The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Prof. Andre Milzarek · Prof. Zizhuo Wang · Prof. Haoxiang Yang
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MAT 3007 – Optimization

Final Exam – Sample

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

Good Luck! Viel Glück!

Exercise 1 (KKT Conditions and Constrained Problems): (24 points)
Consider the nonlinear optimization problem

$$\min_{x\in\mathbb{R}^2} f(x) := \ln(1+x_2) + x_1x_2 - x_1^2x_2^2 \qquad \text{s.t.} \qquad g(x) \le 0, \tag{1}$$

where the constraint function $g : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$g_1(x) := x_1^2 + (x_2-1)^2 - 1, \quad g_2(x) := 1 - (x_1+1)^2 - x_2^2, \quad g_3(x) := x_1 - x_2.$$

Let us further set $\bar{x} := (0,0)^\top$. (*Here,* $\ln$ *denotes the natural logarithm to the base* e).

a) Sketch the feasible set $\Omega := \{x \in \mathbb{R}^2 : g(x) \le 0\}$.

b) Is problem (1) convex? Explain your answer!

c) Determine the active set $\mathcal{A}(\bar{x})$.

d) Is $\bar{x}$ a KKT point of problem (1)? If yes, find all corresponding Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^3$ such that the pair $(\bar{x}, \bar{\lambda})$ is a KKT point of (1). Is the multiplier $\bar{\lambda}$ unique?

Exercise 2 (Convexity): (16 points)
Consider the following tasks:

a) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$ be given. Show that $\Omega := \{x \in \mathbb{R}^n : \|Ax\|^2 - b^\top x \le 1\}$ is a convex set.

b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) := (1+x^2+y^2)\ln(1+x^2+y^2)$.

   Is the mapping f convex on $\mathbb{R}^2$? Explain your answer!

Exercise 3 (Gradient Descent): (10 points)
We consider the unconstrained minimization problem

$$\min_{x\in\mathbb{R}^2} f(x) := \frac{1}{2}x_1^2x_2 + x_1^2x_2^2 - x_2(x_1-2) + \frac{1}{4}x_2^4.$$

The gradient of f is given by (*you don't need to verify this*):

$$\nabla f(x) = \begin{pmatrix} x_1x_2 + 2x_1x_2^2 - x_2 \\ \frac{1}{2}x_1^2 + 2x_1^2x_2 - (x_1-2) + x_2^3 \end{pmatrix}.$$

a) We want to apply the gradient descent method with backtracking to solve $\min_x f(x)$. We choose the initial point x^0 and the Armijo parameter as follows:

$$x^0 = (2, -0.5)^\top, \quad \gamma = 0.1, \quad \sigma = 0.5.$$

   We now select $d_{\mathsf{g}}^0 = -\nabla f(x^0)$ and set $\phi(\alpha) := f(x^0 + \alpha d_{\mathsf{g}}^0) - f(x^0)$. Compute the gradient iterate x_{g}^1 and the stepsize α_0 using backtracking and the plot shown in Figure 1.

b) Let $A \in \mathbb{R}^{m\times 2}$, $m \in \mathbb{N}$, and $x \in \mathbb{R}^2$ be a given with $\nabla f(x) \ne 0$. Verify whether the direction $d = -\nabla f(x) - A^\top A\nabla f(x)$ is a descent direction of f at x.

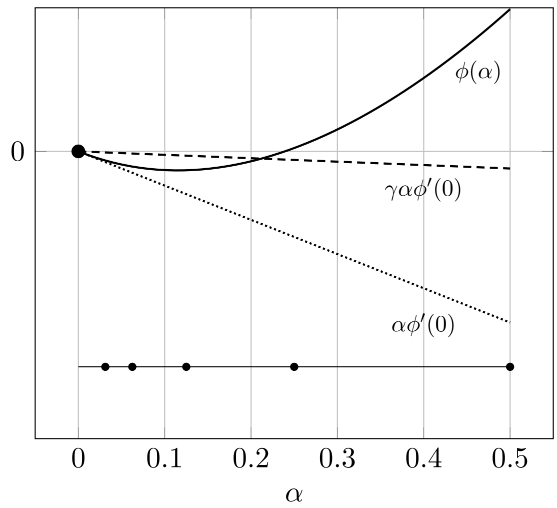

Figure 1: Plot of the functions $\alpha \mapsto \phi(\alpha)$, $\alpha \mapsto \alpha\phi'(0)$, and $\alpha \mapsto \gamma\alpha\phi'(0)$ for $\alpha \in [0, \frac{1}{2}]$.

Exercise 4 (Wine Cellar): (20 points)

Lady Dimitrescu has a wine cellar with three separate temperature controlled areas.

The first area has a temperature of $8°C$ and can store up to 35 bottles (this area is mainly used for sparkling wine and champagne). The second area has a temperature of 10–12$°C$ and can store up to 55 bottles. In the third area, 110 bottles can be stored under a temperature of 16–18$°C$.

Lady Dimitrescu plans to invest in different sorts of wine and to resell the wine after three years of storage and maturation at a higher price. The following table summarizes the data and requirements of the different purchasable sorts of wine.

	Dom Perignon (Champagne)	D.R.M. Grand Cru (White Wine)	Opus One (Red Wine)
Temperature	$< 9°C$	8–12$°C$	$> 15°C$
Price per Bottle	19	14	31
Price per Crate (6 Bottles)	102	80	175
Reselling Price per Bottle (After three years)	20	14.7	33.5

Table 1: Wine Data (prices are in 100 RMB per item/bottle/crate)

Each of the areas also has different fixed operating costs per year in case it is used. The costs are summarized in Table 2.

	Area 1	Area 2	Area 3
Operating Costs (per Year)	6	3	0.5

Table 2: Operating Costs of the Wine Cellar (costs are in 100 RMB)

Formulate an integer program to determine the number of bottles of the different sorts of wine to be purchased so as to maximize the total profit (revenue minus costs) after three years when the wine is stored properly under the correct temperature. You can assume that all stored bottles can be sold after three years (using the prices in Table 1).

Exercise 5 (True and False): (15 points)

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of *True* or *False*, will be graded. Explanations are not required and will not be read.

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth (twice continuously differentiable) and convex function. We want to apply the globalized Newton method to solve the problem $\min_x f(x)$. Let us assume that the method generates a sequence $\{x^k\}$ with $\nabla f(x^k) \neq 0$ for all k. Then, for every iteration k, the Newton direction is well-defined and a descent direction of f at x^k.

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded from below and let x^* be a stationary point of f. Then, x^* is a global minimizer of f.

c) Consider the nonlinear program $\min_{x\in\Omega} f(x)$ with linear constraints $\Omega := \{x \in \mathbb{R}^n : Ax \leq b,\, Cx = d\}$. We assume that $f : \mathbb{R}^n \to \mathbb{R}$ is convex on $\mathbb{R}^n$ and that the set Ω is nonempty. Then, x^* is a global minimizer of this problem, if and only if x^* satisfies the KKT conditions.

d) We use the branch-and-bound method to solve an integer problem (maximization). Suppose we have split the problem into two branches $(S1)$ and $(S2)$ and we continue branching on the subproblem $(S1)$. The method stops branching within $(S1)$ as soon as we recover a feasible integer solution of one of the subproblems. In such a case, the branch-and-bound process continues with the branch $(S2)$.

e) The matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

is totally unimodular.

Exercise 6 (Projection Onto an Integer Set): (15 points)

We consider the integer optimization problem

$$\text{minimize}_x \ \frac{1}{2}\|x - z\|^2 \quad \text{subject to} \quad \mathbb{1}^\top x = 1, \quad x_i \in \{-1, 1\}, \quad \forall\, i, \tag{2}$$

where $z \in \mathbb{R}^n$ is given, $\mathbb{1} \in \mathbb{R}^n$ is the vector of all ones, and n is assumed to be odd.

a) Verify that the problem (2) can be equivalently written as a special linear problem with binary constraints, i.e., as a problem of the form

$$\min_x \ c^\top x \quad \text{subject to} \quad a^\top x = b, \quad x_i \in \{0, 1\}, \quad \forall\, i,$$

with suitable choices of $a \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, and $c \in \mathbb{R}^n$.

Hint: The identity $\|a + b\|^2 = \|a\|^2 + 2a^\top b + \|b\|^2$, $a, b \in \mathbb{R}^n$, can be helpful.

b) Show that there is no integrality gap between the binary problem derived in part a) and its LP relaxation.