BAYESIAN HIERARCHICAL MODELS

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1. *Bayesian Thinking*

2. *Hierarchical Models*

Bayesian Thinking

Probabilistic Modeling

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Bayesian: θ_0 fixed quantity, but model uncertainty by imposing a prob. distr. on Θ

Bayes' Theorem

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$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}} \propto \textit{likelihood} \times \textit{prior}$$

Solving for the Posterior Analytically

Setting: $\{y_i : i = 1, \ldots, n\}$ with $y_i \overset{\text{iid}}{\sim} \mathcal{N}\left(\mu, \sigma^2\right)$
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Why?

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$$\implies \mu \mid y \sim \mathcal{N}\left(\bar{y}, \sigma^2/n\right)$$

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$$= \alpha\boldsymbol{\bar{y}} + (1-\alpha)\boldsymbol{\mu_{0}}$$

Sampling from the Posterior

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Solution: Sampling?

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Quantity of interest: $\mathbb{E}\left[\, h(\theta) \mid \text{data} \,\right] =: \mathbb{E}_\theta[h]$

Estimation: Let $\theta^{(1)}, \ldots, \theta^{(n)} \overset{\text{iid}}{\sim} p(\theta \mid \text{data})$, then

$$\frac{1}{\sqrt{n}} \left(\sum_i h(\theta^{(i)}) - \mathbb{E}_\theta[h] \right) \xrightarrow{\text{d}} \mathcal{N}\left(0, \omega\right)$$

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(iii.) only known up to an integration const.

Markov Chain Monte Carlo

Algorithm Metropolis-Hastings (1953, 1970)

Input: (π, q, T) = (target, proposal, no. of samples)

1: initialize x_0 in supp q

2: **for** $t = 0, \ldots, T$ **do**

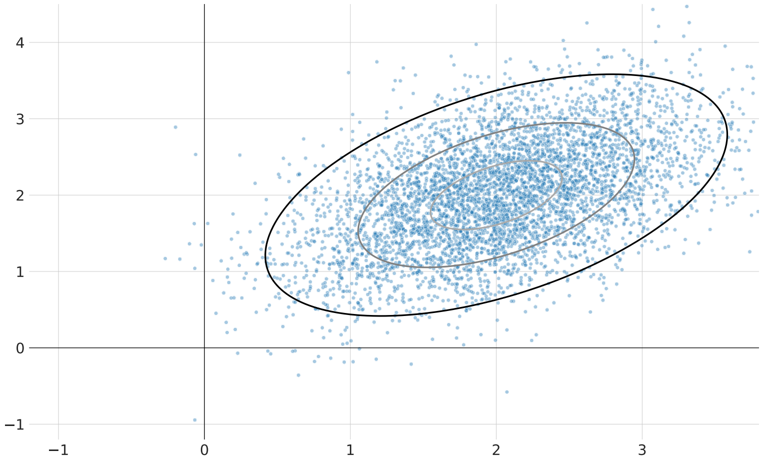
3: candidate: $y \sim q(\cdot \mid x_t)$

4: acceptance prob.: $\mathcal{A} \leftarrow \min\left\{\frac{\pi(y)}{\pi(x_t)}\frac{q(y \mid x_t)}{q(x_t \mid y)}, 1\right\}$

5: update: $x_{t+1} \leftarrow \begin{cases} y & \text{,with prob. } \mathcal{A} \\ x_t & \text{,with remaining prob.} \end{cases}$

6: **return** $\{x_t : t = 1, \ldots, T\}$

Markov Chain Monte Carlo



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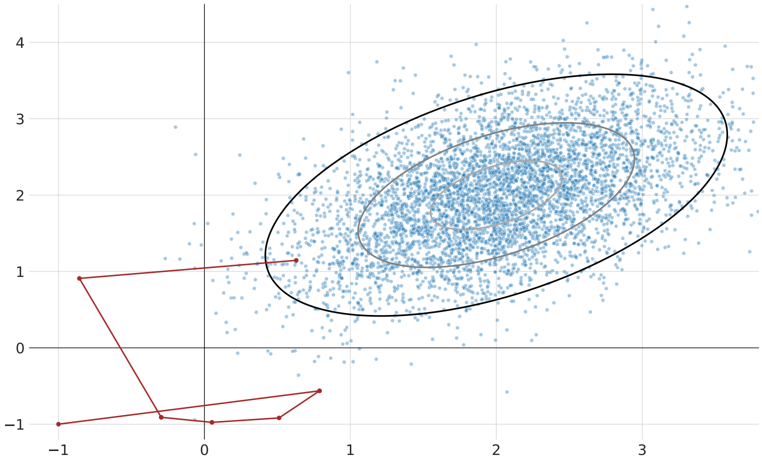

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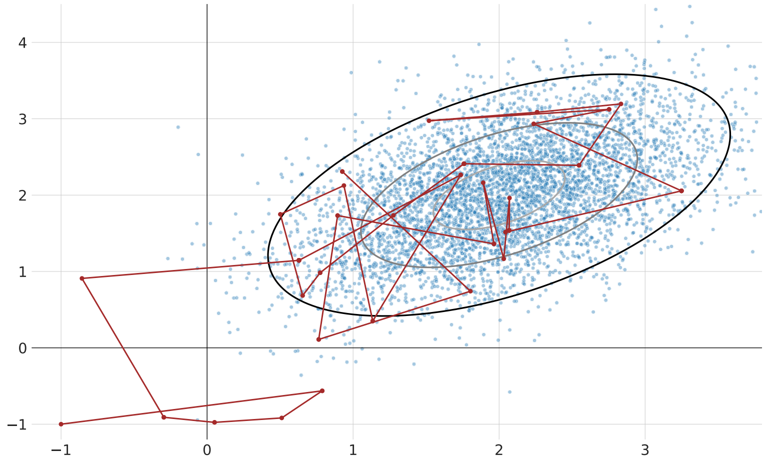

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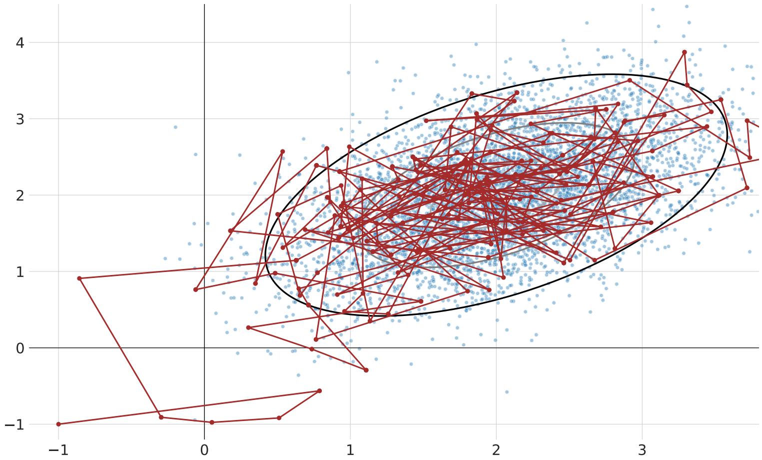

Hierarchical Models

Structure of HM - Setup

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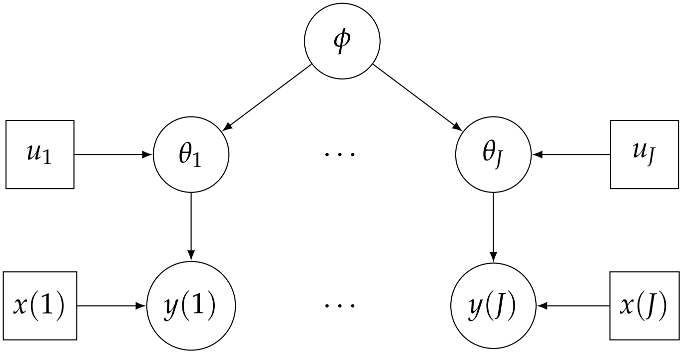
Example:

Test Outcome: y_i

Parental Income: x_i

Num. of Teachers: u_j

Structure of HM



The Prior Revisited

Before:

Model: $p(\text{data} \mid \theta)$

Prior: $p(\theta)$

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Now:

Model: $p(\text{data} \mid \theta, \phi) = p(\text{data} \mid \theta)$

Prior: $p(\theta \mid \phi)$

Hyperprior: $p(\phi)$

The Posterior Revisited

Posterior:

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&\propto p(\text{data} \mid \theta) p(\theta \mid \phi) p(\phi) \\
p(\phi \mid \text{data}) &\propto \int p(\theta, \phi \mid \text{data}) \mathrm{d}\theta
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Varying Slopes, Varying Intercepts

Setup: Individual i in group j

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https://github.com/timmens/
bayesian-hierarchical-models

http://mfviz.com/hierarchical-models/