predmet: Matematička analiza 2

FURIJEOVI REDOVI

Furijeov red funkcije $f : [a,b] \to \mathbb{R}$ na intervalu $[a,b]$ je:
$F(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}\Big(a_n \cos\dfrac{n\pi x}{l} + b_n \sin\dfrac{n\pi x}{l}\Big)$, $l = \dfrac{b-a}{2}$, $a_0 = \dfrac{1}{l}\int\limits_a^b f(x)\,dx$, $a_n = \dfrac{1}{l}\int\limits_a^b f(x)\cos\dfrac{n\pi x}{l}\,dx$,
$b_n = \dfrac{1}{l}\int\limits_a^b f(x)\sin\dfrac{n\pi x}{l}\,dx$.

Ako su zadovoljeni Dirihleovi uslovi (f-ja $f(x)$ je neprekidna i monotona po delovima), tada važi:

- $F(x)$ je definisana nad $\mathbb{R}$
- $F(x)$ je periodična sa periodom $b - a = 2l$
- $F(a) = F(b) = \dfrac{f(a^+) + f(b^-)}{2}$
- $F(x) = \dfrac{f(x^+) + f(x^-)}{2}$, $x \in (a,b)$.

Ako je $f(x)$ neparna na intervalu $[-l,l]$, tada su $a_0 = a_n = 0$, $b_n = \dfrac{2}{l}\int\limits_0^l f(x)\sin\dfrac{n\pi x}{l}\,dx$.

Ako je $f(x)$ parna na intervalu $[-l,l]$, tada su $a_0 = \dfrac{2}{l}\int\limits_0^l f(x)\,dx$, $a_n = \dfrac{2}{l}\int\limits_0^l f(x)\cos\dfrac{n\pi x}{l}\,dx$, $b_n = 0$.

Ako funkciju $f : [0,l] \to \mathbb{R}$ razvijamo u red po "kosinusima", tada pravimo njeno parno produženje nad $[-l,l]$:
$\widetilde{f}(x) = \begin{cases} f(x), & 0 \le x \le l \\ f(-x), & -l \le x < 0 \end{cases}$.

Ako funkciju $f : [0,l] \to \mathbb{R}$ razvijamo u red po "sinusima", tada pravimo njeno neparno produženje nad $[-l,l]$:
$\widetilde{f}(x) = \begin{cases} f(x), & 0 \le x \le l \\ -f(-x), & -l \le x < 0 \end{cases}$.

1. Razviti u Furijeov red funkciju $f(x) = e^{3x}, x \in [0, 2\pi]$ i na osnovu tog razvoja izračunati sumu reda $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+9}$.

   Rešenje:

   $l = \dfrac{2\pi - 0}{2} = \pi$.

   $F(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}\Big(a_n \cos\dfrac{n\pi x}{l} + b_n \sin\dfrac{n\pi x}{l}\Big) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}\Big(a_n \cos(nx) + b_n \sin(nx)\Big)$.

   $a_n = \dfrac{1}{\pi}\int\limits_0^{2\pi} e^{3x}\cos(nx)dx$

   $b_n = \dfrac{1}{\pi}\int\limits_0^{2\pi} e^{3x}\sin(nx)dx$

$a_0 = \frac{1}{\pi}\int\limits_0^{2\pi} e^{3x}dx = \frac{1}{3\pi}e^{3x}|_0^{2\pi} = \frac{1}{3\pi}(e^{6\pi}-1).$

I način računanja a_n i b_n:

$a_n = \frac{1}{\pi}\int\limits_0^{2\pi} e^{3x}\cos(nx)dx$

$\int e^{3x}\cos(nx)dx = \frac{1}{n}e^{3x}\sin(nx) - \int \frac{3}{n}e^{3x}\sin(nx)dx =$

$= \left(\textit{parcijalna integracija}: u = e^{3x},\ dv = \cos(nx)dx,\ du = 3e^{3x}dx,\ v = \frac{1}{n}\sin(nx)\right) =$

$= \frac{1}{n}e^{3x}\sin(nx) - \frac{3}{n}\int e^{3x}\sin(nx)dx =$

$= \left(\textit{parcijalna integracija}: u = e^{3x},\ dv = \sin(nx)dx,\ du = 3e^{3x}dx,\ v = -\frac{1}{n}\cos(nx)\right) =$

$= \frac{1}{n}e^{3x}\sin(nx) - \frac{3}{n}\left(-\frac{1}{n}e^{3x}\cos(nx) + \int \frac{3}{n}e^{3x}\cos(nx)dx\right) =$

$= \frac{1}{n}e^{3x}\sin(nx) + \frac{3}{n^2}e^{3x}\cos(nx) - \frac{9}{n^2}\int e^{3x}\cos(nx)\,dx.$

Dobili smo da je $\int e^{3x}\cos(nx)dx = \frac{1}{n}e^{3x}\sin(nx) + \frac{3}{n^2}e^{3x}\cos(nx) - \frac{9}{n^2}\int (e^{3x}\cos(nx))\,dx$, pa je

$\frac{n^2+9}{n^2}\int e^{3x}\cos(nx)dx = \frac{1}{n}e^{3x}\sin(nx) + \frac{3}{n^2}e^{3x}\cos(nx)$

$\int e^{3x}\cos(nx)dx = \frac{n}{n^2+9}e^{3x}\sin(nx) + \frac{3}{n^2+9}e^{3x}\cos(nx)$

$a_n = \frac{1}{\pi}\left(\frac{n}{n^2+9}e^{3x}\sin(nx) + \frac{3}{n^2+9}e^{3x}\cos(nx)\right)\Bigg|_0^{2\pi} =$

$= \frac{1}{\pi(n^2+9)}e^{3x}\ (n\sin(nx) + 3\cos(nx))|_0^{2\pi} = \frac{1}{\pi(n^2+9)}(3e^{6\pi}-3) = \frac{3(e^{6\pi}-1)}{\pi(n^2+9)}.$

Analogno se dobija da je $b_n = \frac{n(1-e^{6\pi})}{\pi(n^2+9)}$.

II način računanja a_n i b_n:

$a_n + ib_n = \frac{1}{\pi}\int\limits_0^{2\pi} e^{3x}\cos(nx)\,dx + i\frac{1}{\pi}\int\limits_0^{2\pi} e^{3x}\sin(nx)\,dx = \frac{1}{\pi}\int\limits_0^{2\pi}(e^{3x}\cos(nx) + ie^{3x}\sin(nx))\,dx =$

$= \frac{1}{\pi}\int\limits_0^{2\pi} e^{3x}(\cos(nx) + i\sin(nx))\,dx = \frac{1}{\pi}\int\limits_0^{2\pi} e^{3x}e^{inx}\,dx = \frac{1}{\pi}\int\limits_0^{2\pi} e^{(3+in)x}\,dx = \frac{1}{\pi}\ \frac{1}{3+in}e^{(3+in)x}\Bigg|_0^{2\pi} =$

$= \frac{1}{\pi(3+in)}(e^{(3+in)2\pi}-1) = \frac{e^{6\pi+2\pi in}-1}{\pi(3+in)} = \frac{e^{6\pi}-1}{\pi(3+in)}\cdot\frac{3-in}{3-in} = \frac{3(e^{6\pi}-1)-in(e^{6\pi}-1)}{\pi(9+n^2)}$, odavde dobijamo da su

$a_n = \frac{3(e^{6\pi}-1)}{\pi(n^2+9)}$ i $b_n = \frac{n(1-e^{6\pi})}{\pi(n^2+9)}$.

$F(x) = \frac{e^{6\pi}-1}{6\pi} + \sum_{n=1}^{\infty}\Big(\frac{3(e^{6\pi}-1)}{\pi(n^2+9)}\cos(nx) + \frac{n(1-e^{6\pi})}{\pi(n^2+9)}\sin(nx)\Big) =$

$= \frac{e^{6\pi}-1}{\pi}\Big(\frac{1}{6}+\sum_{n=1}^{\infty}\Big(\frac{3}{n^2+9}\cos(nx) - \frac{n}{n^2+9}\sin(nx)\Big)\Big) = \frac{e^{6\pi}-1}{\pi}\Big(\frac{1}{6}+\sum_{n=1}^{\infty}\frac{3\cos(nx)-n\sin(nx)}{n^2+9}\Big).$

Znamo da je: $F(a) = F(b) = \dfrac{f(a^+) + f(b^-)}{2}$ i $f(x) = e^{3x}$, pa je
$F(0) = F(2\pi) = \dfrac{e^0 + e^{6\pi}}{2} = \dfrac{1 + e^{6\pi}}{2}$.
Koristeći ovo i gore dobijen rezultat imamo da je:
$F(0) = \dfrac{e^{6\pi} - 1}{\pi}\left(\dfrac{1}{6} + \sum\limits_{n=1}^{\infty} \dfrac{3}{n^2+9}\right) = \dfrac{1 + e^{6\pi}}{2}$, pa je

$$\frac{1}{6} + \sum_{n=1}^{\infty} \frac{3}{n^2+9} = \frac{\pi(1 + e^{6\pi})}{2(e^{6\pi} - 1)}$$

$$\sum_{n=1}^{\infty} \frac{3}{n^2+9} = \frac{\pi(1 + e^{6\pi})}{2(e^{6\pi} - 1)} - \frac{1}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+9} = \frac{\pi(1 + e^{6\pi})}{6(e^{6\pi} - 1)} - \frac{1}{18}.$$

2. Razviti u Furijeov red funkciju $f(x) = \begin{cases} 2, & -\pi \le x \le 0 \\ 5 - x, & 0 < x \le \pi \end{cases}$.

Rešenje:

$l = \dfrac{\pi - (-\pi)}{2} = \pi$.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos (nx) + b_n \sin (nx)).$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} 2 dx + \int_{0}^{\pi} (5 - x) dx \right) = \frac{1}{\pi} \left(2(0 - (-\pi)) + 5x\Big|_0^\pi - \frac{x^2}{2}\Big|_0^\pi \right) =$$
$$= \frac{1}{\pi} \left(2\pi + (5\pi - 0) - \frac{1}{2}(\pi^2 - 0) \right) = \frac{1}{\pi} \left(7\pi - \frac{\pi^2}{2} \right) = 7 - \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos (nx)\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} 2 \cos (nx)\, dx + \int_{0}^{\pi} (5 - x) \cos (nx)\, dx \right) =$$
$$= \frac{1}{\pi} \left(2\frac{1}{n} \sin (nx)\Big|_{-\pi}^{0} + 5 \int_{0}^{\pi} \cos (nx)\, dx - \int_{0}^{\pi} x \cos (nx)\, dx \right) =$$
$$= \left(u = x, dv = \cos (nx)dx, du = dx, v = \frac{1}{n} \sin (nx) \right) =$$
$$= \frac{1}{\pi} \left(5\frac{1}{n} \sin (nx)\Big|_0^\pi - \left(x\frac{1}{n} \sin (nx)\Big|_0^\pi - \int_{0}^{\pi} \frac{1}{n} \sin (nx)\, dx \right) \right) =$$
$$= \frac{1}{\pi}\frac{1}{n} \int_{0}^{\pi} \sin (nx)\, dx = \frac{1}{n\pi}\frac{1}{n} (-\cos (nx))\,|_0^\pi = \frac{-1}{n^2\pi} (\cos (n\pi) - \cos 0) = \frac{1 - (-1)^n}{n^2\pi}.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (nx)\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} 2 \sin (nx)\, dx + \int_{0}^{\pi} (5 - x) \sin (nx)\, dx \right) =$$
$$= \frac{1}{\pi} \left(2\frac{1}{n}(-\cos (nx))\Big|_{-\pi}^{0} + 5 \int_{0}^{\pi} \sin (nx)\, dx - \int_{0}^{\pi} x \sin (nx)\, dx \right) =$$
$$= \left(u = x, dv = \sin (nx)dx, du = dx, v = -\frac{1}{n} \cos (nx) \right) =$$

$$= \frac{1}{\pi}\left(-\frac{2}{n}(1-(-1)^n) - 5\frac{1}{n}\cos(nx)\Big|_0^\pi - \left(-x\frac{1}{n}\cos(nx)\Big|_0^\pi - \int_0^\pi -\frac{1}{n}\cos(nx)\,dx\right)\right) =$$
$$= \frac{1}{\pi}\left(-\frac{2}{n}(1-(-1)^n) - \frac{5}{n}((-1)^n-1) - \left(-\frac{1}{n}\pi(-1)^n + \frac{1}{n^2}\sin(nx)\Big|_0^\pi\right)\right) =$$
$$= \frac{1}{\pi}\left(3\frac{1-(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right) = \frac{1}{n\pi}\left(3(1-(-1)^n) + (\pi(-1)^n)\right).$$
$$F(x) = \frac{7}{2} - \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{1}{n\pi}\left(3(1-(-1)^n) + (\pi(-1)^n)\right)\sin(nx)\right).$$
$$F(x) = \frac{7}{2} - \frac{\pi}{4} + \sum_{k=1}^{\infty}\frac{\sin(2kx)}{2k} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{2}{(2k-1)^2}\cos((2k-1)x) + \frac{6-\pi}{2k-1}\sin((2k-1)x)\right).$$

3. Razviti u Furijeov red po "sinusima" funkciju $f(x) = 1-2x, x \in [0,1]$.

Rešenje:

Napravićemo neparno produženje funkcije $f(x)$ na intervalu $[-1,1]$:

$$\widetilde{f}(x) = \begin{cases} 1-2x, & 0 \le x \le 1 \\ -1-2x, & -1 \le x < 0 \end{cases}.$$

$$l = \frac{1-(-1)}{2} = 1.$$

$a_n = 0,\ a_0 = 0.$

$$F(x) = \sum_{n=1}^{\infty}(b_n \sin(n\pi x)).$$

$$b_n = 2\int_0^1 \widetilde{f}(x)\sin(n\pi x)\,dx = 2\int_0^1 (1-2x)\sin(n\pi x)\,dx =$$
$$= \left(u = 1-2x, dv = \sin(n\pi x)dx, du = -2dx, v = -\frac{1}{n\pi}\cos(n\pi x)\right) =$$
$$= 2\left(\frac{2x-1}{n\pi}\cos(n\pi x)\Big|_0^1 - \int_0^1 \frac{2}{n\pi}\cos(n\pi x)\,dx\right) = 2\left(\frac{1}{n\pi}\cos n\pi + \frac{1}{n\pi} - \frac{2}{n^2\pi^2}\sin(n\pi x)\Big|_0^1\right) =$$
$$= 2\left(\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi}\right) = \frac{2((-1)^n+1)}{n\pi}.$$
$$F(x) = \sum_{n=1}^{\infty}\left(\frac{2((-1)^n+1)}{n\pi}\sin(n\pi x)\right) = \sum_{k=1}^{\infty}\frac{2\sin(2k\pi x)}{k\pi},\ x \in [0,1].$$

4. Razviti u Furijeov red po "kosinusima" funkciju $f(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$.

Rešenje:

Napravićemo parno produženje funkcije $f(x)$ na intervalu $[-1,1]$:

$$\widetilde{f}(x) = \begin{cases} 0, & -1 \le x < -\frac{1}{2} \\ 1, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1 \end{cases}.$$

$$l = \frac{1-(-1)}{2} = 1.$$

$b_n = 0.$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\pi x)).$$

$$a_0 = \frac{2}{1}\int_0^1 \widetilde{f}(x)dx = 2\left(\int_0^{\frac{1}{2}} dx + \int_{\frac{1}{2}}^1 0\,dx\right) = 1.$$

$$a_n = \frac{2}{1}\int_0^1 \widetilde{f}(x)\cos(n\pi x)\,dx = 2\left(\int_0^{\frac{1}{2}} \cos(n\pi x)\,dx + \int_{\frac{1}{2}}^1 0\,dx\right) = \frac{2}{n\pi}\sin(n\pi x)\Big|_0^{\frac{1}{2}} = \frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

$$F(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos(n\pi x)\right),\ x \in [0,1].$$

5. Razviti u Furijeov red po "sinusima" funkciju $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$.

Rešenje:

Napravićemo neparno produženje funkcije $f(x)$ na intervalu $[-2, 2]$:

$$\widetilde{f}(x) = \begin{cases} x, & -1 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ -2-x, & -2 \le x < -1 \end{cases}.$$

$l = \frac{2-(-2)}{2} = 2.$

$a_n = 0,\ a_0 = 0.$

$$F(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{(n\pi x)}{2}\right).$$

$$b_n = \frac{2}{2}\int_0^2 \widetilde{f}(x)\sin\frac{(n\pi x)}{2}\,dx = \int_0^1 x\sin\frac{(n\pi x)}{2}dx + \int_1^2 (2-x)\sin\frac{(n\pi x)}{2}\,dx =$$

$$= \left(u = x, dv = \sin\frac{(n\pi x)}{2}\,dx, du = dx, v = -\frac{2}{n\pi}\cos\frac{(n\pi x)}{2}\right);$$

$$\left(u_1 = 2-x, dv_1 = \sin\frac{(n\pi x)}{2}\,dx, du_1 = -dx, v_1 = -\frac{2}{n\pi}\cos\frac{(n\pi x)}{2}\right) =$$

$$= \left(\frac{-2x}{n\pi}\cos\frac{(n\pi x)}{2}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{(n\pi x)}{2}\,dx\right) + \left(\frac{2(x-2)}{n\pi}\cos\frac{(n\pi x)}{2}\Big|_1^2 - \frac{2}{n\pi}\int_1^2 \cos\frac{(n\pi x)}{2}\,dx\right) =$$

$$= \left(\frac{-2x}{n\pi}\cos\frac{(n\pi x)}{2} + \frac{4}{n^2\pi^2}\sin\frac{(n\pi x)}{2}\right)\Big|_0^1 + \left(\frac{2(x-2)}{n\pi}\cos\frac{(n\pi x)}{2} - \frac{4}{n^2\pi^2}\sin\frac{(n\pi x)}{2}\right)\Big|_1^2 =$$

$$= \left(\frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$F(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}\right),\ x \in [0,2].$$

6. Razviti u Furijeov red funkciju $f(x) = |2x|, x \in [-\pi, \pi]$, pa zatim izračunati $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ i $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Rešenje:

Primetimo da je $f(x) = |2x|$ parna funkcija.

$$f(x) = \begin{cases} 2x, & 0 \le x \le \pi \\ -2x, & -\pi \le x < 0 \end{cases}.$$

$l = \frac{\pi-(-\pi)}{2} = \pi.$

$b_n = 0.$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx)).$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} 2x\,dx = \frac{4}{\pi} \frac{x^2}{2}\bigg|_0^{\pi} = 2\pi.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} 2x \cos(nx)\,dx = \left(u = x, dv = \cos(nx) dx, du = dx, v = \frac{1}{n} \sin(nx) \right) =$$

$$= \frac{4}{\pi} \left(\frac{x}{n} \sin(nx)\bigg|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin(nx)\,dx \right) = \frac{4}{n\pi} \frac{1}{n} \cos(nx)\bigg|_0^{\pi} = \frac{4}{n^2\pi}((-1)^n - 1).$$

$$F(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{n^2\pi} \cos(nx) = \pi + 4 \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi} \cos(nx).$$

$$F(x) = \pi - \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^2}, x \in [-\pi, \pi].$$

Znamo da je: $F(0) = \dfrac{f(0^+) + f(0^-)}{2} = \dfrac{0+0}{2} = 0$, pa je

$$F(0) = \pi - \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 0.$$

$$\frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \pi$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Znamo da važi: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right) =$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Nazovimo $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = S$, koristeći gornju jednakost i prethodno dobijenu sumu imamo da je:

$$S = \frac{\pi^2}{8} + \frac{1}{4} S.$$

$$\frac{3}{4} S = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}.$$