(T3) Сл.Вел : $p(x) = \begin{cases} e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

причем $\theta > 0$.

По выборке: $n = 3$ найти оценки θ:

$\tilde{\theta}_1 = \overline{x}$, $\tilde{\theta}_3 = X_{(2)}$ (второй член вар. ряда)

$$M[\xi] = \int_0^{+\infty} x \cdot e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta}\,dx = \frac{1}{\theta}\Big(-\theta x e^{-\frac{x}{\theta}}\Big|_0^{+\infty} + \theta\int_0^{+\infty} e^{-\frac{x}{\theta}}\,dx\Big) = \int_0^{+\infty} e^{-\frac{x}{\theta}}\,dx = \theta e^{-\frac{x}{\theta}}\Big|_0^{+\infty} = \theta$$

$$M[\xi^2] = \int_0^{+\infty} x^2 e^{-\frac{x}{\theta}} \frac{1}{\theta}\,dx = \frac{1}{\theta}\Big(-\theta x^2 e^{-\frac{x}{\theta}}\Big|_0^{+\infty} + 2\theta \int_0^{+\infty} x e^{-\frac{x}{\theta}}\,dx\Big) = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = \theta^2$$

а) $\tilde{\theta}_1$: $M[\tilde{\theta}_1(\vec{x}_n)] = M\Big[\Big(\sum_{i=1}^{3} x_i\Big) \cdot \frac{1}{3}\Big] = \frac{1}{3} \cdot 3M[\xi] = \theta$

$\Rightarrow$ несмещенная.

$\tilde{\theta}_3$: $\varphi(y) = n p(y) \cdot C_{n-1}^{k-1} (1 - F(y))^{n-k} (F(x))^{k-1} =$

$= \left\{ \begin{matrix} k=2 \\ n=3 \end{matrix} \right\} = \left(6 \cdot e^{-\frac{2x}{\theta}} - 6 e^{-\frac{3x}{\theta}}\right) \frac{1}{\theta}$

$$M[\tilde{\Theta}_3] = M[X_{(2)}] = \int_0^{\infty} y\,\varphi(y)\,dy =$$

$$= \frac{6}{\theta}\left(-\frac{\theta}{2} y e^{-\frac{2y}{\theta}}\Big|_0^{+\infty} + \frac{\theta}{3} y e^{-\frac{3y}{\theta}}\Big|_0^{+\infty} + \frac{\theta}{2}\int_0^{+\infty} e^{-\frac{3y}{\theta}} dy - \frac{\theta}{3}\int_0^{+\infty} e^{-\frac{3y}{\theta}} dy\right)$$

$$= \frac{5\theta}{6} \Rightarrow \tilde{\Theta}_3 \text{ – смещённая}$$

$$\tilde{\Theta}_3' = \frac{6}{5}\tilde{\Theta}_3 \text{ – несмещённая}$$

в) $\tilde{\Theta}_1$: $D[\tilde{\Theta}_1] = D\left[\frac{1}{3}\sum_{n=1}^{3} x_i\right] = \frac{1}{9}\cdot 3D[\xi] = \frac{\theta^2}{3}$

$\tilde{\Theta}_3$: $M[\tilde{\Theta}_3^2] = M[X_{(2)}^2] = \frac{6}{\theta}\int_0^{+\infty}\left(x^2 e^{-\frac{2x}{\theta}} - x^2 e^{-\frac{3x}{\theta}}\right)dx$

$$= \cancel{\frac{6}{\theta}\cdot\left(\frac{\theta^2}{4}\right.} = \frac{6}{\theta}\left(\int_0^{+\infty} x e^{-\frac{2x}{\theta}} dx - \frac{2}{3}\int_0^{+\infty} x e^{-\frac{3x}{\theta}} dx\right) =$$

$$= \frac{6(27\theta^2 - 8\theta^2)}{2\cdot 3\cdot 2\cdot 9} = \frac{19}{18}\theta^2$$

$$D[\tilde{\Theta}_3] = \frac{13\,\theta^2}{\cancel{4}\,36} \Rightarrow D[\tilde{\Theta}_1] < D[\tilde{\Theta}_3'] \Rightarrow$$

$\Rightarrow \tilde{\Theta}_1$ – более эффективна

г) $I(\theta) = M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial \theta}\right)^2\right] = M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right)^2\right] =$

$$= M\left[\left(\frac{\theta}{\partial\theta}\left(\ln\left(\frac{e^{-\frac{x}{\theta}}}{\theta}\right)\right)\right)^2\right] = M\left[\left(\frac{\theta}{\partial\theta}\left(-\frac{x}{\theta} - \ln\theta\right)\right)^2\right]$$

$$= M\left[\left(\frac{x}{\theta^2}-\frac{1}{\theta}\right)^2\right] = \frac{1}{\theta^4}M[\xi^2] - \frac{2}{\theta^3}M[\xi]$$

$$+\frac{1}{\theta^2} = \frac{2}{\theta^2} - \frac{2}{\theta^2} + \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$\tilde{\theta}_1$: $D[\tilde{\theta}_1] = \frac{\theta^2}{3}$ – огр на $\forall$ отрезке из $(0,+\infty)$; $\left([a,b] \subset (0,+\infty) \Rightarrow \forall \theta \in [a,b] \hookrightarrow \frac{\theta^2}{3} \le \frac{b^2}{3}\right)$

$\Rightarrow \theta_1^2$ – регулярна по дост. условию

$\tilde{\theta}_3'$: $D[\tilde{\theta}_3'] = \frac{13\theta^2}{25}$ – огр. на $\forall$ отрезке из $(0,+\infty)$

$\Rightarrow \tilde{\theta}_3'$ – регулярна по дост. условию.

Модель:

I): $f(x,\theta) = p(x,\theta)$ – непр. дифф по θ на $(0,+\infty)$

II): $\frac{\partial}{\partial\theta} p(x,\theta) = \frac{x e^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2}$

$$\int_0^{+\infty} \frac{\partial}{\partial\theta} p(x,\theta)\,dx = \int_0^{+\infty}\left(\frac{xe^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2}\right)dx = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\int_0^{+\infty} p(x,\theta)\,dx = \int_0^{+\infty}\frac{\partial}{\partial\theta} f(x,\theta)\,dx$$

III): $I(\theta) = \frac{1}{\theta^2} \in C[(0, +\infty)]$,

$I(\theta) > 0 \quad \forall \theta \in (0, +\infty)$

~~I+II+III~~ $I + II + IV \to$ <u>рег. модель</u>

<u>Исследование на эффективность:</u>

$\widetilde{\theta}_1$ — <u>регул. модель</u> $\Rightarrow$

$\forall \theta \in (0, +\infty) \to D[\widetilde{\theta}_1] \geq \frac{1}{nI(\theta)}$

$\frac{\theta^2}{3} \geq \frac{\theta^2}{3}$, т.е. $D[\widetilde{\theta}_1] = \frac{1}{nI(\theta)} \Rightarrow$

$\Rightarrow$ по дост. усл. $\widetilde{\theta}_1$ — <u>эфф.</u>

$\widetilde{\theta}_3{}' \neq \widetilde{\theta}_1 \Rightarrow \widetilde{\theta}_3{}'$ — не <u>эффективна</u>

(<u>но $\exists$ ч! эфф. оценки</u>)