IEOR E4004 HW3 Mark Ma km4054

Question 1

(a) Sets:

1. W: set of workers, indexed by w, where $w = 1, 2 \cdots 100$.
2. D: set of departments, indexed by d (battery, body, assembly, paint and quality control).
3. S: set of shifts, indexed by s (morning, afternoon, night).
4. T: set of days. indexed by t (Mon - Sun)

Parameters

1. $a_{w,d,s,t}$: availability of worker w for department d on shift s and day t (1 if available, 0 otherwise).
2. $P_{w,d,s,t}$: preference score of worker w for department d on shift s and day t. (1 to 10)
3. $e_{w,d,s,t}$: effective score of worker w for department d on shift s and day t. (1 to 10)
4. $min\_staff_{d,s,t}$: Minimum staffing requirement for department d on shift s and day t.
5. $max\_staff_{d,s,t}$: Maximum staffing capacity for department d on shift s and day t.

Decision Variables

$x_{w,d,s,t}$: Binary variable indicating whether worker w is assigned to department d for shift s on day t. (1 if assigned, 0 otherwise)

Objectives

$$\max \sum_{w\in W}\sum_{d\in D}\sum_{s\in S}\sum_{t\in T} P_{w,d,s,t} \times e_{w,d,s,t} \times x_{w,d,s,t}$$

Constraints

1. Single shift per day: $\sum_{d\in D}\sum_{s\in S} x_{w,d,s,t} \le 1$, $\forall w \in W$, $\forall t \in T$
2. Weekly work limit: $\sum_{d\in D}\sum_{s\in S}\sum_{t\in T} x_{w,d,s,t} \le 5$, $\forall w \in W$
3. Availability constraint: $x_{w,d,s,t} \le a_{w,d,s,t}$, where $a_{w,d,s,t} \in \{0,1\}$ $\forall w \in W$, $\forall d \in D$, $\forall s \in S$, $\forall t \in T$.

4. Staffing requirements:

$$\sum_{w\in W} x_{w,d,s,t} \geq min\_staff_{d,s,t} \quad \forall d\in D,\ \forall s\in S,\ \forall t\in T$$

$$\sum_{w\in W} x_{w,d,s,t} \leq max\_staff_{d,s,t} \quad \forall d\in D,\ \forall s\in S,\ \forall t\in T$$

5. Binary decision variable: $x_{w,d,s,t} \in \{0,1\}$

$$\forall w\in W,\ \forall d\in D,\ \forall s\in S,\ \forall t\in T.$$

(b) From above, the problem can be formulated as:

$$\max \sum_{w\in W}\sum_{d\in D}\sum_{s\in S}\sum_{t\in T} P_{w,d,s,t} \times e_{w,d,s,t} \times x_{w,d,s,t}$$

$$\text{s.t.} \quad \sum_{d\in D}\sum_{s\in S} x_{w,d,s,t} \leq 1, \quad \forall w\in W,\ \forall t\in T$$

$$\sum_{d\in D}\sum_{s\in S}\sum_{t\in T} x_{w,d,s,t} \leq 5, \quad \forall w\in W$$

$$x_{w,d,s,t} \leq a_{w,d,s,t}, \quad \forall w\in W,\ \forall d\in D,\ \forall s\in S,\ \forall t\in T.$$

$$\sum_{w\in W} x_{w,d,s,t} \geq min\_staff_{d,s,t}, \quad \forall d\in D,\ \forall s\in S,\ \forall t\in T$$

$$\sum_{w\in W} x_{w,d,s,t} \leq max\_staff_{d,s,t}, \quad \forall d\in D,\ \forall s\in S,\ \forall t\in T$$

$$x_{w,d,s,t} \in \{0,1\}, \quad \forall w\in W,\ \forall d\in D,\ \forall s\in S,\ \forall t\in T.$$

(c) The result is shown below, with codes attached in appendix.

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Solution count 3: 30953 27166 12303

Optimal solution found (tolerance 1.00e-04)
Best objective 3.095300000000e+04, best bound 3.095300000000e+04, gap 0.0000%
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	Worker_ID	Department	Shift	Day
0	1	Body	Afternoon	Sun
1	1	Assembly	Afternoon	Mon
2	1	Assembly	Afternoon	Tue
3	1	Paint	Afternoon	Wed
4	1	Quality	Morning	Thur
...	...	...	...	...
495	100	Assembly	Afternoon	Fri
496	100	Paint	Afternoon	Thur
497	100	Paint	Afternoon	Sat
498	100	Quality	Night	Wed
499	100	Quality	Night	Sun

500 rows × 4 columns

From the result, the maximum total preference-adjusted effectiveness should be 30953, and a snapshot of the optimal solution is shown on the left.

Question 2

For this question, we introduce a binary variable $y \in \{0,1\}$

① If $y=0$, the first constraint is satisfied

② If $y=1$, the second constraint is satisfied

Then, we introduce a large positive number M.

By the big-M method, in order to satisfy at least one constraint, we have the formulation:

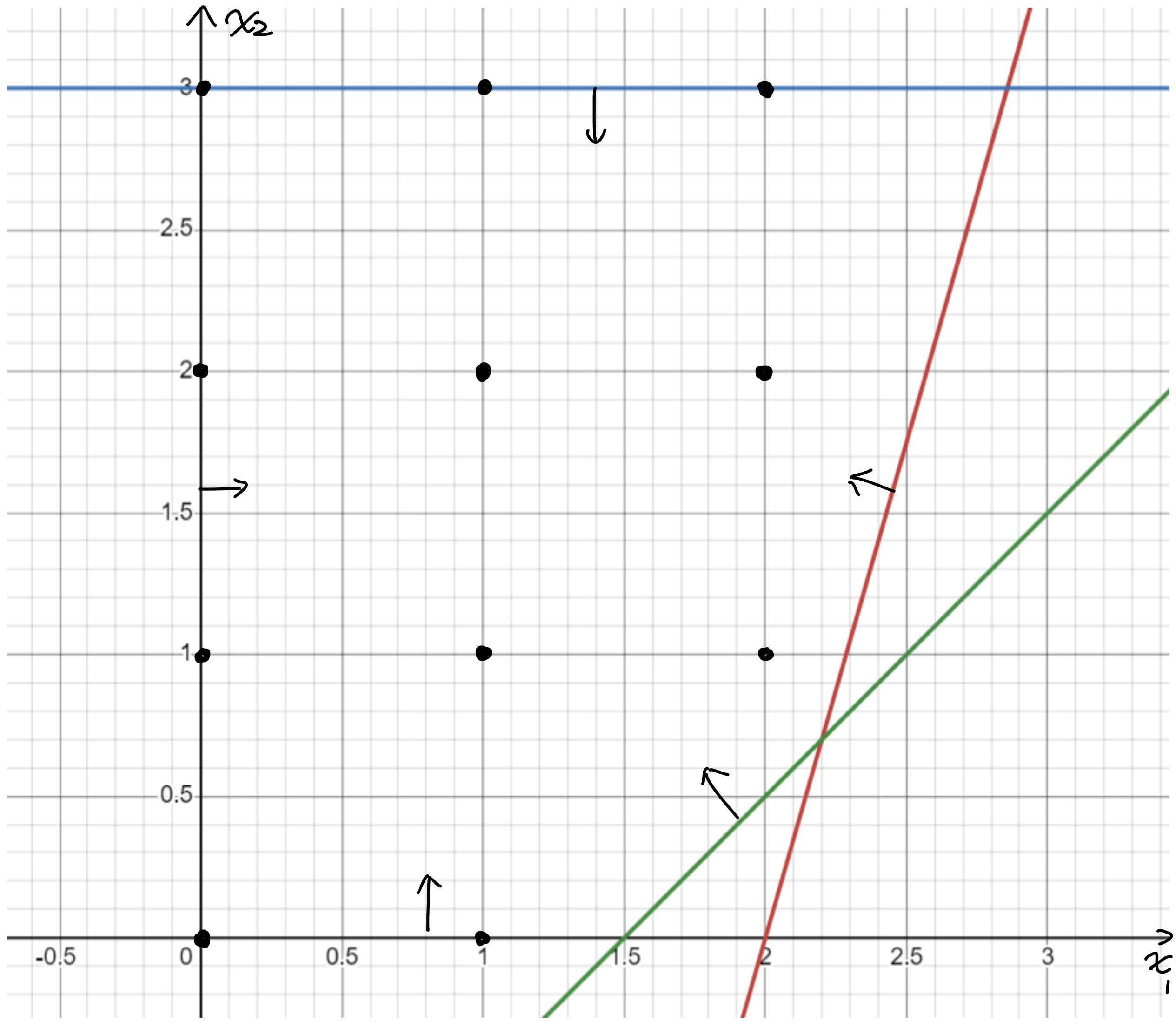
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \geq b_1 - M \cdot y$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \geq b_2 - M(1-y)$$

Therefore we satisfy the requirement.

Question 3

We can draw the feasible region as there're 2 variables below:



The feasible region is the black dots noted in the graph, since it is an integer programming problem. (In total 11 dots).

Question 4

Original LP

$\max \; z = 4x_1 - x_2$

s.t. $7x_1 - x_2 \le 14$

$x_2 \le 3$

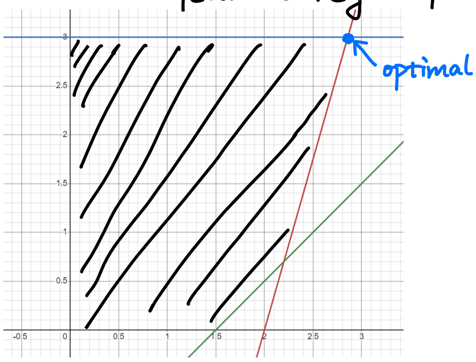
$2x_1 - 2x_2 \le 3$

$x_1, x_2 \ge 0$

$\Rightarrow z = 8.429$

$(x_1, x_2) = (2.857, 3)$

feasible region for LP



Sub-problem 1

$\max \; z = 4x_1 - x_2$

s.t. $7x_1 - x_2 \le 14$

$x_2 \le 3$

$2x_1 - 2x_2 \le 3$

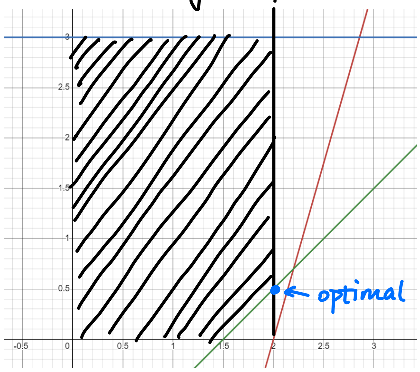
$x_1, x_2 \ge 0$

$x_1 \le 2$

$\Rightarrow z = 7.5$

$(x_1, x_2) = (2, 0.5)$

Feasible Region for sub 1



Sub-problem 2

$\max \; z = 4x_1 - x_2$

s.t. $7x_1 - x_2 \le 14$

$x_2 \le 3$

$2x_1 - 2x_2 \le 3$

$x_1, x_2 \ge 0$

$x_1 \ge 3$

Infeasible

Sub-problem 4

$\max \; z = 4x_1 - x_2$

s.t. $7x_1 - x_2 \le 14$

$x_2 \le 3$

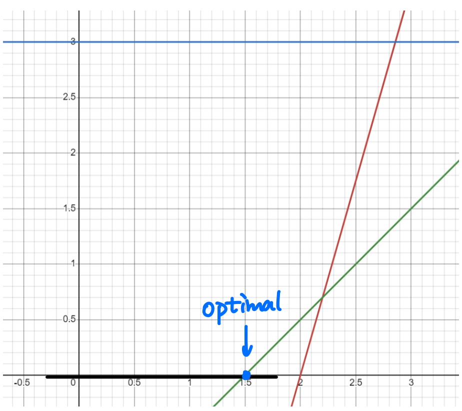
$2x_1 - 2x_2 \le 3$

$x_1, x_2 \ge 0$

$x_1 \le 2, \; x_2 \le 0$

$\Rightarrow z = 6$

$(x_1, x_2) = (1.5, 0)$



The black line is a feasible region.

Sub-problem 3

$\max \; z = 4x_1 - x_2$

s.t. $7x_1 - x_2 \le 14$

$x_2 \le 3$

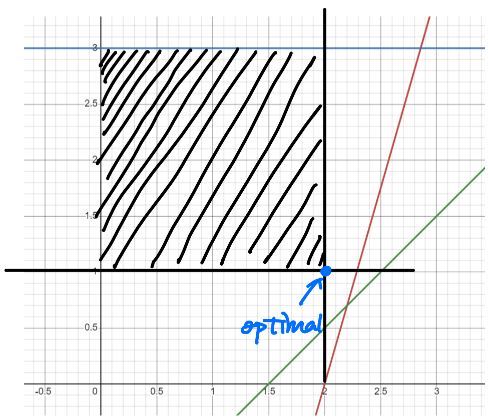
$2x_1 - 2x_2 \le 3$

$x_1, x_2 \ge 0$

$x_1 \le 2, \; x_2 \ge 1$

$\Rightarrow z = 7$

$(x_1, x_2) = (2, 1)$



We can start the branch and bound method by relaxing the original IP to LP first.

1、We solve the LP, and get $z = 8.429$, $(x_1, x_2) = (2.857, 3)$.

we further use x_1 to branch.

2. Sub-problem 1 (add $x_1 \leq 2$)
   $z = 7.5$, $(x_1, x_2) = (2, 0.5)$.

3. Sub-problem 2 (add $x_1 \geq 3$) Infeasible.

4. Sub-problem 3, branch from sub-problem 1 $+ (x_2 \geq 1)$
   $z = 7$, $(x_1, x_2) = (2, 1)$. Optimal.

5. Sub-problem 4, branch from sub-problem 1 $+ (x_2 \leq 0)$.
   $z = 6$, $(x_1, x_2) = (1.5, 0)$.

From the tree, since $z_3 = 7 > 6 = z_4$, we have no need to branch sub-problem 4 since it has provided an upper bound for this branch. So the optimal value should be 7, with $(x_1, x_2) = (2, 1)$.