Izračunavanje krivolinijskog integrala I vrste. $(x(t),y(t))$

$$\int_L P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \left[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\right]dt \quad \begin{array}{l} L: \ x=x(t) \\ \ \ \ y=y(t) \end{array}$$

$(x(t),y(t))$

Pr. Izračunati $I = \int_L y\,dx + x^2\,dy$ ako je

a) $L = \{(x,y) \mid 0 \le x \le 1,\ y = x\}$ orijentisana od $A(0,0)$ do $B(1,1)$.

$$L: \begin{array}{l} x(t)=t \\ y(t)=t \end{array},\ t\in[0,1] \qquad \begin{array}{l} x'(t)=1 \\ y'(t)=1 \end{array}$$

$$I = \int_0^1 (t + t^2 \cdot 1)\,dt = \frac{t^2}{2}\Big|_0^1 + \frac{t^3}{3}\Big|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

b) $L = L_1 \cup L_2$ $\quad L_1 = \{(x,y) \mid 0 \le x \le 1,\ y = 0\}$, $\quad L_2 = \{(x,y) \mid x = 1,\ 0 \le y \le 1\}$

orijentisana od tačke $B(1,0)$ do tačke $A(0,0)$

$$\int_L = \int_{L_1} + \int_{L_2} \qquad L_1: \begin{array}{l} x(t)=t \\ y(t)=0 \end{array},\ t\in[0,1] \qquad \begin{array}{l} x'(t)=1 \\ y'(t)=0 \end{array}$$

$$\int_{L_1} y\,dx + x^2\,dy = \int_1^0 (0 + 1^2 \cdot \boxed{0})\,dt = 0$$

Integral je po promj. t u x-osi

$$L_2: \begin{array}{l} x(t)=1 \\ y(t)=t \end{array},\ t\in[0,1] \qquad \int_{L_2} y\,dx + x^2\,dy = \int_0^1 (\boxed{t} \cdot 0 + 1^2)\,dt = -1$$

Integral je po promj. t u y-osi

Pr. Izračunati $\int_L y^2\,dx + x^2\,dy$ gdje je L gornja polovina centralne elipse sa poluosama $a, b > 0$ orijentisana

$$L: \begin{array}{l} x(t) = a\cos t \\ y(t) = b\sin t \end{array},\ t\in[\pi, 0]$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\int_L y^2\,dx + x^2\,dy = \int_\pi^0 \left[b^2\sin^2 t\,(-a\sin t) + a^2\cos^2 t\,(b\cos t)\right]dt$$

$$= ab^2 \int_0^\pi \sin^3 t\,dt - a^2 b \int_0^\pi \cos^3 t\,dt =$$

$$= ab^2 \int_0^\pi (1-\cos^2 t)\sin t\,dt - a^2 b \int_0^\pi (1-\sin^2 t)\cos t\,dt$$

$\cos t = u$, $-\sin t\,dt = du$, $\sin t\,dt = -du$ $\qquad$ $\sin t = u$, $\cos t\,dt = du$

$$= ab^2 \int_1^{-1} (1-u^2)(-du) = ab^2 \int_{-1}^1 (1-u^2)\,du =$$

$$= ab^2 \left[u - \frac{u^3}{3}\Big|_{-1}^{1}\right] = ab^2\left[2 - \frac{2}{3}\right] = \frac{4}{3}ab^2$$

Informatičarski Informatičari predmet: **Analiza 2**

PREZIME I IME: _____ BROJ INDEKSA: _____

PREDISPITNE OBAVEZE 1

1. Data je divergentna geometrijska suma redu $\sum_{k=1}^{\infty} \left(\frac{1}{3^k} - \frac{1}{3^{k+1}}\right)$, $k \in \mathbb{N}$. Izračunati:

Izračunati ugrubi ćlan reda a_n: $S_n - S_{n-1} = \frac{1}{2}\left(\cancel{1} - \frac{1}{3^n}\right) - \frac{1}{2}\left(\cancel{1} - \frac{1}{3^{n-1}}\right) =$

$= -\frac{1}{2}\cdot\frac{1}{3^n} + \frac{3}{2}\cdot\frac{1}{3^n} = \frac{1}{3^n}$.

Izračunati sumu reda S: $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1}{2}\left(1 - \cancel{\frac{1}{3^k}}^{\,0}\right) = \frac{1}{2}$

2. Poznatiji integrali koriste se kontrolom ili testom.

$a_n = f(n),\ n \in \mathbb{N}$ i f zadovoljava:

1) f je monotono opadajuća na $[n_0, \infty)$

2) $f \ge 0$ i definisana nad $[n_0, \infty)$.

Tada: $\int_{n_0}^{\infty} f(x)\,dx$ (K) $\iff \sum_{n=n_0}^{\infty} a_n$ (K)

Izračunati integralne kriterijum ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \frac{x^{-2}}{-2}\Big|_1^{+\infty} = -\frac{1}{2}\cdot\frac{1}{x^2}\Big|_1^{\infty} = -\frac{1}{2}(0-1) = \frac{1}{2} < \infty$$

(K)

3. Ispitati apsolutnu i uslovnu konvergenciju reda $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^2+1}}$.

$$|a_n| = \left|\frac{(-1)^{n-1}}{\sqrt{n^2+1}}\right| = \frac{1}{\sqrt{n^2+1}} = b_n$$

1) $b_n \to 0, n\to\infty$ ✓

2) $\frac{b_{n+1}}{b_n} \le 1$ ✓ $\quad$ nemamo U.K.

$\sum \frac{1}{n}$ (D) $\Rightarrow$ nemamo A.K.

$\sim \frac{1}{n}, n\to\infty$

4. Ispitati konvergenciju dalećih redova:

a) $\sum_{n=1}^{\infty} \frac{n^2+1-1^{n-1}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

(D) $\quad$ (D) $\quad$ A.K. $\Rightarrow$ (K)

$$\frac{1}{\sqrt{(n+1)^2+1}} \le \frac{1}{\sqrt{n^2+1}} \ (*)$$

$$(*) \iff \sqrt{(n+1)^2+1} \ge \sqrt{n^2+1}\ /^2$$

$$\iff (n+1)^2 \ge n^2$$

$$\iff \cancel{n^2} + 2n + 1 \ge \cancel{n^2} \quad (T)$$

b) $\sum_{n=1}^{\infty} \frac{\operatorname{arctg}(n^2)}{n^{2.5}}$

(K)

$$|a_n| = \frac{\operatorname{arctg}(n^2)}{n^{2.5}} \le \frac{\pi}{2}\cdot\frac{1}{n^{2.5}}$$

$$\sum \frac{1}{n^{2.5}} \ (K)$$

5. Dat je red $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n}$.

a) Odrediti poluprečnik i interval konvergencije.

$$a_n = \frac{1}{3^n} \qquad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} = 3$$

$x+2 \in (-3, 3)$

$x \in (-5, 1)$

b) Ispitati konvergenciju na krajevima intervala.

$x = 1$ $\quad \sum_{n=1}^{\infty} 1$ (D) $\quad 1 \not\to 0, n\to\infty$

$x = -5$ $\quad \sum_{n=1}^{\infty} (-1)^n$ (D) $\quad (-1)^n \not\to 0, n\to\infty$

6. Funkciju $f(x) = \frac{1}{3+x}$ razviti u Taylorov red u okolini tačke $a = -1$ i naći gde dobijeni red konvergira.

$$\frac{1}{3+x} = \frac{1}{2+x+1} = \frac{1}{2\left(1+\frac{x+1}{2}\right)} = \frac{1}{2}\cdot\frac{1}{1-\left(-\frac{x+1}{2}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^{n+1}}$$

$$\left|-\frac{x+1}{2}\right| < 1 \iff |x+1| < 2 \Rightarrow x \in (-3, 1)$$

7. Z ekvivalentnu dvostrukom integralima:

a) pravougaoni ćelijski integrali:

$$\int_0^1 dx \int_1^{2-x} f(x,y)\,dy = \int_0^1 dy \int_0^y f(x,y)\,dx + \int_1^2 dy \int_0^{2-y} f(x,y)\,dx$$

$2 - x = x$

$x = 1$

$y = 2 - x$

$x = 2 - y$

b) preti na polarne koordinate:

$$\int_{-\sqrt{2}}^{\sqrt{2}} dx \int_0^{\sqrt{4-x^2}} f(x,y)\,dy =$$

$$\int_0^2 dr \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(r\cos\varphi, r\sin\varphi)\cdot r\,d\varphi$$

$-y = \sqrt{4-y^2}\ /^2$

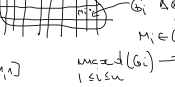
$y^2 = 4 - y^2$

$y^2 = 2$

$y = \pm\sqrt{2}$

8. Definicija dvostrukog integrala.

$$\iint_G f(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta G_i$$



ΔG_i – površina od G_i

$M_i \in G_i$

9. Naćeni tačno naglasak u parametarskom obliku:

a) $x^2 + y^2 = 1,\ x \in [-1, 1]$

$$x(t) = t \qquad y(t) = \sqrt[3]{1-t^2},\ t\in[-1,1]$$

$\operatorname{mc} = \sum d(G_i) \to 0, n\to\infty$

$1 \le i \le n$

b) Kontrolu na centrum $(-2, 2)$ poluprečnika 4

$$(x+2)^2 + (y-2)^2 = 16 \qquad \begin{array}{l} x+2 = 4\cdot\cos t \\ y-2 = 4\cdot\sin t \end{array} \iff \begin{array}{l} x(t) = -2 + 4\cos t \\ y(t) = 2 + 4\sin t \end{array}$$

$t \in [0, 2\pi)$

10. Osnovne karakteristike integrala prve vrste.

1) $\forall \alpha, \beta \in \mathbb{R}\ \ \forall f, g: L \to \mathbb{R}$

$$\int_L (\alpha f + \beta g)\,dl = \alpha\int_L f\,dl + \beta\int_L g\,dl$$

2) L_1 L_2 L

$$\int_L = \int_{L_1} + \int_{L_2}$$

3) $f \le g$

$$\int_L f\,dl \le \int_L g\,dl$$

4) Teorema srednje vrijednosti

. . .

Izračunati $\int_L (x+y)\,dl$ ako je

L dat: A(1, 1), B(1, 0), A(1, 1)

$$L: \begin{array}{l} x(t) = 1 \\ y(t) = t \end{array} \quad t\in[0,1]$$

$$\int_L (x+y)\,dl = \int_0^1 (1+t)\sqrt{0^2+1^2}\,dt = 1 + \frac{t^2}{2}\Big|_0^1 = \frac{3}{2}$$

L dat: C(0, 1), E(2, 3), D(2, 1)

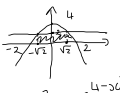
$$L: \begin{array}{l} x(t) = t \\ y(t) = 2 \end{array} \quad t\in[1,2]$$

$$\int_L (x+y)\,dl = \int_1^2 (t+2)\,dt = 2 + \frac{t^2}{2}\Big|_1^2 = 2 + \frac{3}{2} = \frac{7}{2}$$

Pr. $\int_0^2 dy \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\,dx =$

$x = \sqrt{4-y}\ /^2$

$x^2 = 4 - y \Rightarrow y = 4 - x^2$



$4 - x^2 = 2$

$x^2 = 2 \Rightarrow$

$\Rightarrow x = \pm\sqrt{2}$

$$= \int_{-2}^{-\sqrt{2}} dx \int_0^{4-x^2} f(x,y)\,dy + \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_0^2 f(x,y)\,dx + \int_{\sqrt{2}}^{2} dx \int_0^{4-x^2} f(x,y)\,dy.$$