4.1 Landmark-based models

In order to carry out tasks like localization or navigation, a mobile robot has to perceive its workspace. A variety of sensors can be used for that, as well as a number of probabilistic models for managing their behavior.

Typically, the sensors used onboard the robot do not deliver the exact truth of the quantities they are measuring, but a perturbed version. This is due to the working (physical) principles that govern the sensors behavior, and to the conditions of their workspaces (illumination, humidity, temperature, etc.).

As an illustrative example of this, there is a popular European company called Sick, which develops 2D LiDAR sensors (among other devices). One of its most popular sensors is the TiM2xx one (see left part of Fig.1), which can be easily integrable into a robotic platform. If we take a look at the specifications about the performance of such device, we can check how this uncertainty about the sensor measurements is explicitly specified (systematic error and statistical error), as well as how these values depend on environmental conditions (see right part of Fig.1). $\\[10pt]$

Fig. 1: Left, TiM2xx sensor from Sick. Right, performance details of such sensor.

$\\[5pt]$To account for this behavior, sensors' measurements in probabilistic robotics will be modeled by... wait for it... the probability distribution $p(z|v)$, where z models the measurement and v is the ground truth.

4.1.1 Dealing with landmark-based models

In different applications it is interesting for the robot to detect landmarks in its workspace and build internal representations of them, commonly referred to as maps. A landmark can be defined as a distinctive feature present in the environment, that can be used to perform localization, map building, or navigation, since they provide a fixed reference point in the environment. They can be of different nature:

- **Natural landmarks**: mountains, trees, rivers, rocks, etc.
- **Artificial landmarks**: buildings, signs, traffic lights, doors, windows, furniture, etc.
    - ***Purpose-built landmarks***: QR codes, RFID tags, beacons, etc.

In both scenarios there could be also extracted landmark or features like corners, blobs, etc., e.g. using a camera.

In the case of maps consisting of a collection of landmarks $m=\{m_i\}, i=1,\dots,N$, different types of sensors can be used to provide observations z_i of those landmarks:

- **Distance/range** (*e.g.* radio, GPS, etc.): $\\[2pt]$ $\hspace{1cm}$ $z_i = d_i = h_i(x,m)+w_i$ $\\[2pt]$
- **Bearing** (*e.g.* camera): $\\[2pt]$ $\hspace{1cm}$$z_i = \theta_i = h_i(x,m)+w_i$ $\\[2pt]$

- **Distance/range and bearing** (*e.g.* stereo, features in a scan, etc.) $\\[2pt]$ $\hspace{1cm}$$z_i = [d_i,\theta_i]^T = h_i(x,m)+w_i$ *(in this case, $h_i(x,m)$ and w_i are 2D vectors)* $\\[2pt]$

where:

- z_i is an observation, x is the sensor pose, and m is the map of the environment,
- $h(x,m)$ is the Observation (or measurement, or prediction) function: it predicts the value of the observation z_i given the state values x and m, and
- w is an error, modeled by a gaussian distribution as $w=[h(x,m)-z_i]\sim N(O,Q)$, being Q the uncertainty in the observation error.

In this way, the probability distribution $p(z|x,m)$ modeling the sensor measurements results:

$$ p(z|x,m) = K \exp\{-\frac{1}{2}[h(x,m)-z]^T Q^{-1} [h(x,m)-z]\} $$
These types of maps and sensor measurements pose a new problem: **data association**, that is, with which landmark m_i correspond the observation z_i to:

$$h_i(x,m)=h(x,m_i)$$
This problem is usually addressed by applying Chi-squared tests, although for the shake of simplicity in this book we will consider it as solved.

Playing with landmarks and robot poses

In the remaining of this section we will familiarize ourselves with the process of observing landmarks from robots located at certain poses, as well as the transformations needed to make use of these observations, that is, to express those observations into the world frame and backwards.

Some relevant concepts:

- **World frame**: (x, y) coordinates from a selected point of reference $(0, 0)$ used to keep track of the robots pose and landmarks within the map.
- **Observation**: Information from the real world provided by a sensor, from the point of view (*pov*) of a certain robot.
- **Range-bearing sensor**: Sensor model being used in this lesson. This kind of sensors detect how far is an object (d) and its orientation relative to the robot's one (θ).

The main tools to deal with those concepts are:

- the composition of two poses.
- the composition of a pose and a landmark.
- the propagation of uncertainty through the Jacobians of these compositions.

We will address several problems of incremental complexity. In all of them, it is important to have in mind how the composition of a (robot) pose and a landmark point works:

Fig. 1: Composition of a pose and a landmark point.

In [1]:
```
#%matplotlib widget
%matplotlib inline
```

```python
# IMPORTS

import math
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats

import sys
sys.path.append("..")
from utils.PlotEllipse import PlotEllipse
from utils.DrawRobot import DrawRobot
from utils.tcomp import tcomp
from utils.tinv import tinv, jac_tinv1 as jac_tinv
from utils.Jacobians import J1, J2
```

ASSIGNMENT 1: Expressing an observed landmark in coordinates of the world frame

Let’s consider a robot R1 at a perfectly known pose $p_1 = [1, 2, 0.5]^T$ (no uncertainty at this point) which observes a landmark m with a range-bearing (polar) sensor affected by a zero-mean Gaussian error with covariance $W_{1p} = diag\left([0.25, 0.04]\right)$. The sensor provides the measurement $z_{1p} = [4m., 0.7rad.]^T$. The scenario is the one in Fig. 2.

Fig 2. Illustration of the scenario in assignment 1.

You are tasked to compute the Gaussian probability distribution (mean and covariance) of the landmark observation in the world frame (the same as the robot) and plot its corresponding ellipse (in magenta, $\sigma=1$). Concretely, you have to complete the `to_world_frame()` function, and modify the demo code to show the ellipse representing the uncertainty.

Consider the following:

- You can express a sensor measurement in polar coordinates ($z_p=[r,\alpha]^T$) as cartesian coordinates ($z_c=[z_x,z_y]^T)$ by:

  $$ z_c = \begin{bmatrix} z_x \\ z_y \end{bmatrix}

  ```
  = \begin{bmatrix} r \ cos\alpha \\ r \ sin\alpha \end{bmatrix}
  = f(r,\alpha)
  ```
  $$ $\[5pt]$

- While computing the covariance of the landmark observation, you have to start by computing the covariance of the observation in the Cartesian robot $R1$ frame. That is:

  $$ W_{c} = \frac{\partial z_c}{\partial z_p} \ W_{p} \ \frac{\partial z_c}{\partial z_p}^T = \frac{\partial f(r,\alpha)}{\partial \{r,\alpha\}} \ W_{p} \ \frac{\partial f(r,\alpha)}{\partial \{r,\alpha\}}^T $$

Mathematical pill:

$$ F(x_1,\dots,x_n)=\begin{bmatrix}f_1 \\ \vdots \\ f_m\end{bmatrix} \, \Rightarrow \, \frac{\partial F(x_1,\dots,x_n)}{\partial\{x_1,\dots,x_n\}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} &

\frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} $$
Then you can get the convariance in the world frame as:

$$ W_{z\_w} = \frac{\partial f(p,z_c)}{\partial p} \ Q_{p1\_w} \ \left(\frac{\partial f(p,z_c)}{\partial p} \right)^T + \frac{\partial f(p,z_c)}{\partial z_c} \ W_{c} \ \left(\frac{\partial f(p,z_c)}{\partial z_c} \right)^T $$

where $f(p,z_c) = p \oplus z_c$, that is, the composition of the pose and the landmark.

- Note that $\frac{\partial f(p,z_c)}{\partial p}$ and $\frac{\partial f(p,z_c)}{\partial z_c}$ are the same Jacobians as previously used to compose two poses in *robot motion*, but with a reduced size since **while working with landmarks the orientation is meaningless, only the position matters**. The functions `J1()` and `J2()` implement these jacobians for you. $\\[5pt]$
- Note 2: this expression is just a rewriting of: $$ W_{z\_w} = \frac{\partial f(p,z_c)}{\partial {p,z_c}} \ \begin{bmatrix} Q_{p1\_w} & \mathbf{0} \\ \mathbf{0} & W_{c} \end{bmatrix} \ \left(\frac{\partial f(p,z_c)}{\partial {p,z_c}} \right)^T $$

Example: $\\[5pt]$

Fig 3. Pose of a robot (without uncertainty) and position of an observed landmark with its associated uncertainty.

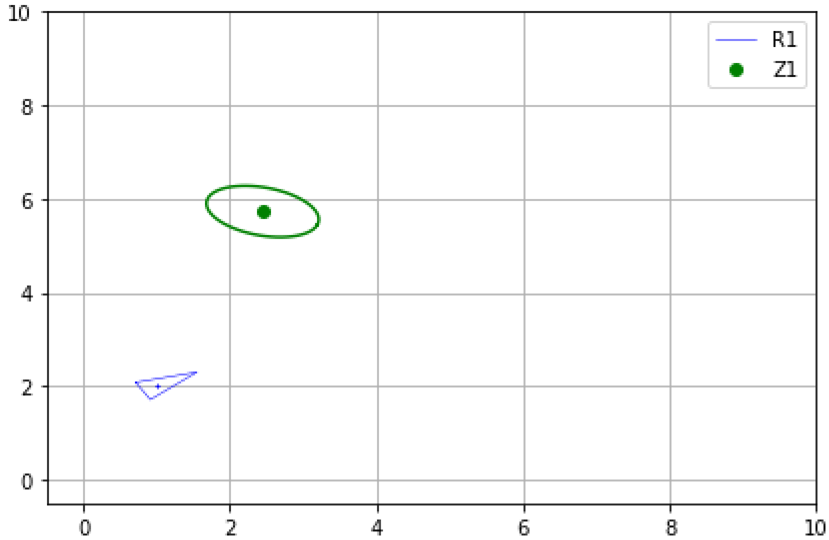
In [2]:
```
def to_world_frame(p1_w, Qp1_w, z1_p_r, W1):
    """ Covert the observation z1_p_r to the world frame

        Args:
            p1_w: Pose of the robot(in world frame)
            Qp1_w: Covariance of the robot
            z1_p_r: Observation to a landmark (polar coordinates) from robots
            W1: Covariance of the sensor in polar coordinates

        Returns:
            z1_w: Pose of landmark in the world frame
            Wz1: Covariance associated to z1_w
    """

    # Definition of useful variables
    r, a = z1_p_r[0,0], z1_p_r[1,0]
    s, c = np.sin(a), np.cos(a)

    # Jacobian to convert the measurement uncertainty from polar to cartesian
    Jac_pol_car = np.array([
        [c, -r*s],
        [s, r*c]
    ])

    # Built a tuple with:
    # z1_car_rel[0]: coordinates of the sensor measurement in cartesian coord
    # z1_car_rel[1]: its associated uncertainty expressed in cartesian coordi
    z1_car_rel = (
            np.vstack([r*c,r*s]), # position
            Jac_pol_car@W1@Jac_pol_car.T # uncertainty
            )
```

```
    z1_ext = np.vstack([z1_car_rel[0], 0]) # Extends z1 for its usage in the

    # Build the jacobians
    Jac_ap = J1(p1_w ,z1_ext)[0:2,:] # Jacobian for expressing the uncertaint
    Jac_aa = J2(p1_w ,z1_ext)[0:2,0:2] # This one expresses the uncertainty i

    z1_w = tcomp(p1_w ,z1_ext)[0:2,[0]] # Compute coordinates of the landmark
    Wz1 = (Jac_ap @ Qp1_w @ Jac_ap.T
          + Jac_aa @ z1_car_rel[1] @ Jac_aa.T) # Finally, propagate the covar

    return z1_w, Wz1
```

In [3]:

```
# Robot
p1_w = np.vstack([1, 2, 0.5]) # Robot R1 pose
Qp1_w = np.zeros((3, 3)) # Robot pose convariance matrix (uncertainty)

# Landmark observation
z1_p_r = np.vstack([4., .7]) # Measurement/Observation
W1 = np.diag([0.25, 0.04]) # Sensor noise covariance

# Express the landmark observation in the world frame (mean and covariance)
z1_w, Wz1 = to_world_frame(p1_w, Qp1_w, z1_p_r, W1)

# Visualize the results
fig, ax = plt.subplots()
plt.xlim([-.5, 10])
plt.ylim([-.5, 10])
plt.grid()
plt.tight_layout()

DrawRobot(fig, ax, p1_w, label='R1', color='blue')

ax.plot(z1_w[0, 0], z1_w[1, 0], 'o', label='Z1', color='green')
PlotEllipse(fig, ax, z1_w, Wz1, color='green')

plt.legend()
print('----\tExercise 4.1.1\t----\n'+
      'z1_w = {}\'\n'.format(z1_w.flatten())
      + 'Wz1_w = \n{}\n'.format(Wz1))
```

```
----    Exercise 4.1.1  ----
z1_w = [2.44943102 5.72815634]'
Wz1_w =
[[ 0.58879177 -0.13171532]
 [-0.13171532  0.30120823]]
```



Expected results for demo:

```
---- Exercise 4.1.1 ----
z1_w = [2.44943102 5.72815634]'
Wz1_w =
[[ 0.58879177 -0.13171532]
 [-0.13171532  0.30120823]]
```

ASSIGNMENT 2: Adding uncertainty to the robot position

Now, let's assume that the robot pose is not known, but it is a RV that follows a Gaussian probability distribution: $p_1 \sim N([1, 2, 0.5]^T, \Sigma_1)$ with $\Sigma_1 = diag([0.08,0.6, 0.02])$.

1. Compute the covariance matrix Σ_{m1} of the landmark in the world frame and plot it as an ellipse centered at the mean m_1 (in blue, $sigma= 1$). Plot also the covariance of the robot pose (in blue, $sigma= 1$).

2. Compare the covariance with that obtained in the previous case. $\\[5pt]$

Example: $\\[5pt]$

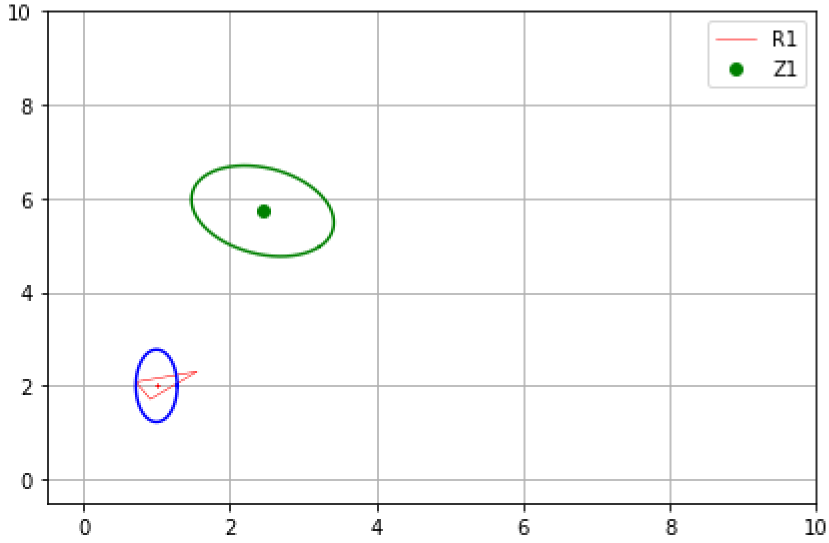
Fig 4. Pose of a robot and position of an observed landmark, along with their associated uncertainties.

In [4]:

```
# Robot
p1_w = np.vstack([1, 2, 0.5]) # Robot R1 pose
Qp1_w = np.diag([0.08,0.6,0.02])  # Robot pose convariance matrix (uncertaint

# Landmark observation
z1_p_r = np.vstack([4., .7]) # Measurement/Observation
W1 = np.diag([0.25, 0.04]) # Sensor noise covariance

# Express the landmark observation in the world frame (mean and covariance)
z1_w, Wz1 = to_world_frame(p1_w, Qp1_w, z1_p_r, W1)

# MATPLOTLIB
fig, ax = plt.subplots()
plt.xlim([-.5, 10])
plt.ylim([-.5, 10])
plt.grid()
plt.tight_layout()

fig.canvas.draw()

DrawRobot(fig, ax, p1_w, label='R1', color='red')
PlotEllipse(fig, ax, p1_w, Qp1_w , color='blue')

ax.plot(z1_w[0, 0], z1_w[1, 0], 'o', label='Z1', color='green')
PlotEllipse(fig, ax, z1_w, Wz1, color='green')

plt.legend()
print('---- Exercise 4.1.2 ----\n'+
      'Wz1_w = \n{}\n'.format(Wz1))
```

```
---- Exercise 4.1.2 ----
Wz1_w =
```

```
[[ 0.94677477 -0.23978943]
 [-0.23978943  0.94322523]]
```



Expected results for demo:

```
---- Exercise 4.1.2 ----
Wz1_w =
[[ 0.94677477 -0.23978943]
 [-0.23978943  0.94322523]]
```

ASSIGNMENT 3: Getting the relative pose between two robots

Another robot `R2` is at pose $p_2 \sim ([6m., 4m., 2.1rad.]^T, \Sigma_2)$ with $\Sigma_2 = diag([0.20,0.09, 0.03])$. Plot `p2` and its ellipse (covariance) in green ($sigma=1$). **Compute the relative pose `p12` between `R1` and `R2`**, including its associated uncertainty. This scenario is shown in Fig. 5.

Fig 5. Illustration of the scenario in this assignment.

This relative pose can be obtained in two different ways:

- **Through the composition of poses**, but using $\ominus p1$ instead of $p1$. Implement it in `inverse_composition1()`.

  Mean:

  $$ p12 = \ominus p1 \oplus p2 = f(\ominus p1, p2) = \begin{bmatrix} x_{\ominus p1} + x_{p2} cos \theta_{\ominus p1} - y_{p2} sin \theta_{\ominus p1} \\ y_{\ominus p1} + x_{p2} sin \theta_{\ominus p1} + y_{p2} cos \theta_{\ominus p1} \\ \theta_{\ominus p1} + \theta_{p2} \end{bmatrix} $$

  Covariance:

  $$ \Sigma_{p12} = \frac{\partial p12}{\partial \ominus p1} \frac{\ominus p1}{\partial p1} \Sigma_{p1} \frac{\ominus p1}{\partial p1}^T \frac{\partial p12}{\partial \ominus p1}^T + \frac{\partial p12}{\partial p2} \Sigma_{p2} \frac{\partial p12}{\partial p2}^T \\ \text{Applying the Chain rule} \rightarrow \Sigma_{p12} = \frac{\partial p12}{\partial \ominus p1}

\Sigma_{\ominus p1} \frac{\partial p12}{\partial \ominus p1}^T + \frac{\partial p12}{\partial p2} \Sigma_{p2} \frac{\partial p12}{\partial p2}^T $$

Being:

$$ \frac{\partial p12}{\partial \ominus p1} = \begin{bmatrix} 1 & 0 & -x_{p2} sin \theta_{\ominus p1} - y_{p2} cos \theta_{\ominus p1} \\ 0 & 1 & x_{p2} cos \theta_{\ominus p1} - y_{p2} sin \theta_{\ominus p1} \\ 0 & 0 & 1 \end{bmatrix} \; \; \; \; \; \frac{\partial p12}{\partial p2} = \begin{bmatrix} cos \theta_{\ominus p1} & -sin \theta_{\ominus p1} & 0\\ sin \theta_{\ominus p1} & cos \theta_{\ominus p1} & 0\\ 0 & 0 & 1 \end{bmatrix} $$ $\\[5pt]$

$$ \frac{\partial \ominus p1}{\partial p1} = \begin{bmatrix} -cos \theta_{p1} & -sin \theta_{p1} & x_{p1} sin \theta_{p1} - y_{p1} cos \theta_{p1} \\ sin \theta_{p1} & -cos \theta_{p1} & x_{p1} cos \theta_{p1} + y_{p1} sin \theta_{p1}\\ 0 & 0 & -1 \\ \end{bmatrix} \; \; \; \; \; \Sigma_{\ominus p1} = \frac{\partial \ominus p1}{\partial p1} \Sigma_{p1} \frac{\partial \ominus p1}{\partial p1}^T $$ $\\[5pt]$

- **Using the inverse composition of poses**, that is $p12 = \ominus p1 \oplus p2 = p2 \ominus p1$. This one is given for you in `inverse_composition2()` .

In [5]:

```
def inverse_composition1(p1_w, Qp1_w, p2_w, Qp2_w):
    jac_inv_p = jac_tinv(p1_w)

    inv_r1 = (
        tinv(p1_w),
        jac_inv_p @ Qp1_w @ jac_inv_p.T
    )

    jac_p12_inv = J1(inv_r1[0], p2_w)
    jac_p12_p2 = J2(inv_r1[0], p2_w)

    p12_w = tcomp(inv_r1[0], p2_w)

    Qp12_w = (
            jac_p12_inv@inv_r1[1]@jac_p12_inv.T
            + jac_p12_p2@Qp2_w@jac_p12_p2.T
        )

    return p12_w, Qp12_w
```

In [6]:

```
def inverse_composition2(p1_w, Qp1_w, p2_w, Qp2_w):
    dx, dy = p2_w[0, 0]-p1_w[0, 0], p2_w[1, 0]-p1_w[1, 0]
    a = p2_w[2, 0] - p1_w[2, 0]
    c, s = np.cos(p1_w[2, 0]), np.sin(p1_w[2, 0])

    p12_w = np.array([
        [dx*c + dy*s],
        [-dx*s + dy*c],
        [a]])

    jac_p12_r1 = np.array([
        [-c, -s, -dx*s + dy*c],
        [s, -c, -dx*c - dy*s],
        [0, 0, -1]
    ])

    jac_p12_r2 = np.array([
```

```
        [c, s, 0],
        [-s, c, 0],
        [0, 0, -1]
    ])

    #jac_p1_pinv = np.linalg.inv(jac_tinv(r1[0]))

    Qp12_w = jac_p12_r1@Qp1_w@jac_p12_r1.T + jac_p12_r2@Qp2_w@jac_p12_r2.T

    return p12_w, Qp12_w
```

In [7]:

```
# Robot R1
p1_w = np.vstack([1., 2., 0.5])
Qp1_w = np.diag([0.08, 0.6, 0.02])

# Robot R2
p2_w = np.vstack([6., 4., 2.1])
Qp2_w = np.diag([0.20, 0.09, 0.03])

# Obtain the relative pose p12 between both robots through the composition of
p12_w, Qp12_w = inverse_composition1(p1_w, Qp1_w, p2_w, Qp2_w)
print( '----\tExercise 4.1.3 with method 1\t----\n'+
        'p12_w = {}\'\n'.format(p12_w.flatten())+
        'Qp12_w = \n{}\n'.format(Qp12_w))

# Obtain the relative pose p12 between both robots through the inverse compos
p12_w, Qp12_w = inverse_composition2(p1_w, Qp1_w, p2_w, Qp2_w)
print( '----\tExercise 4.1.3 with method 2\t----\n'+
        'p12_w = {}\'\n'.format(p12_w.flatten())+
        'Qp12_w = \n{}\n'.format(Qp12_w))
```

```
----    Exercise 4.1.3 with method 1    ----
p12_w = [ 5.34676389 -0.64196257  1.6       ]'
Qp12_w =
[[0.38248035 0.24115    0.01283925]
 [0.24115    1.16751965 0.10693528]
 [0.01283925 0.10693528 0.05      ]]

----    Exercise 4.1.3 with method 2    ----
p12_w = [ 5.34676389 -0.64196257  1.6       ]'
Qp12_w =
[[0.38248035 0.24115    0.01283925]
 [0.24115    1.16751965 0.10693528]
 [0.01283925 0.10693528 0.05      ]]
```

Expected results:

```
p12_w = [ 5.34676389 -0.64196257  1.6       ]'

Qp12_w =
[[0.38248035 0.24115    0.01283925]
[0.24115    1.16751965 0.10693528]
[0.01283925 0.10693528 0.05      ]]
```

ASSIGNMENT 4: Predicting an observation from the second robot

According to the information (provided by R1) that we have about the position of the landmark m in the world coordinates (its location z_{1_w} and its associated uncertainty $W_{z_1\_w}$), compute the *predicted observation* distribution of $z_{2p} =[r, \alpha] \sim

N(z_{2p}, W_{2p})$ as taken by a range-bearing sensor mounted on `R2` . The image below shows this scenario.

Fig 6. Illustration of the scenario in assignment 4.

Consider the following:

- The range-bearing model for taking measurements is *(Note: use `np.arctan2()` for computing the angle. At this point, ignore the noise w_i)*:

  $$ z_i = \begin{bmatrix} r_i \\ \alpha_i \end{bmatrix} = h(x,m_i) + w_i = \begin{bmatrix} \sqrt((x_i-x)^2+(y_i-y)^2) \\ atan(\frac{y_i-y}{x_i-x}) - \theta \end{bmatrix}

- w_i $$ $\[5pt]$

- We need to compute the covariance of the predicted observation in Polar coordinates (W_{2p}). For that, use the following: $\\[10pt]$

  $$ W_{z2_c} = \frac{\partial f(p2,z_{1_w})}{\partial \ominus p2} \frac{\ominus p2}{\partial p2} \ Q_{p2_w} \ \frac{\ominus p2}{\partial p2}^T \ \left(\frac{\partial f(p2,z_{1_w})}{\partial p} \right)^T +

  ```
   \frac{\partial f(p2,z_{1\_w})}{\partial z_{1\_w}} \ W_{z_1\_w} \
  \left( \frac{\partial f(p2,z_{1\_w})}{\partial z_{1\_w}} \right)^T
  ```
  $$

  $$ \text{Applying the Chain rule} \rightarrow W_{z2\_c} = \frac{\partial f(p2,z_{1\_w})}{\partial \ominus p2} \Sigma_{\ominus p2} \frac{\partial f(p2,z_{1\_w})}{\partial \ominus p2}^T + \frac{\partial f(p2,z_{1\_w})}{\partial p2} \ W_{z_1\_w} \ \frac{\partial f(p2,z_{1\_w})}{\partial p2}^T $$

Once you have the covariance expressed in cartesian coordinates, you can express it in polars by means of the following Jacobian:

$$ \frac{\partial{p}}{\partial{c}} = \begin{bmatrix} \cos{(\alpha+\theta)} & \sin{(\alpha+\theta)} \\ -\sin{(\alpha+\theta)} / r & \cos{(\alpha+\theta)} / r \end{bmatrix} $$

In [12]:

```
def predicted_obs_from_pov(p1_w, Qp1_w, z1_w, Wz1_w):
    """ Method to translate a pose+covariance in the world frame to an observ

        This method only translated the landmark to the pov of the robot.
        It does not simulate a new observation.

        Args:
            p1_w: Pose of the robot which acts as pov
            Qp1_w: Covariance of the robot
            z1_w: Landmark observed in cartesian coordinates(world frame)
            Wz1_w: Covariance associated to the landmark.
        Returns:
            z2_pr: Expected observation of z1 from pov of p1_w
            W2_p: Covariance associated to z2_pr
    """

    # Take a measurement using the range-bearing model
    xi = z1_w[0]
```

```
    yi = z1_w[1]
    x = p1_w[0]
    y = p1_w[1]
    theta = p1_w[2]

    z2_pr = np.vstack([
            np.sqrt(np.power(xi-x,2)+np.power(yi-y,2)), # distance
            np.arctan2((yi-y),(xi-x))-theta # angle
        ])

    # Obtain the uncertainty in the R2 reference frame using the composition
    z1_ext = np.vstack([z1_w, 0]) # Prepare position and uncertainty shapes t
    Wz1_w_ext = np.pad(Wz1_w, [(0, 1), (0, 1)], mode='constant')
    _, Wz1_r = inverse_composition1(p1_w, Qp1_w, z1_ext, Wz1_w_ext)
    W2_c = Wz1_r[0:2,0:2]

    # Jacobian from cartesian to polar at z2p_r
    theta = z2_pr[1, 0] + p1_w[2, 0]
    s, c = np.sin(theta), np.cos(theta)
    r = z2_pr[0, 0]

    Jac_car_pol = np.array([
        [c, s],
        [-s/r, c/r]
    ])

    # Finally, propagate the uncertainty to polar coordinates in the
    # robot frame
    W2_p = Jac_car_pol@W2_c@Jac_car_pol.T

    return z2_pr, W2_p
```

In [13]:

```
p2_w = np.vstack([6., 4., 2.1])
Qp2_w = np.diag([0.20, 0.09, 0.03])

z2_pr, W2_p = predicted_obs_from_pov(p2_w, Qp2_w, z1_w, Wz1)
print( '---- Exercise 4.1.4 ----\n'+
    'z2p_r = {}\'\n'.format(z2_pr.flatten())+
    'W2_p = \n{}\n'.format(W2_p)
)
```

```
---- Exercise 4.1.4 ----
z2p_r = [3.94880545 0.58862004]'
W2_p =
[[1.41886714 0.01057848]
 [0.01057848 0.07881227]]
```

Expected output:

```
---- Exercise 4.1.4 ----
z2p_r = [3.94880545 0.58862004]'
W2_p =
[[1.41886714 0.01057848]
 [0.01057848 0.07881227]]
```

ASSIGNMENT 5: Combining observations of the same landmark

Assume now that a measurement $z_2 = [4 m., 0.3 rad.]^T$ of the landmark is taken from R2 with a sensor having the same precision as that of R1 ($W_{2p}= W_{1p}$). **You have to**:

1. Use the previously implemented `to_world_frame()` function to compute the position and uncertainty about both measurements ($z1$ and $z2$) in the world frame.
2. Plot the robots and the two measurements along with their uncertainty (ellipses) in the world frame.
3. Combine both observations within the `combine_pdfs()` function, and show the resultant combined observation along with its associated uncertainty.

Fig. 7: Results from the last exercise.

In [16]:

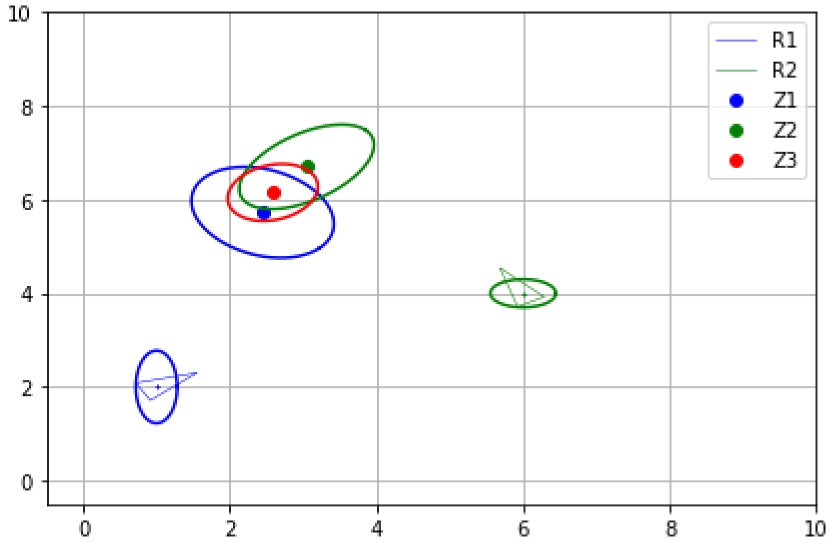
```
def combine_pdfs(z1_w, Wz1_w, z2_w, Wz2_w):
    """ Method to combine the pdfs associated with two observations of the sa

        Args:
            z1_w: Landmark observed in cartesian coordinates(world frame) fro
            Wz1_w: Covariance associated to the landmark.
            z1_w: Landmark observed in cartesian coordinates(world frame) fro
            Wz2_w: Covariance associated to the landmark.
        Returns:
            z: Combined observation
            W_z: Uncertainty associated to z
    """
    Wz1_w_inv = np.linalg.inv(Wz1_w)
    Wz2_w_inv = np.linalg.inv(Wz2_w)

    W_z = np.linalg.inv(Wz1_w_inv+Wz2_w_inv)
    z = W_z@(Wz1_w_inv@z1_w + Wz2_w_inv@z2_w)

    return z, W_z
```

In [17]:

```
z2_p_r = np.vstack([4., .3])
Wz2_p_r = np.diag([0.25, 0.04])

z1_w, Qz1 = to_world_frame(p1_w, Qp1_w, z1_p_r, W1)
z2_w, Qz2 = to_world_frame(p2_w, Qp2_w, z2_p_r, W1)

# Show results
fig, ax = plt.subplots()
plt.xlim([-.5, 10])
plt.ylim([-.5, 10])
plt.grid()
plt.tight_layout()

fig.canvas.draw()

DrawRobot(fig, ax, p1_w, label='R1', color='blue')
PlotEllipse(fig, ax, p1_w, Qp1_w, color='blue')

DrawRobot(fig, ax, p2_w, label='R2', color='green')
PlotEllipse(fig, ax, p2_w, Qp2_w, color='green')

ax.plot(z1_w[0], z1_w[1], 'o', label='Z1', color='blue')
PlotEllipse(fig, ax, z1_w, Qz1, color='blue')

ax.plot(z2_w[0], z2_w[1], 'o', label='Z2', color='green')
PlotEllipse(fig, ax, z2_w, Qz2, color='green')

z_w, Wz_w = combine_pdfs(z1_w, Qz1, z2_w, Qz2)
```

```
ax.plot(z_w[0, 0], z_w[1, 0], 'o', label='Z3', color='red')
PlotEllipse(fig, ax, z_w, Wz_w, color='red')

plt.legend()

# Print results
print( '----\tExercise 4.1.5\t----\n'+
    'z2_w = {}\'\n'.format(z2_w.flatten())+
    'Qz2 = \n{}\n'.format(Qz2)
    )

# Print results
print( '----\tExercise 4.1.5 part 2\t----\n'+
    'z_w = {}\'\n'.format(z_w.flatten())+
    'Wz_w = \n{}\n'.format(Wz_w)
    )
```

```
----    Exercise 4.1.5  ----
z2_w = [3.05042514 6.70185272]'
Qz2 =
[[0.84693794 0.4333316 ]
 [0.4333316  0.81306206]]

----    Exercise 4.1.5 part 2   ----
z_w = [2.58757252 6.15534036]'
Wz_w =
[[0.37966125 0.07773125]
 [0.07773125 0.36999739]]
```



Expected ouputs:

Sensor measurement from R2

```
z2_w = [3.05042514 6.70185272]'
Qz2 =
[[0.84693794 0.4333316 ]
 [0.4333316  0.81306206]]
```

Combined information

```
---- Exercise 4.1.5 parte 2 ----
z_w = [2.58757252 6.15534036]'
Wz_w =
```

```
[[0.37966125 0.07773125]
 [0.07773125 0.36999739]]
```

Thinking about it (1)

Having completed the code above, you will be able to **answer the following questions**:

- When working with landmarks, why do we ignore the information regarding orientation?

  La posición de los landmarks es absoluta en el entorno, por lo que no tiene sentido hablar del ángulo que posea. Lo que sí tiene sentido es hablar de la orientación que tiene el robot con respecto a cierto landmark y el eje de referencia del mundo.

- In the two first assignments we computed the covariance matrix of the observation z_1 captured by robot $R1$ in two different cases: when the $R1$ pose was perfectly known, and having some uncertainty about it. Which covariance matrix was bigger? Is it bigger than that of the robot? Why?

  Evidentemente, la matriz de covarianzas en el caso con incertidumbre en la pose del robot es mayor que en el caso en el que no. Debido a que no sabemos del todo dónde está el robot, la incertidumbre sobre la posición de z_1 también aumentará (siendo mayor que la del robot). Esto es consecuencia de la composición de poses, que a medida que introducimos incertidumbre en la pose del robot, esta operación propagará el error cometido.

- When predicting an observation of m from the second robot $R2$, why did we need to use the Jacobian ${\partial{p}}/{\partial{c}}$?

  Una vez que teníamos la matriz de covarianzas expresada en coordenadas cartesianas, podemos expresarla en polares mediante dicho Jacobiano.

- In the last assignment we got two different pdf's associated to the same landmark. Is that a contradiction? How did you manage two combine these two *pieces of information*?

  Como tenemos dos robots en el entorno, cada uno de ellos va a intentar predecir dónde se encuentra el landmark mediante sus correspondientes pdfs. Técnicamente no hay contradicción entre ambas, sino que además podemos combinarlas mediante la multiplicación de pdfs Gaussianas que aprendimos en la primera práctica.

OPTIONAL

As commented, a number of sensors can be mounted on a mobile robot. In the robotic sensing lecture we discused some of the most popular ones. As an optional exercise, you can look for interesting information about any of them (or any one not listed below) and further describe it here to complete your knowledge.

- Beacons
  - GPS
- Range sensors
  - Sonar
  - Infrared

- Laser scanner
- Cameras
- RGB-D cameras

END OF OPTIONAL PART

OPTIONAL

An alternative to *landmark observation models* are *scan observation* ones, which work with scan-based sensors. Below, the three most popular ones are listed. Surf the internet for some code illustrating any of them, and include it in the notebook with a brief description of how it works and its purpose. You could also implement an example using these models.

Scan observation models

Scan observation models are used when the sensor mounted on the robot provides a scan measuring distance and angle to obstacles in the workspace, *e.g.* a laser range finder. In this case, each element in the map is a cell described by its position (and probably a color representing if its free of obstacles or occupied), and data association is not explicitly addressed.

Beam model

Likelihood field

Scan matching

END OF OPTIONAL PART