T6 $p(x) = \begin{cases} \dfrac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$

а) $L(x,\theta) = \dfrac{\theta-1}{x_1^\theta \cdot \ldots \cdot x_n^\theta}$

$\ln L = n\ln(\theta-1) - \theta\sum_i \ln x_i$

$\dfrac{\partial L}{\partial\theta} = \dfrac{n}{\theta-1} - \sum_i \ln x_i \; ; \quad \dfrac{\partial L}{\partial\theta} = 0$ max

$\dfrac{n}{\theta-1} = \sum_i \ln x_i \qquad \hat{\theta} = 1 + \dfrac{n}{\sum_i \ln x_i}$

$\dfrac{\partial^2 \ln L}{\partial\theta^2} = \dfrac{-n}{(\theta-1)^2} < 0 \Rightarrow L(\theta) \to \sup$

в) Дов. интервал для медианы

$\dfrac{\partial}{\partial\theta}\int \dfrac{\theta-1}{x^\theta}\,dx = x^{1-\theta}\ln x$

$\int \dfrac{\partial}{\partial\theta}\left(\dfrac{\theta-1}{x^\theta}\right)dx = x^{1-\theta}\ln x$

$\Rightarrow$ сильная регулярность

$\int_1^{\hat{x}} \dfrac{\theta-1}{x^\theta}\,dx = -\dfrac{1}{\hat{x}^{\theta-1}} + 1 = \dfrac{1}{2} \Rightarrow \boxed{\hat{x} = \sqrt[\theta-1]{2}}$

$g(\hat{\theta}) = 2^{\frac{1}{\hat{\theta}-1}} \Rightarrow \sqrt{n}\cdot\dfrac{g(\hat{\theta}) - g(\theta)}{\sigma(\theta)} \to N(0,1)$

$\sigma(\theta) \xrightarrow{P} \sigma(\hat{\theta})$

$$\sigma(\tilde{\theta}) = \sqrt{\nabla^T g(\tilde{\theta})\, I^{-1}(\tilde{\theta})\, \nabla g(\tilde{\theta})}$$

$$I(\tilde{\theta}) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = M\left[\left(\frac{\partial\left(\ln(\theta-1) - \theta \ln x\right)}{\partial \theta}\right)^2\right]$$

$$= M\left[\left(\frac{1}{\theta-1} - \ln x\right)^2\right] = \int_1^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 p(x,\theta)\,dx =$$

$$= \int_1^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 \cdot \frac{\theta-1}{x^\theta}\,dx = \frac{1}{(\theta-1)^2} \text{ — } \frac{\text{непр}}{\text{на } \theta > 1}$$

$$\nabla g(\tilde{\theta}) = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{(\tilde{\theta}-1)^2}, \quad \sigma(\tilde{\theta}) = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\tilde{\theta}-1}$$

$$\sqrt{n} \cdot \frac{g(\tilde{\theta}) - g(\theta)}{\sigma(\tilde{\theta})} \rightsquigarrow N(0,1)$$

$$\frac{1.96\,\sigma(\tilde{\theta})}{\sqrt{n}} + g(\tilde{\theta}) < g(\theta) < -\frac{1.96\,\sigma(\tilde{\theta})}{\sqrt{n}} + g(\tilde{\theta})$$

c) $$\sqrt{n} \cdot \frac{\tilde{\theta} - \theta}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$$\sigma(\theta) \xrightarrow{P} \sigma(\tilde{\theta})$$

$$\sigma(\tilde{\theta}) = \theta - 1 \Rightarrow \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\theta - 1} \rightsquigarrow N(0,1)$$

$$-\frac{1.96(\theta-1)}{\sqrt{n}} + 1 + \frac{n}{\sum_i \ln x_i} < \theta < \frac{1.96(\theta-1)}{\sqrt{n}} + 1 + \frac{n}{\sum_i \ln x_i}$$