predmet: Matematička analiza 2

Loranov red

1. Razviti sledeće funkcije u Loranov red po stepenima od z u oblasti konvergencije:

a) $f(z) = \dfrac{1}{1-z}$; b) $f(z) = \dfrac{1}{(z-1)^2}$; c) $f(z) = \dfrac{1}{z+3}$; d) $f(z) = \dfrac{16}{(z-1)^2(z+3)}$.

Rešenje:

a) Singularitet je $z = 1$. Razlikujemo 2 slučaja:

1) za $|z| < 1$:

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

2) za $|z| > 1$:

$$f(z) = \frac{1}{1-z} = \frac{1}{z \cdot \left(\frac{1}{z} - 1\right)} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}.$$

b) Singularitet je $z = 1$. Razlikujemo 2 slučaja:

1) za $|z| < 1$:

$$f(z) = \frac{1}{(z-1)^2} = \left(\frac{1}{1-z}\right)' \overset{a)}{=} \left(\sum_{n=0}^{\infty} z^n\right)' = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n.$$

2) za $|z| > 1$:

$$f(z) = \frac{1}{(z-1)^2} = \left(\frac{1}{1-z}\right)' \overset{a)}{=} \left(-\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}\right)' = -\sum_{n=0}^{\infty} \frac{-n-1}{z^{n+2}} = \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}}.$$

c) Singularitet je $z = -3$. Razlikujemo 2 slučaja:

1) za $\left|\frac{z}{3}\right| < 1$, tj. $|z| < 3$:

$$f(z) = \frac{1}{z+3} = \frac{1}{3 \cdot \left(1 + \frac{z}{3}\right)} = \frac{1}{3} \cdot \sum_{n=0}^{\infty} \left(-\frac{z}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot z^n.$$

2) za $\left|\frac{z}{3}\right| > 1$, tj. $|z| > 3$:

$$f(z) = \frac{1}{z+3} = \frac{1}{z \cdot \left(1 + \frac{3}{z}\right)} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \left(-\frac{3}{z}\right)^n = \sum_{n=0}^{\infty} (-3)^n \cdot \frac{1}{z^{n+1}}.$$

d) Singulariteti su $z = 1$ i $z = -3$. Funkciju $f(z)$ rastavimo na sumu parcijalnih razlomaka:

$$f(z) = \frac{16}{(z-1)^2(z+3)} = -\frac{1}{z-1} + \frac{4}{(z-1)^2} + \frac{1}{z+3} = \frac{1}{1-z} + \frac{4}{(z-1)^2} + \frac{1}{z+3}.$$

Iskoristimo već dobijene razvoje za funkcije $\dfrac{1}{1-z}$, $\dfrac{1}{(z-1)^2}$ i $\dfrac{1}{z+3}$ i tako razlikujemo 3 slučaja:

1) za $|z| < 1$:

$$f(z) = \frac{1}{1-z} + \frac{4}{(z-1)^2} + \frac{1}{z+3} = \sum_{n=0}^{\infty} z^n + 4 \cdot \sum_{n=0}^{\infty} (n+1) z^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot z^n = \sum_{n=0}^{\infty} \left(4n + 5 + \frac{(-1)^n}{3^{n+1}}\right) \cdot z^n.$$

2) za $1 < |z| < 3$:

$$\begin{aligned}
f(z) &= \frac{1}{1-z} + \frac{4}{(z-1)^2} + \frac{1}{z+3} \\
&= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + 4 \cdot \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot z^n \\
&= -\frac{1}{z} - \sum_{n=1}^{\infty} \frac{1}{z^{n+1}} + 4 \cdot \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot z^n \\
&= -\frac{1}{z} - \sum_{n=0}^{\infty} \frac{1}{z^{n+2}} + 4 \cdot \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot z^n \\
&= -\frac{1}{z} + \sum_{n=0}^{\infty} (4n+3) \cdot \frac{1}{z^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot z^n .
\end{aligned}$$

3) za $|z| > 3$:

$$\begin{aligned}
f(z) &= \frac{1}{1-z} + \frac{4}{(z-1)^2} + \frac{1}{z+3} \\
&= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + 4 \cdot \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}} + \sum_{n=0}^{\infty} (-3)^n \cdot \frac{1}{z^{n+1}} \\
&= -\frac{1}{z} - \sum_{n=1}^{\infty} \frac{1}{z^{n+1}} + 4 \cdot \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}} + \frac{1}{z} + \sum_{n=1}^{\infty} (-3)^n \cdot \frac{1}{z^{n+1}} \\
&= -\sum_{n=0}^{\infty} \frac{1}{z^{n+2}} + 4 \cdot \sum_{n=0}^{\infty} \frac{n+1}{z^{n+2}} + \sum_{n=0}^{\infty} (-3)^{n+1} \cdot \frac{1}{z^{n+2}} \\
&= \sum_{n=0}^{\infty} \left(4n+3+(-3)^{n+1}\right) \cdot \frac{1}{z^{n+2}} .
\end{aligned}$$

2. Razviti funkciju $f(z) = \dfrac{1}{z}$ u Loranov red po stepenima od $z - i$.

Rešenje: Singularitet je $z = 0$. Razlikujemo 2 slučaja:
1) za $|z - i| < 1$:
$f(z) = \dfrac{1}{z} = \dfrac{1}{z \pm i} = \dfrac{1}{i\left(1 + \frac{z-i}{i}\right)} = \dfrac{-i}{1 - i(z-i)} = -i \cdot \displaystyle\sum_{n=0}^{\infty} i^n \cdot (z-i)^n = -\sum_{n=0}^{\infty} i^{n+1} \cdot (z-i)^n.$
2) za $|z - i| > 1$:
$f(z) = \dfrac{1}{z} = \dfrac{1}{z \pm i} = \dfrac{1}{(z-i) \cdot \left(1 + \frac{i}{z-i}\right)} = \dfrac{1}{z-i} \cdot \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot i^n}{(z-i)^n} = \sum_{n=0}^{\infty} (-1)^n \cdot i^n \cdot \frac{1}{(z-i)^{n+1}}.$

3. Razviti funkciju $f(z) = \dfrac{z^2+1}{z^2-1}$ u Loranov red po stepenima od $z - 1$.

Rešenje: Funkciju $f(z)$ predstavimo kao zbir polinoma i prave racionalne funkcije koju potom rastavimo na sumu parcijalnih razlomaka:

$$f(z) = \frac{z^2+1}{z^2-1} = \frac{z^2-1+2}{z^2-1} = 1 + \frac{2}{z^2-1} = 1 + \frac{1}{z-1} - \frac{1}{z+1}.$$

Kako je 1 i $\dfrac{1}{z-1}$ već razvijeno po stepenima od $z-1$, ostaje nam da razvijemo $\dfrac{1}{z+1}$. Singulariteti su $z = 1$ i $z = -1$. Razlikujemo 2 slučaja:
1) za $|z - 1| < 2$, $z \neq 1$:
$\dfrac{1}{z+1} = \dfrac{1}{2\left(1 + \frac{z-1}{2}\right)} = \dfrac{1}{2} \cdot \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (z-1)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot (z-1)^n$, odakle dobijamo da je traženi razvoj
$f(z) = 1 + \dfrac{1}{z-1} - \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot (z-1)^n.$

2) za $|z-1|>2$:

$$\frac{1}{z+1} = \frac{1}{z-1+2} = \frac{1}{(z-1)\cdot\left(1+\frac{2}{z-1}\right)} = \frac{1}{z-1}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n}{(z-1)^n} = \sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot\frac{1}{(z-1)^{n+1}},$$ odakle

dobijamo da je traženi razvoj $f(z)=1+\dfrac{1}{z-1}-\displaystyle\sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot\frac{1}{(z-1)^{n+1}}$.

4. Razviti funkciju $f(z)=(z^2+1)\cdot e^{\frac{1}{z}}$ u Loranov red u okolini tačke $z_0=0$.

Rešenje:

$$\begin{aligned}
f(z) &= (z^2+1)\cdot e^{\frac{1}{z}} = z^2\cdot e^{\frac{1}{z}}+e^{\frac{1}{z}} = z^2\cdot\sum_{n=0}^{\infty}\frac{1}{z^n\cdot n!}+\sum_{n=0}^{\infty}\frac{1}{z^n\cdot n!} = \sum_{n=0}^{\infty}\frac{1}{z^{n-2}\cdot n!}+\sum_{n=0}^{\infty}\frac{1}{z^n\cdot n!}\\
&= z^2+z+\sum_{n=2}^{\infty}\frac{1}{z^{n-2}\cdot n!}+\sum_{n=0}^{\infty}\frac{1}{z^n\cdot n!} = z^2+z+\sum_{n=0}^{\infty}\frac{1}{z^n\cdot(n+2)!}+\sum_{n=0}^{\infty}\frac{1}{z^n\cdot n!}\\
&= z^2+z+\sum_{n=0}^{\infty}\frac{1}{z^n}\cdot\left(\frac{1}{(n+2)!}+\frac{1}{n!}\right) = z^2+z+\sum_{n=0}^{\infty}\frac{1+n^2+3n+2}{(n+2)!}\cdot\frac{1}{z^n}\\
&= z^2+z+\sum_{n=0}^{\infty}\frac{n^2+3n+3}{(n+2)!}\cdot\frac{1}{z^n}.
\end{aligned}$$

Ovaj red konvergira za svako $z\neq 0$.

5. Razviti funkciju $f(z)=(2z^2-z+3)\cdot e^{\frac{1}{z-1}}$ u Loranov red u okolini tačke $z_0=1$.

Rešenje:

Uvedemo smenu $\omega=z-1$ i nađemo razvoj funkcije
$g(\omega)=f(\omega+1)=(2(\omega+1)^2-(\omega+1)+3)\cdot e^{\frac{1}{\omega}}=(2\omega^2+4\omega+2-\omega-1+3)\cdot e^{\frac{1}{\omega}}=(2\omega^2+3\omega+4)\cdot e^{\frac{1}{\omega}}$
u okolini tačke $\omega=0$.

$$\begin{aligned}
g(\omega) &= (2\omega^2+3\omega+4)\cdot e^{\frac{1}{\omega}} = 2\omega^2\cdot e^{\frac{1}{\omega}}+3\omega\cdot e^{\frac{1}{\omega}}+4\cdot e^{\frac{1}{\omega}}\\
&= 2\omega^2\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^n}+3\omega\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^n}+4\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^n}\\
&= 2\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^{n-2}}+3\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^{n-1}}+4\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^n}\\
&= 2\omega^2+2\omega+2\cdot\sum_{n=2}^{\infty}\frac{1}{n!\cdot\omega^{n-2}}+3\omega+3\cdot\sum_{n=1}^{\infty}\frac{1}{n!\cdot\omega^{n-1}}+4\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^n}\\
&= 2\omega^2+5\omega+2\cdot\sum_{n=0}^{\infty}\frac{1}{(n+2)!\cdot\omega^n}+3\cdot\sum_{n=0}^{\infty}\frac{1}{(n+1)!\cdot\omega^n}+4\cdot\sum_{n=0}^{\infty}\frac{1}{n!\cdot\omega^n}\\
&= 2\omega^2+5\omega+\sum_{n=0}^{\infty}\frac{1}{\omega^n}\cdot\left(\frac{2}{(n+2)!}+\frac{3}{(n+1)!}+\frac{4}{n!}\right)\\
&= 2\omega^2+5\omega+\sum_{n=0}^{\infty}\frac{1}{\omega^n}\cdot\frac{2+3(n+2)+4(n+2)(n+1)}{(n+2)!}\\
&= 2\omega^2+5\omega+\sum_{n=0}^{\infty}\frac{1}{\omega^n}\cdot\frac{4n^2+15n+16}{(n+2)!}.
\end{aligned}$$

Ako zamenimo ω sa $z-1$ dobijamo $f(z)=2(z-1)^2+5(z-1)+\displaystyle\sum_{n=0}^{\infty}\frac{4n^2+15n+16}{(n+2)!}\cdot\frac{1}{(z-1)^n}$. Ovaj red konvergira za svako $z\neq 1$.

6. Razviti funkciju $f(z)=z^2\cdot\cos\dfrac{1}{z-2}$ u Loranov red u okolini tačke $z_0=2$.

Rešenje:

Uvedemo smenu $\omega = z-2$ i nađemo razvoj funkcije $g(\omega) = f(\omega+2) = (\omega+2)^2 \cdot \cos\frac{1}{\omega} = (\omega^2+4\omega+4) \cdot \cos\frac{1}{\omega}$ u okolini tačke $\omega = 0$.

$$
\begin{aligned}
g(\omega) &= (\omega^2+4\omega+4)\cdot\cos\frac{1}{\omega} = \omega^2\cdot\cos\frac{1}{\omega} + 4\omega\cdot\cos\frac{1}{\omega} + 4\cdot\cos\frac{1}{\omega} \\
&= \omega^2\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n}} + 4\omega\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-2}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-1}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n}} \\
&= \omega^2 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-2}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-1}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n}} \\
&= \omega^2 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!\cdot\omega^{2n}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-1}} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n}} \\
&= \omega^2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{\omega^{2n}}\cdot\left(-\frac{1}{(2n+2)!} + \frac{4}{(2n)!}\right) + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-1}} \\
&= \omega^2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{\omega^{2n}}\cdot\frac{-1+4(2n+2)(2n+1)}{(2n+2)!} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-1}} \\
&= \omega^2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{\omega^{2n}}\cdot\frac{16n^2+24n+7}{(2n+2)!} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot\omega^{2n-1}}.
\end{aligned}
$$

Ako zamenimo ω sa $z-2$ dobijamo $f(z) = (z-2)^2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-2)^{2n}}\cdot\frac{16n^2+24n+7}{(2n+2)!} + 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot(z-2)^{2n-1}}$.
Ovaj red konvergira za svako $z \neq 2$.