$\boxed{T5}$ $\xi \sim R[\theta, 2\theta)$, $\rho(x,\theta) = \frac{1}{\theta}\{[\theta, 2\theta]\}$

а) Выборка n, Найти оценки θ ОММ и ММП

$\boxed{ОММ}$: $\alpha_1 = M_\xi = \int\limits_{-\infty}^{+\infty} x\rho(x)dx = \frac{1}{\theta}\int\limits_{\theta}^{2\theta} x\,dx = \frac{3\theta}{2}$

$\Rightarrow \frac{3\theta}{2} = \overline{X} \Rightarrow \tilde{\theta} = \frac{2}{3}\overline{X}$

• несмещенный: $M[\tilde{\theta}] = M[\frac{2}{3}\overline{X}] = \frac{2}{3}M_\xi = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$

$\Rightarrow$ оценка несмещена

• состоятельность: $D[\tilde{\theta}] = D[\frac{2}{3}\overline{X}] = \frac{4}{9}D[\overline{X}] = \frac{4}{9n}D_\xi$

$M_{\xi^2} = \int\limits_{\theta}^{2\theta}\frac{1}{\theta}x^2dx = \frac{1}{\theta}\frac{x^3}{3}\Big|_{\theta}^{2\theta} = \frac{7\theta^2}{3}$

$D[\tilde{\theta}] = \frac{4}{9n}\left(\frac{7\theta^2}{3} - \frac{9\theta^2}{4}\right) = \frac{4\theta^2}{12\cdot 9n} \xrightarrow[n\to\infty]{} 0$

Оценка $\tilde{\theta}$ – сост, по дост. усл. состоятельности

$\boxed{ОМП}$ $\overset{ф\text{-}я\ правдоподобия}{L} = \frac{1}{\theta^n}\{\theta \le x_i \le 2\theta, \forall x_i\}$

$X_{max} = 2\theta \Rightarrow \tilde{\theta} = \frac{X_{max}}{2}$

$H(x) = (F(x))^n = \left(\int\limits_0^x \frac{1}{\theta}dx\right)^n = \left(\frac{x}{\theta} - 1\right)^n$

$M[\hat{\theta}] = M[\frac{X_{max}}{2}] = \frac{1}{2}\int\limits_{\theta}^{2\theta}\frac{n}{\theta}\left(\frac{x}{\theta}-1\right)^{n-1}x\cdot dx = \frac{1}{2}2\frac{Q_n+\theta}{n+1} = \theta\frac{2n+1}{n+1}$

$$\tilde{\Theta}^* = \frac{n+1}{2n+1}\tilde{\Theta} = \frac{n+1}{2n+1}\frac{X_{max}}{2}$$

$M[\tilde{\Theta}^*] = \Theta$ – исправлен оценка несмещена

$$D[\tilde{\Theta}^*] = D\left[\frac{n+1}{2n+1}\frac{X_{max}}{2}\right] = \left(\frac{n+1}{2n+1}\right)^2 D[X_{max}] =$$

$$= \left(\frac{n+1}{2n+1}\right)^2 \left(M[X^2_{max}] - M^2[X_{max}]\right)\left(\frac{2n+1}{n+1}\right)^2$$

$$M[X^2_{max}] = \int_\Theta^{2\Theta} x^2 \frac{n}{\Theta}\cdot\left(\frac{x}{\Theta}-1\right)^{n-1} dx = 2\Theta^2\frac{2n^2+4n+1}{(n+2)(n+1)}$$

$$D[\tilde{\Theta}^*] = \left(\frac{n+1}{2n+1}\right)^2 \cdot \frac{4n^2+8n+2}{(n+1)(n+2)}\Theta^2 - \Theta^2 = \frac{n\Theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

$\tilde{\Theta}^*$ – состоятельна по дост. условию

$$D[\tilde{\Theta}_1] = \frac{\Theta^2}{27n} \qquad D[\tilde{\Theta}_1] > D[\tilde{\Theta}_2^*]$$

$$D[\tilde{\Theta}_2^*] = \frac{n\Theta^2}{(2n+1)^2(n+2)} \qquad \tilde{\Theta}_2^* \text{ – эффективнее}$$

$$X_i \in [\Theta, 2\Theta], \quad \frac{X_i}{\Theta} \in [1, 2]$$

$$\Phi(X_{max}) = (F(x))^n = \left(\int_1^x dx\right)^n = (x-1)^n$$

$$X = \Theta, \quad \sqrt[n]{0.025} + 1 < x < \sqrt[n]{0.975} + 1$$

$$\sqrt[n]{0.025} + 1 < \frac{X_{max}}{\Theta} < \sqrt[n]{0.975} + 1$$

$$\frac{X_{max}}{\sqrt[n]{0.975}+1} < \Theta < \frac{X_{max}}{\sqrt[n]{0.025}+1}$$

$$\sqrt{n} \cdot \frac{g(\tilde{\alpha}) - g(\alpha)}{\sigma(\alpha)} \rightsquigarrow N(0,1)$$

$$\sigma(\alpha) = \sqrt{\nabla^T g \cdot K \nabla g} = \frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}$$

$$\sqrt{n} \cdot \frac{\frac{2}{3}\tilde{\alpha} - \frac{2}{3}\alpha_1}{\frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}} = \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}} \cdot \underbrace{\frac{\frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}}{\frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}}_{\to 1}$$

$$\rightsquigarrow N(0,1).$$

$$\Rightarrow \quad -1{,}96 < \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}} < 1.96$$

$$-\frac{1.96}{n} \cdot \frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2} + \tilde{\theta} < \theta < \frac{1.96}{\sqrt{n}}\,\frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2} + \tilde{\theta}$$
