香港中文大學（深圳）
The Chinese University of Hong Kong

# CSC3100 Data Structures
# Lecture 12: More sorting algorithms

Yixiang Fang
School of Data Science (SDS)
The Chinese University of Hong Kong, Shenzhen

# Outline

- Comparison-based sorting algorithms
  - ShellSort
- Non-comparison-based sorting algorithms
  - CountingSort
  - BucketSort
  - RadixSort
- A summary of 10 classic sorting algorithms

# ShellSort

- Break the quadratic time barrier by comparing elements that are distant
- The distance between comparisons decreases as the algorithm runs until the last phase, in which adjacent elements are compared (*diminishing increment sort*)
- An increment sequence $h_1, h_2, h_3, \ldots, h_t$, used in reverse order with $h_1=1$

# ShellSort

- After a phase, with an increment $h_k$, A[i] <= A[i + $h_k$]
- All elements spaced $h_k$ apart are sorted (insertion sort)

- Example (1, 3, 5)
  - For each $h_k$, we need to sort $h_k$ subsequences

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Original** | **81** | **94** | **11** | **96** | **12** | **35** | **17** | **95** | **28** | **58** | **41** | **75** | **15** |
| **5-sorted** | **35** | **17** | **11** | **28** | **12** | **41** | **75** | **15** | **96** | **58** | **81** | **94** | **95** |
| **3-sorted** | **28** | **12** | **11** | **35** | **15** | **41** | **58** | **17** | **94** | **75** | **81** | **96** | **95** |
| **1-sorted** | **11** | **12** | **15** | **17** | **28** | **35** | **41** | **58** | **75** | **81** | **94** | **95** | **96** |

Standard insertion sort

# ShellSort with {1,2,4,8,…,n/2}

```
public static void shellSort(int[ ] a) {
    int j;
    for (int gap = a.length/2; gap > 0; gap /=2)
        for (int i = gap; i < a.length; i++) {
            int tmp = a[i];
            for (j = i; j >= gap && tmp < a[j-gap]; j-= gap)
                a[j] = a[j-gap];
            a[j] = tmp;
        }
}
```

# ShellSort

- Analysis of Shellsort
  - Very hard (average-case is a long-standing open problem)
  - Depend on the selection of an increment sequence
    - Difference sequences lead to different time cost!

  - Theorem: the worst-case running time of Shellsort, using some increment, is $\Theta(N^2)$
    - Put the largest N/2 numbers in the even positions

      e.g., 4,12,1,10,3,11,2,9
    - Use the increments {…, 8,4,2,1}
    - Before the last sort, the N/2 largest numbers are still in the even positions, e.g., 1,9,2,10,3,11,4,12
    - The numbers of inversions is 1+2+…+(N-1)/2 = $\Theta(N^2)$

# CountingSort

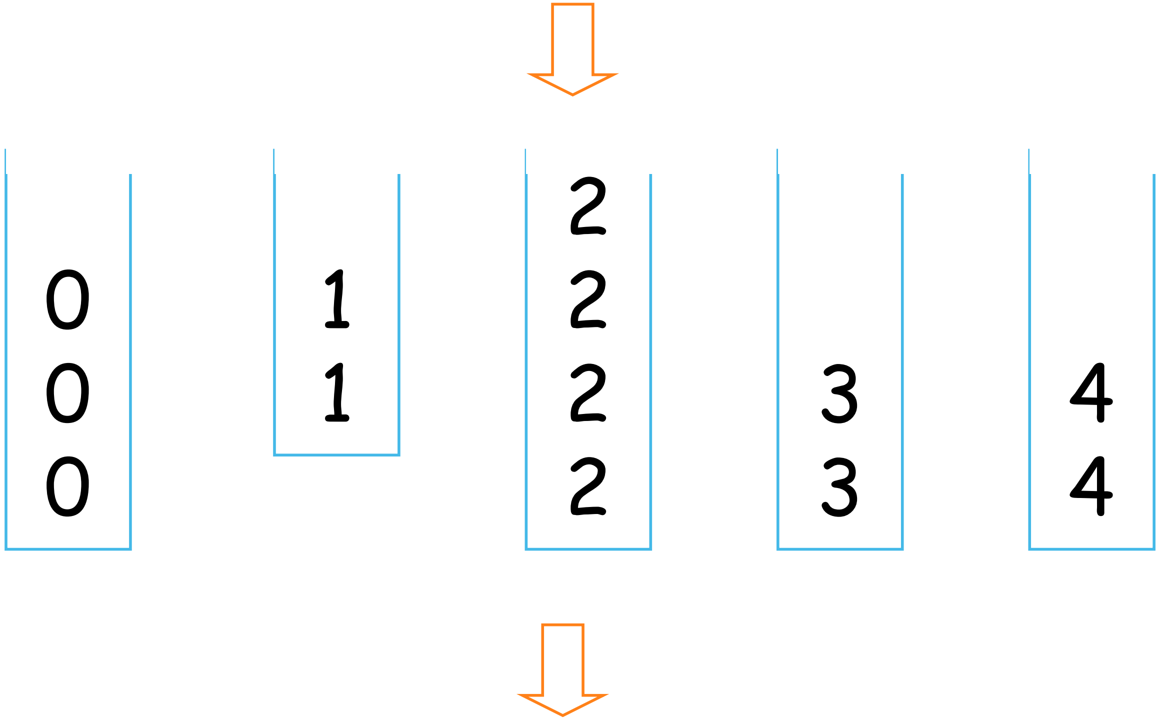
- Idea: suppose the values are integers in [0, m-1]

- Steps
  - Start with m empty buckets numbered 0 to m-1
  - Scan the list and place element s[i] in bucket s[i]
  - Output the buckets in order

- Will need an array of buckets, and the values to be sorted will be the indexes to the buckets
  - No comparisons will be necessary

# CountingSort

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 2 | 1 | 2 | 0 | 3 | 2 | 1 | 4 | 0 | 2 | 3 | 0 |




| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 4 | 4 |

# CountingSort

Algorithm CountingSort( S )
(values in S are between 0 and m-1 )

```
for j ← 0 to m-1 do     // initialize m buckets
   b[j] ← 0
for i ← 0 to n-1 do     // place elements in their appropriate buckets
   b[S[i]] ← b[S[i]]  + 1
i ← 0
for j ← 0 to m-1 do     // place elements in buckets
   for r ← 1 to b[j] do // back in S
         S[i] ← j
         i ← i + 1
```

# Exercise

- Use CountingSort to sort the following sequence of integer values

  4,3,2,1,5,2,4,1,2,0,4,2,0

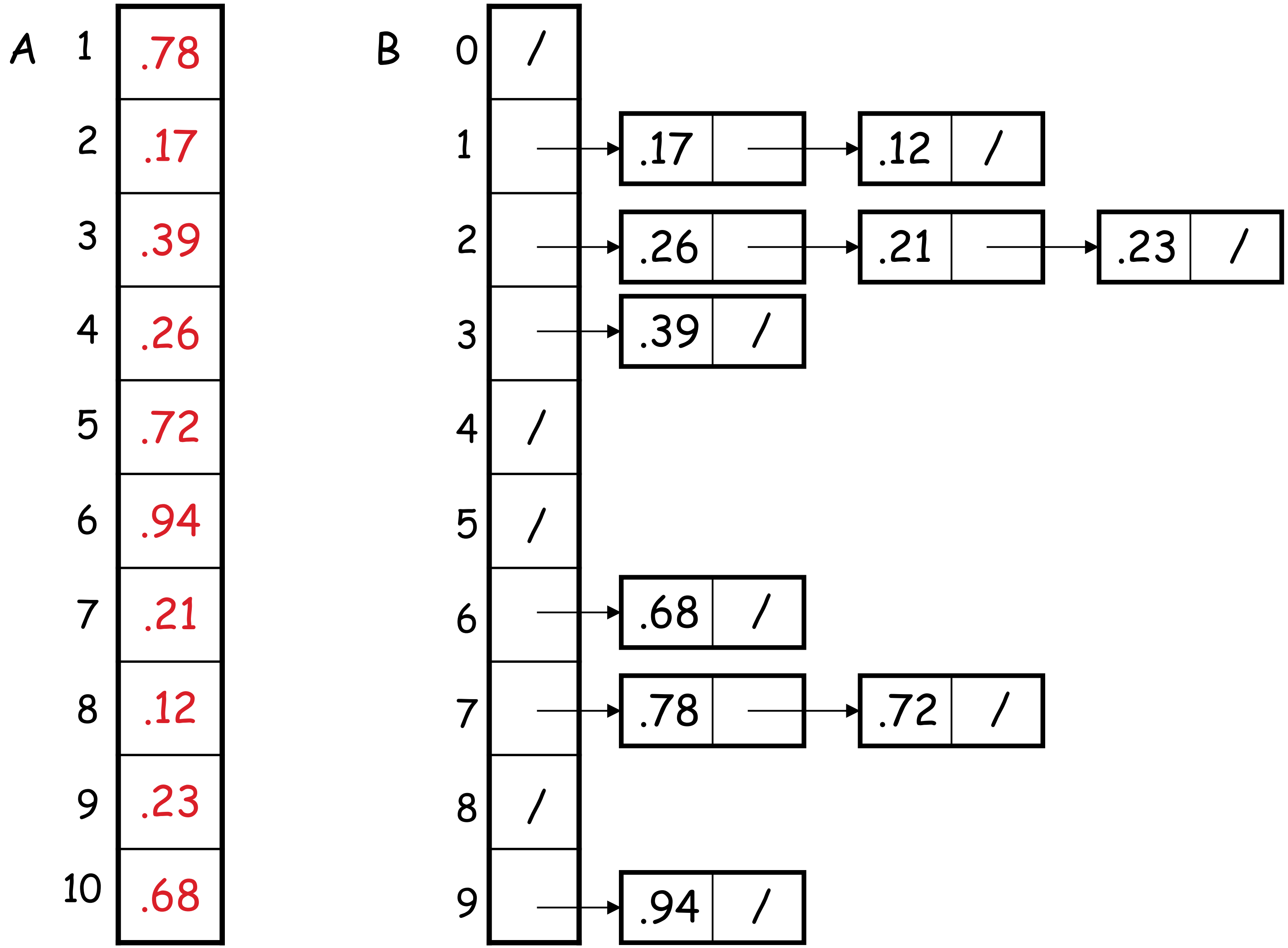
- How to process the case that the minimum value in the input sequence of integers is very large?

# BucketSort

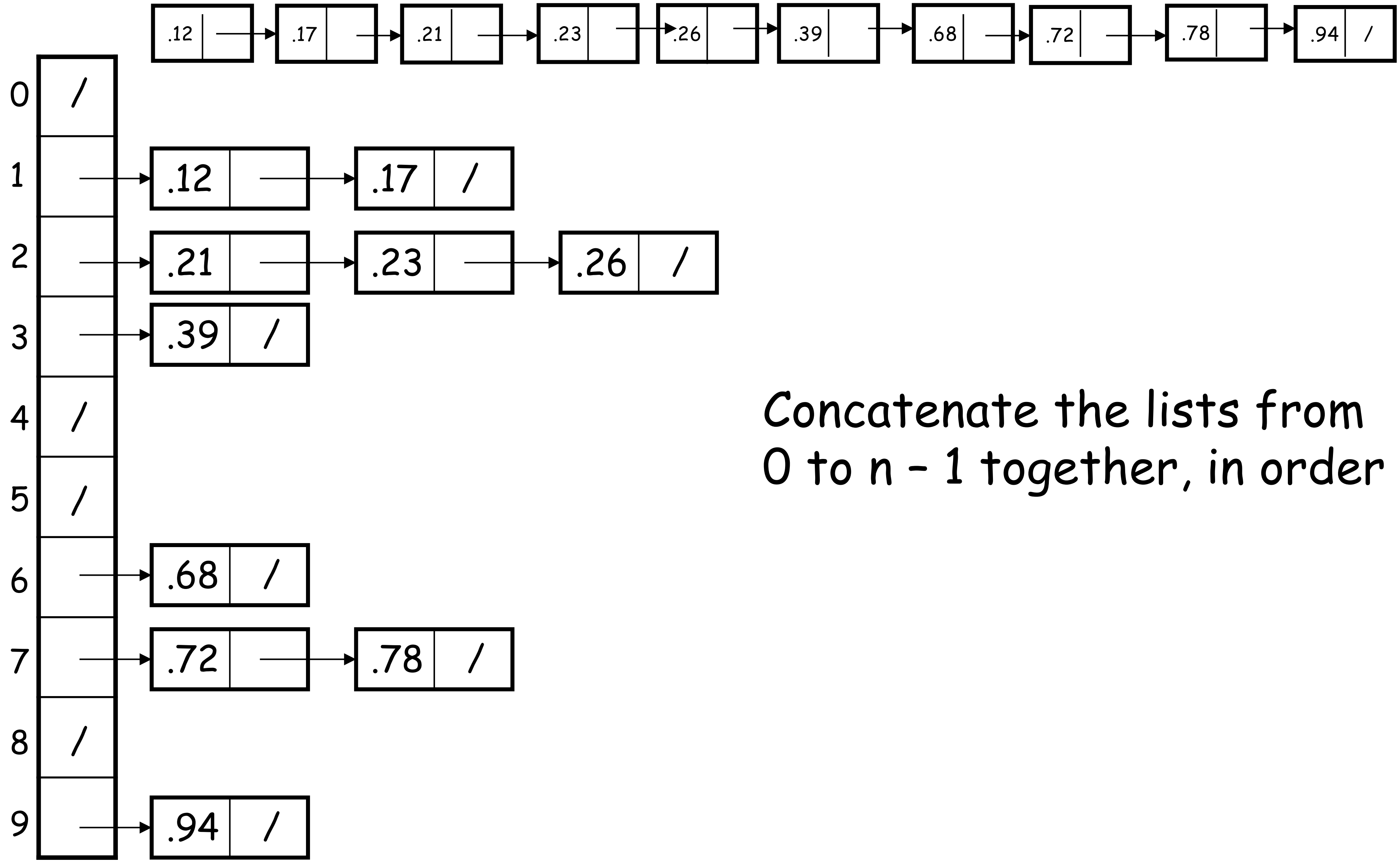
- Assumption:
  - The input is generated by a random process that distributes elements uniformly over [0, 1)

- Idea:
  - Divide [0, 1) into n equal-sized buckets
  - Distribute the n input values into the buckets
  - Sort each bucket (e.g., using QuickSort)
  - Go through buckets in order, listing elements in each one
  - Extra array: B[0 . . n - 1] of linked lists, each of which is initially empty

# Bucket Sort


A
1 .78
2 .17
3 .39
4 .26
5 .72
6 .94
7 .21
8 .12
9 .23
10 .68
B
0 /
1 .17 .12 /
2 .26 .21 .23 /
3 .39 /
4 /
5 /
6 .68 /
7 .78 .72 /
8 /
9 .94 /

# BucketSort




Concatenate the lists from 0 to n – 1 together, in order

# BucketSort

BUCKET-SORT(A, n)

**for** i ← 1 **to** n

    **do** insert A[i] into list B[⌊nA[i]⌋]

**for** i ← 0 **to** n - 1

    **do** sort list B[i] with QuickSort

concatenate lists B[0], B[1], . . . , B[n -1]

together in order

**return** the concatenated lists

$O(n)$ (first loop)

$\Theta(n)$ (second loop)

$O(n)$ (concatenate and return)

$\Theta(n)$ (total)

# Exercise

- Use BucketSort to sort the following sequence of real values

  0.81, 0.65, 0.91, 0.61, 0.55, 0.71

# RadixSort

- CountingSort is not efficient if m is large
- The idea of RadixSort:
  - Apply bucket sort on each digit (from Least Significant Digit to Most Significant Digit)
- A complication:
  - Just keeping the count is not enough
  - Need to keep the actual elements
  - Use a queue for each digit

# RadixSort: example (1/3)

- Input: 170, 045, 075, 090, 002, 024, 802, 066
- The first pass
  - Consider **the least significant digits** as keys and move the keys into their buckets

| | |
|---|---|
| 0 | 170, 090 |
| 1 | |
| 2 | 002, 802 |
| 3 | |
| 4 | 024 |
| 5 | 045, 075 |
| 6 | 066 |
| 7 | |
| 8 | |
| 9 | |

  - Output: 170, 090, 002, 802, 024, 045, 075, 066

# RadixSort: example (2/3)

- **The second pass**
  - Input: 170, 090, 002, 802, 024, 045, 075, 066
  - Consider **the second least significant digits** as keys and move the keys into their buckets

| | |
|---|---|
| 0 | 002, 802 |
| 1 | |
| 2 | 024 |
| 3 | |
| 4 | 045 |
| 5 | |
| 6 | 066 |
| 7 | 170, 075 |
| 8 | |
| 9 | 090 |

  - Output: 002, 802, 024, 045, 066, 170, 075, 090

# RadixSort: example (3/3)

- The **third** pass
  - Input: 002, 802, 024, 045, 066, 170, 075, 090
  - Consider **the third least significant digits** as keys and move the keys into their buckets

| | |
|---|---|
| 0 | 002, 024, 045, 066, 075, 090 |
| 1 | 170 |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 802 |
| 9 | |

  - Output: 002, 024, 045, 066, 075, 090, 170, 802 (Sorted)

# Another example

- Suppose we sort some 2-digit integers
- **Phase 1: Sort by the right digit (the least significant digit)**

| | | | | | |
|---|---|---|---|---|---|
| Initial array: | 25 | 32 | 93 | 22 | 34 |
| Sort by right digit: | 32 | 22 | 93 | 34 | 25 |

# Another example (cont.)

- Phase 2: Sort by the left digit (the second least significant digit)

| | | | | | |
|---|---|---|---|---|---|
| Initial array: | 25 | 32 | 93 | 22 | 34 |
| Sort by right digit: | 32 | 22 | 93 | 34 | 25 |
| Stable sort by left digit: | 22 | 25 | 32 | 34 | 93 |

# Codes (1/2)

```
// items to be sorted are in {0,…,10^d-1},
// i.e., the type of d-digit integers
void radixsort(int A[], int n, int d)
{
    int i;
    for (i=0; i<d; i++)
        bucketsort(A, n, i);
}

// To extract d-th digit of x
int digit(int x, int d)
{
    int i;
    for (i=0; i<d; i++)
        x /= 10; // integer division
    return x%10;
}
```

# Codes (2/2)

```
void bucketsort(int A[], int n, int d)
// stable-sort according to d-th digit
{
   int i, j;
   Queue *C = new Queue[10];
   for (i=0; i<10; i++) C[i].makeEmpty();
   for (i=0; i<n; i++)
      C[digit(A[i],d)].EnQueue(A[i]);
   for (i=0, j=0; i<10; i++)
      while (!C[i].empty())
      { // copy values from queues to A[]
         C[i].DeQueue(A[j]);
         j++;
      }
}
```

# Inductive proof that RadixSort works

- Keys: $k$-digit numbers, base $B$
  - (that wasn't hard!)

- Claim: after $i^{th}$ RadixSort, the least significant $i$ digits are sorted
  - Base case: $i = 0$, implying 0 digits are sorted
  - Inductive step: Assume for $i$, prove for $i+1$

    Consider two numbers: $X$, $Y$. Say $X_i$ is $i^{th}$ digit of $X$:
    - $X_{i+1} < Y_{i+1}$ then $i+1^{th}$ RadixSort will put them in order
    - $X_{i+1} > Y_{i+1}$ , same thing
    - $X_{i+1} = Y_{i+1}$ , order depends on last $i$ digits. Induction hypothesis says already sorted for these digits because RadixSort is **stable**

# Worst-case time complexity

- Assume k digits, each digit comes from {0,…,M-1}

- For each digit,
  - O(M) time to initialize M queues
  - O(n) time to distribute n numbers into M queues

- Total time = O(k(M+n))
- When k is constant and M = O(n), we can make RadixSort run in linear time, i.e., O(n)

# Exercises

- Use RadixSort to sort the following sequence

  123, 251, 369, 278, 451, 222

- Can we start from the most significant digit?

  Now let sort three 3-digit numbers?

  **4**78, **4**30, **3**56

  1st digit:

  **4**, **4**, **3** => **3**, **4**, **4** => **3**56, **4**78, **4**30

  2nd digit:

  **5**, **7**, **3** => **3**, **5**, **7** => 4**3**0, 3**5**6, 4**7**8

  3rd digit:

  **0**, **6**, **8** => **0**, **6**, **8** => 43**0**, 35**6**, 47**8**

# Exercises

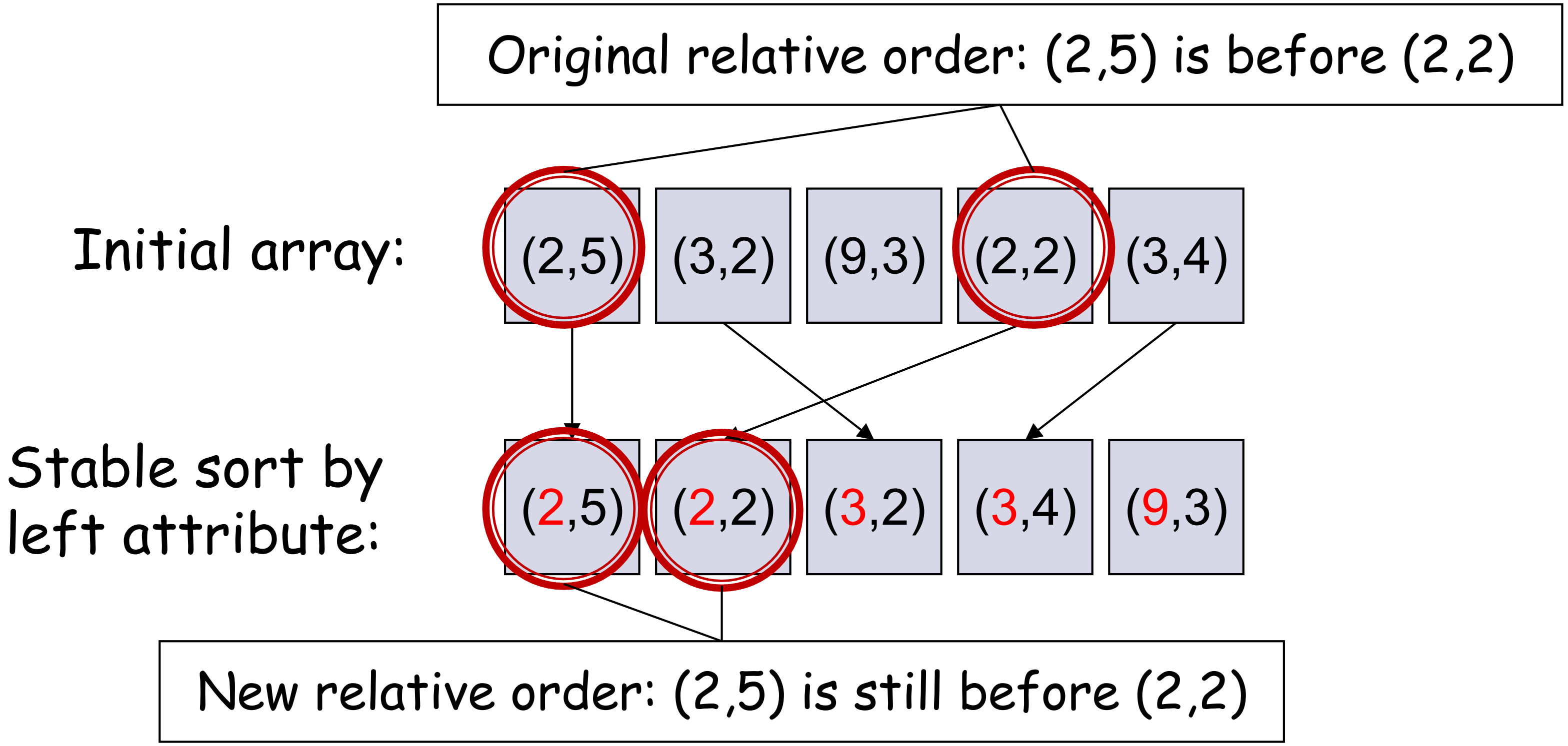
- Since RadixSort is faster than QuickSort, why is QuickSort still preferable in many cases?
  - Although RadixSort runs in $\Theta(n)$ while QuickSort $\Theta(n \lg n)$, QuickSort has a much smaller constant factor c
  - RadixSort requires extra memory, whereas QuickSort works in place

# 10 classic sorting algorithms

| Sorting algorithm | Stability | Time cost | | | Extra space cost |
|---|---|---|---|---|---|
| | | Best | Average | Worst | |
| Bubble sort | √ | O(n) | $O(n^2)$ | $O(n^2)$ | O(1) |
| Insertion sort | √ | O(n) | $O(n^2)$ | $O(n^2)$ | O(1) |
| Selection sort | × | O(n) | $O(n^2)$ | $O(n^2)$ | O(1) |
| MergeSort | √ | O(nlogn) | O(nlogn) | O(nlogn) | O(n) |
| HeapSort | × | O(nlogn) | O(nlogn) | O(nlogn) | O(1) |
| QuickSort | × | O(nlogn) | O(nlogn) | $O(n^2)$ | O(1) |
| ShellSort | × | O(n) | $O(n^{1.3})$ | $O(n^2)$ | O(1) |
| CountingSort | √ | O(n+k) | O(n+k) | O(n+k) | O(k) |
| BucketSort | √ | O(n) | O(n+k) | $O(n^2)$ | O(k) |
| RadixSort | √ | O(nk) | O(nk) | O(nk) | O(n) |

# Concept: stable sorting algorithm

- Definition: A *stable* sorting algorithm is one that preserves the original relative order of elements with equal key
  - E.g., suppose the left attribute is the key attribute



# Recommended reading

- Reading this week
  - Chapter 8, textbook

- Next lecture
  - Tree data structure: Chapter 12