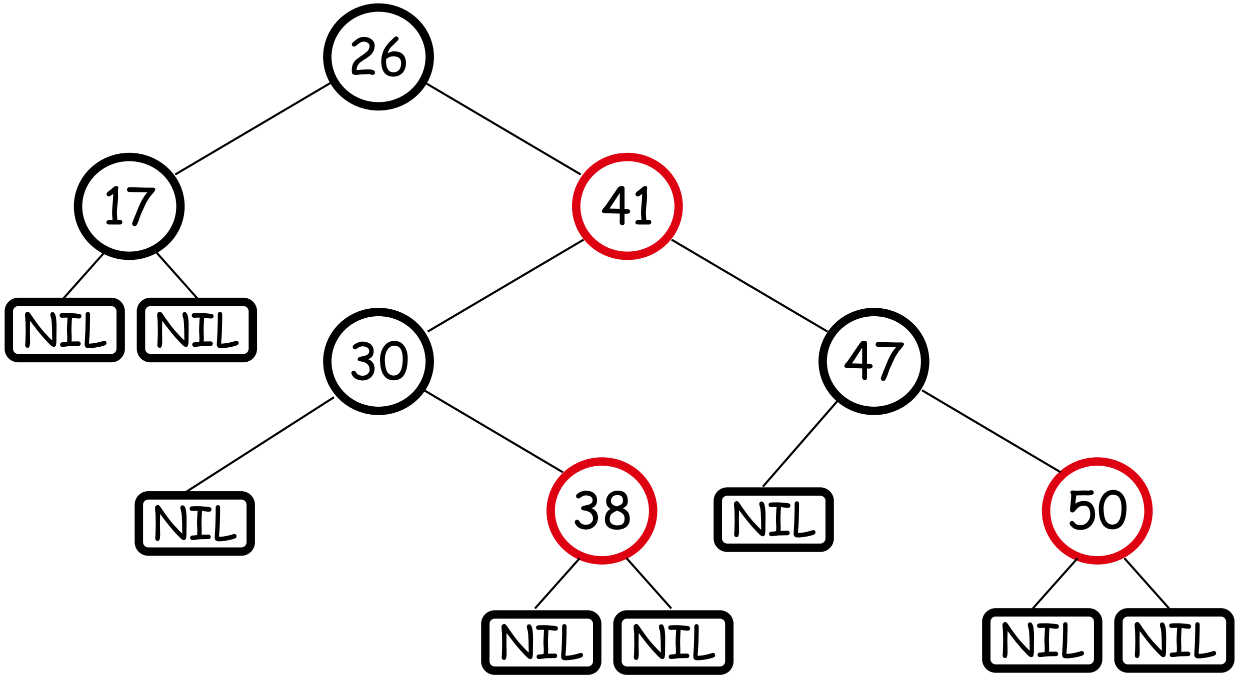
Red-black tree properties

1. Every node is either red or black
2. The root is black
3. Every leaf (NIL) is black
4. If a node is red, then both its children are black

   No two consecutive red nodes on a simple path from the root to a leaf

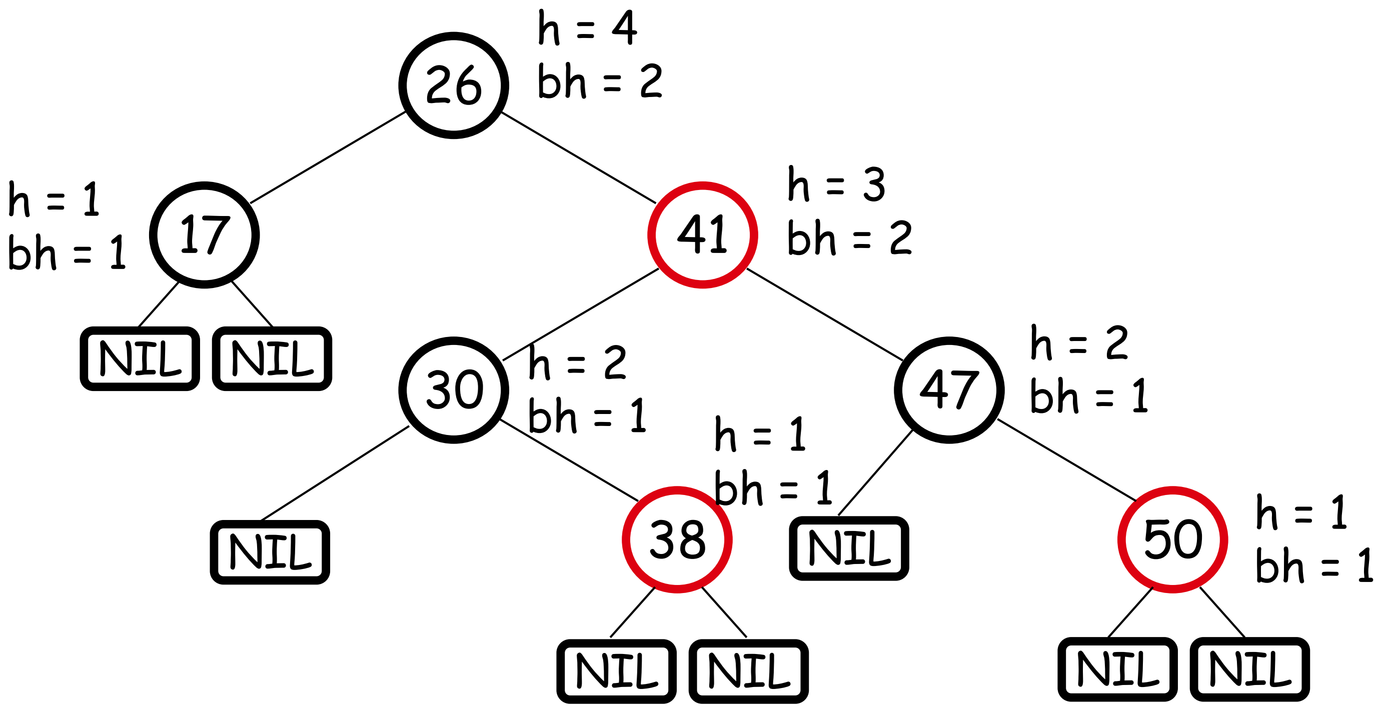
5. For each node, all paths from that node to descendant leaves contain the same number of black nodes

Example



- For convenience we use a sentinel NIL[T] to represent all the NIL nodes at the leaves
  - NIL[T] has the same fields as an ordinary node
  - Color[NIL[T]] = BLACK
  - The other fields may be set to arbitrary values

Black height of a node



- **Height of a node x:**
  - h(x) is the number of edges in the longest path to a leaf
- **Black-height of a node x:**
  - bh(x) is the number of black nodes (including NIL) on the path from x to a leaf, <u>not counting x</u>

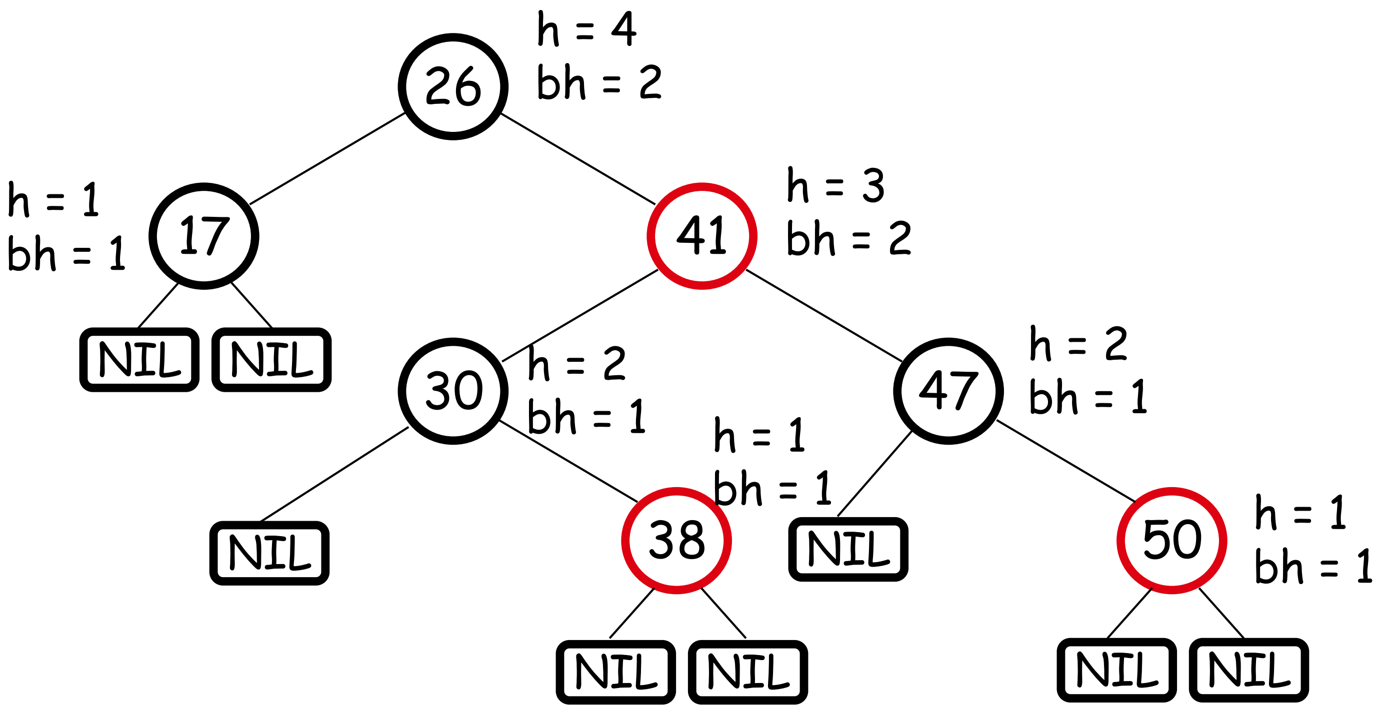
Important property of red-black tree

> A red-black tree with n internal nodes has height at most $2\log(n + 1)$

- Need to prove two claims first …

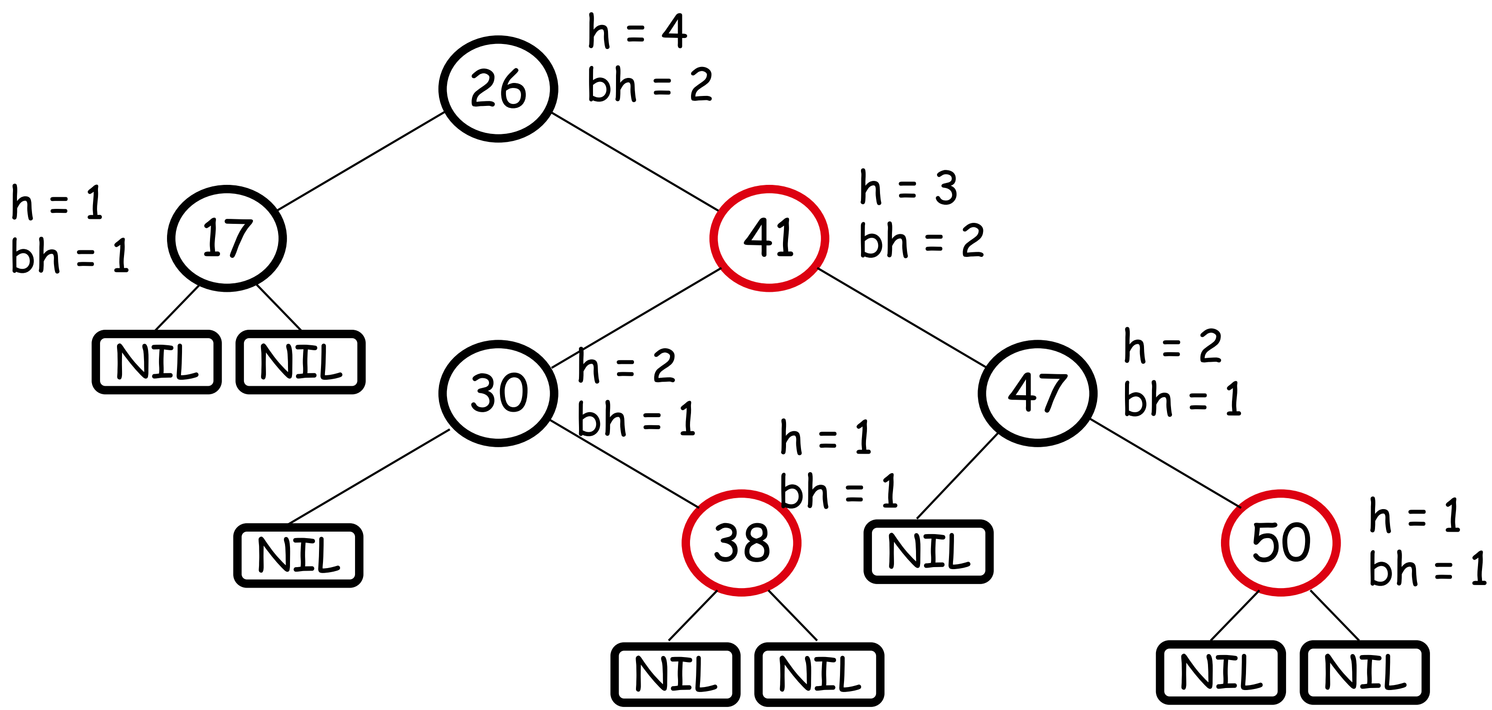
Claim 1

- Any node x with height h(x) has bh(x) ≥ h(x)/2
- **Proof**
  - By property 4, at most h/2 red nodes on the path from the node to a leaf
  - Hence at least h/2 are **black**



Claim 2

- The subtree rooted at any node x contains **at least** $2^{bh(x)} - 1$ internal nodes



Claim 2 (Cont'd)

Proof: By induction on **h[x]**

Basis: h[x] = 0 $\Rightarrow$

x is a leaf (NIL[T]) $\Rightarrow$

bh(x) = 0 $\Rightarrow$

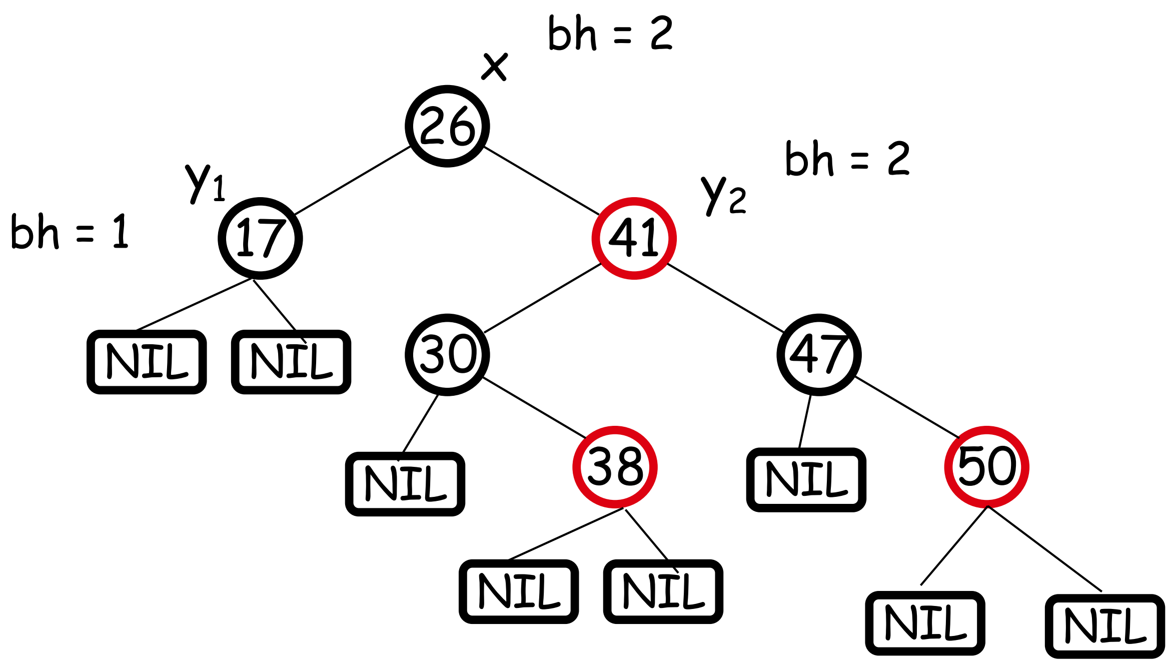
# of internal nodes: $2^0 - 1 = 0$

Inductive hypothesis: assume it is true for h[x] = h-1

Claim 2 (Cont'd)

Inductive step:

- Prove it for h[x] = h
- Let bh(x) = b. Then, any child y of x has:
  - bh (y) = b (if the child is **red**), or
  - bh (y) = b - 1 (if the child is **black**)



Claim 2 (Cont'd)

- Using inductive hypothesis, the number of internal nodes for each child of x is at least if x is black:

$$2^{bh(x)-1} - 1$$

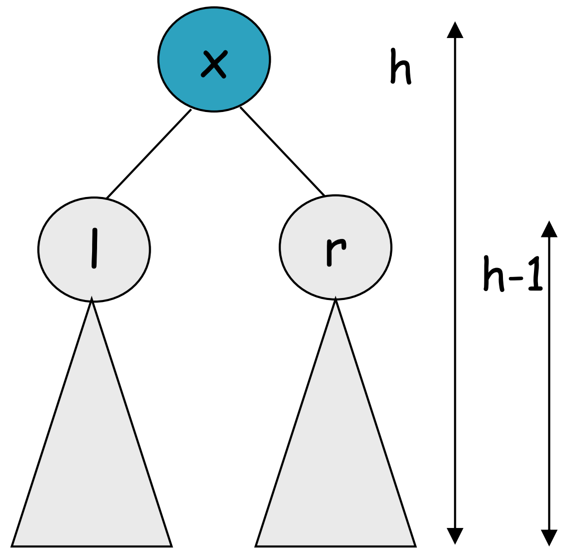
- What if x is red?

- The subtree rooted at x has at least:

$(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1$

$= 2 \cdot (2^{bh(x)-1} - 1) + 1$

$= 2^{bh(x)} - 1$ internal nodes



$bh(l) \geq bh(x)-1$

$bh(r) \geq bh(x)-1$

Important property of red-black tree

A red-black tree with n internal nodes has height at most $2\log(n + 1)$

Proof in the next slides.

- Claim 1: Any node x with height $h(x)$ has $bh(x) \geq h(x)/2$
- Claim 2: The subtree rooted at any node x contains **at least** $2^{bh(x)} - 1$ internal nodes

Height of red-black tree

Lemma: A red-black tree with n internal nodes has height at most $2\log(n + 1)$.

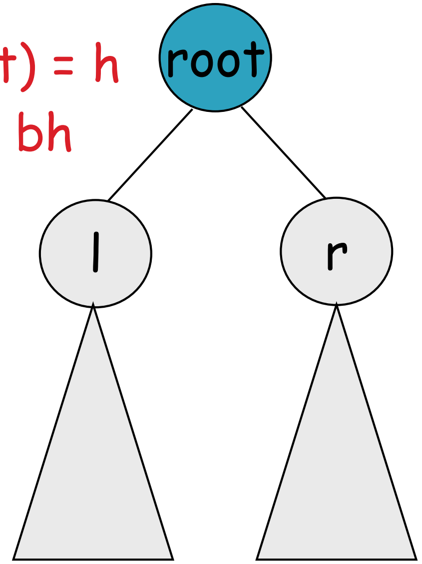
Proof:

$$n \quad \geq 2^{bh} - 1 \quad \geq 2^{h/2} - 1$$

number **n** of internal nodes

since $bh \geq h/2$

height(root) = h
bh(root) = bh



- Add 1 to both sides and then take logs:

$$n + 1 \geq 2^{bh} \geq 2^{h/2}$$

$$\log(n + 1) \geq h/2$$

$$\Rightarrow h \leq 2 \log(n + 1)$$

Operations on red-black tree

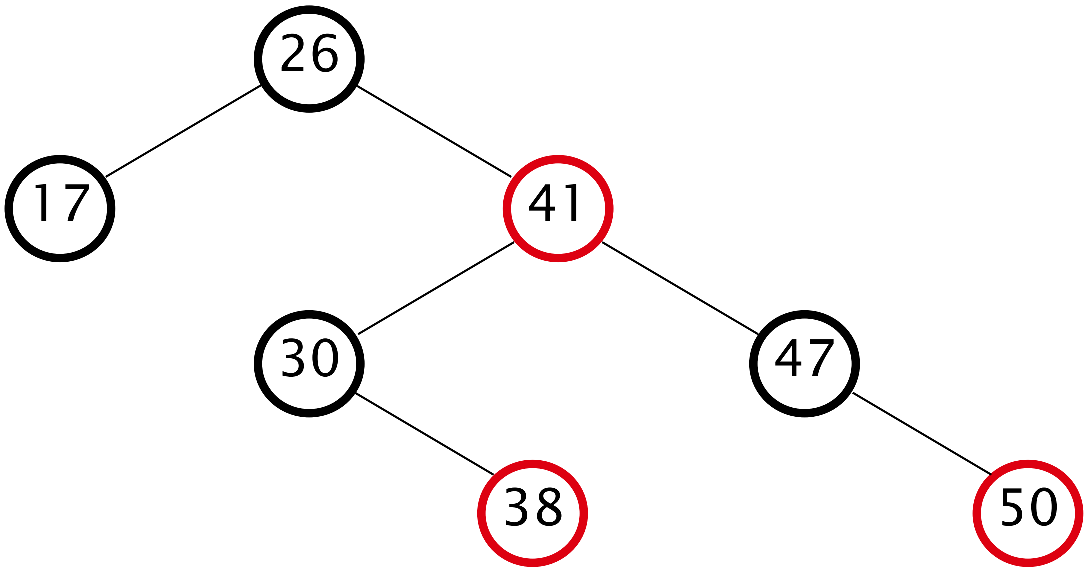
- The non-modifying operations: MINIMUM, MAXIMUM, and SEARCH run in O(h) time
  - They take O(logn) time on red-black trees
  - SEARCH is similar to the search on binary search tree

- What about INSERT and DELETE?
  - We have to guarantee that the modified tree will still be a red-black tree
  - Reconstruction will be too expensive
  - They can still be completed in O(logn) time

INSERT operation

INSERT: Suppose we want to insert 35. What color to make the new node?

- Red?
  - Property 4 is violated: if a node is red, then its children are black
- Black?
  - Property 5 is violated: all paths from a node to its leaves contain the same number of black nodes

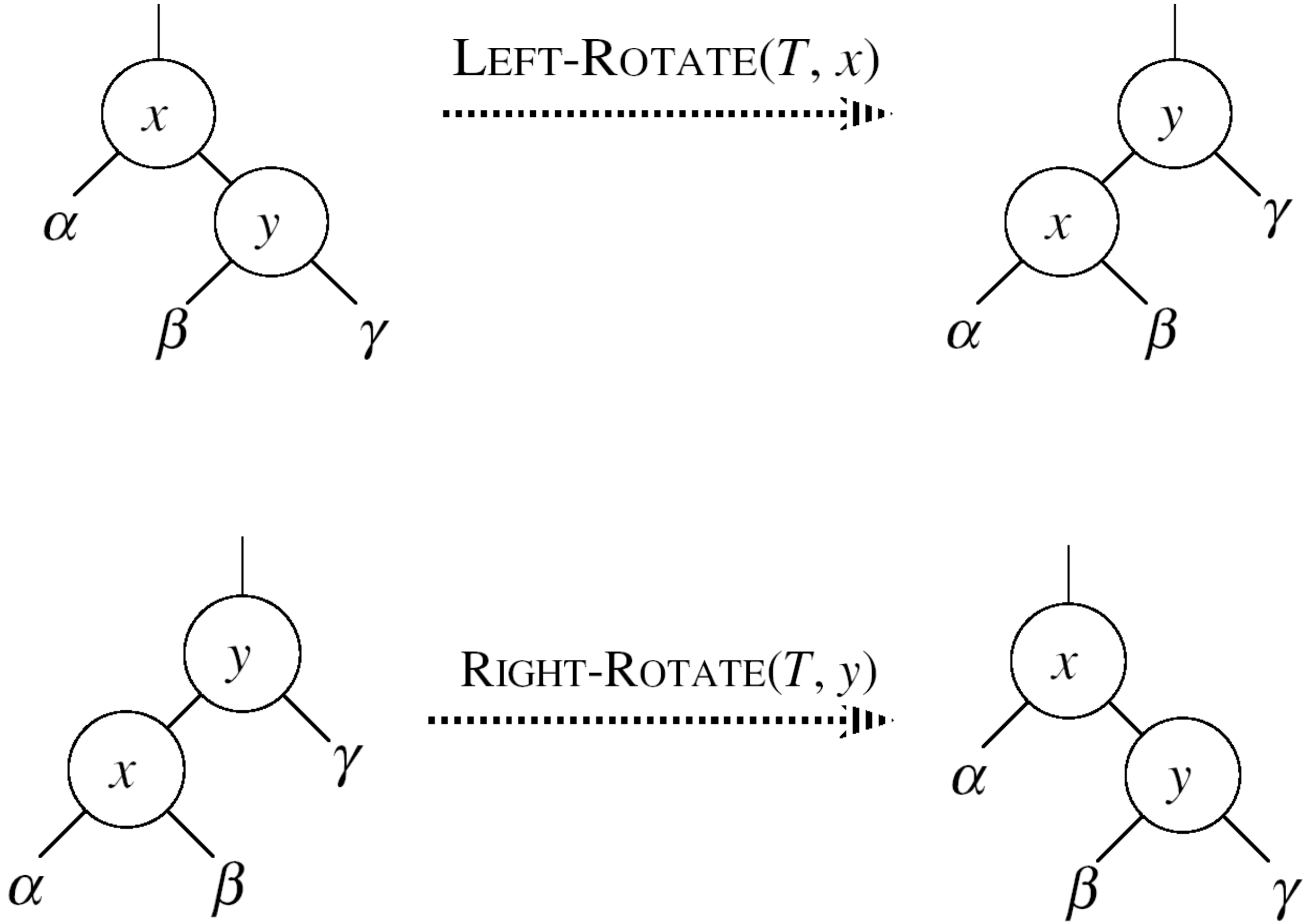


Rotations

- After insertion and deletion on red-black trees, we need to <u>restore</u> the red-black tree properties

- Rotations take a red-black tree and a node within the tree and:
  - Two types of rotations: <u>Left & right rotations</u>
  - Together with some node <u>re-coloring</u> they help restore the red-black tree property
  - Change some of the pointer structure
  - Do not change the binary search tree property

Left and right rotations



INSERT

- Goal:
  - Insert a new node z into a red-black tree
- Idea:
  - Insert node z into the tree as for an ordinary BST
  - Color the node **red**
  - Restore the red-black-tree properties
    - Use an auxiliary procedure RB-INSERT-FIXUP

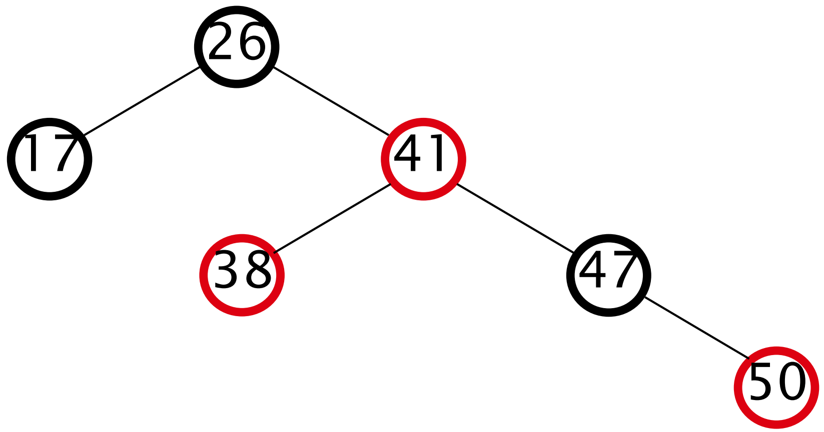
Properties affected by INSERT

1. Every node is either **red** or **black** OK!
2. The root is **black** If the root is changed $\Rightarrow$ May not OK
3. Every leaf (NIL) is **black** OK!
4. If a node is **red**, then both its children are **black**

If p(z) is red $\Rightarrow$ not OK
z and p(z) are both red

OK!

5. For each node, all paths from the node to descendant leaves contain the same number of black nodes



INSERT(T, z)

```
y ← NIL
x ← root[T]
```

- Initialize nodes x and y
- Throughout the algorithm y points to the parent of x

```
while x ≠ NIL
        do y ← x
            if key[z] < key[x]
              then x ← left[x]
              else x ← right[x]
```

- Go down the tree until reaching a leaf
- At that point y is the parent of the node to be inserted

```
p[z] ← y
```

Sets the parent of z to be y

```
if y = NIL
   then root[T] ← z
```

The tree was empty: set the new node to be the root

```
   else if key[z] < key[y]
          then left[y] ← z
          else right[y] ← z
```

Otherwise, set z to be the left or right child of y, depending on whether the inserted node is smaller or larger than y's key

```
left[z] ← NIL
right[z] ← NIL
color[z] ← RED
```

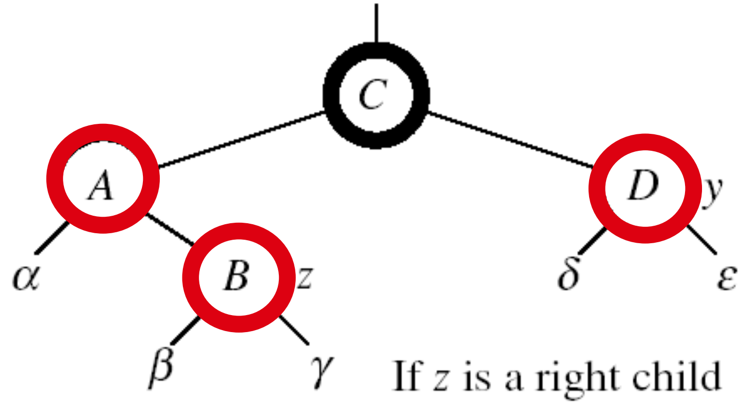
Set the fields of the newly added node

```
RB-INSERT-FIXUP(T, z)
```

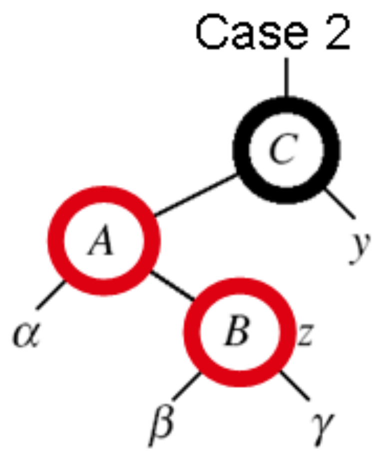
Fix any inconsistencies that could have been introduced by adding this new red node

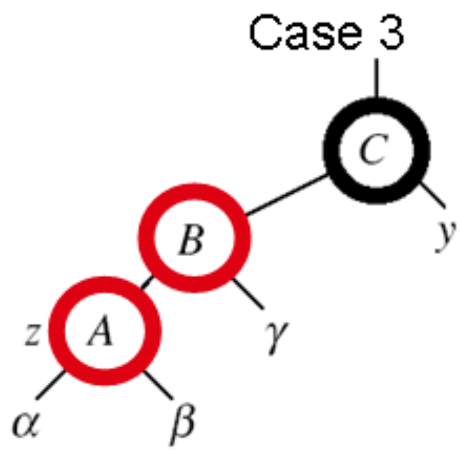
RB-Insert-Fixup(T, z)

- Case 1: z's uncle y is red
  - Solution: recolor



- Case 2: z's uncle y is black and z is a right child
  - Solution: double rotation
  - Can be transferred to Case 3
- Case 3: z's uncle y is black and z is a left child
  - Solution: single rotation





RB-Insert-Fixup(T, z)

The while loop repeats only when Case 1 is executed: O(logn) times

```
while z.p.color == red
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == red
            z.p.color = black                  // case 1
            y.color = black                    // case 1
            z.p.p.color = red                  // case 1
            z = z.p.p                          // case 1
        else if z == z.p.right
                z = z.p                        // case 2
                Left-rotation (T, z)           // case 2
            z.p.color = black                  // case 3
            z.p.p.color = red                  // case 3
            Right-rotation (T, z.p.p)          // case 3
    else (same as then clause with "right" and "left" exchanged)
T.root.color = black
```

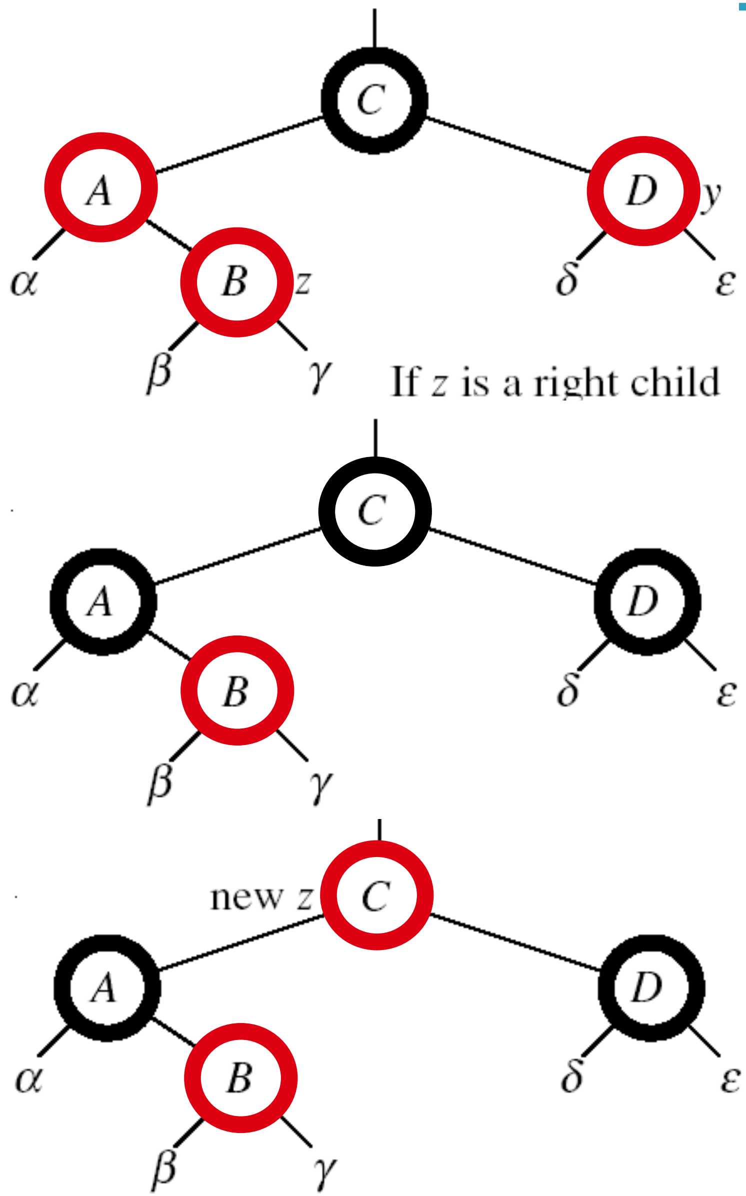
may just insert the root or the red violation reach root

INSERT: case 1

z's "uncle" (y) is red

Idea: (z is a right child)

- p[p[z]] (z's grandparent) must be black: p[z] is red
- Color p[z] black
- Color y black
- Color p[p[z]] red
- z = p[p[z]]
  - Push the "red" violation up the tree



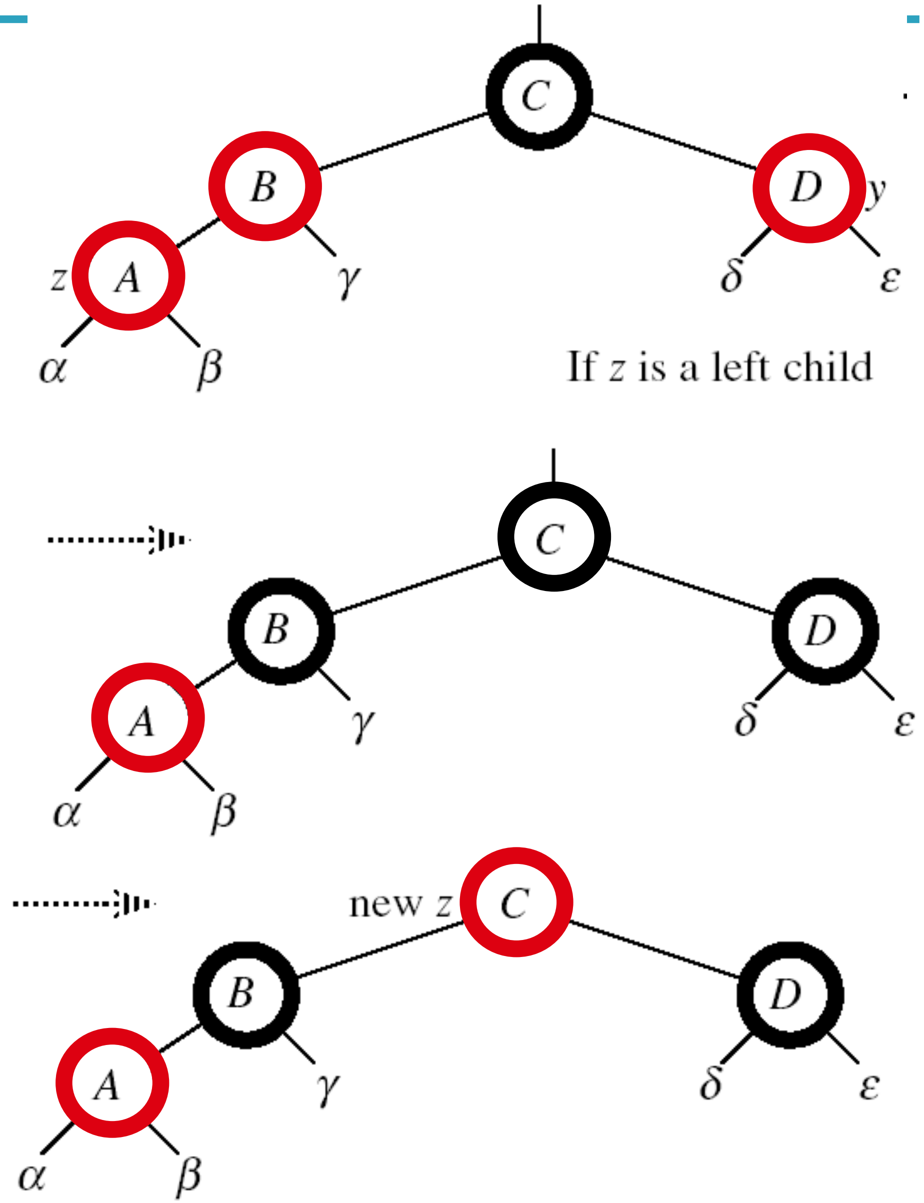
INSERT: case 1

z's "uncle" (y) is red

Idea: (z is a left child)

- p[p[z]] (z's grandparent) must be black: p[z] is red

- Color p[z] ← black
- Color y ← black
- Color p[p[z]] ← red
- z = p[p[z]]
  - Push the "red" violation up the tree



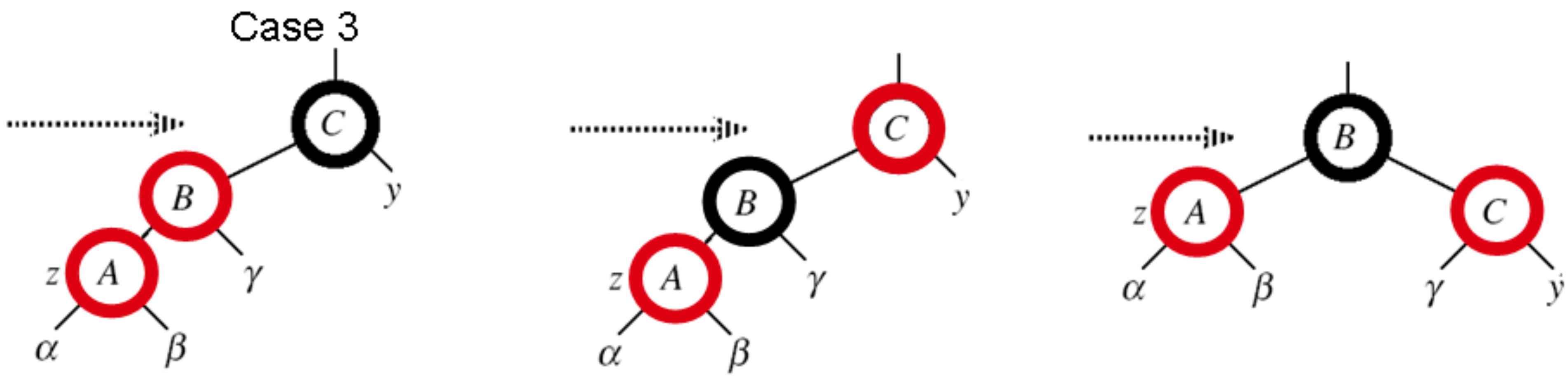
INSERT - case 3

Case 3:

- z's "uncle" (y) is black
- z is a left child

Idea:

- Color p[z] ← black
- Color p[p[z]] ← red
- RIGHT-ROTATE(T, p[p[z]])
- No longer have 2 reds in a row
- p[z] is now black



INSERT - case 2

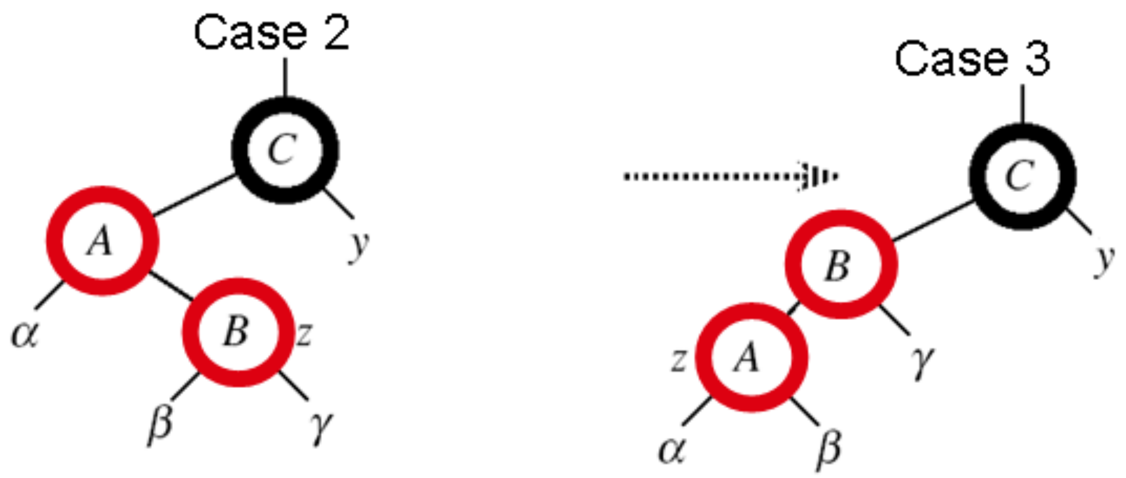
Case 2:

- z's "uncle" (y) is black
- z is a right child

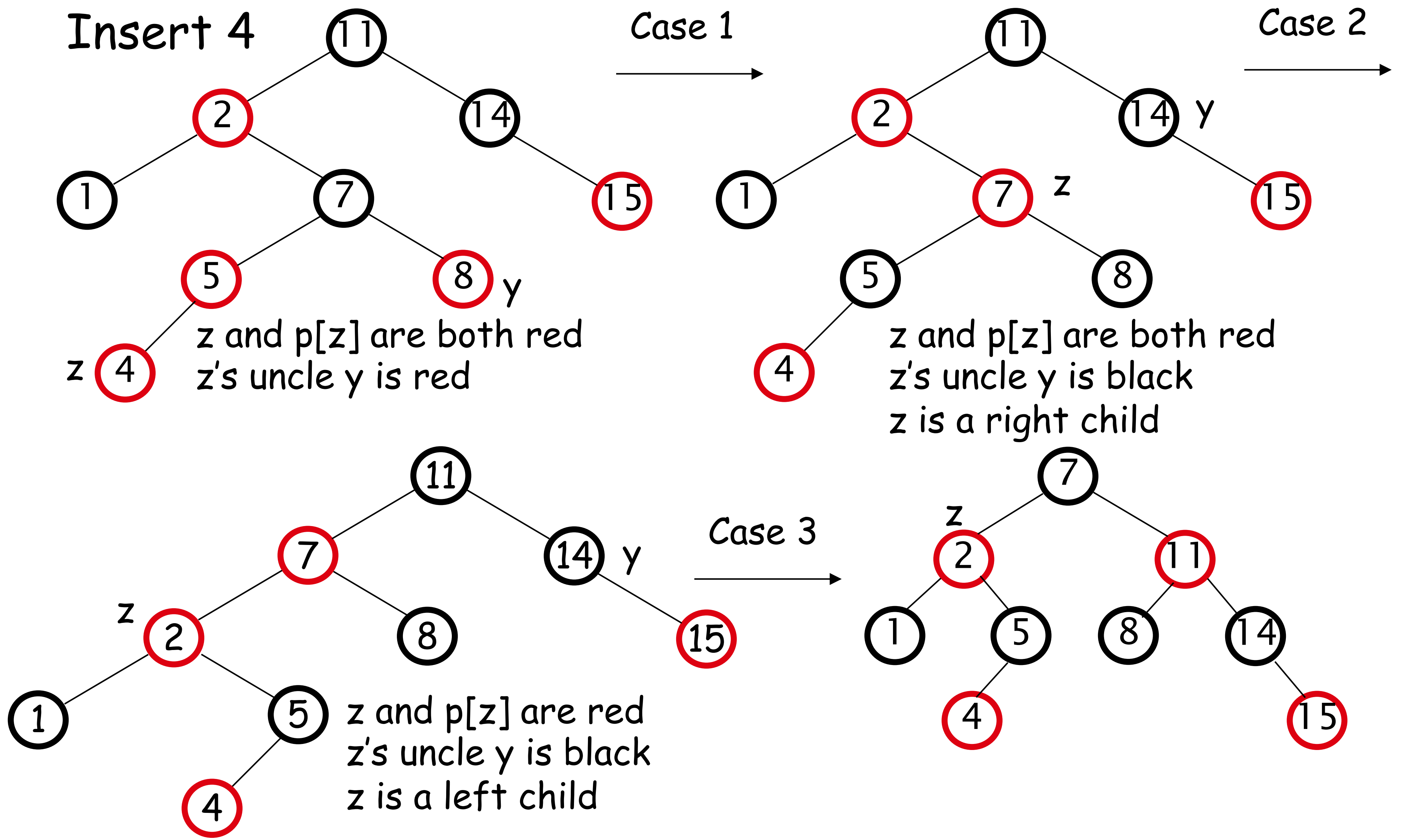
Idea:

- z ← p[z]
- LEFT-ROTATE(T, z)

⇒ now z is a left child, and both z and p[z] are red ⇒ case 3



Example


Insert 4
11
2
14
1
7
15
5
8
y
z
4
z and p[z] are both red
z's uncle y is red
Case 1
11
2
14
y
1
7
z
15
5
8
4
z and p[z] are both red
z's uncle y is black
z is a right child
Case 2
11
7
14
y
z
2
8
15
1
5
4
z and p[z] are red
z's uncle y is black
z is a left child
Case 3
7
z
2
11
1
5
8
14
4
15

Complexity analysis

- Time complexity of detailed steps
  - A red-black tree has $O(\log n)$ height
  - Search for insertion location takes $O(\log n)$ time
  - Addition to the node takes $O(1)$ time

  - The while loop will be executed at most $O(\log n)$ time
    - Each recoloring and each rotation take $O(1)$ time
    - Never performs more than two rotations, since the loop terminates if case 2 or case 3 is executed

  - Hence, an insertion in a red-black tree takes $O(\log n)$ time

What are the advantages of red-black tree over AVL tree?

Exercises

- What is the ratio between the longest path and the shortest path in a red-black tree?
  - The shortest path is at least bh(root)
  - The longest path is equal to h(root)
  - Since h(root) ≤ 2bh(root), the ratio is ≤ 2

- When we insert a node into a red-black tree, we initially set the color of the new node to red. Why didn't we choose to set the color to black?

Red-black trees: summary

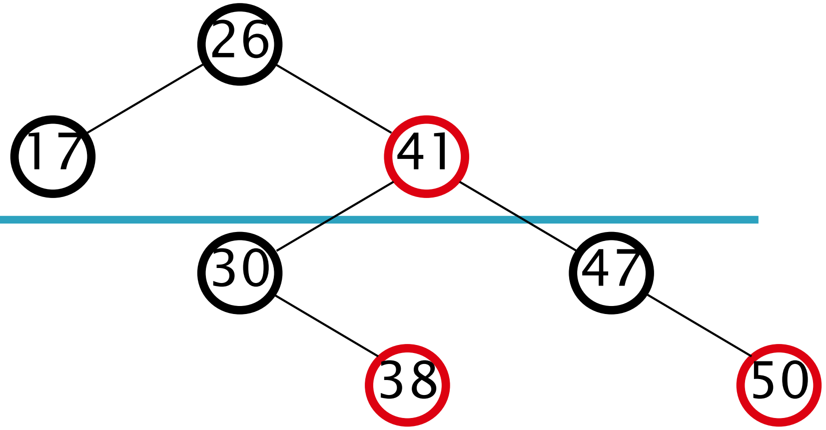
- Red-black trees guarantee that the height of the tree will be O(logn)

- Operations on red-black-trees:
  - SEARCH O(h)
  - PREDECESSOR O(h)
  - SUCCESOR O(h)
  - MINIMUM O(h)
  - MAXIMUM O(h)
  - INSERT O(h)
  - DELETE O(h)

Recommended reading

- Reading
  - Chapter 13, textbook
- Next lectures
  - Heap, chapters 6&12, textbook

DELETE operation

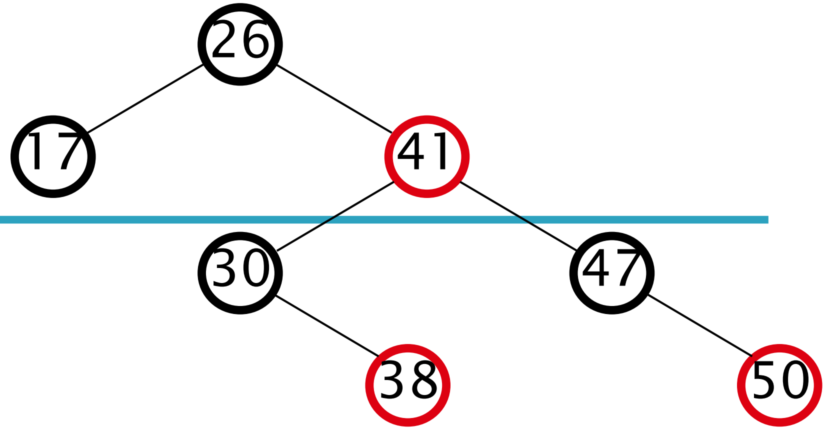


DELETE: the color of the node to be removed -- **red**

1. Every node is either **red** or **black** OK!
2. The root is **black** OK!
3. Every leaf (NIL) is **black** OK!
4. If a node is **red**, then both its children are **black** OK!
5. For each node, all paths from the node to descendant leaves contain the same number of **black** nodes OK!

Note: the deletion of a red node is the same as the deletion of a node in BST

DELETE operation



DELETE: the color of the node to be removed -- **Black**

1. Every node is either <u>red</u> or <u>**black**</u> OK!
2. The root is <u>**black**</u>
   Not OK! If removing the root and the child that replaces it is **red**
3. Every leaf (NIL) is <u>**black**</u> OK!
4. If a node is <u>**red**</u>, then both its children are <u>**black**</u>
   Not OK! Could create two red nodes in a row

   Not OK! Could change the black heights of some nodes
5. For each node, all paths from the node to descendant leaves contain the same number of <u>**black**</u> nodes

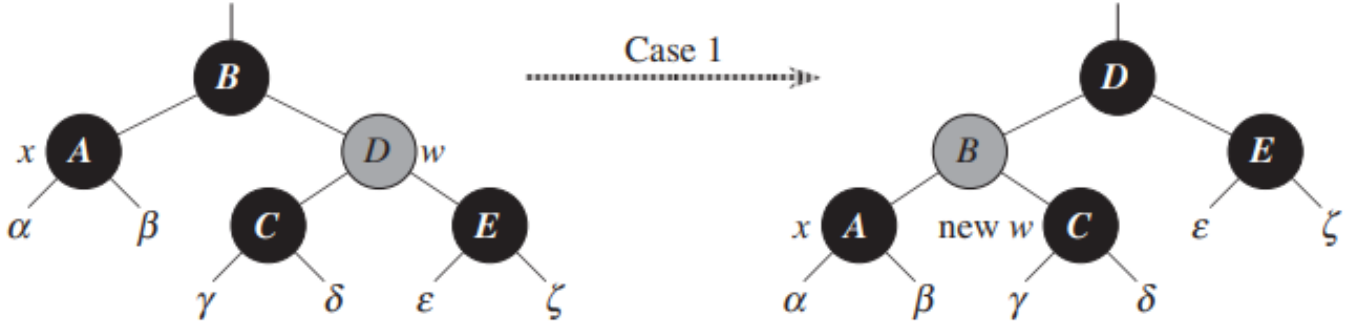
Deletion on red-black tree

- Similar to the deletion on BST, but need to use an auxiliary procedure RB-Delete-Fixup to restore the red-black tree properties

- Four different cases of RB-Delete-Fixup
  - Case 1: x's sibling w is red
  - Case 2:x's sibling w is black, and both of w's children are black
  - Case 3:x's sibling w is black, w's left child is red, and w's right child is black
  - Case 4: x's sibling w is black, and w's right child is red (left child either color)

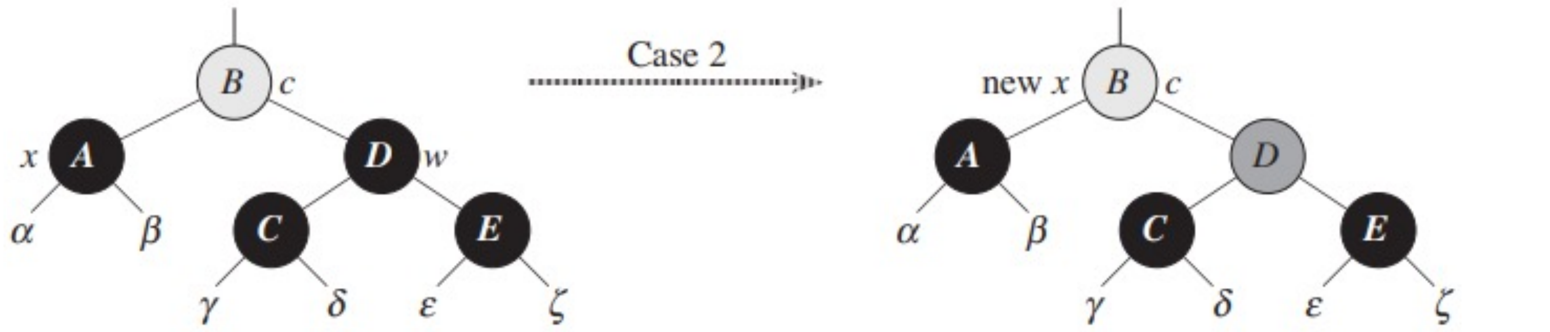
Case 1

- Case 1: x's sibling w is red
  - Solution: rotate and recolor



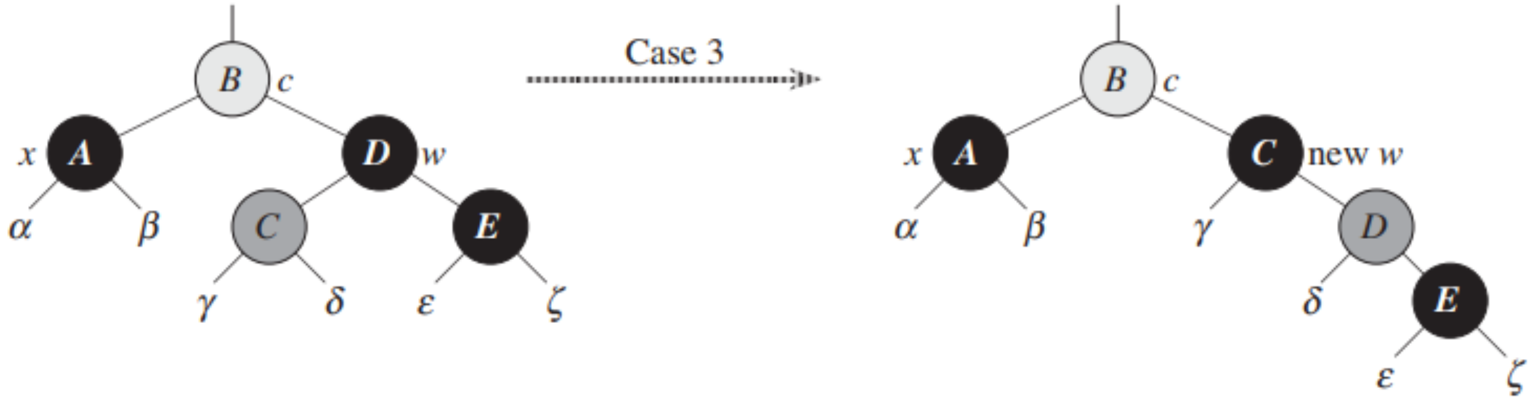
Case 2

- Case 2: x's sibling w is **black**, and both of w's children are **black**
  - Solution: recolor



Case 3

- Case 3: x's sibling w is **black**, w's left child is **red**, and w's right child is **black**



Case 4

- Case 4: x's sibling w is <u>**black**</u>, and w's right child is <u>**red**</u> (left child either color)