The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Andre Milzarek · Zizhuo Wang · Haoxiang Yang · Spring Semester 2022

MAT 3007 – Optimization

Exercise Sheet 7

Problem 1 (Convex Sets): (approx. *25* pts)

In this exercise, we study convexity of various sets.

a) Verify whether the following sets are convex or not and explain your answer!

$$\Omega_1 = \{x \in \mathbb{R}^n : \alpha \le (a^\top x)^3 \le \beta\}, \quad \alpha, \beta \in \mathbb{R},\ \alpha \le \beta,\ a \in \mathbb{R}^n,$$
$$\Omega_2 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x^\top x \le t^2\}.$$

b) Show that the hyperbolic set $\{x \in \mathbb{R}^2_+ : x_1 x_2 \ge 1\}$ is convex, where $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x \ge 0\}$.

Hint: Rewrite the condition "$x_1 x_2 \ge 1$" in a suitable way.

c) Decide whether the following statements are true or false. Explain your answer and either present a proof / verification or a counter-example.

- The intersection of two convex sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ is always a convex set.
- Let $\Omega \subset \mathbb{R}^n$ be a convex set and suppose that the set $S := \{(x,t) \in \Omega \times \mathbb{R} : f(x) \le t\} \subset \mathbb{R}^n \times \mathbb{R}$ is convex. Then, $f : \Omega \to \mathbb{R}$ is a convex function.

Problem 2 (Convex Compositions): (approx. *20* pts)

Either prove or find a counterexample for each of the following statements (you can assume that all functions are twice continuously differentiable if needed):

a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are convex, then the composition $f \circ g : \mathbb{R}^n \to \mathbb{R}$, $(f \circ g)(x) = f(g(x))$ is convex.

b) Let $\Omega \subset \mathbb{R}^n$ be a convex set and suppose that $g : \Omega \to \mathbb{R}$ is convex and $f : I \to \mathbb{R}$ is convex and nondecreasing where $I \supseteq g(\Omega)$ is an interval containing $g(\Omega)$. Then, $f \circ g$ is convex.

c) If $f : \mathbb{R} \to \mathbb{R}$ is increasing, then $x \mapsto |f(x)|$ is a convex function on $\mathbb{R}$.

Problem 3 (Convex Functions): (approx. *30* pts)

In this exercise, convexity properties of different functions are investigated.

a) Let $r : \mathbb{R}^n \to \mathbb{R}$ be a norm on $\mathbb{R}^n$. Show that r is a convex function.

b) Verify that the following functions are convex over the specified domain:

- $f : \mathbb{R}_{++} \to \mathbb{R}$, $f(x) := \sqrt{1 + x^{-2}}$, where $\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$.
- $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \frac{1}{2}\|Ax - b\|^2 + \mu\|Lx\|_\infty$, where $A \in \mathbb{R}^{m\times n}$, $L \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^m$, and $\mu > 0$ are given and $\|y\|_\infty := \max_{i=1,\dots,p} |y_i|$, $y \in \mathbb{R}^p$.
- $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x,y) := \frac{\lambda}{2}\|x\|^2 + \sum_{i=1}^m \max\{0, 1 - b_i(a_i^\top x + y)\}$, where $a_i \in \mathbb{R}^n$ and $b_i \in \{-1, 1\}$ are given data points for all $i = 1, ..., m$ and $\lambda > 0$ is a parameter.

c) Let us set $f(x) = \|x\|^3$ and define $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) := \max_{y\in\mathbb{R}^n} y^\top x - f(y)$.

Show that g is well-defined, i.e, $g(x)$ exists for all x and satisfies $g(x) < \infty$. Calculate $g(x)$ explicitly and verify that the function g is convex.

Problem 4 (Geometric Programming): (approx. *25* pts)

In this exercise, we discuss a class of nonconvex geometric programs that can be reformulated as convex optimization problems.

a) Let $a \in \mathbb{R}^n$ be given with $\sum_{i=1}^n a_i = 1$ and $a \geq 0$. Show that the matrix $A := \mathrm{diag}(a) - aa^\top$ is positive semidefinite. (Here, $\mathrm{diag}(a)$ is a $n \times n$ diagonal matrix with a on its diagonal).

Hint: The Cauchy-Schwarz inequality $x^\top y \leq \|x\|\|y\|$, $x, y \in \mathbb{R}^n$, can be helpful.

b) We define $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \log(\sum_{i=1}^n \exp(x_i))$. Show that f is a convex function.

c) Convert the following optimization problem into a convex problem:

$$
\begin{array}{ll}
\min_{x\in\mathbb{R}^3} & \max\left\{\frac{x_1}{x_2}, \frac{\sqrt{x_3}}{x_2}\right\} \\
\text{subject to} & x_1^2 + \frac{2x_2}{x_3} \leq \sqrt{x_2}, \\
& \frac{x_1}{x_2} \geq x_3^2, \\
& x_1, x_2, x_3 \geq 0.
\end{array} \tag{1}
$$

Hint: Substitute the variables x_i in an appropriate way and apply the result of part b).

d) Use `CVX` (in `MATLAB` or `Python`) to solve problem (1).

Sheet 7 is due on **Apr, 22nd**. Submit your solutions before **Apr, 22nd**, **12:00 pm (noon)**.