# Wigner Fracton

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## 1 Introduction

We look at the effective 1−particle Quantised fracton Hamiltonian; $\mathcal{H}_{tot} = \frac{1}{2}(g(x)p^2 + p^2 g(x))$ where $g$ is the interaction kernel in the reduced space co-ordinate $x$. Let us only solve for $\mathcal{H} = g(x)p^2$ as the rest can be acquired from the properties of $\mathcal{H}_{tot}$. Note that we use Latin symbols for position space and Greek letters for momentum space. We start with the Wigner function in momentum-space;

$$
\begin{aligned}
W(x,p) &= \int \frac{du}{2\pi} \left\langle x + \frac{u}{2} \right| \rho \left| x - \frac{u}{2} \right\rangle e^{-iup} \\
&= \int d\alpha\, d\beta\, \frac{du}{2\pi} \left\langle x + \frac{u}{2} \middle| \alpha \right\rangle \langle \alpha | \rho | \beta \rangle \left\langle \beta \middle| x - \frac{u}{2} \right\rangle e^{-iup} \\
&= \int d\alpha\, d\beta\, \frac{du}{2\pi} \frac{1}{\sqrt{2\pi}} e^{i(x+\frac{u}{2}).\alpha} \langle \alpha | \rho | \beta \rangle \frac{1}{\sqrt{2\pi}} e^{-i(x-\frac{u}{2}).\beta} e^{-iup} \\
&= \int d\alpha\, d\beta\, \frac{\rho(\alpha,\beta)}{2\pi} e^{i(\alpha-\beta).x} \frac{1}{2\pi} \int du\, e^{iu(\frac{\alpha}{2}+\frac{\beta}{2}-p)} \\
&= \int d\alpha\, d\beta\, \frac{\rho(\alpha,\beta)}{2\pi} e^{i(\alpha-\beta).x} \delta\Big(\frac{\alpha}{2} + \frac{\beta}{2} - p\Big) \\
&= \int d\alpha \frac{\rho(\alpha, 2p-\alpha)}{2\pi} e^{2i(\alpha-p).x} \\
&= \int \frac{d\alpha}{2\pi} \rho(p+\alpha, p-\alpha) e^{2i\alpha x} \qquad (1)
\end{aligned}
$$

The evolution of the Wigner function in phase space is of interest to us. We know that $\partial_t \rho = -i[\mathcal{H}, \rho]$ and substituting in the definition of the Wigner function;

$$
\begin{aligned}
\partial_t W &= \frac{1}{2\pi i} \int du \left\langle x + \frac{u}{2} \right| [\mathcal{H}, \rho] \left| x - \frac{u}{2} \right\rangle e^{-iup} \\
&= \frac{1}{2\pi i} \int du \left\langle x + \frac{u}{2} \right| (g(x)p^2\rho - \rho g(x)p^2) \left| x - \frac{u}{2} \right\rangle e^{-iup} \\
&= \frac{1}{2\pi i} \int du\, d\alpha d\beta \Big[ g(x + \frac{u}{2}) \left\langle x + \frac{u}{2} \right| p^2 |\alpha\rangle \langle \alpha | \rho | \beta \rangle \left\langle \beta \middle| x - \frac{u}{2} \right\rangle - \\
&\quad \left\langle x + \frac{u}{2} \middle| \alpha \right\rangle \langle \alpha | \rho | \beta \rangle \langle \beta | g(x) p^2 \left| x - \frac{u}{2} \right\rangle \Big] e^{-iup} \\
&= \frac{1}{2\pi i} \int du\, d\alpha d\beta \Big[ \rho(\alpha,\beta) \left\langle x + \frac{u}{2} \middle| \alpha \right\rangle e^{-iup} \Big( \alpha^2 g\Big(x + \frac{u}{2}\Big) \left\langle \beta \middle| x - \frac{u}{2} \right\rangle - \langle \beta | g(x) p^2 \left| x - \frac{u}{2} \right\rangle \Big) \Big] \\
&= \frac{1}{2\pi i} \int du\, d\alpha d\beta \Big[ \rho(\alpha,\beta) \frac{e^{i\alpha(x+\frac{u}{2})}}{\sqrt{2\pi}} e^{-iup} \Big( \alpha^2 g\Big(x + \frac{u}{2}\Big) \frac{e^{-i\beta(x-\frac{u}{2})}}{\sqrt{2\pi}} - \langle \beta | g(x) p^2 \left| x - \frac{u}{2} \right\rangle \Big) \Big] \qquad (2)
\end{aligned}
$$

We focus our attention to the last unresolved term,

$$
\begin{aligned}
\langle \beta | g(x) p^2 \left| x - \frac{u}{2} \right\rangle &= \int da\, \langle \beta | a \rangle \langle a | g(x) p^2 \left| x - \frac{u}{2} \right\rangle \\
&= \int da \frac{g(a)}{\sqrt{2\pi}} e^{-i\beta a} \langle a | p^2 \left| x - \frac{u}{2} \right\rangle \\
&= \int da d\gamma \frac{g(a)}{\sqrt{2\pi}} e^{-i\beta a} \langle a | \gamma \rangle \langle \gamma | p^2 \left| x - \frac{u}{2} \right\rangle \\
&= \int da d\gamma \frac{g(a)}{2\pi} e^{-i\beta a} e^{i\gamma a} \gamma^2 \left\langle \gamma \middle| x - \frac{u}{2} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \int da d\gamma \frac{g(a)}{(2\pi)^{\frac{3}{2}}} e^{-i\beta a} e^{i\gamma a} \gamma^2 e^{-i\gamma(x-\frac{u}{2})} \\
&= \int da \frac{g(a)}{\sqrt{2\pi}} e^{-i\beta a} \frac{1}{2\pi} \int d\gamma\, \gamma^2 e^{i\gamma(a-x+\frac{u}{2})} \\
&= \int da \frac{g(a)}{\sqrt{2\pi}} e^{-i\beta a} \Big( -\frac{\partial^2}{\partial x^2} \delta\Big(a - x + \frac{u}{2}\Big)\Big) = -\frac{\partial^2}{\partial x^2} \int da \frac{g(a)}{\sqrt{2\pi}} e^{-i\beta a} \delta\Big(a - x + \frac{u}{2}\Big) \\
&= -\frac{\partial^2}{\partial x^2} \Big( \frac{1}{\sqrt{2\pi}} g\Big(x - \frac{u}{2}\Big) e^{-i\beta(x-\frac{u}{2})}\Big) \qquad (3)
\end{aligned}
$$

We put (3) in (2) which looks like;

$$
\begin{aligned}
\partial_t W =& \frac{1}{2\pi i} \int du\, d\alpha\, d\beta \Big[ \frac{\rho(\alpha,\beta)}{2\pi} e^{i\alpha(x+\frac{u}{2})} e^{-i\beta(x-\frac{u}{2})} e^{-iup} \Big( \alpha^2 g\Big(x + \frac{u}{2}\Big) + g''\Big(x - \frac{u}{2}\Big) - 2i\beta g'\Big(x - \frac{u}{2}\Big) \\
& - \beta^2 g\Big(x - \frac{u}{2}\Big)\Big)\Big] \qquad (4)
\end{aligned}
$$

We evaluate all four terms of the integral individually into its most compact recurrent form. Let us start with the first one;

$$
\begin{aligned}
I1 &= \frac{1}{2\pi i} \int du\, d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta)} e^{i\frac{u}{2}(\alpha+\beta)} e^{-iup} g''\Big(x - \frac{u}{2}\Big) \\
&= \frac{1}{2\pi i} \int du\, d\alpha\, d\beta\, dy d\omega \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta)} e^{i\frac{u}{2}(\alpha+\beta)} e^{-iup} \frac{e^{i\omega(y-x+\frac{u}{2})}}{2\pi} g''(y) \\
&= \frac{1}{2\pi i} \int dy\, d\omega \int d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta-\omega)} g''(y) e^{i\omega y} \frac{1}{2\pi} \int du\, e^{i\frac{u}{2}(\alpha+\beta-2p+\omega)} \\
&= \frac{1}{2\pi i} \int dy\, d\omega \int d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta-\omega)} \delta\Big(\frac{\alpha+\beta+\omega}{2} - p\Big) g''(y) e^{i\omega y} \\
&= \frac{1}{2\pi i} \int d\omega \int d\alpha\, \rho(\alpha, 2p - \alpha - \omega) e^{2ix(\alpha-p)} \int \frac{dy}{2\pi} g''(y) e^{i\omega y} \qquad (5)
\end{aligned}
$$

We define $\int \frac{dy}{2\pi} g''(y) e^{i\omega y} \equiv \tilde{g}''(\omega)$ and change $\alpha \to \alpha + p - \frac{\omega}{2}$ in (5);

$$
\begin{aligned}
I1 &= \frac{1}{2\pi i} \int d\omega \int d\alpha\, \tilde{g}''(\omega) \rho\Big(\alpha + p - \frac{\omega}{2}, 2p - \alpha - p + \frac{\omega}{2} - \omega\Big) e^{ix(2\alpha+2p-\omega-2p)} \\
&= \frac{1}{2\pi i} \int d\omega\ \tilde{g}''(\omega) e^{-ix\omega} \int d\alpha\, \rho\Big(\alpha + \Big(p - \frac{\omega}{2}\Big), \Big(p - \frac{\omega}{2}\Big) - \alpha\Big) e^{2i\alpha x} \\
&= -i \int d\omega\ \tilde{g}''(\omega) e^{-ix\omega}\ W\Big(x, p - \frac{\omega}{2}\Big) \qquad (6)
\end{aligned}
$$

The last line comes from substituting the definition of the Wigner function in momentum space derived in (1). We move on to the next integral;

$$
\begin{aligned}
I2 &= \frac{1}{2\pi i} \int du\, d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta)} e^{i\frac{u}{2}(\alpha+\beta)} e^{-iup} \alpha^2 g\Big(x + \frac{u}{2}\Big) \\
&= \frac{1}{2\pi i} \int du\, d\alpha\, d\beta\, dy\, d\omega \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta)} e^{i\frac{u}{2}(\alpha+\beta)} e^{-iup} \frac{\alpha^2}{2\pi} g(y) e^{i\omega(y-x-\frac{u}{2})} \\
&= \frac{1}{2\pi i} \int dy\, d\omega \int d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta-\omega)} \frac{1}{2\pi} \int du\, e^{i\frac{u}{2}(\alpha+\beta-2p-\omega)} \alpha^2 g(y) e^{i\omega y} \\
&= \frac{1}{2\pi i} \int dy\, d\omega \int d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi} e^{ix(\alpha-\beta-\omega)}\ \alpha^2 g(y) e^{i\omega y}\ \delta\Big(\frac{\alpha+\beta-\omega}{2} - p\Big) \\
&= \frac{1}{2\pi i} \int dy\, d\omega \int d\alpha\ \frac{\rho(\alpha, 2p + \omega - \alpha)}{2\pi} e^{2ix(\alpha-p-\omega)}\ \alpha^2 g(y) e^{i\omega y} \\
&= \frac{1}{2\pi i} \int d\omega\, \tilde{g}(\omega) \int d\alpha\, \rho(\alpha, 2p + \omega - \alpha) e^{2ix(\alpha-p-\omega)}\ \alpha^2 \qquad (7)
\end{aligned}
$$

We define $\int \frac{dy}{2\pi} g(y) e^{i\omega y} \equiv \tilde{g}(\omega)$ and change $\alpha \to \alpha + p + \frac{\omega}{2}$ in (7);

$$
\begin{aligned}
I2 &= \frac{1}{2\pi i} \int d\omega\, \tilde{g}(\omega) \int d\alpha\, \rho\Big(\alpha + p + \frac{\omega}{2}, p + \frac{\omega}{2} - \alpha\Big) e^{ix(2\alpha-\omega)}\ \Big(\alpha + p + \frac{\omega}{2}\Big)^2 \\
&= \frac{1}{2\pi i} \int d\omega\, \tilde{g}(\omega) \int d\alpha\, \rho\Big(\alpha + p + \frac{\omega}{2}, p + \frac{\omega}{2} - \alpha\Big) e^{ix(2\alpha-\omega)}\ \frac{1}{4}\Big[(2\alpha - \omega) + (2p + 2\omega)\Big]^2
\end{aligned}
$$

$$= \frac{1}{2\pi i}\int d\omega\, \tilde{g}(\omega)\int d\alpha\, \rho\Big(\alpha + p + \frac{\omega}{2}, p + \frac{\omega}{2} - \alpha\Big)e^{ix(2\alpha-\omega)}\,\frac{1}{4}\Big[(2\alpha-\omega)^2 + 4(p+\omega)^2 + 4(p+\omega)(2\alpha-\omega)\Big]$$
$$= \frac{1}{2\pi i}\int d\omega\, \tilde{g}(\omega)\,\Big[-\partial_x^2 + 4(p+\omega)^2 - 4i(p+\omega)\partial_x\Big]\Big(\int \frac{d\alpha}{4}\rho\Big(\alpha + p + \frac{\omega}{2}, p + \frac{\omega}{2} - \alpha\Big)e^{2i\alpha x}\Big)e^{-i\omega x}$$
$$= -\frac{i}{4}\int d\omega\, \tilde{g}(\omega)\,\Big[-\partial_x^2 + 4(p+\omega)^2 - 4i(p+\omega)\partial_x\Big]W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \tag{8}$$

Let us now look at the third integral,

$$I3 = \frac{1}{2\pi i}\int du\, d\alpha\, d\beta \frac{\rho(\alpha,\beta)}{2\pi}e^{ix(\alpha-\beta)}e^{i\frac{u}{2}(\alpha+\beta)}e^{-iup}(-2i\beta)g'\Big(x - \frac{u}{2}\Big)$$
$$= -\frac{1}{\pi}\int du\, d\alpha\, d\beta\, dy\, d\omega\, \frac{\rho(\alpha,\beta)}{2\pi}e^{ix(\alpha-\beta)}e^{i\frac{u}{2}(\alpha+\beta)}\beta e^{-iup}\frac{1}{2\pi}g'(y)e^{i\omega(y-x-\frac{u}{2})}$$
$$= -\frac{1}{\pi}\int du\, d\alpha\, d\beta\, dy\, d\omega\, \frac{\rho(\alpha,\beta)}{2\pi}e^{ix(\alpha-\beta-\omega)}\beta g'(y)e^{i\omega y}\delta\Big(\frac{\alpha}{2} + \frac{\beta}{2} - p - \frac{\omega}{2}\Big)$$
$$= -\frac{1}{\pi}\int d\omega\, \tilde{g}'(\omega)\int d\beta\, \rho(2p + \omega - \beta, \beta)\,\beta\, e^{2ix(p-\beta)}$$
$$= -\frac{1}{\pi}\int d\omega\, \tilde{g}'(\omega)\int d\beta\, \rho(2p + \omega + \beta, -\beta)\,(-\beta)\, e^{2ix(p+\beta)} \tag{9}$$

We set $\beta \to \beta - p - \frac{\omega}{2}$ in (9) and define $\int \frac{dy}{2\pi}g'(y)e^{i\omega y} \equiv \tilde{g}'(\omega)$ as usual. This gets us;

$$I3 = \frac{1}{\pi}\int d\omega\, \tilde{g}'(\omega)\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, -\beta + p + \frac{\omega}{2}\Big)\,\Big(\beta - p - \frac{\omega}{2}\Big)\, e^{ix(2\beta-\omega)}$$
$$= -\frac{1}{2\pi}\int d\omega\, \tilde{g}'(\omega)\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, -\beta + p + \frac{\omega}{2}\Big)\,(i\partial_x + 2p)\, e^{ix(2\beta-\omega)}$$
$$= -\int d\omega\, \tilde{g}'(\omega)(i\partial_x + 2p)\,\Big(\frac{1}{2\pi}\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, p + \frac{\omega}{2} - \beta\Big)e^{2ix\beta}\Big)e^{-i\omega x}$$
$$= -\int d\omega\, \tilde{g}'(\omega)(i\partial_x + 2p)W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \tag{10}$$

Eq.(10) looks like a recurrent structure as expected and we move to the last remaining term in (4). We note that we directly start after introducing the $\delta-$ integral; $\frac{1}{2\pi}\int dy\, d\omega e^{i\omega(y-x-\frac{u}{2})}g(y) = \int dy\, \delta\Big(y - x - \frac{u}{2}\Big)g(y) = g\Big(x + \frac{u}{2}\Big)$.

$$I4 = \frac{1}{2\pi i}\int du\, d\alpha\, d\beta\, dy d\omega\, \frac{\rho(\alpha,\beta)}{2\pi}e^{ix(\alpha-\beta)}e^{i\frac{u}{2}(\alpha+\beta)}e^{-iup}\frac{e^{i\omega(y-x-\frac{u}{2})}}{2\pi}(-\beta^2)g(y)$$
$$= -\frac{1}{2\pi i}\int d\alpha\, d\beta\, dy\, d\omega\, \frac{\rho(\alpha,\beta)}{2\pi}e^{ix(\alpha-\beta)}e^{i\omega(y-x)}\,\beta^2\, g(y)\delta\Big(\frac{\alpha}{2} + \frac{\beta}{2} - \frac{\omega}{2} - p\Big)$$
$$= -\frac{1}{2\pi i}\int d\omega\, \tilde{g}(\omega)\int d\beta\, \rho(2p + \omega - \beta, \beta)e^{2ix(p-\beta)}\,\beta^2$$
$$= -\frac{1}{2\pi i}\int d\omega\, \tilde{g}(\omega)\int d\beta\, \rho(2p + \omega + \beta, -\beta)e^{2ix(p+\beta)}\,\beta^2 \tag{11}$$

We set $\beta \to \beta - p - \frac{\omega}{2}$ in (11) and define $\int \frac{dy}{2\pi}g(y)e^{i\omega y} \equiv \tilde{g}(\omega)$ as usual. This gets us;

$$I4 = \frac{1}{\pi}\int d\omega\, \tilde{g}(\omega)\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, -\beta + p + \frac{\omega}{2}\Big)\,\Big(\beta - p - \frac{\omega}{2}\Big)^2\, e^{ix(2\beta-\omega)}$$
$$= -\frac{1}{8\pi i}\int d\omega\, \tilde{g}(\omega)\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, -\beta + p + \frac{\omega}{2}\Big)\,[(2\beta-\omega) - 2p]^2\, e^{ix(2\beta-\omega)}$$
$$= -\frac{1}{8\pi i}\int d\omega\, \tilde{g}(\omega)\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, -\beta + p + \frac{\omega}{2}\Big)\,[-\partial_x^2 + 4ip\partial_x + 4p^2]\, e^{ix(2\beta-\omega)}$$
$$= -\frac{1}{2\pi i}\int d\omega\, \tilde{g}(\omega)\frac{1}{4}[-\partial_x^2 + 4ip\partial_x + 4p^2]\,\int d\beta\, \rho\Big(\beta + p + \frac{\omega}{2}, -\beta + p + \frac{\omega}{2}\Big)\, e^{2ix\beta}e^{-i\omega x}$$
$$= \frac{i}{4}\int d\omega\, \tilde{g}(\omega)[-\partial_x^2 + 4ip\partial_x + 4p^2]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \tag{12}$$

We now assemble the whole evolution integral $\partial_t W = I1 + I2 + I3 + I4$. The assembled solution looks like;

$$\partial_t W = -i\int d\omega\ \tilde{g}''(\omega)e^{-ix\omega}\ W\Big(x, p - \frac{\omega}{2}\Big) - \frac{i}{4}\int d\omega\, \tilde{g}(\omega)\,\Big[-\partial_x^2 + 4(p+\omega)^2 - 4i(p+\omega)\partial_x\Big]W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x}$$

$$- \int d\omega\, \tilde{g}'(\omega)(i\partial_x + 2p)W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} + \frac{i}{4}\int d\omega\, \tilde{g}(\omega)[-\partial_x^2 + 4ip\partial_x + 4p^2]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \quad (13)$$

We can simplify this further by realising that the interaction kernel $g(x)$ is an even bounded function vanishing at infinity which reduces the Fourier integrals as $\int \frac{dy}{2\pi} g^{(n)}(y)e^{i\omega y} \equiv \tilde{g}^{(n)}(\omega) = (-i\omega)^n \tilde{g}(\omega)$. Putting this back in (13) and with simple algebra;

$$\begin{aligned}
\partial_t W &= i\int d\omega \Big[\tilde{g}(\omega)(2p+\omega)[i\partial_x - \omega] + i\omega\, \tilde{g}(\omega)(\partial_x - 2ip) - (-i\omega)^2 \tilde{g}(-\omega)e^{2i\omega x}\Big]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \\
&= i\int d\omega \Big[\tilde{g}(\omega)(2p+\omega)[i\partial_x - \omega] + i\omega\, \tilde{g}(\omega)(\partial_x - 2ip) - (-i\omega)^2 \tilde{g}(\omega)e^{2i\omega x}\Big]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \\
&= \int d\omega \tilde{g}(\omega)\Big[-2p\partial_x - \omega\partial_x - i\omega^2 - 2i\omega p - w\partial_x + 2i\omega p + i\omega^2 e^{2i\omega x}\Big]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \\
&= -\int d\omega\, \tilde{g}(\omega)\Big[(2p + 2\omega)\partial_x + i\omega^2(1 - e^{2i\omega x})\Big]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x} \qquad (14)
\end{aligned}$$

We have substituted $\omega \to -\omega$ in the first line of the last integral to take the Wigner term out as a common factor. Now since $g$ is even, the Fourier integral $\tilde{g}$ is also even and we get (14) as a condensed recurrent PDE for the Wigner function for the $2-$ particle quantum fracton. But this is not the total picture; we need to calculate the evolution PDE for $\mathcal{H}_{tot} = \frac{1}{2}(\mathcal{H} + \mathcal{H}^\dagger)$. Since we know the PDE evolution for $H$(doubt);

$$\begin{aligned}
\partial_t W &= \frac{1}{4\pi i}\int du \Big\langle x + \frac{u}{2}\Big|\, [\mathcal{H}, \rho] + [\mathcal{H}^\dagger, \rho]\, \Big| x - \frac{u}{2}\Big\rangle e^{-iup} \\
&= \frac{1}{4\pi i}\int du \Big\langle x + \frac{u}{2}\Big|\, [\mathcal{H}, \rho]\, \Big| x - \frac{u}{2}\Big\rangle e^{-iup} - \frac{1}{4\pi i}\int du \Big\langle x + \frac{u}{2}\Big|\, [\mathcal{H}, \rho]^\dagger\, \Big| x - \frac{u}{2}\Big\rangle e^{-iup} \\
&= \frac{1}{4\pi i}\int du \Big\langle x + \frac{u}{2}\Big|\, [\mathcal{H}, \rho]\, \Big| x - \frac{u}{2}\Big\rangle e^{-iup} + \Big[\frac{1}{4\pi i}\int du \Big\langle x - \frac{u}{2}\Big|\, [\mathcal{H}, \rho]\, \Big| x + \frac{u}{2}\Big\rangle e^{iup}\Big]^* \\
&= \frac{1}{2}\Big(\frac{1}{2\pi i}\int du \Big\langle x + \frac{u}{2}\Big|\, [\mathcal{H}, \rho]\, \Big| x - \frac{u}{2}\Big\rangle e^{-iup} + \Big[\frac{1}{2\pi i}\int du \Big\langle x + \frac{u}{2}\Big|\, [\mathcal{H}, \rho]\, \Big| x - \frac{u}{2}\Big\rangle e^{-iup}\Big]^*\Big) \quad \text{Change } u \to -u \\
&= -\,\mathbb{R}\mathrm{e}\Big[\int d\omega\, \tilde{g}(\omega)\Big[(2p + 2\omega)\partial_x + i\omega^2(1 - e^{2i\omega x})\Big]\, W\Big(x, p + \frac{\omega}{2}\Big)e^{-i\omega x}\Big] \qquad (15)
\end{aligned}$$

Eq.(15) represents the recurrent PDE evolution of the Wigner distribution for the $2-$particle fracton. It is very easy to validate the free particle case is derivable from this if we take $\tilde{g}(\omega) = \delta(\omega)$.