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Compact bright pulse in inhomogeneous and nonlinear medium: Case of the Bose–Einstein Condensate

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ABSTRACT

The extended one-dimensional Gross–Pitaevskii equation which describes the dynamics of localized wave in nonlocal nonlinear media, in particular the Bose–Einstein condensates (BECs) with time-dependent interatomic interactions in a parabolic potential in the presence of feeding or loss of atoms, is investigated. Through analytical methods invoking a modified lens-type transformation, an equivalent nonlocal GP equation with constant coefficients is derived as well as the integrability conditions. In the limit of zero transverse kinetic energy, we show that the nonlocal GP equation exhibits a stationary bright pulse with strictly localized support and with the width independent of the amplitude. However due to the property of conservation of the norm, the BEC amplitude will be a function of both the pulse width and the number of trap atoms. Similarly, the obtained integrability conditions also appear as the conditions under which the compact bright waves describing the BECs can be managed by controlling the parameters of the external potential. Thus, the number of trap atoms can be managed through the functional gain or loss which is not sensitive to the strength of the magnetic trap. It also appears that, in the presence of the nonlocal interaction, the shape of the BEC interpolates between the shape of the compact bright pulse and that of the NLS bright pulse, and the number of condensed atoms participating to the formation of the BEC increases with the strength of the nonlocal interactions.

1. Introduction

Recent advancements in the field of atomic optics have garnered increasing interest and offer promising prospects for research, encompassing both fundamental aspects of physics and emerging technological applications [1–6]. Among the significant developments in this field, the creation of Bose–Einstein condensates (BECs) within dilute atomic gases has paved the way for in-depth investigations over the past decades [7–10]. The study of solitary wave dynamics within BECs has become a focal point of research due to their relevance in various applications, including atomic interferometry [11], quantum phase transitions [12], and nonlinear physics such as nonlinear optics and hydrodynamics [12].

To describe the state of matter at ultra-low temperatures, researchers have developed several models, among which is the Gross–Pitaevskii (GP) equation [13]. Derived from a combination of the Schrödinger equation and mean-field theory, this equation exhibits

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soliton solutions and is used to describe atomic BECs [14,15]. Over the past two decades, the dynamics of BECs described by various forms of the GP equation has been the focus of numerous researchers. For instance, experimentally observed bright and dark matter wave solitons within BECs were subsequently identified as solutions of the nonlinear GP equations, exhibiting positive and negative nonlinear coefficients, respectively [13,16–19].

The behaviour of BECs described by the GP equation with time-varying parameters has been explored under conditions of uniform non-linearity. Exact solutions of bright and dark solitons have been constructed for linear and quadratic potentials [20]. Moreover, bound states of multiple solitons have been formed [21]. In the context of BECs within potential wells, an exact soliton solution of the (1+1)-dimensional GP equation with a linear potential was discovered [22]. Subsequent solutions were obtained by considering more complex potentials in the GP equations [23,24]. However, challenges related to integrating time-dependent coefficients into the GP equation have recently prompted the introduction of a new approach, namely the Lenz-type transformation approach. This approach transforms the initial extended GP equation having a parabolic potential, time-dependent self phase modulation and nonlocal non-linearity coefficients, into a version of the extended NLS equation without potential and featuring constant coefficients including the nonlinear and nonlocal ones. This has facilitated the use of the compact bright pulse from the latter equation to generate some specific types of explicit solutions of GP equations with variants coefficients. Importantly, these works are based on a local GP equation suitable for contact interactions between bosons within BECs. However, this local GP equation cannot precisely describe BECs when long-range interactions between bosons are considered. The concept of nonlocal interaction arises naturally in theoretical mean-field studies of dipolar quantum gases where the interaction between dipoles is described by a suitable function, exhibiting an anisotropic and long-range character [25,26]. Despite its significance, the concept of non-locality has been explored in few studies concerning the behaviour of BECs. For instance, a nonlocal version of the GP equation was derived, and analytical approaches to solve it were studied [20]. Specific solutions were found [21], and the collapsing problem of localized waves described by this equation was discussed [22]. However, due to mathematical complexity, the contribution of non-local interaction between bosons to the behaviour of non-homogeneous BECs has not been deeply explored. Thus, a question remains: How do BECs behave when non-locality is considered in the interaction between bosons, and what form of GP equation can describe this behaviour when accounting for this phenomenon?

In the context of optics, non-locality has been extensively used to create new forms of localized waves. For example, it has recently been demonstrated that the nonlinear nonlocal Schrödinger equation, which describes the propagation of optical pulses in nonlinear optical fibbers, can support stable bright and dark solitons [27], as well as ultrashort and compact bright pulses [28–30]. Unlike classical solitons that extend infinitely, compact bright pulses possess a finite support region and decay outside that region due to a balance between the cubic and nonlocal nonlinearity. These compactons exhibit unique characteristics, such as a width independent of amplitude and the ascents of interaction between adjacent compactons [28,29,31–33]. Consequently, researchers have been keen on studying conditions under which compact structures emerge in various physical systems [34–38].

Our approach is based on the extended derivation of the Gross–Pitaevskii (GP) equation similar to pioneering work [39–43] to establish a robust analytical framework for studying the dynamic properties of BECs. Additionally, we have explored internal interactions within the condensate using a modified Lenz-type transformation, thus, revealing uncharted facets until now. We have also accounted for the time evolution of functional gain, inspired by other researchers [39], and highlighted its influence on BEC dynamics and properties. We emphasize on the crucial significance of transverse kinetic energy, which deeply influences the BEC characteristics [23,24,39]. Moreover, we extend our studies to encompass the combined effects of transverse oscillations and nonlocal interactions, thereby providing a compelling overview of BEC complexity. Through this comprehensive investigation, our goal is to unveil previously unknown fundamental mechanisms, thus, yielding a deeper understanding of the captivating properties of ultracold Bose–Einstein condensates. By addressing limitations in prior work and advancing these concepts, we significantly contribute to the comprehension of nonlocal interactions and their implications in complex quantum systems. In the following sections, we elucidate our methodological approach, present the results obtained from our analyses, and discuss their implications within the broader context of Bose–Einstein condensates and quantum physics.

2. The model and analytical treatment

2.1. Derivation of the extended GP equation

Bose–Einstein condensates (BEC) are the new state of matter that manifest themselves under extremely low temperature conditions in dilute gases. To describe its dynamics and thermodynamic properties, several models have ben built among which the GP equation resulting from the application of the mean field theory, well-known as an increasingly accurate description as one approaches the zero-temperature limit [13]. The system under investigation here consists of a weakly interacting N bosons, cooled below the BEC transition temperature, and confined in the cylindrical trap. The condensate density is described by $n(r,\tilde{t}) = \left|\Psi(r,\tilde{t})\right|^2$ where $\Psi(r,\tilde{t})$ is the condensate wave-function and may be viewed as the order parameter when the classical description limit. In this classical limit, where the s-wave scattering length is much smaller than the average distance between atoms, the time evolution of the condensate wave function, derived using the Heisenberg equation with the many-body interactions, is giving by [13]:

$$i\hbar\frac{\partial\Psi(r,\tilde{t})}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(r,\tilde{t}) + \int V_{int}(r'-r,\tilde{t})\left|\Psi(r',\tilde{t})\right|^2 dr'\right]\Psi(r,\tilde{t}) \tag{1}$$

where m is the atomic mass of atom, $V_{ext}(r,\tilde{t})$ the time-dependent external harmonic potential to the system confined in the axial z direction and defined analytically as follows:

$$V_{ext}(r,\tilde{t}) = \frac{1}{2}m\omega_{\perp}^2\left(x^2+y^2\right) + \frac{1}{2}m\omega_{//}^2 z^2 + \frac{1}{2}i\tilde{\eta}(\tilde{t}) + V_0 \tag{2}$$

with $r(x, y, z)$ the space coordinates, t the time, $\omega_\perp$ and $\omega_{//}$ the oscillator frequencies in the transverse and axial directions, respectively. In Eq. (1), $V_{int}(r' - r, \tilde{t})$ is the interaction potential which describes the two-body interactions between bosons. The interactions between particles obviously play an even more crucial role in the very active field concerning the study of strongly correlated systems realized with ultra-cold atoms [16,44]. For all those reasons, in the last few years, there has been a quest for realizing quantum gases with different, richer interactions, in order to obtain even more interesting properties. Several researchers, including Rzązewski, Shlyapnikov, Zoller, Kurizki, You, DeMille, Baranov, Meystre, Pu and some of us, have pointed out that the dipole–dipole interaction, acting between particles having a permanent electric or magnetic dipole moment, should lead to a novel kind of degenerate quantum gas already in the weakly interacting limit. Its effects should be even more pronounced in the strongly correlated regime. Usually, in the ultra-cold regime characteristic of quantum gases (temperatures in the nano kelvin range), only s-wave scattering between particles can take place. This allows one to replace the real interatomic potential (which at long distances is the usual van der Walls interaction) by a pseudo potential, which is short range, isotropic and characterized by a single parameter, the s-wave scattering length a_s. In this particular case, the interatomic potential takes the following form:

$$V_{int}(r' - r, \tilde{t}) = \frac{4\pi\hbar^2 a_s(\tilde{t})}{m}\delta(r' - r), \tag{3}$$

which is proportional to the scattering length a_s and where $\delta(r' - r)$ is the Dirac function. This expression is consistent only when collision between atoms are considered. Thus, by inserting this expression into Eq. (1), leads to the well-known GP equation given by:

$$i\hbar\frac{\partial\Psi(r,\tilde{t})}{\partial\tilde{t}} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(r,\tilde{t}) + \frac{4\pi\hbar^2 a_s(\tilde{t})}{m}\left|\Psi(r,\tilde{t})\right|^2\right]\Psi(r,\tilde{t}). \tag{4}$$

In the following, we will replace $\delta(r' - r)$ by the Kernel function $K(r' - r)$ in order to take into account the nonlocal and long-range character of the interactions between boson. Similarly, by performing the following change of variables:

$$\Psi\left(r(X,Y,Z),\tilde{t}\right) = \frac{1}{(2\pi a_B)^{1/2}a_\perp}\Phi(z,t)e^{-i\Omega_0\tilde{t}}e^{-\frac{X^2+Y^2}{2a_\perp^2}} \tag{5}$$

with $z = Z/a_\perp$, $t = \Omega_0\tilde{t}$ where $a_\perp = \left(\hbar/m\omega_\perp\right)^{\frac{1}{2}}$ and $\Omega_0 = \omega_\perp + V_0/\hbar$, Eq. (1) in which $V_{ext}(r,\tilde{t})$ is given by Eq. (2) reduces to the following one-dimensional nonlinear partial differential equation:

$$i\frac{\partial\Phi}{\partial t} = -p\frac{\partial^2\Phi}{\partial z^2} + \left[\alpha(t)z^2 + i\eta(t)\right] + q(t)\int\left[K\left(z' - z\right)\left|\Phi(z',t)\right|^2 dz'\right]\Phi(z,t) \tag{6}$$

where the coefficients p, $\alpha(t)$, $\eta(t)$ and $q(t)$ are related to the original system parameters as follows:

$$p = \frac{1}{2}\frac{\omega_\perp}{\omega_\perp + \frac{V_0}{\hbar}}, \quad \alpha(t) = \frac{1}{2}\frac{m\omega_{//}^2}{\hbar\omega_\perp + V_0}, \quad \eta(t) = \frac{1}{2}\frac{\tilde{\eta}(t)}{\hbar\omega_\perp + V_0} \quad \textit{and} \quad q(t) = 2\frac{a_s(t)}{a_B(1 + \frac{V_0}{\hbar\omega_\perp})} \tag{7}$$

In most cases, interactions between two bosons are more relevant when z' is close to z and vanishes when $z' - z$ deviates to zero. In this case and in addition to the symmetric character of these interactions, $K(z' - z)$ will be taken as an odd function and then only odd terms will be non nil. Within these conditions $\left|\Phi(z')\right|^2$ can be satisfactorily approximated by a second order Taylor expansion around z and Eq. (6) can be rewritten in the following form:

$$i\frac{\partial\Phi}{\partial t} = -p\frac{\partial^2\Phi}{\partial z^2} + \left[\alpha(t)z^2 + i\eta(t)\right]\Phi + \gamma(t)|\Phi|^2\Phi + \gamma_{nl}(t)\frac{\partial^2|\Phi|^2}{\partial z^2}\Phi \tag{8}$$

where $\gamma(t)$ and $\gamma_{nl}(t)$ are related to the interaction function $K(z' - z)$ as follows:

$$\gamma(t) = q(t)\int K(z' - z)dz' \quad \textit{and} \quad \gamma_{nl}(t) = \frac{1}{2}q(t)\int (z' - z)^2 K(z' - z)dz', \tag{9}$$

and where the term proportional to $\frac{\partial|\Phi|^2}{\partial z}\Phi$ is missing due to the fact that $\int (z - z')K(z - z')dz' = 0$. However, in some cases where the interaction between bosons are no longer symmetric, the Kernel function is no more an odd function and $\int (z - z')K(z - z')dz' \neq 0$, then, Eq. (8) will include the term proportional to the first derivative usually known as the first moment of the nonlinear interaction. In the context of the nonlinear optical fibre, this term is known as a Raman term and is responsible for the nonlinear frequency-shift of the bright pulse [45] Eq. (8) describes the time and spatial evolution of the condensate wave-function in the case of dilute and cool gases where $\gamma(t)$ and $\gamma_{nl}(t)$ measure respectively the strength of nonlinear and nonlocal interactions between bosons. The parameter p proportional to $\omega_\perp$ accounts for the transverse oscillations of the system while $\alpha(t)$ and $\eta(t)$ account for the intensity of magnetic traps in the z direction and the magnitude of functional gain or loss, respectively. Due to the presence of the nonlocal term, Eq. (8) may be viewed as the nonlocal GP equation and will be used below to study the dynamics of these condensate.

From this equation, some conserved quantities can be obtained namely the norm which verifies the following relation:

$$\frac{\partial}{\partial t}\left(\int |\Phi|^2\right)dz = \eta\int |\Phi|^2 dz \tag{10}$$

describing the time variation of the number of atoms N in the condensate. Generally, N is a conserved quantity. However, due to the presence of gain or loss parameter η, it is no longer a conserved quantity. In fact, if $\eta \neq 0$, the integration of Eq. (10) leads to the expression $N = N_0 e^{h(t)}$, with $h(t) = \int \eta dt$, which may increase or decrease with time according to whether η is a positive or negative quantity. It is obvious that η may describe the gain of atoms if it is a positive quantity or a loss of atoms when it is a negative quantity. Another quantity deriving from the above equation is the Hamiltonian given by:

$$H = \int \left\{ p\left|\frac{\partial \Phi}{\partial z}\right|^2 + \left[\alpha(t)z^2 + i\eta(t)\right]|\Phi|^2 + \frac{1}{2}\gamma(t)|\Phi|^4 - \frac{1}{2}\gamma_{nl}(t)\left[\frac{\partial(|\Phi|^2)}{\partial z}\right]^2 \right\} \tag{11}$$

where the imaginary part, proportional to η, accounts for the dissipation or pumped of the energy in the system.

2.2. Modified lens-type transformation and integrability condition

The nonlocal GP equation (8) is an extension of the non-autonomous nonlinear Schrödinger equation, which has recently received much attention due to its applications in the fields of trapped BECs and optical solitons [46]. Among the problems posed by this equation due to its complexity, one of the prominent challenges is its integrability. To address this challenge, we employ a lens-type transformation approach to transform the GP equation into a nonlocal nonlinear Schrödinger (NLS) equation with constant coefficients, the properties of which are well-understood. Therefore, in order to transform the nonlocal GP equation (8) into an extended NLS equation with constant coefficients, we adopt the following approach.

We begin by introducing the following ansatz equation with constant coefficients, we adopt the following approach. We first use the following ansatz [47]:

$$\Phi(z,t) = D(t)\,\phi(\xi,T)\exp\left[i f(t) z^2 + h(t)\right], \tag{12}$$

where $h(t)$ and $T(t)$ are new variables depending only on time, and $\xi = z/l(t)$. The substitution of Eq. (12) into Eq. (8), yields to:

$$\begin{aligned} il^2\frac{dT}{dt}\frac{\partial\phi(\xi,T)}{\partial T} &= -p\frac{\partial^2\phi(\xi,T)}{\partial\xi^2} + \gamma l^2 e^{2h(t)}D^2(t)|\phi(\xi,T)|^2\phi(\xi,T) + \gamma_{nl}e^{2h(t)}D^2(t)\frac{\partial|\phi(\xi,T)|^2}{\partial\xi^2}\phi(\xi,T) \\ &+ i\left[\frac{dl(t)}{dt} - 4pf(t)l(t)\right]z\frac{\partial\phi(\xi,T)}{\xi} + l^2(t)z^2\left(\frac{df(t)}{dt} + 4pf^2(t) + \alpha_t\right)\phi(\xi,T) \\ &+ i\frac{l^2(t)}{D(t)}\left[\left(-\frac{dD(t)}{dt} - 2pf(t)D(t)\right)\phi(\xi,T) + \left(-\frac{dh(t)}{dt} + \eta(t)\right)D(t)\right]\phi(\xi,T). \end{aligned} \tag{13}$$

In this form,it is obvious that, it can then be reduce to the nonlocal nonlinear Schrödinger equation with constant coefficients, that is,

$$i\frac{\partial\phi(\xi,T)}{\partial T} = -p\frac{\partial^2\phi(\xi,T)}{\partial\xi^2} + g|\phi(\xi,T)|^2\phi(\xi,T) + g_{nl}\frac{\partial^2|\phi(\xi,T)|^2}{\partial\xi^2}\phi(\xi,T). \tag{14}$$

Under the following set of constraints:

$$\frac{dT}{dt} = \frac{1}{l^2(t)} \tag{15}$$

$$\frac{dl(t)}{dt} = 4pf(t)\,l(t) \tag{16}$$

$$\frac{dD(t)}{dt} = -2pf(t)\,D(t) \tag{17}$$

$$\frac{df(t)}{dt} = -\alpha(t) - 4pf^2(t) \tag{18}$$

$$\frac{dh(t)}{dt} = \eta(t) \tag{19}$$

and

$$g = \gamma(t)\,l^2(t)\,D^2(t)\,e^{2h(t)}, \quad g_{nl} = \gamma_{nl}(t)\,D^2(t)\,e^{2h(t)}. \tag{20}$$

From this set of above differential equations, it is possible to get more insight on some properties of the solution of the above non-local non-autonomous GP equation. In fact, from (16) to (17), it appears that $dD/dl = -D/2l$ leading to the following relation between this spatial scale parameter l and the amplitude D when p is non nil : $l(t)/l_0 = (D_0/D)^2$, That is, l is inversely proportional to the square of the amplitude. This property is well-known from the solitary wave theory and is one of the main properties of the pulse solitary waves. The presence of the magnetic traps with magnitude measured by α induces the time-dependence of some characteristic parameter of the condensate among which the amplitude and width. However, due to the dependence of the variation of D with time to the expression of the function $f(t)$, its analytical expression will be obtained only after the integration of Eq. (18). Similarly, the time variation of $f(t)$ is mainly dependent on the parameters $\alpha(t)$. This means that, one can use the characteristic parameter of the trapping potential to control the dynamics of the condensate in the system. In the following, we will focus our attention to some particular cases namely to the case where p is close to zero.

3. Ultracold Bose–Einstein condensate: Zero transverse oscillation limit

3.1. Bose–Einstein condensate with compact support

In the limiting case of ultracold BEC, the kinetic energy is very small compared to other energies of the system such as the interaction energy of atoms and the potential energy so that the parameter p measuring the magnitude of the system oscillations in the transverse direction can be set equal to zero. In this case, Eq. (8) takes the following form:

$$i\frac{\partial\phi(\xi,T)}{\partial T}=g|\phi(\xi,T)|^2\phi(\xi,T)+g_{nl}\frac{\partial^2|\phi(\xi,T)|^2}{\partial\xi^2}\phi(\xi,T). \tag{21}$$

Accordingly, from the integrability conditions (15) to (20), the spatial scale parameter l and the amplitude parameter D are no longer time-dependent, even in the presence of the magnetic traps, that is, $l=l_0$ and $D=D_0$. The others quantities such as T, $f(t)$ and $h(t)$ are explicitly given by

$$T=t/l_0^2,\quad f(t)=-\int\alpha(t)dt\quad and\quad h(t)=\int\eta(t)dt. \tag{22}$$

From the compatibility condition (20), the spatial scale parameter can be explicitly given by:

$$l_0^2=\frac{g}{g_{nl}}\frac{\gamma_{nl}(t)}{\gamma(t)}=cste. \tag{23}$$

This compatibility condition (15) is satisfied since from (9) the ratio between the cubic nonlinear coefficient and the nonlocality coefficient is not time-dependent. It appears that, in the limit $p=0$, the spatial scale parameter l does not longer depend on the amplitude parameter D as it is the case above for $p\neq 0$. This property has been obtained from the nonlinear localized waves with compact support or compactons. In fact, looking for the solution of Eq. (21) in the form:

$$\phi(\xi,T)=A(\xi)\exp\left[i\left(\varphi(\xi)-\Omega_{nl}\mathrm{T}\right)\right]. \tag{24}$$

and substituting into Eq. (21), one obtains the following nonlinear ordinary differential equation

$$A\Omega_{nl}-g^2A^3-2g_{nl}^2\left(A'^2+AA''\right)A=0. \tag{25}$$

By making use of the quadrature method and taking into account the boundary conditions $A\to 0$, $A'\to 0$ when $\xi\to\pm\infty$, one obtains:

$$A'^2=\mu_0^2\left(A_0^2-A^2\right) \tag{26}$$

with $\mu_0=(g/4g_{nl})^{1/2}$ and $A_0^2=2\Omega_{nl}/\gamma$. From this equation, it following that $\frac{dA}{\sqrt{A_0^2-A^2}}=\pm\mu_0 d\xi$. After integration, and taking to account the above stated boundary conditions, one obtain the following cosine function

$$A(\xi)=\begin{cases}A_0\cos\left[\mu_0\left(\xi-\xi_0\right)\right] & \left|\left(\xi-\xi_0\right)\right|\leqslant\pi/2\mu_0\\ 0 & \left|\left(\xi-\xi_0\right)\right|\geqslant\pi/2\mu_0\end{cases} \tag{27}$$

and consequently

$$\phi(\xi,T)=\begin{cases}A_0\cos\left[\mu_0\left(\xi-\xi_0\right)\right]\exp\left[i\left(\varphi(\xi)-\Omega_{nl}\mathrm{T}\right)\right] & \left|\left(\xi-\xi_0\right)\right|\leqslant\pi/2\mu_0\\ 0 & \left|\xi-\xi_0\right|\geqslant\pi/2\mu_0\end{cases} \tag{28}$$

in which μ_0 is the pulse width, A_0 the amplitude, $\Omega_{nl}=\gamma A_0^2/2$ is the pulse nonlinear frequency depending on the pulse amplitude A_0, and φ an arbitrary function of ξ. Due to the strict limitation of the spatial span of this solution, it is usually referred a solution with compact support [28,29,48,49]. Back to the function Φ and with the original variables, one obtains:

$$\Phi(z,t)=\begin{cases}\Phi_m(t)\cos\left[\frac{\mu_0\left(z-z_0\right)}{l_0}\right]\exp\left\{i\left[\varphi\left(z/l_0\right)+f(t)z^2-\left(\Omega_{nl}/l_0^2\right)t\right]\right\} & \left|\left(z-z_0\right)\right|\leqslant\pi l_0/2\mu_0\\ 0 & \left|z-z_0\right|\geqslant\pi l_o/2\mu_0\end{cases} \tag{29}$$

with $\Phi_m(t)=D_0A_0e^{h(t)}$. Thus, when the transverse oscillation of the system become negligible, that is, in the case of ultracold gases, the BEC is described by the compact-like shape given by Eq. (29) where $\Phi_m(t)$ is the amplitude and will depend on the number of the atoms N in the condensate through the relation:

$$N=\int|\Psi|^2dxdydz. \tag{30}$$

By substituting Eq. (29) into this integral after considering the variable changes (5), it appears that,

$$N=\frac{\pi}{4}\frac{a_\perp}{a_B}\frac{l_0}{\mu}\Phi_m^2=N_0e^{2h(t)}\quad with\quad N_0=\frac{\pi}{4}\frac{a_\perp}{a_B}\frac{l_0}{\mu}D_0^2A_0^2\quad. \tag{31}$$

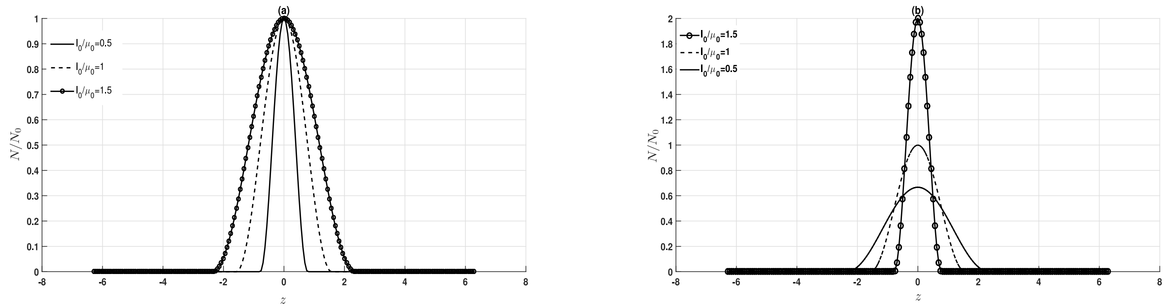

Fig. 1. Density of condensate with compact support as a function of the axial coordinate z for three values of l_0/μ_0: 0.5 (solid line); 1.0 (dash-line) and 1.5 (dot–dot line); The Number of condensed particles being constant.

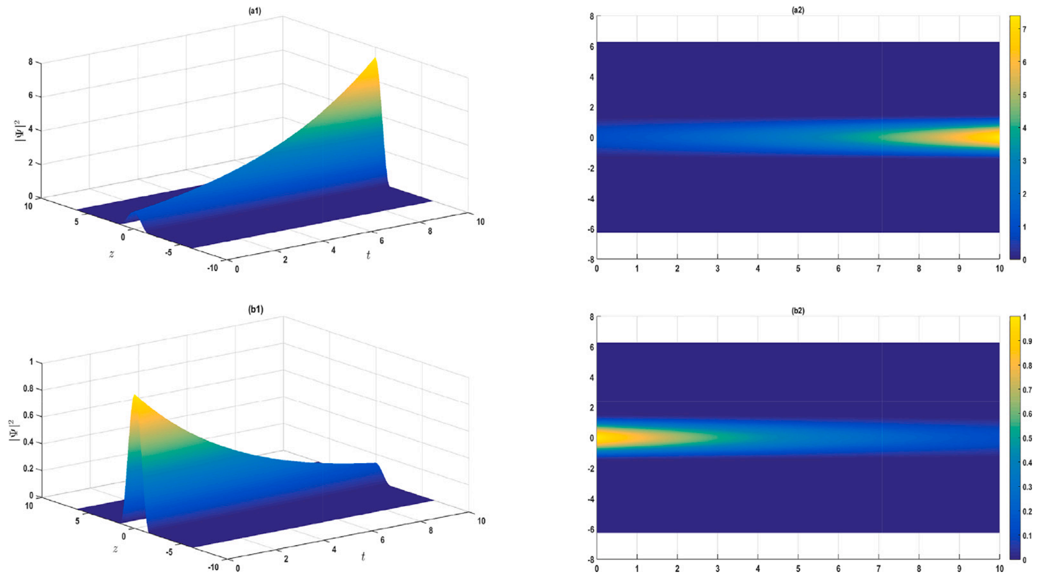

Fig. 2. Temporal evolution of the density of condensate with compact support with constant value of the loss or gain parameter: (a) $\eta_0 = 0.1$ and (b) $\eta_0 = -0.1$. The pulse width is $l_0/\mu_0 = 1.5$.

Thus, the amplitude Φ_m of the compact BEC can then be written in the following form:

$$\Phi_m(t) = \left(\frac{4}{\pi}\frac{a_B}{a_\perp}\frac{\mu_0}{l_0}\right)^{1/2} N(t)^{1/2}. \tag{32}$$

Fig. 1a shows the shape of the density of the condensate as a function of the axial coordinate z at a given time and for several values of l_0/μ_0, that is for several values of γ_{nl}/γ since $l_0^2/\mu_0^2 = 4\gamma_{nl}/\gamma$. The curves are plotted with the constraint $(\mu_0/l_0)N_0 = cste$ in order to keep constant the amplitude of the compact bright pulse. It appears that the pulse width increases with the strength of the nonlocal interactions. As a consequence, the number of atoms participating in the generation of the BEC also increases with this strength of the nonlocal interaction. Now, when the number of condensed atoms is kept constant, the pulse amplitude decreases with increasing values of l_0/μ_0, that is, of the coefficient of the nonlocal interaction as indicated by Fig. 1b.

3.2. Effect of time-dependence of the functional gain and loss on the dynamics of the BEC

The integrability condition (15)–(20) describe the time variation of the pulse parameters. In fact, from Eq. (20), it is obvious that the integrability condition will be satisfied if the cubic nonlinearity coefficient $\gamma(t)$ has the form $\gamma_0(t)\exp[-2h(t)]$. Similarly, the nonlocal coefficient $\gamma_{nl}(t)$ will also exhibit the same form namely $\gamma_{nl0}(t)\exp[-2h(t)]$. Accordingly, the expression of the parameter $\eta(t)$ describing either the feeding or the loss can then be related to this time variation of $h(t)$ by Eq. (19). To illustrate, let us consider two different situations:

- For a constant increasing or decreasing rate of the number of trap atoms, the function $h(t)$ takes the form $h(t) = \eta_0 t$ leading to the following expression of $\eta(t)$: $\eta(t) = \eta_0$. Fig. 2 shows the 3D variation of the shape of the density for two different situations:

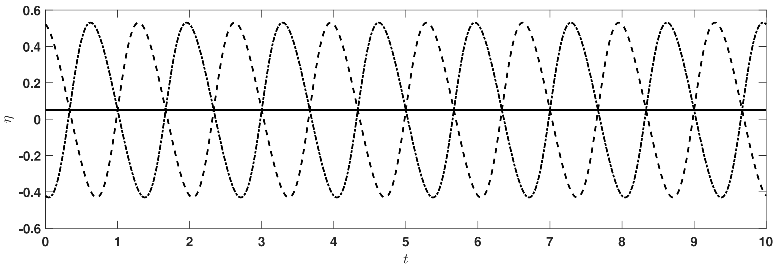

Fig. 3. Time variation of the functional gain or loss: (a) $\eta_0 = 0$ and $\epsilon = 0.2$; (b) $\eta_0 = 0.01$ and $\epsilon = 0.2$; (c) $\eta_0 = -0.1$ and $\epsilon = 0.2$. The pumping frequency ω_p is taking equal to 0.75.

When $\eta_0 = +0.1$, the amplitude of the density increases with time indicating the gain of energy by the system which may result from a constant pump of energy to the system. However, when $\eta_0 = -0.1$, the amplitude of the density decreases with time indicating a loss of the energy by the system and consequently a diminution of the number of condensed atoms in the BEC as the time evolves.

- For the periodic oscillations of the number of trap atoms, that is, taking $\exp[2h(t)] = e^{(2\eta_0 t)}[1 + \epsilon \sin(\omega_p t + \varphi_p)]$, the parameter of the external potential will take the following form:

$$\eta(t) = \eta_0 + \frac{1}{2} \frac{\epsilon \omega_p \cos(\omega_p t + \varphi_p)}{1 + \epsilon \sin(\omega_p t + \varphi_p)} \tag{33}$$

where ϵ and ω_p may be viewed as the amplitude and frequency of the external pumping of the energy to the system, respectively, φ_p a phase-shift parameter, and η_0 a constant which may be positive, negative or zero. Fig. 3 shows the time variation of the parameter $\eta(t)$ for two different cases: $\epsilon = 0$ and $\epsilon \neq 0$ while Fig. 4 shows the time evolution of the corresponding density.

It appears that, the time variation of the compact bright pulse amplitude or the BEC density can be controlled through the functional gain or loss of the external potential. However, due to the fact that the parameter p which acts as a coupling between the variation of the pulse width (see Eq. (16)) and other pulse coefficients such as $f(t)$ and $D(t)$ is equal to zero, the pulse width cannot be sensitive to the external variation of this potential.

3.3. Effect of the transverse kinetic energy (transverse oscillation) in the BEC parameters

In the above subsection, we have derived the solution of the nonlocal GP equation Eq. (8) in the particular case where $p = 0$ which physically corresponds to case of vanishing transverse oscillation ($\omega_\perp \to 0$). In this section we study the influence of this parameter (p) on the properties of the BEC in particular its width. However, in the practical situation, p cannot be exactly equal to zero. In this case, the term proportional to p can be viewed as a perturbation of the system if its magnitude is relatively small. Otherwise, its presence can deeply modify the properties of the system namely the kind of the solution. This case will be considered in the next section (Section 4). So, we search for the conditions under which the compact solution still valid even in the presence of this small values p. For this aim, we assume that, for relatively small values of p, the GP equation still exhibits a CB pulse as a solution, but with a slight modification of its parameters. Accordingly, one can consider the BEC in the following form inspired from the compact expression (29),

$$\Phi(z,t) = \begin{cases} \Phi_0 \cos\left[\dfrac{\mu\left(z - z_0\right)}{L}\right] e^{i\left[f(t)z^2 - (\Omega_{nl}/L^2)t\right]} e^{h(t)} & \left|\left(z - z_0\right)\right| \leqslant \pi L/2\mu \\ 0 & \left|z - z_0\right| \geqslant \pi L/2\mu \end{cases} \tag{34}$$

where $h(t)$ is given by Eq. (19), L/μ is the arbitrary width parameter to be determined. Φ_0 is the amplitude and depends to the number of condensed atoms N through the relation:

$$N = \int |\Psi(r,t)|^2 dx dy dz \tag{35}$$

where $\Psi(r,t)$ is related to $\Phi(z,t)$ through Eq. (5) and which is given here for the compact BEC by Eq. (34). Thus, after integration, one obtains $N = N_0 e^{2h(t)}$ with $N_0 = \frac{\pi}{4}\frac{a_\perp}{a_B}\frac{L}{\mu}\Phi_0^2$. Thus, the amplitude Φ_0 can be written in the following form:

$$\Phi_0^2 = \phi_0^2 \frac{\mu}{L} \quad with \quad \phi_0^2 = \frac{4}{\pi}\frac{a_B}{a_\perp} N_0. \tag{36}$$

By inserting Eq. (34) into the Hamiltonian (11) and after some lengthy integrations, one obtains the energy of the system given by:

$$E = \frac{\pi \phi_0^2}{4} \left\{ 2p\left(\frac{\mu}{L}\right)^2 + \left(\alpha + 4pf^2\right)\left[2z_0^2 + \left(\frac{\pi^2}{6} - 1\right)\left(\frac{L}{\mu}\right)^2\right] + \phi_0^2\left[\frac{3}{4}\gamma\frac{\mu}{L} - \gamma_{nl}\left(\frac{\mu}{L}\right)^3\right] + 2i\eta \right\} e^{2h(t)}. \tag{37}$$

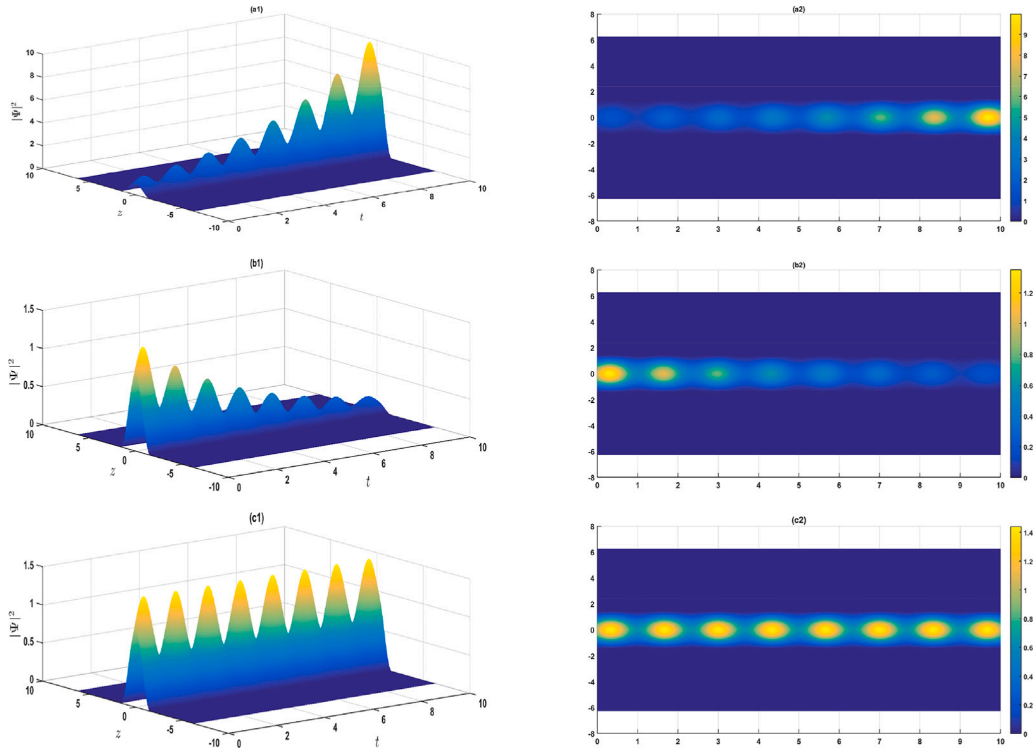

Fig. 4. Temporal evolution of the density of condensate with compact support with time variation of the functional gain or loss: (a) $\eta_0 = 0.01$ and $\epsilon = 0.1$; (b) $\eta_0 = -0.1$ and $\epsilon = 0.01$.

Using the dimensionless width w defined as,

$$w = \frac{L}{\mu w_0}, \quad with \quad w_0^2 = 4\frac{\gamma_{nl}}{\gamma}, \tag{38}$$

Eq. (37) takes the following dimensionless form.

$$\frac{E}{E_0} = \frac{3}{w} - \frac{1}{w^3} + \frac{3p_r}{w^2} + \alpha_r \left[w^2 + \frac{2z_0^2}{w_0^2\left(\frac{\pi^2}{6} - 1\right)} \right] + \frac{2iw_0^3\eta}{\phi_0^2\gamma_{nl}} \tag{39}$$

with

$$E_0 = \frac{\pi}{4}\frac{\gamma_{nl}}{w_0^3}\phi_0^4, \quad p_r = \frac{2pw_0}{3\phi_0^2\gamma_{nl}} \quad and \quad \alpha_r = \left(\frac{\pi^2}{6} - 1\right)\frac{w_0^5\left(\alpha + 4pf^2\right)}{\phi_0^2\gamma_{nl}} \quad . \tag{40}$$

Fig. 5 shows the variation of this energy as a function of w and for several values of the dimensionless parameters of the system namely α_r and p_r. These curves exhibit a local minimum at $w = w_m$ indicating the fact that at this value, the compact expression (29) is the stationary solution of the nonlocal GP equation (8). In fact, the pulse width at the local minimum, satisfies to $\partial E/\partial w = 0$ and $\partial E/\partial z_0 = 0$ which leads to the algebraic equation

$$2\alpha_r w^5 - 3w^2 - 6p_r w + 3 = 0. \tag{41}$$

It is obvious that this equation admits the solution $w = 1$ when $\alpha_r = p_r = 0$ which corresponds to $L/\mu = l_0/\mu_0$, in agreement with the exact analytical solution (34) of the nonlocal GP equation. This width is independent of the number of condensed atoms N. However, when $\alpha_r \neq 0$ and $p_r \neq 0$, the width of condensate become a function of N. This dependence is shown in Fig. 6 in which w is plotted as a function of p_r. Let us note that p_r defined by Eq. (40) is inversely proportional to the number of atoms N. Thus, the increases of w as a function of p_r indicates the fact that the width of the condensate decreases when N increases; This behaviour is consistent with the conservation of the number of condensed atoms.

The plot of the energy as a function of the pulse width w shows also that the shape of the energy depends on the number of atoms N. In fact, when N increases, the depth of the local minimum decreases and disappears when N approaches the threshold value N_{th}. Above this threshold value of N, the effective energy of the condensate does not exhibit a local minimum and as a consequence, the CB pulse (14) is no longer solution of the GP Eq. (8). This situation can occurs when both α_r and p_r are different to zero and

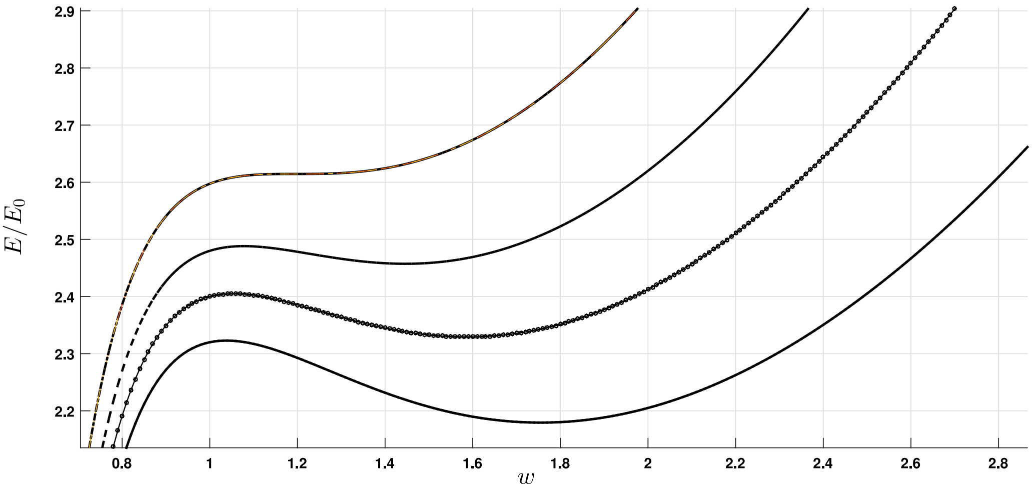

Fig. 5. Variation of this energy as a function of w and for several values of the dimensionless parameters of the system namely α_r and p_r.

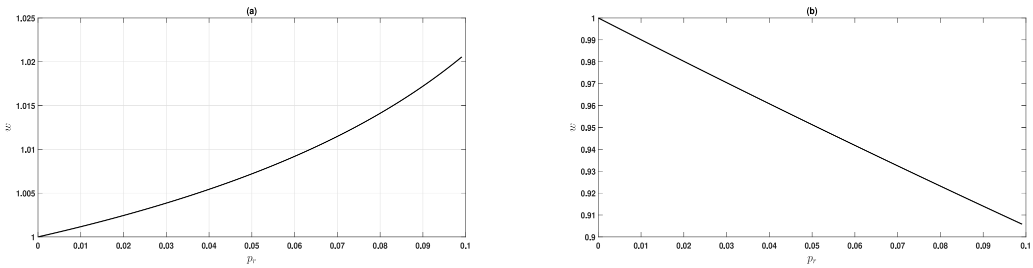

Fig. 6. Variation of the parameter w as a function of p_r.

exceed certain threshold values. In addition to $\partial E/\partial w = 0$, this threshold value of N_{th} satisfies to the constraint $\partial^2 E/\partial w^2 = 0$ which leads to

$$\alpha_r w^5 + 3w^2 + 9p_r w - 6 = 0. \tag{42}$$

From Eq. (41) and (42), one can obtain w_{cr} which satisfies to $w^6 - (5/3)w^4 + \lambda_r w + \lambda_r = 0$ with $\lambda_r = p_r/\alpha_r$ and consequently, the threshold value is given by :

$$\frac{N_{cr}a_\perp}{a_B} = \frac{3\gamma_{nl}}{2pw_0}\frac{8w_{th}}{5 - 3w_{th}{}^2} \tag{43}$$

Fig. 7 shows the variation of w_{cr} as a function of the dimensionless parameter of the system λ_r. For λ_r belonging to $[0.05, 0.3]$, this critical width w_{cr} varies between 1.275 and 1.12 and consequently the quantity $N_{cr}a_\perp/a_B$ can vary in the interval $[94.48, 10.87] \times (\gamma_{nl}/pw_0)$. It appears that in the presence of transverse oscillations of small energy, a, the width of the condensate depends on the number of condensed atoms N. In addition, there exists a threshold critical value of N defined by (43) above which the BEC can be disintegrate.

4. Combined effects of the transverse oscillations and the nonlocal interactions on the BEC dynamics

We now turn our attention to the case where the transverse kinetic energy with the magnitude measured by p is no longer small. As a consequence, the compact-like shape (34) is not also solution of the nonlocal GP equation. In this case, let us search the new solution of Eq. (14), consistent with this nonlocal GP equation, in the following form:

$$\phi(\xi, T) = A(\chi)e^{i[\varphi(\chi)+V_p \mathrm{T}]} \tag{44}$$

where $\chi = \xi - V_e T$ is the moving frame reference of velocity V_e, $A(\chi)$ is the amplitude, $\varphi(\chi)$ the phase and V_p the phase velocity. The substitution of this expression into Eq. (14) leads to the following two coupled differential equations

$$\begin{cases} A'V_e - p\left(A\varphi'' + 2A'\varphi'\right) = 0 \\ A\left(-V_p + V_e\varphi' - p\varphi'^2\right) - gA^3 - 2g_{nl}\left(A'^2 + AA''\right)A + pA'' = 0. \end{cases} \tag{45}$$

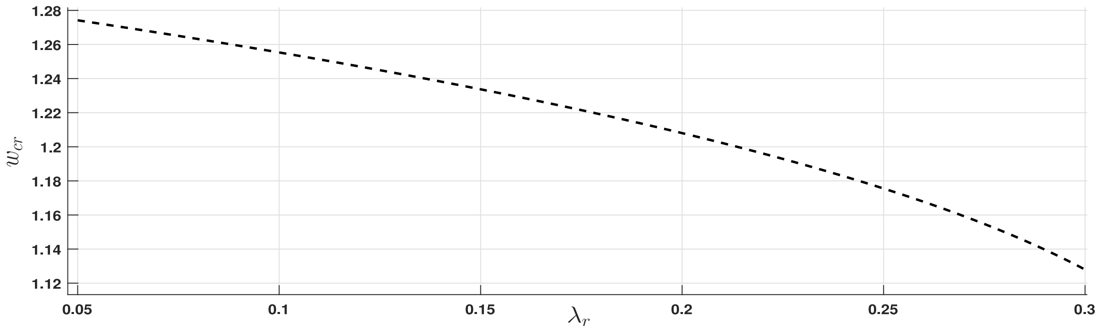

Fig. 7. Variation of w_{cr} as a function of the dimensionless parameter of the system λ_r.

Multiplying the first equation by $A(\chi)$ and integrating once with the boundary conditions of localized waves, that is A and A' vanishes when $\chi \to \pm\infty$, one obtains $A^2(V_e/2 - p\varphi') = 0$ leading to $\varphi' = V_e/2p$, since $A(\chi)$ is different to zero. With this expression of φ, the first integration of the second equation of (45) can be written as follows:

$$\left(-V_p + \frac{V_e^2}{4p}\right)\frac{A^2}{2} + \frac{1}{4}gA^4 - \frac{1}{2}pA'^2 - g_{nl}A'^2A^2 = k, \tag{46}$$

where k is and integration constant. By taking this constant equal to zero, one obtains

$$A'^2 = A^2\mu_0^2\frac{A_0^2 - A^2}{A^2 + {A_{cr}}^2}, \tag{47}$$

with

$$A_0^2 = \frac{2}{g}\left(\frac{V_e^2}{4p} - V_p\right), \quad A_{cr}^2 = \frac{p}{2g_{nl}} \quad \textit{and} \quad \mu_0^2 = \frac{g}{4g_{nl}} \tag{48}$$

provided that $pg_{nl} < 0$, where A_0 and A_{cr} are some critical values of the amplitude of wave. To integrate Eq. (47), let us use the variable change

$$\mathrm{v}^2 = \frac{A_0^2 - A^2}{A^2 + A_{cr}^2} \tag{49}$$

and as a result, Eq. (47) takes the following form:

$$\mathrm{v}' = \pm\mu_0\frac{\left(1+\mathrm{v}^2\right)\left(A_0^2 - A_{cr}^2\mathrm{v}^2\right)}{A_{cr}^2 + A_0^2} \tag{50}$$

which admits the solution:

$$\tan^{-1}\mathrm{v} + \frac{A_{cr}}{A_0}\tanh^{-1}\left(\frac{A_{cr}}{A_0}\mathrm{v}\right) = \pm\mu_0\left(\chi - \chi_0\right). \tag{51}$$

Eq. (48) indicates the fact that, in the presence of p, the phase velocity V_p and the velocity of the pulse envelope V_e are related through the equation $V_p = -gA_0^2/2 + V_e^2/4p$ and in the case of vanishing value of V_p, the envelope velocity is given by $V_e = 2pgA_0^2$. This property is well-known for the solitary waves. For the above Eq. (51), two limiting cases can be considered:

If $p = 0$ and then $A_{cr} = 0$, the term proportional to A_{cr} in Eq. (51) is also equal to zero. Thus, Eq. (51) will lead to the compact form (34) obtained in the preceding section.

If $g_{nl} = 0$, that is $A_{cr} \to \infty$, the first term of Eq. (51) become negligible compared to the second term and consequently one obtains

$$A = A_0 sech\left(\frac{\chi - \chi_0}{L_s}\right), \tag{52}$$

with $L_s = A_{cr}/(\mu_0 A_0) = (2p/g)^{1/2}/A_0$, which is the expression of the well-known Schrödinger bright pulse.

This means that the solution of the nonlocal GP equation interpolates between the compact form (34) when $A_{cr} = 0$ and the sech-type shape (52) when $A_{cr} \to \infty$. Thus, A_{cr} appears a measure of the magnitude of the transverse kinetic energy with regard to the nonlocal interactions energy between atoms of the condensate. Fig. 8 shows the density of atoms in the condensate for several values of the parameter A_{cr}/A_0 which indicates the fact that the shape of the compact-form is just slightly modify by the presence of transverse oscillations. It When $A_{cr} \to \infty$ ($g_{nl} = 0$), the number of condensed atoms is given by Eq. (31) while when $A_{cr} = 0$, the number of these atoms is given by :

$$N_s = \frac{D_0^2 a_\perp}{2a_B}2A_0^2 L_s. \tag{53}$$

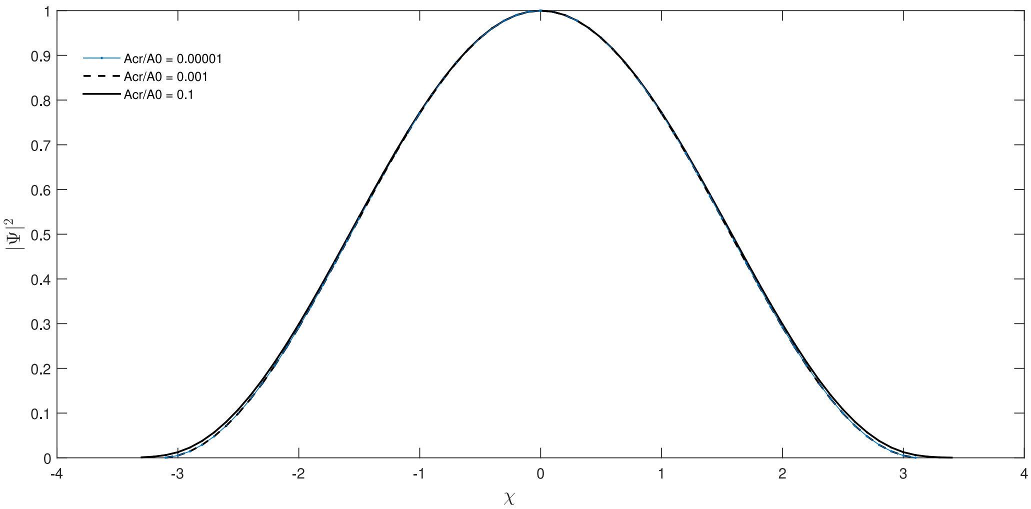

Fig. 8. Density of atoms in the condensate for several values of the parameter A_{cr}/A_0 which indicates the fact that the shape of the compact-form is just slightly modify by the presence of transverse oscillations.

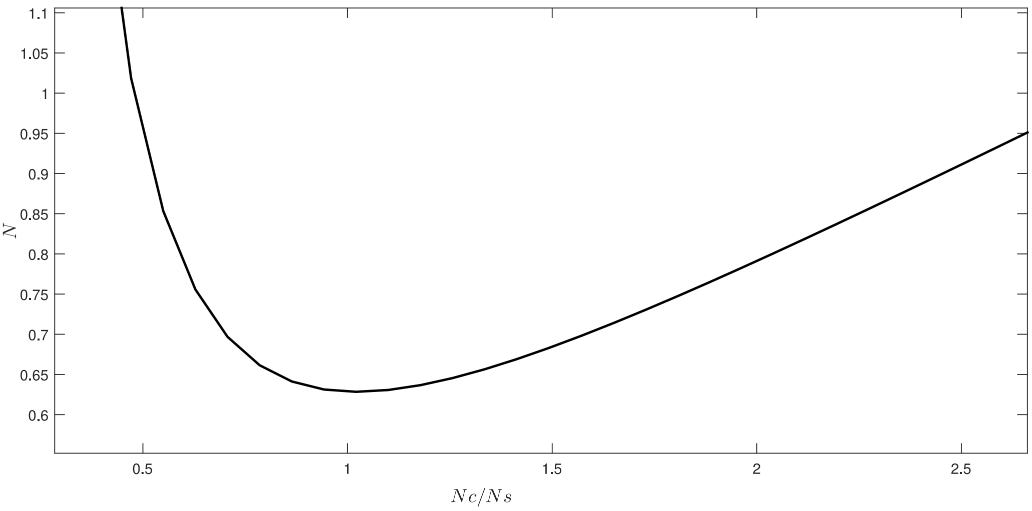

Fig. 9. Variation of quantity N as a function of the ratio N_c/N_s. It appears that N decreases when the ratio N_c/N_s increases and exhibits minimum at the value 1.

The ratio between the two expressions of condensed atoms is given by $N_c/N_s = (\pi/4)A_0/A_{cr}$. This ratio shows that for the same amplitude A_0 for the two waves, N_{CB} will be always less than N_s unless $A_0 > 4A_{cr}/\pi$. For a given A_{cr}, the number of condensed atoms obtained after integration of Eq. (35) with the solution (51) is given by:

$$N = \frac{D_0^2 a_\perp}{2a_B}\frac{A_{cr}^2}{\mu}\left\{\frac{A_0}{A_{cr}} + \frac{1}{2}\left(1 + \frac{A_{cr}^2}{A_0^2}\right)\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{1 + A_0^2/A_{cr}^2}{2A_0/A_{cr}}\right)\right]\right\} \tag{54}$$

Fig. 9 shows the variation of this quantity as a function of the ratio N_c/N_s. It appears that N decreases when the ratio N_c/N_s increases and exhibits minimum at the value 1.

5. Conclusion

The interactions between particles in the BEC are in general more complex and cannot be modelled with good accuracy by a Dirac function consistent only in the case where nonlinear interactions are reduced to collisions between two particles. In other words, The GP equation usually used and which results from the modelling of these interaction will fall when considering the nonlocal interactions between particles. In this paper, we have investigated the properties of the BEC confined in the cylinder trap and subjected to the time-dependent external harmonic potential, and where the interactions between particles are described by a kernel function materializing the existence of nonlinear nonlocal interaction between particles.

Firstly, using the Heisenberg equation with the many-body interactions, an extended GP equation namely the GP equation with nonlocal term has been derived. In order to solve it, the lens-type approach has been used to transform the equation into the nonlocal NLS equation with constant coefficients.

Next, we have focused our attention to the case of very ultracold gases characterized by a small magnitude of transverse kinetic energy compared to other energies of the system such as the interaction energy. We have shown that, the nonlocal GP equation will admit a localized bright pulse with compact support and the corresponding expression of the associated condensed atoms derived. In addition, we have shown that the time-dependence of the functional gain or loss induces a time variation of the number of condensed atoms while the time-dependence of the strength of the magnetic trap acts rather on the phase properties of the wave function of the condensate. In the case of cold gases characterized by the presence both of the transverse kinetic and the nonlocal interaction energy, the shape of the BEC may interpolate between the shape of the compact pulse and that of the well-known Schrödinger bright pulse according to the magnitude of the ratio between the pulse amplitude with regard to the intensity of a certain critical amplitude.

CRediT authorship contribution statement

Blaise Marius Mbiesset Pilah: Investigation, Methodology, Software, Writing – original draft. **Désiré Ndjanfang:** Investigation, Methodology, Software. **Hatou-Yvelin Donkeng:** Conceptualization, Investigation, Methodology, Software, Supervision, Validation, Writing – original draft, Writing – review & editing. **David Yemélé:** Conceptualization, Investigation, Methodology, Software, Supervision, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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