Numerical Methods: 24 Hours Take-home examination, June 2023

This examination consists of 5 exercises. Each question has been assigned a weight (points). The total number of points is 100.

For the exam, you will need to be able to read matrices and vectors from files having the following format for an $M \times N$ matrix:

$$
\begin{array}{r}
M \\
N \\
Row1 \\
\dots \\
RowM
\end{array}
$$

where vectors always have $N = 1$. An example with a 3×2 matrix could be

$$
\begin{array}{ll}
3 & \\
2 & \\
3.04464 & 5.06464 \\
-0.6454 & 0.435435 \\
4.05454 & -1222.933435
\end{array}
$$

You must hand in a zip file containing a report and all your used code. Concerning the report, it needs to be CLEARLY readable (unreadable parts will be assessed as wrong). Whenever the word "state" is used in the questions, the answer MUST be present in the report. Wherever it says "Submit the used code", the used code MUST be handed in. Otherwise, you will get zero points for your answer. Clearly name your code files so that it is easy to see what exercise the code is used for.

Notice the following rules:

- It is allowed to use general purpose methods, text or code that are part of NR or elsewhere publicly available, including methods, text or code that has been uploaded by Kristine or me during the course. However, there must for each use be a CLEAR marking **stated in the report** including where the methods/text/used code was taken from. In all cases, you are solely responsible for the correctness of used methods, text and code. All methods, text and code that **explicitly handles the problems from the exam exercise** MUST be written by yourself.
- It is NOT allowed to share your answers (or parts of these), or to communicate with other people about the exercise. This includes other NM students taking this 24 hours take-home examination.

I wish you all the best for the forthcoming 24 hours!

Best regards, Henrik

Exercise 1 (30 points)

Consider a Linear Least Squares problem with a 40×6 design matrix $\mathbf{A}$ and the right hand side $\mathbf{b}$ that are given in *Ex1A.dat* and *Ex1b.dat* respectively.

i) (5 points) Find the Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T$. State the diagonal elements in $\mathbf{W}$. Submit the used code.

ii) (5 points) Use the Singular Value Decomposition to compute the solution $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{b}$. State the solution $\mathbf{x}$. Submit the used code.

iii) (5 points) State an estimate of the accuracy on the solution $\mathbf{x}$. State an explanation of how you computed the accuracy. Submit the used code.

In the above Linear Least Squares problems, there were no prior knowledge about differences in the uncertainties σ_i in each of the equations, and hence all σ_i's were set to $\sigma_i = \delta \equiv 1$. However there was an expectation of outliers (due to e.g. flawed measurements or large inaccuracies). The rest of the exercise is about an adaptive method to robustly handle such outliers based on residuals.

iv) (5 points) Compute and state the residual vector $\mathbf{r} \equiv \mathbf{A}\mathbf{x} - \mathbf{b}$. Submit the used code.

Using the expectation that large residuals are more likely to be outliers (or to have a large uncertainty), we can adaptively choose new σ_i's as $\sigma_i = \max\{\delta, |(\mathbf{A}\mathbf{x} - \mathbf{b})_i|\}$.

v) (5 points) Compute the new σ_i's and then the new design matrix $\mathbf{A}$ and new right hand side $\mathbf{b}$. State $[\mathbf{A}]_{0,0}$ and $[\mathbf{b}]_6$. Submit the used code.

vi) (5 points) Compute and use the Singular Value Decomposition to compute the solution $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{b}$ with the new design matrix and new right hand side. State the solution $\mathbf{x}$.

Exercise 2 (15 points)

Consider the equation

$$f(x) \equiv -x^3 + 2\cos(x) - \exp(-\sin(x+0.5)) = 0$$

i) (3 points) With $x_0 = -2$ and $y_0 = 2$, state (with at least 6 digits) the values $f(x_0)$ and $f(y_0)$. (HINT: you should get $f(x_0) \simeq 4.46$ and $f(y_0) \simeq -9.38$). Submit the used code.

ii) (6 points) Perform 15 iterations with the Regula Falsi (false position) method starting with $x_0 = -2$ and $y_0 = 2$. State (with at least 6 digits) the values x_{13}, x_{14} and x_{15}. Submit the used code.

ii) (6 points) Assuming that the order is 1, provide a precise estimate of the accuracy of x_{15}. State the estimate together with a clear explanation on how the estimate was arrived at.

Exercise 3 (15 points)

Consider the following system

$$\begin{aligned}
v_1'(t) &= \cos[v_2(t)] + v_3(t)^2 - v_1(t) + \exp(-t); \qquad & v_1(0) = 2 \\
v_2'(t) &= \cos(v_3(t)^2) - v_2(t) - \exp(-t); & v_2(0) = 3 \\
v_3'(t) &= \exp(-v_1(t)^2) - v_3(t); & v_3(0) = 4
\end{aligned} \tag{1}$$

i) (5 points) State the result for $(v_1'(0), v_2'(0), v_3'(0))$ with at least 6 digits. HINT: It should be approximately $(14, -5, -4)$. Submit the used code.

ii) (10 points) Use the Midpoint method with $N = 5, 10, 20, 40, \ldots$; $h = 5.0/N$ to generate solutions for $(v_1(5), v_2(5), v_3(5))$. You may assume that the global order is 2 as expected. Use as few number of subdivisions as possible to reach an estimated accuracy on $v_3(5)$ of better than 10^{-5}. State at which N you terminated, state the estimated accuracy, and state clearly an explanation on how you computed the accuracy estimate. Submit the used code.

Exercise 4 (15 points)

Consider the problem

$$\begin{cases} y''(x) = \cos(y(x)^2) + y'(x) + \exp(-x); \quad 0 < x < 4 \\ y(0) = -1; \quad y(4) = 6 \end{cases}$$

i) (8 points) Use the Finite Difference method to find an approximation to the solution curve $y(x)$. Use $N = 4, 8, 16, 32, \ldots, 32768$). State the numerical estimate of $y(2)$ for each N with at least 10 digits. Submit the used code.

ii) (7 points) Determine the smallest N at which you can obtain an estimated accuracy on $y(2)$ of less than 10^{-6}. State N, state your estimate of the accuracy, and state an explanation on how you found this estimate.

Exercise 5 (25 points)

Consider the integral

$$\int_a^b (x-a)^2(x-b)^4 \exp(-x)dx$$

i) (5 points) For $N=3$, state by hand computations an analytical expression for the approximation of the integral as obtained by the Trapezoidal method.

We now consider the case $a=1$, $b=3$. We wish to approximate the integral using the Trapezoidal method. For this, we as usual split the integration interval into N equidistant subintervals.

ii) (10 points) With $N=3,5,9,17,33,65$ (corresponding to $N-1=2,4,8,16,32,64$) use the Trapezoidal method to approximate the integral. State the results in a table similar to those used during the course. Submit the used code.

iii) (5 points) Use Richardson extrapolation to estimate the order at $N=65$. State the result. If the estimated order is different than the expected order provide an explanation for the difference.

iv) (5 points) State the estimated accuracy on the result at $N=65$ using the estimated order. State clearly how you compute the accuracy estimate.