Weber Problem

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Introduction to the Weber Problem

Weber Problem: Find the "minimum" point (x^*, y^*) which minimizes the sum of weighted distances from itself to n fixed points with co-ordinate $(a_i.b_i)$, where each point i has a weight w_i

Formulation of Weber Problem

$$\min_{x,y}\{W(x,y) = \sum_{t=1}^{n} w_i d_i(x,y)\}$$

$$d_i(x,y) = \sqrt{(x-a_i)^2 + (y-b_i)^2)}$$

Solving the Weber Problem Geometrically

- The Varignon Frame.
- This works because it is essentially the dual of the linear programming formulation.
- The vector (u_i, v_i) is the negative of the force vector exerted on the knot by weight w_i.

Dual of the Weber problem

$$\max_{(U,V)} D(U,V) = -\sum_{i=1}^{n}(a_i u_i + b_i v_i)$$

s.t.

$$\sum_{i=1}^{n} u_i = 0$$

$$\sum_{i=1}^{n} v_i = 0$$

$$\sqrt{u_i^2 + v_i^2} \leq w_i$$

(1)

Weiszfield Algorithm

- Take derivitive of the objective function and set to 0
- Start with a candidate solution
- Determine next iteration
- repeat until convergence
- Note: can have problems if a candidate location is one of the fixed point. (Because the algorithm will only approach the fixed point.)

$$\frac{dW(x,y)}{dx} = \sum_{i=1}^{n} \frac{w_i(x-a_i)}{d_i(x,y)} = 0$$

dW(x,y) $\overline{dy = \sum_{i=1}^{n} \frac{w_i(y-b_i)}{d_i(x,y)} = 0}$

$$x^{(k+1)}, y^{(k+1)} = \left(\frac{\sum_{i=1}^{n} \frac{w_i a_i}{d_i\left(x^{(k)}, y^{(k)}\right)}}{\sum_{i=1}^{n} \frac{w_i}{d_i(x^{(k)}, y^{(k)})}}, \frac{\sum_{i=1}^{n} \frac{w_i b_i}{d_i\left(x^{(k)}, y^{(k)}\right)}}{\sum_{i=1}^{n} \frac{w_i}{d_i\left(x^{(k)}, y^{(k)}\right)}} \right) \quad (2)$$

Centroid formulation

- Minimizing sum of squared euclidean distances
- This is the centroid

$$\min_{x,y} C(x, y) = \sum_{i=1}^{n} w_i d_i^2(x, y)$$

Rectilinear distances

Manhattan distance - only allow travel in a grid e.g. Roads in a grid layout

$$\min_{x,y} R(x,y) = \sum_{i=1}^{n} w_i d_i^R(x,y)$$

$$\mathrm{d}_i^R(x,y) = |x - a_i| + |y - b_i|$$

p-norm Distance

- p-norm are generalizations of Euclidean distances

$$l_{pi} = \sqrt[p]{|x - a_i|^p + |y - b_i|^p} \tag{3}$$

Multiple Facilities

- Where m new facilities will be placed and one new facility y
- Weights between facility j and demand point i is w_{ij}

- Weights between facility j and demand point s is $v$js

$$\min_{(x,y)j=1,\ldots(),m} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \sqrt{(x_j - a_i)^2 + (y_j - b_i)^2} +$$

$\sum_{j=1}^{m-1} \sum_{s=j+1}^{m} v_{js} \sqrt{(x_j - a_s)^2 + (y_j - b_s)^2}$

Restricted regions

- Forbidden regions - Cannot locate in F
- $\min_{x,y} F(x,y) = \sum_{i=1}^{n} w_i d_i(x,y)$
- Barriers - Path cannot cross B
- $\min_{x,y} B(x,y) = \sum_{i=1}^{n} w_i d_i^B(x,y)$

Point along line

- Locate a line l which is as close as possible to the set of points
- $\min_{lines l} L(l) = \sum_{i=1}^{n} w_i \gamma_i(l)$

- $\gamma_i(l)$ is the closest distance to point i to line l