· Nizovi

Niz $a_n : \mathbb{N} \to \mathbb{R}$ $\quad a_n = a(n)$

Elementi niza: $a_1, a_2, a_3, \dots$ $\quad \{a_n\}_{n\in\mathbb{N}}$

<u>Def</u> $\{a_n\}_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{} a \in \mathbb{R}$ akko

$$(\forall \varepsilon>0)(\exists n_0\in\mathbb{N})(\forall n\in\mathbb{N})(n\geq n_0 \Rightarrow |a_n - a| < \varepsilon)$$

ε-beskonačno mnogo

$$\Updownarrow$$
$$-\varepsilon < a_n - a < \varepsilon$$
$$a-\varepsilon < a_n < a+\varepsilon$$

$\dots a_{n_0+1} \quad a-\varepsilon \quad a \quad a+\varepsilon \quad a_{n_0} \dots$

$$\boxed{\lim_{n\to\infty} a_n = a}$$

<u>Def</u> $b \in \mathbb{R}$ je tačka nagomilavanja niza $\{a_n\}_{n\in\mathbb{N}}$ ako u svakoj okolini tačke b imamo beskonačno mnogo članova niza $\{a_n\}_{n\in\mathbb{N}}$.

· Granica niza je i tačka nagomilavanja (jedinstvena).

Pr. $a_n = (-1)^n\left(1+\frac{1}{n}\right)$ $\quad -2, 1+\frac{1}{2}, -(1+\frac{1}{3}), 1+\frac{1}{4}, \dots$

$$a_{2k} = (-1)^{2k}\left(1+\frac{1}{2k}\right) = 1 + \cancel{\frac{1}{2k}}^{0} \xrightarrow[k\to\infty]{} 1$$

$$a_{2k-1} = (-1)^{2k-1}\left(1+\frac{1}{2k-1}\right) = -1 - \frac{1}{2k-1} \xrightarrow[k\to\infty]{} -1$$

$b_1 = 1 = \limsup a_n$

$b_2 = -1 = \liminf a_n$

$\lim\limits_{n\to\infty} a_n$ ne postoji!

· Najveća t.n. niza $\{a_n\}$ se naziva limes superior $\limsup a_n$

· Najmanja t.n. se naziva limes inferior $\liminf a_n$

· Ako $\{a_n\}_{n\in\mathbb{N}}$ konvergentan tada

$$\limsup a_n = \liminf a_n = \lim_{n\to\infty} a_n$$

Brojni redovi

<u>Def</u> Neka je $\{a_n\}_{n\in\mathbb{N}}$ niz u $\mathbb{R}$. Red je izraz oblika

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

ili kraće $\sum\limits_{n=1}^{\infty} a_n$, pri čemu se a_n naziva opšti član reda.

<u>Def</u> k-ta parcijalna suma reda $\sum\limits_{n=1}^{\infty} a_n$ je izraz

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k.$$

$\{S_k\}_{k\in\mathbb{N}}$ je niz parcijalnih suma reda $\sum\limits_{n=1}^{\infty} a_n$.

<u>Def</u> $\sum\limits_{n=1}^{\infty} a_n$ je konvergentan akko je $\{S_k\}_{k\in\mathbb{N}}$ konvergentan

$$\boxed{S = \lim_{k\to\infty} S_k} \quad S - \text{suma reda } \sum_{n=1}^{\infty} a_n$$

$$S = \sum_{n=1}^{\infty} a_n$$

<u>Pr.</u> Izračunati sumu reda $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$, $a_n = \frac{1}{n(n+1)}$

$$S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \tfrac{1}{4} + \dots + \cancel{\tfrac{1}{k}} - \tfrac{1}{k+1}$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \qquad A = 1 \quad B = -1$$

$$S_k = 1 - \cancel{\frac{1}{k+1}}^{0} \xrightarrow[k\to\infty]{} 1 = S \qquad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

<u>Pr.</u> Izračunati sumu reda $\sum\limits_{n=1}^{\infty} \ln\frac{n}{n+1} = -\infty$

$$a_n = \ln\frac{n}{n+1} = \ln n - \ln(n+1)$$

$$S_k = \sum_{n=1}^{k}(\ln n - \ln(n+1)) = \ln 1 - \cancel{\ln 2} + \cancel{\ln 2} - \cancel{\ln 3} + \dots + \cancel{\ln k} - \ln(k+1)$$
$$= -\ln(k+1) \xrightarrow[k\to\infty]{} -\infty = S$$

· Važi: $\boxed{S_k - S_{k-1}} = \cancel{a_1} + \dots + \cancel{a_{k-1}} + a_k - (\cancel{a_1} + \dots + \cancel{a_{k-1}}) \boxed{= a_k}$

<u>Pr.</u> Ako je $S_k = \frac{k}{k+1}$ naći a_n i S.

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k/k}{k+1/k} = 1$$

$$S_k - S_{k-1} = \frac{k}{k+1} - \frac{k-1}{k-1+1} = \frac{\cancel{k^2} - (\cancel{k^2} - 1)}{k(k+1)} = \boxed{\frac{1}{k(k+1)} = a_k}$$

<u>Pr.</u> (Geometrijski red) U zavisnosti od parametra q diskutovati konvergenciju $\sum\limits_{n=0}^{\infty} q^n$.

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \dots$$

$$S_k = \sum_{n=0}^{k-1} q^n = \underbrace{1 + q + q^2 + \dots + q^{k-1}}_{k - \text{sabiraka}} = \frac{1-q^k}{1-q}, \quad q \neq 1$$

geometrijska suma

$$\lim_{k\to\infty} q^k = \begin{cases} 0 & ; |q|<1 \\ \infty & ; q>1 \\ \text{ne postoji} & ; q\leq -1 \\ 1 & ; q=1 \end{cases}$$

Za $|q|<1$: $\quad \lim\limits_{k\to\infty} S_k = \lim\limits_{k\to\infty} \frac{1-\cancel{q^k}^{0}}{1-q} = \frac{1}{1-q}$

$$\boxed{\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad |q|<1}$$

<u>Pr.</u> Sumirati red $\sum\limits_{n=1}^{\infty} \frac{2^n}{3^n}$ $\quad a_n = \left(\frac{2}{3}\right)^n, \ q = \frac{2}{3} < 1$

I način

$n = m+1 \Rightarrow m = n-1$ (promjena brojača)

$n = 1, 2, 3, \dots \to m = 0, 1, 2, \dots$

$$\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^{m+1} = \frac{2}{3}\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^m = \frac{2}{3}\cdot\frac{1}{1-\frac{2}{3}} = \frac{2}{\cancel{3}}\cdot\cancel{3} = 2$$

II način $\quad \sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots = \frac{2}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right) =$

$$= 2\cdot\frac{1}{1-\frac{2}{3}} = 2$$

· Osobine redova

1) Ako su $\sum a_n$ i $\sum b_n$ konvergentni redovi tada

$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$$

2) Ako je $\sum a_n$ konvergentan i $\lambda \neq 0$ tada

$$\sum \lambda a_n = \lambda \sum a_n$$

· Napomena: Konačno mnogo sabiraka ne utiče na konvergenciju reda

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \dots + a_{n_0}}_{\text{konačan broj}} + \sum_{n=n_0+1}^{\infty} a_n \qquad n_0 \in \mathbb{N}$$

T (Košijev kriterijum konvergencije)

Red $\sum a_n$ konvergira akko

$$(\forall\varepsilon>0)(\exists k_0\in\mathbb{N})(\forall p, k\in\mathbb{N})\left(k\geq k_0 \Rightarrow \left|\sum_{n=k+1}^{k+p} a_n\right| < \varepsilon\right)$$

$$S_{k+p} - S_k = (\cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_k} + a_{k+1} + \dots + a_{k+p}) - (\cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_k}) =$$
$$= \sum_{n=k+1}^{k+p} a_n$$

$\sum a_n$ je konvergentan akko je $\{S_k\}_{k\in\mathbb{N}}$ Košijev!

<u>Pr.</u> Dokazati da je red $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ divergentan. (harmonijski red)

Pokazaćemo (na osnovu Košijevog kriterijuma)

$$(\exists\varepsilon>0)(\forall k_0\in\mathbb{N})(\exists p, k\in\mathbb{N})\left(k\geq k_0 \Rightarrow \sum_{n=k+1}^{k+p}\frac{1}{n} \geq \varepsilon\right)$$

(slično ...).