Map building

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Reference Books:

- Probabilistic Robotics. S. Thrun, W. Burgard, D. Fox. MIT Press. 2001
- Simultaneous Localization and Mapping for Mobile Robots: Introduction and Methods. Juan-Antonio Fernández-Madrigal and José Luis Blanco Claraco. IGI-Global. 2013.

Content

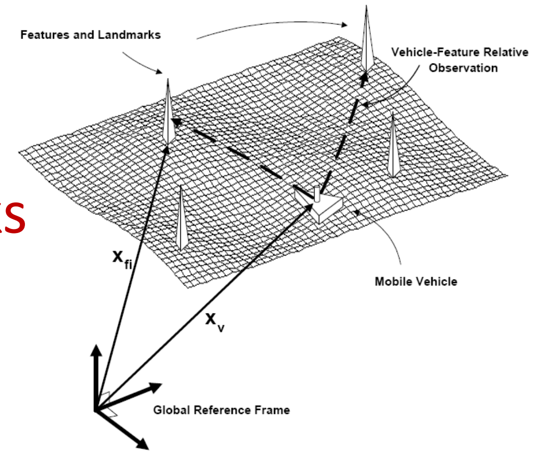
- Type of Maps for mobile robots
- Estimation of Landmark Maps
  - Least Squares
  - Extended Kalman filter
    - Range-bearing mapping
- Estimation of Occupancy Grid Maps

Types of Maps

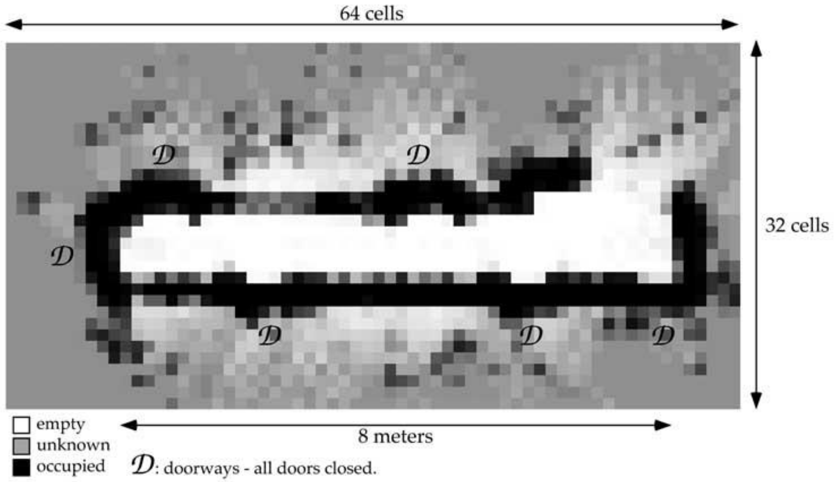
Geometric :

NEXT

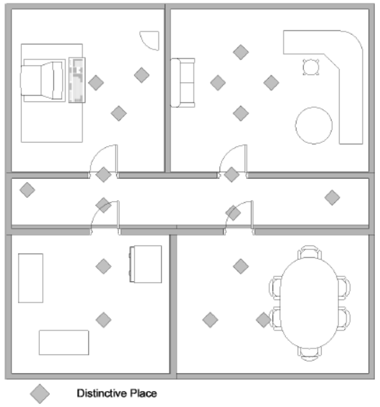
Landmarks

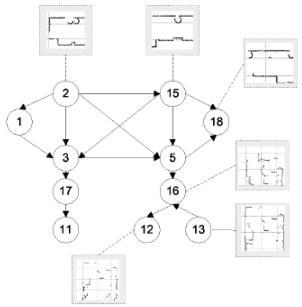


Grid

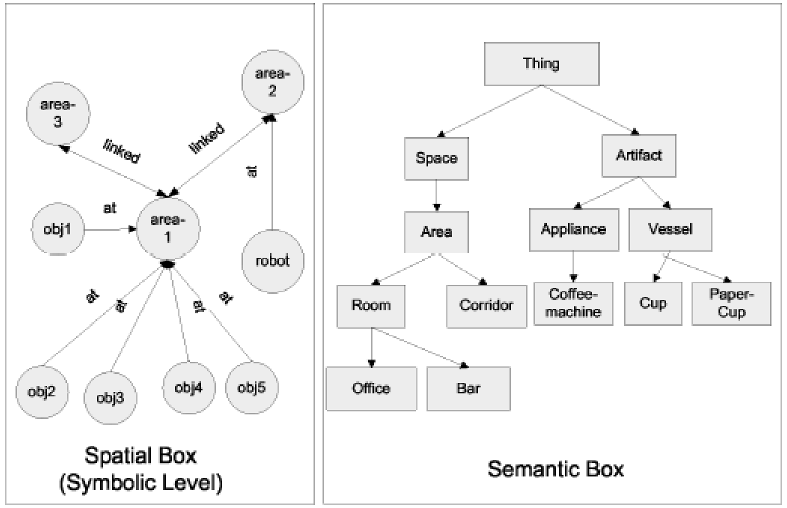


Topological





Semantic



Landmark Mapping

Problem: Given a set of measurements z = {z_k}, k=1,..,M that are related to the *pose* x and *map* m by the observation function

$$z_k = h(x,m) + e_k \qquad\qquad e_k \sim N(0,Q)$$

find the map distribution $p(m|z,x)$ that "best" explains z

m = {(x_i, y_i), i=1,...,N}

Notice:

- the pose x is not a random variable here, but a known 3x1 (or 6x1) vector
- z_k is a sample from a distribution (typically, Gaussian) with mean $h(x,m)$

$$z_k \sim p(z_k|x,m) \; = K \cdot \exp\left\{-\frac{1}{2}[h(x,m) - z_k]^T Q^{-1}[h(x,m) - z_k]\right\}$$

DIFFERENT SOLUTIONS for the mapping problem:

- **Least Squares** = MLE if $p(z_k|x,m)$ is Gaussian (same as in localization)
- **Kalman Filter**

Landmark Mapping. Least Squares

Least Squares (LS): The best estimate of the landmark is the weighted average of all observations (Similar to Localization)

$$Q = \begin{bmatrix} Q_1 & 0 & 0 \\ & \ddots & \\ 0 & 0 & Q_N \end{bmatrix}$$

$$\hat{m} = \arg\min_m e^T Q^{-1} e = \arg\min_m [(z - h(m))^T Q^{-1} (z - h(m))]$$

$$e_k = [z_k - h(x, m)] \sim N(0, Q_k)$$

Solution if h(m) linear: h(m) = H m

$$\hat{m} = (H^T Q^{-1} H)^{-1} H^T Q^{-1} z$$
$$\Sigma_{\hat{m}} = (H^T Q^{-1} H)^{-1}$$

Closed form solution

Solution if h(m) not linear

$$\delta = -\left(J^T Q^{-1} J\right)^{-1} J^T Q^{-1} [z - h(x_0)]$$

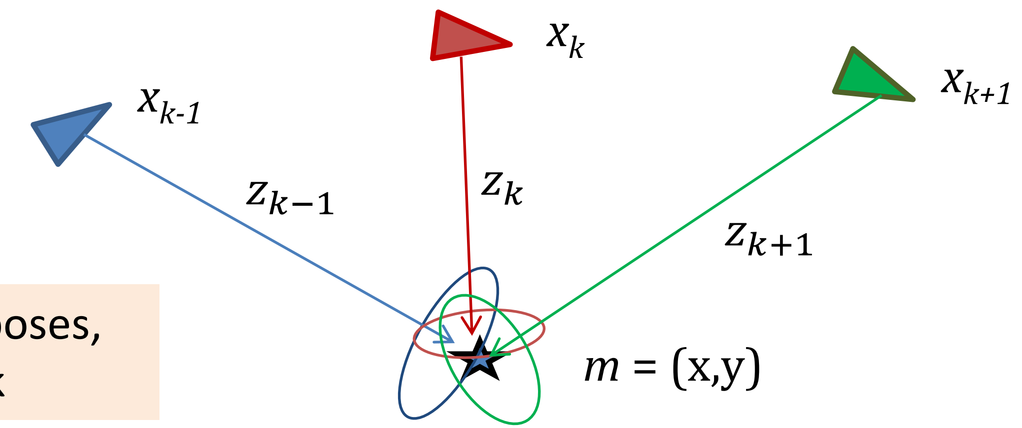
Gauss-Newton iterative algorithm

Example: 3 measures z_k to a landmark m

Three errors: $e = [e_1, e_2, e_3]^T$

$z_k = [r_k, \theta_k] \longrightarrow \dim(e_k) = 2, \dim(e) = 6$

This could be the case of 1 robot from 3 known poses, or 3 robots estimating the position of a landmark



Landmark Mapping. Least Squares

Example: Least squares in 1D with two range observations z_1, z_2

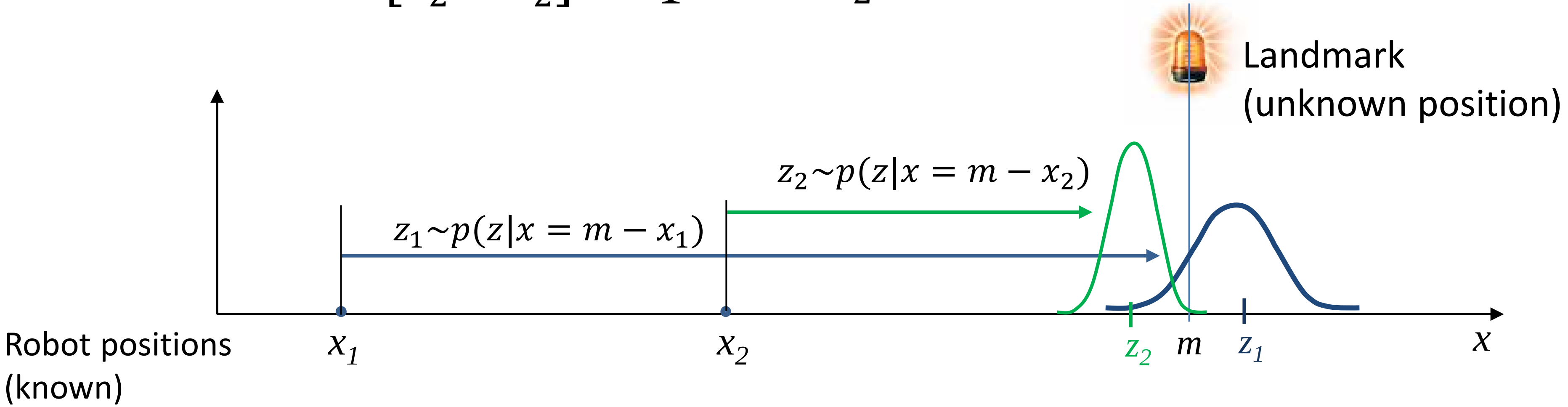
$$\left.\begin{aligned} z_1 &= h(x_1, m) + e_1 = (m - x_1) + e_1 \\ z_2 &= h(x_2, m) + e_2 = (m - x_2) + e_2 \end{aligned}\right\} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} m - x_1 \\ m - x_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \sim N(0, \overbrace{\begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}}^{Q})$$

To write this expresión as a linear form $z = Hm$:

We "create" an observation $z' = x + z$ that measures the position of the landmark (not the distance)

$z_k + x_k = m + e_k$

$$z' = \begin{bmatrix} x_1 + z_1 \\ x_2 + z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} m + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = Hm + e$$



Example (cont): Least squares in 1D with two range observations z_1, z_2

$$z' = Hm + e \begin{cases} \hat{m} = (H^T Q^{-1} H)^{-1} H^T Q^{-1} z' \\ \Sigma_m = (H^T Q^{-1} H)^{-1} \end{cases}$$

Most probable value for m (the *mode*)

Covariance of the estimation (tell us how precise is the estimated $\hat{m}$)

$$\hat{m} = \left(\begin{bmatrix}1\\1\end{bmatrix}^T \begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}^{-1} \begin{bmatrix}1\\1\end{bmatrix}\right)^{-1} \begin{bmatrix}1\\1\end{bmatrix}^T \begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}^{-1} \begin{bmatrix}z'_1\\z'_2\end{bmatrix} = \left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^{-1} \begin{bmatrix}\frac{1}{\sigma_1^2} & \frac{1}{\sigma_2^2}\end{bmatrix} \begin{bmatrix}z'_1\\z'_2\end{bmatrix} = \frac{\sigma_2^2 z'_1 + \sigma_1^2 z'_2}{\sigma_1^2+\sigma_2^2}$$

Quite intuitive!

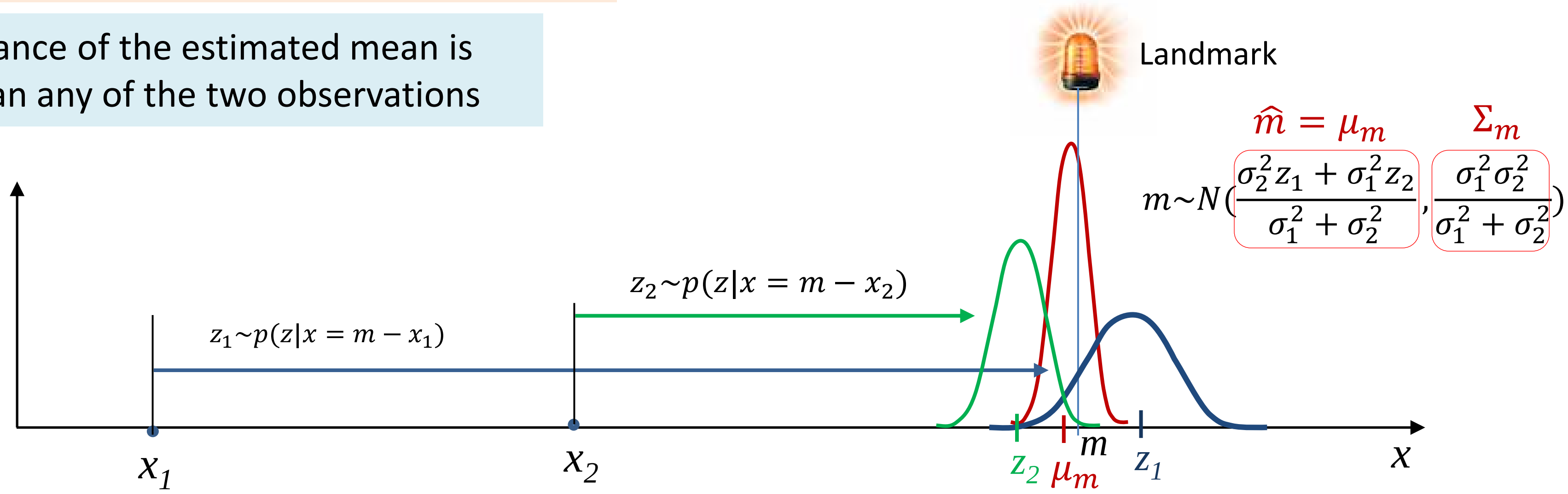
$\sigma_1 >> \sigma_2 \rightarrow m = z_2$,

$\sigma_2 >> \sigma_1 \rightarrow m = z_1$,

The estimated mean is closer to the observation having the smaller variance

$$\Sigma_m = \left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^{-1} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \sigma_1^2 \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

The covariance of the estimated mean is smaller than any of the two observations



Landmark Mapping. Least Squares

Least Squares is not the best method when:

1. the observations come to the robot sequentially (not all at once)
2. the observation does not get the full state of m (e.g. only distance to a 2D landmark) or
3. some prior information (model) might exist for the map

SOLUTION: to apply Bayes to recursively integrate measurements

$$\underbrace{p(m|z_t, x_t)}_{\text{posterior}} = K p(z_t|m, x_t) \underbrace{p(m|z_{t-1}, x_{t-1})}_{\text{prior}}$$

pdf are Gaussians

(EXTENDED) KALMAN FILTER

Note: the subindices k and t are used indistinctly

Landmark Mapping. EKF

Extended Kalman Filter

Map m given by N landmarks:

$$m = \{m_1, \ldots, m_i, \ldots, m_N\}$$

Measures given by the sensor at each pose

Problem: Estimate the probability: $p(m|z_{1:t}, x_{1:t})$

Known robot poses

Assumptions:

1. The estimation of landmarks is independent one to another, and each one depends only on its observations

$$p(m_i|z_{1:t}, x_{1:t}) = p(m_i|z_{1:t}^i, x_{1:t})$$

2. The map is static → state transition model is $m_t = m_{t-1}$

$$m_t = A_t m_{t-1} + B_t u_t + \varepsilon_t \qquad (A=I,\ B=0,\ \varepsilon=0)$$

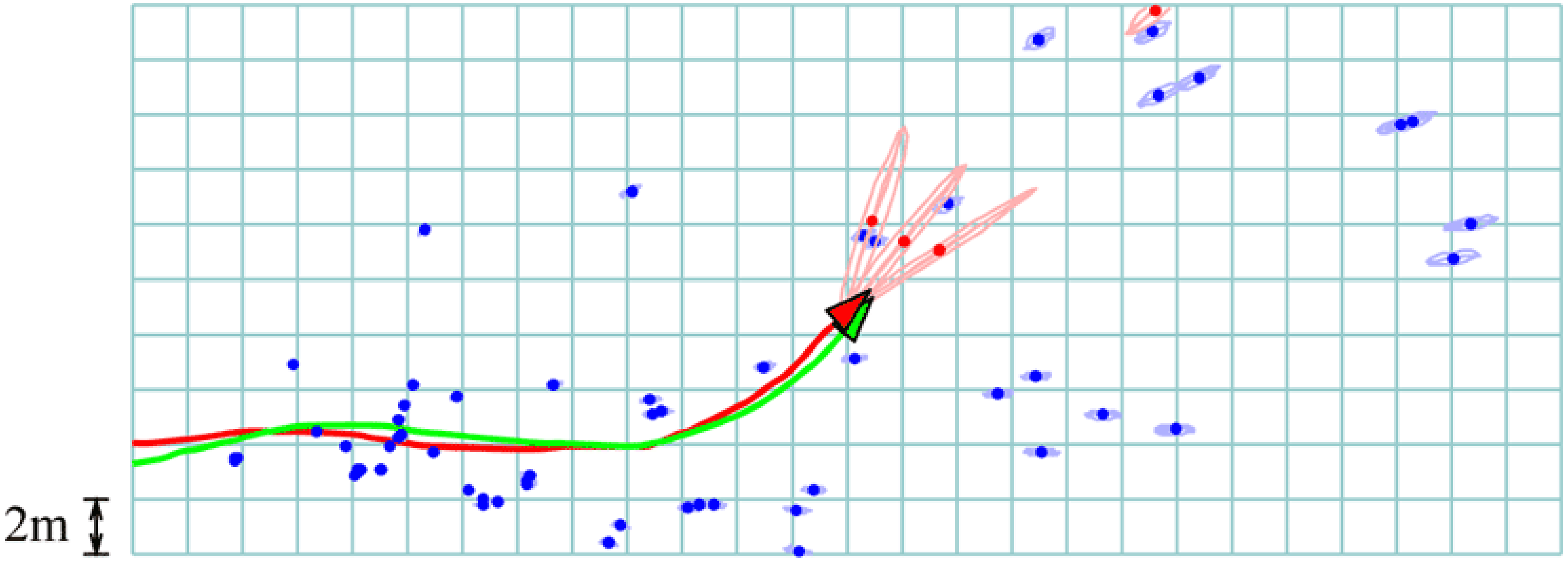
No need of a prediction step in the KF!!

Landmark Mapping. EKF

Extended Kalman Filter

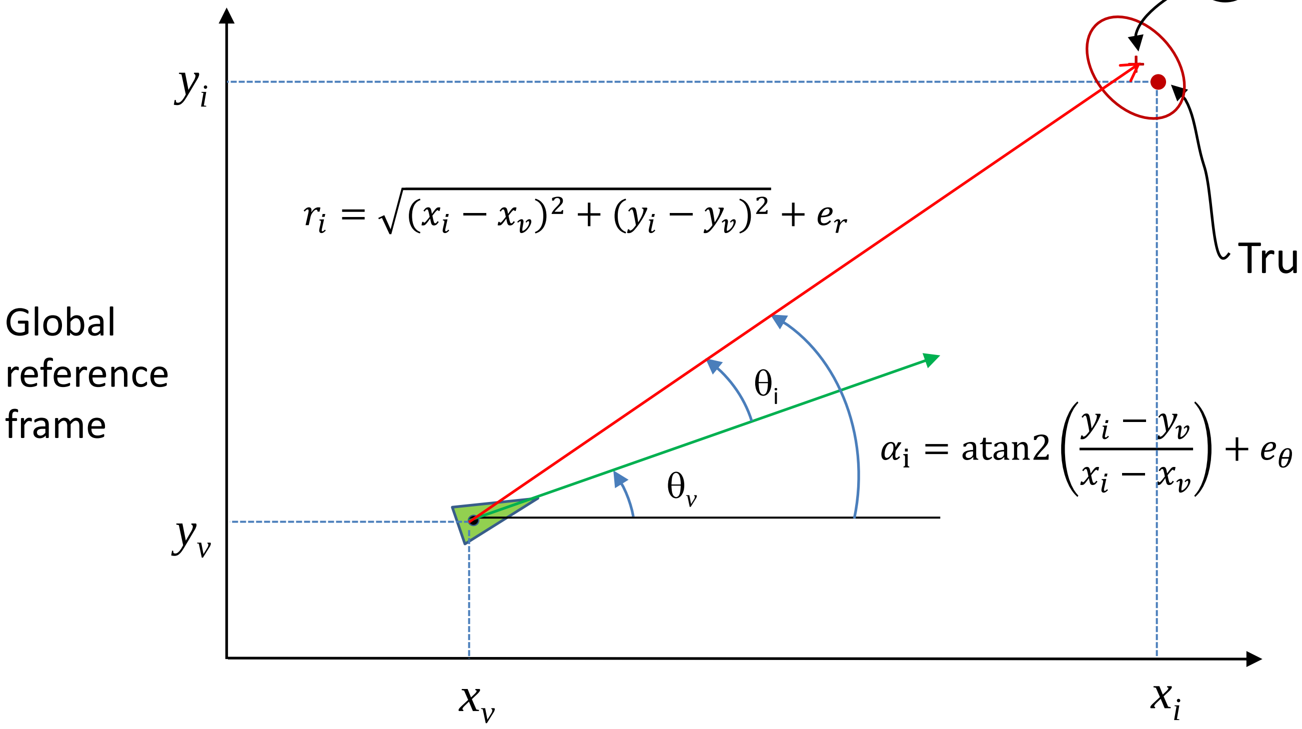
Features may be provided by:

- **Laser scanners** or **depth cameras**: at the intersection of lines, corners of rectangles, edges of objects, …
- **Vision systems**: at key points, lines, planes



Range-Bearing mapping (RECALL)

Range-bearing observations: $z = [r, \theta]$



z_i : observed position of m_i

$$z_i = \begin{bmatrix} r_i \\ \theta_i \end{bmatrix} \sim N\left(\begin{bmatrix} h_r(x_v, m_i) \\ h_\theta(x_v, m_i) \end{bmatrix}, R\right)$$

True position of $m_i = \begin{bmatrix} x \\ y_i \end{bmatrix}$ (to be estimated)

Known vehicle pose: $[x_v, y_v, \theta_v]^T$

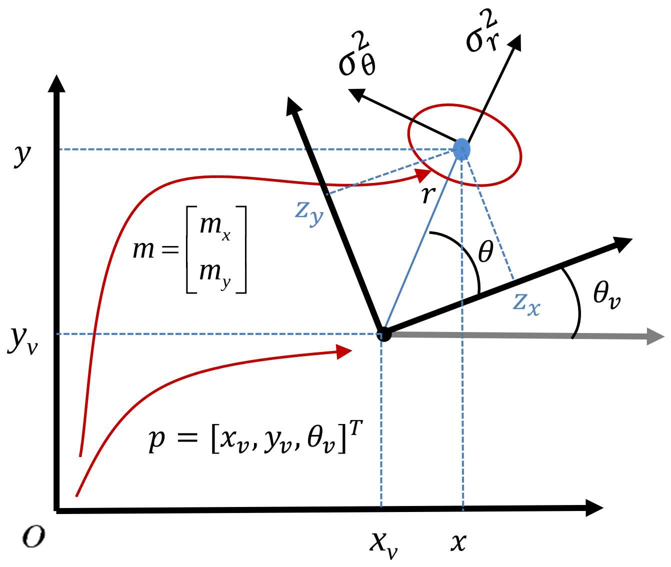
$$z_i \equiv \begin{bmatrix} r_i \\ \theta_i \end{bmatrix} = \begin{bmatrix} h_r(x_v, m_i) \\ h_\theta(x_v, m_i) \end{bmatrix} + \begin{bmatrix} e_r \\ e_\theta \end{bmatrix} = \begin{bmatrix} \sqrt{(x_i - x_v)^2 + (y_i - y_v)^2} \\ \text{atan2}\left(\frac{y_i - y_v}{x_i - x_v}\right) - \theta_v \end{bmatrix} + \begin{bmatrix} e_r \\ e_\theta \end{bmatrix}$$

$$\begin{bmatrix} e_r \\ e_\theta \end{bmatrix} \sim N(0, R) \text{ with } R = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$$

The observation function is non-linear –> EKF

RECALL

How is the covariance $Q = \Sigma_{z_p} = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$ seen from the global frame (Σ_{xy})?



Sensor pose: $p = [x_v, y_v, \theta_v]^T$

Observation in Cartesian : $z_c = [z_x, z_y]^T$

Observation in polar: $z_p = [r, \theta]^T$

$$m = p \oplus z_c = f(p, z_c) = \begin{bmatrix} x_v + z_x \cos\theta_v - z_y \sin\theta_v \\ y_v + z_x \sin\theta_v + z_y \cos\theta_v \end{bmatrix} = \begin{bmatrix} x_v + r\cos(\theta + \theta_v) \\ y_v + r\sin(\theta + \theta_v) \end{bmatrix}$$

Covariance in the **moving system** (Σ_z):

$$z_c = \begin{bmatrix} z_x \\ z_y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = f(r, \theta) \Rightarrow \Sigma_{z_c} = \frac{\partial z_c}{\partial z_p} \Sigma_{z_p} \left(\frac{\partial z_c}{\partial z_p}\right)^T \quad \text{with} \quad \frac{\partial z_c}{\partial z_p} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Covariance in the **global system** (Σ_{xy}):

$$\Sigma_{xy} = \frac{\partial m}{\partial z_c}\frac{\partial z_c}{\partial z_p} \Sigma_{z_p} \left(\frac{\partial m}{\partial z_c}\frac{\partial z_c}{\partial z_p}\right)^T = \frac{\partial m}{\partial z_c} \overbrace{\boxed{\frac{\partial z_c}{\partial z_p} \Sigma_{z_p} \left(\frac{\partial z_c}{\partial z_p}\right)^T}}^{\Sigma_{z_c}} \left(\frac{\partial m}{\partial z_c}\right)^T = \frac{\partial m}{\partial z_c} \Sigma_{z_c} \left(\frac{\partial m}{\partial z_c}\right)^T$$

Chain rule for derivative of functions

Landmark Mapping. EKF

Range-Bearing mapping:

$$\begin{bmatrix} z_x \\ z_y \end{bmatrix} = r_k \begin{bmatrix} \cos\alpha_k \\ \sin\alpha_k \end{bmatrix}$$

- m (map): matrix of landmarks (also named features)

$$m = xEst = \left[x_1, y_1, x_2, y_2, \cdots, x_{nf}, y_{nf}\right]^T$$

nf:#features

Coordinates of the landmark 2

- The estimation of each landmark can be done independently of the others, that is, we could use a different KF for each

$$p(m|z,x) = p(m_1, m_2, \ldots m_{nf}|z,x) = \prod_{k=1}^{nf} p(m_k|z_k, x)$$

Data association association given, i.e. we know which observation is from the landmark k

for example: the estimation of 3 landmarks simultaneously is the same as if they were estimated with 3 concurrent KFs

- Coordinates of a new map landmark: $m = x \oplus z_c = \begin{bmatrix} x_v \\ y_v \end{bmatrix} + \begin{bmatrix} z_x\cos\theta - z_y\sin\theta \\ z_x\sin\theta + z_y\cos\theta \end{bmatrix}$

- Observations: $[r_k, \theta_k, f]$

index of the observed landmark (known, data association given)

z_k : observation k

EKF Algorithm

Given: Robot pose, measurement convariance Q

Initially the map m is empty

For all the robot poses

Get an observation z_k to landmark k

If it is a new landmark (not in the map yet)

- Extend the mean vector $mean_m$ and covariance Σ_m as follows:
  - landmark mean: $m_k = x \oplus z_k$
  - covariance matrix $\Sigma_m = J\Sigma_z J^T$ (*recall*: the pose has no uncertainty)
- The covariance matrices of the other landmarks are not changed

else

- do the update (correction) step of the EKF for the landmark k

end_if

end_For

EKFMapping details:

New observation mean added $[\bar{x} \quad \bar{y}]^T$

New observation: No update step

- Mean State vector extended: $mean_m = [-, -, \dots, -, [\bar{x} \quad \bar{y}]]^T$
- Covariance matrix extended:
- $$\Sigma_m = \begin{bmatrix} \begin{bmatrix} - & - \\ - & - \end{bmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}_{2x2} \end{bmatrix}$$

Observation covariance projected to the global system added: $Qest = J\Sigma_{r\theta}J^T$

$$\Sigma_{r\theta} = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$$

No correlation between the landmarks because the pose is known

Observed landmark k already in the map: do update for that landmark

Observation Jacobian: $J = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdots \underbrace{[J_{mk}]_{2x2}}_{} \cdots \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right]$

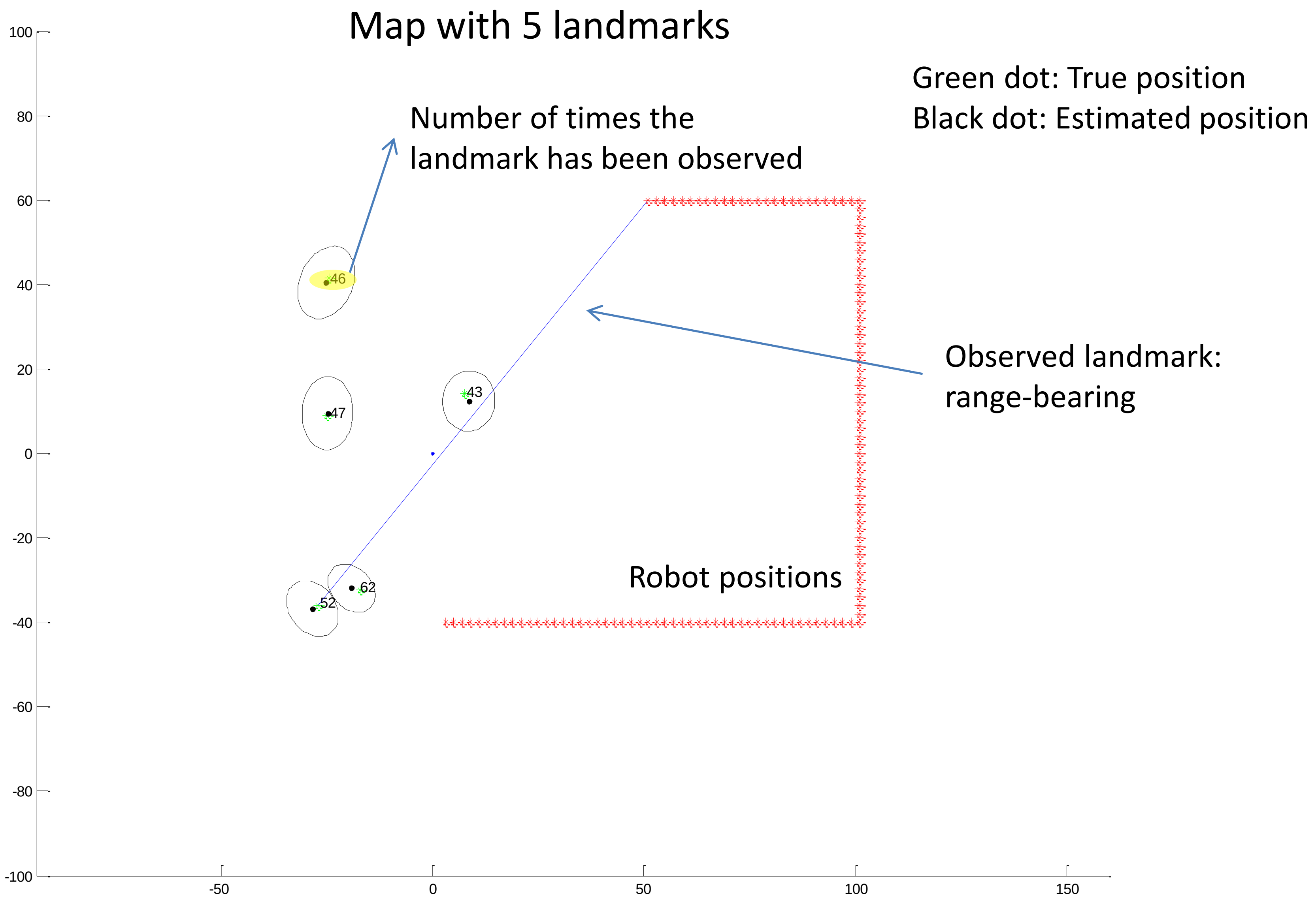
Jacobian of the landmark m_k that has been observed, 0_{2x2} for the rest

This step is equivalent to combine the previous Gaussian with the new one → product of gaussians

$$\left.\begin{matrix} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{matrix}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\Sigma_{12}(\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2), \quad \Sigma_{12} = (\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}\right)$$

Landmark Mapping. EKF

Example

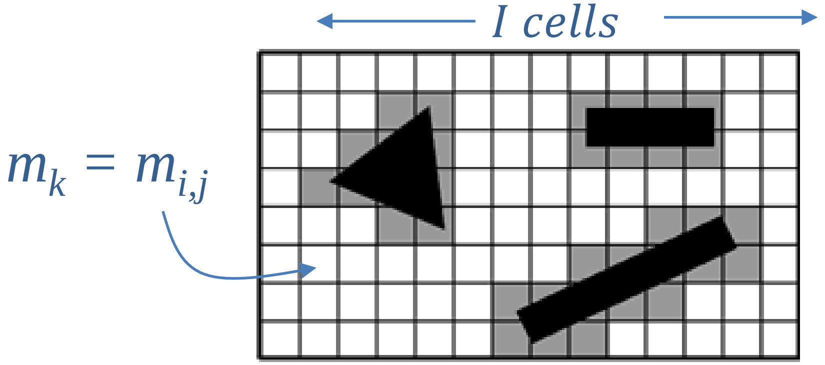


Occupancy Grid Mapping

Map m represented by a grid $m = \{m_1, \ldots, m_k, \ldots, m_N\}$

- Each element of the map m_k can be either occupied (1) or empty (0) → binary random variables!
- **The location of each cell $\boldsymbol{m_k}$ is known**, i.e. $m_k = m_{i,j}$

No need to estimate the coordinates, only if the cell is occupied (1) or not (0)



occupancy grid map (OGM)

with $k = i * I + j$

Mapping Problem: Estimate the probability that a cell $m_k = m_{i,j}$ is occupied by an obstacle: $P(m_k \mid z_{1:t}, x_{1:t})$

Assumption: Occupancy of individual cells (m_k) is independent of each other

This map is used for navigation and localization by scan-based sensor: **sonar and laser**

Occupancy Grid Mapping

PROBLEM: Estimation of

$(assuming\ m_i \perp m_j | \mathbf{z}_{1:t}, \mathbf{x}_{1:t}\ for\ any\ i \neq j)$

$$P(m|z_{1:t}, x_{1:t}) = P(\{m_i\}_{i=1}^{N}|z_{1:t}, x_{1:t}) \approx \prod_{i=1}^{N} \boxed{P(m_i|z_{1:t}, x_{1:t})}$$

Bel(m_i)

Notice:

1. No index *t* in *m* since the **map does not change over time** (static)
2. The probability of each cell *Bel(m_i)* is estimated independently of the others

Bayes for z_t

$$P(m_i|z_{1:t}, x_{1:t}) = P(m_i|z_t, z_{1:t-1}, x_{1:t}) = \frac{\boxed{P(z_t|m_i, z_{1:t-1}, x_{1:t})} P(m_i|z_{1:t-1}, x_{1:t})}{P(z_t|z_{1:t-1}, x_{1:t})}$$

Depends only on 1:t-1→ recursivity

$$\boxed{P(z_t|m_i, z_{1:t-1}, x_{1:t})} = P(z_t|m_i, x_t)$$

Markov

$$= \frac{P(m_i|z_t,x_t)P(z_t|x_t)}{P(m_i|x_t)}$$

Bayes for m_i

m_i independent of x_t

$$= \boxed{\frac{P(m_i|z_t, x_t)P(z_t|x_t)}{P(m_i)}}$$

$$= \boxed{\frac{P(m_i|z_t, x_t)p(z_t|x_t)}{P(m_i)}} \frac{\boxed{P(m_i|z_{1:t-1}, x_{1:t-1})}}{\boxed{p(z_t|z_{1:t-1}, x_{1:t})}} \quad (1)$$

Scale factor that will be eliminated next

Depends only on t

An expression hard to apply, but …

Occupancy Grid Mapping

Probability that the cell m_i is occupied (from previous slide -repeated-):

(1)

$$P(m_i|z_{1:t}, x_{1:t}) = \frac{P(m_i|z_t, x_t)p(z_t|x_t)}{P(m_i)} \frac{P(m_i|z_{1:t-1}, x_{1:t-1})}{\boxed{p(z_t|z_{1:t-1}, x_{1:t})}}$$

Scale factor

Now, we introduce the probability that the cell m_i is empty (not occupied):

(2)

$$P(\neg m_i|z_{1:t}, x_{1:t}) = \frac{P(\neg m_i|z_t, x_t)p(z_t|x_t)P(\neg m_i|z_{1:t-1}, x_{1:t-1})}{P(\neg m_i)\boxed{p(z_t|z_{1:t-1}, x_{1:t})}}$$

Scale factor

Of course: $P(\neg m_i|z_{1:t}, x_{1:t}) = 1 - P(m_i|z_{1:t}, x_{1:t})$

Occupancy Grid Mapping

Dividing (1) by (2) to elimiante the scale factor, and taken ln to simplify:

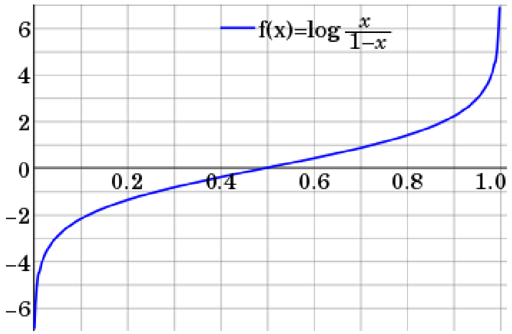
This is called *odds(m)*

$$\underbrace{\ln \frac{P(m_i|z_{1:t}, x_{1:t})}{P(\neg m_i|z_{1:t}, x_{1:t})}}_{\mathrm{l}_t(m_i)} = \underbrace{\ln \frac{P(m_i|z_t, x_t)}{P(\neg m_i|z_t, x_t)}}_{\tau_t(m_i)} + \underbrace{\ln \frac{P(m_i|z_{1:t-1}, x_{1:t-1})}{P(\neg m_i|z_{1:t-1}, x_{1:t-1})}}_{\mathrm{l}_{t-1}(m_i)} - \underbrace{\ln \frac{P(m_i)}{P(\neg m_i)}}_{\mathrm{l}_0(m_i)}$$

Recursive expression for computing the OGM:

$$\mathrm{l}_t(m_i) = \tau_t(m_i) + \mathrm{l}_{t-1}(m_i) + \mathrm{l}_0(m_i)$$

This is called *Log_odds(m)*



Log_odd shape

From WIKIPEDIA:

In statistics, odds are an expression of relative probabilities: is the ratio of the probability that the event will happen to the probability that the event will not happen.

$$\mathrm{l}_t(m_i) = \tau_t(m_i) + \mathrm{l}_{t-1}(m_i) + \underbrace{\mathrm{l}_0(m_i)}^{\text{constant}}$$

We need to provide values for $\mathrm{l}_{t-1}(m_i)$ (previous estimation) and $\tau_t(m_i)$ at each step, which requires computing $P(m_i)$ and $P(m_i|z_t, x_t)$

$P(m_i|z_{1:t}, x_{1:t}) \in (0,1)$ but $\mathrm{l}_t(m_i) \in (-\infty, +\infty)$ *Log_odds* never saturate!

To recover the probability: $\mathrm{l}_t(m_i) = \ln \frac{P(m_i|z_{1:t}, x_{1:t})}{P(\neg m_i|z_{1:t}, x_{1:t})}$

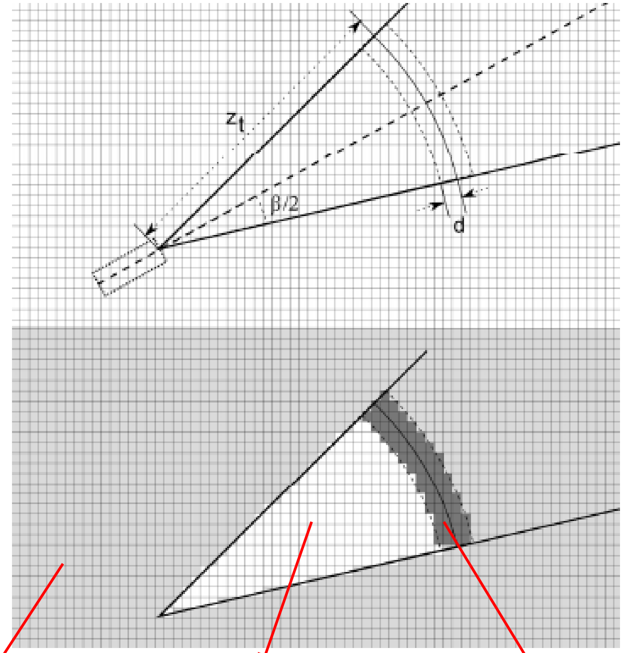
$$e^{\mathrm{l}_t(m_i)} = \frac{P(m_i|z_{1:t}, x_{1:t})}{1 - P(m_i|z_{1:t}, x_{1:t})} \Rightarrow e^{\mathrm{l}_t(m_i)}\left(1 - P(m_i|z_{1:t}, x_{1:t})\right) = P(m_i|z_{1:t}, x_{1:t})$$

$$P(m_i|z_{1:t}, x_{1:t}) = \frac{e^{\mathrm{l}_t(m_i)}}{1 + e^{\mathrm{l}_t(m_i)}} = 1 - \frac{1}{1 + e^{\mathrm{l}_t(m_i)}}$$

Occupancy Grid Mapping

For a sonar:

Sensor model

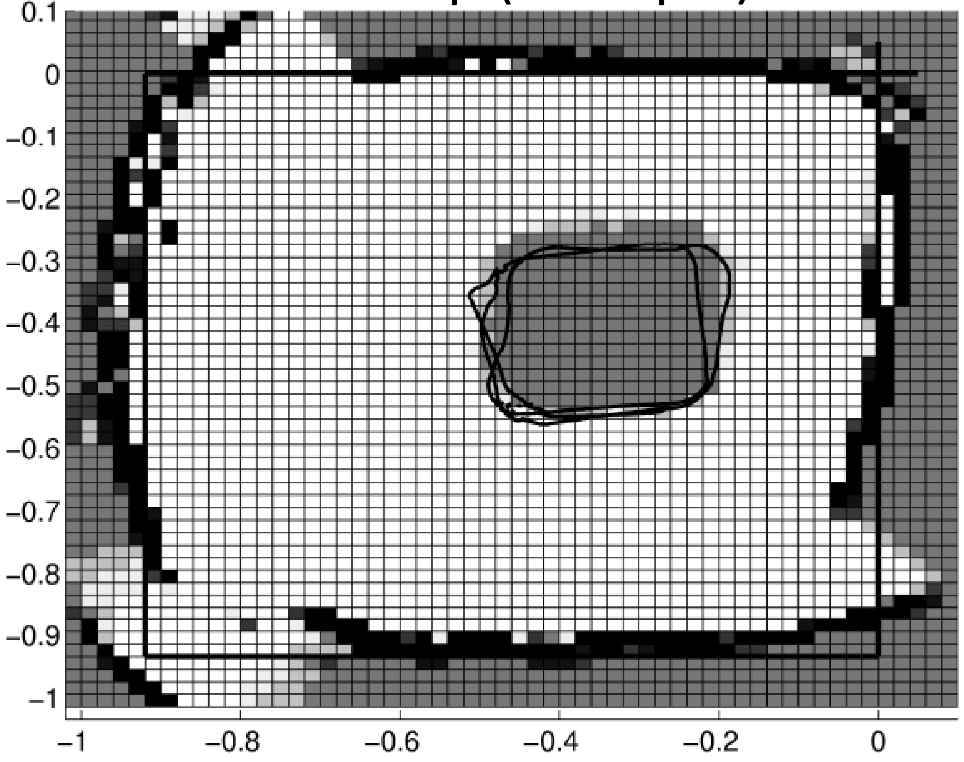


$p(m_i \,|z_t, x_t) = 0.25$

$p(m_i \,|z_t, x_t) = 0.5$

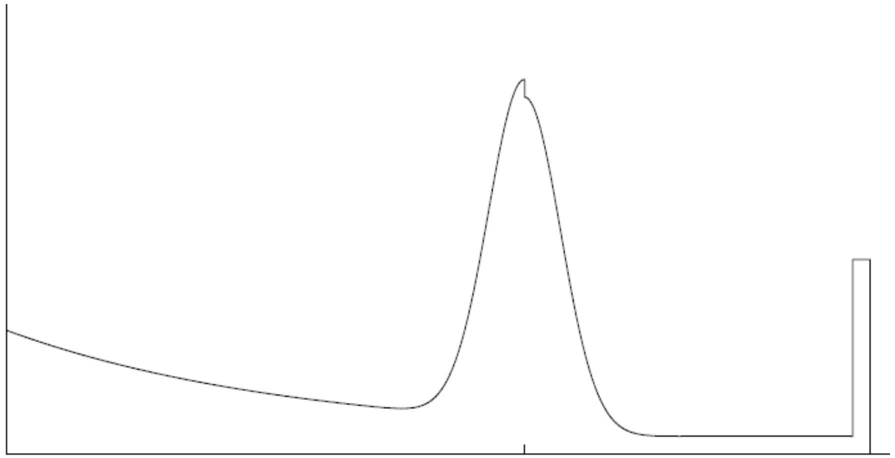
$p(m_i \,|z_t, x_t) = 0.75$

Map (example)

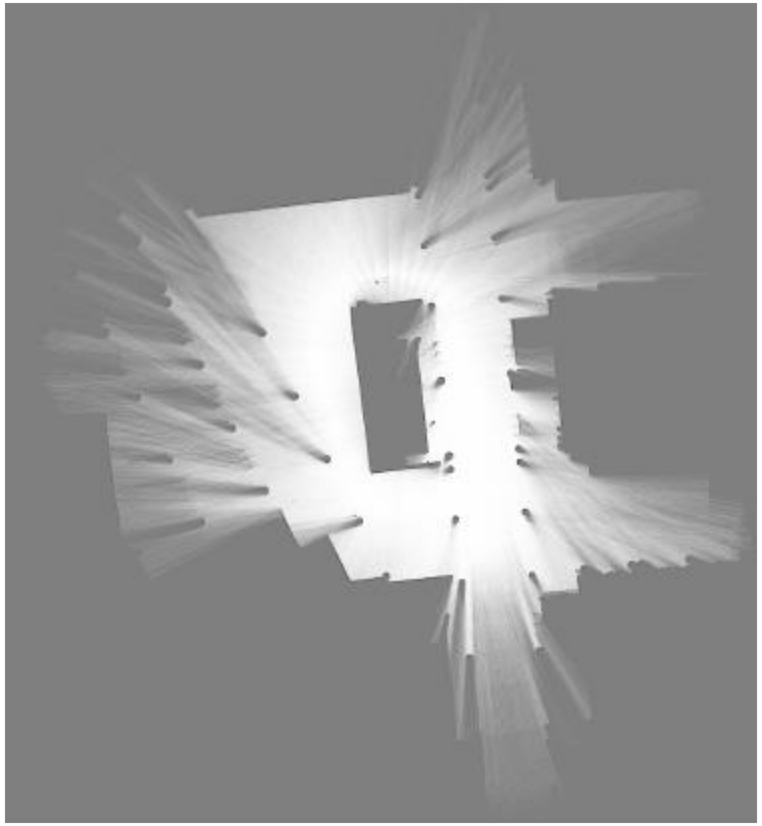


Occupancy Grid Mapping

For a laser: Beam sensor model

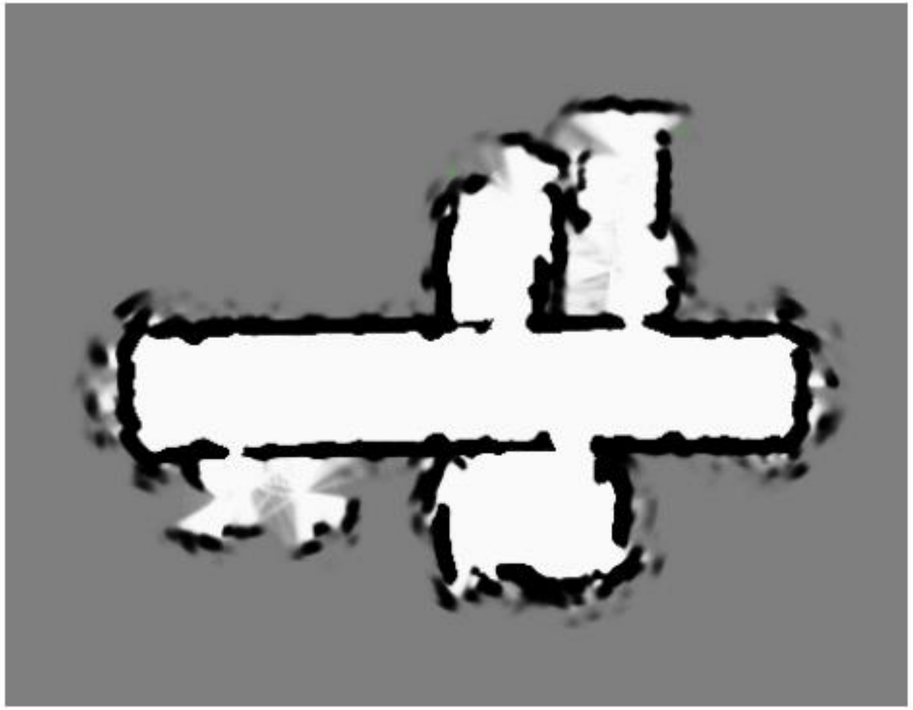


$P(z|x,m) = \alpha_{\text{hit}} P_{\text{hit}}(z|x,m) + \alpha_{\text{unexp}} P_{\text{unexp}}(z|x,m) +$
$\alpha \text{max}_{\text{rand}} \text{rand}_{\text{max}}$

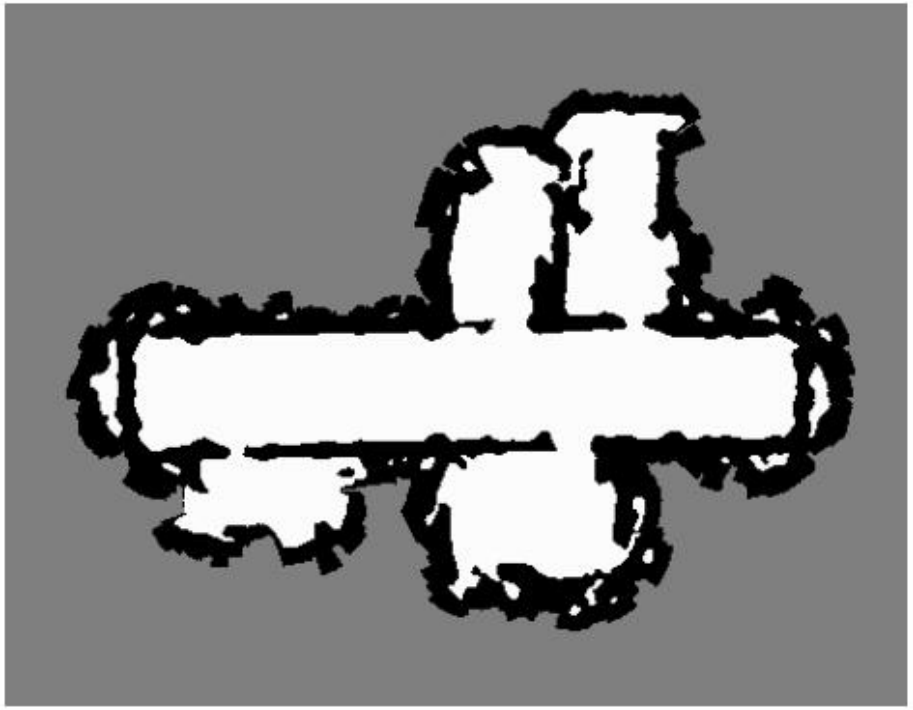


Occupancy Grid Mapping

OGM (Gray level map): Value of a pixel indicates the probability of being occupied



Binarizing the occupancy grid map at a threshold of 0.5



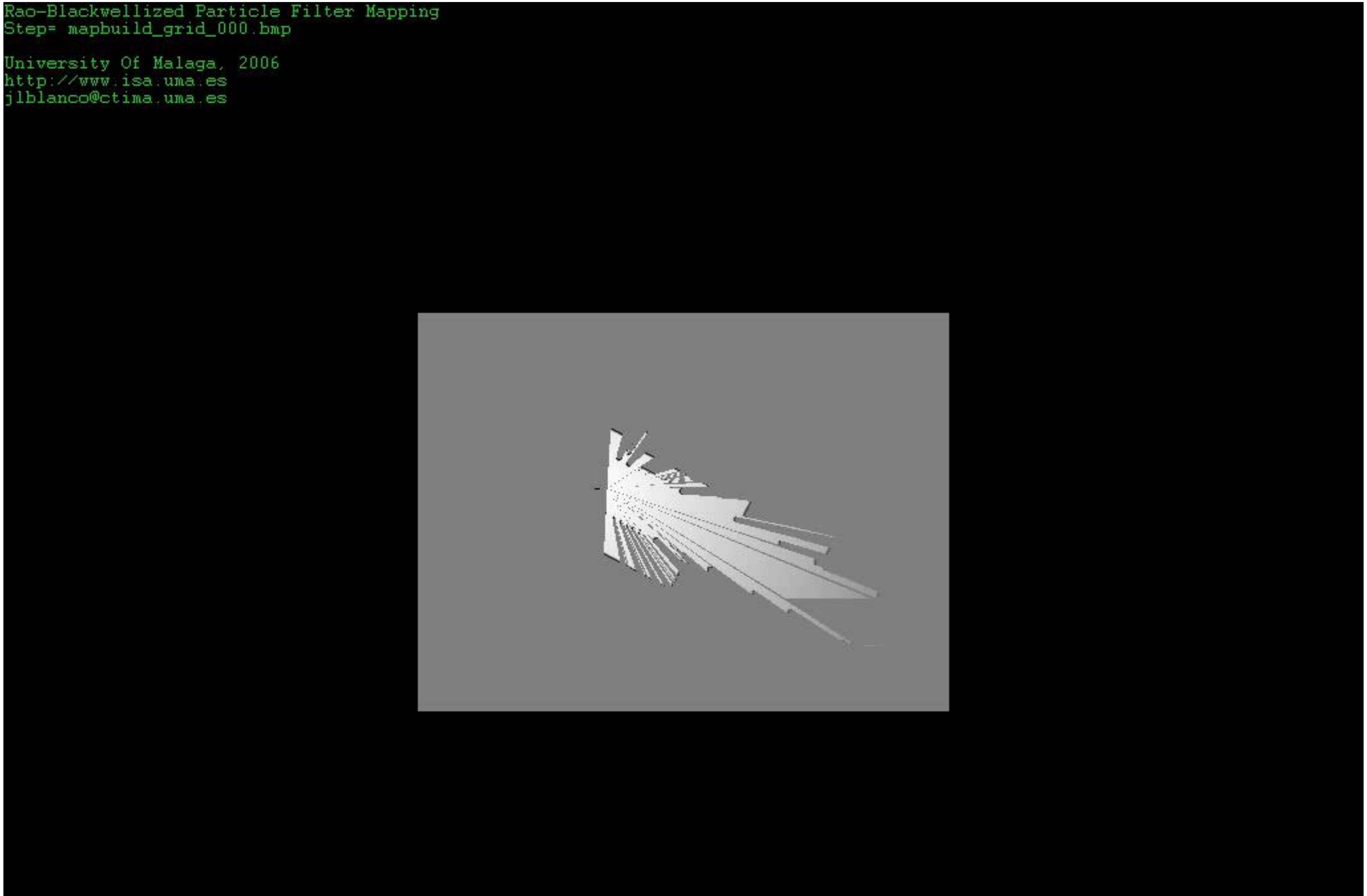
Three-level map:

0 (black): occupied

1 (white): empty

0.5 (gray): unobserved (initialization value)

Occupancy Grid Mapping: ETSI Informatica – Univ. Malaga

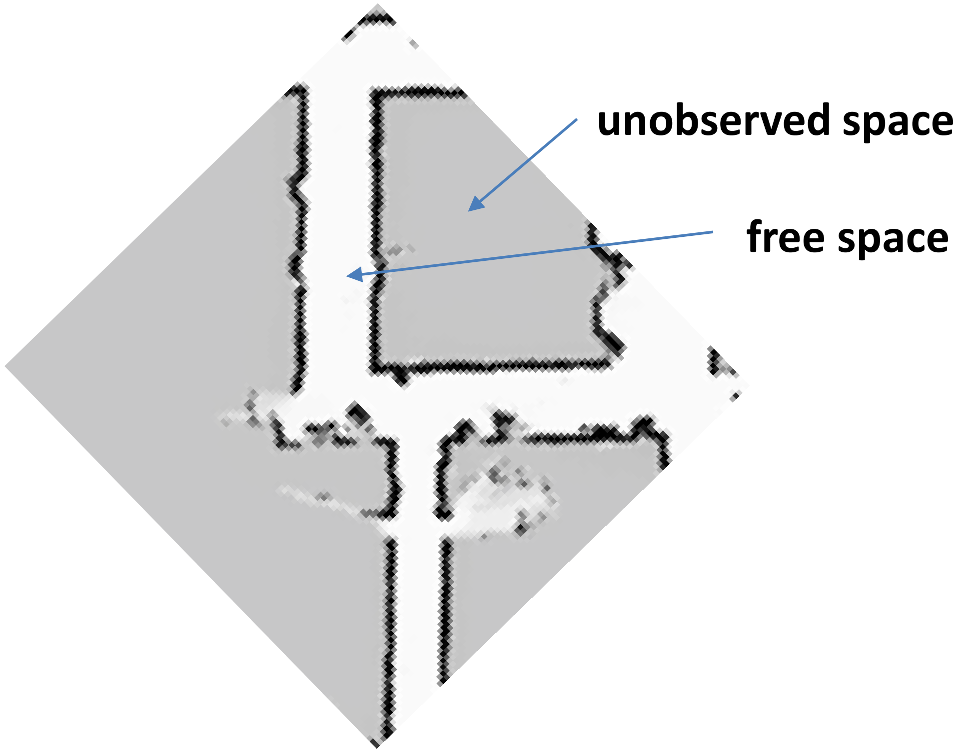


Occupancy Grid Mapping

Difference between a **point-based map** and an occupancy grid map

Point-based map

Occupancy Grid Map (OGM)



In OGM we can distinguish between **free space** and **unobserved space,** while in Point-based maps we don't!!

Summary

- Typical geometric maps can be represented either by a set of landmarks or a rectangular grid (cells)
- The probabilistic map consists in obtaining $p(m|z,x)$

LANDMARKS

  - Landmarks are given by their position in space plus a descriptor to distinguish one to another.
  - If the correspondence between landmark and observation is given, we only care about the landmark position, i.e. $m = (x, y)$ (not its description)
  - Two techniques: Least Squares (batch) and Extended Kalman Filter (recursive)

OGM

  - Grid maps are composed of cells representing the probability of being occupied, i.e. m_i ={1,0}, $P(m_i|z_{1:t}, x_{1:t}) \in (0,1)$
  - Implemented with a recursive algorithm: $\mathrm{l}_t(m_i) = \tau_t(m_i) + \mathrm{l}_{t-1}(m_i) + \mathrm{l}_0(m_i)$

    Only $\tau_t(m_i)$ needs to be estimated at each step (new observation)