6.7.2024.

① $\vdash (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow (\neg A \wedge (A \Rightarrow C)))$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$B \vdash B$}
\RightLabel{$\Rightarrow L$}
\BinaryInfC{$A, A \Rightarrow B \vdash B$}
\RightLabel{$\neg L$}
\UnaryInfC{$A, \neg B, A \Rightarrow B \vdash$}
\RightLabel{$\neg R$}
\UnaryInfC{$\neg B, A \Rightarrow B \vdash \neg A$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$B \vdash B$}
\RightLabel{$\Rightarrow L$}
\BinaryInfC{$A, A \Rightarrow B \vdash B$}
\RightLabel{W_R}
\UnaryInfC{$A, A \Rightarrow B \vdash C, B$}
\RightLabel{$\neg L$}
\UnaryInfC{$A, \neg B, A \Rightarrow B \vdash C$}
\RightLabel{$\Rightarrow R$}
\UnaryInfC{$\neg B, A \Rightarrow B \vdash A \Rightarrow C$}
\RightLabel{$\wedge R$}
\BinaryInfC{$\neg B, A \Rightarrow B \vdash \neg A \wedge (A \Rightarrow C)$}
\RightLabel{$\Rightarrow R$}
\UnaryInfC{$A \Rightarrow B \vdash \neg B \Rightarrow (\neg A \wedge (A \Rightarrow C))$}
\RightLabel{$\Rightarrow R$}
\UnaryInfC{$\vdash (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow (\neg A \wedge (A \Rightarrow C)))$}
\end{prooftree}
$$

② $(\forall x)(\exists y)(\forall z)(P(f(x,y),a) \wedge Q(y,z)) \vee (\forall x)(\forall y)(R(x,b) \Rightarrow R(g(x,y),b))$

$\boxed{\text{model}}$

$D = \mathbb{N}_0$ $\qquad I = \begin{pmatrix} P & Q & R & f & g & a & b \\ \neq & < & \geq & \cdot & + & 0 & 1 \end{pmatrix}$

$(\forall x)(\forall y)(x \geq 1 \Rightarrow x+y \geq 1)$ ova formula je tačna

jer ako je $x \geq 1$, y u najgorem slučaju može biti 0

ali i tada je $x+y \geq 1$.

Onda je i cela formula tačna jer je disjunkcija

Kontramodel

$$D = \mathbb{N}_0 \qquad I = \begin{pmatrix} P & Q & R & f & g & a & b \\ \neq & < & \leq & \cdot & + & 0 & 1 \end{pmatrix}$$

$(\forall x)(\exists y)(\forall z)(x \cdot y \neq 0 \land y < z)$ nije točno jer za $x = 0$ ne postoji y tako da je $x \cdot y \neq 0$

$(\forall x)(\forall y)(x \leq 1 \Rightarrow x + y \leq 1)$ nije točno jer je npr. za $x = 0$ i $y = 2$ lijeva strana implikacije točna, a desna netočna

$\Rightarrow$ disjunkcija ovih formula je netočna.

(3)

$$(\exists x)(\forall y)(P(f(x,y),a) \Rightarrow Q(x,a)) \Rightarrow (\exists z)((\forall x)Q(x,z) \Rightarrow (\forall y)(Q(z,y) \lor Q(b,y)))$$

$$\equiv \neg(\exists x)(\forall y)(\neg P(f(x,y),a) \lor Q(x,a)) \lor (\exists z)(\neg(\forall x)Q(x,z) \lor (\forall y)(Q(z,y) \lor Q(b,y)))$$

$$\equiv (\forall x)(\exists y)\neg(\neg P(f(x,y),a) \lor Q(x,a)) \lor (\exists z)((\exists x)\neg Q(x,z) \lor (\forall y)(Q(z,y) \lor Q(b,y)))$$

$$\equiv (\forall x)(\exists y)(P(f(x,y),a) \land \neg Q(x,a)) \lor (\exists z)((\exists x)\neg Q(x,z) \lor (\forall y)(Q(z,y) \lor Q(b,y)))$$

$$[x \mapsto u] \qquad [y \mapsto v]$$

$$\equiv (\forall x)(\exists y)(P(f(x,y),a) \land \neg Q(x,a)) \lor (\exists z)((\exists u)\neg Q(u,z) \lor (\forall v)(Q(z,v) \lor Q(b,v)))$$

$$\equiv (\forall x)(\exists y)(P(f(x,y),a) \land \neg Q(x,a)) \lor (\exists z)(\exists u)(\neg Q(u,z) \lor (\forall v)(Q(z,v) \lor Q(b,v)))$$

$$\equiv (\forall x)(\exists y)(P(f(x,y),a) \land \neg Q(x,a)) \lor (\exists z)(\exists u)(\forall v)(\neg Q(u,z) \lor Q(z,v) \lor Q(b,v))$$

$$\equiv (\exists z)((\forall x)(\exists y)(P(f(x,y),a) \land \neg Q(x,a)) \lor (\exists u)(\forall v)(\neg Q(u,z) \lor Q(z,v) \lor Q(b,v)))$$

$$\equiv (\exists z)(\exists u)(\forall v)((\forall x)(\exists y)(P(f(x,y),a) \land \neg Q(x,a)) \lor (\neg Q(u,z) \lor Q(z,v) \lor Q(b,v)))$$

nastavak

$$\equiv (\exists z)(\exists u)(\forall v)(\forall x)(\exists y)\Big(\big(P(f(x,y),a) \wedge \neg Q(x,a)\big) \vee \big(\neg Q(u,z) \vee Q(z,v) \vee Q(b,v)\big)\Big)$$

$[z \mapsto c]\ [u \mapsto d]$

$$(\forall v)(\forall x)(\exists y)\Big(\big(P(f(x,y),a) \wedge \neg Q(x,a)\big) \vee \big(\neg Q(d,c) \vee Q(c,v) \vee Q(b,v)\big)\Big)$$

$[y \mapsto g(v,x)]$

$$(\forall v)(\forall x)\Big(\big(P(f(x,g(v,x)),a) \wedge \neg Q(x,a)\big) \vee \big(\neg Q(d,c) \vee Q(c,v) \vee Q(b,v)\big)\Big)$$

$$\equiv (\forall v)(\forall x)\Big(\big(P(f(x,g(v,x)),a) \vee \neg Q(d,c) \vee Q(c,v) \vee Q(b,v)\big) \wedge \big(\neg Q(x,a) \vee \neg Q(d,c) \vee Q(c,v) \vee Q(b,v)\big)\Big)$$

4. $\models \big((\exists x)A \vee B\big) \Leftrightarrow (\exists x)(A \vee B)$, x nije slobodna u B

Neka je $\langle D, I\rangle$ proizv. $\mathcal{L}$-struktura i v proizv. valuacija

$I_v\big(((\exists x)A \vee B) \Leftrightarrow (\exists x)(A \vee B)\big) = 1$ akko

$I_v((\exists x)A \vee B) = I_v((\exists x)(A \vee B))$

pretp. $I_v((\exists x)A \vee B) = 1$ akko $I_v((\exists x)A) = 1$ ili $I_v(B) = 1$

$I_v((\exists x)A) = 1$ akko postoji valuacija $w \sim_x v$ tako da $I_w(A) = 1$

Pošto x nije slobodna u B, onda je $I_w(B) = I_v(B)$

$I_v((\exists x)A \vee B) = 1$ akko postoji valuacija $w \sim_x v$ tako da $I_w(A) = 1$ ili $I_w(B) = 1$

Pretp. $I_v((\exists x)(A \vee B)) = 1$ akko postoji $w \sim_x v$ $I_w(A \vee B) = 1$

akko postoji $w \sim_x v$ tako da $I_w(A) = 1$ ili $I_w(B) = 1$

17.6.2023

(1) $\vdash ((D \Rightarrow C) \vee (C \wedge E)) \Rightarrow ((D \wedge B) \Rightarrow ((A \Rightarrow B) \wedge C))$

$B \vdash B$ Ax

$B, (D \Rightarrow C) \vee (C \wedge E) \vdash B$ WL

$A, B, (D \Rightarrow C) \vee (C \wedge E) \vdash B$ WL

$A, D \wedge B, (D \Rightarrow C) \vee (C \wedge E) \vdash B$ ∧L

$D \wedge B, (D \Rightarrow C) \vee (C \wedge E) \vdash A \Rightarrow B$ ⇒R

$D \vdash D$ Ax

$D \wedge B \vdash D$ ∧L

$C \vdash C$ Ax

$D \wedge B, D \Rightarrow C \vdash C$ ⇒L

$C \vdash C$ Ax

$D \wedge B, C \vdash C$ WL

$D \wedge B, C \wedge E \vdash C$ ∧L

$D \wedge B, (D \Rightarrow C) \vee (C \wedge E) \vdash C$ ∨L

$D \wedge B, (D \Rightarrow C) \vee (C \wedge E) \vdash (A \Rightarrow B) \wedge C$ ∧R

$(D \Rightarrow C) \vee (C \wedge E) \vdash (D \wedge B) \Rightarrow ((A \Rightarrow B) \wedge C)$ ⇒R

$\vdash ((D \Rightarrow C) \vee (C \wedge E)) \Rightarrow ((D \wedge B) \Rightarrow ((A \Rightarrow B) \wedge C))$ ⇒R

④ $\vdash (\forall x)(P(x) \wedge R(x)) \Rightarrow ((\exists x)((R(x) \Rightarrow Q(x)) \vee (P(x) \Rightarrow Q(x))) \Rightarrow (\exists x) Q(x))$

1. $(\forall x)(P(x) \wedge R(x))$ pretp
2. $(\exists x)((R(x) \Rightarrow Q(x)) \vee (P(x) \Rightarrow Q(x)))$ pretp
3. $(R(x') \Rightarrow Q(x')) \vee (P(x') \Rightarrow Q(x'))$ pretp
4. $P(x') \wedge R(x')$ $\forall E$ (1)
5. $P(x')$ $\wedge E_1$ (4)
6. $R(x')$ $\wedge E_2$ (4)
7. $R(x') \Rightarrow Q(x')$ pretp.
8. $Q(x')$ $\Rightarrow E$ (6,7)
9. $P(x') \Rightarrow Q(x')$ pretp.
10. $Q(x')$ $\Rightarrow E$ (5,9)
11. $Q(x')$ $\vee E$ (3, 7-8, 9-10)
12. $(\exists x) Q(x)$ $\exists I$ (11)
13. $(\exists x) Q(x)$ $\exists E$ (2, 3-12)
14. $(\exists x)((R(x) \Rightarrow Q(x)) \vee (P(x) \Rightarrow Q(x))) \Rightarrow (\exists x) Q(x)$ $\Rightarrow I$ (2-13)
15. F $\Rightarrow I$ (1-14)