$\boxed{Д/З}$

Т1 Сл. вел распределена равномерно на $[0, \theta]$.
По выборке n найдены оценки параметра θ:

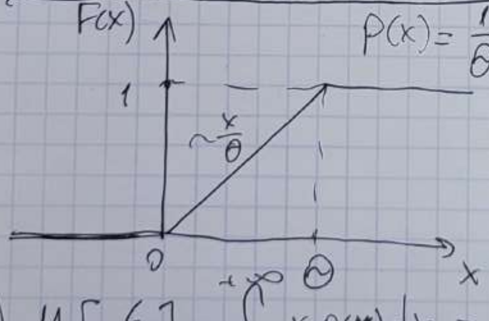
$$\tilde\theta_1 = 2\bar{x}, \quad \tilde\theta_2 = x_{min}, \quad \tilde\theta_3 = x_{max},$$

$$\tilde\theta_5 = \left(x_1 + \frac{\sum\limits_{k=2}^{n} x_k}{(n-1)} \right)$$

а) Проверить оценки на несмещенность/состоятельность

в) Найти наиболее эффективную оценку

$$p(x) = \frac{1}{\theta}\{(0, \theta)\}$$



$X_i \sim \xi$

а) $$M[\xi] = \int_{-\infty}^{+\infty} x\, p(x)\, dx = \int_0^{\theta} \frac{x}{\theta} dx = \frac{\theta}{2}$$

$$M[\xi^2] = \int_{-\infty}^{+\infty} x^2 p(x)\, dx = \int_0^{\theta} \frac{x^2}{\theta} dx = \frac{\theta^2}{3}$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12},$$

$$= \Theta \cdot \left[1 - \frac{1}{n+1}\right] = \frac{\Theta}{n+1} \text{ — } \underline{\text{смещенная}}$$

$$\tilde{\Theta}_2' = (n+1)\, x_{min} \qquad M[\tilde{\Theta}_2'] = \Theta$$

$$M[\tilde{\Theta}_2^2] = M[\tilde{\Theta}_2^2(\vec{X}_n)] = \int_0^\theta n\left(1 - \frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} y^2 \, dy =$$

$$= \left\{ t = 1 - \frac{y}{\theta} \right\} = -\int^{1} n\Theta^2 \left(t^{n-1} - 2t^n + t^{n+1}\right) dt =$$

$$= n\Theta^2 \left(\frac{1}{n} - 2\frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{2\Theta^2}{(n+1)(n+2)}$$

$$D[\tilde{\Theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\Theta^2}{(n+1)^2} = \frac{\Theta^2 n}{(n+1)^2(n+2)}$$

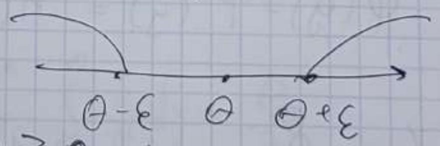
$$D[\tilde{\Theta}_2'] = (n+1)^2 D[\tilde{\Theta}_2] = \frac{\theta^2 n}{n+2} \not\to 0$$

дост. усл не работает

$\tilde{\Theta}_2'$ — по определению:

$$\forall \Theta > 0 \ \forall \varepsilon > 0 \quad P(|\tilde{\Theta}_2' - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|\tilde{\Theta}_2' - \Theta| \geq \varepsilon) \geq$$



$$\geq P(\tilde{\Theta}_2' \geq \Theta + \varepsilon) = P((n+1)x_{min} \geq \Theta + \varepsilon) =$$

$$= P\left(x_{min} \geq \frac{\Theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\Theta+\varepsilon}{n+1}\right) =$$

$$= 1 - \left(1 - \left(1 - F\left(\frac{\Theta+\varepsilon}{n-1}\right)\right)^n\right) = \left(1 - F\left(\frac{\Theta+\varepsilon}{n+1}\right)\right)^n =$$

$$= \left\{\varepsilon : \frac{\Theta+\varepsilon}{n+1} < \Theta\right\} = \left(1 - \frac{\Theta+\varepsilon}{\Theta} \cdot \frac{1}{n+1}\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\Theta+\varepsilon}{\Theta}} \text{ —}$$

— не является состоятельной

$\tilde{\theta}_2: P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) = P(\tilde{\theta}_2 < \theta - \varepsilon) +$
$+ \underbrace{P(\tilde{\theta}_2 > \theta + \varepsilon)}_{=0} = P(x_{min} < \theta - \varepsilon) =$
$= \Phi(\theta - \varepsilon) = 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n = 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n \to \infty]{} 1$

$\Rightarrow$ несовместная.

3) $\tilde{\theta}_3 = X_{max}$. $\tilde{\Phi}(y) = (F(y))^n$

$\tilde{\varphi}(y) = n(F(y))^{n-1} p(y) = n\left(\frac{y}{\theta}\right)^{n-1} \theta^{-1} = \frac{n y^{n-1}}{\theta^n}$

$M[\tilde{\theta}_3(\vec{x}_n)] = \int_{-\infty}^{+\infty} y \cdot \tilde{\varphi}(y) dy = \int_0^{\theta} n \cdot \frac{y^n}{\theta^n} dy = \frac{n}{n+1}\theta$

— смещенная.

$\tilde{\theta}'_3 = \frac{n+1}{n} X_{max}$

$M[\tilde{\theta}^2(\vec{x}_n)] = \frac{n y^{n+2}}{n+2\, \theta^n}\Big|_0^{\theta} = \frac{n}{n+2}\theta^2$

$D[\tilde{\theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$

$D[\tilde{\theta}'_3] = \frac{(n+1)^2}{n^2} \cdot D[\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \xrightarrow{n \to \infty} 0$

$\Rightarrow$ по достат. усл.: $\tilde{\theta}'_3$ — сост.

Поопр.: $\tilde{\theta}_3$:

$\forall \theta > 0 \ \forall \varepsilon > 0 \quad P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) =$
$= P$

$= P(x_{max} \le \theta - \varepsilon) + \underbrace{P(x_{max} \ge \theta + \varepsilon)}_{=0} =$

$= (F(\theta - \varepsilon))^n$;

$0 < \varepsilon < \theta$: $\quad P(|\tilde{\theta}_3 - \theta| \ge \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n =$

$= \left(1 - \frac{\varepsilon}{\theta}\right)^n \xrightarrow{n \to \infty} 0$

$\varepsilon \ge 0$: $\quad P(|\tilde{\theta}_3 - \theta| \ge \varepsilon) = 0^n = 0 \xrightarrow{n \to \infty} 0$

$\Rightarrow \tilde{\theta}_3$ – по опр. состоятельн

$\tilde{\theta}_3'$ – по определению: (ПОДАРОК)

$P(|\tilde{\theta}_3' - \theta| \ge \varepsilon) = P\left(\left(\frac{n+1}{n}\right) x_{max} \le \theta + \varepsilon\right) +$

$+ P\left(x_{max}\left(\frac{n+1}{n}\right) \ge \theta + \varepsilon\right) =$

$= 1 - P\left(x_{max} < \frac{\theta + \varepsilon}{n+1}\right) + P\left(x_{max} \le \frac{\theta - \varepsilon}{n+1} \cdot n\right) =$

$= \underbrace{1 - \left(F\left((\theta + \varepsilon) \cdot \frac{n}{n+1}\right)\right)^n}_{\mathrm{I}} + \underbrace{\left(F\left((\theta - \varepsilon) \cdot \frac{n}{n+1}\right)\right)^n}_{\mathrm{II}}$

$\boxed{\mathrm{I}}$: • при $\varepsilon \ge \frac{\theta}{n}$: $\quad 1 - \left(F\left((\theta - \varepsilon) \frac{n}{n+1}\right)\right)^n = 0 \Rightarrow$

$\Rightarrow \lim_{n \to \infty} \mathrm{I} = 0$

$\boxed{\mathrm{II}}$: $\left(F\left((\theta - \varepsilon) \cdot \frac{n}{n+1}\right)\right)^n = 0, \quad \varepsilon \ge 0$

$0 < \varepsilon < \theta$: $\quad \left(\frac{\theta - \varepsilon}{\theta} \cdot \frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n \cdot \left(1 - \frac{\varepsilon}{\theta}\right)^n \xrightarrow{n \to \infty} e^{-1} \cdot$

$= 0 \Rightarrow \lim\limits_{n\to\infty} \text{II} = 0$

$\Rightarrow P\left(\left|\widetilde{\theta}_3' - \theta_3\right| \geq \varepsilon\right) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0$

$\Rightarrow \theta_3'$ – <u>состоятелен по определению</u>

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4) $\widetilde{\theta}_5 = X_1 + \frac{1}{n-1} \sum\limits_{k=2}^{n} X_k$ $\qquad M[\xi]$ $\, \parallel$

$M[\widetilde{\theta}_5] = M\left[X_1 + \frac{1}{n-1} \sum\limits_{k=2}^{n} X_k\right] = M[X_1] + \frac{1}{n-1} \sum\limits_{k=2}^{n} M[X_k] =$

$= \frac{\theta}{2} + \frac{1}{n-1} \cdot (n-1) \cdot \frac{\theta}{2} = \theta$ – <u>несмещ</u>

$D[\widetilde{\theta}_5] = D\left[X_1 + \frac{1}{n-1} \sum\limits_{k=2}^{n} X_k\right] = D[\xi] + \frac{1}{n-1} D[\xi] =$

$= \frac{\theta^2}{12} \cdot \frac{n}{n-1} \not\xrightarrow[n\to\infty]{} 0$

$\xi_n \xrightarrow{P} \xi, \; \eta_n \xrightarrow{P} \eta \Rightarrow \xi_n + \eta_n \xrightarrow{P} \xi + \eta$

$X_1 \xrightarrow{P} \xi = X_1, \quad \frac{1}{n-1} \sum\limits_{k=2}^{n} X_k \longrightarrow M[\xi] = \frac{\theta}{2}$

по ЗБЧ Хинчина $\{X_k\}_{k=2}^{n}$ – н.з., <u>одинрасп</u>

$\Rightarrow \widetilde{\theta}_4 \longrightarrow X_1 + \frac{\theta}{2} \Rightarrow$ не явл. состоятел

<u>сл.вел</u>

б) $\tilde{\theta}_1 = 2\bar{x} \to D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$

$\tilde{\theta}_3' = \frac{n+1}{n} x_{max} \to D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)}$

$\frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n}$, $n>1$ - выполнено

$\Rightarrow \tilde{\theta}_3'$ - эффективнее θ_1.