Design and Analysis of Algorithms

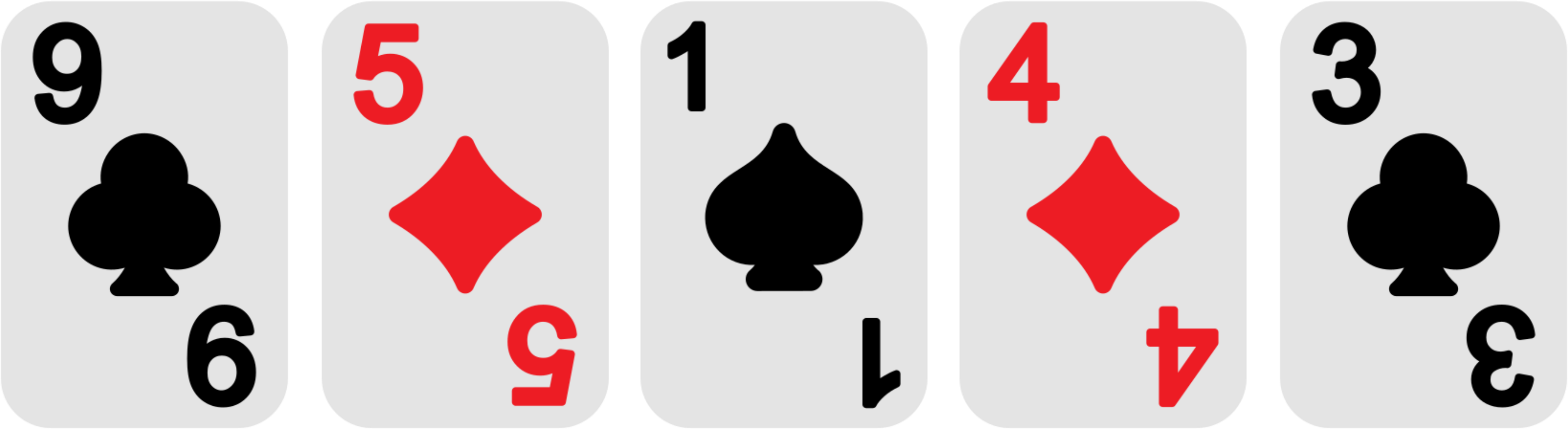
Algorithm Design Techniques

- **Brute Force and Exhaustive Search**
- Divide-and-Conquer
- **Decrease-and-Conquer**
- Transform-and-Conquer
- Space and Time Trade-Offs
- Dynamic Programming
- Greedy Technique
- Iterative Improvement
- Backtracking
- Branch-and-Bound

Decrease-and-Conquer

- **Reduce** original problem instance to smaller instance of the same problem.
- **Solve** smaller instance.
- **Extend** solution of smaller instance to obtain solution to original instance.

Sorting Playing Cards



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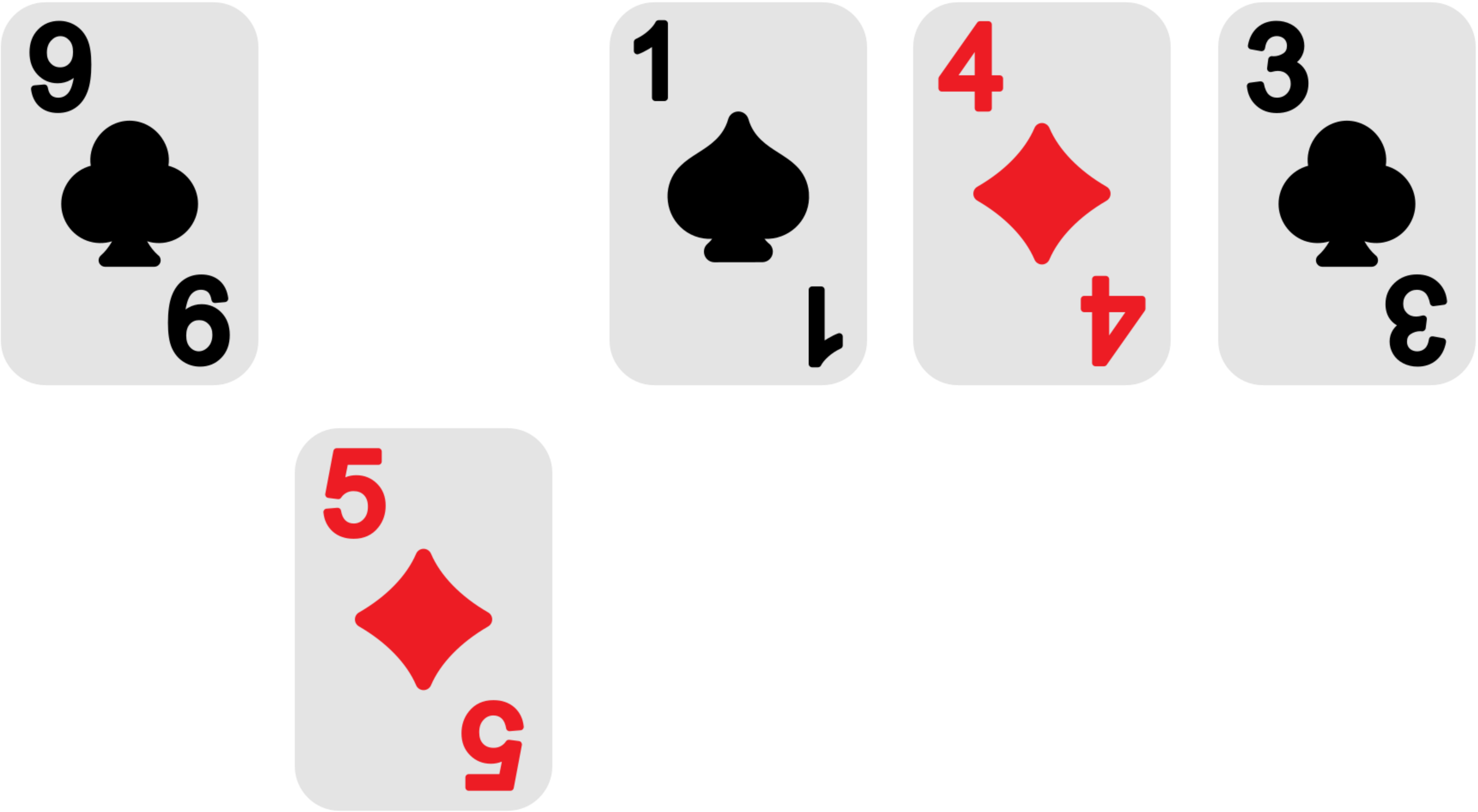

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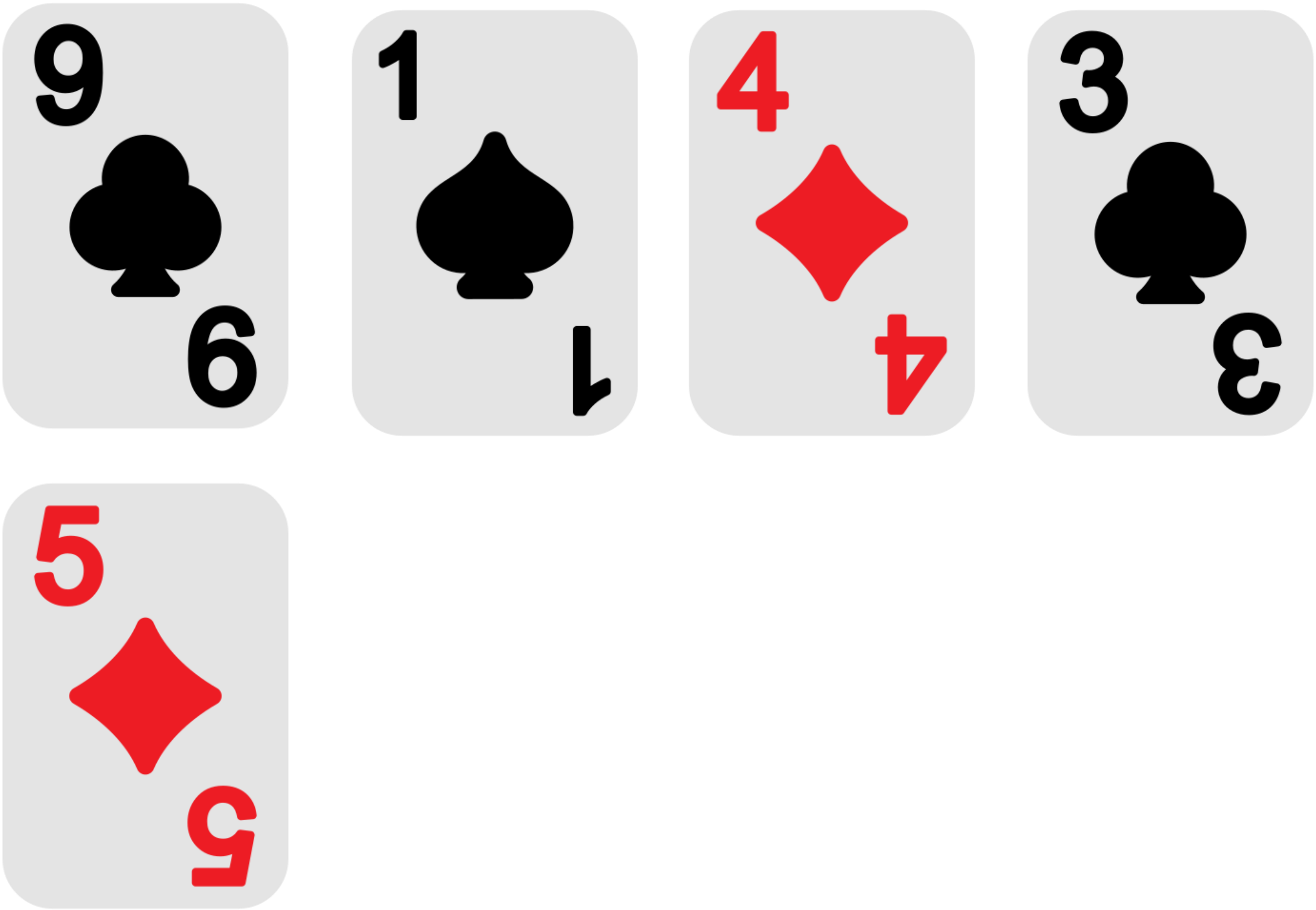

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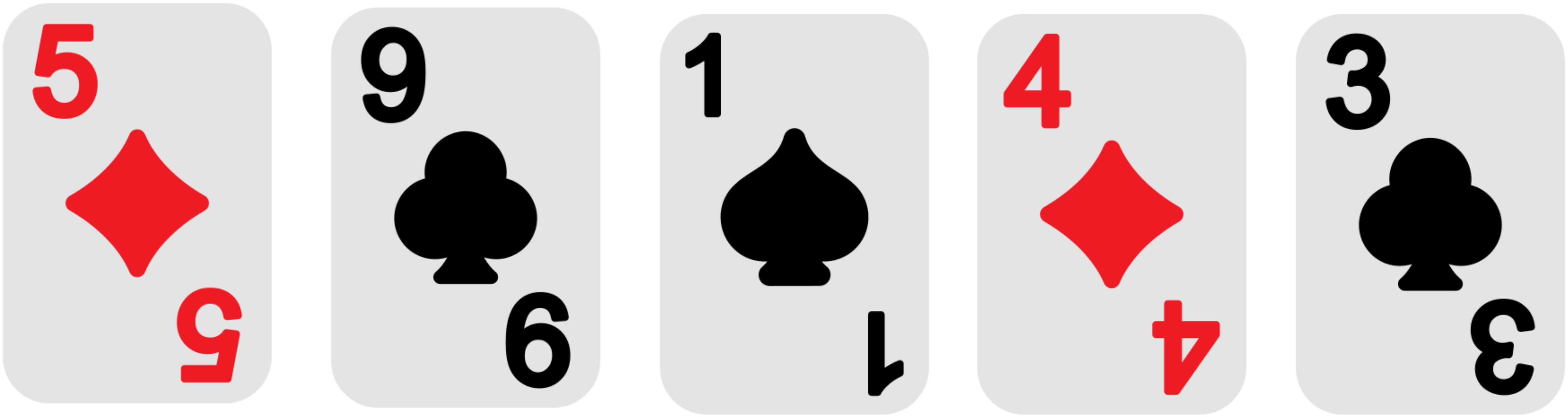

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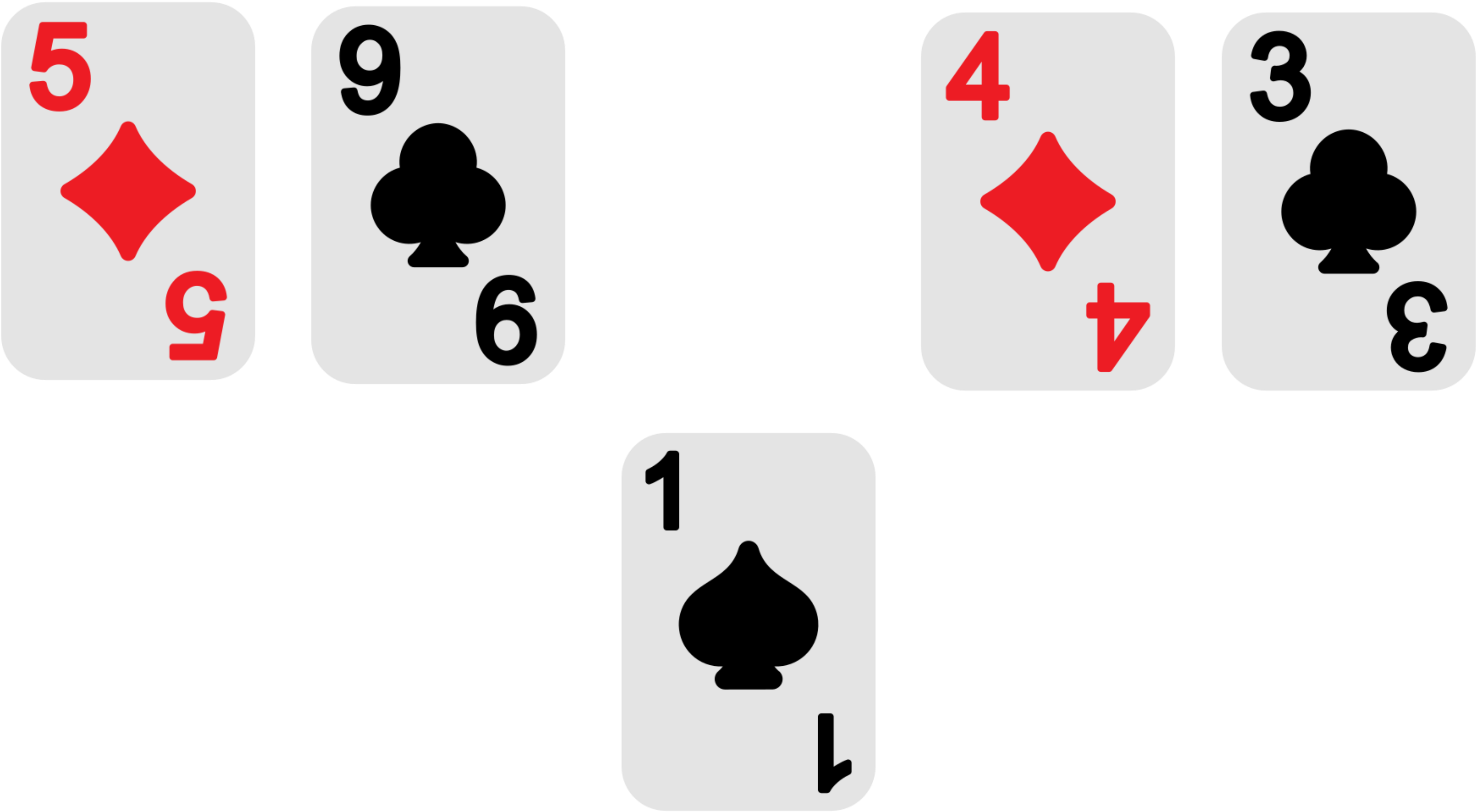

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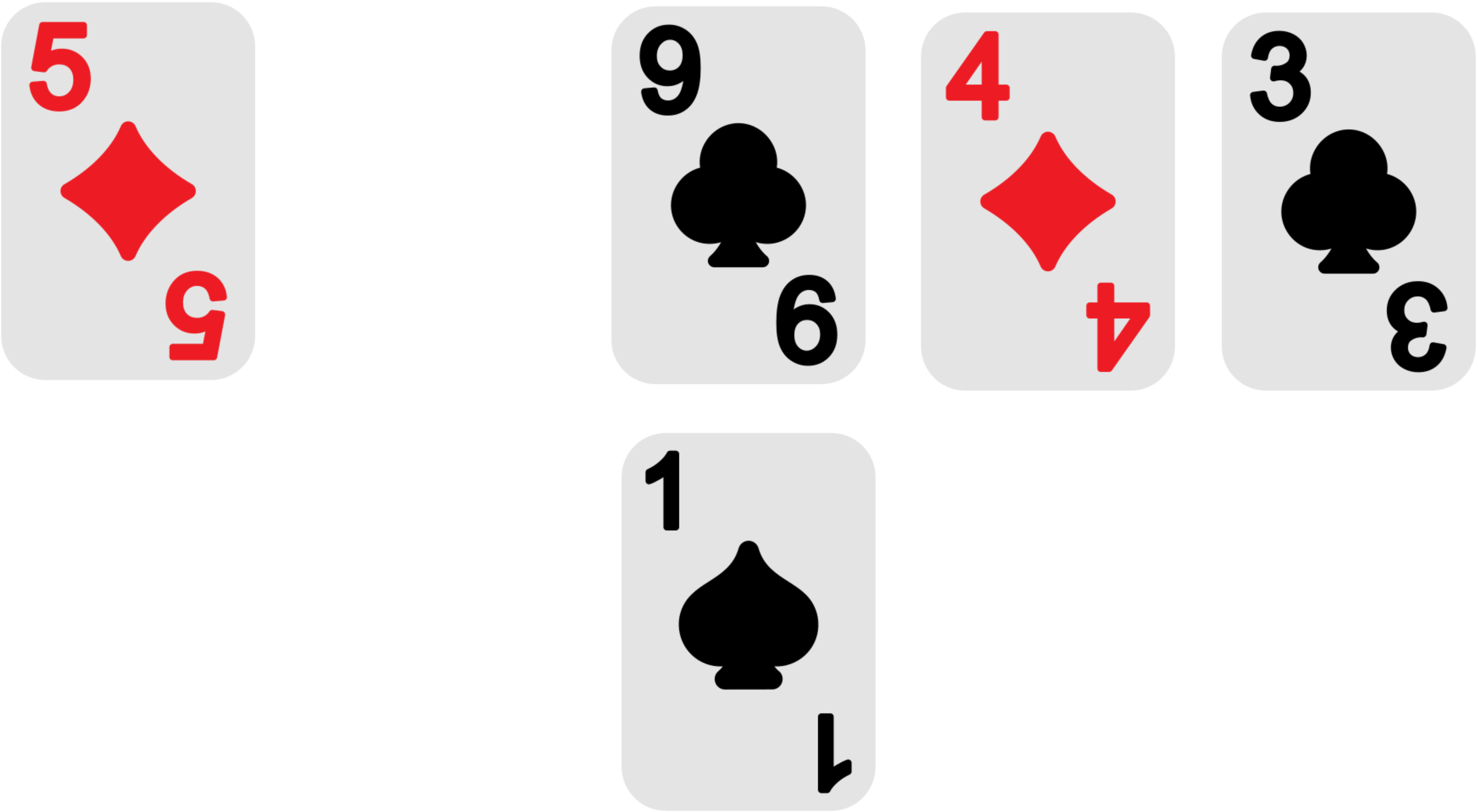

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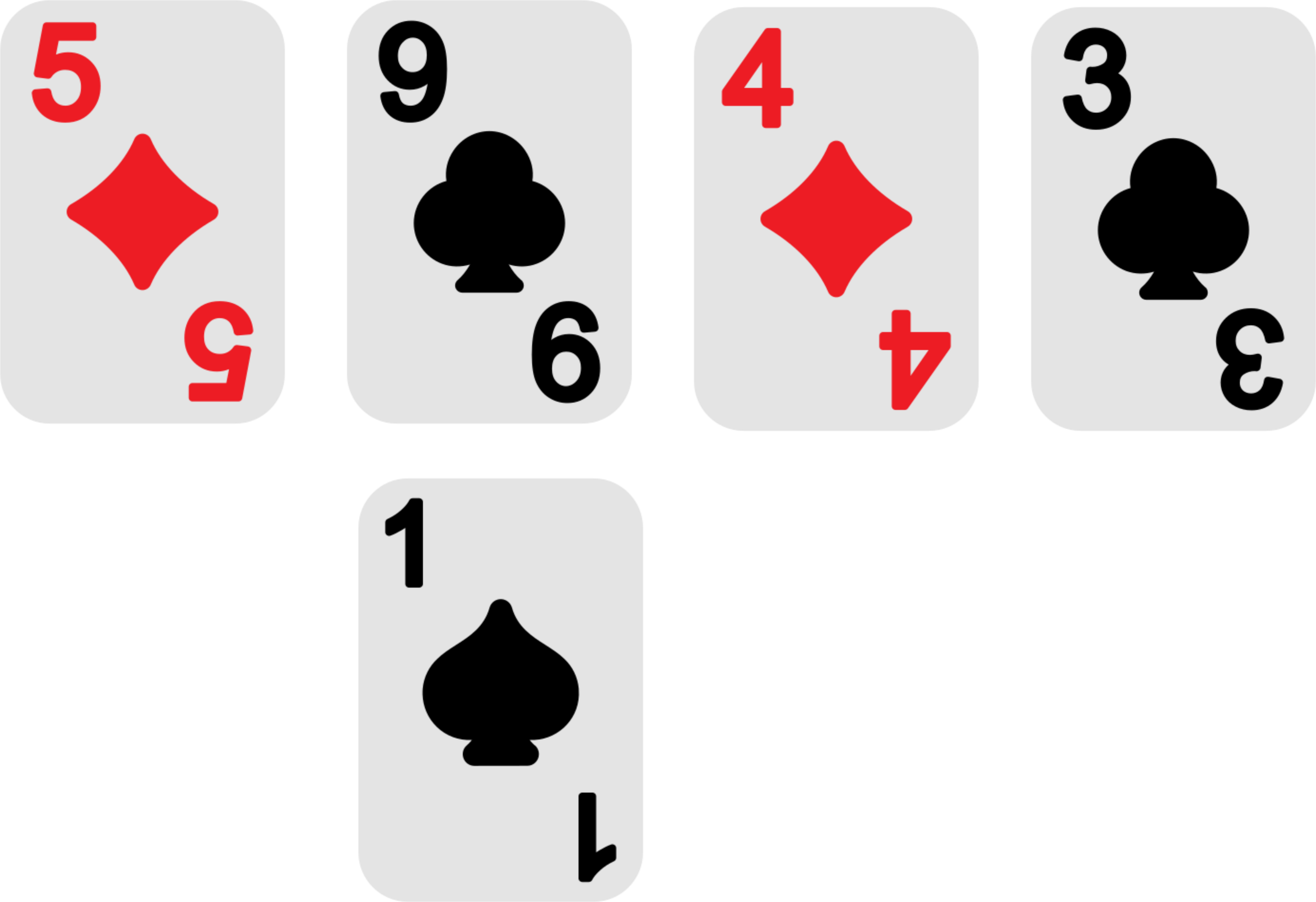

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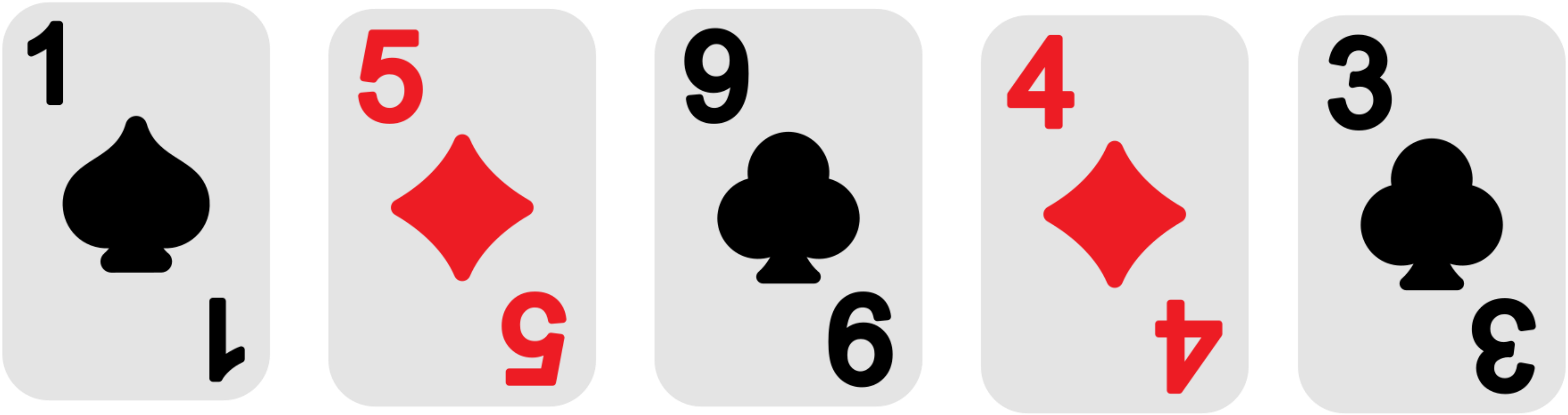

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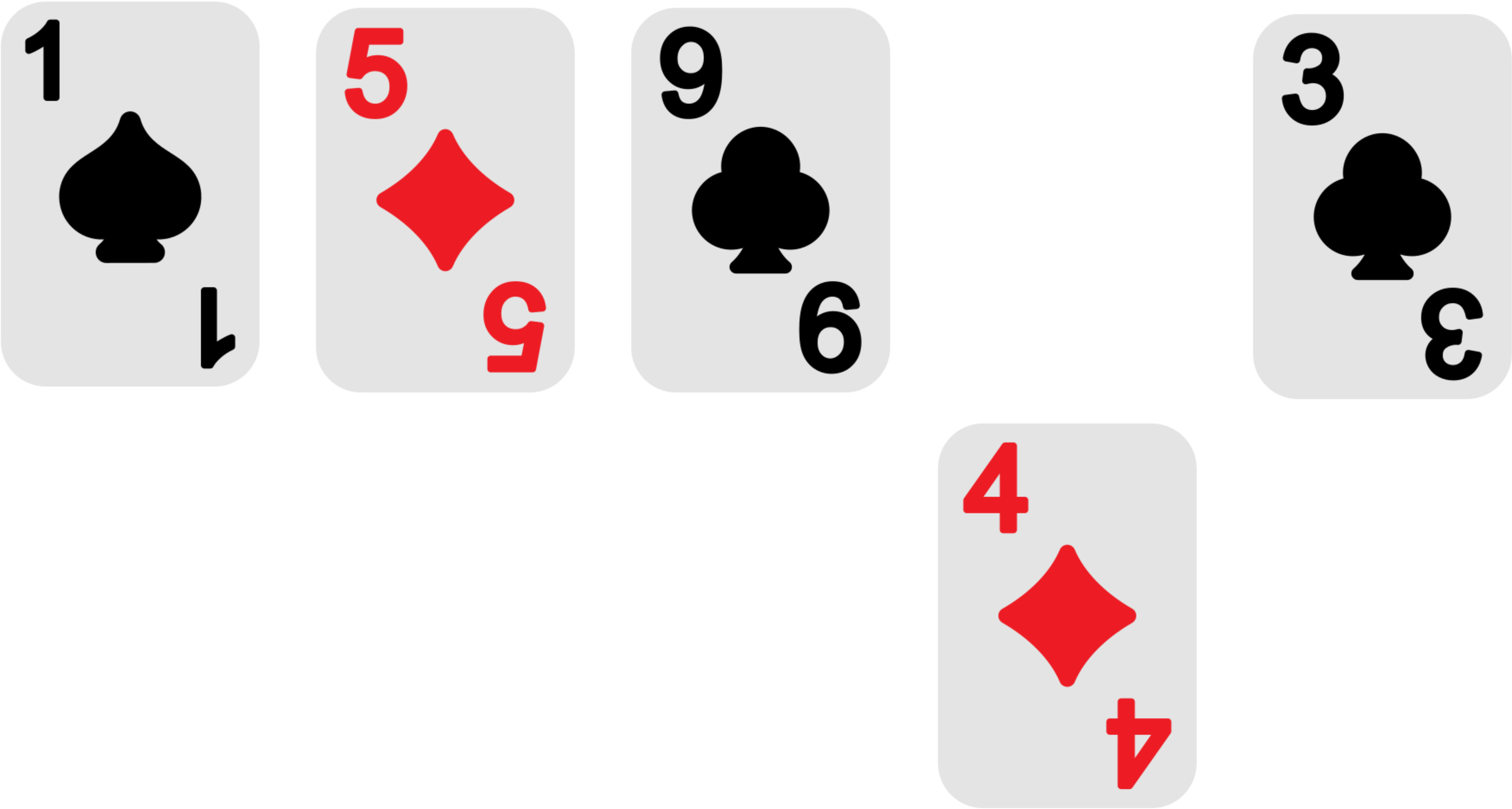

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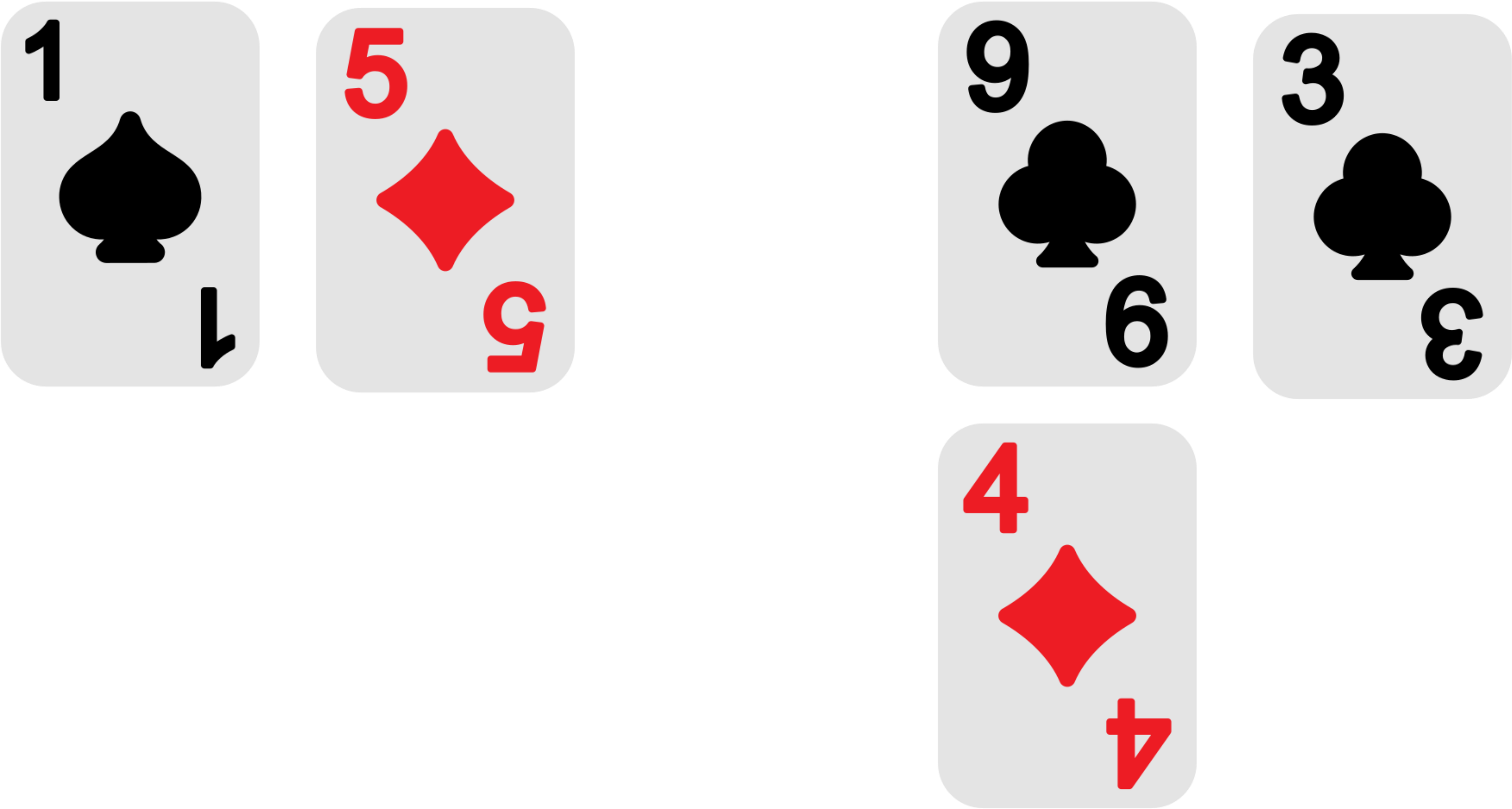

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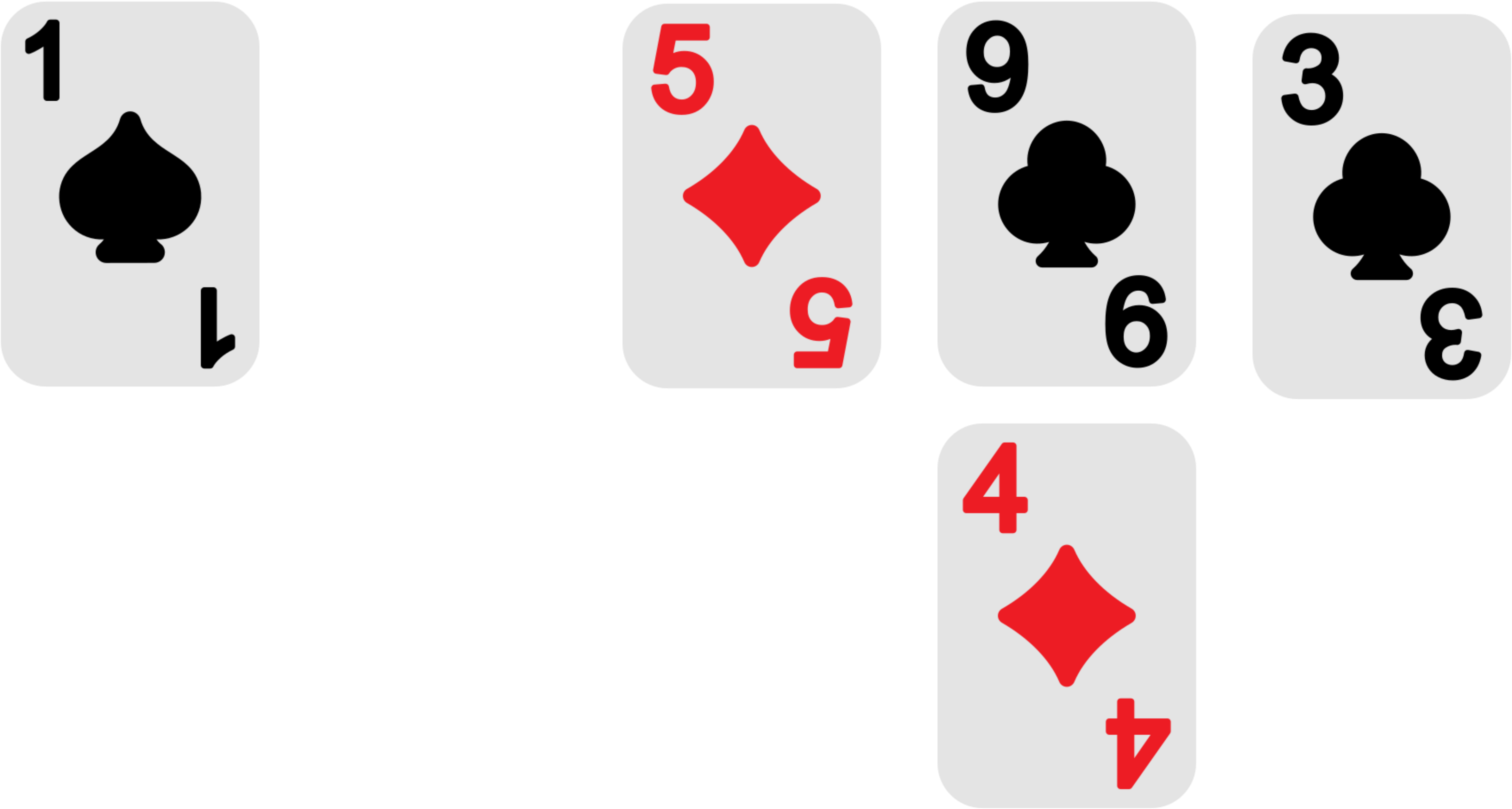

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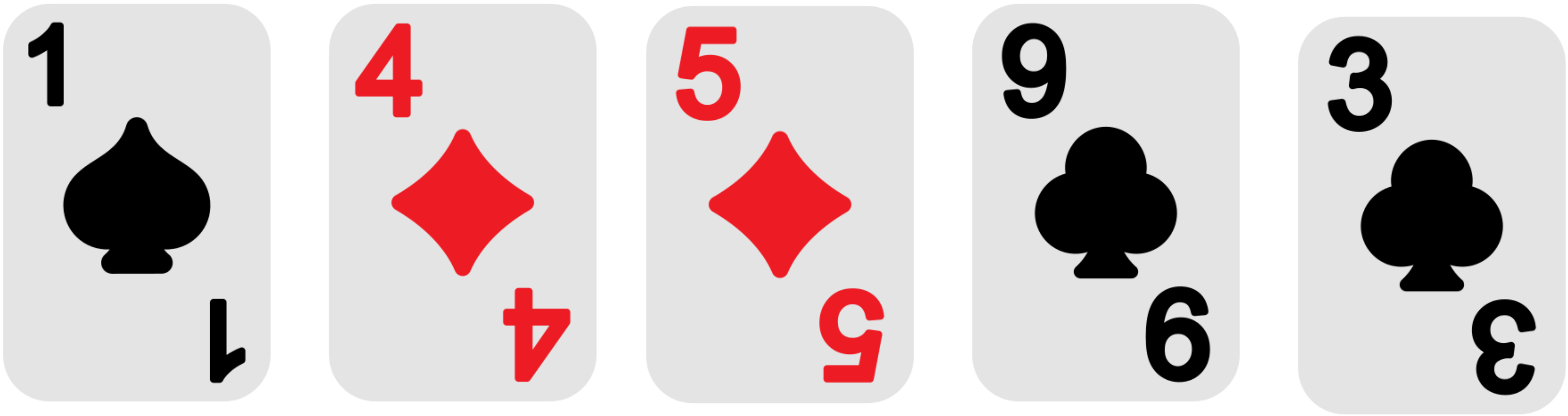

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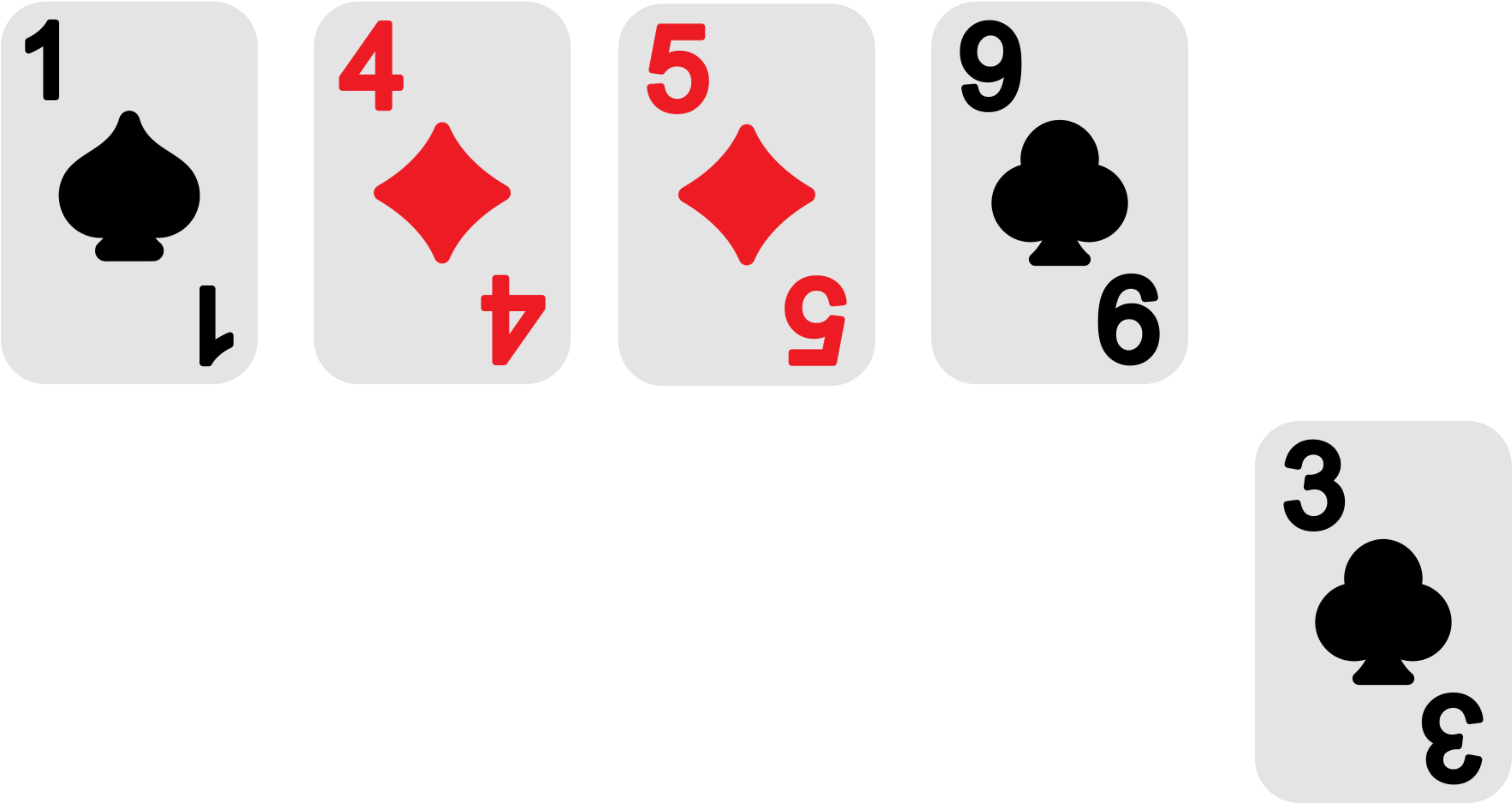

Sorting Playing Cards

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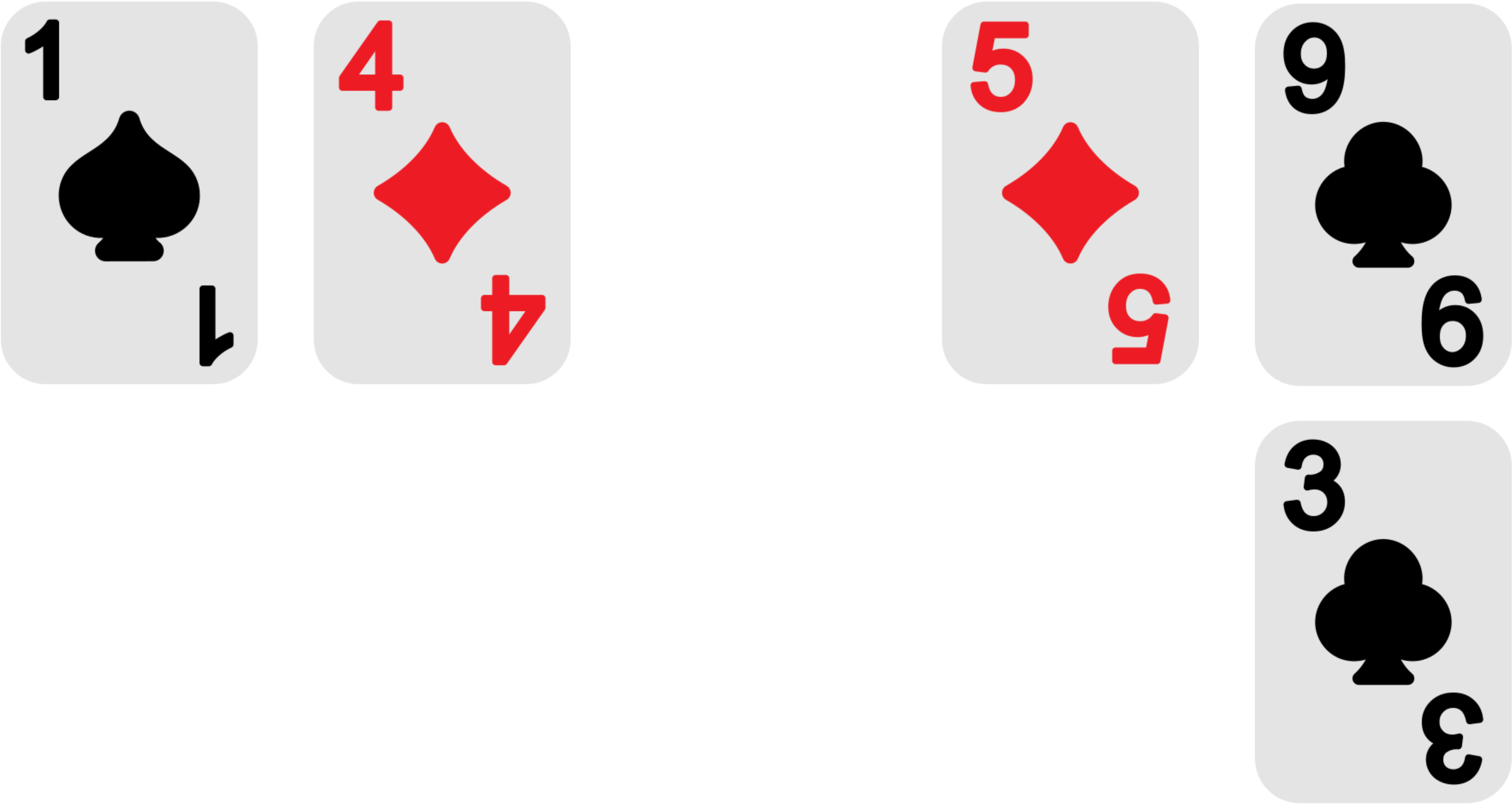

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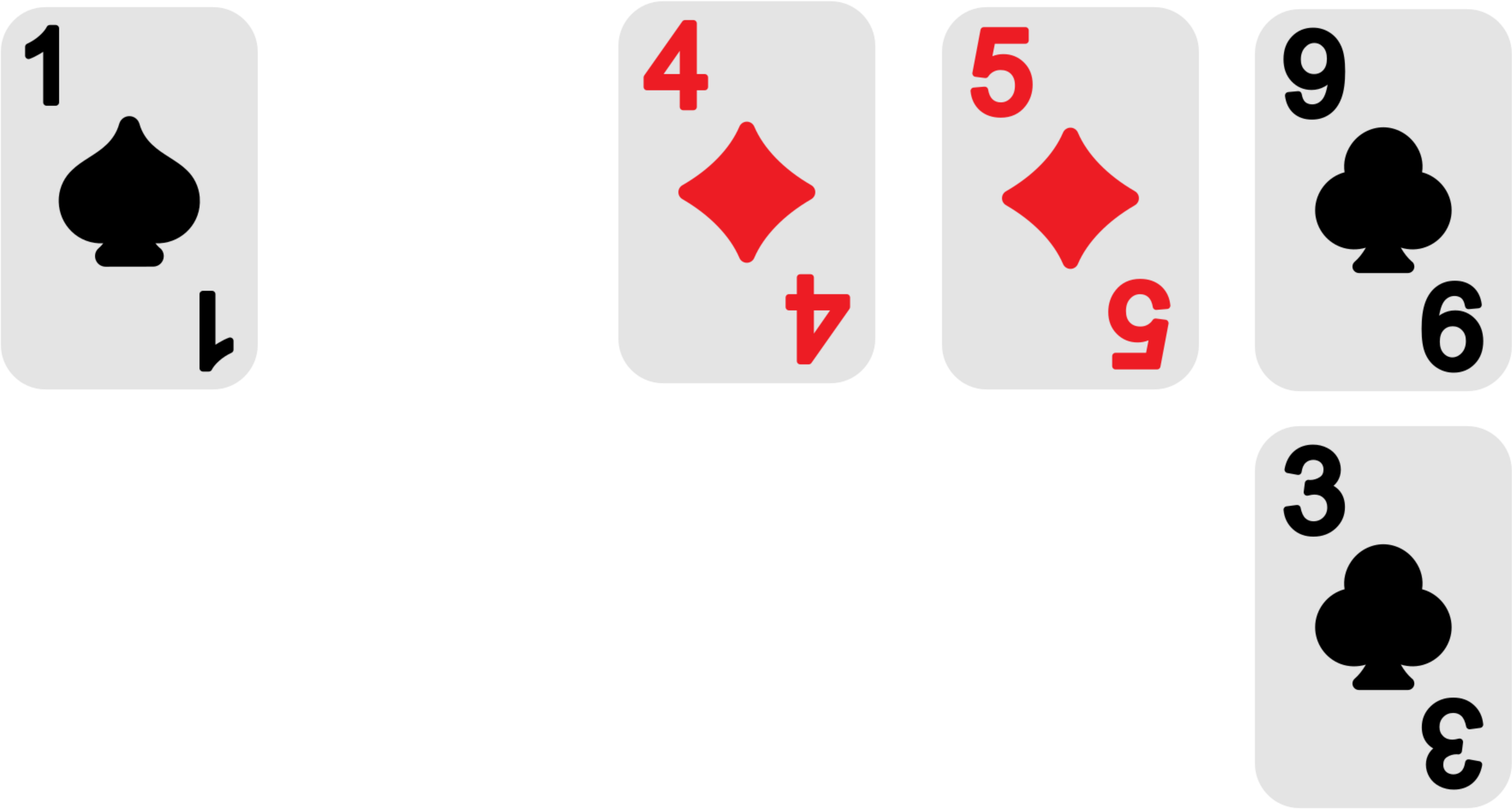


Sorting Playing Cards

Insertion Sort: Pseudocode

ALGORITHM *InsertionSort*($A[0..n-1]$)
//Sorts a given array by insertion sort
//Input: An array $A[0..n-1]$ of n orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
for $i \leftarrow 1$ **to** $n-1$ **do**
    key $\leftarrow A[i]$
    j $\leftarrow i - 1$
    while j ≥ 0 **and** A[j] $>$ key **do**
        A[j + 1] $\leftarrow A$[j]
        j $\leftarrow$ j $-$ 1
    A[j + 1] $\leftarrow$ key

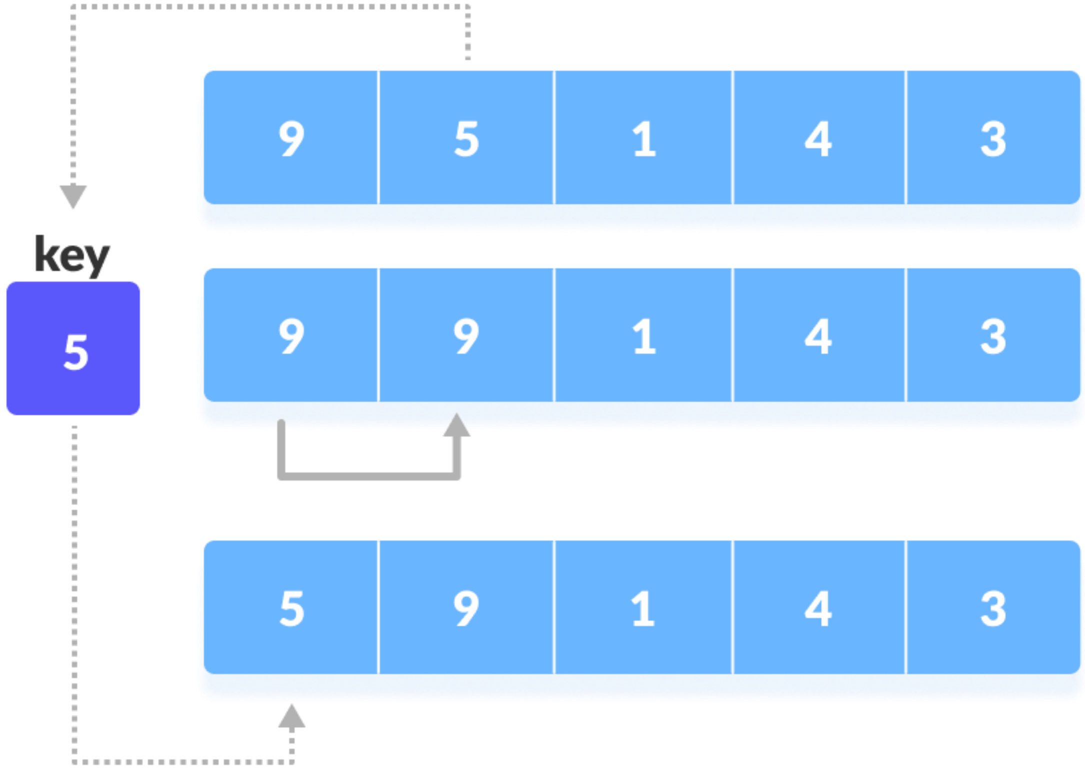
Working of Insertion Sort

- Suppose we need to sort the following array.

9	5	1	4	3

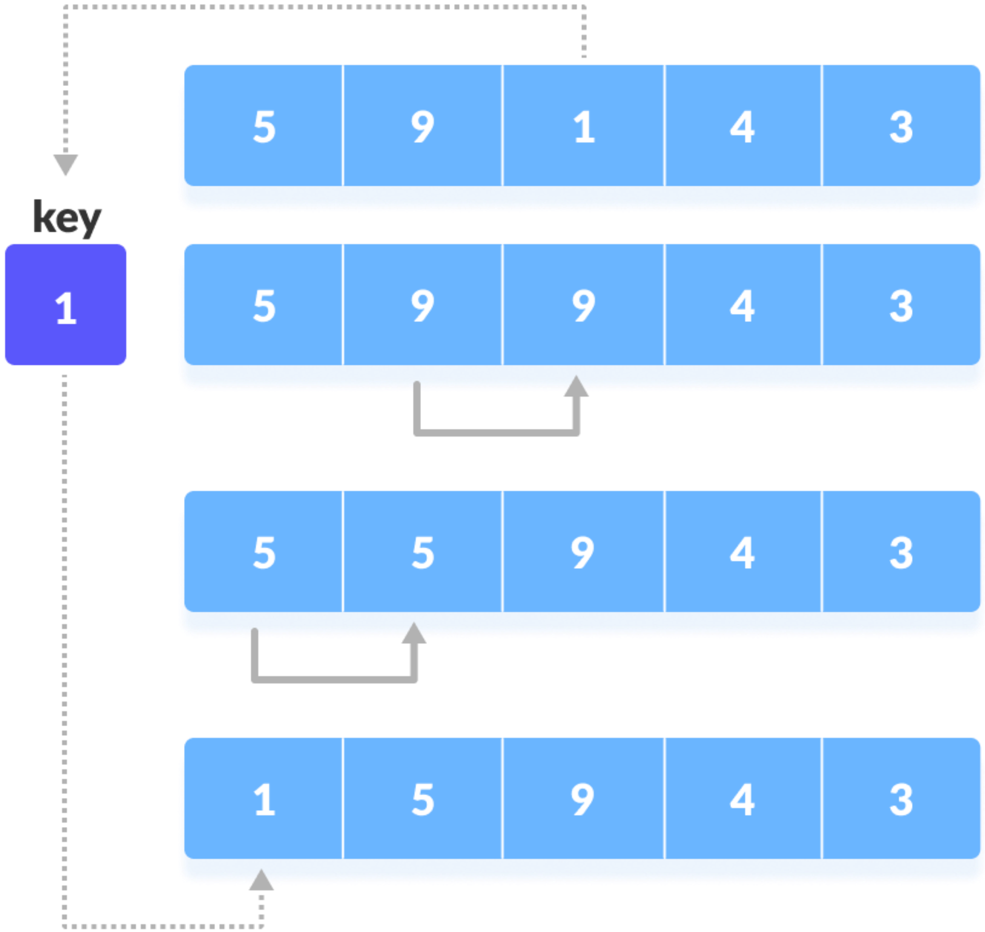
Working of Insertion Sort

- The first element in the array is assumed to be sorted.
- Take the second element and store it separately in key.
- Compare key with the first element.



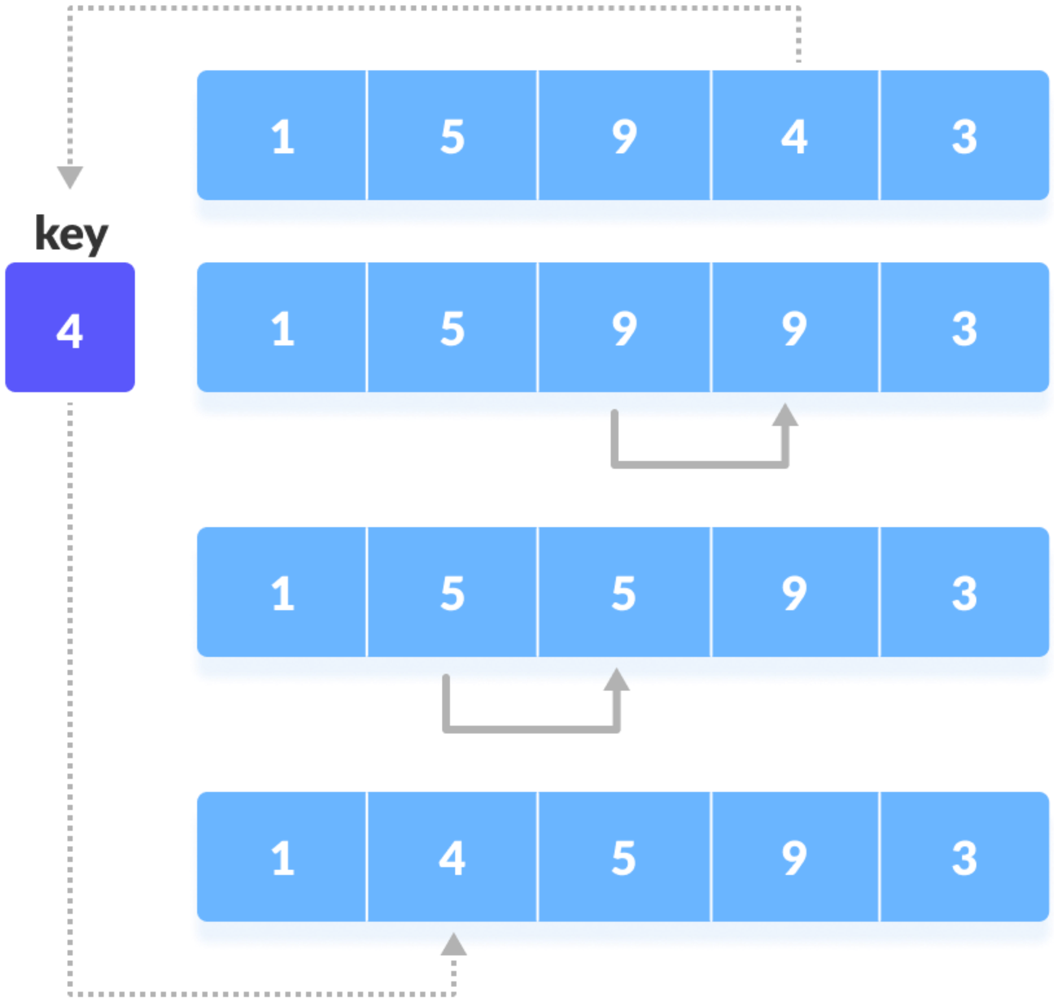
Working of Insertion Sort

- Now, the first two elements are sorted.
- Take the third element and compare it with the elements on the left.



Working of Insertion Sort

- Similarly, place every unsorted element at its correct position.



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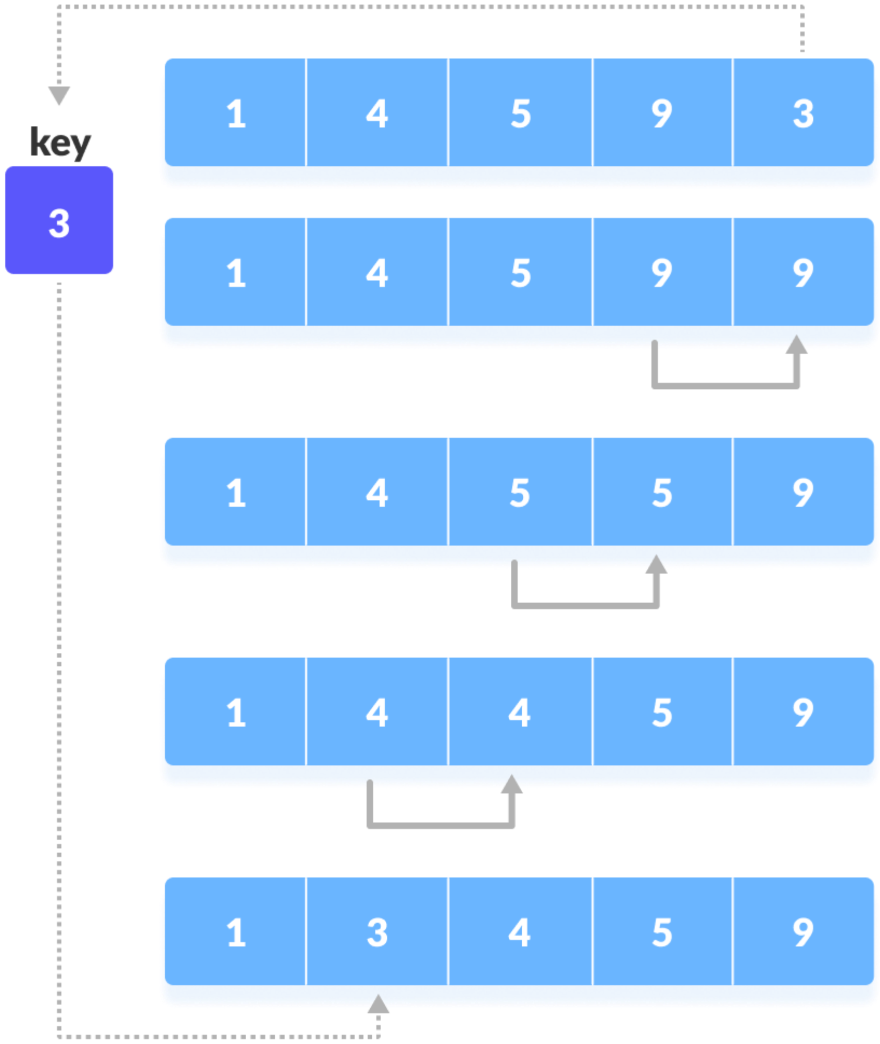


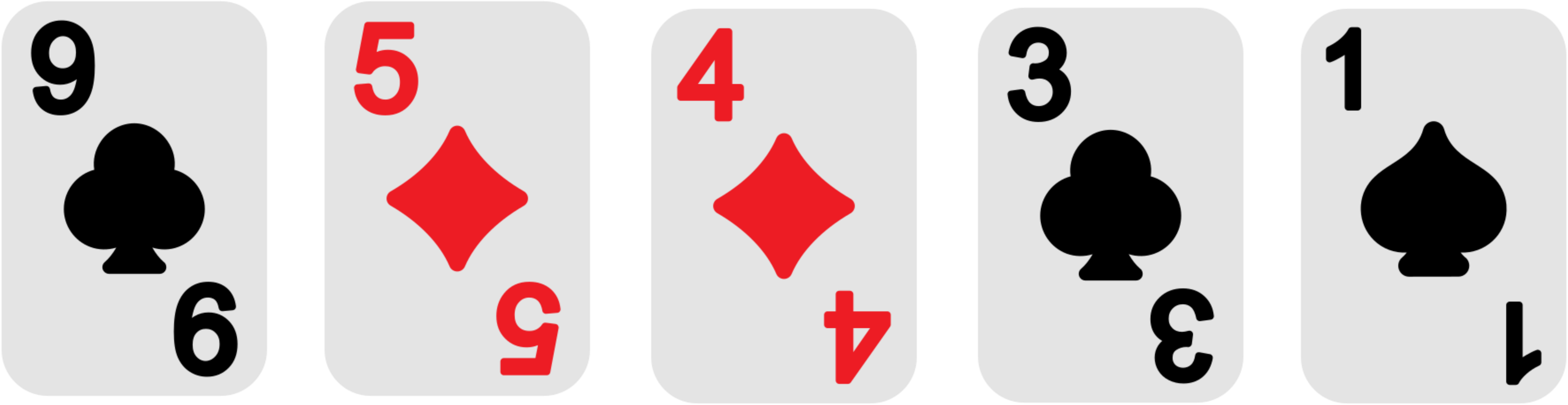
Example Of Sorting With Insertion Sort

Sort 6, 4, 1, 8, 5

6 |	**4**	1	8	5
4	6 |	**1**	8	5
1	4	6 |	**8**	5
1	4	6	8 |	**5**
1	4	5	6	8

Insertion Sort: Best Case

Insertion Sort: Worst Case



Insertion Sort Complexity

Time Complexity	
Best	$O(n)$
Worst	$O(n^2)$
Average	$O(n^2)$
Space Complexity	$O(1)$
Stability	Yes

Sorting Algorithms

- Bubble Sort
- Selection Sort
- **Insertion Sort**
- Merge Sort
- Quicksort
- Counting Sort
- Radix Sort
- Bucket Sort
- Heap Sort
- Shell Sort

Summations

$$\sum_{i=m}^{n} 1 = n - m + 1$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} i^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

Mystery 1

a. What does this algorithm compute?
b. What is its basic operation?
c. How many times is the basic operation executed?

ALGORITHM *Mystery*(x, y)

//Input: Two numbers x and y

$S \leftarrow x + y$

return S

Mystery 1 – Solution

a. What does this algorithm compute? The sum of x and y
b. What is its basic operation? Addition
c. How many times is the basic operation executed? $T(n) = 1$

ALGORITHM *Add*(x, y)

//Input: Two numbers x and y

//Output: The sum of x and y

$S \leftarrow x + y$

return S

Mystery 2

a. What does this algorithm compute?
b. What is its basic operation?
c. How many times is the basic operation executed?

ALGORITHM *Mystery*(n)

//Input: A nonnegative integer n

$S \leftarrow 0$

for $i \leftarrow 1$ **to** n **do**

$\quad S \leftarrow S + i$

return S

Mystery 2 – Solution

a. What does this algorithm compute? The sum of integers from 1 to n
b. What is its basic operation? Addition
c. How many times is the basic operation executed? $T(n) = n$

ALGORITHM $Sum(n)$

$S \leftarrow 0$

for $i \leftarrow 1$ **to** n **do**

$S \leftarrow S + i$

return S

Mystery 2 – Analysis

- The algorithm makes one addition on each execution of the loop.
- Let us denote $T(n)$ the number of times this addition is executed.
- We try to find a formula expressing it as a function of size n.
- Which is repeated for each value of the loop's variable i within the bounds 1 and n.
- Therefore, we get the following sum for $T(n)$:

$$T(n) = \sum_{i=1}^{n} 1 = n$$

Mystery 3

a. What does this algorithm compute?
b. What is its basic operation?
c. How many times is the basic operation executed?

ALGORITHM *Mystery*(A[0..$n-1$], K)
    //Input: An array A[0..n − 1] and a key K
    $i \leftarrow 0$
    for $i \leftarrow 0$ **to** $n-1$ **do**
        if A[i] = K
        return i
    return −1

Mystery 3 – Solution

a. What does this algorithm compute?

Searches for a given value in a given array by sequential search, and returns the index of the matching element, or −1 if there are no matching elements

b. What is its basic operation? Comparison

c. How many times is the basic operation executed? $T(n) = n$

ALGORITHM *SequentialSearch*(A[0..n − 1], K)

$i \leftarrow 0$

for $i \leftarrow 0$ **to** $n - 1$ **do**

  if A[i] = K

  return i

return −1

Mystery 4

a. What does this algorithm compute?

b. What is its basic operation?

c. How many times is the basic operation executed?

ALGORITHM *Mystery*(A[0..$n-1$])

//Input: An array A[0..$n-1$] of real numbers

$maxval \leftarrow A[0]$

for $i \leftarrow 1$ **to** $n-1$ **do**

  if $A[i] > maxval$

    $maxval \leftarrow A[i]$

return *maxval*

Mystery 4 – Solution

a. What does this algorithm compute?

   Determines the value of the largest element in a given array

b. What is its basic operation? Comparison

c. How many times is the basic operation executed? $T(n) = n - 1$

ALGORITHM *MaxElement*(A[0..$n - 1$])

    maxval ← A[0]
    for i ← 1 to n − 1 do
        if A[i] > maxval
            maxval ← A[i]
    return maxval

Mystery 4 – Analysis

- Let us denote $C(n)$ the number of times this comparison is executed.
- We try to find a formula expressing it as a function of size n.
- The algorithm makes one comparison on each execution of the loop.
- Which is repeated for each value of the loop's variable i within the bounds 1 and $n-1$.
- Therefore, we get the following sum for $C(n)$:

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 \in \Theta(n)$$

Mystery 5

a. What does this algorithm compute?

b. What is its basic operation?

c. How many times is the basic operation executed?

ALGORITHM *Mystery*(A[0..$n-1$, 0..$n-1$], B[0..$n-1$, 0..$n-1$])

//Input: Two n × n matrices A and B

for $i \leftarrow 0$ **to** $n-1$ **do**

  for $j \leftarrow 0$ **to** $n-1$ **do**

    C[i, j] $\leftarrow$ A[i, j] + B[i, j]

return C

Mystery 5 – Solution

a. What does this algorithm compute? Adds two square matrices of order n

b. What is its basic operation? Addition

c. How many times is the basic operation executed? $T(n) = n^2$

ALGORITHM *MatrixAddition*(A[0..$n-1$, 0..$n-1$], B[0..$n-1$, 0..$n-1$])

//Input: Two n × n matrices A and B

//Output: Matrix C = A + B

for $i \leftarrow 0$ **to** $n - 1$ **do**

  for $j \leftarrow 0$ **to** $n - 1$ **do**

    $C[i, j] \leftarrow A[i, j] + B[i, j]$

return C

Mystery 6

a. What does this algorithm compute?
b. What is its basic operation?
c. How many times is the basic operation executed?

ALGORITHM *Mystery*(A[0..$n-1$])
//Input: An array A[0..n − 1]
for $i \leftarrow 0$ **to** $n - 2$ **do**
  for $j \leftarrow i + 1$ **to** $n - 1$ **do**
    if A[i] = A[j]
      return false
return true

Mystery 6 – Solution

a. What does this algorithm compute?

   Determines whether all the elements in a given array are distinct

b. What is its basic operation? Comparison

c. How many times is the basic operation executed? $T(n) = n(n - 1)/2$

ALGORITHM *UniqueElements*(A[0..$n - 1$])

    for $i \leftarrow 0$ **to** $n - 2$ **do**

        for $j \leftarrow i + 1$ **to** $n - 1$ **do**

            if $A[i] = A[j]$

                return false

    return true

Mystery 6 – Analysis

- Only one comparison is made for each repetition of the innermost loop.
- For each value of the loop variable j between its limits $i + 1$ and $n - 1$.
- This is repeated for each value of the outer loop.
- For each value of the loop variable i between its limits 0 and $n - 2$.

$$C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$

$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{(n-1)n}{2} \approx \frac{1}{2} n^2 \in \Theta(n^2).$$

Mystery 7

a. What does this algorithm compute?

b. What is its basic operation?

c. How many times is the basic operation executed?

```
ALGORITHM Mystery(A[0..n − 1, 0..n − 1], B[0..n − 1, 0..n − 1])
    //Input: Two n × n matrices A and B
    for i ← 0 to n − 1 do
        for j ← 0 to n − 1 do
            C[i, j] ← 0
            for k ← 0 to n − 1 do
                C[i, j] ← C[i, j] + A[i, k] ∗ B[k, j]
    return C
```

Mystery 7 – Solution

a. What does this algorithm compute? Multiplies two square matrices of order n

b. What is its basic operation? Multiplication

c. How many times is the basic operation executed? $T(n) = n^3$

ALGORITHM *MatrixMultiplication*(A[0..$n-1$, 0..$n-1$], B[0..$n-1$, 0..$n-1$])

```
    //Input: Two n × n matrices A and B
    for i ← 0 to n − 1 do
        for j ← 0 to n − 1 do
            C[i, j] ← 0
            for k ← 0 to n − 1 do
                C[i, j] ← C[i, j] + A[i, k] ∗ B[k, j]
    return C
```

Mystery 7 – Analysis

- Obviously, there is just one multiplication executed on each repetition of the algorithm's innermost loop.
- The total number of multiplications $M(n)$ is expressed by the following triple sum:

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1 = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} n^2 = n^3.$$