predmet: Matematička analiza 2

BROJNI REDOVI

1. Ispitati konvergenciju sledećih redova:

a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$, b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n=1+\frac{1}{7}+\frac{1}{7^2}+\cdots$,

c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+\cdots$, d) $\displaystyle\sum_{n=1}^{\infty}(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n})$.

Rešenje:

a) Opšti član datog reda je $a_n=\dfrac{1}{n(n+1)}=\dfrac{1}{n}-\dfrac{1}{n+1}$, a n-ta parcijalna suma

$$S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}.$$

Kako je $\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$, dati red konvergira i njegova suma je $\displaystyle S=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$.

b) Opšti član datog reda je $a_n=\left(\dfrac{1}{7}\right)^n$, a n-ta parcijalna suma

$$S_n=\sum_{k=0}^{n-1}\frac{1}{7^k}=1+\frac{1}{7}+\frac{1}{7^2}+\cdots+\frac{1}{7^{n-1}}=\frac{1-\left(\frac{1}{7}\right)^n}{1-\frac{1}{7}}.$$

Kako je $\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1-\left(\frac{1}{7}\right)^n}{1-\frac{1}{7}}=\frac{1}{\frac{6}{7}}=\frac{7}{6}$, dati red konvergira i njegova suma je $\displaystyle S=\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n=\frac{7}{6}$.

c) Za dati red $\displaystyle\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+\cdots$, parcijalne sume su redom: $S_1=1$, $S_2=0$, $S_3=1$, $S_4=0$, itd. odnosno $S_{2n}=0$, a $S_{2n+1}=1$. Niz parcijalnih suma ima dva konvergentna podniza sa različitim granicama, pa $\displaystyle\lim_{n\to\infty}S_n$ ne postoji i dati red $\displaystyle\sum_{n=0}^{\infty}(-1)^n$ divergira.

d) Opšti član datog reda je

$a_n=\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}=\sqrt{n+2}-\sqrt{n+1}-\sqrt{n+1}+\sqrt{n}=\sqrt{n+2}-\sqrt{n+1}-(\sqrt{n+1}-\sqrt{n})$,

a n-ta parcijalna suma

$$\begin{aligned}S_n&=\sum_{k=1}^{n}(\sqrt{k+2}-\sqrt{k+1})-\sum_{k=1}^{n}(\sqrt{k+1}-\sqrt{k})=\\&=(\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\cdots+\sqrt{n+2}-\sqrt{n+1})-(\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\cdots+\sqrt{n+1}-\sqrt{n})=\\&=\sqrt{n+2}-\sqrt{2}-\sqrt{n+1}+1.\end{aligned}$$

Kako je

$$\begin{aligned}\lim_{n\to\infty}S_n&=\lim_{n\to\infty}(\sqrt{n+2}-\sqrt{n+1})+1-\sqrt{2}=\\&=1-\sqrt{2}+\lim_{n\to\infty}(\sqrt{n+2}-\sqrt{n+1})\cdot\frac{\sqrt{n+2}+\sqrt{n+1}}{\sqrt{n+2}+\sqrt{n+1}}=\\&=1-\sqrt{2}+\lim_{n\to\infty}\frac{n+2-n-1}{\sqrt{n+2}+\sqrt{n+1}}=1-\sqrt{2}+\lim_{n\to\infty}\frac{1}{\sqrt{n+2}+\sqrt{n+1}}=1-\sqrt{2},\end{aligned}$$

dati red konvergira i njegova suma je $\displaystyle S=\sum_{n=1}^{\infty}(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n})=1-\sqrt{2}$.

Kriterijum divergencije

2. Ispitati konvergenciju sledećih redova:

a) $\sum\limits_{n=1}^{\infty}\frac{n}{n+1}$, b) $\sum\limits_{n=1}^{\infty}\left(1+\frac{2}{n+1}\right)^n$, c) $\sum\limits_{n=0}^{\infty}(-1)^n n^2$, d) $\sum\limits_{n=1}^{\infty}\cos\left(\frac{n\pi}{2}\right)$,
e) $\sum\limits_{n=1}^{\infty}\frac{1}{(an+b)^p}, a,b>0, p\in\mathbb{R}$, f) $\sum\limits_{n=1}^{\infty}\frac{n}{\sqrt[n]{n!}}$.

Rešenje:

a) $\lim\limits_{n\to\infty}\frac{n}{n+1}=1\neq 0 \implies \sum\limits_{n=1}^{\infty}\frac{n}{n+1}$ divergira;

b) $\lim\limits_{n\to\infty}\left(1+\frac{2}{n+1}\right)^n=\lim\limits_{n\to\infty}\left(1+\frac{1}{\frac{n+1}{2}}\right)^{\frac{n+1}{2}\frac{2}{n+1}n}=\lim\limits_{n\to\infty}e^{\frac{2n}{n+1}}=e^2\neq 0 \implies \sum\limits_{n=1}^{\infty}\left(1+\frac{2}{n+1}\right)^n$ divergira;

c) $\lim\limits_{n\to\infty}(-1)^n n^2$ ne postoji $\implies \sum\limits_{n=0}^{\infty}(-1)^n n^2$ divergira;

d) $\lim\limits_{n\to\infty}\cos\left(\frac{n\pi}{2}\right)$ ne postoji $\implies \sum\limits_{n=1}^{\infty}\cos\left(\frac{n\pi}{2}\right)$ divergira;

e) $\lim\limits_{n\to\infty}\frac{1}{(an+b)^p}=\begin{cases}0, & p>0\\ 1, & p=0\\ \infty, & p<0\end{cases}$ pa za $p\leq 0$ red $\sum\limits_{n=1}^{\infty}\frac{1}{(an+b)^p}$ divergira (opšti član mu ne teži nuli), a za $p>0$ ne može se ništa reći o konvergenciji datog reda (potrebna su dalja ispitivanja).

f) Kad $n\to\infty$, $n!\sim\sqrt{2\pi n}\cdot\frac{n^n}{e^n}$(Stirlingova formula). Odatle je $\frac{n}{\sqrt[n]{n!}}\sim\frac{n}{\sqrt[n]{\sqrt{2\pi n}\cdot\frac{n^n}{e^n}}}=\frac{n}{\sqrt[2n]{2\pi n}\frac{n}{e}}=\frac{e}{\sqrt[2]{\sqrt[n]{2\pi}\sqrt[n]{n}}}$ i $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\lim\limits_{n\to\infty}\frac{e}{\sqrt[2]{\sqrt[n]{2\pi}\sqrt[n]{n}}}=e\neq 0$, pa sledi da $\sum\limits_{n=1}^{\infty}\frac{n}{\sqrt[n]{n!}}$ divergira.

Uporedni kriterijum I vrste

3. Uporednim kriterijumom prve vrste ispitati konvergenciju sledećih redova:

a) $\sum\limits_{n=1}^{\infty}\frac{2^n}{n\cdot 5^n}$;

b) $\sum\limits_{n=2}^{\infty}\frac{\ln n}{n}$.

Rešenje:

a) $a_n=\frac{2^n}{n\cdot 5^n}=\frac{1}{n}\left(\frac{2}{5}\right)^n\leq\left(\frac{2}{5}\right)^n, \forall n\in\mathbb{N}$, pa s obzirom na to da red $\sum\limits_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$ konvergira, i početni red $\sum\limits_{n=1}^{\infty}\frac{2^n}{n\cdot 5^n}$ konvergira;

b) $a_n=\frac{\ln n}{n}>\frac{1}{n}, n\geq 3$, pa s obzirom na to da red $\sum\limits_{n=2}^{\infty}\frac{1}{n}$ divergira, i početni red $\sum\limits_{n=2}^{\infty}\frac{\ln n}{n}$ divergira.

Uporedni kriterijum II vrste

4. Uporednim kriterijumom druge vrste ispitati konvergenciju sledećih redova:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-\sqrt{n}}{\sqrt{n^5+3n^2}}$, b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^5-2n+3}}{n^2}$, c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+n}{n^2+5^n}$, d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$,
e) $\displaystyle\sum_{n=1}^{\infty} (\sqrt[3]{n+1}-\sqrt[3]{n})$, f) $\displaystyle\sum_{n=1}^{\infty} \Big(1-\cos\frac{\pi}{n}\Big)$, g) $\displaystyle\sum_{n=1}^{\infty} \ln\Big(\frac{n^2+1}{n^2}\Big)$, h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}\ln\Big(\frac{n+1}{n-1}\Big)$,
j) $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{1+n^3}\cdot\sin\frac{1}{n}}$.

Rešenje:

a) Kad $n\to\infty$, $\displaystyle\frac{2n-\sqrt{n}}{\sqrt{n^5+3n^2}} \sim \frac{2n}{\sqrt{n^5}} = \frac{2}{n^{\frac{3}{2}}}$. Kako red $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^{\frac{3}{2}}}$ konvergira $\Big(\frac{3}{2}>1\Big)$, i početni red $\displaystyle\sum_{n=1}^{\infty}\frac{2n-\sqrt{n}}{\sqrt{n^5+3n^2}}$ konvergira.

b) Kad $n\to\infty$, $\displaystyle\frac{\sqrt[3]{n^5-2n+3}}{n^2} \sim \frac{\sqrt[3]{n^5}}{n^2} = \frac{1}{n^{\frac{1}{3}}}$. Kako red $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{3}}}$ divergira $\Big(\frac{1}{3}<1\Big)$, i početni red $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[3]{n^5-2n+3}}{n^2}$ divergira.

c) Kad $n\to\infty$, $\displaystyle\frac{2^n+n}{n^2+5^n} \sim \frac{2^n}{5^n} = \Big(\frac{2}{5}\Big)^n$. Kako red $\displaystyle\sum_{n=0}^{\infty}\Big(\frac{2}{5}\Big)^n$ konvergira, i početni red $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+n}{n^2+5^n}$ konvergira.

d) Kad $n\to\infty$,

$$\begin{aligned}\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n} &= \frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}\cdot\frac{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}} = \\ &= \frac{n^2+n+1-n^2+n-1}{n(\sqrt{n^2+n+1}+\sqrt{n^2-n+1})} = \frac{2}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}} \sim \\ &\sim \frac{2}{n+n} = \frac{1}{n}.\end{aligned}$$

Kako red $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ divergira, i početni red $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$ divergira.

e) Kad $n\to\infty$,

$$\begin{aligned}\sqrt[3]{n+1}-\sqrt[3]{n} &= (\sqrt[3]{n+1}-\sqrt[3]{n})\cdot\frac{\sqrt[3]{(n+1)^2}+\sqrt[3]{n+1}\sqrt[3]{n}+\sqrt[3]{n^2}}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n+1}\sqrt[3]{n}+\sqrt[3]{n^2}} = \\ &= \frac{n+1-n}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n+1}\sqrt[3]{n}+\sqrt[3]{n^2}} \sim \frac{1}{3n^{\frac{2}{3}}}.\end{aligned}$$

Kako red $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3n^{\frac{2}{3}}}$ divergira $\Big(\frac{2}{3}\le 1\Big)$, i početni red $\displaystyle\sum_{n=1}^{\infty}(\sqrt[3]{n+1}-\sqrt[3]{n})$ divergira.

f) Kad $n\to\infty$, $\displaystyle 1-\cos\frac{\pi}{n} = 2\sin^2\frac{\pi}{2n} \sim 2\Big(\frac{\pi}{2n}\Big)^2 = \frac{\pi^2}{2n^2}$. Kako red $\displaystyle\sum_{n=1}^{\infty}\frac{\pi^2}{2n^2}$ konvergira $(2>1)$, i početni red $\displaystyle\sum_{n=1}^{\infty}\Big(1-\cos\frac{\pi}{n}\Big)$ konvergira.

g) Kad $n \to \infty$, $\ln\left(\frac{n^2+1}{n^2}\right) = \ln\left(1+\frac{1}{n^2}\right) \sim \frac{1}{n^2}$. Kako red $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira $(2 > 1)$, i početni red $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{n^2}\right)$ konvergira.

h) Kad $n \to \infty$, $\frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right) = \frac{1}{\sqrt{n}} \ln\left(1+\frac{2}{n-1}\right) \sim \frac{2}{\sqrt{n}(n-1)} \sim \frac{2}{n^{\frac{3}{2}}}$. Kako red $\sum_{n=2}^{\infty} \frac{2}{n^{\frac{3}{2}}}$ konvergira $\left(\frac{3}{2} > 1\right)$, i početni red $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)$ konvergira.

j) Kad $n \to \infty$, $\frac{n+3}{\sqrt{1+n^3} \cdot \sin\frac{1}{n}} \sim \frac{n}{n^{\frac{3}{2}}\frac{1}{n}} = \frac{n}{n^{\frac{1}{2}}} = n^{\frac{1}{2}} = \sqrt{n}$. Kako red $\sum_{n=1}^{\infty} \sqrt{n}$ divergira (opšti član ne teži nuli), i početni red $\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{1+n^3} \cdot \sin\frac{1}{n}}$ divergira.

Dalamberov kriterijum

5. Dalamberovim kriterijumom ispitati konvergenciju sledećih redova:

a) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$, b) $\sum_{n=0}^{\infty} \frac{n!}{3^n}$, c) $\sum_{n=1}^{\infty} \frac{n^p}{n!}$, d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$, e) $\sum_{n=1}^{\infty} \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$, $a > 0$.

Rešenje:

a) Opšti član datog reda je $a_n = \frac{2^n n!}{n^n}$. Kako je $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} \frac{\frac{2(n+1)n!}{(n+1)\cdot(n+1)^n}}{\frac{n!}{n^n}} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = 2\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$, red $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ konvergira.

b) Opšti član datog reda je $a_n = \frac{n!}{3^n}$. Kako je $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \lim_{n\to\infty} \frac{n+1}{3} = \infty$, red $\sum_{n=0}^{\infty} \frac{n!}{3^n}$ divergira.

c) Opšti član datog reda je $a_n = \frac{n^p}{n!}$. Kako je $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^p}{(n+1)!}}{\frac{n^p}{n!}} = \lim_{n\to\infty} \frac{1}{n+1}\left(\frac{n+1}{n}\right)^p = \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{1}{n}\right)^p = 0 \cdot 1 = 0 < 1$, red $\sum_{n=1}^{\infty} \frac{n^p}{n!}$ konvergira.

d) Opšti član datog reda je $a_n = \frac{(n!)^2}{(2n)!}$. Kako je $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{(n+1)^2(n!)^2}{(2n+2)(2n+1)(2n)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$, red $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ konvergira.

e) Opšti član datog reda je $a_n = \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$. Kako je $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{a^{n+1}}{(1+a)(1+a^2)\cdots(1+a^n)(1+a^{n+1})}}{\frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}} = \lim_{n\to\infty} \frac{a}{1+a^{n+1}} = \begin{cases} 0, & a > 1 \\ \frac{1}{2}, & a = 1 \\ a, & a < 1 \end{cases}$, red $\sum_{n=1}^{\infty} \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$ konvergira za $a > 0$.

Košijev (korenski) kriterijum

6. Košijevim kriterijumom ispitati konvergenciju sledećih redova:

a) $\displaystyle\sum_{n=1}^{\infty} n\frac{2^n}{3^n}$, b) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{3}{n}\right)^{n^2}$, c) $\displaystyle\sum_{n=1}^{\infty} a^{\frac{n^2+2}{n+1}}$, $a>0$, d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{\left(a+\frac{1}{n}\right)^n}$, $a>0$.

Rešenje:

a) Opšti član datog reda je $a_n = n\dfrac{2^n}{3^n}$. Kako je $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{n\frac{2^n}{3^n}} = \lim_{n\to\infty}\sqrt[n]{n}\frac{2}{3} = \frac{2}{3} < 1$, red $\displaystyle\sum_{n=1}^{\infty} n\frac{2^n}{3^n}$ konvergira.

b) Opšti član datog reda je $a_n = \left(1+\dfrac{3}{n}\right)^{n^2}$. Kako je $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(1+\frac{3}{n}\right)^{n^2}} = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}\cdot\frac{3}{n}\cdot n} = e^3 > 1$, red $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{3}{n}\right)^{n^2}$ divergira.

c) Opšti član datog reda je $a_n = a^{\frac{n^2+2}{n+1}}$, $a>0$. Kako je $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{a^{\frac{n^2+2}{n+1}}} = \lim_{n\to\infty} a^{\frac{n^2+2}{n^2+n}} = a$. Konvergencija datog reda zavisi od vrednosti parametra a. Za $a<1$, red $\displaystyle\sum_{n=1}^{\infty} a^{\frac{n^2+2}{n+1}}$ konvergira. Za $a>1$, red $\displaystyle\sum_{n=1}^{\infty} a^{\frac{n^2+2}{n+1}}$ divergira. Za $a=1$ dobija se red $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$ koji divergira.

d) Opšti član datog reda je $a_n = \dfrac{n^3}{\left(a+\frac{1}{n}\right)^n}$, $a>0$. Kako je $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^3}{\left(a+\frac{1}{n}\right)^n}} = \lim_{n\to\infty}\frac{(\sqrt[n]{n})^3}{a+\frac{1}{n}} = \frac{1}{a}$. Konvergencija datog reda zavisi od vrednosti parametra a. Za $a>1$, red $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{\left(a+\frac{1}{n}\right)^n}$ konvergira. Za $a<1$, red $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{\left(a+\frac{1}{n}\right)^n}$ divergira. Za $a=1$, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^3}{\left(1+\frac{1}{n}\right)^n} = \infty$ pa kako opšti član ne teži nuli, red divergira.