

Recursive Formulae for Trotter-Suzuki Decompositions

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Introduction

In this document we gather the results of our derivations within the Mathematica notebook (Efficient_Trotterizations.nb). First we define our basis for each order n , which is written in terms of right-nested commutators as described by Arnal *et al.* (2021). We also symmetrize the basis according to the symmetry $A \leftrightarrow B$, which allows us to derive the recursive formulae for one of the steps only (we choose the step with the A operator). For orders $n \geq 6$ there also exist identities, which help reduce the total number of basis commutators. We also write down these relations.

In the second part we present the recursive formulae for the scheme coefficients $\alpha_A^{(i,i-1)}$, $\beta_A^{(i,i-1)}$, $\gamma_{j,A}^{(i,i-1)}$, ... These are written down in terms of the step done with the commutator A . By replacing the parameters a_i with b_i and the scheme coefficients with their respective counterparts, it is possible to find the formulae for the step in the B operator.

Basis

2nd order

$$\begin{aligned} C_1 &= [A^2B], \quad \tilde{\alpha} = -\frac{1}{24}, \\ C_2 &= [B^2A], \quad \tilde{\beta} = \frac{1}{12}. \end{aligned} \tag{1}$$

4th order

$$\begin{aligned} D_1 &= [A^4B] = [A^2C_1], & \tilde{\gamma}_1 &= \frac{7}{5760}, \\ D_2 &= [A^2BAB] = -[A^2C_2] = [ABA^2B] = [ABC_1], & \tilde{\gamma}_2 &= \frac{1}{480}, \\ D_3 &= [BA^3B] = [BAC_1], & \tilde{\gamma}_3 &= \frac{1}{360}, \\ D_4 &= [AB^3A] = [ABC_2], & \tilde{\gamma}_4 &= \frac{1}{360}, \\ D_5 &= [B^2ABA] = -[B^2C_1] = [BAB^2A] = [BAC_2], & \tilde{\gamma}_5 &= -\frac{1}{120}, \\ D_6 &= [B^4A] = [B^2C_2], & \tilde{\gamma}_6 &= -\frac{1}{720}. \end{aligned} \tag{2}$$

6th order

$$\begin{aligned} E_1 &= [A^6B] = [A^2D_1], & \tilde{\delta}_1 &= -\frac{31}{967680}, \\ E_2 &= [A^2BA^3B] = [A^2D_3], & \tilde{\delta}_2 &= -\frac{19}{80640}, \\ E_3 &= [ABA^4B] = [ABD_1], & \tilde{\delta}_3 &= \frac{23}{161280}, \\ E_4 &= [BA^5B] = [BAD_1], & \tilde{\delta}_4 &= -\frac{1}{10080}, \\ E_5 &= [A^2BABAB] = [A^2B^2A^2B] = -[A^2D_5], & \tilde{\delta}_5 &= -\frac{1}{10080}, \\ E_6 &= [ABA^2BAB] = [ABABA^2B] = [ABD_2], & \tilde{\delta}_6 &= -\frac{1}{20160}, \\ E_7 &= [AB^2A^3B] = [ABD_3], & \tilde{\delta}_7 &= \frac{13}{60480}, \\ E_8 &= [BABA^3B] = [BAD_3], & \tilde{\delta}_8 &= -\frac{1}{5040}, \\ E_9 &= [BA^3BAB] = [BA^2BA^2B] = [BAD_2], & \tilde{\delta}_9 &= -\frac{1}{3360}, \end{aligned} \tag{3}$$

$$\begin{aligned}
E_{10} &= [AB^3ABA] = [AB^2AB^2A] = [ABD_5], & \tilde{\delta}_{10} &= -\frac{19}{40320}, \\
E_{11} &= [ABAB^3A] = [ABD_4], & \tilde{\delta}_{11} &= \frac{1}{6720}, \\
E_{12} &= [BA^2B^3A] = [BAD_4], & \tilde{\delta}_{12} &= -\frac{1}{7560}, \\
E_{13} &= [BAB^2ABA] = [BABAB^2A] = [BAD_5], & \tilde{\delta}_{13} &= \frac{1}{1008}, \\
E_{14} &= [B^2ABABA] = [B^2A^2B^2A] = -[B^2D_2], & \tilde{\delta}_{14} &= -\frac{1}{10080}, \\
E_{15} &= [AB^5A] = [ABD_6], & \tilde{\delta}_{15} &= -\frac{1}{10080}, \\
E_{16} &= [BAB^4A] = [BAD_6], & \tilde{\delta}_{16} &= \frac{1}{10080}, \\
E_{17} &= [B^2AB^3A] = [B^2D_4], & \tilde{\delta}_{17} &= \frac{1}{5040}, \\
E_{18} &= [B^6A] = [B^2D_6], & \tilde{\delta}_{18} &= \frac{1}{30240}.
\end{aligned} \tag{4}$$

$$\begin{aligned}
[A^4BAB] &= [A^3BA^2B] = [A^2D_2] = 2E_2 - E_3, \\
[A^3B^2AB] &= -[A^2D_4] = 3E_5 - 3E_6 + E_7, \\
[B^2A^4B] &= [B^2D_1] = 2E_8 - E_9, \\
[A^2B^4A] &= [A^2D_6] = 2E_{11} - E_{10}, \\
[B^3A^2BA] &= -[B^2D_3] = 3E_{14} - 3E_{13} + E_{12}, \\
[B^4ABA] &= [B^3AB^2A] = [B^2D_5] = 2E_{17} - E_{16}.
\end{aligned} \tag{5}$$

Coefficient recursive formulae

2nd order

$$\alpha_A^{(i,i-1)} = \alpha_B^{(i-1,i-1)} + \tilde{\alpha} a_i^2 \sigma^{(i-1)} - \tilde{\beta} a_i \nu^{(i-1)} \sigma^{(i-1)}, \quad (6)$$

$$\beta_A^{(i,i-1)} = \beta_B^{(i-1,i-1)} + \tilde{\beta} a_i (\sigma^{(i-1)})^2. \quad (7)$$

4th order

$$\begin{aligned} \gamma_{1,A}^{(i,i-1)} &= \gamma_{1,B}^{(i-1,i-1)} + \tilde{\gamma}_1 a_i^4 \sigma^{(i-1)} + (\tilde{\gamma}_2 + \tilde{\gamma}_3) a_i^3 \nu^{(i-1)} \sigma^{(i-1)} - (\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 (\nu^{(i-1)})^2 \sigma^{(i-1)} - \\ &\quad - \tilde{\gamma}_6 a_i (\nu^{(i-1)})^3 \sigma^{(i-1)} + \tilde{\alpha} a_i^2 \alpha_B^{(i-1,i-1)} - \tilde{\beta} a_i \nu^{(i-1)} \alpha_B^{(i-1,i-1)}, \end{aligned} \quad (8)$$

$$\begin{aligned} \gamma_{2,A}^{(i,i-1)} &= \gamma_{2,B}^{(i-1,i-1)} + \tilde{\gamma}_2 a_i^3 (\sigma^{(i-1)})^2 - (2\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 \nu^{(i-1)} (\sigma^{(i-1)})^2 - \\ &\quad - 2\tilde{\gamma}_6 a_i (\nu^{(i-1)})^2 (\sigma^{(i-1)})^2 - \tilde{\alpha} a_i^2 \beta_B^{(i-1,i-1)} + \tilde{\beta} a_i \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} + \\ &\quad + \tilde{\beta} a_i \nu^{(i-1)} \beta_B^{(i-1,i-1)}, \end{aligned} \quad (9)$$

$$\begin{aligned} \gamma_{3,A}^{(i,i-1)} &= \gamma_{3,B}^{(i-1,i-1)} + \tilde{\gamma}_3 a_i^3 (\sigma^{(i-1)})^2 - \tilde{\gamma}_5 a_i^2 \nu^{(i-1)} (\sigma^{(i-1)})^2 - \tilde{\gamma}_6 a_i (\nu^{(i-1)})^2 (\sigma^{(i-1)})^2 - \\ &\quad - 2\tilde{\beta} a_i \sigma^{(i-1)} \alpha_B^{(i-1,i-1)}, \end{aligned} \quad (10)$$

$$\gamma_{4,A}^{(i,i-1)} = \gamma_{4,B}^{(i-1,i-1)} + \tilde{\gamma}_4 a_i^2 (\sigma^{(i-1)})^3 + \tilde{\gamma}_6 a_i \nu^{(i-1)} (\sigma^{(i-1)})^3 + \tilde{\beta} a_i \sigma^{(i-1)} \beta_B^{(i-1,i-1)}, \quad (11)$$

$$\gamma_{5,A}^{(i,i-1)} = \gamma_{5,B}^{(i-1,i-1)} + \tilde{\gamma}_5 a_i^2 (\sigma^{(i-1)})^3 + 2\tilde{\gamma}_6 a_i \nu^{(i-1)} (\sigma^{(i-1)})^3 - 2\tilde{\beta} a_i \sigma^{(i-1)} \beta_B^{(i-1,i-1)}, \quad (12)$$

$$\gamma_{6,A}^{(i,i-1)} = \gamma_{6,B}^{(i-1,i-1)} + \tilde{\gamma}_6 a_i (\sigma^{(i-1)})^4. \quad (13)$$

6th order

$$\begin{aligned} \delta_{1,A}^{(i,i-1)} &= \delta_{1,B}^{(i-1,i-1)} + \tilde{\delta}_1 a_i^6 \sigma^{(i-1)} + (\tilde{\delta}_2 + \tilde{\delta}_3 + \tilde{\delta}_4) a_i^5 \nu^{(i-1)} \sigma^{(i-1)} + \\ &\quad + (\tilde{\delta}_5 + \tilde{\delta}_6 + \tilde{\delta}_7 + \tilde{\delta}_8 + \tilde{\delta}_9) a_i^4 (\nu^{(i-1)})^2 \sigma^{(i-1)} - \\ &\quad - (\tilde{\delta}_{10} + \tilde{\delta}_{11} + \tilde{\delta}_{12} + \tilde{\delta}_{13} + \tilde{\delta}_{14}) a_i^3 (\nu^{(i-1)})^3 \sigma^{(i-1)} - (\tilde{\delta}_{15} + \tilde{\delta}_{16} + \tilde{\delta}_{17}) a_i^2 (\nu^{(i-1)})^4 \sigma^{(i-1)} - \\ &\quad - \tilde{\delta}_{18} a_i (\nu^{(i-1)})^5 \sigma^{(i-1)} + \tilde{\gamma}_1 a_i^4 \alpha_B^{(i-1,i-1)} + (\tilde{\gamma}_2 + \tilde{\gamma}_3) a_i^3 \nu^{(i-1)} \alpha_B^{(i-1,i-1)} - \\ &\quad - (\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 (\nu^{(i-1)})^2 \alpha_B^{(i-1,i-1)} - \tilde{\gamma}_6 a_i (\nu^{(i-1)})^3 \alpha_B^{(i-1,i-1)} + \tilde{\alpha} a_i^2 \gamma_{1,B}^{(i-1,i-1)} - \\ &\quad - \tilde{\beta} a_i \nu^{(i-1)} \gamma_{1,B}^{(i-1,i-1)}, \end{aligned} \quad (14)$$

$$\begin{aligned}
\delta_{2,A}^{(i,i-1)} = & \delta_{2,B}^{(i-1,i-1)} + \tilde{\delta}_2 a_i^5 \left(\sigma^{(i-1)} \right)^2 + \left(3\tilde{\delta}_5 + 2\tilde{\delta}_6 + \tilde{\delta}_7 + \tilde{\delta}_8 + 2\tilde{\delta}_9 \right) a_i^4 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 - \\
& - \left(3\tilde{\delta}_{10} + 4\tilde{\delta}_{11} + 4\tilde{\delta}_{12} + 3\tilde{\delta}_{13} + 2\tilde{\delta}_{14} \right) a_i^3 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^2 - \\
& - \left(5\tilde{\delta}_{15} + 5\tilde{\delta}_{16} + 4\tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^2 - 5\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^4 \left(\sigma^{(i-1)} \right)^2 - \\
& - 2\tilde{\gamma}_1 a_i^4 \beta_B^{(i-1,i-1)} + \tilde{\gamma}_3 a_i^3 \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} - 2(\tilde{\gamma}_2 + \tilde{\gamma}_3) a_i^3 \nu^{(i-1)} \beta_B^{(i-1,i-1)} - \\
& - \tilde{\gamma}_5 a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} + 2(\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 \left(\nu^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - \\
& - \tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^2 \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} + 2\tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^3 \beta_B^{(i-1,i-1)} + 2\tilde{\alpha} a_i^2 \gamma_{2,B}^{(i-1,i-1)} + \\
& + \tilde{\alpha} a_i^2 \gamma_{3,B}^{(i-1,i-1)} - 2\tilde{\beta} a_i \nu^{(i-1)} \gamma_{2,B}^{(i-1,i-1)} - \tilde{\beta} a_i \nu^{(i-1)} \gamma_{3,B}^{(i-1,i-1)} - \tilde{\beta} a_i \left(\alpha_B^{(i-1,i-1)} \right)^2,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\delta_{3,A}^{(i,i-1)} = & \delta_{3,B}^{(i-1,i-1)} + \tilde{\delta}_3 a_i^5 \left(\sigma^{(i-1)} \right)^2 - \left(\tilde{\delta}_5 - \tilde{\delta}_7 + \tilde{\delta}_9 \right) a_i^4 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 + \\
& + \left(\tilde{\delta}_{11} + 2\tilde{\delta}_{12} + \tilde{\delta}_{13} \right) a_i^3 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^2 + \left(\tilde{\delta}_{15} + 2\tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^2 + \\
& + \tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^4 \left(\sigma^{(i-1)} \right)^2 + \tilde{\gamma}_1 a_i^4 \beta_B^{(i-1,i-1)} + (\tilde{\gamma}_2 - 2\tilde{\gamma}_3) a_i^3 \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} + \\
& + (\tilde{\gamma}_2 + \tilde{\gamma}_3) a_i^3 \nu^{(i-1)} \beta_B^{(i-1,i-1)} - (2\tilde{\gamma}_4 - \tilde{\gamma}_5) a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} - \\
& - (\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 \left(\nu^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - \tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^3 \beta_B^{(i-1,i-1)} - \tilde{\alpha} a_i^2 \gamma_{2,B}^{(i-1,i-1)} + \\
& + \tilde{\beta} a_i \sigma^{(i-1)} \gamma_{1,B}^{(i-1,i-1)} + \tilde{\beta} a_i \nu^{(i-1)} \gamma_{2,B}^{(i-1,i-1)} + 2\tilde{\beta} a_i \left(\alpha_B^{(i-1,i-1)} \right)^2,
\end{aligned} \tag{16}$$

$$\begin{aligned}
\delta_{4,A}^{(i,i-1)} = & \delta_{4,B}^{(i-1,i-1)} + \tilde{\delta}_4 a_i^5 \left(\sigma^{(i-1)} \right)^2 + \left(\tilde{\delta}_8 + \tilde{\delta}_9 \right) a_i^4 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 - \\
& - \left(\tilde{\delta}_{12} + \tilde{\delta}_{13} + \tilde{\delta}_{14} \right) a_i^3 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^2 - \left(\tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^2 - \\
& - \tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^4 \left(\sigma^{(i-1)} \right)^2 + 2\tilde{\gamma}_3 a_i^3 \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} - 2\tilde{\gamma}_5 a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} - \\
& - 2\tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^2 \sigma^{(i-1)} \alpha_B^{(i-1,i-1)} - 2\tilde{\beta} a_i \sigma^{(i-1)} \gamma_{1,B}^{(i-1,i-1)} - \tilde{\beta} a_i \left(\alpha_B^{(i-1,i-1)} \right)^2,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\delta_{5,A}^{(i,i-1)} = & \delta_{5,B}^{(i-1,i-1)} + \tilde{\delta}_5 a_i^4 \left(\sigma^{(i-1)} \right)^3 - \left(\tilde{\delta}_{10} + 3\tilde{\delta}_{11} + 3\tilde{\delta}_{12} + \tilde{\delta}_{13} \right) a_i^3 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^3 - \\
& - \left(5\tilde{\delta}_{15} + 5\tilde{\delta}_{16} + 3\tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^3 - 5\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^3 + \\
& + \tilde{\gamma}_2 a_i^3 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - (\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} - \\
& - (2\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - 3\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} - \\
& - 2\tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^2 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - 3\tilde{\alpha} a_i^2 \gamma_{4,B}^{(i-1,i-1)} - \tilde{\alpha} a_i^2 \gamma_{5,B}^{(i-1,i-1)} + 3\tilde{\beta} a_i \nu^{(i-1)} \gamma_{4,B}^{(i-1,i-1)} + \\
& + \tilde{\beta} a_i \nu^{(i-1)} \gamma_{5,B}^{(i-1,i-1)} + \tilde{\beta} a_i \alpha_B^{(i-1,i-1)} \beta_B^{(i-1,i-1)},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\delta_{6,A}^{(i,i-1)} = & \delta_{6,B}^{(i-1,i-1)} + \tilde{\delta}_6 a_i^4 \left(\sigma^{(i-1)} \right)^3 - \left(\tilde{\delta}_{10} - \tilde{\delta}_{11} - 3\tilde{\delta}_{12} + \tilde{\delta}_{14} \right) a_i^3 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^3 + \\
& + \left(\tilde{\delta}_{15} + 3\tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^3 + \tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^3 - \\
& - 3\tilde{\gamma}_2 a_i^3 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} + (3\tilde{\gamma}_4 + 2\tilde{\gamma}_5) a_i^2 \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} + \\
& + (6\tilde{\gamma}_4 + 3\tilde{\gamma}_5) a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \beta_B^{(i-1,i-1)} + 7\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} + \\
& + 6\tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^2 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} + 3\tilde{\alpha} a_i^2 \gamma_{4,B}^{(i-1,i-1)} + \tilde{\beta} a_i \sigma^{(i-1)} \gamma_{2,B}^{(i-1,i-1)} - \\
& - 3\tilde{\beta} a_i \nu^{(i-1)} \gamma_{4,B}^{(i-1,i-1)} - 2\tilde{\beta} a_i \alpha_B^{(i-1,i-1)} \beta_B^{(i-1,i-1)}, \tag{19}
\end{aligned}$$

$$\begin{aligned}
\delta_{7,A}^{(i,i-1)} = & \delta_{7,B}^{(i-1,i-1)} + \tilde{\delta}_7 a_i^4 \left(\sigma^{(i-1)} \right)^3 - \left(\tilde{\delta}_{10} + \tilde{\delta}_{11} + \tilde{\delta}_{12} \right) a_i^3 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^3 - \\
& - \left(2\tilde{\delta}_{15} + \tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^3 - 2\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^3 + \\
& + (\tilde{\gamma}_2 + \tilde{\gamma}_3) a_i^3 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - 3\tilde{\gamma}_4 a_i^2 \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} - \\
& - 2(\tilde{\gamma}_4 + \tilde{\gamma}_5) a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - 3\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} - \\
& - 3\tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^2 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - \tilde{\alpha} a_i^2 \gamma_{4,B}^{(i-1,i-1)} + \tilde{\beta} a_i \sigma^{(i-1)} \gamma_{3,B}^{(i-1,i-1)} + \\
& + \tilde{\beta} a_i \nu^{(i-1)} \gamma_{4,B}^{(i-1,i-1)} - \tilde{\beta} a_i \alpha_B^{(i-1,i-1)} \beta_B^{(i-1,i-1)}, \tag{20}
\end{aligned}$$

$$\begin{aligned}
\delta_{8,A}^{(i,i-1)} = & \delta_{8,B}^{(i-1,i-1)} + \tilde{\delta}_8 a_i^4 \left(\sigma^{(i-1)} \right)^3 - \left(\tilde{\delta}_{13} + 2\tilde{\delta}_{14} \right) a_i^3 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^3 - \\
& - \left(\tilde{\delta}_{16} + 2\tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^3 - 3\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^3 - \\
& - 2\tilde{\gamma}_5 a_i^2 \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} - 4\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \alpha_B^{(i-1,i-1)} - 2\tilde{\beta} a_i \sigma^{(i-1)} \gamma_{3,B}^{(i-1,i-1)}, \tag{21}
\end{aligned}$$

$$\begin{aligned}
\delta_{9,A}^{(i,i-1)} = & \delta_{9,B}^{(i-1,i-1)} + \tilde{\delta}_9 a_i^4 \left(\sigma^{(i-1)} \right)^3 - \left(2\tilde{\delta}_{12} + \tilde{\delta}_{13} \right) a_i^3 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^3 - \\
& - \left(2\tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^3 - \tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^3 \left(\sigma^{(i-1)} \right)^3 - 2\tilde{\gamma}_3 a_i^3 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - \\
& + 2\tilde{\gamma}_5 a_i^2 \nu^{(i-1)} \sigma^{(i-1)} \beta_B^{(i-1,i-1)} + 2\tilde{\gamma}_6 a_i \left(\nu^{(i-1)} \right)^2 \sigma^{(i-1)} \beta_B^{(i-1,i-1)} - 2\tilde{\beta} a_i \sigma^{(i-1)} \gamma_{2,B}^{(i-1,i-1)} + \\
& + 2\tilde{\beta} a_i \alpha_B^{(i-1,i-1)} \beta_B^{(i-1,i-1)}, \tag{22}
\end{aligned}$$

$$\begin{aligned}
\delta_{10,A}^{(i,i-1)} = & \delta_{10,B}^{(i-1,i-1)} + \tilde{\delta}_{10} a_i^3 \left(\sigma^{(i-1)} \right)^4 + \left(\tilde{\delta}_{15} - \tilde{\delta}_{16} \right) a_i^2 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^4 + \tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^4 - \\
& - (2\tilde{\gamma}_4 - \tilde{\gamma}_5) a_i^2 \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - \tilde{\gamma}_6 a_i \left(\sigma^{(i-1)} \right)^3 \alpha_B^{(i-1,i-1)} - \tilde{\alpha} a_i^2 \gamma_{6,B}^{(i-1,i-1)} + \\
& + \tilde{\beta} a_i \sigma^{(i-1)} \gamma_{5,B}^{(i-1,i-1)} + \tilde{\beta} a_i \nu^{(i-1)} \gamma_{6,B}^{(i-1,i-1)} - \tilde{\beta} a_i \left(\beta_B^{(i-1,i-1)} \right)^2, \tag{23}
\end{aligned}$$

$$\begin{aligned}
\delta_{11,A}^{(i,i-1)} = & \delta_{11,B}^{(i-1,i-1)} + \tilde{\delta}_{11} a_i^3 \left(\sigma^{(i-1)} \right)^4 + \left(3\tilde{\delta}_{15} + 2\tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^4 + \\
& + 3\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^4 + \tilde{\gamma}_4 a_i^2 \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} + \tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} + \\
& + 2\tilde{\alpha} a_i^2 \gamma_{6,B}^{(i-1,i-1)} + \tilde{\beta} a_i \sigma^{(i-1)} \gamma_{4,B}^{(i-1,i-1)} - 2\tilde{\beta} a_i \nu^{(i-1)} \gamma_{6,B}^{(i-1,i-1)}, \tag{24}
\end{aligned}$$

$$\begin{aligned}\delta_{12,A}^{(i,i-1)} &= \delta_{12,B}^{(i-1,i-1)} + \tilde{\delta}_{12} a_i^3 \left(\sigma^{(i-1)} \right)^4 + \left(\tilde{\delta}_{16} + \tilde{\delta}_{17} \right) a_i^2 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^4 + 2\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^4 + \\ &+ \tilde{\gamma}_5 a_i^2 \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} + 4\tilde{\gamma}_6 a_i \left(\sigma^{(i-1)} \right)^3 \alpha_B^{(i-1,i-1)} + 2\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - \\ &- 2\tilde{\beta} a_i \sigma^{(i-1)} \gamma_{4,B}^{(i-1,i-1)},\end{aligned}\quad (25)$$

$$\begin{aligned}\delta_{13,A}^{(i,i-1)} &= \delta_{13,B}^{(i-1,i-1)} + \tilde{\delta}_{13} a_i^3 \left(\sigma^{(i-1)} \right)^4 + 2\tilde{\delta}_{16} a_i^2 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^4 - \tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^4 - \\ &- 4\tilde{\gamma}_5 a_i^2 \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - 8\tilde{\gamma}_6 a_i \left(\sigma^{(i-1)} \right)^3 \alpha_B^{(i-1,i-1)} - \\ &- 8\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - 2\tilde{\beta} a_i \sigma^{(i-1)} \gamma_{5,B}^{(i-1,i-1)} + 2\tilde{\beta} a_i \left(\beta_B^{(i-1,i-1)} \right)^2,\end{aligned}\quad (26)$$

$$\begin{aligned}\delta_{14,A}^{(i,i-1)} &= \delta_{14,B}^{(i-1,i-1)} + \tilde{\delta}_{14} a_i^3 \left(\sigma^{(i-1)} \right)^4 + 2\tilde{\delta}_{17} a_i^2 \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^4 + 5\tilde{\delta}_{18} a_i \left(\nu^{(i-1)} \right)^2 \left(\sigma^{(i-1)} \right)^4 + \\ &+ \tilde{\gamma}_5 a_i^2 \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} + 6\tilde{\gamma}_6 a_i \left(\sigma^{(i-1)} \right)^3 \alpha_B^{(i-1,i-1)} + 2\tilde{\gamma}_6 a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^2 \beta_B^{(i-1,i-1)} - \\ &- \tilde{\beta} a_i \left(\beta_B^{(i-1,i-1)} \right)^2,\end{aligned}\quad (27)$$

$$\begin{aligned}\delta_{15,A}^{(i,i-1)} &= \delta_{15,B}^{(i-1,i-1)} + \tilde{\delta}_{15} a_i^2 \left(\sigma^{(i-1)} \right)^5 + \tilde{\delta}_{18} a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^5 + \tilde{\gamma}_6 a_i \left(\sigma^{(i-1)} \right)^3 \beta_B^{(i-1,i-1)} + \\ &+ \tilde{\beta} a_i \sigma^{(i-1)} \gamma_{6,B}^{(i-1,i-1)},\end{aligned}\quad (28)$$

$$\delta_{16,A}^{(i,i-1)} = \delta_{16,B}^{(i-1,i-1)} + \tilde{\delta}_{16} a_i^2 \left(\sigma^{(i-1)} \right)^5 - \tilde{\delta}_{18} a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^5 - 2\tilde{\beta} a_i \sigma^{(i-1)} \gamma_{6,B}^{(i-1,i-1)}, \quad (29)$$

$$\delta_{17,A}^{(i,i-1)} = \delta_{17,B}^{(i-1,i-1)} + \tilde{\delta}_{17} a_i^2 \left(\sigma^{(i-1)} \right)^5 + 5\tilde{\delta}_{18} a_i \nu^{(i-1)} \left(\sigma^{(i-1)} \right)^5 - 2\tilde{\gamma}_6 a_i \left(\sigma^{(i-1)} \right)^3 \beta_B^{(i-1,i-1)}, \quad (30)$$

$$\delta_{18,A}^{(i,i-1)} = \delta_{18,B}^{(i-1,i-1)} + \tilde{\delta}_{18} a_i \left(\sigma^{(i-1)} \right)^6. \quad (31)$$